Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 27, 2023

1. (15p) Find the general solution of
(a) the difference equation $x_{k+2} - \sqrt{3}\, x_{k+1} + x_k = 0$;
(b) the differential equation $x'' - \sqrt{3}\, x' + x = 0$.

2. We consider the ideal pendulum system
$\dot{x} = y, \quad \dot{y} = -9 \sin x$.
(a) (10p) Find the expression of a global first integral. Check it using the corresponding first order partial differential equation.
(b) (10p) If $(2\pi, 0)$ is an equilibrium point, is it hyperbolic? Is it stable?

3. (15p) Find the values of $h > 0$ such that the attractor equilibrium point of $x' = x^2 + 5x + 6$ is also an attractor fixed point of the discrete dynamical system associated to the Euler's numerical formula with stepsize $h > 0$ for the given differential equation.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 27, 2023

1. (15p) Find the values of $h > 0$ such that the attractor equilibrium point of $x' = x^2 - x - 6$ is also an attractor fixed point of the discrete dynamical system associated to the Euler's numerical formula with stepsize $h > 0$ for the given differential equation.

2. (15p) Find the general solution of
(a) the difference equation $x_{k+2} + \sqrt{2}\, x_{k+1} + x_k = 0$;
(b) the differential equation $x'' + \sqrt{2}\, x' + x = 0$.

3. We consider the pray-predator system
$\dot{x} = x(1 - y), \quad \dot{y} = -y(2 - x).$
(a) (10p) Find the expression of a first integral in $(0, \infty) \times (0, \infty)$. Check it using the corresponding first order partial differential equation.
(b) (10p) If $(2, 1)$ is an equilibrium point, is it hyperbolic? Is it stable?

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 28, 2023

1. For each $k > 0$ we consider the differential equation $\dot{x} = -k\,(x - 21)$, which is the model of Newton for cooling processes, here $x(t)$ being the temperature of a cup of tea at time t. The time is measured in minutes, and the temperature in Celsius degrees.

(a) (5p) Find its flow.

(b) (5p) An experiment revealed the following fact: a cup of tea with initial temperature of $49°C$ has, after 10 minutes, $37°C$. Find the initial temperature of a cup of tea such that after 20 minutes the tea has $37°C$.

2. (10p) Find the solution of the IVP

$$\theta'' + 9\theta = 0, \quad \theta(0) = \frac{\pi}{2}, \quad \theta'(0) = 0,$$

which models the motion of a simple gravity pendulum. After how much time the pendulum will return to the initial state? Here $\theta(t)$ is the angle at time t between the rod and the vertical. The time is measured in minutes, and the angle in radians.

3. (15p) Let $g : I \to \mathbb{R}$ be a C^1 map such that $g'(x) \neq 0$ for all x in the interval I. Assume that there exists $r \in I$ such that $g(r) = 0$. Prove that for $\eta \in I$ sufficiently close to r the unique solution $(x_k)_{k\geq 0}$ of the IVP

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}, \quad x_0 = \eta$$

satisfies $\lim_{k\to\infty} x_k = r$.

4. (15p) We consider the IVP $y' = 1 + xy^2$, $y(0) = 0$. Write the Euler numerical formula on the interval $[0, 1]$ with step-size $h = 0.02$. Specify the initial values and the number of steps necessary to find the approximate value of $\varphi(0.5)$ and, respectively, of $\varphi(1)$. Compute the approximate value of $\varphi(0.04)$. Here with φ is denoted the exact solution of the given IVP.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 14, 2023

1.

(a) (5p) Let $A \in \mathcal{M}_n(\mathbb{R})$ and $\eta \in \mathbb{R}^n$. Write a representation formula for the solution of the IVP

$$X' = AX, \quad X(0) = \eta.$$

(b) (10p) Let $t \in \mathbb{R}$. Using the definition of the matrix exponential, compute

$$e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}, \quad e^{t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \quad e^{t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \quad e^{t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}.$$

(c) (10p) Let $A, J, P \in \mathcal{M}_n(\mathbb{R})$ and assume that P is invertible and $A = PJP^{-1}$. Prove that $e^A = Pe^JP^{-1}$.

2. We consider the planar Lotka-Volterra system

$\dot{x} = x(1-y), \quad \dot{y} = y(2-x)$.

(a) (10p) Find the expression of a first integral in $(0,\infty) \times (0,\infty)$. Check it using the corresponding first order partial differential equation.

(b) (10p) If $(2,1)$ is an equilibrium point, is it hyperbolic? Is it stable?

3. (5p) Find the solution of each of the following IVPs:

(a) $x' + 3x = 2,\ x(0) = 0$;

(b) $x_{k+1} + 3x_k = 2,\ x_0 = 0$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Seminar Tests on Dynamical Systems
June 14, 2023

T1. (10p) Find the solution of the IVP

$$\theta'' + 9\theta = 0, \quad \theta(0) = \frac{\pi}{2}, \quad \theta'(0) = 0.$$

Here the unknown function θ has the variable t, and t is the time variable measured in minutes. Describe the motion of a simple gravity pendulum if $\theta(t)$ is the angle at time "t" (measures in radians) between the rod and the vertical. After how much time the pendulum will return to the initial state?

T2. (10p) We consider the scalar differential equation

$$\dot{x} = 2x(2 - x),$$

whose unknown is the function x of variable t.

a) Study the stability of its equilibria using the linearization method.
b) Depict its phase portrait.
b) Describe the properties of $\varphi(t;1)$, $\varphi(t;2)$ and $\varphi(t;5)$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 6, 2022

1. (1p) Consider the map

$$f:(0,\infty)\to\mathbb{R},\quad f(x)=\frac{x^2+5}{2x}\ .$$

(a) Prove that it has a unique fixed point, denoted η^*.
(b) Using the linearization method, prove that η^* is an attractor.
(c) Using the stair-step (cob-web) diagram, estimate the basin of attraction of η^*.

2. (2p) Consider the following planar system

$$\dot{x}=-y(x^2+y^2),\quad \dot{y}=x(x^2+y^2).$$

a) Find its equilibria.
b) Decide whether the equilibrium point $(0,0)$ is hyperbolic or not.
c) Verify that $\varphi(t,1,0)=(\cos t,\sin t)$, $\varphi(t,2,0)=(2\cos 4t,2\sin 4t)$ for all $t\in\mathbb{R}$. Find $\varphi(t,3,0)$.
d) Find a first integral.
e) Represent its phase portrait.
f) What remarkable property have the solutions of this system?

3. (2p) Consider the system

$$\dot{x}=ax-5y,\ \dot{y}=x-2y\ .$$

(a) For what values of the real parameter a the system has a center at the origin?
(b) For $a=0$ find the general solution of this system and specify its type and stability.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 20, 2022

1. (1p) Consider the difference equation

$$x_{k+2} - 6x_{k+1} + 9x_k = 12k.$$

(a) Find $a, b \in \mathbb{R}$ such that $(x_k)_p = ak + b$ is a particular solution.
(b) Find the general solution of the given linear nonhomogeneous difference equation.
(c) Find the solution of the IVP $x_{k+2} - 6x_{k+1} + 9x_k = 0$, $x_0 = 0$ and $x_1 = 0$.

2. (1.5p) Let $g : I \to \mathbb{R}$ be a C^1 map such that $g'(x) \neq 0$ for all x in the interval I. Assume that there exists $r \in I$ such that $g(r) = 0$. Prove that for $\eta \in I$ sufficiently close to r the unique solution $(x_k)_{k\geq 0}$ of the IVP

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}, \quad x_0 = \eta$$

satisfies

$$\lim_{k\to\infty} x_k = r.$$

3. (1.5p) Let $\omega > 0$ be a fixed real parameter. Find a global first integral of the following nonlinear planar system $\ddot{\theta} + \omega^2 \sin\theta = 0$.

4. (1p) Find the solution of the IVP $x'' + tx' = 1$, $x(0) = x'(0) = 0$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 21, 2022

1. (2p) Find the values of $h > 0$ such that the attractor equilibrium point of $x' = x^2 - x - 6$ is also an attractor fixed point of the discrete dynamical system associated to the Euler's numerical formula with stepsize $h > 0$ for the given differential equation.

2. (1.5p) Find the general solution of
(a) the difference equation $x_{k+2} + \sqrt{2}\, x_{k+1} + x_k = 0$;
(b) the differential equation $x'' + \sqrt{2}\, x' + x = 0$.

3. (1.5p) Let $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ be such that $a_{12} \neq 0$. Show that the roots of the characteristic equation corresponding to the second-order difference equation obtained by reducing the linear system

$$x_{k+1} = a_{11}x_k + a_{12}y_k, \quad y_{k+1} = a_{21}x_k + a_{22}y_k$$

are the eigenvalues of the matrix associated to this system.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 21, 2022

1. (1.5p) Find the general solution of
(a) the difference equation $x_{k+2} - x_{k+1} + x_k = 0$;
(b) the differential equation $x'' - x' + x = 0$.

2. (1.5p) Let $b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{R}$ be such that $b_{12} \neq 0$. Show that the roots of the characteristic equation corresponding to the second-order difference equation obtained by reducing the linear system

$$x_{k+1} = b_{11}x_k + b_{12}y_k, \quad y_{k+1} = b_{21}x_k + b_{22}y_k$$

are the eigenvalues of the matrix associated to this system.

3. (2p) Find the values of $h > 0$ such that the attractor equilibrium point of $x' = x^2 - 5x + 4$ is also an attractor fixed point of the discrete dynamical system associated to the Euler's numerical formula with stepsize $h > 0$ for the given differential equation.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 21, 2022

1. (1.5p) Let $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ be such that $a_{21} \neq 0$. Show that the roots of the characteristic equation corresponding to the second-order difference equation obtained by reducing the linear system

$$x_{k+1} = a_{11}x_k + a_{12}y_k, \quad y_{k+1} = a_{21}x_k + a_{22}y_k$$

are the eigenvalues of the matrix associated to this system.

2. (2p) Find the values of $h > 0$ such that the attractor equilibrium point of $x' = x^2 + 5x + 6$ is also an attractor fixed point of the discrete dynamical system associated to the Euler's numerical formula with stepsize $h > 0$ for the given differential equation.

3. (1.5p) Find the general solution of
(a) the difference equation $x_{k+2} - \sqrt{3}\, x_{k+1} + x_k = 0$;
(b) the differential equation $x'' - \sqrt{3}\, x' + x = 0$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
June 21, 2022

1. (2p) Find the values of $h > 0$ such that the attractor equilibrium point of $x' = -2x^2 - 10x - 12$ is also an attractor fixed point of the discrete dynamical system associated to the Euler's numerical formula with stepsize $h > 0$ for the given differential equation.

2. (1.5p) Let $b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{R}$ be such that $b_{21} \neq 0$. Show that the roots of the characteristic equation corresponding to the second-order difference equation obtained by reducing the linear system

$$x_{k+1} = b_{11}x_k + b_{12}y_k, \quad y_{k+1} = b_{21}x_k + b_{22}y_k$$

are the eigenvalues of the matrix associated to this system.

3. (1.5p) Find the general solution of
(a) the difference equation $x_{k+2} - \sqrt{2}\, x_{k+1} + x_k = 0$;
(b) the differential equation $x'' - \sqrt{2}\, x' + x = 0$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
July 7, 2022

1. (1p) Find the solution of the IVP

$$\theta'' + 9\theta = 0, \quad \theta(0) = \frac{\pi}{2}, \quad \theta'(0) = 0.$$

Here the unknown function θ has the variable t, and t is the time variable measured in minutes. Describe the motion of a pendulum governed by this IVP. After how much time the pendulum will return to the initial position?

2. (1.5p)

(a) Write the Euler's numerical formula with stepsize $h = 0.01$ to approximate the solution of the IVP $y' = y, \quad y(0) = 1$. The unknown of the differential equation is the function denoted by y of variable x.

(b) Using (a) find a rational approximation of the Euler's constant e. (Hint: First find the expression of the solution of the IVP from (a)).

3. (2p) We consider the map

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{50}x(100 - x).$$

(a) Find its fixed points and study their stability.

(b) Using the stair-step diagram estimate the basin of attraction of the asymptotically stable fixed point.

4. (0.5p) Find the linear homogeneous differential equation with constant coefficients that has the general solution

$$c_1e^{2t} + c_2e^{-2t}, \quad c_1, c_2 \in \mathbb{R}.$$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Tests on Dynamical Systems
July 7, 2022

T1. (1p) We consider the differential equation

$$t^2x'' + 2tx' - 2x = 0, \quad t \in (0, \infty).$$

a) Find solutions of the form $x(t) = t^r$ where $r \in \mathbb{R}$ has to be determined.
b) Specify its type and find its general solution.

T2. (1p) Consider the following planar system

$$\dot{x} = 2x - x^2, \quad \dot{y} = y - xy^2.$$

a) Find the equilibrium points.
b) Study the stability of each equilibrium point (apply the linearization method).

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems

1. (1.75p)

(a) Let $a, f \in C(\mathbb{R})$ and $\eta \in \mathbb{R}$. Find an integral representation of the solution of the IVP

$$x' + a(t)x = f(t), \quad x(0) = \eta.$$

(b) Find the general solution of the IVP

$$x' + 2tx = t, \quad x(0) = 0.$$

2. (2p)

(a) Using the Euler's formula compute e^{it}.

(b) Represent in the complex plane the curves

$$\{2e^{it} \ : \ t \in [0, \pi/2]\}, \quad \{2e^{it} \ : \ t \in [0, \pi]\}, \quad \{2e^{it} \ : \ t \in [0, 2\pi)\}.$$

(c) We consider the planar system $\dot{x} = -y, \ \dot{y} = x$. Denote, as usual, by $\varphi(t, \eta_1, \eta_2)$ the flow of this system. Compute $\varphi(t, 2, 0)$ and represent the corresponding orbit.

(d) Find a first integral of the planar system at (c) and represent its phase portrait.

3. (1.75p) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (-x^2 + y/5, \, x)$.

(a) Find the fixed points of f and study their stability.

(b) In case that you found an attracting fixed point, write the consequence of this fact for the sequence $(f^k(\eta))_{k\geq 0}$ where $\eta \in \mathbb{R}^2$ is properly chosen. As usual, f^k denotes the k-th iterate of f.

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Exam on Dynamical Systems

1. (1.75p)
(a) Let $t \in \mathbb{R}$. Using the Euler's formula compute

$$e^{it}, \quad e^{i\pi}, \quad e^{i\pi/2}, \quad e^{(-1+i)t}.$$

(b) Find the linear homogeneous differential equation with constant coefficients of minimal order that has the general solution $c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$, where $c_1, c_2 \in \mathbb{R}$.

2. (2.5p) We consider the map

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{50}x(100 - x).$$

(a) Find its fixed points and study their stability.
(b) Using the stair-step diagram estimate the basin of attraction of the asymptotically stable fixed point.
(c) If $(x_k)_{k\geq 0}$ represent the number of fish in some lake at month k and

$$x_{k+1} = \frac{1}{50}x_k(100 - x_k), \quad x_0 = \eta$$

try to predict the fate of the fish in the case $\eta = 80$ and also in the case $\eta = 10$.

3. (1.25p) We consider the scalar differential equation

$$\dot{x} = 2x(2 - x),$$

whose unknown is the function x of variable t.
a) Study the stability of its equilibria using the linearization method.
b) Depict its phase portrait.
b) Describe the properties of $\varphi(t; 1)$, $\varphi(t; 2)$ and $\varphi(t; 5)$.

Universitatea Babeş–Bolyai
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Exam on Dynamical Systems

1. (1p) Let $A, J, P \in \mathcal{M}_n(\mathbb{R})$ be invertible matrices such that $A = PJP^{-1}$. Prove that $e^A = Pe^J P^{-1}$.

2. (2p) We consider the IVP $\quad y' = -200y, \quad y(0) = 1,$
where the unknown is the function $y(t)$.

a) Find the solution and its limit as $t \to \infty$.

b) Write the Euler's numerical formula with constant step-size h.

c) For $h = 0.001$, and, respectively, $h = 0.01$ find the solution $(y_k)_{k\geq 0}$ of the difference equation found at b) and decide if it satisfies $\lim\limits_{k\to\infty} y_k = 0$.

d) Find a range of values for the step-size h such that the solution $(y_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} y_k = 0$.

3. (2.5p) We consider the planar differential system

$$x' = x - y, \quad y' = x + y.$$

a) Find the type and stability of this linear system.

b) Pass to polar coordinates, i.e. find the differential system in the unknowns $(\rho(t), \theta(t))$ when

$$x(t) = \rho(t) \cos\theta(t), \quad y(t) = \rho(t) \sin\theta(t).$$

c) Represent the phase portrait.

d) Find the second order linear homogeneous equation which is equivalent to this system.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems

1. (2p) Let $\mathcal{L} : C^2(\mathbb{R}) \to C(\mathbb{R})$ be defined by $\mathcal{L}x = x'' - 2x' + x$ for any $x \in C^2(\mathbb{R})$.

(a) Prove that $\mathcal{L}$ is a linear map. What is the dimension of its kernel?

(b) Find the general solution of the equation $x'' - 2x' + x = \cos 2t$ knowing that it has a particular solution of the form $a \cos 2t + b \sin 2t$, for some $a, b \in \mathbb{R}$.

(c) Let $f_1(t) = e^{2t}$ and $f_2(t) = e^{-2t}$ for all $t \in \mathbb{R}$. Find a particular solution of the equation $\mathcal{L}x = 3f_1 + 5f_2$.

2. (1.5p)

(a) Write the Euler's numerical formula with stepsize $h = 0.01$ to approximate the solution of the IVP $y' = y, \quad y(0) = 1$.

(b) Using (a) find a rational approximation of the Euler's constant e.

3. (2p) Let $g : I \to \mathbb{R}$ be a C^1 map such that $g'(x) \neq 0$ for all x in the interval I. Assume that there exists $r \in I$ such that $g(r) = 0$. Prove that for $\eta \in I$ sufficiently close to r the unique solution $(x_k)_{k \geq 0}$ of the IVP

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}, \quad x_0 = \eta$$

satisfies

$$\lim_{k \to \infty} x_k = r.$$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems

1. (1.5p) Represent the phase portrait of the scalar dynamical system $\dot{\rho} = \rho(1 - \rho^2)$. Find $\varphi(t, 1)$ and justify. Specify the properties of $\varphi(t, 2)$ and, respectively, $\varphi(t, 0.5)$.

2. (0.25p) Find the polar coordinates of the points whose cartesian coordinates are: $(1, 0)$, $(0, 1)$, $(-2, 0)$ and $(0, -0.5)$, respectively.

3. (2.5p) We consider the planar system
$\dot{x} = -y + x(1 - x^2 - y^2),\ \dot{y} = x + y(1 - x^2 - y^2)$.

a) Study the type and stability of the equilibrium point $(0, 0)$ using the linearization method.

b) Check that $\varphi(t, 1, 0) = (\cos t, \sin t)$ for any $t \in \mathbb{R}$. Represent the corresponding orbit.

c) Transform the given system to polar coordinates.

d) Sketch the phase portrait of this planar system. There is an attractor?

4. (1.25p) Let $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ be such that $a_{12} \neq 0$. Show that the roots of the characteristic equation corresponding to the second-order difference equation obtained by reducing the linear system

$$x_{k+1} = a_{11}x_k + a_{12}y_k, \quad y_{k+1} = a_{21}x_k + a_{22}y_k$$

are the eigenvalues of the matrix associated to this system.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems

1. (0.75p) Find the expression of the Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k,\ x_0 = 0,\ x_1 = 1.$$

2. (1.5p) Find the solution of the IVP

$$\theta'' + 9\theta = 0, \quad \theta(0) = \frac{\pi}{2}, \quad \theta'(0) = 0.$$

Here $\theta = \theta(t)$, and t is the time variable measured in minutes. Describe the motion of a pendulum governed by this IVP. After how much time the pendulum will return to the initial position?

3. (0.5p) Find the linear homogeneous differential equation with constant coefficients that has the general solution

$$c_1 e^{2t} + c_2 e^{-2t}, \quad c_1, c_2 \in \mathbb{R}.$$

4. (2.75p) We consider the map

$$T : \mathbb{R} \to \mathbb{R}, \quad T(x) = 1 - |2x - 1|.$$

a) Represent the graph of T. Find its fixed points.
b) Let $n \in \mathbb{N}$, $n \geq 2$. Find the orbit of the initial state $\frac{3}{2^n}$.
c) Find the 2-periodic points of T.
d) Represent the graphs of T^2 and T^3. How many fixed points they have?
e) The map T has a 2-periodic orbit? Or a 3-periodic orbit? T has a 917-periodic orbit?

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems

1. (2p) Let $c \in \{20, 200\}$ be a parameter and consider the scalar dynamical system $\dot{x} = 100x - x^2 - cx$.

a) Find its equilibria and study their stability using the linearization method.

b) Represent its phase portrait.

c) When $x(t) > 0$ is considered to be the number of fish in some lake, and c to be the rate of fishing, try to predict the fate of the fish from the lake interpreting the theoretical results obtained at a) and b).

2. (1.5p) Find the solution of each of the following IVPs.

(a) $x' + tx = 1, \quad x(0) = 0$; (b) $x'' + 4x = 1, \quad x(0) = 1, \quad x'(0) = 0$.

3. (2p) Let $g : I \to \mathbb{R}$ be a C^1 map such that $g'(x) \neq 0$ for all x in the interval I. Assume that there exists $r \in I$ such that $g(r) = 0$. Prove that for $\eta \in I$ sufficiently close to r the unique solution $(x_k)_{k\geq 0}$ of the IVP

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}, \quad x_0 = \eta$$

satisfies

$$\lim_{k\to\infty} x_k = r.$$

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems
July, 2017

1. (1.5p) Find the solution of the IVP

$$\theta'' + 9\theta = 0, \quad \theta(0) = \frac{\pi}{2}, \quad \theta'(0) = 0.$$

Here $\theta = \theta(t)$, and t is the time variable measured in minutes. Describe the motion of a pendulum governed by this IVP. After how much time the pendulum will return to the initial position?

2. (1.5p)
(a) Write the Euler's numerical formula with stepsize $h = 0.01$ to approximate the solution of the IVP $y' = y, \quad y(0) = 1$.
(b) Using (a) find a rational approximation of the Euler's constant e.

3. (2.5p) We consider the map

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{50}x(100 - x).$$

(a) Find its fixed points and study their stability.
(b) Using the stair-step diagram estimate the basin of attraction of the asymptotically stable fixed point.
(c) If $(x_k)_{k\geq 0}$ represent the number of fish in some lake at month k and

$$x_{k+1} = \frac{1}{50}x_k(100 - x_k), \quad x_0 = \eta$$

try to predict the fate of the fish in the case $\eta = 80$ and also in the case $\eta = 10$.

Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Tests on Dynamical Systems
July, 2017

T1. (1p) We consider the differential equation

$$t^2x'' + 2tx' - 2x = 0, \quad t \in (0, \infty).$$

a) Find solutions of the form $x(t) = t^r$ where $r \in \mathbb{R}$ has to be determined.
b) Specify its type and find its general solution.

T2. (1p) Consider the following planar system

$$\dot{x} = 2x - x^2, \quad \dot{y} = y - xy^2.$$

a) Find the equilibrium points.
b) Indicate the type and stability character of every equilibrium point.

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Exam on Dynamical Systems

1. (1.5p) For each $k > 0$ we consider the differential equation $\dot{x} = -k\,(x - 21)$, which is the model of Newton for cooling processes, here $x(t)$ being the temperature of a cup of tea at time t.

(a) Find its flow.

(b) An experiment revealed the following fact. A cup of tea with initial temperature of $49°C$ has a temperature of $37°C$ after 10 minutes. Find the initial temperature of a cup of tea such that after 20 minutes the tea has $37°C$.

2. (2.5p) We consider the map

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{50} x(100 - x).$$

(a) Find its fixed points and study their stability.

(b) Using the stair-step diagram estimate the basin of attraction of the asymptotically stable fixed point.

(c) If $(x_k)_{k\geq 0}$ represent the number of fish in some lake at month k and

$$x_{k+1} = \frac{1}{50} x_k(100 - x_k), \quad x_0 = \eta$$

try to predict the fate of the fish in the case $\eta = 80$ and also in the case $\eta = 10$.

3. (1p) Find the general solution of each of the following equations.

(a) $x''' = \sin t$; (b) $x_{k+1} = 2x_k$; (c) $x_{k+1} = Ax_k$, where $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

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Exam on Dynamical Systems

1. (1.5p) We consider the differential equation

$$x'' + 4x = \cos 2t.$$

a) Find a solution of the form $x_p = t(a\cos 3t + b\sin 3t)$, with $a, b \in \mathbb{R}$.
b) Find its general solution.
c) Describe the motion of a spring-mass system governed by this equation.

2. (1p) Find the solution of

$$x_{k+2} - 6x_{k+1} + 9x_k = 12k, \quad x_0 = 0, \quad x_1 = 0.$$

Hint: look for $a, b \in \mathbb{R}$ such that $(x_k)_p = ak + b$ is a particular solution of the difference equation.

3. (1.5p) We consider the IVP $x' = -200x, \quad x(0) = 1$.
a) Find the solution and its limit as $t \to \infty$.
b) Write the Euler's numerical formula with constant step-size h.
c) Find a range of values for the step-size h such that the solution $(x_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} x_k = 0$.

4. (1.5p) Find the fixed points and the 2–periodic points of the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$. Study their stability.

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Exam on Dynamical Systems

1. (1p)

a) Write the statement of the Fundamental Theorem for linear homogeneous second order differential equations.

b) The following proposition is true or false? Justify. We remind you that $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$.

"The general solution of the differential equation $x'' - x = 0$ is
$x(t) = c_1 \cosh t + c_2 \sinh t$, where c_1, c_2 are arbitrary real constants."

2. (1p) Find a first integral in $\mathbb{R}^2$ of the pendulum equation

$$\ddot{\theta} + \omega^2 \sin \theta = 0.$$

Hint: Write first the planar system equivalent to this equation.

3. (0.75p) Find the expression of the Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k,\ x_0 = 0,\ x_1 = 1.$$

4. (2p) We consider the following nonlinear planar systems
$\dot{x} = -x + xy, \quad \dot{y} = -2y + 3y^2.$

a) Find its equilibria and study their stability using the linearization method.

b) Find $\varphi(t, 0, 2/3)$, $\varphi(t, 4, 0)$ and $\varphi(t, 1, 2/3)$.

5. (0.75p) We consider the IVP $y' = 1 + xy^2,\ y(0) = 0$. Write the Euler numerical formula on the interval $[0, 1]$ with step-size $h = 0.02$. Specify the initial values and the number of steps necessary to find the approximate value of $\varphi(0.5)$ and, respectively, of $\varphi(1)$. Here with φ is denoted the exact solution of the given IVP.

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Exam on Dynamical Systems

1. (1.5p) For each $k > 0$ we consider the differential equation $\dot{x} = -k\,(x-21)$, which is the model of Newton for cooling processes, here $x(t)$ being the temperature of a cup of tea at time t.

(a) Find its flow.

(b) An experiment revealed the following fact. A cup of tea with initial temperature of $49°C$ has a temperature of $37°C$ after 10 minutes. Find the initial temperature of a cup of tea such that after 20 minutes the tea has $37°C$.

2. (2.5p) We consider the map

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{50}x(100 - x).$$

(a) Find its fixed points and study their stability.

(b) Using the stair-step diagram estimate the basin of attraction of the asymptotically stable fixed point.

(c) If $(x_k)_{k\geq 0}$ represent the number of fish in some lake at month k and

$$x_{k+1} = \frac{1}{50}x_k(100 - x_k), \quad x_0 = \eta$$

try to predict the fate of the fish in the case $\eta = 80$ and also in the case $\eta = 10$.

3. (1p) Find the general solution of each of the following equations.

(a) $x''' = \sin t$; (b) $x_{k+1} = 2x_k$; (c) $x_{k+1} = Ax_k$, where $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

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Exam on Dynamical Systems

1. (2p) We consider the difference equation

$$x_{k+2} + x_k = \cos\frac{k\pi}{2}\ .$$

a) Find a solution of the form $(x_k)_p = ak\cos\frac{k\pi}{2}$, with $a \in \mathbb{R}$. (Hint: we recall that $\cos(t+\pi) = -\cos t$ for any $t \in \mathbb{R}$)

b) Find its general solution.

c) Find the solution with $x_0 = x_1 = 0$ and describe its long-term behavior (is it periodic? is it bounded? is it oscillatory around 0?).

2. (1.5p) We consider the linear difference system

$$x_{k+1} = \frac{3}{5}x_k + \frac{1}{5}y_k, \quad y_{k+1} = \frac{1}{5}x_k + \frac{3}{5}y_k.$$

a) Study the stability of this system.

b) Find its general solution.

3. (2p) Let $c \in \{20, 200\}$ be a parameter and consider the scalar dynamical system $\dot{x} = 100x - x^2 - cx$.

a) Find its equilibrium points and study their stability using the linearization method.

b) Represent its phase portrait.

c) When $x(t) > 0$ is considered to be the number of fish in a lake, and c to be the rate of fishing, try to predict the fate of the fish from the lake interpreting the theoretical results obtained at a) and b).

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Exam on Dynamical Systems

1. (1.5p) For what values of the real parameter a the system $\dot{x} = ax - 5y,\ \dot{y} = x - 2y$ has a center at the origin? For $a = 2$ find the general solution of this system and specify its type and stability.

2. (1p) Find the solution of $x_{k+2} - 6x_{k+1} + 9x_k = 12k$, $x_0 = 0$, $x_1 = 0$. *Hint*: look for $a, b \in \mathbb{R}$ such that $(x_k)_p = ak + b$ is a particular solution of the difference equation.

3. (1.5p) We consider the IVP $x' = -10^3 x, \quad x(0) = 1.$

a) Find the solution and its limit as $t \to \infty$.

b) Write the Euler's numerical formula with constant step-size h.

c) Find a range of values for the step-size h such that the solution $(x_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} x_k = 0$.

4. (1.5p) We consider the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x - \frac{1}{4}(x^2 - 2)$ and, given $x_0 \in \mathbb{R}$, consider the sequence $(x_k)_{k\geq 0}$ satisfying the recurrence

$$x_{k+1} = f(x_k)\ .$$

a) Find the fixed points of f, and study their stability.

b) Find $(x_k)_{k\geq 0}$ when $x_0 = \sqrt{2}$.

c) There exists an $x_0 \in \mathbb{R} \setminus \{\sqrt{2}\}$ such that $\lim\limits_{k\to\infty} x_k = \sqrt{2}$?

d) There exists an $x_0 \in \mathbb{R}$ such that $\lim\limits_{k\to\infty} x_k = 2$?

T1. (1p) Find the linear homogeneous differential equation with constant real coefficients, of minimal order, which has as solution the function $\cos 3t$.

T2. (1p) We consider the planar differential system $x' = -4y, \quad y' = x$.

a) Find its general solution.

b) Specify the type and stability of this linear system.

c) Represent its phase portrait.

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Exam on Dynamical Systems

1. (1.5p) Let $a_1, a_2 \in \mathbb{R}$. Prove the following propositions.

(a) Any solution of the differential equation $x'' + a_1x' + a_2x = 0$ satisfies $\lim_{t\to\infty} x(t) = 0$ if and only if $Re(\lambda) < 0$ for any root $\lambda \in \mathbb{C}$ of the characteristic equation. Here $Re(\lambda)$ denotes the real part of λ.

(b) Any solution of the difference equation $x_{k+2} + a_1x_{k+1} + a_2x_k = 0$ satisfies $\lim_{k\to\infty} x_k = 0$ if and only if $|\lambda| < 1$ for any root $\lambda \in \mathbb{C}$ of the characteristic equation. Here $|\lambda|$ denotes the modulus of λ.

2. (1p)

(a) Find the general solution of $x' - 2x = 3t$.

(b) Check that $x = \frac{1}{t^2}$ is a solution of $x' + \frac{2x}{t} = 0$. Write the general solution of this differential equation. Is it a linear equation?

(c) Find the general solution of $x' + \frac{2x}{t} = 1$.

3. (1.5p) Let $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$.

(a) Prove that the matrix A is diagonalizable.

(b) Using the characteristic equation method, find the general solution of $X' = AX$.

4. (1.5) Let $\lambda \in (1, 4)$ be a real parameter and consider the family of maps $f_\lambda : \mathbb{R} \to \mathbb{R}$, $f_\lambda(x) = \lambda x(1 - x)$. Find the fixed points of f_λ and study their stability.

T1. (1p) Prove that $x(t) = c_1 \cosh(t) + c_2 \sinh(t)$, $c_1, c_2 \in \mathbb{R}$, is the general solution of the differential equation $x'' - x = 0$. Recall the notations $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$ and $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$.

T2. (1p) Find a first integral in $(0, \infty) \times (0, \infty)$ for the planar Lotka-Volterra system $\dot{x} = x(1 - y), \quad \dot{y} = y(2 - x)$.

T1. (1p) Find the linear homogeneous differential equation with constant real coefficients, of minimal order, which has as solution the function $\cos 3t$.

T2. (1p) We consider the planar differential system $x' = -4y, \quad y' = x$.

a) Find its general solution.

b) Specify the type and stability of this linear system.

c) Represent its phase portrait.