Seminar 3

1. Find the sum for each of the following series:

   ~~(a)~~ $\sum_{n\geq 1} \frac{2}{3^n}$.

   ~~(b)~~ $\sum_{n\geq 1} \frac{2n+1}{n!}$.

   ~~(c)~~ $\sum_{n\geq 1} \frac{1}{4n^2-1}$.

   ~~(d)~~ $\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$.

   $\boxed{\cancel{(e)}\ \sum_{n\geq 1} \frac{n}{2^n}.}$ midterm

   (f) $\sum_{n\geq 1} \frac{1}{\sqrt[3]{n}}$.

2. ★ Find the sum for each of the following series:

   (a) $\sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right)$.

   (b) $\sum_{n\geq 1} \frac{n+1}{3^n}$.

   (c) $\sum_{n\geq 1} \frac{n}{n^4+n^2+1}$.

3. Study if the following series are convergent or divergent:

   (a) $\sum_{n\geq 2} \frac{1}{\ln n}$.

   (b) $\sum_{n\geq 1} \frac{1}{n\sqrt{n+1}}$.

   (c) $\sum_{n\geq 1} \frac{\ln\left(1+\frac{1}{n}\right)}{n}$.

   (d) $\sum_{n\geq 1} \frac{n!}{n^n}$.

   (e) $\sum_{n\geq 1} \left(\frac{n}{n+1}\right)^{n^2}$.

   (f) $\sum_{n\geq 2} \frac{1}{n\ln(n)}$.

4. ★ Study if the following series are convergent or divergent:

   (a) $\sum_{n\geq 1} \frac{x^n}{n^p}$, $x>0, p\in\mathbb{N}$.

   (b) $\sum_{n\geq 2} \frac{1}{(\ln n)^{\ln n}}$.

   (c) $\sum_{n\geq 1} (\sqrt[n]{n}-1)$.

5. ★ Start with an equilateral triangle of side 1. For each side, remove the middle third and add there another equilateral triangle. Repeat this process at each iteration (see the figure). How many sides are there at iteration n? What is the limit of the perimeter and the area?

Compute the parameter and the area

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.

1. (a) $\sum_{n\geq 1} \frac{2}{3^n} = \frac{2}{3^1} + \frac{2}{3^2} + \dots$

$$= 2\cdot\left(\frac{1}{1-\frac{1}{3}} - 1\right) = 2\left(\frac{1}{\frac{2}{3}} - 1\right) = 2\left(\frac{3}{2} - 1\right) = 2\cdot\frac{1}{2} = 1$$

$$\boxed{\begin{aligned} 1+q+q^2+\dots &= \frac{1}{1-q} \\ q+q^2+\dots &= \frac{1}{1-q} - 1 \end{aligned}}$$

(b) $\sum_{n\geq 1} \frac{2n+1}{n!} = \sum_{n\geq 1} \frac{2n}{n!} + \frac{1}{n!} =$ $\boxed{\sum_{n=0}^{\infty} \frac{1}{n!} = e}$

$$= 2\sum_{n\geq 1} \frac{1}{(n-1)!} + \sum_{n\geq 1} \frac{1}{n!} = 2e + e - 1 = 3e - 1$$

(c) $\sum_{n\geq 1} \frac{1}{4n^2-1} = \sum_{n\geq 1} \frac{\overset{2}{1}}{(2n-1)(2n+1)} = \frac{1}{2}\left(\sum_{n\geq 1} \frac{1}{2n-1} - \sum \frac{1}{2n+1}\right) = \lim_{n\to\infty} \frac{1}{2}\left(1 + \cancel{\frac{1}{3}} + \cancel{\frac{1}{5}} + \dots + \cancel{\frac{1}{2n-1}} - \left(\cancel{\frac{1}{3}} + \cancel{\frac{1}{5}} + \dots + \frac{1}{2n+1}\right)\right)$

$$= \frac{1}{2}\lim_{n\to\infty}\left(\overset{2n+1}{1} - \frac{1}{2n+1}\right) = \frac{1}{2}\lim_{n\to\infty} \frac{2n}{2n+1} = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

(d) $\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{\overset{n(n+1)}{C}}{n+2} = \frac{A(n+1)(n+2) + B(n+2)\cdot n + C(n+1)\cdot n}{n(n+1)(n+2)} = \frac{A(n^2+3n+2) + B(n^2+2n) + C(n^2+n)}{n(n+1)(n+2)}$$

$$\begin{cases} 2A = 1 \Rightarrow A = \frac{1}{2} \\ 3A + 2B + C = 0 \Rightarrow 2B + C = -\frac{3}{2} \\ A + B + C = 0 \Rightarrow \quad B + C = -\frac{1}{2} \end{cases}$$

$(-)$

$B = -1 \Rightarrow C = -\frac{1}{2} + 1 = \frac{1}{2}$

$$\Rightarrow \frac{1}{2}\sum_{n\geq 1}\frac{1}{n} - \frac{1}{2}\sum_{n\geq 1}\frac{1}{n+1} - \frac{1}{2}\sum_{n\geq 1}\frac{1}{n+1} + \frac{1}{2}\sum_{n\geq 1}\frac{1}{n+2} = \frac{1}{2}\sum_{n\geq 1}\frac{1}{n} - \frac{1}{n+1} - \frac{1}{2}\sum_{n\geq 1}\frac{1}{n+1} - \frac{1}{n+2} =$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(\overset{n+1}{1} - \frac{1}{n+1}\right) - \frac{1}{2}\lim_{n\to\infty}\left(\overset{n+2}{\frac{1}{2}} - \frac{\overset{2}{1}}{n+2}\right) = \frac{1}{2}\underbrace{\lim_{n\to\infty}\frac{n}{n+1}}_{=1} - \frac{1}{2}\underbrace{\lim_{n\to\infty}\frac{\cancel{n+2}-\cancel{2}}{2(n+2)}}_{=\frac{1}{2}} = \overset{2}{\cancel{\frac{1}{2}}} - \frac{1}{4} = \frac{1}{2}$$

(e) $\sum_{n\geq 1} \frac{n}{2^n} \overset{\text{ratio}}{\Longrightarrow} \lim_{n\to\infty} \frac{x_{n+1}}{x_n} < 1 \Rightarrow \lim_{n\to\infty} x_n = 0$

$$\lim_{n\to\infty} \frac{(n+1)}{\cancel{2^{n+1}}_{2}} \cdot \frac{\cancel{2^n}}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1 \Rightarrow \lim_{n\to\infty} x_n = 0$$

$$S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n} = \sum_{n\geq 0}\frac{n}{2^n} - \sum_{n\geq 1}\frac{n}{2^n} + \sum_{n\geq 0}\frac{1}{2^n} = 1 + \frac{1}{2^2} + \frac{1}{2^3} + \dots = \frac{\left(\frac{1}{2}\right)^n - \frac{2^n}{1}}{-\frac{1}{2}} = \frac{1-2^n}{2^n}\cdot(-2) \to 2$$

$$\frac{n}{2^n} = \frac{2n-2-n+2}{2^n} = \frac{n-1}{2^{n-1}} - \frac{n}{2^n} + \frac{1}{2^{n-1}}$$

$$\sum_{n=1}^{N} n x^n = x \cdot \sum_{n=1}^{N} n x^{n-1}$$

$$(x^n)' = n x^{n-1}$$

$$\Bigg\} = \boxed{x \sum_{n=1}^{N} (x^n)' = x \left(\sum_{n=1}^{N} x^n \right)'}$$

$$\sum_{n=1}^{N} x^n = x(1 + x + x^2 + \dots + x^{N-1}) = x \cdot \frac{x^N - 1}{x - 1} = \frac{x^{N+1} - x}{x - 1}$$

$$\left(\sum_{n=1}^{N} x^n \right)' = \frac{[(N+1)x^N - 1](x-1) - (x^{N+1} - x)}{(x+1)^2} = \frac{\overset{\to 0}{N x^N} - \overset{\to 0}{(N+1) x^N} + 1}{(x-1)^2}$$

$$\left(\frac{f}{g} \right)' = \frac{f'g - fg'}{g^2}$$

Let $N \to \infty$

$$N \cdot x^N \to 0 \quad \text{if } x < 1$$

$$N \cdot \frac{1}{2^N} = \frac{N}{2^N} \to 0$$

$$\Bigg\} \; x \cdot \left(\sum_{n=1}^{N} x^n \right)' \longrightarrow \frac{x}{(x-1)^2}$$

$$S = \sum_{n \ge 1} \frac{n}{2^n} = 2 \cdot \sum_{n \ge 1} \frac{n+1}{2^{n+1}} - \sum_{n \ge 1} \frac{1}{2^n} = 2\left(S - \frac{1}{2}\right) - 1 = 2S - 2 \Rightarrow \boxed{S = 2}$$

2.(b) from HW

Ratio test $\Rightarrow S < \infty$

1 f) $\sum_{\ge 1} \frac{1}{\sqrt[3]{n}} = \sum_{\ge 1} \frac{1}{n^{\frac{1}{3}}} = \sum \frac{1}{n^{\frac{1}{3}}}$

$$\sum_{n \ge 1} \frac{1}{n^{\frac{1}{3}}} > \underbrace{\sum_{n \ge 1} \frac{1}{n}}_{\infty} \longrightarrow \sum_{n \ge 1} \frac{1}{\sqrt[3]{n}} \to \infty$$

3. (a) $\sum_{n \ge 2} \frac{1}{\ln n}$

$$\sum_{n \ge 2} \frac{1}{\ln n} > \sum_{n \ge 2} \frac{1}{n} \longrightarrow \infty \Rightarrow \sum_{n \ge 2} \frac{1}{\ln n} \to \infty$$

or use ratio test:

$$\frac{x_{n+1}}{x_n} = \frac{1}{\ln(n+1)} \cdot \frac{\ln(n)}{n} \to 1$$

(b) $\sum_{n \ge 1} \frac{1}{n\sqrt{n+1}}$

$$n\sqrt{n+1} < n^2$$

$$\frac{1}{n\sqrt{n+1}} > \frac{1}{n^2} \text{ — not helpful}$$

Compare with $\sum \frac{1}{n^{\frac{3}{2}}}$

$$n\sqrt{n+1} > n^{\frac{3}{2}} \quad |()^{-1}$$

$$\frac{1}{n\sqrt{n+1}} < \frac{1}{n^{\frac{3}{2}}}$$

$\Rightarrow \lim_{n \to \infty} \frac{n^{\frac{3}{2}}}{n\sqrt{n+1}} = 1 \Rightarrow x_n$ and y_n have the same nature and $\sum \frac{1}{n^p}$ is convergent if $p > 1$, $\frac{3}{2} > 1 \Rightarrow$ CONVNIVERGENT

(c) $\sum_{n\geq 1} \frac{\ln(1+\frac{1}{n})}{n} = \sum_{n\geq 1} \frac{\ln(n+1)-\ln n}{n}$

$\sum_{n\geq 1} \frac{n\ln(1+\frac{1}{n})}{n^2} = \sum_{n\geq 1} \frac{\overbrace{\ln(1+\frac{1}{n})^n}^{\to 1}}{n^2} = \sum_{n\geq 1} \frac{1}{n^2} \to$ converges to 0

(d) $\sum_{n\geq 1} \frac{n!}{n^n}$

$\lim\left(\frac{x_{n+1}}{x_n} = \frac{\cancel{(n+1)!}}{(n+1)^{\cancel{n+1}}} \cdot \frac{n^n}{\cancel{n!}} = \left(\frac{n}{n+1}\right)^n = \left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-n}{n+1}}\right) = e^{-1} = \frac{1}{e} < 1 \Rightarrow$ Convergent

(e) $\sum_{n\geq 1}\left(\frac{n}{n+1}\right)^{n^2} =$

$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = (1^\infty) = \lim_{n\to\infty}\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-n}{n+1}} = e^{\lim_{n\to\infty}\frac{-n}{n+1}} = e^{-1} = \frac{1}{e} < 1 \Rightarrow (x_n)$ convergent

(f) $\sum_{n\geq 2} \frac{1}{n\ln(n)}$

$\frac{x_{n+1}}{x_n} = \frac{1}{(n+1)\ln(n+1)} \cdot \frac{n\ln(n)}{1} = \frac{n\ln(n)}{(n+1)\ln(n+1)} \longrightarrow 1$ so we can't say anything about ~~of~~

Cauchy condensation test

$\sum_{n\geq 1} x_n$ "like" $\sum 2^n x_{2^n}$

$\sum_{n\geq 2} \cancel{2^n} \frac{1}{\cancel{2^n}\cdot \ln 2^n} = \sum_{n\geq 2} \frac{1}{n\ln 2} = \frac{1}{\ln 2} \underbrace{\sum_{n\geq 2} \frac{1}{n}}_{\infty \to \text{divergent}} \Rightarrow \sum_{n\geq 2} \frac{1}{n\ln(n)}$ has the same nature $\Rightarrow$ divergent

Mathematical Analysis
1st Year

Babeș-Bolyai University
Bachelor in Computer Science

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   (d) $\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$.

   $\boxed{\text{(e) } \sum_{n\geq 1} \frac{n}{2^n}.}$ $\to$ midterm

   (f) $\sum_{n\geq 1} \frac{1}{\sqrt[3]{n}}$.

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(d) $\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$.

(e) $\sum_{n\geq 1} \frac{n}{2^n}$.

(f) $\sum_{n\geq 1} \frac{1}{\sqrt[3]{n}}$.

(a) $$\sum_{n\geq 1} \frac{2}{3^n} = \frac{2}{3} + \frac{2}{3^2} + \dots = 2\cdot\underbrace{\left(\frac{1}{3} + \frac{1}{3^2} + \dots\right)}_{\frac{1}{1-q}} = 2\cdot\left(\frac{1}{1-\frac{1}{3}} - 1\right) =$$

$$= 2\left(\frac{3}{2} - 1\right) = 2\cdot\frac{1}{2} = 1$$

(b) $$\sum_{n\geq 1} \frac{2n+1}{n!} = \sum_{n\geq 1}\left(\frac{2n}{n!} + \underbrace{\frac{1}{n!}}_{e-1}\right) = 2\cdot\sum_{n\geq 1}\frac{1}{(n-1)!} + \sum_{n\geq 1}\frac{1}{n!} = 2e + e - 1 = 3e - 1$$

(c) $$\sum_{n\geq 1} \frac{1}{4n^2-1} = \sum_{n\geq 1}\frac{1}{(2n+1)(2n-1)} = \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\lim_{n\to\infty}\left(1 - \frac{1}{2n+1}\right) =$$

$$= \frac{1}{2}\lim_{n\geq 1}\frac{2n}{2n+1} = \frac{1}{2}$$

(d) $$\sum_{n\geq 1}\frac{1}{n(n+1)(n+2)}$$

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} = \frac{A(n+1)(n+2) + B(n+2)\cdot n + C(n+1)\cdot n}{n(n+1)(n+2)}$$

$$\Rightarrow \begin{cases} 2A = 1 & \Rightarrow A = \frac{1}{2} \\ 3A + 2B + C = 0 & 2B + C = -\frac{3}{2} \\ A + B + C = 0 & B + C = -\frac{1}{2} \end{cases} \Bigg\} \Rightarrow B = -1 \quad C = \frac{1}{2}$$

$$\Rightarrow \sum_{n\geq 1}\frac{1}{2n} + \frac{-1}{n+1} + \frac{1}{2(n+2)} = \frac{1}{2}\sum_{n\geq 1}\frac{1}{n} - \frac{1}{n+1} - \frac{1}{2}\sum_{n\geq 1}\frac{1}{n+1} - \frac{1}{n+2} =$$

$$= \frac{1}{2}\lim\left(1 - \frac{1}{n+1}\right) - \frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2}\lim_{n\to\infty}\left(1 - \frac{1}{n+1}_{2(n+2)} - \frac{1}{2}_{(n+1)(n+2)} + \frac{1}{n+2}_{2(n+1)}\right) = \frac{1}{2}\lim_{n\to\infty}\frac{2(n+1)(n+2) - 2(n+2) - (n+1)(n+2) + 2(n+1)}{2(n+1)(n+2)} =$$

$$= \lim_{n\to\infty}\frac{2n^2 + 6n + \cancel{4} - \cancel{2n} - \cancel{4} - n^2 - 3n - \cancel{2} + \cancel{2n+2}}{2n^2 + 6n + 4} = \lim_{n\to\infty}\frac{n^2 + 3n}{2n^2 + 6n + 4} = \frac{1}{2}$$

(e) $\sum_{n\ge1}\frac{n}{2^n} \Longrightarrow \frac{x_{n+1}}{x_n} = \frac{n+1}{\cancel{2^{n+1}}_{2}}\cdot\frac{\cancel{2^n}}{n} = \frac{n+1}{2n} \to \frac12 < 1 \Rightarrow$ convergent $\lim_{n\to\infty} x_n = 0$

$$\left.\begin{array}{l}\sum_{n=1}^{N} n x^n = x\cdot\sum_{n=1}^{N} n x^{n-1} \\ (x^n)' = n x^{n-1}\end{array}\right\} = \boxed{x\sum_{n=1}^{N}(x^n)' = x\left(\sum_{n=1}^{N}x^n\right)'} \longrightarrow \frac{x}{(x-1)^2}$$

$$\sum_{n=1}^{N} x^n = x(1+x+x^2+\dots+x^{N-1}) = x\cdot\frac{x^N-1}{x-1} = \frac{x^{N+1}-x}{x-1}$$

$$\left(\sum_{n=1}^{N}x^n\right)' = \frac{[(N+1)x^N-1](x-1)-(x^{N+1}-x)}{(x+1)^2} = \frac{\underbrace{Nx^N}_{\to 0}-\underbrace{(N+1)x^N}_{\to 0}+1}{(x-1)^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$$

$$S = \sum_{n\ge1}\frac{n}{2^n} = 2\cdot\sum_{n\ge1}\frac{n+1}{2^{n+1}} - \sum_{n\ge1}\frac{1}{2^n} = 2\left(S-\frac12\right)-1 = 2S-2 \Rightarrow S=2$$

$$S = \frac12+\frac{2}{2^2}+\frac{3}{2^3}+\dots+\frac{n}{2^n} = \underbrace{\sum_{n\ge0}\frac{n}{2^n} - \sum_{n\ge1}\frac{n}{2^n}} + \sum_{n\ge0}\frac{1}{2^n} = 1+\frac{1}{2^2}+\frac{1}{2^3}+\dots =$$

$$= \frac{\left(\frac12\right)^n - \frac{2^n}{1}}{\frac12-1} = \frac{1-2^n}{2^n}\cdot(-2) \longrightarrow 2$$

$$\frac{n}{2^n} = \frac{2n-\cancel{2}-n+\cancel{2}}{2^n} = \frac{n-1}{2^{n-1}} - \frac{n}{2^n} + \frac{1}{2^{n-1}}$$

(f) $\sum_{n\ge1}\frac{1}{\sqrt[3]{n}} = \sum_{n\ge1}\frac{1}{n^{\frac13}} \longrightarrow \infty$ $\quad\sum_{n\ge1}\frac{1}{n^p}$ converges iff $p>1$

2. ★ Find the sum for each of the following series:

(a) $\sum_{n\ge2}\ln\left(1-\frac{1}{n^2}\right)$. (b) $\sum_{n\ge1}\frac{n+1}{3^n}$. (c) $\sum_{n\ge1}\frac{n}{n^4+n^2+1}$.

(a) $\sum_{n\ge2}\ln\left(\overset{n^2}{1}-\frac{1}{n^2}\right) = \sum_{n\ge2}\ln\frac{(n+1)(n-1)}{n^2} = \sum_{n\ge2}\left[\ln(n+1)+\ln(n-1)-2\ln n\right] = \sum_{n\ge2}\left(\ln\frac{n+1}{n}+\ln\frac{n-1}{n}\right) = \sum_{n\ge2}\left(\ln\frac{n-1}{n}-\ln\frac{n}{n+1}\right)$

$= \ln\frac12 - \ln\frac{n}{n+1} = \lim_{n\to\infty}\ln\frac{n+1}{2n} = \ln\frac12 = \ln1-\ln2 = -\ln2$

(b) $\sum_{n\ge1}\frac{n+1}{3^n} \Rightarrow \lim_{n\to\infty}\frac{n+2}{\cancel{3^{n+1}}_{3}}\cdot\frac{\cancel{3^n}}{n+1} = \frac13 < 1 \Rightarrow \lim_{n\to\infty}x_n = 0$

$$S = \sum_{n\geq 1} \frac{n}{3^n} + \sum_{n\geq 1} \frac{1}{3^n} = \frac{1}{3}\sum_{n\geq 1} \frac{n}{3^{n-1}} + \sum_{n\geq 1} \frac{1}{3^n} = \frac{1}{3} S + \frac{1}{1-\frac{1}{3}} - 1 = \frac{1}{3} S + \frac{3}{2} - 1 = \frac{1}{3} S + \frac{1}{2}$$

$$S = \frac{1}{3} S + \frac{1}{2} \quad |\cdot 6$$

$$6S = 2S + 3 \Rightarrow S = \frac{3}{4}$$

(c) $$\sum_{n\geq 1} \frac{n}{n^4+n^2+1} = \sum_{n\geq 1} \frac{n}{(n^2-n+1)(n^2+n+1)} = \frac{1}{2}\sum_{n\geq 1} \frac{2n}{(n^2-n+1)(n^2+n+1)} =$$

$$= \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right) = \frac{1}{2}\lim_{n\to\infty}\left(1 - \frac{1}{n^2+n+1}\right) = \frac{1}{2}\lim_{n\to\infty} \frac{n^2+n}{n^2+n+1} = \frac{1}{2}$$

3. Study if the following series are convergent or divergent:

(a) $\sum\limits_{n\geq 2} \frac{1}{\ln n}$.

(b) $\sum\limits_{n\geq 1} \frac{1}{n\sqrt{n+1}}$.

(c) $\sum\limits_{n\geq 1} \frac{\ln\left(1+\frac{1}{n}\right)}{n}$.

(d) $\sum\limits_{n\geq 1} \frac{n!}{n^n}$.

(e) $\sum\limits_{n\geq 1} \left(\frac{n}{n+1}\right)^{n^2}$.

(f) $\sum\limits_{n\geq 2} \frac{1}{n\ln(n)}$.

(a) $\ln n < n \quad |()^{-1}$

$\frac{1}{\ln n} > \frac{1}{n}$

$\sum_{n\geq 1} \frac{1}{\ln n} > \sum_{n\geq 1} \frac{1}{n} \to \infty \Rightarrow \sum_{n\geq 1} \frac{1}{\ln n} \to \infty$

(b) $\sum_{n\geq 1} \frac{1}{n\sqrt{n+1}}$

$n\sqrt{n+1} > n^{\frac{3}{2}} \quad |()^{-1}$

$\frac{1}{n\sqrt{n+1}} < \frac{1}{n^{\frac{3}{2}}}$

$\Rightarrow \lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{n\sqrt{n+1}} = 1 \Rightarrow \frac{1}{n^{\frac{3}{2}}}$ and $\frac{1}{n\sqrt{n+1}}$ have the same nature

we know that $\sum \frac{1}{n^p}$ - conv iff $p>1$ $\Bigg\} \Rightarrow$ convergent

(c) $\sum_{n\geq 1} \frac{\ln(1+\frac{1}{n})}{n} = \sum_{n\geq 1} \frac{\overbrace{n\ln(1+\frac{1}{n})}^{\ln e = 1}}{n^2} = \sum_{n\geq 1} \frac{1}{n^2} \longrightarrow$ convergent to 0

(d) $\sum \frac{n!}{n^n} \Rightarrow \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n = \left(1 + \frac{-1}{n+1}\right)^n = e^{\lim_{n\to\infty}\frac{-n}{n+1}} = \frac{1}{e} < 1 \Rightarrow$ convergent

(e) $\sum_{n\geq 1} \underbrace{\left(\frac{n}{n+1}\right)^{n^2}}_{x_n} \Rightarrow$

root test $\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = 1^\infty = e^{-1} = \frac{1}{e} < 1 \Rightarrow$ convergent

(f) $\sum_{n\geq 2} \frac{1}{n\ln n} \Rightarrow \frac{1}{(n+1)\ln(n+1)} \cdot \frac{n\ln n}{1} = \frac{n\ln n}{(n+1)\ln(n+1)} \longrightarrow 1 \Rightarrow$ Cauchy

$$\sum_{n\geq 1} x_n = \sum_{n\geq 1} 2^n x_{2^n}$$

$$\sum_{n\geq 2} \cancel{2^n} \frac{1}{\cancel{2^n} \ln 2^n} = \sum_{n\geq 2} \frac{1}{n \ln 2} = \frac{1}{\ln 2} \cdot \underbrace{\sum_{n\geq 2} \frac{1}{n}}_{\to \text{ divergent}} \Rightarrow \text{divergent}$$

4. ★ Study if the following series are convergent or divergent:

(a) $\sum_{n\geq 1} \frac{x^n}{n^p}$, $x > 0, p \in \mathbb{N}$. (b) $\sum_{n\geq 2} \frac{1}{(\ln n)^{\ln n}}$. (c) $\sum_{n\geq 1} (\sqrt[n]{n} - 1)$.

(a) ratio test:

$$\frac{\cancel{x^{n+1}}\, x}{(n+1)^p} \cdot \frac{n^p}{\cancel{x^n}} = x \underbrace{\left(\frac{n}{n+1}\right)}_{\to 1}^p \longrightarrow x, \quad x \text{ is fixed} \Rightarrow \text{convergent}$$

(b) $$\sum_{n\geq 2} \frac{1}{(\ln n)^{\ln n}}$$

$$\sum_{n\geq 2} 2^n \cdot \frac{1}{(\ln 2^n)^{\ln 2^n}} = \sum_{n\geq 2} \frac{2^n}{((n \ln 2)^{\ln 2})^n} = \sum_{n\geq 2} \left(\frac{2}{(n \ln 2)^{\ln 2}}\right)^n$$

root test

$$\lim_{n\to\infty} \frac{2}{(n \ln 2)^{\ln 2}} = \frac{2}{\ln^2 2} \cdot \underbrace{\frac{1}{n}}_{\to 0} \Rightarrow \text{convergent}$$

(c) $$\sum_{n\geq 1} (\sqrt[n]{n} - 1) = \sum_{n\geq 1} (n^{\frac{1}{n}} - 1) = \sum_{n\geq 1} (e^{\frac{\ln n}{n}} - 1)$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{e^x - 1}{x} \to 1$$

$$\lim_{n\to\infty} \overbrace{\frac{e^{\frac{\ln n}{n}} - 1}{\frac{\ln n}{n}}}^{\to 1} \cdot \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\ln n}{n}$$

$$e^{\frac{\ln n}{n}} - 1 \simeq \frac{\ln n}{n}$$

$$\left.\begin{array}{l} \ln n > 1 \\ \frac{\ln n}{n} > \frac{1}{n} \\ \sum \frac{1}{n} \text{ divergent} \end{array}\right\} \Rightarrow \sum_{n\geq 1} \frac{\ln n}{n} \text{ divergent} \Rightarrow \sum_{n\geq 1} \sqrt[n]{n} - 1 \text{ divergent}$$