

Ch. 9.

1, 2, 4, 6, 7, 8.

$$Q: Q_{11}x^2 + 2Q_{12}xy + Q_{22}y^2 + 2Q_{10}x + 2Q_{01}y + Q_{00} = 0$$

Isometry (preserves length)

$\tilde{Q} \rightsquigarrow$

(rotation & translation)

$$\begin{aligned} \lambda_1 x^2 + \lambda_2 y^2 &= k \\ \text{or} \\ \lambda_1 x^2 + \lambda_2 y &= k \\ \downarrow \text{rescaling} \\ k, \lambda_1, \lambda_2 &= \pm 1 \end{aligned}$$

$n = \text{rank } Q$	$(p, n-p)$	equation	name
2	$(0,2) \text{ or } (2,0)$	$x^2 + y^2 + 1 = 0$	imaginary ellipse
2	$(q_{12}, q_{10}) \text{ or } (2,-2)$	$x^2 + y^2 - 1 = 0$	circle (ellipse)
2	$(1,1)$	$x^2 - y^2 - 1 = 0$	hyperbola
2	$(0,2) \text{ or } (2,0)$	$x^2 + y^2 = 0$	two complex lines (a point)
2	$(1,1)$	$x^2 - y^2 = 0$	two real lines
1	$(0,1) \text{ or } (1,0)$	$x^2 + 1 = 0$	two complex lines
1	$(1,0)$	$x^2 - 1 = 0$	two real lines
1	$(1,0)$	$x^2 = 0$	a real, double line
1	$(0,1) \text{ or } (1,0)$	$x^2 - y^2 = 0$	parabola

$$M_Q = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

2. Write down a quadratic eq with associated matrix A and find the matrix $M \in SO(2)$ which diagonalizes A.

$$A = \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$$

$$P_A(x) = \det(A - xJ_2)$$

$$= \begin{vmatrix} 6-x & 2 \\ 2 & 9-x \end{vmatrix} = 0$$

$$\Rightarrow (6-x)(9-x) - 4 = 0$$

$$54 - 15x + x^2 - 4 = 0$$

$$x^2 - 15x + 50 = 0$$

$$\Delta = 225 - 200 = 25$$

$$\Rightarrow x_{1,2} = \frac{15 \pm \sqrt{25}}{2}$$

10
5

$$\lambda_1 = 10, \lambda_2 = 5$$

$$S(x) = \left\{ (x, y) \mid A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ (x, y) \mid (A - xJ_2) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$S(x_1) = \{ (x_1 y) \mid \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

$$\begin{cases} -x + 2y = 0 \\ 2x - y = 0 \end{cases} \Rightarrow S(x_1) = \{(x, 2x) \mid x \in \mathbb{R}\} = \langle (1, 2) \rangle$$

Choose $v_1 = \frac{1}{\sqrt{5}} \cdot (1, 2)$

$$S(x_2) = \{(-2y, y) \mid y \in \mathbb{R}\} = \langle (-2, 1) \rangle$$

Choose $v_2 = \frac{1}{\sqrt{5}} \cdot (-2, 1)$

The base-change matrix is $M = M_{B'B} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$

$\det M_{B'B} = -1$, but we want it to be 1.

\Rightarrow flip one of the vectors you chose

$$\Rightarrow v_2 = -\frac{1}{\sqrt{5}} (-2, 1)$$

$$\Rightarrow M = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$\underbrace{M^{-1} \cdot A \cdot M}_{= M^T \text{ (bc. } M \text{ is orthogonal)} \text{ (NC012)}} = M^T \cdot A \cdot M$$

$$= \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 10 & 20 \\ -10 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 30 & 0 \\ 0 & 25 \end{pmatrix} =$$

$$= \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$$

Q: $(x, y) \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$

Q: $(bx^2 + 2y \quad 2x + gy) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + x + 2y - 1 = 0$

Q: $bx^2 + 2xy + 2x + gy^2 + x + 2y - 1 = 0$

Q: $bx^2 + 2xy + gy^2 + x + 2y - 1 = 0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$xg = \begin{pmatrix} x \\ y \end{pmatrix}^T = \left(M \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} \right)^T = (x' \ y') \cdot M^T$$

Q: $(x' \ y') \cdot \underbrace{M^T \cdot A \cdot M}_{=} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + (x' \ y') \cdot M^T \begin{pmatrix} 1 \\ -1 \end{pmatrix} - 1 = 0$

Q: $(x' \ y') \cdot \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} + (x' \ y') \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - 1 = 0$

Q: $10x'^2 + 5y'^2 + (x' \ y') \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} - 1 = 0$

Q: $10x'^2 + 5y'^2 + \sqrt{5}x' - 1 \Rightarrow$

$$Q: (10x^2 + \sqrt{5}x) + 5y^2 - 1 = 0$$

$$10(x^2 + \frac{1}{2}\sqrt{5}x + \frac{1}{400}) + 5y^2 - 1 - \frac{1}{8} = 0$$

$$Q: 10(x + \frac{\sqrt{5}}{20})^2 + 5y^2 - \frac{9}{8} = 0$$

$$\begin{cases} x'' = x + \frac{\sqrt{5}}{20} \\ y'' = y \end{cases}$$

$$10x''^2 + 5y''^2 = \frac{9}{8}$$

$$\Rightarrow \frac{x''^2}{\frac{9}{8}} + \frac{y''^2}{\frac{9}{50}} = 1$$

$$\begin{cases} x''' = \frac{x''}{\frac{3}{\sqrt{80}}} \\ y''' = \frac{y''}{\frac{3}{\sqrt{50}}} \end{cases} \Rightarrow Q: x'''^2 + y'''^2 = 1$$

2nd approach (Lagrange)

$$Q: 6x^2 + 4xy + 5y^2 + x + 2y - 1 = 0$$

$$\Rightarrow (6x^2 + 4xy) + 5y^2 + x + 2y - 1 = 0$$

$$(6x^2 + 2 \cdot \sqrt{6}x \cdot \frac{2}{\sqrt{6}}y + \frac{2}{3}y^2)$$

$$+ \frac{3}{3}y^2 - \frac{2}{3}y^2 + x^2 + 2y - 1 = 0$$

$$= (\sqrt{6}x + \frac{2}{\sqrt{6}}y)^2 + \frac{25}{3}y^2 + x^2 + 2y - 1 = 0$$

$$x' = \sqrt{6}x + \frac{2}{\sqrt{6}}y$$

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{6} & \frac{2}{\sqrt{6}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = \frac{x' - \frac{2}{\sqrt{6}}y}{\sqrt{6}}$$

$$\textcircled{1}: x'^2 + \frac{25}{3}y^2 + \frac{1}{6}x' - \frac{1}{3}y + 1y - 1 = 0$$

$$Q: \left(x'^2 + \frac{1}{6}x'\right) + \left(\frac{25}{3}y^2 + \frac{7}{3}y\right) - 1 = 0$$

$$Q: \left(x'^2 + 2 \cdot \frac{1}{12}x' + \frac{1}{144}\right) +$$

$$+ \left(\frac{25}{3}y^2 + 2 \cdot \frac{5}{\sqrt{3}}y \cdot \frac{\sqrt{3}}{6} + \frac{1}{12}\right)$$

$$- \frac{1}{144} - \frac{1}{12} - 1 = 0$$

$$\textcircled{2}: \left(x' + \frac{1}{12}\right)^2 + \left(\frac{5}{\sqrt{3}}y + \frac{\sqrt{3}}{6}\right)^2$$

$$= \frac{157}{144}$$

$$x'' = x' + \frac{1}{2}$$

$$y'' = y' + \frac{\sqrt{3}}{6}$$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{pmatrix}$$

$$Q: x''^2 + y''^2 = \frac{157}{144}$$

$$x''' = x'' \cdot \sqrt{\frac{144}{157}}$$

$$y''' = y'' \cdot \sqrt{\frac{144}{157}}$$

$$Q: x'''^2 + y'''^2 = 1$$

$$\text{f. g. } Q: -x^2 + xy - y^2 = 0$$

Bring Q to the canonical form

$$(x^2 - xy) + y^2 = 0$$

$$\left(x^2 - 2 \cdot \frac{1}{2} \cdot x \cdot y + \frac{y^2}{4}\right) + \frac{3}{4}y^2 = 0$$

$$Q: \left(x - \frac{y}{2}\right)^2 + \frac{3}{4}y^2 = 0$$

$$\begin{cases} x' = x - \frac{y}{2} \\ y' = y \end{cases}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Q. $x^2 + \bar{z}^3 y^3 = 0$ Two complex lines (a point)