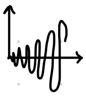
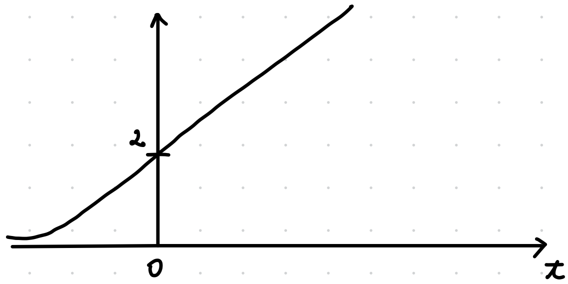
1) Let $\varphi(t) = 2\cdot e^{3t}$. Show that φ is the solution of the differential equation

$$\begin{cases} \varphi'(t) = 3\cdot\varphi(t) \\ \varphi(0) = 2 \end{cases}$$

; Plot the integral curve (the solution) and study the long term behaviour $\Rightarrow$ $\begin{cases} \text{periodicity} \\ \text{oscilation around a point} \\ \text{monotonicity} \\ \text{values at } \pm\infty \end{cases}$ ex: $\frac{\sin x}{x}$



$$\left.\begin{array}{l} \varphi'(t) = 6e^{3t} \\ 3\cdot\varphi(t) = 3\cdot 2e^{3t} = 6e^{3t} \end{array}\right\} \Rightarrow \varphi'(t) = 3\varphi(t) \Rightarrow \varphi(t) \text{ is the solution}$$



$\Rightarrow$ exponential asscending

$\lim_{x\to-\infty} \varphi(t) = \frac{1}{e^{\infty}} = 0$

$\lim_{x\to\infty} \varphi(t) = \infty$

2) Let μ be fixed. Show that $\varphi : \mathbb{R} \to \mathbb{R}$ $\varphi(t) := \mu\cdot\sin t$ is a solution of the diff. eq.

$$\begin{cases} x''(t) + x(t) = 0 \\ x(0) = 0 \\ x'(0) = \mu \end{cases}$$

; Plot the sol and study the long term behaviour.

$\varphi(t) = \mu \sin t$

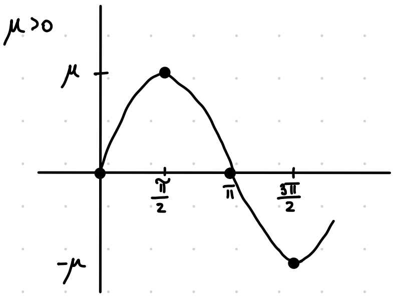
$\varphi'(t) = \mu \cos t$

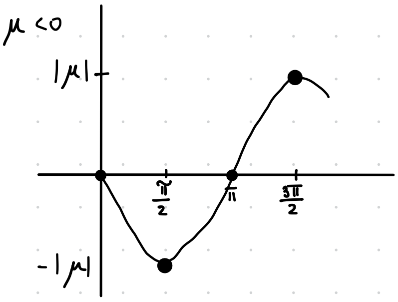
$\varphi''(t) = -\mu \sin t$

$\varphi''(t) + \varphi(t) = -\mu\sin t + \mu \sin t = 0$

$\varphi(0) = \mu\cdot\sin 0 = 0$

$\varphi'(0) = \mu\cdot\cos 0 = \mu$





$\to$ periodicity 2π

$x''(t) + \sin t = 0$ – eq. pendulului

3) Show that $\varphi(t) = e^{-2t}\cdot\cos t$ is a solution of the diff eq.

$$\begin{cases} x''(t) + 4x'(t) + 5x(t) = 0 \\ x(0) = 1 \\ x'(0) = 2 \end{cases}$$

$\varphi'(t) = -2e^{-2t}\cdot\cos t + e^{-2t}\cdot\sin t$

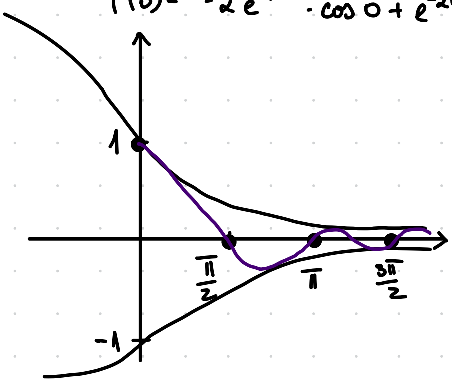
$\varphi''(t) = 4e^{-2t}\cdot\cos t - 2e^{-2t}\sin t - e^{-2t}\cos t - 2e^{-2t}\sin t = 3e^{-2t}\cos t - 4e^{-2t}\sin t$

$\varphi''(t) + 4\varphi'(t) + 5\varphi(t) = 0$

$3\cancel{e^{-2t}\cos t} - 4e^{-2t}\sin t - 8\cancel{e^{-2t}\cos t} + 4e^{-2t}\sin t + 5\cancel{e^{-2t}\cos t} = 0 \Rightarrow 0 = 0$

$\varphi(0) = e^{-2\cdot 0}\cdot\cos 0 = 1 \quad 1 = 1$

$\varphi'(0) = -2e^{-2\cdot 0}\cdot\cos 0 + e^{-2t}\cdot\sin 0 = -2\cdot 1 + 1\cdot 0 = -2$



behaviour: $\lim_{t\to\infty} \varphi(t) = 0$

$\nexists \lim_{t\to-\infty}$

oscilations around 0

4) Let us consider the functions: $\begin{cases} x_1(t) = \cos t \\ x_2(t) = \sin t \\ x_3(t) = e^t \end{cases}$; Prove that $\{x_1, x_2, x_3\}$ are liniarly independent in the space of functions.

$\forall a,b,c$ a.î. $ax+by+cz=0 \Rightarrow a=b=c=0$

For functions to be liniarly independent: if a,b,c are such that

$$\boxed{a \cdot x_1(t) + b \cdot x_2(t) + c \cdot x_3(t) = 0 \ \boxed{\forall t}} \Rightarrow a=b=c=0$$

* we can choose an arbitrary t, since this relation must be true for <u>all</u> t

Let a,b,c s.t. $a \cdot \cos t + b \cdot \sin t + c \cdot e^t = 0$

Let $t=0$: $\quad a+c=0 \Rightarrow a=-c$

$t=\frac{\pi}{2}$ $\quad b + c \cdot e^{\frac{\pi}{2}} = 0$

$t=2\pi$ $\quad a + c \cdot e^{2\pi} = 0$

$\hookrightarrow -c + c \cdot e^{2\pi} = 0 \Rightarrow c(e^{2\pi}-1) = 0 \Rightarrow c=0 \Rightarrow a=0 \Rightarrow b=0$

5) Find all α, β, γ s.t. $x(t) = \alpha \sin t + \beta \cos t + \gamma e^t$ is a solution of:

a) $x''(t) + x(t) = -3\sin t$

$x'(t) = \alpha \cos t - \beta \sin t + \gamma e^t$

$x''(t) = -\alpha \sin t - \beta \cos t + \gamma e^t$

$x''(t) + x(t) = -\alpha \sin t - \beta \cos t + \gamma e^t + \alpha \sin t + \beta \cos t + \gamma e^t = 2\gamma e^t = -3\sin t \Rightarrow \nexists \gamma$ since e^t and $\sin t$ are liniarly independent

b) $x' + x = -3\sin t + 2e^t$

$x' + x = \alpha \cos t - \beta \sin t + \gamma e^t + \alpha \sin t + \beta \cos t + \gamma e^t =$

$= (\alpha+\beta)\cos t + (\alpha-\beta)\sin t + 2\gamma e^t = -3\sin t + 2e^t$

$\Rightarrow (\alpha+\beta)\cos t + (\alpha-\beta+3)\sin t + 2(\gamma-1)e^t = 0$
$\cos t, \sin t, e^t$ are lin. indep

$\Rightarrow \begin{cases} \alpha+\beta=0 \Rightarrow \alpha=-\beta \Rightarrow \beta=\frac{3}{2} \\ \alpha-\beta+3=0 \Rightarrow 2\alpha+3=0 \Rightarrow \alpha=\frac{-3}{2} \\ 2\gamma-1=0 = \gamma=1 \end{cases}$

$\Rightarrow (\alpha, \beta, \gamma) = \left(\frac{-3}{2}, \frac{3}{2}, 1\right)$

$\Rightarrow x(t) = -\frac{3}{2}\sin t + \frac{3}{2}\cos t + e^t$

c) $x' + 4x = -3\sin t \Rightarrow x(t) = -\sin t$

6) Find r s.t. $x(t) = t^r$ is the solution of $t^2 x''(t) - 4t x'(t) + 6x(t) = 0$ on $(0,\infty)$

$x_1(t), x_2(t)$ 2 solutions lin. indepenent $\Rightarrow c_1 x_1(t) + c_2 x_2(t)$ is also a solution $\forall c_1, c_2 \in \mathbb{R}$

$x'(t) = r t^{r-1}$

$x''(t) = r(r-1)t^{r-2}$

$r(r-1)t^2 \cdot t^{r-2} - 4 \cdot t \cdot r \cdot t^{r-1} + 6t^r = 0$

$r(r-1)t^r - 4rt^r + 6t^r = 0$

$t^r(r(r-1) - 4r + 6) = 0, \ \forall t$

$\Rightarrow r^2 - 5r + 6 = 0$

$\Delta = 25 - 24 = 1 \Rightarrow r_{1,2} = \frac{5 \pm 1}{2} \left\langle \begin{array}{l} r_1 = 3 \\ r_2 = 2 \end{array} \right.$

$\Rightarrow \quad x_1(t) = t^2$

$x_2(t) = t^3$

$a \cdot t^2 + b \cdot t^3 = 0 \Rightarrow t^2(a + bt) = 0$

$t = 1 \Rightarrow a + b = 0$

$t = 2 \Rightarrow 4(a + 2b) = 0$

$\Rightarrow a = b = 0 \Rightarrow$ liniarly indep.

$\Rightarrow x(t) := c_1 x_1(t) + c_2 x_2(t) = c_1 t^2 + c_2 t^3$ is also a solution