Topic 2

From Complex Fourier Series to Fourier Transforms

2.1 Introduction

In the previous lecture you saw that complex Fourier Series and its coefficients were defined by as

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega t} \qquad \text{where} \qquad C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-in\omega t}dt \ .$$

However, we noted that this did not extend Fourier analysis beyond periodic functions and discrete frequency spectra. What would fix this?

2.1.1 What happens if we let T get very large? A thought ...

Suppose the period T tended ¡to infinity. The fundamental frequency ω would become so small that it might properly be called an infinitesimal $d\omega$. To reach any finite frequency ω, m would have to tend to infinity. Then $m\, d\omega \to \omega$, and the C_m might best be called $C(\omega)d\omega$. (This point is tricky. It is to preserve the integral as the discrete spectrum morphs into a continuous one!)

So in this regime,

$$C(\omega)d\omega = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-i\omega t}dt \ .$$

But as $T \to \infty$, $T = 2\pi/d\omega$

$$C(\omega)d\omega = \frac{d\omega}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \ , \qquad \Rightarrow \qquad C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \ .$$

The expression for $f(t)$ has to be changed too. First change $C_n \to C(\omega)\mathrm{d}\omega$, then change from a sum to an integral:

$$f(t) = \int_{-\infty}^{\infty} C(\omega)\mathrm{d}\omega \mathrm{e}^{\mathrm{i}\omega t} \quad = \quad \int_{-\infty}^{\infty} C(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega \quad .$$

2.1.2 We (NEARLY) get to the Fourier Transform!

Our last thought was a good one. By allowing T to tend to infinity, we seem to have a method of going from the non-periodic time domain signal $f(t)$ to a frequency domain spectrum $C(\omega)$. Everything is correct, EXCEPT the actual Fourier transform is, by modern convention, $2\pi C(\omega)$.

So forget the thinking, let's use the definition.

2.2 The definition of the Fourier Transform

The **Fourier Transform** of a temporal signal $f(t)$ is the frequency spectrum

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\ \mathrm{e}^{-\mathrm{i}\omega t}\ \mathrm{d}t\ .$$

Given a frequency spectrum, the equivalent temporal signal is given by the

Inverse Fourier Transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\ \mathrm{e}^{\mathrm{i}\omega t}\ \mathrm{d}\omega\ .$$

Because Fourier **Series** have a continuous time signal but discrete frequency spectrum, it is rather natural to think about the signal, $f(t)$, as the "proper thing", and the coefficents A_n and B_n (or C_n) as slightly "derived".

With Fourier **Transforms**, however, there is a real equality between the representations. Indeed the time signal and frequency spectrum are described as a Fourier transform pair.

A Fourier Transform Pair is denoted by

$$f(t) \Leftrightarrow F(\omega)\ .$$

This means

$$F(\omega) = \mathcal{FT}\left[f(t)\right] \quad \text{and} \quad f(t) = \mathcal{FT}^{-1}\left[F(\omega)\right] .$$

2.3 ♣ Example

[Q] Determine the Fourier transform $F(\omega)$ of the function shown in Figure 2.1(a), $f(t) = u(t)\mathrm{e}^{-bt}$, where $u(t)$ is the Heaviside step function, and plot the amplitude spectrum $|F(\omega)|$.

[A] The Heaviside step function is zero for $t < 0$ and unity thereafter, so

$$\begin{aligned} \mathcal{FT}\left[u(t)\mathrm{e}^{-bt}\right] &= \int_0^\infty \mathrm{e}^{-bt}\ \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \\ &= \frac{-1}{b+\mathrm{i}\omega}\ \mathrm{e}^{-(b+\mathrm{i}\omega)t}\Big|_0^\infty = \frac{1}{b+\mathrm{i}\omega}\ . \end{aligned}$$

The frequency spectrum $F(\omega)$ is complex, and so it has amplitude and phase. The amplitude spectrum is $|F(\omega)| = 1/\sqrt{b^2+\omega^2}$, and is plotted in Figure 2.1(b).

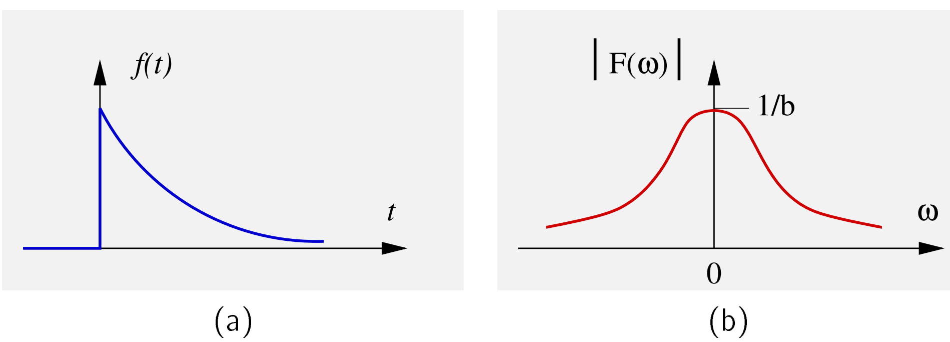


Figure 2.1: (a) $f(t) = u(t)\mathrm{e}^{-bt}$ (b) Amplitude spectrum $|F(\omega)| = |1/(b+\mathrm{i}\omega)|$.

Notice that negative values of ω are permitted. There is nothing deep happening: recall that $\cos(-\omega t) = \cos(\omega t)$ and $\sin(-\omega t) = -\sin(\omega t)$.

2.4 ♣ Example

[Q] Determine the Fourier transform of the unit "top hat" function of width a:

$$f_\sqcap(t) = \begin{cases} 1 & |t| < a/2 \\ 0 & \text{otherwise} \end{cases}$$

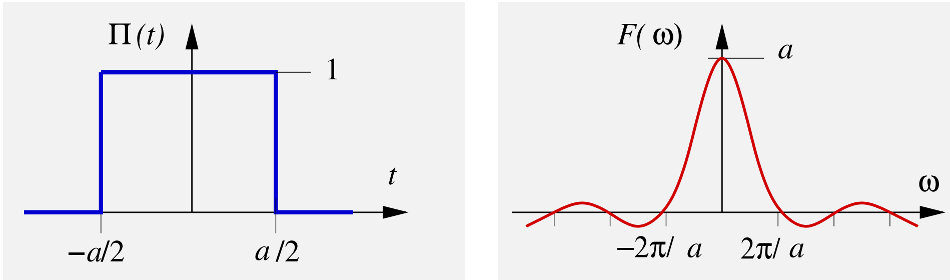


Figure 2.2:

[A] The Fourier transform is

$$
\begin{aligned}
F(\omega) &= \int_{-a/2}^{a/2} \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t = \frac{-1}{\mathrm{i}\omega}\left(\mathrm{e}^{-\mathrm{i}\omega a/2} - \mathrm{e}^{\mathrm{i}\omega a/2}\right) \\
&= \frac{2}{\omega}\sin(\omega a/2) \\
&= a\frac{\sin(\omega a/2)}{\omega a/2} = a\frac{\sin(\pi\omega a/2\pi)}{\pi\omega a/2\pi} \\
&= a\ \mathrm{sinc_N}(\omega a/2) = a\ \mathrm{sinc_H}(\omega a/2\pi)
\end{aligned}
$$

The Fourier Transform is real, and the frequency spectrum, rather than the amplitude spectrum, is shown in Figure 2.2.

Why do we write the answer in two ways? Unfortunately the $\mathrm{sinc}(\cdot)$ function has two definitions in common use.

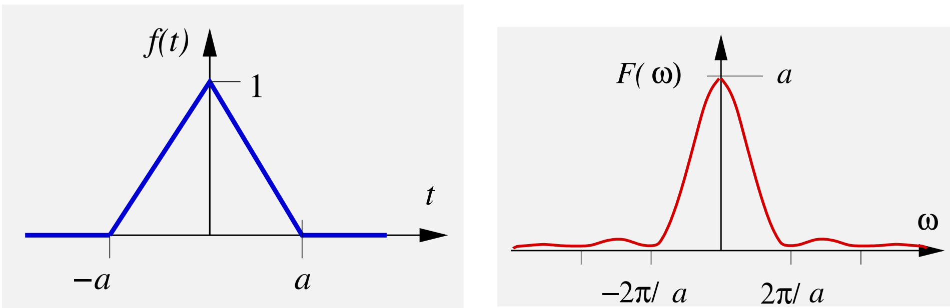
The HLT definition is $\mathrm{sinc_H}(x) = \sin(\pi x)/(\pi x)$, but often elsewhere you'll see sinc defined as $\mathrm{sinc_N}(x) = \sin(x)/x$, where the N stands for Not-HLT.

Because of the confusion we will try to avoid using it entirely, but if it is used we should try to stick to the HLT definition. Conversion is easy. If you read $\mathrm{sinc}(\xi)$ in non-HLT-compliance-mode replace it with $\mathrm{sinc}(\xi/\pi)$.

2.5 ♣ Example

[Q] Determine the Fourier transform of the unit "triangle" function of width $2a$:

$$f_\wedge(t) = \begin{cases} 1 + t/a & -a \le t < 0 \\ 1 - t/a & 0 \le t \le a \end{cases}$$



(a) (b)

Figure 2.3:

[A] The Fourier transform is

$$\begin{aligned} F(\omega) &= \int_{-a}^{0} (1 + t/a) e^{-i\omega t} dt + \\ &\qquad \int_0^a (1 - t/a) e^{-i\omega t} dt \\ &= 2\int_0^a (1 - t/a) \cos\omega t \; dt \\ &= \text{grind using integration by parts} \\ &= a\frac{\sin^2(\omega a/2)}{(\omega a/2)^2} \end{aligned}$$

2.6 Further examples

There are further examples of the Fourier transforms of important functions later in the notes. But a more complete list is in HLT.

2.7 More realistic signals

In case you had started to think that Fourier Transforms were merely for signals that could be described in as functions, Figure 2.4 shows a computational example (using methods described later) of the transform of signal which has equal amplitude 50Hz and 120Hz harmonic components but is corrupted with noise.

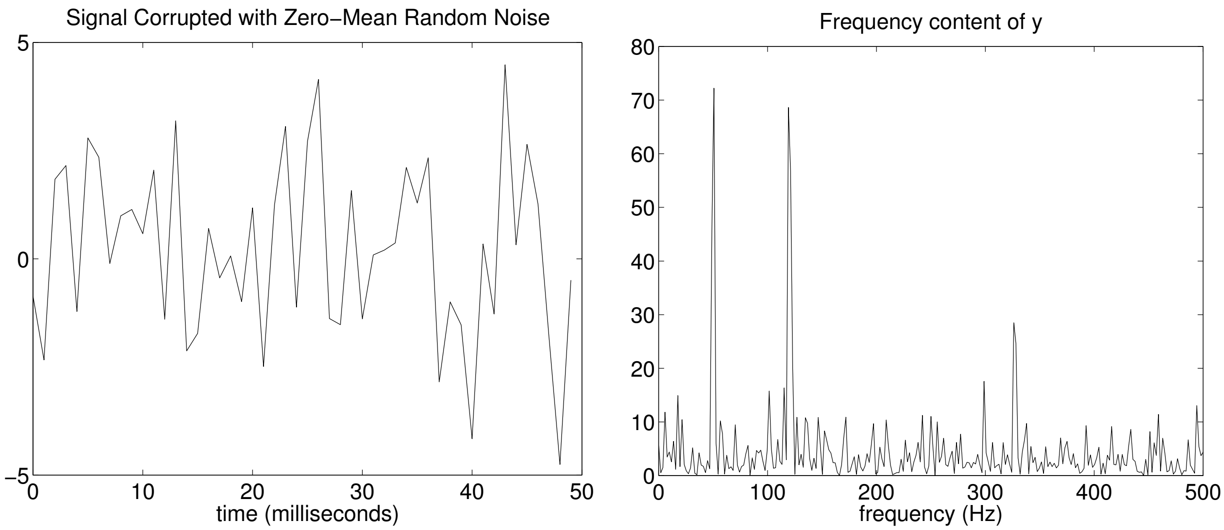


Figure 2.4:

2.8 "Routine" Properties of the FT

To start applying the Fourier transform to engineering problems will require a few more mathematical tools to be in place.

However, we can immediately explore some of the basic properties of the Fourier transform — those that can be derived from straightforward mathematical properties of the integral.

These are all in HLT. Deriving them is a bit tricky in places, and might seem a slightly tedious, but it does provide practice at recognizing the Fourier integral.

2.8.1 Property #1: Linearity

Linearity Property: If $\mathcal{FT}\left[f(t)\right] = F(\omega)$ and $\mathcal{FT}\left[g(t)\right] = G(\omega)$, then

$$\mathcal{FT}\left[\alpha f(t) + \beta g(t)\right] = \alpha F(\omega) + \beta G(\omega)$$

Proof: Integration is a linear operation. That is
$\int_{-\infty}^{\infty}[\alpha f(t) + \beta g(t)]\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t = \alpha \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t + \beta \int_{-\infty}^{\infty} g(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$.

2.8.2 Property #2: Duality

Dual Property If $\mathcal{FT}[f(t)] = F(\omega)$, then

$$\mathcal{FT}[F(t)] = 2\pi f(-\omega) \qquad \textit{Yes, weird! Watch out for the } -\omega$$

Proof: We know that $f(t)$ is the $\mathcal{FT}^{-1}[F(\omega)]$ — that is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

But t and ω are just symbols, so replace t with $(-\omega)$ and ω with t.
NB! Replacing ω with t does NOT flip the limits of integration. So

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{it(-\omega)}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt$$

$\Rightarrow 2\pi f(-\omega) = \mathcal{FT}[F(t)]$ *"comme il faut", as Mrs Fourier would say*

(If you did not like the swapping of symbols in one go, try $t \to p$ and $\omega \to q$, then $p \to (-\omega)$ and $q \to t$:

$$\begin{array}{ccl}
f(t) & = & \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\, d\omega \\
\downarrow & & \qquad\quad \downarrow \quad \downarrow \quad \downarrow \\
f(p) & = & \frac{1}{2\pi}\int_{-\infty}^{\infty} F(q)e^{iqp}\, dq \\
\downarrow & & \qquad\quad \downarrow \quad \downarrow \quad \downarrow \\
f(-\omega) & = & \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{it(-\omega)}dt
\end{array}$$

then rearrange ...)

2.8.3 Property #3: Similarity Property

Parameter Scaling or Similarity Property: If $\mathcal{FT}[f(t)] = F(\omega)$, then

$$\mathcal{FT}[f(at)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Proof: The appearance of $|a|$ hints that this proof needs to consider the ranges $a > 0$ and $a < 0$ separately.
For $a > 0$:

$$\mathcal{FT}[f(at)] = \int_{-\infty}^{\infty} f(at) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t \ .$$

Write $p = at$, and note that the signs on the limits do NOT change because a is positive. Then

$$\mathcal{FT}[f(at)] = \frac{1}{a} \int_{-\infty}^{\infty} f(p) \mathrm{e}^{-\mathrm{i}(\omega/a)p} \mathrm{d}p = \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

For $a < 0$:
Substitute $p = -|a|t$, and remember to change signs on the limits — when $t = \infty$, $p = -\infty$:

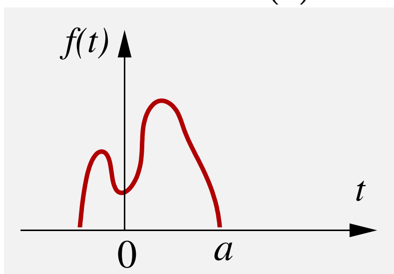
$$\begin{aligned} \mathcal{FT}[f(at)] &= \frac{1}{-|a|} \int_{\infty}^{-\infty} f(p) \mathrm{e}^{-\mathrm{i}(\omega/a)p} \mathrm{d}p = \frac{1}{|a|} \int_{-\infty}^{\infty} f(p) \mathrm{e}^{-\mathrm{i}(\omega/a)p} \mathrm{d}p \\ &= \frac{1}{|a|} F\left(\frac{\omega}{a}\right) \end{aligned}$$

So, for both cases, one can write

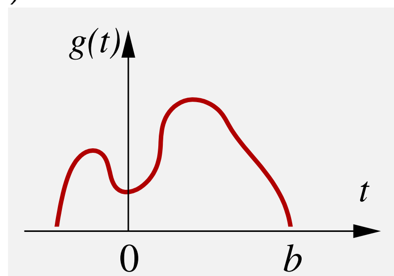
$$\mathcal{FT}[f(at)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

2.8.4 ♣ Property #3: Example

[Q] The Fourier transform of $f(t)$ is $F(\omega)$. What the the FT of $g(t)$?



(a)



(b)

[A]

$$\mathcal{FT}\left[g(t)\right] = \mathcal{FT}\left[f(St)\right] = \frac{1}{|S|}F\left(\frac{\omega}{S}\right)$$

So the real question is what is the scale S?

VERY tempting to say that $g(t)$ looks wider so that $S > 1$. Now $b > a$ so that

$$\Rightarrow S = b/a.\textbf{WRONG}$$

Something interesting happens to $g(t)$ when $t = b$.

This point matches to $f(a)$.

So $Sb = a$, and

$$\Rightarrow S = a/b.\ \textbf{CORRECT}$$

2.8.5 Property #4: Parameter Shifting

Parameter Shifting Property: If $\mathcal{FT}\left[f(t)\right] = F(\omega)$, then

$$\mathcal{FT}\left[f(t-a)\right] = \exp(-\mathrm{i}\omega a)F(\omega)$$

Proof:

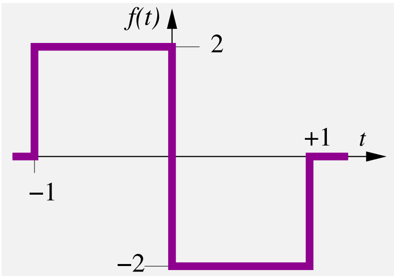
$$\mathcal{FT}\left[f(t-a)\right] = \int_{-\infty}^{\infty} f(t-a)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t\ .$$

Substitute $p = t - a$. Then

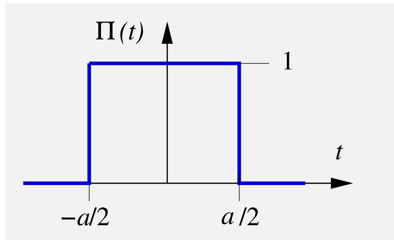
$$\begin{aligned}\mathcal{FT}\left[f(t-a)\right] &= \int_{-\infty}^{\infty} f(p)\mathrm{e}^{-\mathrm{i}\omega(p+a)}\mathrm{d}p = \mathrm{e}^{-\mathrm{i}\omega a}\int_{-\infty}^{\infty} f(p)\mathrm{e}^{-\mathrm{i}\omega p}\mathrm{d}p \\ &= \mathrm{e}^{-\mathrm{i}\omega a}F(\omega)\end{aligned}$$

2.8.6 ♣ Properties #1,3,4: Example

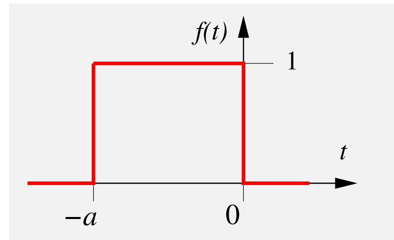
[Q] Find the Fourier Transform of the function shown (a) below.



(a)



(b)



(c)

[A] We know the FT of the tophat in (b) is $\frac{1}{\omega}\sin(\frac{\omega a}{2})$.
The version in (c) is time-shifted "to the left" and involves $t+\frac{a}{2}$, so its FT is

$$\exp\left(\frac{\mathrm{i}\omega a}{2}\right)\frac{1}{\omega}\sin\left(\frac{\omega a}{2}\right)$$

Subtract a copy time-shifted to the right ... and set $a=1$ ...

$$\begin{aligned} & \left[\exp\left(\frac{\mathrm{i}\omega a}{2}\right)-\exp\left(\frac{-\mathrm{i}\omega a}{2}\right)\right]\frac{1}{\omega}\sin\left(\frac{\omega a}{2}\right) \\ = & \left[2\mathrm{i}\sin\left(\frac{\omega a}{2}\right)\right]\frac{1}{\omega}\sin\left(\frac{\omega a}{2}\right)=\frac{2\mathrm{i}}{\omega}\sin^2\left(\frac{\omega}{2}\right) \end{aligned}$$

Now scale the amplitude by a factor of 2 ...

$$\textbf{Ans}=\frac{4\mathrm{i}}{\omega}\sin^2\left(\frac{\omega}{2}\right)$$

We've done the scaling in time by setting parameter $a=1$. Suppose you applied the scaling property. What would be the correct scaling — a or $(1/a)$?

2.8.7 Property #5: Frequency Shifting

Frequency Shifting Property: If $\mathcal{FT}[f(t)]=F(\omega)$, then

$$\mathcal{FT}[f(t)\exp(\pm\mathrm{i}\omega_S t)]=F(\omega\mp\omega_S)$$

Proof:

$$\begin{aligned}
\mathcal{FT}\left[f(t)\exp(\pm \mathrm{i}\,\omega_S t)\right] &= \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\pm \mathrm{i}\,\omega_S t}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}(\omega \mp \omega_S)t}\mathrm{d}t \\
&= F(\omega \mp \omega_S)
\end{aligned}$$

2.8.8 Property #6: Amplitude modulation by a cosine

Amplitude modulation by a cosine If $\mathcal{FT}\left[f(t)\right] = F(\omega)$ then

$$\mathcal{FT}\left[f(t)\cos\omega_0 t\right] = \frac{1}{2}\left[F(\omega-\omega_0)+F(\omega+\omega_0)\right]$$

Proof: Write

$$\cos\omega_0 t = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\omega_0 t}+\mathrm{e}^{-\mathrm{i}\omega_0 t}\right)$$

then use the Frequency shifting property.

2.8.9 Property #7: Differentiation wrt time

Differentiation Property in time: If $\mathcal{FT}\left[f(t)\right] = F(\omega)$, then

$$\mathcal{FT}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t)\right] = (\mathrm{i}\omega)^n F(\omega)$$

Proof: It is so tempting to start by writing $\mathcal{FT}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t)\right] = \int_{-\infty}^{\infty}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t)\right]\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$. Resist. Instead, write

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$$

Now differentiate w.r.t. t. Because the integral is w.r.t. ω, the differentation can move through the integral sign:

$$\begin{aligned}\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t) &= \frac{1}{2\pi}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{-\infty}^{\infty}F(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)\frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)(\mathrm{i}\omega)^n\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega\end{aligned}$$

This last expression says that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t) = \mathcal{FT}^{-1}\left[F(\omega)(\mathrm{i}\omega)^n\right] \quad\Rightarrow \mathcal{FT}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t)\right] = F(\omega)(\mathrm{i}\omega)^n\ .$$

2.8.10 Property #8: Differentiation wrt frequency

Differentiation Property in frequency: If $\mathcal{FT}[f(t)] = F(\omega)$, then

$$\mathcal{FT}[(-\mathrm{i}t)^n f(t)] = \frac{\mathrm{d}^n}{\mathrm{d}\omega^n} F(\omega)$$

Proof: For you to colour in.

2.9 The Fourier Transform of Complex Fourier Series

This requires knowledge of the δ-function. See Lecture 3.

2.10 Summary

By allowing the period T to tend to infinity, and the fundamental frequency tend to an infinitesimal $\mathrm{d}\omega$, we have given a reasoned argument for the origin of the Fourier Transform which allows non-periodic functions to be represented in the continuous (and complex) frequency domain.

However, there is no need to reproduce the argument and you might just as well take the definition for granted.

We have worked out the Fourier Transform for a couple of signals.

We have gone through the proofs of several basic properties of the Fourier Transform.

We are nearly at the point of being able to do things, but not quite!

In the next lecture we must learn about the δ-function and the general technique of convolution, and then applications begin to open up for us.