2. ★ For $x, y \in \mathbb{R}^n$ prove that the following statements are equivalent:

(a) $\langle x, y\rangle = 0$. (b) $\|x+y\| = \|x-y\|$. (c) $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Assume $\langle x,y\rangle = 0 \Rightarrow$ $\|x+y\|^2 = (x+y)\cdot(x+y) = x\cdot x + 2xy + y\cdot y =$

$$= \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle =$$

$$= \|x\|^2 + 2\cdot 0 + \|y\|^2 =$$

$$= \|x\|^2 + \|y\|^2 \quad (1)$$

$$\|x-y\|^2 = (x-y)(x-y) = x\cdot x - 2xy + y\cdot y =$$

$$= \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle =$$

$$= \|x\|^2 - 2\cdot 0 + \|y\|^2 =$$

$$= \|x\|^2 + \|y\|^2 \quad (2)$$

from (1) and (2) $\Rightarrow$ $\|x+y\| = \|x-y\|$ (b)

Assume that $\|x+y\| = \|x-y\| \Rightarrow$

$$\|x+y\|^2 = \|x-y\|^2 \Rightarrow (x+y)(x+y) = (x-y)(x-y)$$

$$\|x\|^2 + 2\langle x,y\rangle + \|y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2$$

$$\Rightarrow 2\langle x,y\rangle = -2\langle x,y\rangle \Rightarrow \langle x,y\rangle = 0$$

$$\Rightarrow \|x\|^2 + \|y\|^2 = \|x+y\|^2 \quad (c)$$

Assume that $\|x+y\|^2 = \|x\|^2 + \|y\|^2 \Rightarrow$

$$(x+y)(x+y) = \|x\|^2 + \|y\|^2$$

$$\|x\|^2 + 2\langle x,y\rangle + \|y\|^2 = \|x\|^2 + \|y\|^2 \Rightarrow \langle x,y\rangle = 0 \quad (a)$$

Since
$$\left.\begin{array}{l}(a) \Rightarrow (b)\\ (b) \Rightarrow (c)\\ (c) \Rightarrow (a)\end{array}\right\} \Rightarrow (a),(b),(c) \text{ are equivalent}$$