Seminar 7

1. Compute the following limits using Riemann integrals:

   (a) $\lim\limits_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$.

   (b) ★ $\lim\limits_{n\to\infty} \frac{\sqrt[n]{e}^{\frac{1}{n}} + 2\sqrt[n]{e^2}^{\frac{2}{n}} + \cdots + n\sqrt[n]{e^n}^{\frac{n}{n}}}{n^2}$.

   $= \sum_{k=1}^{n} \frac{k}{n^2} e^{\frac{k}{n}} = \sum_{k=1}^{n} \frac{1}{n} \left(\frac{k}{n} e^{\frac{k}{n}} \right) = \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx = \int_0^1 x e^x\,dx = x e^x \Big|_0^1 - e^x \Big|_0^1 = e^x(x-1)\Big|_0^1 = -1$

   (c) $\lim\limits_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$.

   (d) ★ $\lim\limits_{n\to\infty} \sqrt[n]{\sin\frac{\pi}{2n} \sin\frac{2\pi}{2n} \ldots \sin\frac{(n-1)\pi}{2n}}$.

2. Study the Riemann integrability of the function $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

3. Compute the following improper integrals:

   (a) $\int_1^2 \frac{1}{x(x-2)}\,\mathrm{d}x$.

   (b) $\int_0^\infty x e^{-x^2}\,\mathrm{d}x$.

   (c) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$.

   (d) ★ $\int_0^\infty e^{-x}\sin x\,\mathrm{d}x$.

4. Study the convergence of the following improper integrals:

   (a) $\int_1^\infty \frac{1}{x\sqrt{1+x^2}}\,\mathrm{d}x$.

   (b) $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\,\mathrm{d}x$.

   (c) $\int_1^\infty \frac{\ln x}{x\sqrt{x^2-1}}\,\mathrm{d}x$.

5. Using the integral test, study the convergence of the following series:

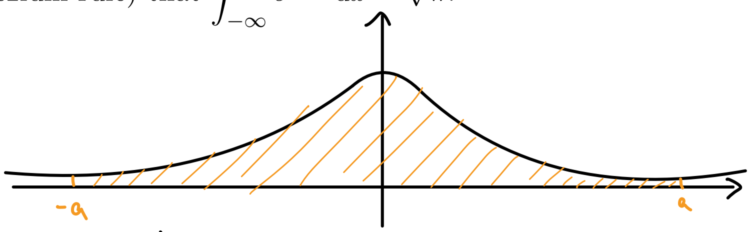
   (a) $\sum\limits_{n\geq 1} \frac{1}{n^p}$, $p > 0$.

   (b) $\sum\limits_{n\geq 2} \frac{1}{n(\ln n)^2}$.

   (c) $\sum\limits_{n\geq 2} \frac{\ln n}{n^2}$.

6. ★ [Python] The integral $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$ represents the area under the bell curve $y = e^{-x^2}$ and it is related to the normal (Gaussian) probability distribution. It is essential in probability theory and has a wide range of applications. Considering intervals of the form $[-a, a]$, for increasing $a > 0$, show numerically (e.g. trapezium rule) that $\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$.

   curba lui Gauss



Homework questions are marked with ★.

$\int e^{-x^2} dx$ is impossible to compute with primitive functions

take $-a$ & a and approx the area (trapezium rule or Riemann)

→ as many points as poss.

↳ larger and larger so the interval gets closer to $\sqrt{\pi}$

Definition 6.8 (Trapezium rule)**.** Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and consider $a = x_0 < x_1 < \ldots < x_n = b$. The area below the curve $y = f(x)$ can be approximated by

$$\sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2}(x_k - x_{k-1}).$$

Note that $\frac{f(x_{k-1})+f(x_k)}{2}(x_k - x_{k-1})$ is the area of the trapezium determined by $x_{k-1}, x_k, f(x_{k-1}), f(x_k)$. In the case of a uniform partition with $x_k - x_{k-1} = \frac{b-a}{n}$, $\forall k \in \overline{1,n}$, we have that

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{n}\left(\tfrac{1}{2}f(a) + \sum_{i=1}^{n-1} f(x_k) + \tfrac{1}{2}f(b)\right).$$

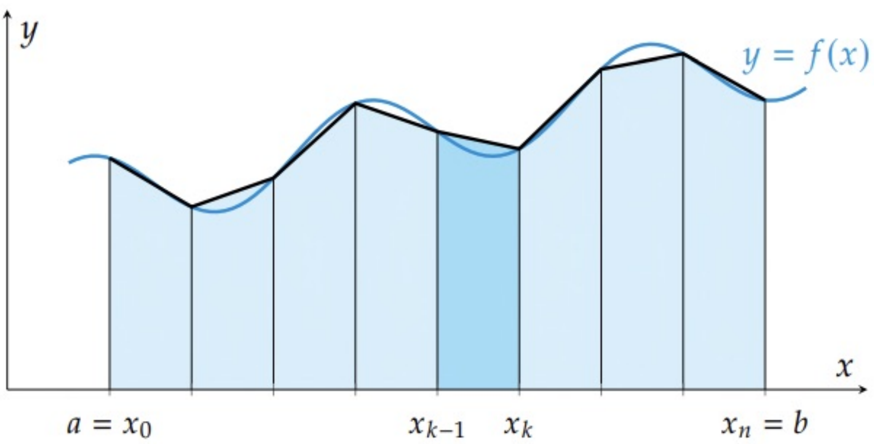


Figure 3: Trapezium rule.

1. (a) $0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n-1}{n} < \frac{n}{n} = 1$

$$\boxed{\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \longrightarrow \int_0^1 f(x)\, dx}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{1}{1+\frac{k}{n}} = \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \longrightarrow \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{1+x}\,dx = \ln(x+1)\Big|_0^1 = \ln 2$$

$$f(x) = \frac{1}{1+x}$$

Definition 6.1. For $f : [a,b] \to \mathbb{R}$ and a partition $\mathcal{P}$ of $[a,b]$, the Riemann sum is given by

$$\sigma(f,\mathcal{P}) := \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}).$$

Remark 6.2. The Riemann sum collects the areas of the rectangles defined by the partition $\mathcal{P}$ (and the intermediate points). In the limit one obtains the area below the curve $y = f(x)$.

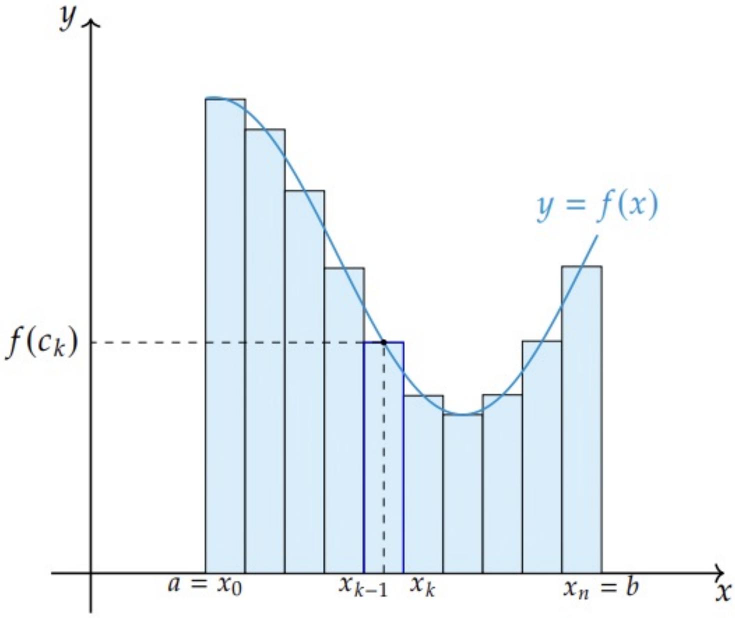


Figure 1: Area under a curve approximated through rectangles. Riemann sum.

Definition 6.3. We say that $f : [a,b] \to \mathbb{R}$ is *Riemann integrable* if there exists $I \in \mathbb{R}$ s.t. for any partition $\mathcal{P}$ of $[a,b]$ the Riemann sum $\sigma(f,\mathcal{P})$ converges to I as $\|\mathcal{P}\| \to 0$, i.e.

$$\lim_{\|\mathcal{P}\|\to 0} \sigma(f,\mathcal{P}) = I =: \int_a^b f(x)\,dx.$$

(c) $$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \left(\frac{n!}{n^n}\right)^{\frac{1}{n}} = e^{\frac{1}{n}\ln\frac{n!}{n^n}} = \frac{1}{e}$$

$$\frac{1}{n} \ln\frac{n!}{n^n} = \frac{1}{n} \ln \frac{1\cdot 2\cdot \ldots \cdot n}{n\cdot n\cdot \ldots \cdot n} = \frac{1}{n}\sum_{k=1}^{n} \overbrace{\ln\frac{k}{n}}^{f(\frac{k}{n})} \longrightarrow \int_0^1 \ln x\,dx = x\ln x\Big|_0^1 - 1 = -1$$

$$\lim_{x\to 0} x\ln x = \lim_{x \searrow 0} \frac{\ln x}{\frac{1}{x}} \overset{l'H}{=} \lim_{x\searrow 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\searrow 0}(-x) = 0$$

or: let $b_n = \ln a_n = \ln \frac{\sqrt[n]{n!}}{n}$ and compute in a similar way

2. $f:[0,1] \to \mathbb{R}$ $\qquad f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

Let $a = x_0 < x_1 < \dots < x_n = b$ a partition on $[0,1]$

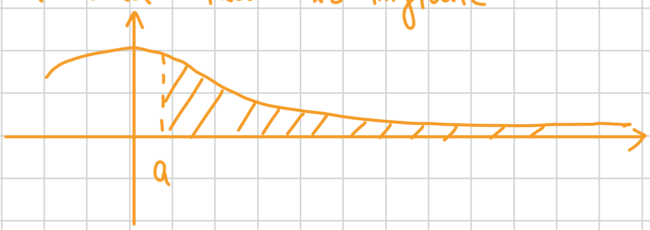
Let $\overline{c_k} \in \mathbb{Q}$, $\underline{c_k} \in \mathbb{R}\setminus\mathbb{Q}$, $c_k \in [x_{k-1}, x_k]$

$$\overline{\sigma}(f, P, \overline{c_k}) = \sum_{k=1}^{n} 1\cdot(x_k - x_{k-1}) = 1$$

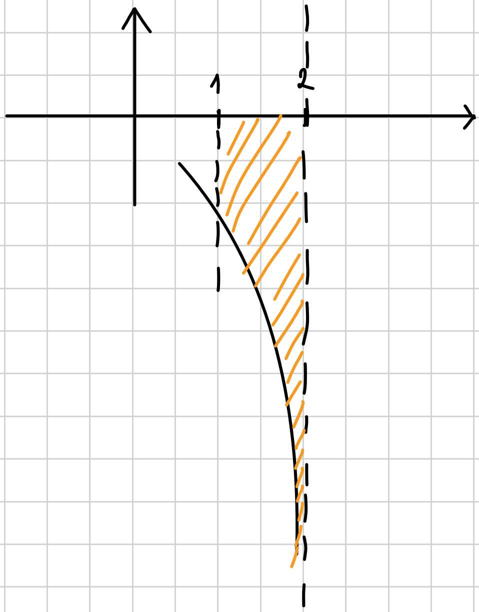
$$\underline{\sigma}(f, P, \underline{c_k}) = \sum_{k=1}^{n} 0\,(x_k - x_{k-1}) = 0$$

3. Improper integrals compute the area that is infinite

$$\int_a^\infty f(x)\,dx = \lim_{t \nearrow \infty} \int_a^t f(x)\,dx$$



(a) $\displaystyle\int_1^2 \frac{1}{x(x-2)}\,dx$



$$\int_1^2 \frac{1}{x(x-2)}\,dx = \lim_{t\nearrow 2}\int_1^t \frac{1}{x(x-2)}\,dx =$$

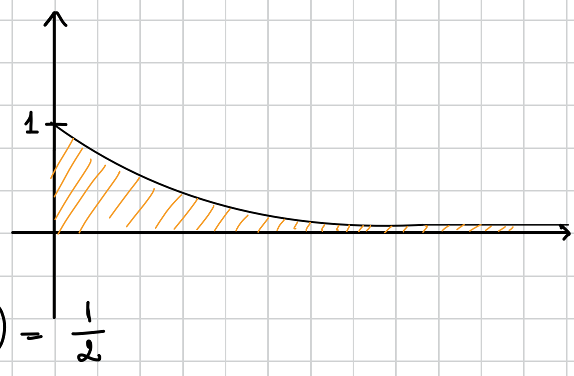
$$= \int_1^t \frac{1}{x(x-2)}\,dx = -\frac{1}{2}\int_1^t \left(\frac{1}{x} - \frac{1}{x-2}\right)dx =$$

$$= -\frac{1}{2}\ln x\Big|_1^t + \frac{1}{2}\ln|x-2|\Big|_1^t =$$

$$= -\frac{1}{2}\ln t + \frac{1}{2}\ln(2-t)$$

$$= \lim_{t\nearrow 2}\left(-\frac{1}{2}\ln t + \frac{1}{2}\ln(2-t)\right) = -\frac{1}{2}\ln 2 - \infty = -\infty$$

(b) $t = x^2$, $\dfrac{dt}{dx} = 2x$, $dt = 2x\,dx$



$$\int_0^\infty x e^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-x^2}2x\,dx = \frac{1}{2}\int_0^\infty e^{-t}\,dt$$

$$= \frac{1}{2}\lim_{n\to\infty}\int_0^n e^{-t}\,dt = \lim_{n\to\infty}\frac{1}{2}\left(-e^{-t}\right)\Big|_0^n = \lim_{n\to\infty}\left(-\frac{1}{2}e^{-n} + \frac{1}{2}\right) = \frac{1}{2}$$

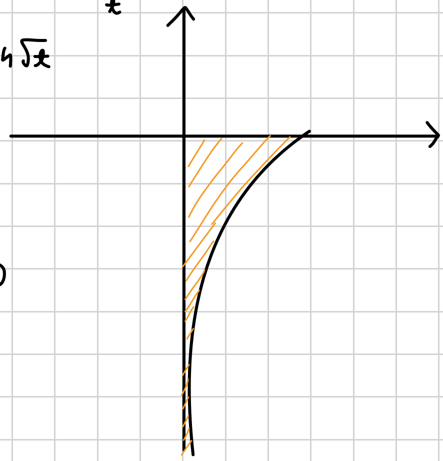
We can write : $\frac{1}{2}\int_0^\infty e^{-t}dt = \frac{-1}{2}e^{-t}\Big|_0^\infty = -\frac{1}{2}e^{-\infty}+\frac{1}{2}=\frac{1}{2}$

$\left(\frac{1}{u}\right)' = \frac{-u'}{u^2}$

(c) $\int_0^1 \frac{\ln x}{\sqrt{x}}dx$ improper because $\lim_{x\searrow 0}\frac{\ln x}{\sqrt{x}} = \frac{-\infty}{0_+} = -\infty\cdot\infty = -\infty$

$\lim_{t\searrow 0}\int_t^1 \frac{\ln x}{\sqrt{x}}dx$, $\int_t^1 \frac{\ln x}{\sqrt{x}}dx = 2\int_t^1 \ln x\cdot(\sqrt{x})'dx = 2\ln x\sqrt{x}\Big|_t^1 - 2\int_t^1\frac{\sqrt{x}}{x}dx$

$= -2\ln t\sqrt{t} - 4\sqrt{x}\Big|_t^1 = -2\ln t\sqrt{t} - 4 + 4\sqrt{t}$



$= \lim_{t\searrow 0}\left(-2\ln t\sqrt{t} - 4 + 4\sqrt{t}\right) = -4$

$\lim_{t\searrow 0}\sqrt{t}\cdot\ln t = \lim_{t\searrow 0}\frac{\ln t}{\frac{1}{\sqrt{t}}} \overset{l'H}{=} \lim_{t\searrow 0}\frac{\frac{1}{\cancel{t}}}{-\frac{\frac{1}{2\sqrt{t}}}{\cancel{t}}} = \lim_{t\searrow 0} -2\sqrt{t} = 0$

Example 6.11. Let $a > 0$ and $p \in \mathbb{R}$. The improper integral

$$\int_a^\infty \frac{1}{x^p}\,dx \longrightarrow$$ good to check if an $\int$ converges

converges when $p > 1$ and diverges when $p \leq 1$. Indeed, for $p = 1$ the integral diverges ($\ln(\infty)$) and for $p \neq 1$,

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx = \lim_{t\to\infty}\frac{t^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1},$$

which converges when $-p+1 < 0$, i.e. $p > 1$, and diverges when $p < 1$.

4. (a) $\int_1^\infty \frac{1}{x\sqrt{1+x^2}}dx$ $\qquad \sqrt{1+x^2} \approx x$, $\frac{1}{x\sqrt{1+x^2}} \approx \frac{1}{x^2}$,

$\int_1^\infty \frac{1}{x^2}dx = -\frac{1}{x}\Big|_1^\infty = -\frac{1}{\infty}+1 = 1$

$p > 1 \Rightarrow$ converges

$\int_1^\infty \frac{1}{x\sqrt{1+x^2}}dx$ "like" $\int_1^\infty \frac{1}{x^2}dx < \infty \Rightarrow$ converges

(c) $\int_1^\infty \frac{\ln x}{x\sqrt{x^2-1}}dx$

$\left.\begin{array}{l}\frac{\ln x}{x\sqrt{x^2-1}} \approx \frac{\ln x}{x^2} < \frac{x^\alpha}{x^2} = \frac{1}{x^{2-\alpha}} , \quad \alpha\in(0,1) \\ 2-\alpha > 1 \Rightarrow \int_1^\infty \frac{1}{x^{2-\alpha}}dx \text{ converges}\end{array}\right\} \Rightarrow \int_1^\infty \frac{\ln x}{x\sqrt{x^2-1}} < \int_1^\infty \frac{1}{x^{2-\alpha}} < \infty$

(b) **Example 6.12.** Let $0 < a < b$ and $p \in \mathbb{R}$. The improper integrals

$$\int_a^b \frac{1}{(b-x)^p}\,\mathrm{d}x, \int_a^b \frac{1}{(x-a)^p}\,\mathrm{d}x$$

converge when $p < 1$ and diverge when $p \geq 1$. Indeed, for $p = 1$ the integrals diverge ($\ln(0)$) and for $p \neq 1$ the first integral, for example, is

$$\int_a^b \frac{1}{(b-x)^p}\,\mathrm{d}x = -\lim_{t \nearrow b} \frac{(b-t)^{-p+1}}{-p+1} + \frac{(b-a)^{-p+1}}{-p+1},$$

which converges when $-p+1 > 0$, i.e. $p < 1$, and diverges when $p > 1$.

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\,dx$$

as $x \nearrow \frac{\pi}{2}$ $\frac{1}{\cos x} \approx \frac{1}{(\frac{\pi}{2}-x)^p}$ for some $p = 1$

$$\frac{1}{\cos x} = \frac{1}{\sin(\frac{\pi}{2}-x)} \approx \frac{1}{\frac{\pi}{2}-x} \quad \text{as } x \nearrow \frac{\pi}{2}$$

$$\lim_{x \nearrow \frac{\pi}{2}} \frac{\sin(\frac{\pi}{2}-x)}{\frac{\pi}{2}-x} = \lim_{x \searrow 0} \frac{\sin t}{t} = 1$$

$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\,dx$ "like" $\int_0^{\frac{\pi}{2}} \frac{1}{\frac{\pi}{2}-x}\,dx = +\infty \Rightarrow$ divergence

Theorem 6.14 (Integral test for series)**.** Let $f : [1,\infty) \to [0,\infty)$ be decreasing, then $\int_1^\infty f(x)\,\mathrm{d}x$ and $\sum_{n=1}^\infty f(n)$ have the same nature.

5. (a) $\sum_{n\geq 1} \frac{1}{n^p}$, $p>0$

$\sum_{n\geq 1} \frac{1}{n^p}$ "like" $\int_1^\infty \frac{1}{x^p}\,dx$

(b) $\sum_{n\geq 2} \frac{1}{n \ln^2 n}$ "like" $\int_2^\infty \frac{1}{x \ln^2 x}\,dx \overset{t=\ln x}{=\!=} \int_{\ln 2}^\infty \frac{1}{t^2}\,dx = \frac{-1}{t}\Big|_{\ln 2}^\infty$ $\qquad \frac{dt}{dx} = \frac{1}{x}$, $dt = \frac{1}{x}dx$

$= \frac{1}{\ln 2} < \infty \to$ conv.

(c) $\int_2^\infty \frac{\ln x}{x^2}\,dx = \int_2^\infty \ln x \cdot \left(\frac{-1}{x}\right)' dx = \underbrace{\ln x \left(-\frac{1}{x}\right)\Big|_2^\infty}_{=\frac{\ln 2}{2}} + \underbrace{\int_2^\infty \frac{1}{x^2}\,dx}_{<\infty} \to$ conv.