Final 2024-2025 Fall

1

Let V be an inner product space over $\mathbb{R}$, and let T be a function (not necessarily linear) such that $T(0) = 0$ and $\|T(v) - T(w)\| = \|v - w\|$ for all $v, w \in V$. Let $\dim(V) = n$.

(a) Show that $\|T(v)\| = \|v\|$ for all $v \in V$. (5 pts)

(b) Show that $\langle T(v), T(w)\rangle = \langle v, w\rangle$ for all $v, w \in V$. (5 pts)

(c) If $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V, show that $\{T(e_1), \ldots, T(e_n)\}$ is an orthonormal basis as well. (5 pts)

(d) Show that T is linear. (5 pts)

2

Perform the Gram-Schmidt process on the following basis for $\mathbb{R}^3$:

$$\{(1,1,1)^t, (0,1,1)^t, (1,3,0)^t\}$$

(8 pts)

3

Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & 8 & 0 \\ 3 & -1 & 6 & 0 \\ -1 & 0 & -5 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \in M_{4\times 4}(\mathbb{R}).$$

- Determine whether the matrix is diagonalizable. (6 pts)
- If it is not diagonalizable, find its Jordan canonical form. Specifically, find an invertible matrix Q and a Jordan matrix B such that $Q^{-1}AQ = B$. (10 pts)

4

Let $B \in M_{n\times n}(\mathbb{C})$ be a matrix where the i-th row is e_k for some integers $1 < i \leq n$ and $1 \leq k \leq n$. Prove that

$$\det(B) = (-1)^{i+k} \det(\tilde{B}_{ik}),$$

You are only allowed to use the definition of the determinant that applies expansion along the first row. (16 pts)

5

Consider the matrix

$$A = \begin{pmatrix} a & 0 & 0 & -c \\ 0 & -d & b & 0 \\ 0 & c & a & 0 \\ -b & 0 & 0 & -d \end{pmatrix}.$$

Find an invertible matrix $S \in M_{4\times 4}(\mathbb{R})$ and matrices $H_1, H_2 \in M_{2\times 2}(\mathbb{R})$ such that

$$SAS^{-1} = H_1 \oplus H_2.$$

(12 pts)

6

Apply an orthogonal transformation to transform the quadratic form

$$xy + yz + zx = 1$$

into a standard form. (8 pts)

7

Let $A \in M_{m\times n}(\mathbb{C})$ and $B \in M_{n\times m}(\mathbb{C})$. Prove that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible. (8 pts)

8

For a matrix $A \in M_{n\times n}(\mathbb{C})$ with distinct eigenvalues $\lambda_1, \dots, \lambda_n$, consider the linear operator T_A on $M_{n\times n}(\mathbb{C})$ defined by

$$T_A(B) = AB - BA.$$

Prove that $\lambda_i - \lambda_j$ is an eigenvalue of T_A for $1 \leq i < j \leq n$. (12 pts)