Seminar 13

1. Consider a $(63, 56)$-code.
(i) What is the number of digits in the message before coding?
(ii) What is the number of check digits?
(iii) What is the information rate?
(iv) How many different syndromes are there?

2. Using the parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and the syndromes and coset leaders

Syndrome	000	001	010	011
Coset leader	000000	001000	010000	000010

Syndrome	100	101	110	111
Coset leader	100000	000110	000100	000001

decode the following words: 101110, 011000, 001011, 111111, 110011.

3. A (7,4)-code is defined by the equations $u_1 = u_4 + u_5 + u_7$, $u_2 = u_4 + u_6 + u_7$, $u_3 = u_4 + u_5 + u_6$, where u_4, u_5, u_6, u_7 are the message digits and u_1, u_2, u_3 are the check digits. Write its generator matrix and parity check matrix. Decode the received words 0000111 and 0001111.

4. Find the syndromes of all the received words in the (3,2)-parity check code and in the (3,1)-repeating code.

5. Construct a table of coset leaders and syndromes for the (7,4)-code with parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

6. Determine the parity check matrix and all syndromes and coset leaders of the (5,3)-code with generator matrix $G = \left(\frac{P}{I_3}\right) \in M_{5,3}(\mathbb{Z}_2)$, where:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

7. Construct a table of coset leaders and syndromes for the (3,1)-code generated by $p = 1 + X + X^2 \in \mathbb{Z}_2[X]$.

8. Construct a table of coset leaders and syndromes for the (7,3)-code generated by $p = 1 + X^2 + X^3 + X^4 \in \mathbb{Z}_2[X]$.

k - length of message
n - ?

2. Using the parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & \boxed{1 & 0 & 1} \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

upper part of gen. matrix

and the syndromes and coset leaders

Syndrome	000	001	010	011
Coset leader	000000	001000	010000	000010

Syndrome	100	101	110	111
Coset leader	100000	000110	000100	000001

decode the following words: 101110, 011000, 001011, 111111, 110011.

S - syndrome is what you obtain when multiplying a vector with the parity matrix

$[S] = H \cdot [v]$

most likely error pattern

Decoding steps:

1) Multiply the vector $[v]$ by H to get the syndrome
2) Use the table to identify coset leader 'e' associated to S
3) Correct v using e

$v \rightsquigarrow v+e$

4) Extract the message from $v+e$

101110

$$\begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1\\0\\1\\1\\1\\0 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 1+1 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 0\\0\\0 \end{pmatrix} \Rightarrow e = 000000$$

$v + e = v$

From $H \in \mathcal{M}_{n,k} \Rightarrow n=6 \quad k=3 \Rightarrow \quad m = 110$

011000

$$\begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 0\\1\\1\\0\\0\\0 \end{pmatrix} = \begin{pmatrix} 0\\1\\1 \end{pmatrix} \Rightarrow e = 000010$$

$$v + e = \begin{pmatrix} 0\\1\\1\\0\\0\\0 \end{pmatrix} + \begin{pmatrix} 0\\0\\0\\0\\1\\0 \end{pmatrix} = \begin{pmatrix} 0\\1\\1\\0\\1\\0 \end{pmatrix} \Rightarrow m = 010$$

001011

$$\begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 0\\0\\1\\0\\1\\1 \end{pmatrix} = \begin{pmatrix} 1\\0\\1 \end{pmatrix} \Rightarrow e = 000110$$

$v + e = 001101 \Rightarrow m = 101$

111111

$$\begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1\\1\\1\\1\\1\\1 \end{pmatrix} = \begin{pmatrix} 1\\0\\1 \end{pmatrix} \Rightarrow e = 000110$$

$v + e = 111001 \Rightarrow m = 001$

110011

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow e = 010000$$

$$v + e = 100011 \Rightarrow m = 011$$

000110

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow e = 000110$$

$$v + e = 000\,000 \Rightarrow m = 000$$

Syndrome	000	001	010	011
Coset leader	000000	001000	010000	000010

always, you don't correct anything

Syndrome	100	101	110	111
Coset leader	100000	000110	000100	000001

how do we get them?

1) weight (no. of 1)
2) the closest the bits are

* for exam the only thing grades is the table

$$H \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

* add the column corresp. to the 1s

* we're done with the 1s, but still have 101 to fill ⇒ we go to 2 bits flipped

⇒ they should be as close together as possible

go first next to each other, next 1 spot away, 2 and so on ....

Ex:

$$H = \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

S	CL	
0000	000000	
0001	000100	
0010	001000	
0011	001100	$C_3 + C_4$
0100	010000	
0101	010100	
0110	000001	
0111	000101	
1000	100000	
1001	100100	
1010	101000	
1011	000011	
1100	110000, 000110	$(C_1 + C_2,\ C_4 + C_5)$
1101	000010	
1110	100001	
1111	001010	

* exam

8. Construct a table of coset leaders and syndromes for the (7,3)-code generated by $p = 1 + X^2 + X^3 + X^4 \in \mathbb{Z}_2[X]$.

- for $m = 100 \Rightarrow f = 1$

$f_m = f \cdot x^4 = x^4$

$$\begin{array}{l|l} x^4 & x^4+x^3+x^2+1 \\ \underline{x^4+x^3+x^2+1} & \overline{1} \\ x^3+x^2+1 & \end{array}$$

$R_m = x^3+x^2+1$

$g_m = x^4+x^3+x^2+1 \rightsquigarrow v = 1011100$

- $m = 010 \Rightarrow f = x$

$f_m = x^5$

$$\begin{array}{l|l} x^5 & x^4+x^3+x^2+1 \\ \underline{x^5+x^4+x^3+x} & \overline{x+1} \\ x^4+x^3+x & \\ \underline{x^4+x^3+x^2+1} & \\ R_m = x^2+x+1 & \end{array}$$

$\Rightarrow g_m = x^5+x^2+x+1 \rightsquigarrow v = 1110010$

- $m = 001 \Rightarrow f = x^2$

$f_m = x^6$

$$\begin{array}{l|l} x^6 & x^4+x^3+x^2+1 \\ \underline{x^6+x^5+x^4+x^2} & \overline{x^2+x} \\ x^5+x^4+x^2 & \\ \underline{x^5+x^4+x^3+x} & \\ R_m\ x^3+x^2+x & \end{array}$$

$\Rightarrow g_m = x^6+x^3+x^2+x \rightsquigarrow v = 0111001$

$$H = (I_{n-k} \mid P) = \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

S	CL
0000	0000000
0001	0001000
0010	0010000
0011	0011000
0100	0100000
0101	0000110
0110	0110000
0111	0000001
1000	1000000
1001	0000011
1010	0001100
1011	0000100
1100	1100000
1101	1101000
1110	0000010
1111	0001010