Seminar 6

1. Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:
(*i*) v_1, v_2, v_3 are linearly dependent and determine a dependence relationship.
(*ii*) v_1, v_2 are linearly independent.

2. Prove that the following vectors are linearly independent:
(*i*) $v_1 = (1, 0, 2)$, $v_2 = (-1, 2, 1)$, $v_3 = (3, 1, 1)$ in ${}_{\mathbb{R}}\mathbb{R}^3$.
(*ii*) $v_1 = (1, 2, 3, 4)$, $v_2 = (2, 3, 4, 1)$, $v_3 = (3, 4, 1, 2)$, $v_4 = (4, 1, 2, 3)$ in ${}_{\mathbb{R}}\mathbb{R}^4$.

3. Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly independent.

4. Let $v_1 = (1, -2, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (0, a, 1, 2)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^4$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly dependent.

5. Let $v_1 = (1, 1, 0)$, $v_2 = (-1, 0, 2)$, $v_3 = (1, 1, 1)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$.
(*i*) Show that the list (v_1, v_2, v_3) is a basis of the real vector space $\mathbb{R}^3$.
(*ii*) Express the vectors of the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$ as a linear combination of the vectors v_1, v_2 and v_3.
(*iii*) Determine the coordinates of $u = (1, -1, 2)$ in each of the two bases.

6. Let $n \in \mathbb{N}^*$. Show that the vectors

$$v_1 = (1, \dots, 1, 1), v_2 = (1, \dots, 1, 2), v_3 = (1, \dots, 1, 2, 3), \dots, v_n = (1, 2, \dots, n-1, n)$$

form a basis of the real vector space $\mathbb{R}^n$ and write the coordinates of a vector $(x_1, \dots, x_n)$ in this basis.

7. Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the lists (E_1, E_2, E_3, E_4) and (A_1, A_2, A_3, A_4) are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in each of the two bases.

8. Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X - a, (X - a)^2)$ $(a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial $f = a_0 + a_1 X + a_2 X^2 \in \mathbb{R}_2[X]$ in each basis.

9. Determine the number of bases of the vector space $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$.

10. Determine the number of elements of the general linear group $(GL_3(\mathbb{Z}_2), \cdot)$ of invertible 3×3-matrices over $\mathbb{Z}_2$.

V K - vector space

$v_1, v_2, \dots, v_n \in V$ linearly independent if

$\forall \alpha_1, \dots, \alpha_2 \in K$ $\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

$\alpha_1 v_1 + \alpha_1 v_2 + \dots + \alpha_n v_n = 0$

If $V = K^n$ (standard K vector space)

then # linearly independent vectors in a certain list in $(v_1, \dots v_n)$

is the rank $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ = rank $(v_1^t \mid v_2^t \mid \dots \mid v_n^t)$

So the vectors $v_1, \dots, v_n$ are linearly independent $\Leftrightarrow$ rank $\begin{pmatrix} v_1 \\ v_n \\ \vdots \\ v_n \end{pmatrix} = n$

Ex: $v_1 = (1,2,3,4)$ $\quad v_2 = (5,6,7,8)$ $\quad v_3 = (9,10,11,12)$

$$\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline v_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \qquad \left(v_1^t \mid v_2^t \mid v_3^t \right) = \begin{pmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{pmatrix}$$

! Can be used only if $V = K^n$ (standard vector space) → not on matrices or polynomials

1. Let $v_1 = (1,-1,0)$, $v_2 = (2,1,1)$, $v_3 = (1,5,2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:

(*i*) v_1, v_2, v_3 are linearly dependent and determine a dependence relationship.

(*ii*) v_1, v_2 are linearly independent.

(i) v_1, v_2, v_3 linearly dependent $\iff \exists\, k_1, k_2, k_3$ not all zeros s.t. $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$$k_1(1,-1,0) + k_2(2,1,1) + k_3(1,5,2) = (0,0,0)$$

$$(k_1 + 2k_2 + k_3,\ -k_1 + k_2 + 5k_3,\ k_2 + 2k_3) = (0,0,0)$$

$$\begin{cases} k_1 + 2k_2 + k_3 = 0 \\ -k_1 + k_2 + 5k_3 = 0 \quad \Rightarrow -k_1 + 3k_3 = 0 \\ \qquad\qquad\qquad\qquad\qquad\quad 3k_3 = k_1 \\ k_2 + 2k_3 = 0 \\ k_2 = -2k_3 \Rightarrow k_3 = -\frac{1}{2}k_2 \end{cases}$$

$$\bar{A} = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 1 & 5 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

let $k_3 = \alpha \Rightarrow k_2 = -2\alpha \quad k_1 = 3\alpha$

$\Rightarrow \quad 3\alpha \cdot v_1 - 2\alpha v_2 + \alpha v_3 = 0$

if $\alpha = 1 \Rightarrow \quad 3v_1 - 2v_2 + v_3 = 0$

(ii) v_1, v_2 linearly independent

$$\alpha_1 v_1 + \alpha_2 v_2 = 0 \Rightarrow \alpha_1 = \alpha_2 = 0$$

$$\alpha_1(1,-1,0) + \alpha_2(2,1,1) = 0$$

$$\begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ -\alpha_1 + \alpha_2 = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0 \Rightarrow \text{linearly independent}$$

3. Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly independent.

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = (0,0,0)$$

$$(\alpha_1, a\alpha_1, 0) + (a\alpha_2, \alpha_2, \alpha_2) + (\alpha_3, 0, a\alpha_3) = (0,0,0)$$

$$\begin{cases} \alpha_1 + a\alpha_2 + \alpha_3 = 0 \\ a\alpha_1 + \alpha_2 = 0 \\ \alpha_2 + a\alpha_3 = 0 \end{cases}$$

$$\alpha_2 = -a\alpha_3$$

$$A = \begin{pmatrix} 1 & a & 1 \\ a & 1 & 0 \\ 0 & 1 & a \end{pmatrix}$$

$$a + a + 0 - 0 - 0 - a^3 = -a^3 + 2a = a(2 - a^2) \neq 0 \Rightarrow \begin{cases} a \neq 0 \\ 2 - a^2 \neq 0 \Rightarrow a \neq \pm\sqrt{2} \end{cases} \Rightarrow a \in \mathbb{R} \setminus \{\pm\sqrt{2}, 0\}$$

$V \to K$ vector space

$B = (v_1, \dots, v_n)$

B basis for $V \Leftrightarrow$ (i) $\langle v_1, \dots, v_n \rangle = V$ (system of generators) $\Leftrightarrow$

(ii) $v_1, \dots, v_n$ linearly independent

$$\boxed{\forall\ v \in V \quad \exists! \ \alpha_1, \dots, \alpha_n \in K : \quad v = \alpha_1 v_1 + \dots + \alpha_n v_n}$$
unique

$\leftarrow$ use this one probably

$\ast$ if you know the dymention of the vectorspace, you can prove the linearly independence & note that n (from v_n) $=$ dymention

8. Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X - a, (X - a)^2)$ $(a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial $f = a_0 + a_1 X + a_2 X^2 \in \mathbb{R}_2[X]$ in each basis. $[f]_E$ and $[f]_B$

! ~~not~~ vectors of polynom

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ $\qquad u = \alpha_1 + \alpha_2 X + \alpha_3 X^2$

$$\Rightarrow \begin{cases} \alpha_1 = a_0 \\ \alpha_2 = a_1 \\ \alpha_3 = a_2 \end{cases} \Rightarrow E = (1, x, x^2) \text{ is a base of } \mathbb{R}_2[x]$$

• Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ s.t. $u = \alpha_1 \cdot 1 + \alpha_2(X-a) + \alpha_3(X-a)^2$

$u = \alpha_1 + \alpha_2 X - \alpha_2 \cdot a + \alpha_3 X^2 - 2\alpha_3 \cdot aX + \alpha_3 a^2$

$u = (\alpha_1 - \alpha_2 a - \alpha_3 a^2) + (\alpha_2 - 2\alpha_3 a) X + \alpha_3 X^2$

$$\Rightarrow \begin{cases} \alpha_1 - \alpha_2 a + \alpha_3 a^2 = a_0 \\ \alpha_2 - 2\alpha_3 a = a_1 \\ \alpha_3 = a_2 \end{cases}$$

$$\begin{cases} \alpha_3 = a_2 \\ \alpha_2 = a_1 + 2a \cdot a_2 \\ \alpha_1 = a_0 - a_2 a^2 + a_1 \cdot a + 2a^2 a_2 \end{cases}$$

unique solution $\Rightarrow \mathcal{B}$ is a basis

$$[f]_{\mathcal{B}} = \begin{pmatrix} a_0 + a \cdot a_1 + a^2 a_2 \\ a_1 + 2a \cdot a_2 \\ a_2 \end{pmatrix}$$

7. Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the lists (E_1, E_2, E_3, E_4) and (A_1, A_2, A_3, A_4) are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in each of the two bases.

show $\mathcal{B} = (A_1, A_2, A_3, A_4)$ basis of $\mathcal{M}_2(\mathbb{R})$ find $\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{\mathcal{B}}$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$

$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = M$, $M \in \mathcal{M}_2(\mathbb{R})$

$\alpha_1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = M$

$$\begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_3 + \alpha_4 \\ \alpha_3 + \alpha_4 & \alpha_4 \end{pmatrix} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = x & \Rightarrow \alpha_1 = x - y \\ \alpha_2 + \alpha_3 + \alpha_4 = y & \Rightarrow \alpha_2 = y - z \\ \alpha_3 + \alpha_4 = z & \Rightarrow \alpha_3 = z - t \\ \alpha_4 = t & \alpha_4 = t \end{cases}$$

unique sol $\Rightarrow$

$\Rightarrow \mathcal{B}$ basis

$$\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_B = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x-y \\ y-z \\ z-t \\ t \end{pmatrix} = \begin{pmatrix} 2-1 \\ 1-1 \\ 1-0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} ?$$

$$= \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$