$\|x\|^2 = x\cdot x$

1. $\forall\ x, y \in \mathbb{R}^n$

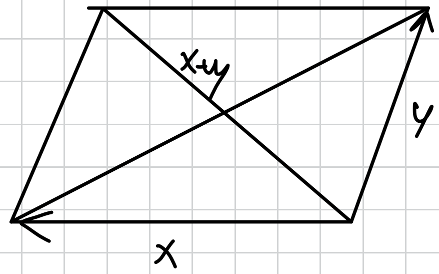
(a) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$ (parallelogram identity)

$\|x+y\|^2 = (x+y)\cdot(x+y) + (x-y)\cdot(x-y) =$

$= \|x\|^2 + \|y\|^2 + \cancel{2x\cdot y} + \|x\|^2 + \|y\|^2 - \cancel{2x\cdot y} =$

$= 2\cdot\|x\|^2 + 2\|y\|^2 =$

$= 2(\|x\|^2 + \|y\|^2)$



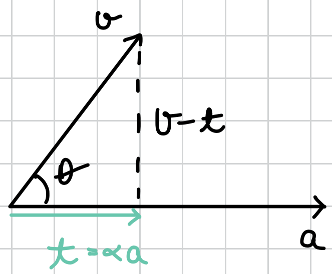
(b) $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) =$

$= \frac{1}{4}((x+y)\cdot(x+y) - (x-y)\cdot(x-y)) =$

$= \frac{1}{4}(\cancel{\|x\|^2} + \cancel{\|y\|^2} + 2x\cdot y \cancel{- \|x\|^2} - \cancel{\|y\|^2} + 2x\cdot y)$

$= x\cdot y$

3. $v \in \mathbb{R}^2$ $\quad a \in \mathbb{R}^2$



Let $t = \text{pr}_a v = \alpha\cdot a$, $\alpha = ?$

$t\cdot(v-t) = 0$ (perpendicular)

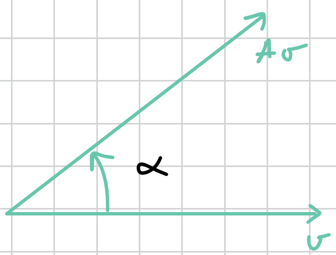
$\cos\theta = \frac{\|\alpha a\|}{\|v\|} = \frac{\alpha\|a\|}{\|v\|}$

$v\cdot a = \|v\|\cdot\|a\|\cdot\cos\theta$

$\Rightarrow \frac{\alpha\|a\|}{\cancel{\|v\|}} = \frac{v\cdot a}{\cancel{\|v\|}\cdot\|a\|} \Rightarrow \alpha = \frac{v\cdot a}{\|a\|^2}$

$\Rightarrow t = \alpha\cdot a = \underbrace{\frac{v\cdot a}{\|a\|^2}}_{\text{this is just a num!}}\cdot a = \frac{v\cdot a}{\|a\|}\cdot\underbrace{\frac{1}{\|a\|}\cdot a}_{\text{unit vector } = 1}$

4. $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ – rotation matrix with angle θ in $\mathbb{R}^2$



So prove that $\widehat{v, Av} = \alpha \overset{?}{=} \theta$

Let $v = \begin{bmatrix} v_1\\ v_2\end{bmatrix}$, $\|v\| = \sqrt{v_1^2 + v_2^2}$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$a \cdot b = a_1 b_1 + a_2 b_2$$

$$A \cdot v = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \cdot v_1 - \sin\theta\, v_2 \\ \sin\theta\, v_1 + \cos\theta\, v_2 \end{bmatrix}$$

$$v \cdot Av = v_1 (\cos\theta \cdot v_1 - \sin\theta \cdot v_2) + v_2 (\sin\theta\, v_1 + \cos\theta\, v_2) =$$

$$= \cos\theta \cdot v_1^2 - \cancel{\sin\theta\, v_1 v_2} + \cancel{\sin\theta\, v_1 v_2} + \cos\theta + v_2^2 =$$

$$= \cos\theta\, (v_1^2 + v_2^2)$$

$$= \cos\theta \cdot \|v\| \|Av\| \Rightarrow \theta = \widehat{v, Av}$$

$$\|Av\|^2 = (\cos\theta\, v_1 - \sin\theta\, v_2)^2 + (\sin\theta\, v_1 + \cos\theta\, v_2)^2$$

$$= \underline{\cos^2\theta\, v_1^2} + \sin^2\theta\, v_2^2 + \underline{\sin^2\theta\, v_1^2} + \cos^2\theta\, v_2^2 =$$

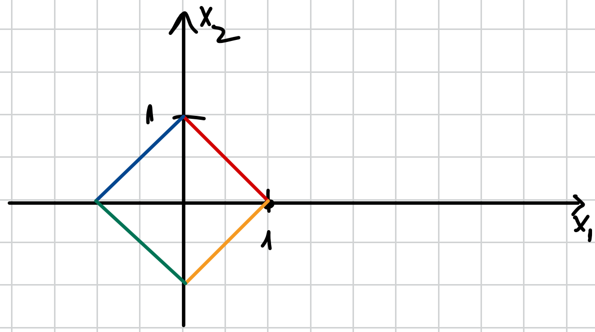
$$= v_1^2 (\cos^2\theta + \sin^2\theta) + v_2^2 (\sin^2\theta + \cos^2\theta) =$$

$$= v_1^2 + v_2^2 = \|v\|^2 \Rightarrow \|Av\| = \|v\|$$

$$A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ → matrix is doing the rotation in reverse ↘

5. $\|x\|_p := (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$

unit ball : $\|x\| = 1$



$p = 1 : \|x\|_1 = |x_1| + |x_2|$

$\|X\|_1 = |X_1| + |X_2|$

if ★ $x_1, x_2 > 0 \Rightarrow x_1 + x_2 = 1$

$x_2 = 1 - x_1$

★ $x_1 > 0, x_2 < 0 \Rightarrow x_1 - x_2 = 1$

$x_2 = x_1 - 1$

★ $x_1 < 0, x_2 > 0 \Rightarrow -x_1 + x_2 = 1$

$x_2 = 1 + x_1$

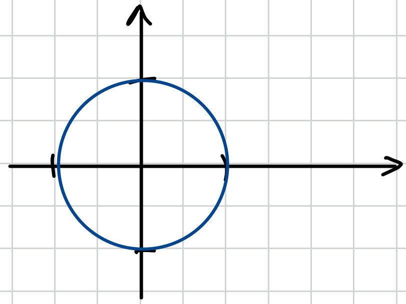
★ $x_1, x_2 < 0 \Rightarrow -x_1 - x_2 = 1$

$x_2 = -x_1 - 1$

$p=2 \qquad \|x\|_2 = \sqrt{x_1^2 + x_2^2}$

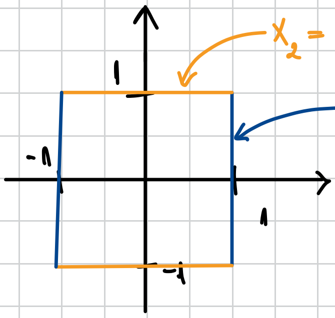
$\|x\|_2 = 1 \Rightarrow x_1^2 + x_2^2 = 1$

$x_1 = \cos\alpha$
$x_2 = \sin\alpha$

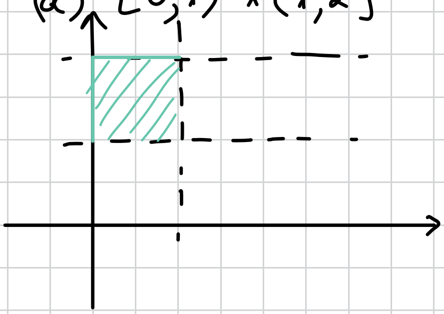


$p=\infty \qquad \|x\|_\infty = \max\{\|x_1\|, \|x_2\|\}$

$\|x\|_\infty = 1 \Leftrightarrow \max\{\|x_1\|, \|x_2\|\} = 1 \Rightarrow x_1 \in [-1,1]$
$x_2 \in [-1,1]$



7. (a) $[0,1) \times (1,2]$

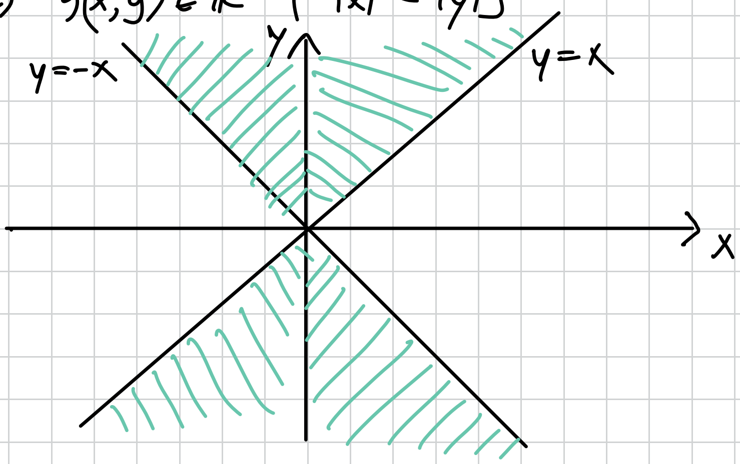


$cl([0,1)\times(1,2]) = [0,1] \times [1,2]$

$int([0,1)\times(1,2]) = (0,1)\times(1,2)$

$bd([0,1)\times(1,2]) = ([0,1]\times\{1\}) \cup ([0,1]\times\{2\}) \cup (\{0\}\times[1,2])$
$\cup(\{1\}\times[1,2])$

(b) $\{(x,y)\in\mathbb{R}^2 \mid |x| < |y|\}$



$x<0, y<0 \Rightarrow -x < -y \Rightarrow x > y$
$x>0, y<0 \Rightarrow x < -y \Rightarrow y < -x$

$x>0, y>0 \Rightarrow x<y$
$x<0, y>0 \Rightarrow -x<y \Rightarrow x>-y$

$int(A) = A$
$cl(A) = A \cup \{(x,x)\} \cup \{(x,-x)\}$
$= \{(x,y)\in\mathbb{R}^2 \mid |x| \le |y|\}$

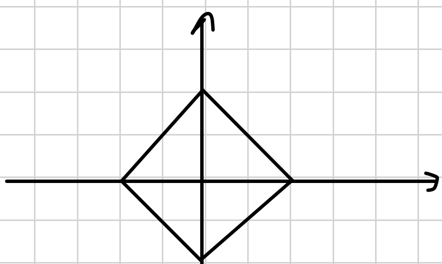
$bd(A) = \{(x,x) \mid x\in\mathbb{R}\} \cup \{(x,-x)\}$

(c) $\{(x,y)\in\mathbb{R}^2 \mid |x|+|y| < 1\}$

$x<0, y<0 \Rightarrow -x-y<1 \Rightarrow -x < 1+y$

$x<0, y>0 \Rightarrow -x+y<1 \Rightarrow x > y-1$

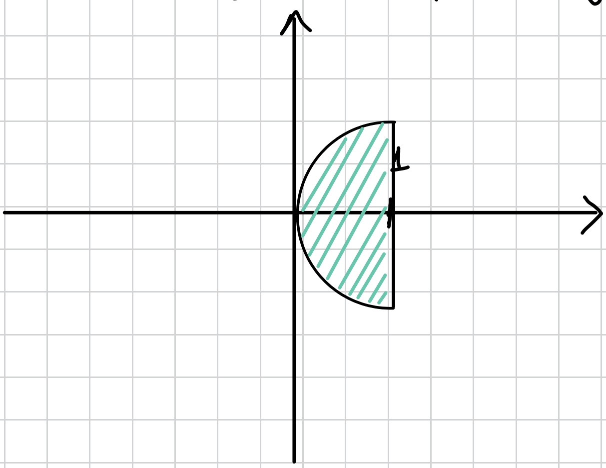
$\vdots$



$int\, A = A$
$cl\, A = \{(x,y)\in\mathbb{R}^2 \mid |x|+|y| \le 1\}$

$bd\ A = \{(x,y) \in \mathbb{R}^2 \mid |x| + |y| = 1\}$

(d) $A = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1,\ x \leq 1\}$



$(x-x_0)^2 + (y-y_0)^2 = R^2$

circle centered around (x_0, y_0)

$int(A) = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 < 1,\ x < 1\}$

$cl(A) = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1,\ x \leq 1\}$

$bd(A) = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y = 1,\ x = 1\}$

8. $L_c = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$

(a) $f(x,y) = x + y$ and $c = \{0, \pm 1\}$

* $c = 0 \Rightarrow x + y = 0 \Rightarrow x = -y$

$c = -1 \Rightarrow x + y = -1 \Rightarrow x = -1 - y$

$c = 1 \Rightarrow x + y = 1 \Rightarrow x = 1 - y$