Mathematical Analysis Babeș-Bolyai University
1st Year Bachelor in Computer Science

Midterm Test

1. Find $\inf, \sup, \min, \max$, the interior and the closure of the set $\{0.1, 0.11, 0.111, \ldots\}$.

2. Study the convergence of the following series:

   (a) $\sum_{n\geq 1} \frac{(n+1)^{n-1}}{n^{n+1}}$. $\frac{1}{n(n+1)}$ $\sqrt[n]{\left(\frac{n+1}{n}\right)^n} \cdot \sqrt[n]{\frac{1}{n(n+1)}} =$

   (b) $\sum_{n\geq 1} \frac{a^n (n!)^2}{(2n)!}$, $a > 0$. $= \left(1+\frac{1}{n}\right) \cdot \sqrt[n]{\frac{1}{n(n+1)}}$
   $= \left(1+\frac{1}{n}\right) \left(n(n+1)\right)^{-\frac{1}{n}}$

   (c) $\sum_{n\geq 1} \frac{\ln n}{n^2}$.

   (d) $\sum_{n\geq 1} n! \sin x \sin \frac{x}{2} \ldots \sin \frac{x}{n}$, $x \in (0, \pi)$.

3. Study the convergence and the absolute convergence of the series $\sum_{n\geq 1} (-1)^n (\underline{\sqrt{n} - \sqrt{n+1}})$.

4. Using power series, find the sum of the following series:

   (a) $\sum_{n\geq 0} \frac{n+1}{4^n}$.

   (b) $\sum_{n\geq 2} \frac{n(n-1)}{2^n}$.

   (c) $\sum_{n\geq 0} \frac{(-1)^n}{2n+1}$. $= \sum_{n\geq 0} (-1)^n \frac{1}{2n+1} x^{2n+1}$

5. Find the radius of convergence and the convergence set for the power series

$$\sum_{n\geq 1} \frac{x^n}{n^p},\ p \in \mathbb{R}.$$

$\Rightarrow |x| < R$ absolutely conv
$|x| > R$ divergent

$\sum \frac{1}{n^p} (x-0)^n$ $\Rightarrow$ divergent for $p \leq 1 \Rightarrow L = \infty \Rightarrow R = 0$
conv for $p > 1 \Rightarrow L \in (0, \infty) \Rightarrow R = \frac{1}{L}$

6. Given the data points (x_i, y_i), $i \in \{1, \ldots, n\}$, the line of best fit f minimizes $\sum_{i=1}^{n} (y_i - \underline{f(x_i)})^2$. $\Rightarrow$ extinzi
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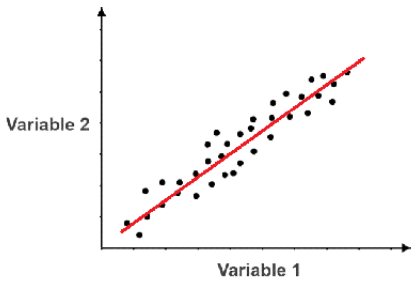
   Find the line of best fit that passes through the origin (and explain its uniqueness).



18th November 2022

(a)? $\displaystyle\sum_{n\geq 1} \frac{(n+1)^{n-1}}{n^{n+1}}$.

Comp with $\frac{1}{n^2}$

ratio : $\dfrac{(n+2)^n}{(n+1)^{n+2}} \cdot \dfrac{n^{n+1}}{(n+1)^{n-1}} =$

$= \left(\dfrac{n+2}{n+1}\right)^n \cdot \left(\dfrac{n}{n+1}\right)^{n+1} = \left(1+\dfrac{1}{n+1}\right)^n \cdot \left(1+\dfrac{-1}{n+1}\right)^{n+1} = e \cdot \dfrac{1}{e} = 1 \rightarrow$ incond

R-D : $\lim\limits_{n\to\infty} n\left(\dfrac{x_n}{x_{n+1}} - 1\right) = \lim\limits_{n\to\infty} n \cdot \left(\left(\dfrac{n+1}{n}\right)^{n+1} \cdot \left(\dfrac{n+1}{n+2}\right)^n - 1\right)$

(b) $\displaystyle\sum_{n\geq 1} \frac{a^n (n!)^2}{(2n)!}$, $a > 0$.

ratio $\dfrac{\cancel{a^{n+1}}^{\,a}\, \cancel{(n+1)!}^{\,2}_{\,(n+1)^2}}{\cancel{(2n+2)!}_{\,(2n+1)(2n+2)}} \cdot \dfrac{\cancel{2n!}}{\cancel{a^n}\,\cancel{(n!)^2}} = a \cdot \dfrac{(n+1)^2}{(2n+1)(2n+2)} \longrightarrow \dfrac{a}{4}$

$a = 4 \Rightarrow \displaystyle\sum_{n\geq 1} \frac{4^n (n!)^2}{(2n)!} \Rightarrow$ R-D: $\lim\limits_{n\to\infty} n\left(\dfrac{x_n}{x_{n+1}} - 1\right) = \lim\limits_{n\to\infty} n \cdot \left(\dfrac{(2n+1)(2n+2)}{4(n+1)^2} \overset{4(n+1)^2}{-} 1\right) =$

$= \lim\limits_{n\to\infty} n \cdot \dfrac{\cancel{4n^2} + 6n + 2 - \cancel{4n^2} - 8n - 4}{4n^2 + 8n + 1} = \lim\limits_{n\to\infty} \dfrac{-2n+1}{4n^2 \ldots} = -\dfrac{1}{2}$

(c) $\displaystyle\sum_{n\geq 1} \frac{\ln n}{n^2}$.

ratio: $\dfrac{\ln(n+1)}{(n+1)^2} \cdot \dfrac{n^2}{\ln(n)} = \dfrac{n^2}{(n+1)^2} \cdot \dfrac{\ln n(1+\frac{1}{n})}{\ln n} = \dfrac{n^2}{(n+1)^2} \cdot \dfrac{\ln n + \ln(1+\frac{1}{n})}{\ln n}$

Cond: $2^n \cdot x_{2^n}$

$\displaystyle\sum_{n\geq 1} \cancel{2^n} \cdot \frac{\ln 2^n}{\cancel{(2^n)^2}} = \sum_{n\geq 1} \frac{n \ln 2}{2^n}$ has the same

ratio: $\dfrac{(n+1)\cancel{\ln 2}}{\cancel{2^{n+1}}_{\,2}} \cdot \dfrac{\cancel{2^n}}{n \cancel{\ln 2}} = \dfrac{n+1}{2n} = \dfrac{1}{2} < 1 \Rightarrow$ convergent

(d) $\sum_{n\geq 1} n!\sin x \sin\frac{x}{2}\ldots\sin\frac{x}{n},\ x\in(0,\pi)$.

Ratio : $\frac{\cancel{(n+1)!}\cdot \cancel{\sin x \ldots \sin\frac{x}{n}}\cdot \sin\frac{x}{n+1}}{\cancel{n!}\ \cancel{\sin x \ldots \sin\frac{x}{n}}} = (n+1)\sin\frac{x}{n+1} \to \frac{\sin\frac{x}{n+1}}{\frac{1}{n+1}}$

$\sum_{n\geq 1}(-1)^n\underbrace{(\sqrt{n}-\sqrt{n+1})}$. conv. & abs. conv

Leibniz

$x_n \to 0$

$x_{n+1} = \sqrt{n+1} - \sqrt{n+2}$

$x_n = \sqrt{n} - \sqrt{n+1}$ (-)

$-\sqrt{n} - 2\sqrt{n+1} - \sqrt{n+2} < 0 \Rightarrow$ decreasing

~~Ratio: $\frac{\sqrt{n+1}-\sqrt{n+2}}{\sqrt{n}-\sqrt{n+1}} = \frac{(\sqrt{n+1}-\sqrt{n+2})(\sqrt{n}+\sqrt{n+1})}{n-n-1} = -(\sqrt{n(n+1)}+n+1-\sqrt{n(n+2)}-\sqrt{n(n+1)})$~~

$\sqrt{n} - \sqrt{n+1} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \in (0,1] \Rightarrow$ bounded

$\Rightarrow$ convergent

$|x_n| = |\sqrt{n}-\sqrt{n+1}| = \sqrt{n+1}-\sqrt{n}$

$\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \in (0,1]$ $\Rightarrow \sum(-1)^k x_n$ - conv.

decreasing

(a) $\sum_{n\geq 0}\frac{n+1}{4^n} = \sum_{n\geq 0}\frac{n}{4^n} + \frac{1}{4^n} = \sum_{n\geq 1}\frac{n}{4^n} + \sum_{n\geq 0}\left(\frac{1}{4}\right)$ $\to \frac{1}{1-\frac{1}{4}}$

Let $\sum_{\geq 0} x^n = \frac{1}{1-x}$ $|x|<1$

$\sum_{n\geq 1}\frac{n}{4^n} \to \frac{\frac{1}{4}}{(1-\frac{1}{4})^2}$

$\boxed{\frac{x}{(x-1)^2}}$

$\sum x^n = \frac{1}{1-x}$ $|()'$

$\sum n x^{n-1} = \frac{-1}{(1-x)^2}$ $|\cdot x$

$\sum n x^n = \frac{-x}{(1-x)^2}$

$x \to \frac{1}{4}$

(b) $\displaystyle\sum_{n\geq 2} \frac{n(n-1)}{2^n}$.

$$\sum_{n\geq 2} x^n = \frac{1}{1-x} \qquad |x|<1$$

$$\sum n\, x^{(n-1)} = \frac{-1}{(1-x)^2}$$

$$\sum n(n-1)\, x^{n-2} = \frac{2}{(1-x)^4} \quad \Big|\cdot x^2$$

$$\sum n(n-1)\, x^n = \frac{2x^2}{(1-x)^4}$$

$$x \to \frac{1}{2} \Rightarrow \sum \frac{n(n-1)}{2^n} = \frac{2\cdot\frac{1}{4}}{\left(1-\frac{1}{2}\right)^4} = \frac{\frac{1}{2}}{\frac{1}{2^4}} = 2^3$$

(c) $\displaystyle\sum_{n\geq 0} \frac{(-1)^n}{2n+1}$.

$$\sum x^n = \frac{1}{1-x}$$

$$x \to -x : \quad \sum (-1)^n x^n = \frac{1}{1+x}$$

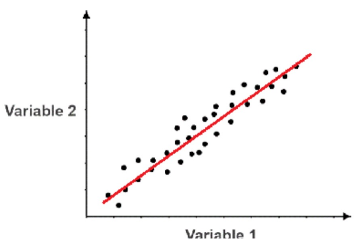
$$x \to x^2 : \quad \sum (-1)^n \cdot x^{2n} = \frac{1}{1+x^2} \quad \Big| \int_0^x$$

$$\sum \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan x$$

$$x = 1 \Rightarrow \sum \frac{(-1)^n}{2n+1} = \arctan 1 = \frac{\pi}{4}$$

6. Given the data points (x_i, y_i), $i \in \{1, \ldots, n\}$, the line of best fit f minimizes $\displaystyle\sum_{i=1}^{n} (y_i - \underline{f(x_i)})^2$. $\Rightarrow$ extinzi dev. de 2 ori

Find the line of best fit that passes through the origin (and explain its uniqueness).



$$\sum_{i=1}^{n} \left(y_i^2 - 2y_i f(x_i) + f(x_i)^2\right)'' = \sum_{i=1}^{n} \left(2y_i + 2f(x_i)\right)' = \sum_{i=1}^{n} 2$$

Midterm Test Retake

1. Find inf, sup, min, max, the interior and the closure of the set $\{\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots\}$. **1.5p (6×0.25p)**

2. Study the convergence of the following series:

(a) $\sum_{n\geq1} \frac{a^n n!}{n^n}$, with $a > e$. **1p**

(b) $\sum_{n\geq1} \frac{1}{n\sqrt[n]{n}}$. **1p**

(c) $\sum_{n\geq0} \frac{a(a+1)\dots(a+n)}{n!}$, with $a > 0$. **1p**

(d) $\sum_{n\geq1} \frac{(\ln n)^k}{n^2}$, with $k > 1$. **1p**

3. Find the sum and the radius of convergence for the following power series:

(a) $\sum_{n\geq0}(-1)^n \frac{x^{2n+1}}{2n+1}$. **1p**

(b) $\sum_{n\geq1} \frac{n}{x^n}$. **1p**

4. Find the Taylor series around zero and its radius of convergence for the following functions:

(a) $\sinh(x) := \frac{1}{2}(e^x - e^{-x})$. **1.25p**

(b) $(1+x)^\alpha$, with $\alpha \in \mathbb{R}\setminus\mathbb{Z}$. **1.25p**

$\lim_{n\to0} \frac{\ln(1+n)}{n} = 1$

$\lim_{x\to\infty} \frac{\sin x}{x} = 0$

$\frac{a^x - 1}{x} = \ln a$

1. $A = \{\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \dots\}$

$\inf(A) = -1$

$\sup(A) = 1$

$\mathrm{int}(A) = A$

$\mathrm{cl}(A) = [-1, 1]$

$\nexists$ min, $\nexists$ max

2. (a) $\sum_{n\geq1} \frac{a^n n!}{n^n}$ $a > e$

ratio: $\frac{\cancel{a^{n+1}} \cancel{(n+1)!}}{(n+1)^{n+1}} \cdot \frac{n^n}{\cancel{a^n} \cancel{(n)!}} = a\left(\frac{n}{n+1}\right)^n = a\left(1 + \frac{-1}{n+1}\right)^n = \frac{a}{e} > 1 \Rightarrow$ div

(b) $\sum_{n\geq1} \frac{1}{n\sqrt[n]{n}} = \sum_{n\geq1} \frac{1}{n \cdot n^{\frac{1}{n}}} = \sum_{n\geq1} \frac{1}{n^{\frac{n+1}{n}}}$

$\frac{n+1}{n} > 1 \Rightarrow p > 1 \Rightarrow$ conv.

ratio: $\frac{n\sqrt[n]{n}}{(n+1)\sqrt[n+1]{n}}$

$\sum_{n\geq1} \cancel{2^n} \cdot \frac{1}{\cancel{2^n} \sqrt[2^n]{2^n}} = \sum_{n\geq1} \frac{1}{\sqrt[2^n]{2^n}} = \sum_{n\geq1}$ ———

(c) $\sum_{n\geq0} \frac{a(a+1)\dots(a+n)}{n!}$

ratio: $\frac{a\cancel{(a+1)\dots(a+n)}(a+n+1)}{\cancel{(n+1)!}} \cdot \frac{n!}{\cancel{a(a+1)\dots(a+n)}} = \frac{a+n+1}{n+1} = 1$

R-D: $\lim_{n\to\infty} n\left(\frac{n+1}{a+n+1} - 1\right) = \lim_{n\to\infty} n \cdot \frac{-a}{a+n+1} = -a < 1 \Rightarrow$ divergent

(d) $\sum \frac{(\ln n)^k}{n^2}$

like: $\sum \frac{(\ln 2^n)^k}{2^{2n}}$

ratio: $\frac{(\cancel{(n+1)\ln 2})^k}{(2^n)^2} \cdot \frac{\cancel{2^{2n}}}{(n \cancel{\ln 2})^k} = \left(\frac{n+1}{n}\right)^k \cdot \frac{1}{2} \to \frac{1}{2} < 1 \Rightarrow$ div

3. (a) $\sum_{n \geq 0} (-1)^n \frac{x^{2n+1}}{2n+1}$

$$\sum x^n = \frac{1}{1-x} \qquad |x| < 1$$

$$x \to -x \Rightarrow \sum (-1)^n \cdot x^n = \frac{1}{1+x}$$

$$\sum (-1)^n \cdot x^{2n} = \frac{1}{1+x^2} \quad \Big| \int_0^x$$

$$\frac{\sum (-1)^n \cdot x^{2n+1}}{2n+1} = \arctan x$$

radius of conv.

$$\frac{(-1)^n}{2n+1} = a_n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}}{2n+3} \cdot \frac{2n+1}{(-1)^n}\right| = 1 \Rightarrow (-1,1)$$

(b) $\sum_{n \geq 1} \frac{n}{x^n}$

$$\sum x^n = \frac{1}{1-x} \qquad |x| < 1$$

$$x \to \frac{1}{x} \Rightarrow \sum \left(\frac{1}{x}\right)^n = \frac{1}{1-\frac{1}{x}} = \frac{x}{x-1} \quad \Big| \text{ deriv}$$

$$\sum n \cdot \frac{1}{x^{n-1}} = \frac{\cancel{x}-1-\cancel{x}}{(x-1)^2} \quad \Big| \cdot \frac{1}{x}$$

$$\sum n \cdot \frac{1}{x^n} = \frac{-1}{x \cdot (x-1)^2}$$

$C = (-1, 1)$

4. (a) $\sinh(x) := \frac{1}{2}(e^x - e^{-x})$

$\sinh'(x) = \frac{1}{2}(e^x + e^{-x}) \qquad \sinh'(0) = 1$

$\sinh''(x) = \frac{1}{2}(e^x - e^{-x}) \qquad \sinh''(0) = 0$

$\vdots$

$\sinh^{n}(x) = \frac{1}{2}(e^x + (-1)^{n+1} e^{-x})$

$\Rightarrow \sinh^{(2n+1)}(0) = 1$
$\sinh^{(2n)}(0) = 0$

Taylor Series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{1}{(2n+1)!}}_{a_n} \underbrace{x^{2n+1}}_{x^n}$$

$$L = \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\cancel{(2n-1)}!}{\cancel{(2n+1)}!} \right| = \lim_{n\to\infty} \frac{1}{2n(2n+1)} = 0 \Rightarrow R = \infty \Rightarrow \text{convergent every where}$$

(b) $f(x) = (1+x)^{\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Z}$

$f'(x) = \alpha (1+x)^{\alpha-1}$ $\qquad f'(0) = \alpha$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$ $\qquad f''(0) = \alpha(\alpha-1)$

$\vdots$

$f^{(n)}(x) = \underbrace{\alpha(\alpha-1)\dots(\alpha-n+1)}_{A_\alpha^n}(1+x)^{\alpha-n}$ $\qquad f^{(n)}(0) = A_\alpha^n$

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \sum_{n=0}^{\infty} \underbrace{\frac{A_\alpha^n}{n!}}_{a_n} x^n$$

$$L = \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{A_\alpha^{n+1}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{A_\alpha^n} \right| =$$

$$= \left| \frac{\cancel{\alpha(\alpha-1)\dots}(\alpha-n-1)}{\cancel{\alpha(\alpha-1)\dots(\alpha-n)}} \cdot \frac{1}{n+1} \right| = \left| \frac{\alpha-n-1}{n+1} \right| = 1 \in (0,\infty) \Rightarrow R = \frac{1}{L} = 1 \Rightarrow \text{conv. } (-1,1) \checkmark$$

(d) $\sum_{n\geq 1} n! \sin x \sin \frac{x}{2} \dots \sin \frac{x}{n}, x \in (0,\pi)$.

Ratio: $\dfrac{\cancel{(n+1)!}\ \cancel{\sin x \cdot \sin \frac{x}{2} \cdot \ldots \cdot \sin\frac{x}{n}} \cdot \sin\frac{x}{n+1}}{\cancel{n!}\ \cancel{\sin x \cdot \sin\frac{x}{2} \cdot \ldots \cdot \sin\frac{x}{n}}} = (n+1)\sin\frac{x}{n+1} = \dfrac{\sin\frac{x}{n+1}}{\frac{x}{n+1}} \cdot x = x$ $\Rightarrow$ for $x \in (0,1) \Rightarrow$ conv; $x \in (1,\pi) \Rightarrow$ div

$\to 1$

for $x = 1$: R-D: $\lim_{n\to\infty} n\left(\dfrac{1}{(n+1)\sin\frac{1}{n+1}} - 1\right) = \lim_{n\to\infty} n \cdot \dfrac{1-(n+1)\cdot\sin\frac{1}{n+1}}{(n+1)\sin\frac{1}{n+1}} = \lim_{n\to\infty} \dfrac{n - n\cdot\cancel{(n+1)}\cdot\dfrac{\sin\frac{1}{n+1}}{\frac{1}{n+1}}\cdot\frac{1}{n+1}}{\cancel{(n+1)}\cdot\dfrac{\sin\frac{1}{n+1}}{\frac{1}{n+1}}\cdot\frac{1}{n+1}}$

$(n+1)\cdot\sin\frac{1}{n+1}$ $\qquad \to 0$

$$= \frac{n - n\cdot\dfrac{\sin\frac{1}{n+1}}{\frac{1}{n+1}}}{\dfrac{\sin\frac{1}{n+1}}{\frac{1}{n+1}}} = 0 < 1 \Rightarrow \text{div.}$$