2.1 2.2 2.5 2.10 2.11 2.14 2.18

2.19 2.22 2.27, 2.30

3.1, 3.2 3.3 3.4 3.7 3.9 3.10

3.12, 3.18, 3.20

Dot product (scalar product)

$\vec{v}, \vec{w} \in V$ vectors

$$\vec{v}\cdot\vec{w} = \begin{cases} 0, & \text{if } \vec{v}=\vec{0} \text{ or } \vec{w}=\vec{0} \\ |\vec{v}|\cdot|\vec{w}|\cdot\cos|\vec{v},\vec{w}|, & \text{otherwise} \end{cases}$$

Notation: $\vec{v}\cdot\vec{w} = \langle\vec{v},\vec{w}\rangle$

Properties

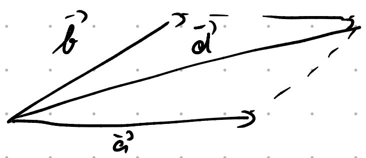
- bilinearity: $\forall v_1, v_2, w \in V$ $\langle\alpha\vec{v_1}+\beta\vec{v_2},\vec{w}\rangle = \alpha\langle\vec{v_1},\vec{w}\rangle + \beta\langle\vec{v_2},\vec{w}\rangle$
  $\forall \alpha, \beta \in \mathbb{R}$:
  $(\alpha\vec{v_1}+\beta\vec{v_2})\cdot\vec{w} = \alpha\vec{v_1}\cdot\vec{w} + \beta\vec{v_2}\cdot\vec{w}$
- symmetry: $\forall \vec{v}, \vec{w} \in V$
  $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$
- positive definiteness: $\forall v \in V$
  $\vec{v}\cdot\vec{v} \in \mathbb{R}_{\geq 0}$, if $\vec{v} \neq \vec{0}$, $\vec{v}\cdot\vec{v} > 0$

Consequences

- $\vec{v}\cdot\vec{v} = |\vec{v}|^2$
- $\vec{v}^2 - \vec{w}^2 = (\vec{v}-\vec{w})(\vec{v}+\vec{w})$

1. $\vec{m}, \vec{n}$ unit vectors s.t. $\sphericalangle(m, n) = 60^\circ \Rightarrow \cos(m, n) = \frac{1}{2}$

Det. lengths of the parallelogram spanned by the vectors $\vec{a} = 2\vec{m} + \vec{n}$, $\vec{b} = \vec{m} - 2\vec{n}$



$\vec{d} = \vec{a} + \vec{b}$
$= 2\vec{m} + \vec{n} + \vec{m} - 2\vec{n}$
$= 3\vec{m} - \vec{n}$

$|\vec{d}|^2 = \langle d, d\rangle = \langle \vec{a}+\vec{b}, \vec{a}+\vec{b}\rangle$
$= |\vec{a}+\vec{b}|^2$
$= |3\vec{m} - \vec{n}|^2$
$= 9\langle m, m\rangle - 6\langle m, n\rangle + \langle n, n\rangle$
$= 9|\vec{m}|^2 - 3|\vec{m}||\vec{n}| + |\vec{n}|^2$
$= 10 - 3 = 7$

$\Rightarrow |\vec{d_1}| = \sqrt{7}$

$\vec{d_2} = \vec{a} - \vec{b} = \vec{m} + 3\vec{n}$

$\Rightarrow \langle d_2, d_2\rangle = |\vec{m} + 3\vec{n}|^2$
$= \langle m, m\rangle + 3\vec{m}\vec{n} + 3\vec{n}\vec{m} + 9\langle n, n\rangle$
$= 13$

$\Rightarrow |\vec{d_2}| = \sqrt{13}$

If we have an orthonormal basis of V, $B = (\vec{v_1}, \dots, \vec{v_n})$ (i.e. $\forall i \neq j : \vec{v_i} \cdot \vec{v_j} = 0$ $\forall i |v_i| = 1$)

and $[\vec{w}]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, $[\vec{w}']_B = \begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix}$

then:

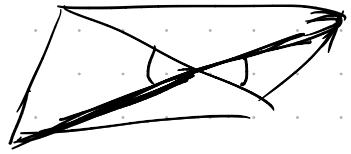
$\vec{w} \cdot \vec{w}' = a_1 a_1' + a_2 \cdot a_2' + \dots + a_n \cdot a_n'$

So

$|\vec{w}|^2 = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$

3. We are given $\vec{a}(2, 1, 0)$; $\vec{b}(0, -2, 1)$ w/ respect to an orthonormal basis,

Det. the angle between the diags. of the parallelog. spanned by $\vec{a}$ and $\vec{b}$.



$\vec{a} + \vec{b} = (2, -1, 1)$

$\vec{a} - \vec{b} = (2, 3, -1)$

$\langle d_1, d_1\rangle = \langle a+b, a+b\rangle = 4 + 1 + 1 = 6 \Rightarrow |\vec{d_1}| = \sqrt{6}$

$\langle d_2, d_2 \rangle = 4 + 9 + 1 = 14 \Rightarrow |\vec{d_2}| = \sqrt{14}$ 4

$\langle d_1, d_2 \rangle = \sqrt{6} \cdot \sqrt{14} \cdot \cos(d_1, d_2)$

$= \sqrt{84} \cdot \cos(d_1 d_2) = 4 - 3 - 1 = 0$

$\Rightarrow \measuredangle d_1, d_2 = \arccos(0) = \frac{\pi}{2}.$

6. ΔABC

$\vec{AB}^2 + \vec{AC}^2 - \vec{BC}^2 = 2 \cdot \vec{AB} \cdot \vec{AC}$ - law of cosines.

Proof:

$\vec{AB}^2 - \vec{BC}^2 + \vec{AC} = (\vec{AB} - \vec{BC}) \cdot (\vec{AB} + \vec{BC}) + \vec{AC}^2$

$= (\vec{AB} - \vec{BC}) \cdot \vec{AC} + \vec{AC}^2$

$= (\vec{AB} - \vec{BC} + \vec{AC}) \cdot \vec{AC}$

$= (\vec{r_B} - \vec{r_A} - \cancel{\vec{r_C}} + \vec{r_B} + \cancel{\vec{r_C}} - \vec{r_A}) \cdot \vec{AC}$

$= (2\vec{r_B} - \vec{r_A}) \cdot \vec{AC}$

$= 2 \vec{AB} \cdot \vec{AC}.$

7. $ABCD$ - tetrahedron,

$\cos \measuredangle (\vec{AB}, \vec{CD}) = \dfrac{\vec{AD}^2 + \vec{BC}^2 - \vec{AC}^2 - \vec{BD}^2}{2 AB \cdot CD.}$

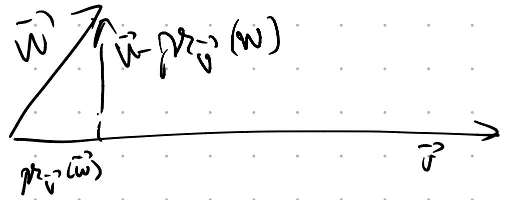
$\vec{AD}^2 - \vec{AC}^2 + \vec{BC}^2 - \vec{BD}^2 = (\vec{AD} - \vec{AC})(\vec{AD} + \vec{AC}) + (\vec{BC} - \vec{BD})(\vec{BC} + \vec{BD})$

$= \vec{CD}(\vec{AD} + \vec{AC} - \vec{BC} - \vec{BD})$

$= \vec{CD}(\vec{AD} + \vec{AC} + \vec{CB} + \vec{DB})$

$= \vec{CD}(\vec{AB} + \vec{AB})$

$= 2\vec{AB} \cdot \vec{CD} \Rightarrow$ true.



$$|\vec{w}| \cdot \cos(\vec{v}, \vec{w}) = |pr_{\vec{v}}(\vec{w})| =$$

$$= \vec{v} \cdot \frac{|\vec{w}| \cdot \cos(\vec{v}, \vec{w})}{|\vec{v}|} =$$

$$= \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|^2} \cdot \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\vec{v} \cdot \vec{w}} \cdot \vec{v}$$

Gramm Schmidt

$B = (v_1, \dots, v_n)$ basis of V

I Orthogonalization

1. $\vec{v}_1' := \vec{v}_1$

2. $\vec{v}_2' := \vec{v}_2 - pr_{\vec{v}_1'}(\vec{v}_2)$

3. $\vec{v}_3' := \vec{v}_3 - pr_{span(\vec{v}_1', \vec{v}_2')}(\vec{v}_3)$

$$= \vec{v}_3 - pr_{\vec{v}_1'}(\vec{v}_3) - pr_{\vec{v}_2'}(\vec{v}_3)$$

$$= \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{v}_1'}{\vec{v}_1' \cdot \vec{v}_1'} \vec{v}_1' - \frac{\vec{v}_3 \cdot \vec{v}_2'}{\vec{v}_2' \cdot \vec{v}_2'} \cdot \vec{v}_2'$$

- repeat process until we have a basis.

$B' = (\vec{v}_1', \dots, \vec{v}_n')$ - orthogonal.

II Normalization

$$\vec{v}_i'' := \frac{\vec{v}_i'}{|\vec{v}_i'|}$$

$\Rightarrow B'' = (\vec{v}_1'', \dots, \vec{v}_n'')$

orthonormal basis.

3.10. In an orthonormal basis consider the vectors $\vec{v}_1 (0,1,0)$, $\vec{v}_2 (2,1,0)$, $\vec{v}_3 (-1,0,1)$

Use G-Schmidt to find an orthonormal basis containing $\vec{v}_1$

1. $\vec{v}_1{}' = (0,1,0)$

$$\vec{v}_2{}' = (2,1,0) - pr_{\vec{v}_1{}'}(v_2) = (2,1,0) - \frac{(2,1,0)\cdot(0,1,0)}{(0,1,0)\cdot(0,1,0)}\cdot(0,1,0)$$

$$= (2,1,0) - (0,1,0)$$

$$= (2,0,0)$$

$$\vec{v}_3{}' = (-1,0,1) - \frac{(-1,0,1)\cdot(0,1,0)}{(0,1,0)\cdot(0,1,0)}(0,1,0) - \frac{(-1,0,1)\cdot(2,0,0)}{(2,0,0)\cdot(2,0,0)}(2,0,0)$$

$$= (-1,0,1) - 0 - \frac{-2}{4}\cdot(2,0,0)$$

$$= (-1,0,1) + (1,0,0)$$

$$= (0,0,1).$$

$$\Rightarrow v_1{}'' = (0,1,0)$$

$$v_2{}'' = (1,0,0)$$

$$v_3{}'' = (0,0,1).$$