2, 8, 9, 10, 18, 20, 26, 27, 28 — CHAPTER 6

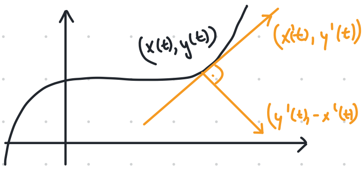
6.2. For a circle $\mathcal{C}$ of radius R:

a) Use the parametrization $x \mapsto (x, \pm\sqrt{R^2 - x^2})$ to deduce a parametrization of tangent lines to $\mathcal{C}$.

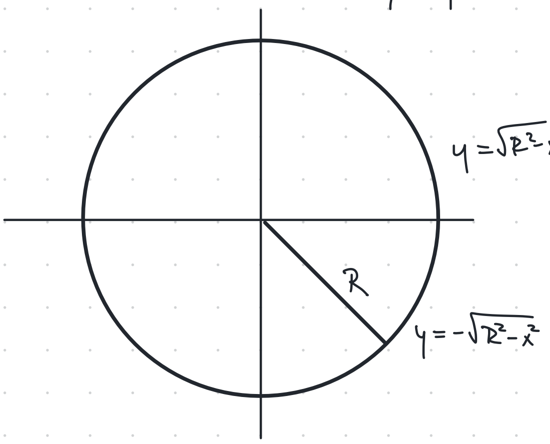
b) Use the parametrization $\theta \mapsto (R\cos(\theta), R\sin(\theta))$ to deduce a parametrization of tangent lines to $\mathcal{C}$.

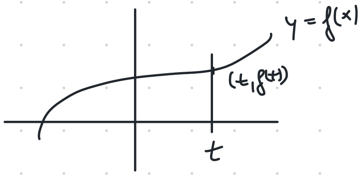
c) Compare these to the equation of the tangent line $xx_0 + yy_0 = R^2$ where $(x_0, y_0) \in \mathcal{C}$.

$(C)\ \begin{cases} x = x(t) \\ y = y(t) \end{cases}$



$T_{(x(t),y(t))}(C):\ \dfrac{x - x(t)}{x'(t)} = \dfrac{y - y(t)}{y'(t)}$





$(x'(t), y'(t)) = (1, f'(t)) \neq (0,0)$

general tangent

$T_{(t,f(t))}(G_f)\ \dfrac{x-t}{1} = \dfrac{y - f(t)}{f'(t)}$

$y - f(t) = f'(t)(x-t)$

$y = f'(t)(x-t) + f(t)$

$\cancel{(\sqrt{R^2-x^2})' =}$

a) $f(t) = \sqrt{R^2 - t^2}$, $\quad f'(t) = \dfrac{-2t}{2\sqrt{R^2-t^2}} = \dfrac{-t}{\sqrt{R^2-t^2}}$

$T_{(t,f(t))}G_f:\ y = \sqrt{R^2-t^2} - \dfrac{t}{\sqrt{R^2-t^2}}(x-t)$

$\sqrt{R^2-t^2}\cdot y = R^2 \cancel{-t^2} - tx \cancel{+t^2}$

$y\sqrt{R^2-t^2} = R^2 - tx$

$t \in (-R, R)$

$T_{(t,g(t))}G_g:\ y = -\sqrt{R^2-t^2} + \dfrac{t}{\sqrt{R^2-t}}(x-t)$

b) $(C)\ \begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases} \quad \begin{array}{|l} x'(\theta) = -R\sin\theta \\ y'(\theta) = R\cos\theta \end{array}$

$T_{(x(\theta),y(\theta))}(C):\ \dfrac{x - R\cos\theta}{-R\sin\theta} = \dfrac{y - R\sin\theta}{R\cos\theta}$

$-yR\sin\theta + R^2\sin^2\theta = xR\cos\theta - R^2\cos^2\theta$

$R(x\cos\theta + y\sin\theta) = R^2$

c) $x^2 + y^2 = R^2 \qquad (x_0, y_0) \in C$

$T_{(x_0,y_0)}:\ x_0x + y_0y = R^2$

6.8. Consider the family of ellipses $\mathcal{E}_a : \frac{x^2}{a^2} + \frac{y^2}{16} = 1$. For what value $a \in \mathbb{R}$ is $\mathcal{E}_a$ tangent to the line $\ell : x - y + 5 = 0$?

6.9. Consider the family of lines $\ell_c : \sqrt{5}x - y + c = 0$. For what values $c \in \mathbb{R}$ is ℓ_c tangent to the ellipse $\mathcal{E} : x^2 + \frac{y^2}{4} = 1$?

6.10. Determine the common tangents to the ellipses

$$\frac{x^2}{45} + \frac{y^2}{9} = 1 \quad \text{and} \quad \frac{x^2}{9} + \frac{y^2}{18} = 1.$$

6.9. $\ell : \sqrt{5}x - y + c = 0 \Rightarrow y = \sqrt{5}x + c$

$\mathcal{E} : x^2 + \frac{y^2}{4} = 1$

$\Rightarrow x^2 + \frac{(\sqrt{5}x + c)^2}{4} = 1$

$4x^2 + 5x^2 + 2\sqrt{5}xc + c^2 = 4$

$9x^2 + 2\sqrt{5}xc + c^2 = 4$

ℓ tangent to the elipse iff $\ell \cap \mathcal{E}$ unique points

$\Rightarrow \Delta = 0$

$\Delta = 20c^2 - 4 \cdot 9(c^2 - 4)$

$\Delta = 20c^2 - 36c^2 + 144$

$$\left.\begin{array}{l}\Delta = -16c^2 + 144 \\ \Delta = 0\end{array}\right\} \Rightarrow c = \pm 3$$

6.10 $\frac{x^2}{45} + \frac{y^2}{9} = 1$ $\qquad \frac{x^2}{9} + \frac{y^2}{18} = 1$

elipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

tangent if $m^2 = a^2k^2 + b^2$

$$\left.\begin{array}{l}m^2 = 45k^2 + 9 \\ m^2 = 9k^2 + 18\end{array}\right\} \Rightarrow 36k^2 = 9$$

$k^2 = \frac{9}{36} \Rightarrow k = \pm\frac{1}{4}$

$m^2 = 45 \cdot \frac{1}{16} + 9 \Rightarrow m^2 = \frac{45 + 144}{16} = \frac{189}{16}$

6.18. Determine the tangents to the hyperbola $\mathcal{H} : x^2 - y^2 = 16$ which contain the point $M(-1, 7)$.

$\mathcal{H} : x^2 - y^2 = 16$

$M(-1, 7) \in T_{(x_0, y_0)}$

$\mathcal{H} : x^2 - y^2 = 16 \Leftrightarrow \frac{x^2}{16} - \frac{y^2}{16} = 1$

Let ℓ_1, ℓ_2 be tangents to the $\mathcal{H} \Rightarrow$

$$\left.\begin{array}{l}T_{(x_0, y_0)} : \frac{xx_0}{16} - \frac{yy_0}{16} = 1 \\ \text{but } M \in T_{(x_0, y_0)}\end{array}\right\} \Rightarrow T(x_0, y_0) : \begin{array}{l}-x_0 - 7y_0 = 16 \\ x_0^2 - y_0^2 = 16\end{array}$$

$$\Rightarrow \begin{cases} x_0 = -16 - 7y_0 \\ x_0^2 - y_0^2 = 16 \end{cases} \Rightarrow (16 + 7y_0)^2 - y_0^2 = 16$$

$$240 + 224 y_0 + 48 y_0^2 = 0 \quad | :16$$

$$15 + 14 y_0 + 3y_0^2 = 0$$

$$\Delta = 14^2 - 4 \cdot 15 \cdot 3 = 196 - 180 = 16 > 0$$

$$\begin{cases} y_{0_1} = \frac{-14+4}{6} = \frac{-5}{3} \Rightarrow x_{0_1} = \frac{-13}{3} \\ y_{0_2} = \frac{-14-4}{6} = -3 \qquad x_{0_2} = -4 \end{cases} \Rightarrow D'\left(\frac{-5}{3}, \frac{-13}{3}\right) \text{ \& } \mathcal{T}(-3,-4)$$

$$\ell_1 : \frac{-5x}{3} + \frac{13}{3} y = 16$$

$$\ell_2 : -3x + 4y = 16$$