# Open Issue: Mathematical Foundation of Multi-level Economic System of Heterogeneous Interacting Agents

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**Abstract**

We consider the mathematical abstraction of multi-level dynamical system for complex economic system with heterogeneous interacting agents (HIA). We are concerned with the mathematical representation of micro-level states and dynamics as well as heterogeneity among economic agents. In particular, under the presumption of local time-invariance for economic system, such that the structure of economic system remains unchanged within a short time window, we explore the relationship between micro-level and macro-level dynamics, and then provide inference of macro-level dynamics from micro-level information.

Keywords: multi-level dynamical system, economic system, heterogeneous interacting agent, mathematical foundation

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## 1 Introduction

Our economy accepts diversification among economic agents (firms, households, etc.). Economic agents are not isolated, instead, they connect and interact with each other in one way or another along time horizon. Thus, it is natural to understand our economy from the perspective of dynamical system of heterogeneous interacting agents (HIA). This perspective implies complexity that agents' interaction in micro-level leads to macro-level dynamics. In other words, economic system can be regarded as multi-level dynamical system where micro-level interaction among heterogeneous agents results in macro-level complex dynamics.

Economic studies that admit the viewpoint of multi-level economic system start to prevail in different strands in economics. To name a few, existing economic literature demonstrates studies in complexity economics and econophysics [8, 9], economic networks [1, 3], monetary policy [11, 2, 7], fiscal policy [4], and economic forecasting [6]. The prevailing of economic studies under the framework of multi-level economic system calls for theoretical support and formalization that provide systemic understanding on multi-level economic system, which, to our knowledge, is still a missing piece in economic research. In view of this, our work aims at providing, at the first attempt, a formal framework of multi-level dynamical system for economic system, with particular concern on the linkage between micro-level and macro-level dynamics.

From a mathematical perspective, our work can be regarded as an application and concretization of current mathematical study of multi-level dynamical system, e.g., see [10]. Mathematical theories are usually somewhat abstract in the sense that a direct application to economic study is not immediate. In this regard, our work contributes to filling the gap between mathematical theory and its application in economics.

Our paper is organized as following. Section 2 provides as background the mathematical setting and notations used in our work. Section 3 investigates properties related to micro-level dynamics, which lays the foundation for deriving macro-level dynamics. Section 4 then provides formulation of macro-level dynamics derived from micro-level dynamics. Section 5 concludes.

## 2 Mathematical Background

Consider an economic system with $N$ economic agents for discrete-time dynamics of $t \in \{1, \dots, T\}$ periods. Suppose at the beginning of period $t$, each agent $i \in \{1, \ldots, N\}$ has its state[1] $a_{i,t} \in \mathbb{R}$ carried on from the end of period $t-1$. Agent $i$ conducts activities at period $t$. As a result, the agent has its updated state $a_{i,t+1} \in \mathbb{R}$ at the end of period $t$ or at the beginning of period $t+1$. Denote agent $i$'s micro-level dynamics from $a_{i,t}$ to $a_{i,t+1}$ by the mapping $f_{i,t} : a_{i,t} \mapsto a_{i,t+1}$.

The collection of the state of each agent is regarded as the micro-level state of the system, denoted by the column vector $\mathbf{a_t} := (a_{1,t}, \ldots, a_{N,t})^\top \in \mathbb{R}^N$ at the beginning of period $t$. The collection of agents' micro-level dynamics is regarded as the micro-level dynamics of the system, denoted by $\mathbf{f}_t := (f_{1,t}, \ldots, f_{i,t}, \ldots, f_{N,t})^\top$.

The corresponding macro-level state of the system at the beginning of period $t$, denoted by $\widehat{a}_t \in \mathbb{R}$, is derived from the micro-level state $\mathbf{a}_t$ by some form of aggregation, e.g., a summation of agents' states such that $\widehat{a}_t = \sum_{k=1}^N a_{k,t}$. We assume this form of aggregation is fixed for all periods and denote it by $\pi : \mathbf{a_t} \mapsto \widehat{a}_t$ for any period $t$. Denote the dynamics of the macro-level state from $\widehat{a}_t$ to $\widehat{a}_{t+1}$ by the mapping $\widehat{f}_t : \widehat{a}_t \mapsto \widehat{a}_{t+1}$.

The following diagram illustrates the dynamics of this multi-level economic system along time horizon.

$$
\begin{tikzcd}
\cdots \arrow[r] & \widehat{a}_{t-1} \arrow[r, "\widehat{f}_{t-1}"] & \widehat{a}_t \arrow[r, "\widehat{f}_t"] & \widehat{a}_{t+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathbf{a}_{t-1} \arrow[r, "\mathbf{f}_{t-1}"] \arrow[u, "\pi"'] & \mathbf{a}_t \arrow[r, "\mathbf{f}_t"] \arrow[u, "\pi"] & \mathbf{a}_{t+1} \arrow[r] \arrow[u, "\pi"'] & \cdots
\end{tikzcd}
$$

Figure 1: dynamics of multi-level economic system.

As shown in Figure 1, the macro-level state is linked with the micro-level state, and the macro-level dynamics is linked with the micro-level dynamics. Understanding the relationship between micro-level dynamics and macro-level dynamics gives us hints on how to utilize micro-level information to obtain inference on macro-level systemic behavior.

## 3 Properties of Micro-level Dynamics

Consider the micro-level dynamics at period $t$, where individual agent $i$'s dynamics has the mapping $f_{i,t} : a_{i,t} \mapsto a_{i,t+1}$. Here we explore fundamental

[1]Agent's state can be regarded as economic property that can be quantified, e.g., household's income, firm's revenue, etc.

properties of $f_{i,t}$, which can be used to classify micro-level dynamics into different categories.

### 3.1 Time-variance and Time-invariance

The mapping $f_{i,t}$ can be *time-variant* or non-autonomous, i.e., $f_{i,t}$ changes along the time horizon such that in general $f_{i,t} \neq f_{i,t'}$ for $t \neq t'$. If $f_{i,t}$ remains unchanged along the time horizon, i.e., $f_{i,t}(a) = f_{i,t'}(a)$ for any $t, t'$, and $a$, we call it *time-invariant* or autonomous, see [5]. In this case we often omit the explicit time in the notation and write $f_i$ for simplicity.

The recent financial crisis at 2007 and the following "great recession" convince us that our economy is time-variant in structure for the time scale of decades. On the other hand, when the time scale is relatively small, it is reasonable to argue the main structure of the economy remains stable. To make a combination between these two aspects, we presume that economic system is globally time-variant in the medium or long run, while it is locally time-invariant in the short run. To be specific, given a time window $[t_0, t_1]$, the mapping $f_{i,t}$ is called *locally time-invariant*, if $f_{i,t}$ and $f_{i,t'}$ have the same form for any $t, t' \in [t_0, t_1]$.

### 3.2 Nonlinearity and Linearity

Suppose $f_{i,t}$ has a general form such that

$$a_{i,t+1} = f_{i,t}(a_{1,t}, \dots, a_{N,t}), \tag{1}$$

in which $f_{i,t}$ can be linear or nonlinear. We consider linearity as a special case, with the specific form such that

$$a_{i,t+1} = f_{i,t}(a_{1,t}, \dots, a_{N,t}) = \sum_{j=1}^{N} f_{i,j}^{t} \cdot a_{j,t}, \tag{2}$$

where $f_{i,j}^{t} \in \mathbb{R}$.

### 3.3 Weak Heterogeneity and Strong Heterogeneity

In economic system of heterogeneous interacting agents, scenarios of heterogeneity emerge on the agents' states and on the agents' behavioral rules. In general, agents may have different micro-level states, i.e. $a_{i,t} \neq a_{j,t}$, for $i, j \in \{1, \dots, N\}$. This heterogeneity in agents' states are called *weak heterogeneity*, which is implicitly assumed in economic system of heterogeneous interacting agents.

In our work we highlight another type of heterogeneity for diversifying agents' micro-level dynamics. We define it as *(strong) heterogeneity* at period $t$ if $f_{i,t} \neq f_{j,t}$, for some $i, j \in \{1, \dots, N\}$.

The concept of *homogeneity* at period $t$, as a special case of heterogeneity, emerges when $f_{i,t} = f_{j,t}$, for any $i, j \in \{1, \dots, N\}$. This depreciated concept is mainly related with the assumption of representative agent in main-stream economic models, which has been well recognized on its failure to understand market fluctuation and economic crisis.

One remark on heterogeneity is that strong heterogeneity implies weak heterogeneity: Suppose at period $t$ that $a_{i,t} = a_{j,t}$, for $i, j \in \{1, \dots, N\}$ and strong heterogeneity with $f_{i,t} \neq f_{j,t}$, for some $i, j \in \{1, \dots, N\}$. Then there exists weak heterogeneity afterwards such that $a_{i,t+1} = f_{i,t}(a_{i,t}) \neq a_{j,t+1} = f_j(a_{j,t})$, for some $i, j \in \{1, \dots, N\}$, and so on.

**Example 1** (Production Firm)**.** Assume firm $i$ in a production sector, with $a_{i,t}$ as its equity (net asset) at the beginning of period $t$. The firm conducts its production based on its current equity $a_{i,t}$ and earns profit $y_{i,t}$ at period $t$ that is accumulated with its equity by the end of period $t$. The firm's micro-level dynamics on equity for period $t$ can be termed as $a_{i,t+1} = a_{i,t} + y_{i,t} = a_{i,t} \cdot (1 + r_{i,t})$, where $y_{i,t}$ and $r_{i,t}$ are the firm's profit and profit rate respectively.

We may start with a naive economic model such that firms do not impact each other by their behavioral rules. By assuming that the firm's production is dependent on its equity on hand, one may regard the firm's profit $y_{i,t}$ and profit rate $r_{i,t}$ as functions of its equity $a_{i,t}$, with the form of $y_{i,t}(a_{i,t})$ and $r_{i,t}(a_{i,t})$ respectively. Under this assumption, the firm's micro-level dynamics on equity can be termed as $a_{i,t+1} = f_{i,t}(a_{i,t}) = a_{i,t} + y_{i,t}(a_{i,t}) = a_{i,t} \cdot [1 + r_{i,t}(a_{i,t})]$. We may assume that the firm's profit function is linear such that:

$$a_{i,t+1} = (1 + r_i^t) \cdot a_{i,t}, \tag{3}$$

where $r_i^t \in \mathbb{R}$ is the firm's rate of profit at period $t$. We can calculate $r_i^t$ by the data of firm's financial statements.

Here we show by a sample of 870 firms listed in Tokyo Stock Exchange for the year of 1980 to 2012 the micro-level and macro-level dynamics of firm's equity. Figure 2 shows the macro-level dynamics of firms' aggregate equity, while Figure 3(a) and 3(b) show the distribution of the annual profit rates $r_i^t$ for a sample of 870 firms listed in Tokyo Stock Exchange for the year of 1980 to 2012 and for the year of 1980 to 1982 respectively.

Figure 3(a) demonstrates that the distribution for $r_i^t$ does not have exactly the same form. This observation implies time-variance in the micro-level dynamics under the linear form. That is, when linearity is preserved

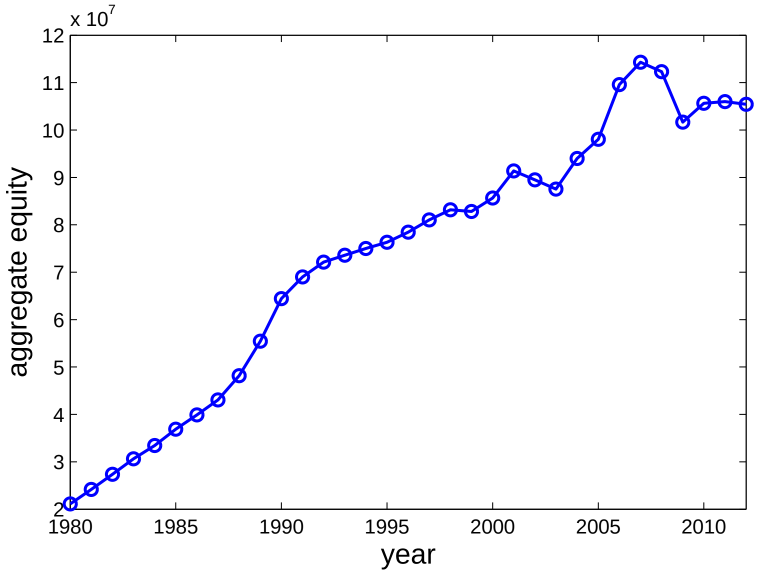
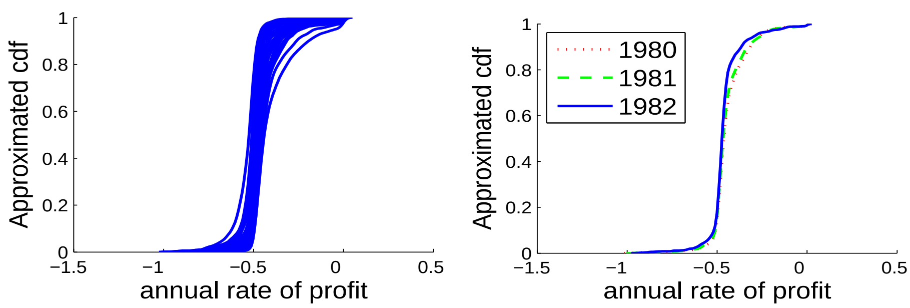



Figure 2: Firms' aggregate equity (in million Yen): year 1980 to 2012.

in the micro-level dynamics, values of $r_i^t$ and thus coefficients $f_{i,j}^t$ in (2) change along the time. In this case, we have micro-level dynamics with time-variance, linearity, and strong heterogeneity. The time-variance works on the coefficients under the linear form of the micro-level dynamics.

In particular, the shape-preserving shown in Figure 3(b) suggests the change on the coefficients is relatively small among the year of 1980, 1981, and 1982. Thus, we may approximate the time-variant micro-level dynamics with linearity and heterogeneity by time-invariant micro-level dynamics under a short time span (of 3 time points) with linearity and heterogene-




(a) 1980 to 2012

(b) 2010 to 2012

Figure 3: Firms' annual profit rates

ity. Notice this approximation contains implicitly the trade-off between tractability of time-invariance and accuracy of time-variance, which requires cautions to handle when dealing with complex dynamics in economics. To obtain tractability, we assume micro-level dynamics is with time-invariance, linearity, and heterogeneity in the following context to derive macro-level dynamics.

## 4 Macro-level Dynamics

Macro-level state is derived from micro-level states by aggregation $\pi$. We focus on the following form of aggregation:

**Assumption 1** (Linear Aggregation)**.** *Assume the aggregation $\pi$ is fixed at each period and has the linear form such that*

$$\widehat{a}_t = \sum_{i=1}^{N} \pi_i \cdot a_{i,t}, \quad \textit{with} \quad \pi_i \in \mathbb{R}.$$

For the reason of tractability, we focus on the following form of micro-level dynamics:

**Assumption 2** (Micro-level Dynamics)**.** *Assume micro-level dynamics is with time-invariance, linearity, and heterogeneity.*

Next, we deliver from micro-level states and dynamics mathematical inference of macro-level dynamics. Without loss of generality, in the following context we assume the time point to consider is at period $t$, with period $t+1$ as the one-period-ahead future. From the perspective of economic forecasting, we observe the data for the micro-level dynamics up to period $t$ and are interested in deriving the forecast on the one-period-ahead out-of-sample macro-level dynamics.

Suppose agent $i$'s micro-level dynamics following the linear form in (2). Denote the column vector $\mathbf{a_t} = (a_{1,t}, \dots a_{N,t})^\top$, the row vector $\boldsymbol{\pi} = (\pi_1, \dots, \pi_N)$, and the matrix $\mathbf{F} = [f_{i,j}]$, the micro-level dynamics has the condense form:

$$\mathbf{a_{t+1}} = \mathbf{F} \cdot \mathbf{a_t}, \tag{4}$$

and the macro-level dynamics has the condense form:

$$\hat{a}_{t+1} = \boldsymbol{\pi} \cdot \mathbf{a_{t+1}} = \boldsymbol{\pi} \cdot \mathbf{F} \cdot \mathbf{a_t}. \tag{5}$$

The question is whether we can derive from micro-level the function $\widehat{f}$ such that $\widehat{a}_{t+1} = \widehat{f}(\widehat{a}_t)$. [2]

[2] As the time point to consider is at period $t$, we drop the time index and use the shorthand $\widehat{f}$.

Since $\widehat{a}_t$ and $\widehat{a}_{t+1}$ are scalar, and everything is linear, the function $\widehat{f}$ will also be a scalar function, i.e., it will be equivalent to multiplication by some number $\lambda$. The following theorem gives a criterion to derive $\widehat{f}$.

**Theorem 1.** *There exists a closed-form description $\hat{a}_{t+1} = \hat{f}(\hat{a}_t)$ for the macro-level dynamics if and only if $\boldsymbol{\pi}$ is a left eigenvector of the micro-level dynamics $\mathbf{F}$, i.e. $\boldsymbol{\pi} \cdot \mathbf{F} = \lambda \boldsymbol{\pi}$ for some scalar $\lambda$. In this case the macro-level dynamics obeys $\hat{a}_{t+1} = \lambda \hat{a}_t$.*

The proof of Theorem 1 is straightforward, and is thus omitted here.

Consider Example 1, the firm's micro-level dynamics on equity is assumed to follow the form $a_{i,t+1} = (1 + r_i) \cdot a_{i,t}$ derived from (3). The matrix $\mathbf{F} = [f_{i,j}]$ by (5) is diagonal with the explicit form:

$$\mathbf{F} = \begin{bmatrix} 1+r_1 & 0 & \dots & 0 \\ 0 & 1+r_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1+r_N \end{bmatrix} \tag{6}$$

The condition for applying Theorem 1 is not satisfied in this example. To see why, notice that profit rates for firms at each year are heterogeneous. The linear aggregation $\boldsymbol{\pi}$ is with the form of aggregation for all firms' equity, i.e., $\boldsymbol{\pi} = (1, 1, \dots, 1)_{1 \times N}$. A simple computation verifies that $\boldsymbol{\pi} \cdot \mathbf{F} \neq \lambda \boldsymbol{\pi}$ for any $\lambda$, when there exists $r_i \neq r_j$ for some $i, j \in \{1, \dots, N\}$, which is the case for firms' profit rates at each year shown in Example 1 [3]. Thus, in this case $\boldsymbol{\pi}$ is not a left eigenvector of $\mathbf{F}$.

By this consideration, we define the matrices

$$\mathcal{O}_k = \begin{pmatrix} \boldsymbol{\pi} \\ \boldsymbol{\pi} \cdot \mathbf{F} \\ \boldsymbol{\pi} \cdot \mathbf{F}^2 \\ \vdots \\ \boldsymbol{\pi} \cdot \mathbf{F}^{k-1} \end{pmatrix} \in \mathbb{R}^{k \times N}.$$

Assume $\boldsymbol{\pi} \neq 0$. Let $r$ be the smallest positive integer for which

$$\operatorname{rank}(\mathcal{O}_{r+1}) = \operatorname{rank}(\mathcal{O}_r). \tag{7}$$

Clearly, $r \leq N$. Then the next theorem shows that the macro-level dynamics follows a $r$th-order closed-form dynamics.

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[3] An alternative verification is that, with the form $\boldsymbol{\pi} = (1, 1, \dots, 1)_{1 \times N}$, $\boldsymbol{\pi} \cdot \mathbf{F} = \lambda \boldsymbol{\pi}$ if and only if $r_i = r_j$ for all $i, j \in \{1, \dots, N\}$.

**Theorem 2.** *The macro-level dynamics obeys the closed-form description*

$$\hat{a}_{t+1} = \lambda_{r-1}\hat{a}_t + \lambda_{r-2}\hat{a}_{t-1} + \cdots + \lambda_0\hat{a}_{t-r+1} \tag{8}$$

*where the quantities $\lambda_i$ are given by*

$$\begin{bmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_{r-1}\end{bmatrix} = \pi \cdot \mathbf{F}^r \mathcal{O}_r^\top (\mathcal{O}_r \mathcal{O}_r^\top)^{-1}. \tag{9}$$

*Proof.* By (7), $\pi \cdot \mathbf{F}^r$ is linearly dependent on the set $\{\boldsymbol{\pi} \cdot \mathbf{F}^i \mid i = 0, \ldots, r-1\}$, so we can write

$$\boldsymbol{\pi} \cdot \mathbf{F}^r = \sum_{i=0}^{r-1} \lambda_i \boldsymbol{\pi} \cdot \mathbf{F}^i \tag{10}$$

for some scalars $\lambda_i$. Multiplying both sides on the right by $\mathbf{a}_t$, and noting that $\boldsymbol{\pi} \cdot \mathbf{F}^i \mathbf{a_t} = \boldsymbol{\pi} \cdot \mathbf{a_{t+i}} = \hat{a}_{t+i}$, we obtain (8). Note that, since $\mathcal{O}_r$ has rank $r$, the matrix $\mathcal{O}_r \mathcal{O}_r^\top \in \mathbb{R}^{r \times r}$ has full rank, i.e., is invertible. To calculate the coefficients $\lambda_i$ explicitly, we rewrite (10) as

$$\boldsymbol{\pi} \cdot \mathbf{F}^r = \begin{bmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_{r-1}\end{bmatrix} \mathcal{O}_r.$$

Now multiplying on the right by $\mathcal{O}_r^\top$ and inverting yields (9). $\square$

Consider once again firms in year 2011 by Example 1, where $\mathbf{F}$ takes the form of (6) and $\pi = (1, 1, \ldots, 1)^\top$ . To apply Theorem 2, we first compute $r = 14$ such that $\text{rank}(\mathcal{O}_{r+1}) = \text{rank}(\mathcal{O}_r)$ by (7). Then by (9), we calculate:

$$\begin{array}{llll}
\lambda_0 = -1.36, & \lambda_1 = 21.13, & \lambda_2 = -137.13, & \lambda_3 = 493.14, \\
\lambda_4 = -1088.27, & \lambda_5 = 1519.09, & \lambda_6 = -1297.07, & \lambda_7 = 562.10, \\
\lambda_8 = 10.96, & \lambda_9 = -98.58, & \lambda_{10} = -26.35, & \lambda_{11} = 74.13, \\
\lambda_{12} = -40.99, & \lambda_{13} = 10.20 & &
\end{array}$$

In order to work with (8), we employ from Example 1 the information that

$$\begin{array}{lll}
\hat{a}_{2011} = 105986077, & \hat{a}_{2010} = 105655226, & \hat{a}_{2009} = 101670392, \\
\hat{a}_{2008} = 112299907, & \hat{a}_{2007} = 114293833, & \hat{a}_{2006} = 109579491, \\
\hat{a}_{2005} = 98037798, & \hat{a}_{2004} = 93986915, & \hat{a}_{2003} = 87532069, \\
\hat{a}_{2002} = 89483210, & \hat{a}_{2001} = 91368288, & \hat{a}_{2000} = 85644493, \\
\hat{a}_{1999} = 82823584, & \hat{a}_{1998} = 83154299 &
\end{array}$$

Then we compute the value of aggregate equity at year 2012, denoted by $\tilde{a}_{2012}$, with $\tilde{a}_{2012} = -5001656655$ by (8). We may compare with the true value $\hat{a}_{2012} = 105437547$ from the data and find a large discrepancy between our calculation (a negative number) and the real value.

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**Theorem 3** (Theorem 2′)**.** *The macro-level dynamics obeys the closed-form description*

$$\hat{a}_{t+r} = \lambda_{r-1}\hat{a}_{t+r-1} + \lambda_{r-2}\hat{a}_{t+r-2} + \cdots + \lambda_0\hat{a}_t. \tag{11}$$

*where the quantities $\lambda_i$ are given by*

$$\begin{bmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_{r-1}\end{bmatrix} = \pi \cdot \mathbf{F}^r \mathcal{O}_r^\top (\mathcal{O}_r \mathcal{O}_r^\top)^{-1}. \tag{12}$$

*Proof.* By (7), $\pi \cdot \mathbf{F}^r$ is linearly dependent on the set $\{\boldsymbol{\pi} \cdot \mathbf{F}^i \mid i = 0, \ldots, r-1\}$, so we can write

$$\boldsymbol{\pi} \cdot \mathbf{F}^r = \sum_{i=0}^{r-1} \lambda_i \boldsymbol{\pi} \cdot \mathbf{F}^i \tag{13}$$

for some scalars $\lambda_i$. Multiplying both sides on the right by $\mathbf{a}_t$, and noting that $\boldsymbol{\pi} \cdot \mathbf{F}^i \mathbf{a_t} = \boldsymbol{\pi} \cdot \mathbf{a_{t+i}} = \hat{a}_{t+i}$, we obtain (11). Note that, since $\mathcal{O}_r$ has rank $r$, the matrix $\mathcal{O}_r \mathcal{O}_r^\top \in \mathbb{R}^{r \times r}$ has full rank, i.e., is invertible. To calculate the coefficients $\lambda_i$ explicitly, we rewrite (13) as

$$\boldsymbol{\pi} \cdot \mathbf{F}^r = \begin{bmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_{r-1}\end{bmatrix} \mathcal{O}_r.$$

Now multiplying on the right by $\mathcal{O}_r^\top$ and inverting yields (12). □

Consider the dynamics of firms' equities under an arbitrary year of 1982 by Example 1. Here $\mathbf{F}$ takes the form of (6) and $\pi = (1, 1, \ldots, 1)^\top$. To apply Theorem 3, we first compute $r = 11$ such that $\mathrm{rank}(\mathcal{O}_{r+1}) = \mathrm{rank}(\mathcal{O}_r)$ by (7). Then by (12), we calculate:

$$\begin{array}{llll}
\lambda_0 = -1094.33, & \lambda_1 = 13651.75, & \lambda_2 = -55554.38, & \lambda_3 = 107253.42, \\
\lambda_4 = -112801.57, & \lambda_5 = 67360.70, & \lambda_6 = -21789.74, & \lambda_7 = 2714.25, \\
\lambda_8 = 429.16, & \lambda_9 = -189.95, & \lambda_{10} = 23.40. &
\end{array}$$

In order to work with (11), we employ from Example 1 the information that

$$\begin{array}{lll}
\hat{a}_{1982} = 27380142, & \hat{a}_{1983} = 30636225, & \hat{a}_{1984} = 33424058, \\
\hat{a}_{1985} = 36874105, & \hat{a}_{1986} = 39896338, & \hat{a}_{1987} = 43066014, \\
\hat{a}_{1988} = 48171571, & \hat{a}_{1989} = 55453473, & \hat{a}_{1990} = 64425979, \\
\hat{a}_{1991} = 69010613, & \hat{a}_{1992} = 72128906. &
\end{array}$$

Then we compute the value of aggregate equity at year 1993, denoted by $\tilde{a}_{1993} = 3,979,551,369$ by (8). We may compare with the true value $\hat{a}_{1993} = 73,579,856$ from the data and find large discrepancy between our calculation and the real value, the computation is 53.08 larger than the true value from the data. Such a large discrepancy is due to the fact that the micro-level dynamics in firms' equity at the year of 1982 to 1992 violates the presumption of time-invariance, which pales the effectiveness of Theorem 3.

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## 5 Concluding Remark

By employing mathematical theory of multi-level dynamic system, we have considered in this work the mathematical framework for complex economic system with heterogeneous interacting agents. We have discussed properties of micro-level dynamics for time-invariance, linearity, heterogeneity. Under the assumption that the micro-level dynamics are with time-invariance, linearity, and heterogeneity, we have derived from micro-level dynamics the macro-level dynamics under linear aggregation, by Theorem 1 and 3. As an experiment, we have applied to real-world economic dynamics of equities by firms listed in Tokyo Stock Exchange for the year of 1980 to 2012, to infer macro-level dynamics from micro-level dynamics. The experiment results have shown limited capability of applying to study real-world economic dynamics with Theorem 1 and 3. This implies that the assumption of linearity, time-invariance, and heterogeneity, which are the prerequisite of Theorem 1 and 3, might not fit in with real-world economic phenomena. Further study is thus necessary to consider scenario more than linearity and time invariance.

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