# Mock Exam (2h)

1. Draw the interior and the boundary of the unit ball in $\mathbb{R}^2$ for the norm $\|(x,y)\|_1 = |x| + |y|$.

2. Study the continuity and the differentiability (partial and total) of the function $f : \mathbb{R}^2 \to \mathbb{R}$,

   prove that is discontinuous

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

3. (a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and let $v \in \mathbb{R}^n$. Write the definition for the directional derivative $D_v f(x)$ and prove that it equals $\nabla f(x) \cdot v$.

   (b) Let $A \in \mathbb{R}^{2\times 2}$ be a symmetric matrix and let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x^T A x$. In which direction does $f$ decrease the most at the point $(1,1)$?

   gotta prove that $H(x) = A$, $\forall f(x) = Ax$
   * true only if $A$ is symmetric

4. Let $f(x,y) = x^2 e^{-xy}$. Find the second order Taylor expansion of $f$ around $(1,1)$.

5. Find and classify all the critical points of $f(x,y) = (x^2 + 3y^2)e^{1-x^2-y^2}$.

6. Find the extrema of $3x + 2y$ subject to $2x^2 + 3y^2 = 1$.

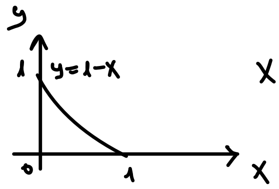
7. Compute the following integrals:

   (a) $\displaystyle\int_0^\infty e^{-2x^2}\,dx$. like $\int_0^\infty e^{-x^2}dx$

   $\lim_{t\to\infty} \int_0^t e^{-2x^2} dx$ * double integral and polar coordinates 11.16 lecture notes

   (b) $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy. = \int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dx\, dy = ?$

8. Let $D$ be the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Compute $\displaystyle\iint_D (x^2 - y^2)\,dx\,dy. = \int_0^1 \left( \int_0^x (x^2 - y^2) dy \right) dx$




$x \in [0,1]$ $y \in [0, 1-x]$

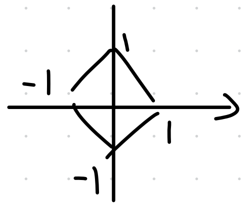
$2^{nd}$ order Taylor pol: $T_2(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0)\cdot(x-x_0, y-y_0) + \frac{1}{2}(x-x_0, y-y_0)\cdot H\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$

$\to$ Quadratic in x and y

This is a mock exam for January 2024 intended to give you an idea about what to expect.

try 2

1) $\|(x,y)\|_1 = |x| + |y|$




$\operatorname{int} A = \{(x,y) \in \mathbb{R}^2 \mid |x|+|y| < 1\}$

$\operatorname{bd} A = \{ \text{—}\|\text{—} \mid |x|+|y| = 1\}$

2) cont: $f(x,y) \begin{cases} \dfrac{xy^3}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(my^3,y)\to(0,0)} \frac{my^6}{m^2y^6+y^6} = \frac{m}{m^2+1}$$ dep on $m \Rightarrow$ not cont

$Df(0,0) = \nabla f(0,0)$

diff $\iff \exists \lim_{(x,y)\to(0,0)} \dfrac{f(x,y) - f(0,0) - (0,0)\cdot(x,y)}{\|xy\|}$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{0}{y} = 0$$

$\Rightarrow \nabla f(0,0) = (0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \overset{=0}{f(0,0)} - \overset{=0}{(0,0)}\cdot(x,y)}{\|xy\|} = \lim_{(x,y)\to(0,0)} \frac{xy^3}{(x^2+y^6)\sqrt{x^2+y^2}} = \lim_{\substack{y\to 0 \\ x = my^3}} \frac{\cancel{y^6}}{\cancel{y^6}(m^2+1)\sqrt{m^2y^6+y^2}} =$$

$$= \lim_{y\to 0} \frac{1}{(m^2+1)\sqrt{y^2(m^2y^4+1)}}$$ $\nexists$ lim ?

3) $f: \mathbb{R}^n \to \mathbb{R}$ $\quad D_v f(x) = \lim_{h\to 0} \dfrac{f(x+hv) - f(x)}{h}$

$$\lim_{h\to 0} \frac{f(x+hv) - f(x) - \nabla f(x)\cdot hv}{\|hv\|} = 0$$

(b) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x) = x^T A x$

$f(x+h) = (x+h)^T A (x+h) = \frac{1}{2}(x^TAx + x^TAh + h^TAx + h^TAh) =$

$h^TAx = \langle h, Ax\rangle = h\cdot Ax = \langle Ax, h\rangle = x^TA^Th = x^TAh = \langle x, Ah\rangle$

$\Rightarrow f(x+h) = f(x) + Ax\cdot h + \frac{1}{2} h^TAh$

Taylor's exp:

$f(x+h) = f(x) + \nabla f(x)\cdot h + \frac{1}{2} h^T H(x) h$

$\begin{cases} \nabla f(x) = Ax \\ H(x) = A \end{cases}$

* greatest decrease $\iff -\nabla f(x_0, y_0)$

$$Ax=\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1+bx_2 \\ bx_1+ax_2 \end{bmatrix} = \nabla f(x) = (ax_1+bx_2, bx_1+ax_2)$$

4) $f(x,y) = x^2 e^{-xy}$ $\qquad T_2(1,1) = ?$

$$T_2(x,y) = f(x_0,y_0) - \nabla f(x_0,y_0)\cdot(x-x_0, y-y_0) - \frac{1}{2}(x-x_0, y-y_0)\, H(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2xe^{-xy} - x^2 e^{-xy} \\ \frac{\partial f}{\partial y} &= -x^2 e^{-xy} \end{aligned}\right\} \Rightarrow \nabla f(x,y) = (xe^{-xy}(2-x), -x^2e^{-xy})$$

$$\nabla f(1,1) = (e^{-1}, -e^{-1})$$

$$\frac{\partial^2 f}{\partial x^2} = 2e^{-xy} - 2xe^{-xy} - 2xe^{-xy} - x^2e^{-xy} = e^{-xy}(x^2-4x+2)$$

$$\frac{\partial^2 f}{\partial y^2} = x^2 e^{-xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2xe^{-xy} + x^2 e^{-xy}$$

$$H(1,1) = \begin{bmatrix} 2e^{-1}-e^{-1} & -2e^{-1}+e^{-1} \\ -2e^{-1}+e^{-1} & e^{-1} \end{bmatrix} = \begin{bmatrix} -e^{-1} & -e^{-1} \\ -e^{-1} & e^{-1} \end{bmatrix}$$

$$T_2(x,y) = f(1,1) + \nabla f(1,1)(x-1, y-1) + \frac{1}{2}[x-1 \;\; y-1]\cdot H(1,1)\begin{bmatrix} x-1 \\ y-1 \end{bmatrix} =$$

$$= e^{-1} + (e^{-1}, -e^{-1})(x-1, y-1) + \frac{1}{2}[x-1 \;\; y-1]\begin{bmatrix} -e^{-1} & -e^{-1} \\ -e^{-1} & e^{-1} \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$= \frac{1}{e} + \frac{x}{e} - \cancel{\frac{1}{e}} - \frac{y}{e} + \cancel{\frac{1}{e}} + \frac{1}{2}[x-1 \;\; y-1]\begin{bmatrix} -e^{-1}(x-1) - e^{-1}(y-1) \\ -e^{-1}(x-1) + e^{-1}(y-1) \end{bmatrix} =$$

$$= \frac{1+x-y}{e} + \frac{1}{2}[x-1 \;\; y-1]\begin{bmatrix} \frac{-x-y+2}{e} \\ \frac{-x+y}{e} \end{bmatrix} =$$

$$= \frac{1+x-y}{e} + \frac{1}{2}\left[(x-1)\cdot\frac{-x-y+2}{e} + (y-1)\cdot\frac{-x+y}{e}\right] =$$

$$= 1+x-y + \frac{1}{2}\left(-x^2 - xy + 2x + x + \cancel{y} - 2 - xy + y^2 + x - \cancel{y}\right) =$$

$$= 1+x-y + \frac{1}{2}\left(-x^2 + y^2 - 2xy + 4x - 2\right)$$

6) $f(x,y) = 3x+2y \qquad g(x) = 2x^2+3y^2-1$

$L(x,y,\lambda) = 3x+2y+\lambda(2x^2+3y^2-1)$

$\frac{\partial L}{\partial x} = 3+4\lambda x = 0 \Rightarrow x = \frac{-3}{4\lambda}$

$\frac{\partial L}{\partial y} = 2+6\lambda y = 0 \Rightarrow y = \frac{-1}{3\lambda}$

$\frac{\partial L}{\partial \lambda} = 2x^2+3y^2-1 = 0$

$\cancel{2}\cdot\frac{9}{\cancel{16}_8\lambda^2} + \cancel{3}\cdot\frac{1}{\cancel{9}_3\lambda^2} = 1$

$\frac{24+8}{24\lambda^2} = 1 \Rightarrow \lambda = \pm\sqrt{\frac{35}{24}}$

$\lambda_1 = \sqrt{\frac{35}{24}} \Rightarrow x = \frac{-3}{4\sqrt{\frac{35}{24}}}$ ?

4) (a) $\int_0^\infty e^{-2x^2}dx$ Let $D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le R^2\}$

$\iint_D e^{-2(x^2+y^2)}dx\,dy = \int_0^R\int_0^{2\pi} e^{-2r^2} r\,d\theta\,dr = 2\pi\int_0^R e^{-2r^2} r\,dr = -\frac{\pi}{2}e^{-2r^2}\Big|_0^R =$

polar coord: $x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow r^2 \le R^2 \Rightarrow r\in[0,R]$, $\theta\in[0,2\pi]$

$= -\frac{\pi}{2}e^{-2R^2} + \frac{\pi}{2} = \frac{\pi}{2}(1-e^{-2R^2}) \simeq \frac{\pi}{2}$

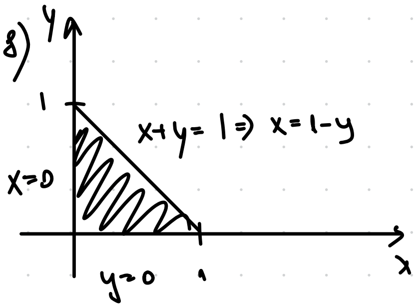
$\Rightarrow \frac{\pi}{2} = \int_0^\infty\int_0^\infty e^{-2(x^2+y^2)}dx\,dy = \int_0^\infty e^{-2x^2}dx\int_0^\infty e^{-2y^2}dy \Rightarrow \int_0^\infty e^{-2x^2}dx = \sqrt{\frac{\pi}{2}}$

(b) $\int_0^1\int_{\sqrt{y}}^1 e^{x^3}dx\,dy =$

$\sqrt{y} \le x \le 1$, $0 \le y \le 1$ $\Leftrightarrow$ $0 \le y \le x^2$, $0 \le x \le 1$

$\Rightarrow \int_0^1\int_0^{x^2} e^{x^3}dy\,dx = \int_0^1 x^2e^{x^3}dx = \frac{1}{3}e^{x^3}\Big|_0^1 = \frac{e}{3}-\frac{1}{3} = \frac{e-1}{3}$

$(e^{x^3})' = e^{x^3}\cdot 3x^2$

8) 


$x \in [0,1]$
$y \in [0, 1-x]$

$0 \leq x \leq 1$
$0 \leq y \leq 1-x$

$$\iint_D (x^2-y^2)\,dx\,dy = \int_0^1 \int_0^{1-x} (x^2-y^2)\,dy\,dx = \int_0^1 \left(x \cdot y - \frac{y^3}{3}\right)\Big|_0^{1-x} dx = \int_0^1 \Big(\underbrace{x-x^2}_{x(1-x)} - \frac{(1-x)^3}{3}\Big)\,dx =$$

let $t = 1-x \Rightarrow x = 1-t)$

$dt = -dx$

$$= -\int_0^1 t(1-t) - \frac{t^3}{3}\,dt = -\int_0^1 t - t^2 - \frac{t^3}{3}\,dt = -\frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{12}\Big|_0^1 =$$

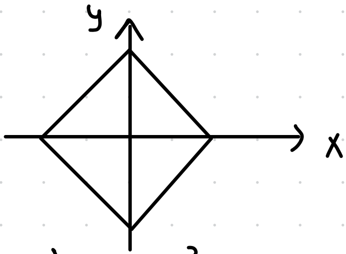
$$= \frac{-1}{2} + \frac{1}{3} + \frac{1}{12} = \frac{-6+4+1}{12} = -\frac{1}{12}$$

try 1

1) $\|(x,y)\|_1 = |x|+|y|$

$A = \{(x,y)\in\mathbb{R}^2 \mid \|(x,y)\|_1 = |x|+|y|\}$

$\text{int } A = \{(x,y)\in\mathbb{R}^2 \mid \|(x,y)\| < 1\}$




$\text{bd } A = \{(x,y)\in\mathbb{R}^2 \mid \|x,y\|_1 = |x|+|y|\}$

$|x|+|y| = 1 \Rightarrow |x| = 1-|y|$

2) $f:\mathbb{R}^2\to\mathbb{R}$

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$$

a) continuity: $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{y\to 0} \frac{my^4}{m^2y^2+y^6} = \lim_{y\to 0}\frac{my^4y^2}{\cancel{y^2}(m^2+y^4)} = 0$

$\lim_{x\to 0} \frac{m^3x^4}{x^2+m^6x^6} = \lim_{x\to 0}\frac{m^3x^2x^2}{\cancel{x^2}(1+m^6x^4)} = 0$

we could have $\lim_{(x,y)\to(0,0)}$

* mean equality

$x = my^3$

$\lim_{(my^3,y)\to(0,0)} \frac{my^6}{m^2y^6+y^6} = \lim_{(my^3,y)\to(0,0)} \frac{my^6}{y^6(m^2+1)} = \frac{m}{m^2+1}$ depends on $m \Rightarrow$ not continuous

b) diff. at $(0,0)$

$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\frac{\frac{x\cdot 0}{x^2+0} - 0}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$

$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0}\frac{\frac{0\cdot y^3}{y^6}-0}{y} = 0$

$\Rightarrow \nabla f(0,0) = (0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \overbrace{f(0,0)}^{=0} - \overbrace{(0,0)\cdot(x,y)}^{=0}}{\|xy\|} = \lim_{(x,y)\to(0,0)}\frac{f(x,y)}{\|xy\|} = \lim_{(x,y)\to(0,0)}\frac{\frac{xy^3}{x^2+y^6}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)}\frac{xy^3}{(x^2+y^6)\sqrt{x^2+y^2}}$$

let $y = mx \Rightarrow \lim_{x\to 0}\frac{\cancel{x}\cdot 0}{x^2\sqrt{x^2}} = \lim_{x\to 0}\frac{0}{x^2} = 0 \Rightarrow \exists \lim_{(x,y)\to(0,0)} \Rightarrow f$ is diff

3) (a) $D_v f(x) = Df(x)(v) = \partial_v f(x) = v\cdot\nabla f(x) = v\cdot\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+hv)-f(x)}{h}$

**Definition 9.15.** Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and a vector $v\in\mathbb{R}^n$. The derivative of $f$ in the direction of $v$ at $x\in A$ (*directional derivative*) is given by

$$Df_v(x) := \lim_{h\to 0}\frac{f(x+hv)-f(x)}{h}.$$

Note that here $h$ is a scalar. The directional derivative $Df_v(x)$ is also denoted by $\partial_v f(x)$.

If $f$ differentiable $\Rightarrow D_v f(x) = \nabla f(x) \cdot v$

$$\lim_{h\to 0} \frac{f(x+hv) - f(x) - \nabla f(x) \cdot hv}{\|hv\|} = 0$$

$$\|hv\| = |h| \|v\| \Rightarrow \lim_{h\to 0} \frac{f(x+hv) - f(x) - \nabla f(x) \cdot hv}{h} = 0$$

$$D f_v(x) = \lim_{h\to 0} \frac{f(x+hv) - f(x) - \nabla f(x) \cdot hv}{h} = \nabla f(x) \cdot v$$

(b) $A \in \mathbb{R}^{2\times 2}$ - symmetric

$f: \mathbb{R}^2 \to \mathbb{R} \quad f(x) = x^T A x$

$$Ax = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ bx_1 + ax_2 \end{bmatrix}$$

$$x^T A x = [x_1 \; x_2] \begin{bmatrix} ax_1 + bx_2 \\ bx_1 + ax_2 \end{bmatrix} = [x_1(ax_1 + bx_2) + x_2(bx_1 + ax_2)] = ax_1^2 + bx_1x_2 + bx_1x_2 + ax_2^2$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} &= 2ax_1 + 2bx_2 \\ \frac{\partial f}{\partial x_2} &= 2ax_2 + 2bx_1 \end{aligned}\right\} \Rightarrow \nabla f(x) = (2ax_1 + 2bx_2, \; 2ax_2 + 2bx_1)$$

$$-\nabla f(1,1) = -(2a+2b, \; 2a+2b) = (-2(a+b), -2(a+b))$$

4) $f(x,y) = x^2 e^{-xy}$

$T_2(x,y) - ?$ at $(1,1)$

$\nabla f(1,1) = \left(\frac{1}{e}, -\frac{1}{e}\right)$

$$\frac{\partial f}{\partial x} = 2x e^{-xy} - x^2 e^{-xy} \Rightarrow \frac{\partial f}{\partial x}(1,1) = 2e^{-1} - e^{-1} = \frac{1}{e}$$

$$\frac{\partial f}{\partial y} = -x^2 e^{-xy} \Rightarrow \frac{\partial f}{\partial y}(1,1) = -e^{-1}$$

$$\frac{\partial^2 f}{\partial x^2} = 2e^{-xy} - 2xe^{-xy} - 2xe^{-xy} + x^2 e^{-xy} = e^{-xy}(x^2 - 4x + 2) \Rightarrow \frac{\partial^2 f}{\partial x^2}(1,1) = \frac{1}{e}(1 - 4 + 2) = -\frac{1}{e}$$

$$\frac{\partial^2 f}{\partial y^2} = x^2 e^{-xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2x e^{-xy} + x^2 e^{-xy} \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = -2 \cdot \frac{1}{e} + \frac{1}{e} = -\frac{1}{e}$$

$$H(1,1) = \begin{bmatrix} -\frac{1}{e} & -\frac{1}{e} \\ -\frac{1}{e} & \frac{1}{e} \end{bmatrix}$$

$$T_2(x,y) = f(1,1) + \nabla f(1,1)(x-1,y-1) + \frac{1}{2}(x-1,y-1)\,H(1,1)\cdot\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} =$$

$$= \frac{1}{e} + \frac{1}{e}(x-1) - \frac{1}{e}(y-1) + \frac{1}{2}(x-1,y-1)\begin{pmatrix} -\frac{1}{e} & -\frac{1}{e} \\ -\frac{1}{e} & \frac{1}{e} \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} =$$

$$= \frac{1}{e} + \frac{x}{e} - \cancel{\frac{1}{e}} - \frac{y}{e} + \cancel{\frac{1}{e}} + \frac{1}{2}(x-1,y-1)\cdot\begin{pmatrix} -\frac{1}{e}(x-1) - \frac{1}{e}(y-1) \\ -\frac{1}{e}(x-1) + \frac{1}{e}(y-1) \end{pmatrix} =$$

$$= \frac{x-y+1}{e} + \frac{1}{2}(x-1,y-1)\cdot\begin{pmatrix} -\frac{1}{e}\cdot x - \frac{1}{e}y - \frac{2}{e} \\ -\frac{1}{e}x + \frac{1}{e}y \end{pmatrix} =$$

$$= \frac{x-y+1}{e} + \frac{1}{2}\left[(x-1)\cdot\frac{-x-y-2}{e} + (y-1)\cdot\frac{-x+y}{e}\right] =$$

$$= x-y+1 + \frac{1}{2}\left(-x^2 - xy - \cancel{2x} + \cancel{x} + \cancel{y} + 2 - xy + y^2 + \cancel{x} - \cancel{y}\right) =$$

$$= x-y+1 + \frac{1}{2}\left(-x^2+y^2-2xy+2\right)$$

5) $f(x,y) = (x^2+3y^2)e^{1-x^2-y^2}$

$$\frac{\partial f}{\partial x} = 2x\,e^{1-x^2-y^2} - 2x(x^2+3y^2)e^{1-x^2-y^2} = 2xe^{1-x^2-y^2}(1-x^2-3y^2) = 0 \Rightarrow \begin{cases} x=0 \\ \text{or} \\ x^2+3y^2=1 \Rightarrow x=\sqrt{1-3y^2} \end{cases}$$

$$\frac{\partial f}{\partial y} = 6ye^{1-x^2-y} - 2y(x^2+3y^2)e^{1-x^2-y^2} = 2ye^{1-x^2-y^2}(3-x^2-3y^2) = 0 \Rightarrow \begin{cases} y=0 \\ \text{or} \\ x^2+3y^2=3 \Rightarrow x=\sqrt{3-3y^2} \\ y=\sqrt{\frac{3-x^2}{3}} \end{cases}$$

critical points: $(x,y) = \{(0,0),(0,1),(1,0)\}$ $\qquad x=\sqrt{1-\cancel{3}\cdot\frac{3-x^2}{\cancel{3}}} = \sqrt{1-3+x^2} = \sqrt{x^2+2}$ not poss

$$\frac{\partial^2 f}{\partial x^2} =$$

Compute

$$H(x,y) = [\quad]$$

$\det(H(x,y) - \lambda I_n) = 0$ find $\lambda$ and classify

6) $f(x,y) = 3x+2y \qquad g(x,y) = 2x^2+3y^2-1$

$$L(x,y,\lambda) = 3x+2y+\lambda(2x^2+3y^2-1)$$

$$\frac{\partial L}{\partial x} = 3+4\lambda x = 0 \Rightarrow x = \frac{-3}{4\lambda}$$

$$\frac{\partial L}{\partial y} = 2+6\lambda y = 0 \Rightarrow y = \frac{-2}{6\lambda} = \frac{-1}{3\lambda}$$

$$\frac{\partial L}{\partial \lambda} = 2x^2 + 3y^2 - 1 = 0$$

$$\cancel{2} \cdot \frac{9}{\cancel{16}_{8} \lambda^2} + \cancel{3} \cdot \frac{1}{\cancel{9}_{3} \lambda^2} = 1$$

$$\frac{9}{8\lambda^2} + \frac{1}{3\lambda^2} = 1$$

$$\frac{1}{\lambda^2}\left( {}^{3/}\frac{9}{8} + {}^{8/}\frac{1}{3} \right) = 1$$

$$\frac{1}{\lambda^2}\left( \frac{27}{24} + \frac{8}{24} \right) = 1$$

$$\frac{1}{\lambda^2} \cdot \frac{35}{24} = 1 \Rightarrow \lambda = \pm \sqrt{\frac{35}{24}}$$

# Final Exam #1

1. Study the convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^k}$, with $k > 1$. **1p**

2. (a) Draw the interior and the boundary of the set $\{(x, y) \in \mathbb{R}^2 \mid |x| < |y| < 1\}$. **1p**
   (b) Let $x, y \in \mathbb{R}^n$ be orthogonal vectors. Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. **1p**

3. Find the second order Taylor polynomial for $f(x, y) = \sqrt{x^2 + y^2}$ around $(1, 1)$. **1p**

4. Find and classify all the critical points of $f(x, y) = x^3 - 3x + y^2$. **1p**

5. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and let $b \in \mathbb{R}^n$. Consider the function

$$f : \mathbb{R}^n \to \mathbb{R},\ f(x) = \frac{1}{2}x^T Ax - b^T x.$$

   (a) Prove that $f$ has a unique minimum, which satisfies the equation $Ax = b$. **1p**
   (b) Write a gradient descent method for finding the minimum of $f$. **1p**

6. Let the probabilities $p_1, p_2, p_3 \in (0, 1)$ with $p_1 + p_2 + p_3 = 1$. Consider the function

$$f : \mathbb{R}^3 \to \mathbb{R},\ f(p_1, p_2, p_3) = -\sum_{i=1}^{3} p_i \log_2(p_i),$$

   known as information entropy (a measure of uncertainty for the probability distribution).

   (a) Using Lagrange multipliers, find $p_1, p_2, p_3$ that maximize the entropy function $f$. **0.75p**
   (b) Generalize to $n$ probabilities $p_1, \dots, p_n$. **0.25p**

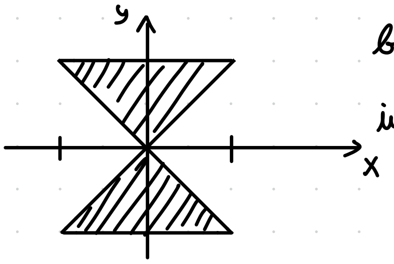
7. Consider the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $a, b > 0$.

   (a) Find the equation of the tangent line to the ellipse at a point $(x_0, y_0)$. **1p**
   (b) Find the area enclosed by the ellipse, for example by using a double integral. **1p**

Time: 2h. The marks in the final exam add up to **10p**.

2) a) $\{(x,y) \in \mathbb{R}^2 \mid |x| < |y| < 1\} = A$

$\text{int}(A) = ?$ $\quad$ $\text{bd}(A) = ?$




$\text{bd}(A) = \{(x,x) \in \mathbb{R}^2\} \cup \{(x,-x) \in \mathbb{R}^2 \mid |x| \leq 1\}$

$\text{int}(A) = \{(x,y) \in \mathbb{R}^2 \mid |x| < |y| < 1\}$

b) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

$$\left.\begin{array}{l} (x+y)(x+y) = \|x\|^2 + \|y\|^2 + 2xy \\ \text{but since } x \perp y \Rightarrow x \cdot y = 0 \end{array}\right\} \Rightarrow \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

3) $T_2(x,y) = ?$ $\quad f(x,y) = \sqrt{x^2+y^2} \quad (1,1)$

$$T_2(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0)(x-x_0, y-y_0) + \frac{1}{2}(x-x_0, y-y_0) \cdot H(x_0,y_0) \cdot \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \Rightarrow \nabla f(1,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{d}{dx}\frac{\partial f}{\partial x} = \frac{\overset{\sqrt{x^2+y^2}}{\cancel{\sqrt{x^2+y^2}}} - x \cdot \frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{\cancel{x^2}+y^2 - \cancel{x^2}}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = y \cdot \left((x^2+y^2)^{-\frac{1}{2}}\right)'_x = \frac{-y}{\cancel{2}} \frac{\cancel{2}x}{\sqrt{(x^2+y^2)^3}} = -\frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$H(1,1) = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} \\ \frac{-1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & -2\sqrt{2} \\ -2\sqrt{2} & 2\sqrt{2} \end{bmatrix}$$

$f(1,1) = \sqrt{2}$

$$T_2(x,y) = \sqrt{2} + \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot (x-1, y-1) + \frac{1}{2}(x-1,y-1) \cdot \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$= \cancel{\sqrt{2}} + x\frac{\sqrt{2}}{2} - \cancel{\frac{\sqrt{2}}{2}} + y\frac{\sqrt{2}}{2} - \cancel{\frac{\sqrt{2}}{2}} + \frac{1}{2}(x-1, y-1) \cdot \begin{bmatrix} \frac{x}{2\sqrt{2}} - \cancel{\frac{1}{2\sqrt{2}}} - \frac{y}{2\sqrt{2}} + \cancel{\frac{1}{2\sqrt{2}}} \\ -\frac{x}{2\sqrt{2}} + \cancel{\frac{1}{2\sqrt{2}}} + \frac{y}{2\sqrt{2}} - \cancel{\frac{1}{2\sqrt{2}}} \end{bmatrix}$$

$$= \frac{\sqrt{2}}{2}(x+y) + \frac{1}{2}(x-1, y-1) \cdot \begin{bmatrix} \frac{x-y}{2\sqrt{2}} \\ \frac{-x+y}{2\sqrt{2}} \end{bmatrix} =$$

$$= \frac{\sqrt{2}}{2}(x+y) + \frac{1}{2}\left( (x-1)\cdot\frac{x-y}{2\sqrt{2}} + (y-1)\frac{-x+y}{2\sqrt{2}}\right) =$$

$$= \frac{\sqrt{2}}{2}(x+y) + \frac{1}{2}\cdot\frac{x^2 - xy \cancel{-x} \cancel{+y} - xy + y^2 \cancel{+x} \cancel{-y}}{2\sqrt{2}} =$$

$$= \frac{4(x+y) + x^2+y^2 - 2xy}{4\sqrt{2}} =$$

$$= \frac{x^2+y^2-2xy+4(x+y)}{4\sqrt{2}}$$

4) $f(x,y) = x^3 - 3x + y^2$

$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

$\frac{\partial^2 f}{\partial x^2} = 6x$

$\frac{\partial f}{\partial y} = 2y = 0 \Rightarrow y = 0$

$\Rightarrow (1,0)$ and $(-1,0)$ critical points

$\frac{\partial^2 f}{\partial y^2} = 2$

$\frac{\partial^2 f}{\partial x \partial y} = 0 \quad \Rightarrow H(x,y) = \begin{bmatrix} 6x & 0 \\ 0 & 2 \end{bmatrix}$

$H(1,0) = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow$ eigenval. $> 0 \Rightarrow f(1,0)$ loc. min

$H(-1,0) = \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow f(-1,0)$ saddl point

5) A - symmetric and pos def.

a) $f(x) = \frac{1}{2}x^T A x - b^T x$

$\nabla f(x) = Ax - b$

$\nabla f(x) = 0 \Leftrightarrow Ax - b = 0$

$Ax = b \quad | \, A^{-1}$

A-sym $\quad x = A^{-1}b$ critical point, unique

$H(x) = A$

A is pos def $\Rightarrow f$ has a min $= x = A^{-1}b$

b) $x_{k+1} = x_k - s_k \nabla f(x_k)$

$$x_{k+1} = x_k - \alpha A x_k + \alpha b$$

$$x_{k+1} = (\mathcal{I} - \alpha A) x_k + \alpha b$$

6) $p_1 + p_2 + p_3 = 1$

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad f(p_1, p_2, p_3) = -\sum_{i=1}^{3} p_i \log_2(p_i)$$

a) $$L(p_1, p_2, p_3, \lambda) = -p_1 \log_2(p_1) - p_2 \log_2(p_2) - p_3 \log_2(p_3) + \lambda(p_1 + p_2 + p_3 - 1)$$

$$\frac{\partial L}{\partial p_1} = -\log_2(p_1) - \cancel{p_1} \cdot \frac{1}{(\ln 2)\cdot \cancel{p_1}} + \lambda = 0 \Rightarrow \log_2(p_1) = \lambda - \frac{1}{\ln 2}$$

$$p_1 = 2^{\lambda - \frac{1}{\ln 2}} =$$

$$\frac{\partial L}{\partial p_2} = -\log_2(p_2) - \frac{1}{\ln 2} + \lambda$$

$$p_2 = 2^{\lambda - \frac{1}{\ln 2}}$$

$$\frac{\partial L}{\partial p_3} = -\log_2(p_3) - \frac{1}{\ln 2} + \lambda$$

$$p_3 = 2^{\lambda - \frac{1}{\ln 2}}$$

$$\frac{\partial L}{\partial \lambda} = p_1 + p_2 + p_3 - 1 = 0$$

$$3 \cdot 2^{\lambda - \frac{1}{\ln 2}} = 1$$

$$2^{\lambda - \frac{1}{\ln 2}} = \frac{1}{3}$$

$$\lambda - \frac{1}{\ln 2} = -\log_2 3$$

$$\lambda = -\log_2 3 + \underbrace{\frac{1}{\ln 2}}_{\frac{\log_2 2}{\log_2 e}} = -\log_2 3 + \frac{\log_2 e}{\log_2 2} = \log_2 \frac{e}{3} = \frac{1 - \ln 3}{\ln 2} > 0$$

OR

$$\frac{\partial L}{\partial p_1} = -\log_2(p_1) - 1 + \lambda = 0 \Rightarrow p_1 = 2^{\lambda - 1}$$

$$\log_2(p_1 \cdot p_2 \cdot p_3) = \lambda - 3$$

$$p_1 \cdot p_2 \cdot p_3 = 2^{\lambda - 3} = \frac{2^\lambda}{3} = \frac{2^\lambda}{8}$$

$$p_1 + p_2 + p_3 = 1$$

$$p_1 = p_2 = p_3 = \frac{1}{3}$$

7. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) tan $(x_0, y_0) = ?$

(b) $\iint_D \frac{x^2}{a^2}+\frac{y^2}{b^2}\,dx\,dy \qquad D=\{(x,y)\in\mathbb{R}^2 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1\}$

$\frac{x}{a} = r\cos\theta \qquad r^2 = 1 \Rightarrow r\in[0,1]$

$\frac{y}{b} = r\sin\theta \qquad \theta\in[0,2\pi]$

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{bmatrix}$$

$$|\det J| = abr\cos^2\theta + abr\sin^2\theta = abr$$

$$\int_0^{2\pi}\int_0^1 r^2 abr\,dr\,d\theta = 2\pi ab\int_0^1 r^3\,dr = 2\pi ab\,\frac{r^4}{4}\Big|_0^1 = \frac{\pi}{2}ab \; ?$$

$$\int_0^1\int_0^{2\pi} abr\,d\theta\,dr = \; ?$$

(a) $f(x,y) = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

$$\left.\begin{aligned}\frac{\partial f}{\partial x} &= \frac{2x}{a^2}\\ \frac{\partial f}{\partial y} &= \frac{2y}{b^2}\end{aligned}\right\} \Rightarrow \nabla f(x_0,y_0) = \left(\frac{2x_0}{a^2}, \frac{2y_0}{b^2}\right)$$

tangent line: $\nabla f(x_0,y_0)\cdot(x-x_0, y-y_0) = 0$

$$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)=0$$

$$\frac{2xx_0}{a^2}-\frac{2x_0^2}{a^2}+\frac{2yy_0}{b^2}-\frac{2y_0^2}{b^2}=0$$

$$a^2y_0(y-y_0)+b^2x_0(x-x_0)=0$$

$$a^2y_0(y-y_0) = -b^2x_0(x-x_0)$$

Mathematical Analysis<br>1st Year | Babeș-Bolyai University<br>Bachelor in Computer Science

# Final Exam #2

1. Study the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^k}$, with $k > 1$. **1p**

2. (a) Draw the interior and the boundary of the set $\{(x, y) \in \mathbb{R}^2 \mid |y| < |x| < 1\}$. **1p**
   (b) Let $x, y \in \mathbb{R}^n$ with $\|x\| = \|y\|$. Prove that $x + y$ and $x - y$ are orthogonal. **1p**

3. Find the second order Taylor polynomial for $f(x, y) = e^{-x^2-y^2}$ around $(1, 1)$. **1p**

4. Find and classify all the critical points of $f(x, y) = x^3 + y^3 - 6xy$. **1p**

5. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and let $b \in \mathbb{R}^n$. Consider the function

$$f : \mathbb{R}^n \to \mathbb{R},\ f(x) = \frac{1}{2}x^T Ax - \underbrace{b^T x.}_{\text{const}}$$

   (a) Prove that $f$ has a unique minimum, which satisfies the equation $Ax = b$. **1p**
   (b) Write a gradient descent method for finding the minimum of $f$. **1p**

6. Let the probabilities $p_1, p_2, p_3 \in (0, 1)$ with $p_1 + p_2 + p_3 = 1$. Consider the function

$$f : \mathbb{R}^3 \to \mathbb{R},\ f(p_1, p_2, p_3) = -\sum_{i=1}^{3} p_i \log_2(p_i),$$

   known as information entropy (a measure of uncertainty for the probability distribution).

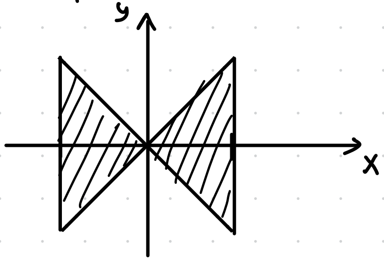
   (a) Using Lagrange multipliers, find $p_1, p_2, p_3$ that maximize the entropy function $f$. **0.75p**
   (b) Generalize to $n$ probabilities $p_1, \ldots, p_n$. **0.25p**

7. Consider the ellipse with equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

   (a) Find the equation of the tangent line to the ellipse at a point $(x_0, y_0)$. **1p**
   (b) Find the area enclosed by the ellipse, for example by using a double integral. **1p**

Time: 2h. The marks in the final exam add up to **10p**.

2) a) $A = \{ (x,y) \in \mathbb{R}^2 \mid |y| < |x| < 1 \}$




$bd(A) = \{ (x,x) \in \mathbb{R}^2 \mid |x| \leq 1 \} \cup \{ (x,-x) \in \mathbb{R}^2 \mid |x| < 1 \}$

$int(A) = \{ (x,y) \in \mathbb{R}^2 \mid |y| < |x| < 1 \}$

b) $\|x\| = \|y\|$

$(x+y) \perp (x-y) \Rightarrow (x+y)(x-y) = 0$

$\|x\|^2 - \|y\|^2 + \cancel{xy} - \cancel{xy} = 0$

$\|x\|^2 - \|y\|^2 = 0 \Rightarrow \|x\| = \|y\|$

3) $f(x,y) = e^{-x^2-y^2}$ at $(1,1)$

$$T_2(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0)(x-x_0, y-y_0) + \frac{1}{2}(x-x_0, y-y_0)\cdot H(x_0,y_0)\cdot \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

$\frac{\partial f}{\partial x} = -2x e^{-x^2-y^2}$

$\frac{\partial f}{\partial y} = -2y e^{-x^2-y^2}$

$f(1,1) = e^{-1-1} = \frac{1}{e^2}$

$\nabla f(1,1) = \left(-2\cdot\frac{1}{e^2}, -2\cdot\frac{1}{e^2}\right)$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= -2e^{-x^2-y^2} + 4x^2 e^{-x^2-y^2} \\ \frac{\partial^2 f}{\partial y^2} &= -2e^{-x^2-y^2} + 4y^2 e^{-x^2-y^2} \\ \frac{\partial^2 f}{\partial x \partial y} &= 4xy e^{-x^2-y^2} \end{aligned}\right\} \Rightarrow H(1,1) = \begin{bmatrix} -2\cdot\frac{1}{e^2} + 4\cdot\frac{1}{e^2} & \frac{4}{e^2} \\ \frac{4}{e^2} & \frac{2}{e^2} \end{bmatrix}$$

$$T_2(x,y) = \frac{1}{e^2} + \left(\frac{-2}{e^2}, \frac{-2}{e^2}\right)\cdot(x-1, y-1) + \frac{1}{2}(x-1, y-1)\cdot \begin{bmatrix} \frac{2}{e^2}(x-1) + \frac{4}{e^2}(y-1) \\ \frac{4}{e^2}(x-1) + \frac{2}{e^2}(y-1) \end{bmatrix}$$

$$= \frac{1}{e^2} - \frac{2x}{e^2} + \frac{2}{e^2} - \frac{2y}{e^2} + \frac{2}{e^2} + \frac{1}{2}\left(\frac{2}{e^2}(x-1)^2 + \frac{4}{e^2}(x-1)(y-1) + \frac{4}{e^2}(x-1)(y-1) + \frac{2}{e}(y-1)^2\right) =$$

$$= 1 - 2x + 2 - 2y + 2 + \frac{1}{2}\left(\underline{2x^2} - 4x + 2 + 4xy - 4x - 4y + 4 + 4xy - 4x - 4y + 4 + \underline{2y^2} - 4y + 2\right)$$

$$= x^2 + y^2 - 8x - 8y + 10 + 4xy$$

4) $f(x,y) = x^3 + y^3 - 6xy$

$$\frac{\partial f}{\partial x} = 3x^2 - 6y = 0 \Rightarrow x^2 = 2y \Rightarrow \frac{y^4}{4} = 2y \Rightarrow \frac{y^3}{8} = 1 \Rightarrow y^3 = 8 \Rightarrow y = 2$$

$$\frac{\partial f}{\partial y} = 3y^2 - 6x = 0 \Rightarrow y^2 = 2x \Rightarrow x = \frac{y^2}{2} \Rightarrow x = 2$$

$$\Rightarrow x = y = 2$$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 6x \\ \frac{\partial^2 f}{\partial y^2} &= 6y \\ \frac{\partial^2 f}{\partial x \partial y} &= -6 \end{aligned}\right\} \Rightarrow H(2,2) = \begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix}$$

$$\det(H - \lambda I_n) = 0$$

$$\begin{vmatrix} 12-\lambda & -6 \\ -6 & 12-\lambda \end{vmatrix} = (12-\lambda)^2 - 36 = 144 - 24\lambda + \lambda^2 - 36 = 0$$

$$\lambda^2 - 24\lambda + 108 = 0$$

$$D = 576 - 432 = 144$$

$$\lambda_{1,2} = \frac{24 \pm 12}{2} = \left\langle \begin{aligned} \lambda_1 &= 18 \\ \lambda_2 &= 6 \end{aligned} \right\} > 0 \Rightarrow \text{loc. min}$$

7) $f(x,y) = \frac{x^2}{9} + \frac{y^2}{4} - 1$

a) $$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{2}{9}x \\ \frac{\partial f}{\partial y} &= \frac{1}{2}y \end{aligned}\right\} \Rightarrow \nabla f(x,y) = \left(\frac{2}{9}x, \frac{1}{2}y\right)$$

$$\nabla f(x,y) \cdot (x-x_0, y-y_0) = 0$$

$$\left(\frac{2}{9}x, \frac{1}{2}y\right)(x-x_0, y-y_0) = 0$$

$$\frac{2x^2 - 2xx_0}{9} + \frac{y^2 - yy_0}{2} = 0$$

$$4x(x-x_0) + 9y(y-y_0) = 0$$

b) $$\iint_D \frac{x^2}{9} + \frac{y^2}{4} \, dx\, dy \qquad D = \left\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{9} + \frac{y^2}{4} \leq 1\right\}$$

$$\frac{x}{3} = r\cos\theta \qquad r^2 \leq 1 \Rightarrow r \in [0,1]$$

$$\frac{y}{2} = r\sin\theta \qquad \theta \in [0, 2\pi]$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 3\cos\theta & -3r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{bmatrix}$$

$$\Rightarrow |\det J| = 6r\cos^2\theta + 6r\sin^2\theta = 6r$$

$$\int_0^{2\pi}\int_0^1 r^2\, 6r\, dr\, d\theta = \cancel{2}\pi \cdot \overset{3}{\cancel{6}}\,\frac{r^4}{\cancel{4}}\Big|_0^1 = 3\pi$$