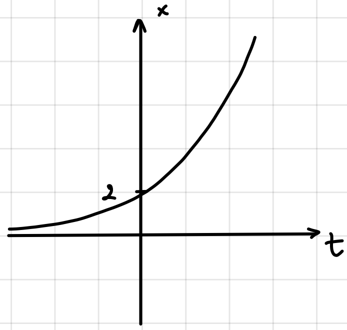
1.1.1. $f: \mathbb{R} \to \mathbb{R}$, $f(t) = 2e^{3t}$ - sol. of IVP $\begin{cases} x' = 3x \\ x(0) = 2 \end{cases}$ $\iff$ $x'(t) = 3x(t)$

$x = x(t) \to$ unknown sol.

$$\left.\begin{array}{l} f'(t) = 6e^{3t} \\ 3f(t) = 3 \cdot 2e^{3t} = 6e^{3t} \end{array}\right\} \Rightarrow f'(t) = 3f(t)$$

$f(0) = 2e^0 = 2$



f is not periodic

$\hookrightarrow f(t) \neq f(t+T)$, $\forall t \in \mathbb{R}$

f is increasing

$\lim_{t \to -\infty} f(t) = 0$

$\lim_{t \to \infty} f(t) = \infty$

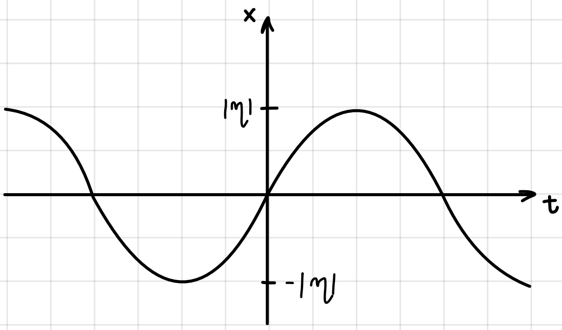
1.1.2 $\eta \in \mathbb{R}^*$

$f: \mathbb{R} \to \mathbb{R}$, $f(t) = \eta \sin t$ - sol of IVP: $\begin{cases} x'' + x = 0 \\ x(0) = 0 \\ x'(0) = \eta \end{cases}$

$$\left.\begin{array}{l} f''(t) = -\eta \sin t \\ f(t) = \eta \sin t \end{array}\right\} f''(t) + f(t) = 0$$

$f(0) = \eta \sin 0 = 0$

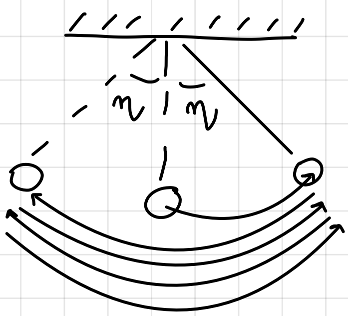
$f'(0) = \eta \cos 0 = \eta$



f is periodic

f is bounded, $f(t) \in [-|\eta|, |\eta|]$

oscill + const ampl $\iff$ periodic



pendulum

1.1.3 $f: \mathbb{R} \to \mathbb{R}$, $f(t) = e^{-2t}\cos t$ - sol. of IVP $\begin{cases} x'' + 4x' + 5x = 0 \\ x(0) = 1 \\ x'(0) = -2 \end{cases}$

$$f'(t) = -2e^{-2t} \cdot \cos t - e^{-2t}\sin t = -e^{-2t}(2\cos t + \sin t)$$

$$f''(t) = 2e^{-2t}(2\cos t + \sin t) - e^{-2t}(-2\sin t + \cos t) = e^{-2t}(3\cos t + 4\sin t)$$

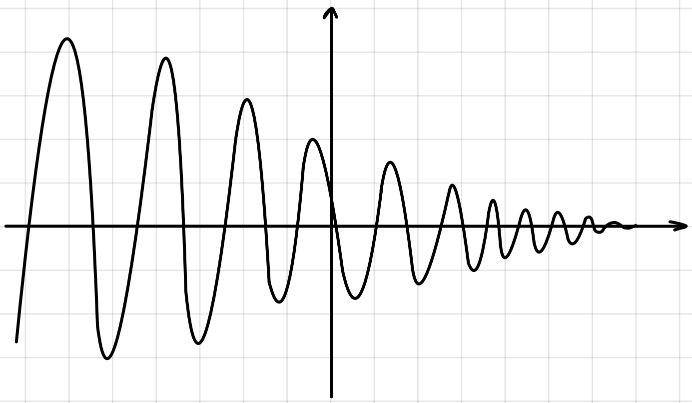
$$f''(t) + 4f'(t) + 5f(t) \stackrel{?}{=} 0$$

$$e^{-2t}(3\cos t + 4\sin t) - 4e^{-2t}(2\cos t + \sin t) + 5e^{-2t}\cos t \stackrel{!}{=} 0$$

$$e^{-2t}3\cos t - 8e^{-2t}\cos t + 5e^{-2t}\cos t + 4e^{-2t}\sin t - 4e^{-2t}\sin t = 0 + 0 = 0$$

$$f(0) = e^0 \cdot \cos 0 = 1$$

$$f'(0) = -e^0(2\cos 0 + \sin 0) = -2$$



$G_f \cap O_t$ equals an infinite nr. of sol.

$\Rightarrow f$ is oscill. so $\lim\limits_{t \to -\infty} f(t) = \pm\infty$ (diverg.)

$\lim\limits_{t \to \infty} f(t) = 0$

1.1.7 $f: \mathbb{R} \to \mathbb{R}$, $f(t) = \cos t$, $x'' - x = 0$

$$f'(t) = -\sin t$$

$$f''(t) = -\cos t$$

$f''(t) - f(t) = -\cos t - \cos t = -2\cos t \Rightarrow f(t)$ it is not a solution

$$x''' + x' = 0$$

$$f'''(t) = \sin t$$

$f'''(t) + f'(t) = \sin t - \sin t = 0 \Rightarrow f(t)$ is a solution

1.1.8. a) $x' = x - x^3 \iff 0 = c - c^3$

$\iff c(1 - c^2) = 0 \Rightarrow c_1 = 0$

$(1-c)(1+c) = 0$

$c_2 = 1$

$c_3 = -1$

$f(t) = c$

$f'(t) = 0$

b) $x' = \sin x \iff \sin c = 0$

$\varphi(t) = c$ $\qquad$ $c = \{k\pi / k \in \mathbb{Z}\}$ $\qquad$ c_1, c_2, c_3 are sol. for ex 1.

$\varphi'(t) = 0$

1.1.11 $\quad r = ?$ $r \in \mathbb{R}$ st. $x(t) = e^{rt}$ sol of $x'' - 5x' + 6x = 0$

$x'(t) = re^{rt}$

$x''(t) = r^2 e^{rt}$

$x''(t) - 5x'(t) + 6x(t) = 0 \iff r^2 e^{rt} - 5re^{rt} + 6e^{rt} = 0$

$$\left.\begin{array}{l} e^{rt}(r^2 - 5r + 6) = 0 \\ e^{rt} > 0 \end{array}\right\} \Rightarrow \boxed{r^2 - 5r + 6 = 0}$$

$(r-2)(r-3) = 0$

$r \in \{2, 3\}$

$x_1(t) = e^{2t}$ and $x_2(t) = e^{3t}$

$\Rightarrow x(t) = c_1 x_1 + c_2 x_2 = c_1 e^{2t} + c_2 e^{3t} \leftarrow$ general solution

1.1.12. $\quad r \in \mathbb{R}$

$x(t) = t^r$ sol. on $(0, \infty)$ for: $t^2 x'' - 4tx' + 6x = 0$

$x' = (t^r)' = rt^{r-1}$

$x'' = r(r-1)t^{r-2}$

$t^2 x'' - 4tx' + 6x = t^2 r(r-1)t^{r-2} - 4t\,rt^{r-1} + 6t^r \stackrel{?}{=} 0$

$t^r(r^2 - r - 4r + 6) = 0$

$t^r \neq 0 \Rightarrow r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$$\left.\begin{array}{l} r_1 = 3 \\ r_2 = 2 \end{array}\right| \Rightarrow \left.\begin{array}{l} x_1 = t^3 \\ x_2 = t^2 \end{array}\right\} \Rightarrow x = c_1 t^3 + c_2 t^2$$

1.1.4. $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \sin 2.2t - \sin 2t$ solution of the IVP $\left\{\begin{array}{l} x'' + 4.84x = -0.84\sin 2t \\ x(0) = 0 \\ x'(0) = 0.2 \end{array}\right.$

$\varphi''(t) = (2.2\cos 2.2t - 2\cos 2t)' = -4.84\sin 2.2t + 4\sin 2t$

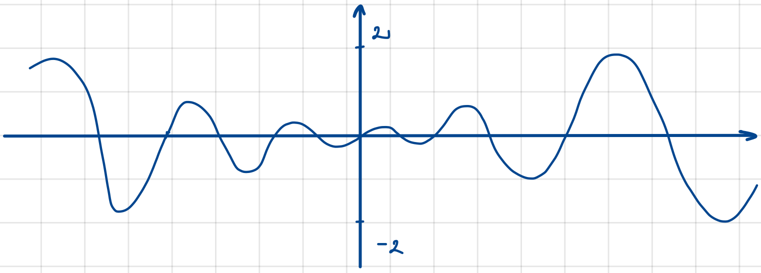
$\varphi''(t) + 4.84\varphi(t) \stackrel{?}{=} -0.84\sin 2t$

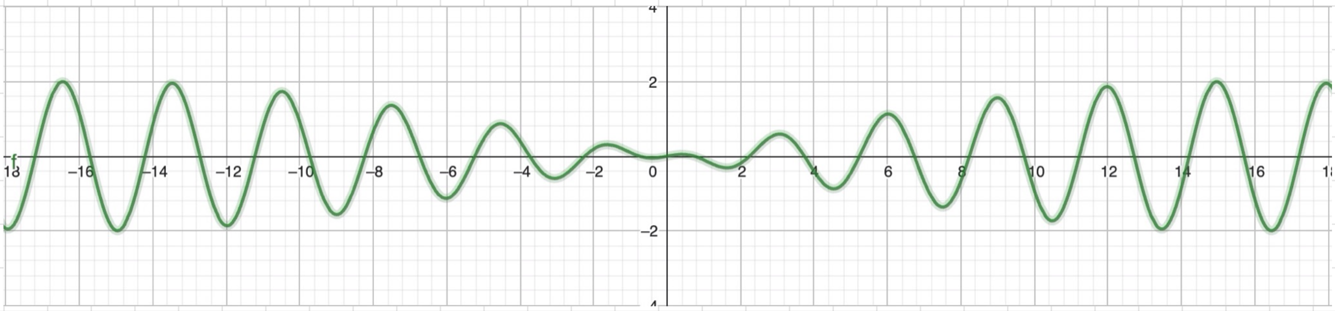
$\cancel{-4.84\sin 2.2t} + 4\sin 2t + \cancel{4.84\sin 2.2t} - 4.84\sin 2t = -0.84\sin 2t$

$f(0) = \sin 2.2\cdot 0 + \sin 2\cdot 0 = 0+0=0$

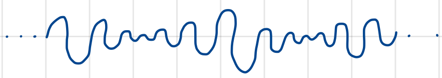
$f'(0) = 2.2\cos 2.2\cdot 0 - 2\cos 2\, 0 = 2.2 - 2 = 0.2$

$\Rightarrow f(t)$ is a solution





f is periodic

f is oscillating 

f is bounded $[-2,2]$

1.1.5. $f:\mathbb{R}\to\mathbb{R}$, $f(t) = \sin\sqrt{6}t - \sin 2t$ solution of the IVP $\begin{cases} x'' + 6x = -2\sin 2t \\ x(0) = 0 \\ x'(0) = \sqrt{6} - 2 \end{cases}$

$f''(t) = (\sqrt{6}\cos\sqrt{6}t - 2\cos 2t)' = -6\sin t\sqrt{6} + 4\sin 2t$

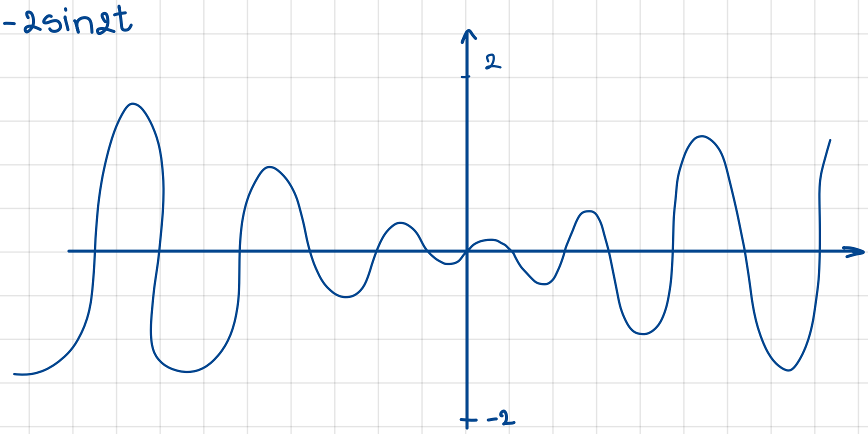
$f''(t) + 6f(t) \overset{?}{=} -2\sin 2t$

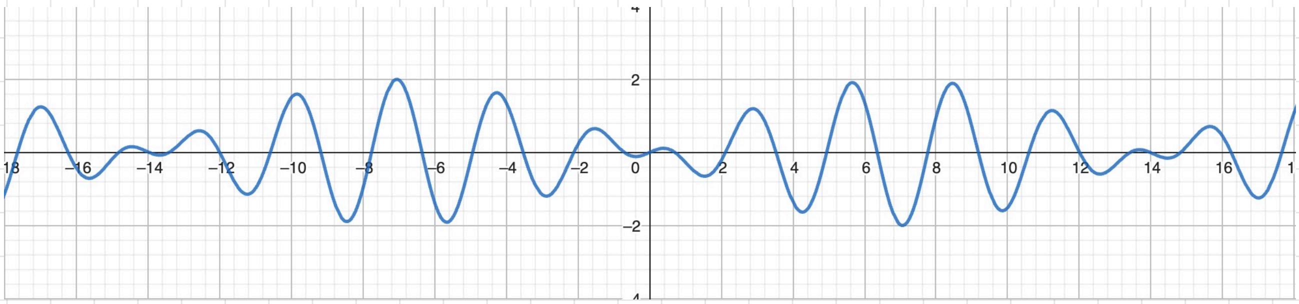
$\cancel{-6\sin\sqrt{6}t} + 4\sin 2t + \cancel{6\sin\sqrt{6}t} - 6\sin 2t = -2\sin 2t$

$f(0) = \sin\sqrt{6}\cdot 0 + \sin 2\cdot 0 = 0+0=0$

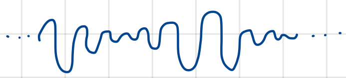
$f'(0) = \sqrt{6}\cos\sqrt{6}\cdot 0 - 2\cos 2\cdot 0 = \sqrt{6} - 2$

$\Rightarrow f(t)$ is a solution





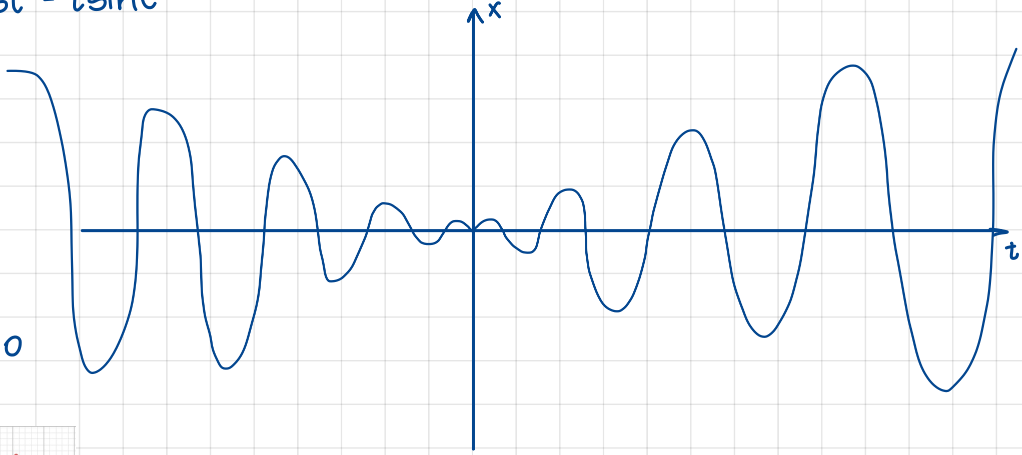
f is periodic

f is oscillating 

f is bounded $[-2,2]$

1.1.6. $f: \mathbb{R} \to \mathbb{R}$, $f(t) = t\sin t$ solution of the IVP $\begin{cases} x'' + x = 2\cos t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

$$f''(t) = (\sin t + t\cos t)' = \cos t + \cos t - t\sin t$$
$$= 2\cos t - t\sin t$$

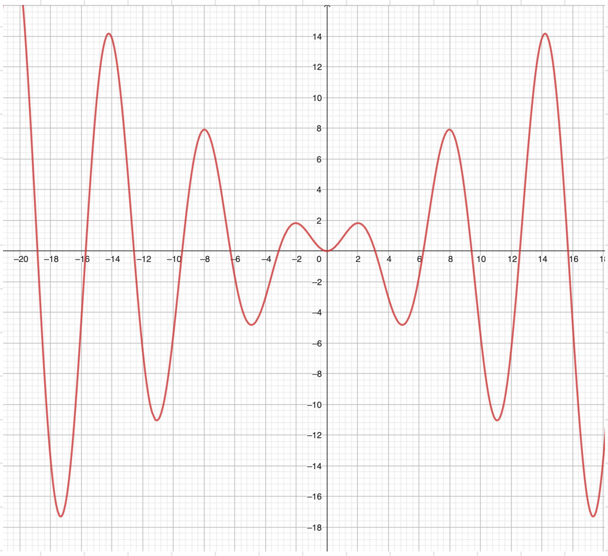


$$f''(t) + f(t) \overset{?}{=} 2\cos t$$

$$2\cos t - \cancel{t\sin t} + \cancel{t\sin t} = 2\cos t$$

$$f(0) = 0 \cdot \sin 0 = 0$$

$$f'(0) = \sin 0 + 0 \cos 0 = 0 + 0 \cdot 1 = 0$$



f is ocillating : $\lim_{t \to \pm\infty} f(t) = \pm\infty$

$\lim_{t \to 0} f(t) = 0$

$G_f \cap O_t$ = infinite nr. of sol

1.1.7. $f: \mathbb{R} \to \mathbb{R}$, $f(t) = \cos t$ solution of : $x' + x = 0$ or $x^{(4)} + x'' = 0$

$$f'(t) = -\sin t$$

$f'(t) + f(t) = -\sin t + \cos t \Rightarrow f(t)$ is not a solution

$$f''(t) = -\cos t$$

$$f'''(t) = \sin t$$

$$f^{(4)}(t) = \cos t$$

$f^{(4)}(t) + f''(t) = \cos t - \cos t = 0 \Rightarrow f(t)$ is a solution

1.1.8 Find all constant solutions for the d.e :

c) $x' = \dfrac{x+1}{2x^2+5} \Rightarrow 0 = \dfrac{c+1}{\underbrace{2c^2+5}_{>0}} \Leftrightarrow c+1 = 0 \Leftrightarrow c_1 = -1$

$c_2 = i^2$

d) $x' = x^2 + x + 1 \Rightarrow 0 = c^2 + c + 1$

$\Delta = -3 \Rightarrow \sqrt{\Delta} = i\sqrt{3} \Rightarrow c_1 = \frac{-1 + i\sqrt{3}}{2}$

$c_2 = \frac{-1 - i\sqrt{3}}{2}$

e) $x' = x + 4x^3 \Rightarrow 0 = c + 4c^3$

$0 = c\underbrace{(1 + 4c^2)} \Rightarrow c_1 = 0$

$\hookrightarrow c_2 = \frac{i}{2}$

$c_2 = \frac{-i}{2}$

f) $x' = -1 + x + 4x^3 \Rightarrow 0 = -1 + c + 4c^3$

$(c-p)(c-q)(c-r) = 0$

$p + q + r = 0$

$pq + qr + pr = \frac{1}{4} \rightarrow pq + q(-p-q) + p(-p-q) = \frac{1}{4}$

$pqr = -\frac{1}{4}$ $\qquad \cancel{pq} - \cancel{pq} - q^2 - p^2 - pq = \frac{1}{4}$

$p^2 + pq + q^2 = \frac{1}{4}$

1.1.9

LDECC → subchategory of LDE
$\underset{\sim}{C}$onstant coefficients

$\underbrace{x'' + x' + x}_{\text{linear combination}} = t$ $x = x(t)$

1.4.1. ?gen. solution

a) $x' + 6x = 0$
b) $x'' + 4x' + 4x = 0$
g) $x'' + x' + x = 0$
h) $x^{(4)} - x = 0$

→ step 1: the characteristic eq. ⇒ $r_1, \dots, r_n$ roots $\in \mathbb{R} \vee \mathbb{C}$
↳ instead of derivative we have the degree

→ step 2: a) $r_1 \in \mathbb{R}$ simple root ⇒ $x_1 = e^{r_1 t}$
b) $r_1 = r_2 = \dots = r_k \in \mathbb{R}$ multiple roots ⇒ $\begin{cases} x_1 = e^{r_1 t} \\ x_2 = te^{r_1 t} \\ \dots \\ x_k = t^{k-1} e^{r_1 t} \end{cases}$

c) $r_{1,2} = \alpha \pm i\beta \in \mathbb{C} \setminus \mathbb{R}$ ⇒ $\begin{cases} x_1 = e^{\alpha t} \cos \beta t \\ x_2 = e^{\alpha t} \sin \beta t \end{cases}$

d) $r_{1,2,3,4} = \alpha \pm i\beta$ ⇒ $\begin{cases} x_1 = e^{\alpha t} \cos \beta t \\ x_2 = e^{\alpha t} \sin \beta t \\ x_3 = te^{\alpha t} \cos \beta t \\ x_4 = te^{\alpha t} \sin \beta t \end{cases}$

→ step 3: $\{x_1, \dots, x_n\}$ - fundamental system of solution ⇒ $x = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$, $c_1 \dots c_n \in \mathbb{R}$

a) $x' + 6x = 0$

step 1: the characteristic eq: $r + 6 = 0$
$r_1 = -6$ ⇒ $x = e^{-6t}$

step 3: $x = c \cdot e^{-6t}$, $c \in \mathbb{R}$

b) $x'' + 4x' + 4 = 0$
$r^2 + 4r + 4 = 0$
$(r+2)^2 = 0$ ⇒ $r_1 = r_2 = -2$ → $x_1 = e^{-2t}$
$x_2 = te^{-2t}$

⇒ $x = c_1 e^{-2t} + c_2 te^{-2t}$, $c_1, c_2 \in \mathbb{R}$

g) $x'' + x' + x = 0$

$r^2 + r + 1 = 0$

$\Delta = -3 \Rightarrow r_1 = \dfrac{-1 + i\sqrt{3}}{2} = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$

$r_2 = \dfrac{-1 - i\sqrt{3}}{2} = -\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$

$\Rightarrow x_1 = e^{-\frac{t}{2}} \cos \dfrac{\sqrt{3}}{2}t$

$x_2 = e^{-\frac{t}{2}} \sin \dfrac{\sqrt{3}}{2}t$

$\Rightarrow x = c_1 e^{-\frac{t}{2}} \cos \dfrac{\sqrt{3}}{2}t + c_2 e^{-\frac{t}{2}} \sin \dfrac{\sqrt{3}}{2}t, \; c_1, c_2 \in \mathbb{R}$

h) $x^{(4)} - x = 0$

$r^4 - 1 = 0 \Leftrightarrow (r^2 - 1)(r^2 + 1) = 0 \longrightarrow r_1 = 1, \; r_2 = -1, \; r_3 = i, \; r_4 = -i$

$x_1 = e^{r_1 t} = e^t$ $\qquad x_3 = e^{\alpha t} \cos \beta t = \cos t$

$x_2 = e^{r_2 t} = e^{-t}$ $\qquad x_4 = e^{\alpha t} \sin \beta t = \sin t$

$\Rightarrow x = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t, \; c_1, c_2, c_3, c_4 \in \mathbb{R}$

1.4.2

a) e^{-3t} and e^{5t} — Minimul order : the least solutions

b) $5e^{-3t}$ and $-3e^{5t}$

d) $5te^{-3t}$ and $-3e^{5t}$

g) $(5-3t)e^{-3t}$

e) $(t-1)^2$

j) $-t\sin 3t$

a) $\left.\begin{array}{l} x_1 = e^{-3t} \Rightarrow r_1 = -3 \\ x_2 = e^{5t} \Rightarrow r_2 = 5 \end{array}\right\} \Rightarrow (r+3)(r-5) = 0$

$\Downarrow$

$r^2 - 2r - 15 = 0$

$x'' - 2x' - 15x = 0$

$x = c_1 e^{-3t} + c_2 e^{5t}$ - gen. sol.

b) $\left.\begin{array}{l} x_1 = 5e^{-3t} \Rightarrow r_1 = -3 \\ x_2 = -3e^{5t} \Rightarrow r_2 = 5 \end{array}\right\} \Rightarrow (r+3)(r-5) = 0$

$\Downarrow$

$$r^2 - 2r - 15 = 0$$
$$x'' - 2x' - 15x = 0$$
$$x = c_1 5e^{-3t} + c_2(-3)e^{5t} \quad , c_1, c_2 \in \mathbb{R}$$
$$x = c_3 e^{-3t} + c_4 e^{5t} \quad , c_3, c_4 \in \mathbb{R}$$

d) $x_1 = 5te^{-3t} \Rightarrow r_{1,2} = -3 \Rightarrow x_2 = e^{-3t} \Rightarrow r_2 = -3$

$x_3 = -3e^{5t} \Rightarrow r_3 = 5$

$$\Rightarrow (r+3)^2(r-5) = 0 \Rightarrow (r^2+6r+9)(r-5) = 0 \Rightarrow r^3 + r^2 + 21r - 45 = 0$$
$$\Rightarrow x''' + x'' + 21x' - 45x = 0$$
$$\Rightarrow x = c_1 te^{-3t} + c_2 e^{-3t} + c_3 e^{5t} \quad , c_1, c_2, c_3 \in \mathbb{R}$$

f) $(5-3t)e^{-3t}$

$5e^{-3t} - 3te^{-3t}$

$5e^{-3t}$ and $-3te^{-3t}$

$x_1 = 5e^{-3t} \Rightarrow r_1 = -3$

$x_2 = -3te^{-3t} \Rightarrow r_2 = -3$

$(r+3)^2 = 0 \Rightarrow r^2 + 6r + 9 = 0$

$$x'' + 6x' + 9x = 0$$
$$\Rightarrow x = c_1 e^{-3t} + c_2 te^{-3t} \quad , c_1, c_2 \in \mathbb{R}$$

e) $(t-1)^2 = t^2 - 2t + 1$

$x_1 = e^{0t} t^2 = t^2 \Rightarrow r_1 = 0$

$x_2 = e^{0 \cdot t} t = t \Rightarrow r_2 = 0$

$x_3 = e^{0 \cdot t} \cdot 1 = 1 \Rightarrow r_3 = 0$

$t''' = 0$

$x = c_1 t^2 + c_2 t + c_3 \quad , c_1, c_2, c_3 \in \mathbb{R}$

j) $x_1 = -t\sin 3t \cdot e^{0t} \Rightarrow x_2 = -t\cos 3t \cdot e^{0t}$

$x_3 = \sin 3t \cdot e^{0t}$

$x_4 = \cos 3t \cdot e^{0t}$

$\Rightarrow r_{1,3} = 0 + 3i$

$r_{2,4} = 0 - 3i$

$\Rightarrow$ characteristic eq : $(r-3i)^2(r+3i)^2 = 0$

$(r^2+9)^2 = 0$

$r^4 + 18r^2 + 81 = 0$

$\Rightarrow x^{(4)} + 18x'' + 81x = 0$

$\Rightarrow x = c_1 t\sin 3t + c_2 t\cos 3t + c_3 \sin 3t + c_4 \cos 3t$, $c_1, c_2, c_3, c_4 \in \mathbb{R}$

1.4.4.

$$\text{IVP} = \begin{cases} x'' + \tilde{\pi}^2 x = 0 \\ x(0) = 0 \\ x'(0) = \eta, \ \eta \in \mathbb{R} \end{cases}$$

1.4.5

$$\text{BVP} = \begin{cases} x'' + x = 0 \\ x(0) = 0 \\ x(\tilde{\pi}) = 0 \end{cases}$$

$x'' + \tilde{\pi}^2 x = 0$

$r^2 + \tilde{\pi}^2 = 0 \Rightarrow r_{1,2} = \pm\tilde{\pi}$

$\Rightarrow x_1 = e^{0t}\cos\tilde{\pi}t$

$x_2 = e^{0t}\sin\tilde{\pi}t$

- gen. sol. of the eq $x = c_1\cos\tilde{\pi}t + c_2\sin\tilde{\pi}t$, $c_1, c_2 \in \mathbb{R}$

$x(0) = 0 \Leftrightarrow x = c_1\cos\tilde{\pi}\cdot 0 + c_2\sin\tilde{\pi}\cdot t = 0$

$c_1 + 0 = 0 \Rightarrow c_1 = 0$

$x'(0) = -\tilde{\pi}c_1\sin\tilde{\pi}t + \tilde{\pi}c_2\cos\tilde{\pi}t = \eta$

$0 + \tilde{\pi}c_2 = \eta \Rightarrow c_2 = \frac{\eta}{\tilde{\pi}}$

- gen. sol. of the IVP : $x = \frac{\eta}{\tilde{\pi}}\sin(\tilde{\pi}t)$

$x'' + x = 0$

$r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i \Rightarrow x_1 = e^{0t}\cos 1\cdot t$

$x_2 = e^{0t}\sin 1\cdot t$

• gen. sol. of the eq: $x = c_1 \cos t + c_2 \sin t$, $c_1, c_2 \in \mathbb{R}$

$$x(0) = 0 \iff c_1 + 0 = 0 \Rightarrow c_1 = 0$$

$$x(\pi) = 0 \iff -c_1 + 0 = 0 \Rightarrow c_1 = 0$$

• gen. sol. of the BVP. $x = c_2 \sin t$, $c_2 \in \mathbb{R}$

1.4.6 $\lambda = ?$ s.t. $\exists\, x \neq 0$, $x = 2\pi$ periodic, $x =$ sol. of $x'' + \lambda x = 0$

$x'' + \lambda x = 0$

$r^2 + \lambda = 0$

I $\lambda < 0 \Rightarrow r^2 = \underbrace{-\lambda}_{>0}$

$$\Rightarrow r_1, r_2 \in \mathbb{R},\ \left.\begin{array}{l} r_1 = \sqrt{-\lambda} \Rightarrow x_1 = e^{\sqrt{-\lambda}\,t} \\ r_2 = -\sqrt{-\lambda} \Rightarrow x_2 = e^{-\sqrt{-\lambda}\,t} \end{array}\right\} \Rightarrow x = c_1 e^{\sqrt{-\lambda}\,t} + c_2 e^{-\sqrt{-\lambda}\,t}$$

→ this function can't be periodic

II $\lambda > 0 \Rightarrow r^2 = \underbrace{-\lambda}_{<0}$

$$\Rightarrow r_1, r_2 \in \mathbb{R},\ r_1 = i\sqrt{\lambda} \Rightarrow x_1 = \sin\sqrt{\lambda}\,t$$
$$r_2 = -i\sqrt{\lambda} \Rightarrow x_2 = \cos\sqrt{\lambda}\,t$$

$$\Rightarrow x = c_1 \sin\sqrt{\lambda}\,t + c_2 \cos\sqrt{\lambda}\,t$$

the main period $= 2\pi$

$$\Rightarrow \sqrt{\lambda}\,t = 2\pi \Rightarrow t = \frac{2\pi}{\sqrt{\lambda}}$$

any period is $n t = n \cdot \frac{2\pi}{\sqrt{\lambda}}$

$$\Rightarrow n \cdot \frac{2\pi}{\sqrt{\lambda}} = 2\pi \Rightarrow \lambda = n^2,\ n \in \mathbb{Z}$$

III $\lambda = 0 \Rightarrow r^2 = 0$

$$r_{1,2} = 0 \Rightarrow x_1 = e^{0t},\quad x_2 = te^{0t}$$

$$\Rightarrow x = c_1 e^{0t} + c_2 e^{0t} t = c_1 + c_2 t,\ c_1, c_2 \in \mathbb{R}$$

not a periodic function

LNDECC : $\underbrace{x'' + a_1 x' + a_2 x}_{L[x]=0} = f(t)$, $a_{1,2} \in \mathbb{R}$

$\downarrow$

$x = x_h + x_p \rightarrow$ gen. sol. of LNDECC

x_h = gen. sol. of LNDEcc

x_p = a particular solution of LNDEcc

$r_1 \neq 0 \neq r_2 \Rightarrow q=0$

$r_1 = 0 \Rightarrow q = 1$, $r_1 = r_2 = 0 \Rightarrow q = 2$

1) $f(t) = P_m(t) \Rightarrow x_p = t^q Q_m(t)$, q = multiple order of "0" as root of the char. eq

2) $f(t) = P_m(t) e^{\alpha t} \Rightarrow x_p = t^q Q_m(t) e^{\alpha t}$, q = —" —"α" —" —

3) $\left.\begin{array}{l} f(t) = P_m(t) e^{\alpha t} \cos\beta t \\ \text{or } f(t) = P_m(t) e^{\alpha t} \sin\beta t \end{array}\right\} \Rightarrow x_p = t^q e^{\alpha t} [Q_m(x) \sin\beta t + R_m(x) \cos\beta t]$,

q = —" —"$\alpha + i\beta$" —" —

1.5.1 Decide whether the following statements are T/F

a) All the solutions of $x'' + 3x' + x = 1$ satisfy $\lim_{t\to\infty} x(t) = 1$

b) The solution of the IVP : $\begin{cases} x'' + 4x = 1 \\ x(0) = \frac{5}{4} \\ x'(0) = 0 \end{cases}$ satisfies $x(\pi) = \frac{5}{4}$

c) The equation $x' = 3x + t^3$ admits a polynomial sol. (of degree 3)

a) Step 1 : solve the associated homogeneous equation

$x'' + 3x' + x = 0$

$r^2 + 3r + 1 = 0$

$\Delta = 9 - 4 = 5 \Rightarrow r_{1,2} = \frac{-3 \pm \sqrt{5}}{2}$

$r_{1,2}$ simple solutions $\Rightarrow$ $e^{\frac{-3-\sqrt{5}}{2}t}$, $e^{\frac{-3+\sqrt{5}}{2}t}$ solutions

$$\Rightarrow \quad x_h = c_1 e^{\frac{-3-\sqrt{5}}{2}t} + c_2 e^{\frac{-3+\sqrt{5}}{2}t}, \quad c_{1,2} \in \mathbb{R}$$

step 2 : $x_p = ?$

$x'' + 3x' + x = 1$

$\underbrace{f(t) = 1}_{=P_0(t)} \Rightarrow x_p = Q_0(t) = k$, $k \in \mathbb{R}$

$q = 0$ bc. we don't have r_1 and/or $r_2 = 0$
if $r_1 = 0$, $r_2 = 5 \rightarrow q = 1$
if $r_1 = r_2 = 0 \rightarrow q = 2$

$\underbrace{k''}_{=0} + \underbrace{3k'}_{=0} + k = 1 \Leftrightarrow k = 1 \Rightarrow x_p = 1 \Rightarrow x = x_h + x_p = c_1 e^{\frac{-3-\sqrt{5}}{2}t} + c_2 e^{\frac{-3+\sqrt{5}}{2}t} + 1$

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \underbrace{c_1 e^{\frac{-3-\sqrt{5}}{3}t}}_{=0} + \underbrace{c_2 e^{\frac{-3+\sqrt{5}}{3}t}}_{=0} + 1 = \lim_{t\to\infty} x(t) = 1$$

b) $x'' + 4x = 0$

$r^2 + 4 = 0$

$(r-2i)(r+2i) = 0 \Rightarrow r_{1,2} = \pm 2i \in \mathbb{C}\setminus\mathbb{R} \Rightarrow x_1 = e^{0\cdot t}\sin 2t$

$x_2 = e^{0\cdot t}\cos 2t$

$\Rightarrow x_h = c_1 \sin 2t + c_2 \cos 2t$, $c_{1,2} \in \mathbb{R}$

$x_p = ?$ (step 2) a sol. of $x'' + 4x = 1$

$f(t) = 1 = P_0(t) \Rightarrow x_p = Q_0(t) = k$, k - constant

$\underbrace{k''}_{=0} + 4k = 1 \Rightarrow 4k = 1 \Rightarrow k = \frac{1}{4} \Rightarrow x_p = \frac{1}{4}$

(step 3) $x = x_k + x_p = c_1 \sin 2t + c_2 \cos 2t + \frac{1}{4}$

$x(0) = \frac{5}{4}$

$x(0) = c_1 \sin 0 + c_2 \cos 0 + \frac{1}{4} \Leftrightarrow c_2 + \frac{1}{4} = \frac{5}{4} \Rightarrow c_2 = 1$

$x'(0) = 0$

$x'(t) = 2c_1 \cos 2t - 2c_2 \sin 2t$

$x'(0) = 2c_1 = 0 \Leftrightarrow c_1 = 0$

$\Rightarrow x = \cos 2t + \frac{1}{4} \leftarrow$ gen. sol. of the IVP

$x(\pi) = \cos 2\pi + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4}$

c) $x' = 3x + t^3$

$x' - 3x = t^3$

(step 1) $x' - 3x = 0$

$r - 3 = 0 \Rightarrow r = 3 \Rightarrow x = e^{3t}$

$x_h = c \cdot e^{3t}$, $c \in \mathbb{R}$

(step 2) $x_p = ?$ sol. of $x'' - 3x = t^3$

$x_p = at^3 + bt^2 + ct + d$

$$3at^2+2bt+c-3at^3-3bt^2-3ct-3d=t^3$$

$$(-3a-1)t^3+(3a-3b)t^2+(2b-3c)t+c-3d=0,\forall t$$

$$\Rightarrow \begin{cases} -3a-1=0 \Rightarrow a=-\frac{1}{3} \\ 3a-3b=0 \Rightarrow b=-\frac{1}{3} \\ 2b-3c=0 \Rightarrow 3c=-\frac{2}{3} \Rightarrow c=-\frac{2}{9} \\ c-3d=0 \Rightarrow 3d=-\frac{2}{9} \Rightarrow d=-\frac{2}{27} \end{cases}$$

we equal to 0, because it has to work for EVERY t

$$\Rightarrow x_p=-\frac{1}{3}t^3-\frac{1}{3}t^2-\frac{2}{9}t+\frac{2}{27}$$

(step 3) $x=x_h+x_p=c_1e^{3t}-\frac{1}{3}t^3-\frac{1}{3}t^2-\frac{2}{9}t-\frac{2}{27}$ ← GENERAL SOLUTION

1.5.2. $\lambda\in\mathbb{R}$, Find the general solution of $x''-x=e^{\lambda t}$ knowing that, depending on λ it has a particular solution, either of the form $ae^{\lambda t}$ or of the form $ate^{\lambda t}$.

(step 1) $x''-x=0$

$$r^2-1=0$$

$$(r-1)(r+1)=0 \Leftrightarrow r_{1,2}=\pm 1 \Rightarrow x_1=e^t,\quad x_2=e^{-t}$$

$$\Rightarrow x_h=c_1e^t+c_2e^{-t},\quad c_{1,2}\in\mathbb{R}$$

(step 2) $\left.\begin{matrix} x''-x=e^{\lambda t} \\ x_p=ae^{\lambda t}\end{matrix}\right\} \Rightarrow a\lambda^2e^{\lambda t}-ae^{\lambda t}=e^{\lambda t}$

$$\lambda^2ae^{\lambda t}-ae^{\lambda t}-e^{\lambda t}=0$$

$$\underbrace{e^{\lambda t}}_{>0}(\lambda^2a-a-1)=0$$

$$\lambda^2a-a=1 \Leftrightarrow a(\lambda^2-1)=1 \Rightarrow a=\frac{1}{(\lambda^2-1)},\quad (\lambda^2-1)\neq 0$$

→ case 1: $\lambda\neq\pm 1 \Rightarrow x_p=\dfrac{1}{\lambda^2-1}e^{\lambda t}$

→ case 2: $\lambda=1 \Rightarrow x_p=ate^t$ sol. of the eq $x''-x=e^t$

$$x_p=ate^t \Rightarrow x_p'=ae^t+ate^t$$

$$x_p''=ae^t+ae^t+ate^t$$

$$x''-x=e^t \Leftrightarrow 2ae^t+ate^t-ate^t=e^t$$

$$2ae^t=e^t \Rightarrow a=\frac{1}{2}$$

$$\Rightarrow x_p=\frac{1}{2}te^t$$

$$\Rightarrow x = x_h + x_p = c_1e^t + c_2e^{-t} + \frac{1}{2}te^t$$

$\rightarrow$ case 3 : $\lambda = -1 \Rightarrow x_p = ate^{-t}$ sol. of the eq. $x'' - x = e^{-t}$

$$x_p = ate^{-t} \Rightarrow x_p' = ae^{-t} - ate^{-t}$$

$$x_p'' = -ae^{-t} + ate^{-t} - ae^{-t} = ate^{-t} - 2ae^{-t}$$

$$x'' - x = e^{-t} \Leftrightarrow -2ae^{-t} + ate^{-t} - ate^{-t} = e^{-t}$$

$$-2ae^{-t} = e^{-t} \Rightarrow a = -\frac{1}{2}$$

$$\Rightarrow x_p = -\frac{1}{2}te^{-t}$$

$$\Rightarrow x = c_1e^t + c_2e^{-t} - \frac{1}{2}te^{-t}$$

1.5.3 $w > 0$, $f(\cdot, w)$ - sol of the IVP $\begin{cases} x'' + x = \cos wt \\ x(0) = x'(0) = 0 \end{cases}$

i) $w \neq 1$ find a sol $x_p = a\cos wt + b\sin wt$ of the eq

ii) $x_p = ?$ st. $x_p = t(a\cos t + b\sin t)$ sol of $x'' + x = \cos t$ $(w = 1)$

iii) $f(\cdot, w) = ?$ $w > 0$

iv) Prove $\lim_{w \to 1} f(t, w) = f(t, 1)$, $\forall t \in \mathbb{R}$

i) $(a\cos wt + b\sin wt)'' + a\cos wt + b\sin wt = \cos wt$

$$-aw^2\cos wt - bw^2\sin wt + a\cos wt + b\sin wt = \cos wt$$

$$\cos wt(-aw^2 + a - 1) + \sin wt(-bw^2 + b) = 0$$

$$\begin{cases} -aw^2 + a - 1 = 0 \Rightarrow a = \frac{1}{1-w^2}, \quad w > 0, w \neq 1 \\ -bw^2 + b = 0 \Rightarrow b = 0 \end{cases}$$

$$\Rightarrow x_p = \frac{1}{1-w^2}\cos wt, \quad r^2 + 1 = 0, \ r = \pm i$$

(ii) If : $(a = 0, b = \frac{1}{2})$! $w = 1$

$$x_p = \frac{1}{2}t\sin t$$

$$x(t, w) = c_1\cos t + c_2\sin t + \frac{1}{1-w^2}\cos wt$$

$$\left.\begin{array}{l} x(0) = 0 \\ x'(0) = 0 \end{array}\right| \Rightarrow f(t, w) = \ldots$$

(iii) From (ii) $\Rightarrow x(t, 1) = c_1\cos t + c_2\sin t + \frac{1}{2}t\sin t$ $\Big| \Rightarrow f(t, 1) = \ldots$

$x(0) = 0$; $x'(0) = 0$

(iv) $\lim_{w \to 1} f(t, w) = \ldots = f(t, 1)$

Seminar 4.

1. $k > 0$, we consider the differential eq.: $x' = -k(x - 21)$ (the cooling proam model - Newton)

$x(t)$ = the temperature of a cup of tea at the same time t.

a) flow = ?

b) experiment: a cup of tea with the initial temperature of 49°C has a temperature of 37°C in 10'. Find the in temperature of a cup of tea such that after 20' the tea has 37°C.

a) IVP $\begin{cases} x' = -k(x - 21) \\ x(0) = \eta, \ \eta > 0 \end{cases}$ => solution $\varphi(t, \eta)$ - flow

$x' = -kx + 21k$

$x' + kx = 21k$ (non-homogeneous linear differential equation or non-hom. c.c. diff eq.)

step 1: $x' + kx = 0$ hom. eq.

$x' = -kx$ $\left(= \underbrace{f(t)}_{-x} \cdot \underbrace{g(x)}_{k}\right)$ - only for first order and when we can separate the coeff. from the function

→ when we can use this method.

→ separate variable diff. eq.

→ separate variables: $\frac{dx}{x} = -k\,dt$

→ integrate with respect to each variable

$$\int \frac{dx}{x} = -k \int dt$$

$$\ln|x| = -kt + \underbrace{\ln|c|}_{\text{constant.}}$$

$$\ln|x| = \ln e^{-kt} + \ln|c|$$

$$\ln|x| = \ln|c \cdot e^{-kt}|$$

$$x_h = c \cdot e^{-kt}$$

step2: $x' + kx = 21k$ (non-hom)

$x_p = ?$

- we will find $\underbrace{\quad}_{?}$ with the method of Lagrange (variation of constant method): we will take the constant from step 1 and mak

$$x_p = c(t)\cdot e^{-kt}$$

$$x_p' = c'(t)e^{-kt} + c(t)\cdot(-k)e^{-kt}$$

$$\underbrace{c'(t)e^{-kt} - \cancel{kc(t)e^{-kt}}}_{x_p'} + \underbrace{\cancel{kc(t)e^{-kt}}}_{x_p} = 21k$$

if we did it write, the ones with $c(t)$ have to cancel out.

$$c'(t)e^{-kt} = 21k \Rightarrow c'(t) = 21ke^{+kt} \quad \Big| \int dt$$

$$\Rightarrow c(t) = 21k\int e^{+kt}dt = 21k\cdot\frac{1}{+k}e^{+kt}$$

$$\Rightarrow c(t) = 21e^{+kt} \Rightarrow x_p = \underbrace{21e^{kt}}_{c(t)}\cdot e^{-kt}$$

$$\Rightarrow x_p = 21$$

step 3: $x = x_h + x_p$

$x = ce^{-kt} + 21$ ← sol. of the diff. eq.

$x(0) = \eta$

$c\cdot e^0 + 21 = \eta \Rightarrow c = \eta - 21$ flow

$\Rightarrow$ solution of the IVP : $\varphi(t,\eta) = (\eta - 21)e^{-kt} + 21$, $\eta \in \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$

b) $x(0) = \eta =$ initial temperature, $t_0 = 0$

$\varphi(0,49) \xrightarrow{10'} \varphi(10',49) = 37$ $(\eta = 49)$

$\varphi(0,49) = 49$; $\varphi(10,49) = (49-21)e^{-k\cdot 10} + 21 = 37$

$\Rightarrow 28e^{-k10} = 16 \Leftrightarrow e^{-k10} = \frac{4}{7} \,\Big|\, \ln \Leftrightarrow -k10 = \ln\frac{4}{7} \Rightarrow k = -\dfrac{\ln\frac{4}{7}}{10}$

$\tilde{\eta} = ?$ $\varphi(0,\tilde{\eta}) = \tilde{\eta}$, such that $\varphi(20',\tilde{\eta}) = 37$

$\varphi(20,\tilde{\eta}) = (\tilde{\eta} - 21)e^{-k20} + 21 = (\tilde{\eta} - 21)e^{2\ln\frac{4}{7}} + 21 = 37$

$$(\tilde{\eta}-21)\, e^{\ln\frac{16}{49}} = 16$$

$$(\tilde{\eta}-21)\cdot\frac{16}{49} = 16 \iff \tilde{\eta}-21 = 49 \iff \tilde{\eta} = 70$$

2. $0 < c < 1$; $x' = x(1-x) - cx$

a) find the equilibria, study their stability using the linearitation method.
b) represent the phase portrait
c) a) → b) if $x(t) > 0$ is the density of fish in a lake and $c \in (0,1)$ is the rate of fishing, try to predict the fate of the fish from the lake.

Th. (lin method)

$f \in C^1(\mathbb{R})$, η^* st. $f(\eta^*) = 0$ (eq. point) → $\eta^* =$

a) If $f'(\eta^*) > 0 \Rightarrow \eta^* =$ repeller (unstable)

b) If $f'(\eta^*) < 0 \Rightarrow \eta^* =$ attractor (is stable)

- an equilibrium point (an equilibria) is a solution that is constant in time.

Denote: equilibrium of diff. eq: $x^* =$ solution constant in time $\Rightarrow (x^*)' = 0$

$$0 = x(1-x) - cx$$

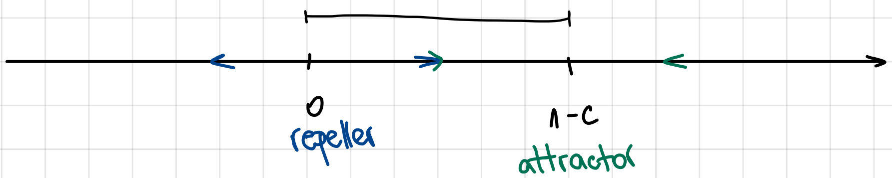
R: equilibrium point of diff. eq. $x' = f(x) \Rightarrow f(x) = 0$

$f(x) = 0 \iff x(1-x) - cx = 0 \iff x(1-x-c) = 0 \iff x_1^* = 0$, $x_2^* = 1-c$ (eq. points)

$f(x) = x - x^2 - cx \Rightarrow f'(x) = 1-2x-c \Rightarrow f'(x_1^*) = 1-c > 0 \Rightarrow x_1^*$ - repeller

$c \in (0,1) \Rightarrow -c \in (-1,0) +1 \Rightarrow 1-c \in (0,1)$

$f'(x_2^*) = 1-2(1-c)-c = 1-2+2c-c = c-1 < 0 \Rightarrow x_1^*$ - attractor



The orbits : $(-\infty, 0)$; $\{0\}$; $(0, 1-c)$; $\{1-c\}$; $(1-c, \infty)$

N - optimal density $\Rightarrow N = 1-c$
$\Rightarrow c = 1-N$

Conclusion: we have to choose any value between 0 and 1-N, because with a value from this interval the density of fish will increase in time.

3. Represent the face portrait of $x' = x - x^3$; and from this representation find: $\varphi(t,-1)$ and $\varphi(t,0)$; the find the properties of the functions: $\varphi(t,-2)$, $\varphi(t,3)$ and $\varphi(t,-1/2)$

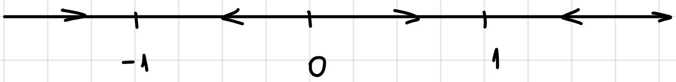
x^* - eq. point $\Leftrightarrow$ $(x^*)' = 0$ $\Rightarrow$ $x - x^3 = 0$
x^* - sol. $\quad x(1-x^2) = 0 \Rightarrow$ $x^*_1 = 0$, $x^*_2 = 1$, $x^*_3 = -1$

$(x - x^3)' = 1 - 3x^2$

$1 - 3x_1^{*2} = 1 - 3 \cdot 0 = 1 > 0 \Rightarrow x_1^* = 0 \rightarrow$ repeller

$1 - 3x_2^{*2} = 1 - 3 = -2 < 0 \Rightarrow x_2^* = 1 \rightarrow$ attractor

$1 - 3x_3^{*2} = 1 - 3 = -2 < 0 \Rightarrow x_3^* = -1 \rightarrow$ attractor



Orbits: $(-\infty,-1)$, $\{-1\}$, $(-1,0)$, $\{0\}$, $(0,1)$, $\{1\}$, $(1,\infty)$

IVP: $\begin{cases} x' = x - x^3 \\ x(0) = \eta \end{cases}$

solution: $\varphi(t,\eta)$

$\varphi(t,\eta^*) = \eta^*$, η^* - eq. point.

$\varphi(t,-1) = -1$
$\varphi(t,0) = 0$; -1 and 0 are eq. points.

$\varphi(t,-2) \in (-\infty,-1) \Rightarrow$ the function is increasing
$\lim\limits_{t\to-\infty} \varphi(t,-2) = -\infty$, $\lim\limits_{t\to -1} \varphi(t,-1) = -1$

$\varphi(t,3) \in (1,-\infty) \Rightarrow \varphi(t,3)$ - decreasing

$\lim\limits_{t\to\infty} \varphi(t,3) = 1$; $\lim\limits_{t\to 1} \varphi(t,3) = \infty$

$\varphi(t,-\frac{1}{2}) \in (-1,0) \Rightarrow \varphi(t,-\frac{1}{2})$ -decreasing

$\lim\limits_{t\to\infty} \varphi(t,-\frac{1}{2}) = -1$; $\lim\limits_{t\to -1} \varphi(t,-\frac{1}{2}) = \infty$

4. Represent the face portrait of $\cdot\ x' = x - x^3 + 1$

eq. p: $x - x^3 + 1 = 0$

$x_1 \in \mathbb{R}$, $x_{2,3} \in \mathbb{C} \setminus \mathbb{R}$; $x_{1,2,3} \in \mathbb{R}$ $\Rightarrow \exists x_1^* \in \mathbb{R}$

$f'(x) = 1 - 3x^2$

$f\left(-\frac{\sqrt{3}}{3}\right) = \frac{27 - 6\sqrt{3}}{27} > 0$

$f\left(\frac{\sqrt{3}}{3}\right) = \frac{6\sqrt{3} + 27}{27} > 0$

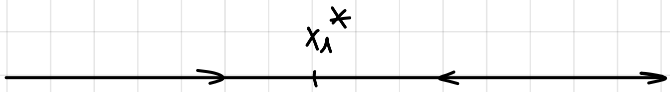
x		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	1	x_1^*	2
$f'(x)$	$-\ -$	0	$+\ +\ +$	0	$-$		
$f(x)$	$+\infty \searrow$	$+$	$\nearrow$	$+$	$\searrow$	0	$\searrow -\infty$
$sgn(x)$	$+\ +\ +$	$+$	$+\ +\ +$	$+$	$+$	0	$-\ -$

- from the table we conclude that we'll have only one real root, so only one equilibrium point x_1^*

$f(1) = 1 - 1 + 1 = 1 > 0$

$f(2) = 2 - 8 + 1 = -5 < 0$ $\Rightarrow x_1^* \in (1, 2)$

$f'(x_1^*) < 0 \Rightarrow x_1^*$ - attractor



Orbits: $(-\infty, x_1^*)$; $\{x_1^*\}$; (x_1^*, ∞)

$A_{x_1^*}$ - basin of attraction (bazinul de atractie)

- toate orbitele din vecinatatea bazinului

$A_{x_1^*} = \mathbb{R}$

Seminar 5.

1. a) $\begin{cases} \dot{x} = -y \\ \dot{y} = -5x \end{cases}$

i) eq. point is at $(0,0)$, we study it's stability

$X = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad X' = AX \Rightarrow A = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}$

$\det(A - \lambda I_2) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & -1 \\ 5 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 5 = 0$

$\Rightarrow \lambda_{1,2} = \pm i\sqrt{5} \in \mathbb{C} \setminus \mathbb{R}$

$\Rightarrow$ center, stable

$\frac{dy}{dx} = \frac{5x}{-y} \Leftrightarrow -y\,dy = 5x\,dx \,\Big|\int$

$-\int y\,dy = 5\int x\,dx \Leftrightarrow \frac{1}{2}y^2 + c_1 = \frac{5}{2}x^2 + c_2 \qquad c_1, c_2 \in \mathbb{R}$

$2(c_1 - c_2) = c \in \mathbb{R} \Rightarrow 5x^2 + y^2 = c$

$H: \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = 5x^2 + y^2$

$f_1 = -y$, $f_2 = 5x$

$\frac{dH}{dx}(x,y) \cdot f_1(x,y) + \frac{dH}{dy}(x,y) \cdot f_2(x,y) = 0$

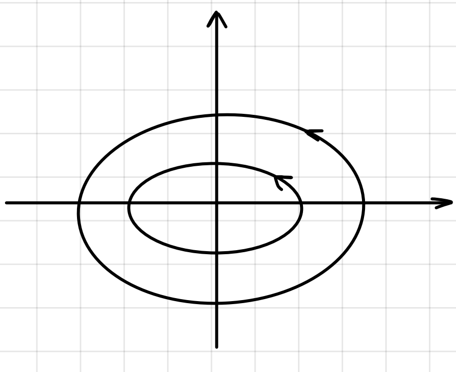
$10x \cdot (-y) + 2y \cdot 5x = 0$

$-10xy + 10xy = 0 \quad \forall t \in \mathbb{R}$

$\Rightarrow$ H is a global integral

iv) $H(x,y) = c \in \mathbb{R}$ fixed and arbitrary.

$\Leftrightarrow \quad 5x^2 + y^2 = c$



$x < 0 \Rightarrow y' > 0$

$x > 0 \Rightarrow y' < 0$

$y < 0 \Rightarrow x' < 0$

$y > 0 \Rightarrow x' > 0$

1. b) $\begin{cases} x' = -x \\ y' = 5y \end{cases} \Rightarrow A = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$ (diag. m) $\Rightarrow \lambda_{1,2} = -1, 5 \Rightarrow \lambda_1 < 0 < \lambda_2$

$\Rightarrow$ saddle point

i) eq. point $(0,0)$

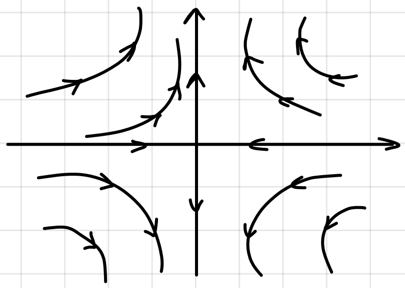
(ii)(iii) $\frac{dH}{dt} = \frac{dH}{dx} x' + \frac{dH}{dy} \cdot y' = 0$

$\begin{cases} \frac{dx}{dt} = -x \\ \frac{dy}{dt} = 5y \end{cases} \Leftrightarrow \frac{dx}{dy} = \frac{-x}{5y} \Leftrightarrow \frac{dy}{5y} = \frac{dx}{-x} \Rightarrow \int \frac{dy}{y} = -5 \int \frac{dx}{x}$

$\Leftrightarrow \ln|y| + 5\ln|x| = \ln|c|, \; c \in \mathbb{R}$

$\Leftrightarrow yx^5 = c, \; H(x,y) = yx^5$

$5x^4 \cdot y(-x) + x^5 \cdot 5y = 0$, hence H is a first integral



$x > 0, \; x' < 0$

$x < 0, \; x' > 0$

$y > 0, \; y' > 0$

$y < 0, \; y' < 0$

Problem for grade: Seminar 5: 1. c) $\dot{x} = -3x, \; \dot{y} = -2y$

(i) Decide the type and stability of the echilibrium point at the origin

(ii) Decide whether it has a global first integral.

(iii) Find a first integral (global or not)

(iv) Represent the phrase portrait (using the expression of the first integral)

diagonal matrix

$\begin{cases} x' = -3x \\ y' = -2y \end{cases} \Rightarrow A = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \begin{matrix} \lambda_1 = -3 \\ \lambda_2 = -2 \end{matrix}, \; \lambda_1 \text{ and } \lambda_2 \in \mathbb{R}$

$\lambda_1, \lambda_2 < 0 \Rightarrow$ node (global atractor) because the eigenvalues have the same sign

because they are both less than zero

- if we have a global atractor, we don't have a global first integral
- we try to find in a region U

$\Rightarrow \frac{dx}{dy} = \frac{-2y}{-3x} \Rightarrow 3\frac{dy}{y} = 2 \cdot \frac{dx}{x}$

$\Rightarrow \ln|y^3| = \ln|x^2| + \ln c \Rightarrow \ln\left|\frac{y^3}{x^2}\right| = \ln c, \; c \in \mathbb{R}$

$\Rightarrow \frac{y^3}{x^2} = c, \; c \in \mathbb{R}$

- let's take $H(x,y) = \frac{y^3}{x^2}$, $x \neq 0$

$$\mathcal{U}_1 = \{(x,y) \in \mathbb{R}^2 \,/\, x > 0\}$$
$$\mathcal{U}_2 = \{(x,y) \in \mathbb{R}^2 \,/\, x < 0\}$$

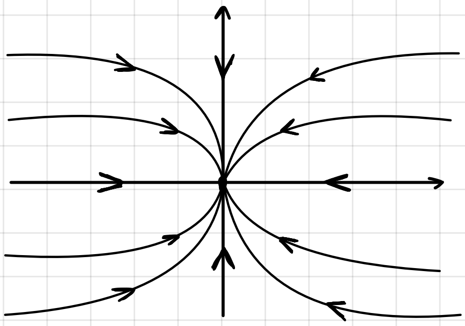
$$\frac{\partial H}{\partial x} \cdot f_1 + \frac{\partial H}{\partial y} \cdot f_2 = 0 \quad \text{in } \mathcal{U}_1, \mathcal{U}_2$$

$$\Leftrightarrow \quad y^3(-2x^{-3}) \cdot (-3x) + x^{-2} \cdot 3y^2 \cdot (-2y) = 0$$
$$y^3 \cdot 6x^{-2} + x^{-2} \cdot (-6y^3) = 0 \quad \text{"T"}$$

- shape of the orbits: $\frac{y^3}{x^2} = c$, $c \in \mathbb{R}$ $\Rightarrow$ $y = c \cdot x^{\frac{2}{3}}$, $c \in \mathbb{R}$

$x' = -3x < 0$
$y' = -2y < 0$

$x' = -3x < 0$
$y' = -2y > 0$



- flow: $\varphi(t, \eta_1, \eta_2) = (\eta_1 \cdot e^{-3t}, \eta_2 \cdot e^{-2t})$

$\Rightarrow \varphi(t, 0, \eta_2) = (0, \eta_2 \cdot e^{-2t})$

- $\eta_2 > 0 \Rightarrow \gamma = (0, \infty)$
- $\eta_2 < 0 \Rightarrow \gamma = (-\infty, 0)$
- $\eta_2 = 0 \Rightarrow \gamma = (0, 0)$

1. d) $\begin{cases} x' = x - y \\ y' = x + y \end{cases}$ $\qquad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

i) eq. point $(0,0)$

The charc. eq.: $\det(A - \lambda I_2) = 0$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \iff (1-\lambda)^2 + 1 = 0$$

$$\iff \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = 1 \pm i \in \mathbb{C} \setminus \mathbb{R}$$

$$\Rightarrow \begin{cases} Re(\lambda_{1,2}) \neq 0 & \Rightarrow \text{focus} \\ Re(\lambda_{1,2}) > 0 & \Rightarrow \text{global repeller} \end{cases}$$

$\Rightarrow$ NO global integral

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

$\Rightarrow$ not separable

$\Rightarrow$ polar coordinates

$x = \rho\cos\theta,\ y = \rho\sin\theta$

$x^2 + y^2 = \rho^2\cos^2\theta + \rho^2\sin^2\theta$

$x^2 + y^2 = \rho^2(\cos^2\theta + \sin^2\theta)$

$x^2 + y^2 = \rho^2$

$\tan\theta = \frac{y}{x}$

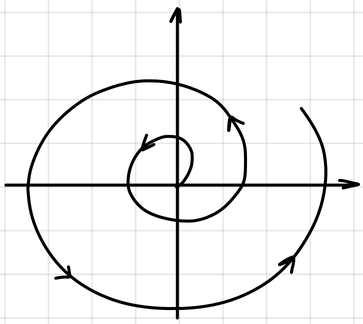
$$\frac{\dot\theta}{\cos^2\theta} = \frac{x\dot y - \dot x y}{x^2} \iff \frac{\dot\theta}{\cos^2\theta} = \frac{x(x+y) - (x-y)y}{x^2}$$

$$\iff \frac{\dot\theta}{\cos^2\theta} = \frac{x^2 \cancel{+xy} \cancel{-xy} + y^2}{x^2}$$

$$\iff \frac{\dot\theta}{\cos^2\theta} = \frac{x^2+y^2}{x^2} \iff \frac{\theta}{\cos^2\theta} = \frac{\rho^2}{x^2}$$

$$\Rightarrow \dot\theta = \cos^2\theta\left(\frac{\rho}{x}\right)^2 = \cancel{\cos^2\theta}\,\frac{\cancel{\rho^2}}{\cancel{\rho^2}\cancel{\cos^2\theta}}$$

$\Rightarrow \theta = 1 > 0 \Rightarrow$ trigonometric sens



$x^2 + y^2 = \rho^2$

$x \cdot x' + y \cdot y' = \rho \cdot \rho' \iff x(x-y) + (x+y)y = \rho \cdot \rho'$

$\iff x^2 - xy + xy + y^2 = \rho \cdot \rho'$

$<=> x^2 + y^2 = f \cdot f' <=> f^2 = f \cdot f \Rightarrow f' = f$

2. $\begin{cases} \dot{x} = x(1-x) = x - x^2 \\ \dot{y} = y(3-y) = 3y - y^2 \end{cases}$

$\left.\begin{array}{l} \dot{x} = 0 <=> x(1-x) = 0 \Rightarrow x \in \{0, 1\} \\ \dot{y} = 0 <=> y(3-y) = 0 \Rightarrow y \in \{0, 3\} \end{array}\right\} \Rightarrow$ eq. point $\in \{(0,0), (0,3), (1,0), (1,3)\}$

$$J = \begin{pmatrix} \frac{d\dot{x}}{dx} & \frac{d\dot{x}}{dy} \\ \frac{d\dot{y}}{dx} & \frac{d\dot{y}}{dy} \end{pmatrix} = \begin{pmatrix} 1-2x & 0 \\ 0 & 3-2y \end{pmatrix}$$

I $(0,0) \Rightarrow J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = 3 \rightarrow$ node repeller

II $(0,3) \Rightarrow J(0,3) = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = -3 \rightarrow$ saddle

III $(1,0) \Rightarrow J(1,0) = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \Rightarrow \lambda_1 = -1, \lambda_2 = 3 \rightarrow$ saddle

IV $(1,3) \Rightarrow J(1,3) = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \Rightarrow \lambda_1 = -1, \lambda_2 = -3 \rightarrow$ ~~saddle~~, node attractor

3. $\begin{cases} \dot{x} = ax - 5y \\ \dot{y} = x - 2y \end{cases}$ has a center

i) $a = 2$

$$A = \begin{pmatrix} a & -5 \\ 1 & -2 \end{pmatrix}$$

$\det(A - \lambda I_2) = 0 <=> \begin{vmatrix} a - \lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = 0 <=> (a - \lambda)(-2-\lambda) + 5 = 0$

$<=> -2a - a\lambda + 2\lambda + \lambda^2 + 5 = 0$

$<=> \lambda^2 + (2-a)\lambda + 5 - 2a^2 = 0$

$$\Delta = (2-a)^2 - 4(5-2a)$$

$$\Delta = a^2 + 4a - 16 =$$

$$\lambda_{1,2} = \frac{a-2}{2} \cdot \left(\pm \frac{\sqrt{a^2+4a-16}}{2} \right)$$

$$\mathrm{Re}(\lambda) = 0 \iff \frac{a-2}{2} = 0 \iff a = 2$$

For $a=2 \Rightarrow \Delta = 4+8-16 = -4 \Rightarrow \lambda_{1,2} = \pm \frac{2i}{2} = \pm i$

ii) $\det(A) = 0 \iff \begin{vmatrix} a & -5 \\ 1 & -2 \end{vmatrix} = 0 \iff -2a+5 = 0$

$$a = \frac{5}{2}$$