Seminar 5

1. Find the accumulation points for each of the following sets: $[0,1) \cup \{2\}$, $\mathbb{Z}$, $\{0.1, 0.11, \ldots\}$.

2. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous everywhere, with $|f|$ continuous everywhere.

3. If $f : [a,b] \to [a,b]$ is continuous, then it has at least one fixed point x^* s.t. $x^* = f(x^*)$.

4. Study the continuity and the differentiability for f and f', where $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

5. Prove (from scratch) that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ and then that $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$.

6. Compute the following limits:

(a) $\lim\limits_{x \to \infty} \dfrac{\lfloor x \rfloor}{x}$.

(b) $\lim\limits_{x \to \infty} x\big(\ln(x+2) - \ln(x+1)\big)$.

(c) $\lim\limits_{x \to 0} \big(\cos x\big)^{\frac{1}{x^2}}$. $= e^{\frac{1}{x^2} \ln \cos x}$ $\left[\left(1 + (\cos x - 1)\right)^{\frac{1}{\cos x - 1}}\right]^{\frac{\cos x - 1}{x^2}}$

(d) $\lim\limits_{\substack{x \to 0 \\ x > 0}} x^x$. $= e^{x \ln x} = e^{\frac{\ln x}{\frac{1}{x}}}$

(e) $\lim\limits_{\substack{x \to 0 \\ x > 0}} (\sin x)^x$. $= e^{x \ln \sin x} = e^{\frac{\ln \sin x}{\wedge}}$

(f) $\lim\limits_{x \to \infty} x\big((1 + \frac{1}{x})^x - e\big)$.

7. Find the n^{th} derivative of the following functions:

(~~a~~) $f : (-1, \infty) \to \mathbb{R}$, $f(x) = \ln(1+x)$.

(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$.
$\cos x$
$-\sin x$
$-\cos x$

(c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 \sin x$. $2x \cdot \sin x + x^2 \cdot \cos x$
$\underline{2 \sin x + 4x \cos x}$
$2 \sin x + 2x \cos x + 2x \cos x + x^2(-\sin x)$
$2\cos x + 4 \cos x + 4x(-\sin x) + 2x(-\sin x) + x^2(-\cos x)$
$6 \cos x - 6x \sin x$

(d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{2x} x^3$.

8. ★ Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. To minimize f, consider the *gradient descent* method

$$x_{n+1} = x_n - \eta f'(x_n),$$

where $x_1 \in \mathbb{R}$ and $\eta > 0$ (learning rate). Use Python (numerics or graphics) for the following:
(a) Take a convex f and show that for small η the method converges to the minimum of f.
(b) Show that by increasing η the method can converge faster (in fewer steps).
(c) Show that taking η too large might lead to the divergence of the method.
(d) Take a nonconvex f and show that the method can get stuck in a local minimum.

Homework questions are marked with ★.

$\boxed{A \subseteq \mathbb{R},\ x_0 \text{ is an accumulation point if } \forall\, V \in \mathcal{V}(x_0)}$ $\quad V \cap (A \setminus \{x_0\}) \neq \emptyset$

1. (i) $A = [0,1) \cup \{2\}$

A' = the set of all accumulation points

$A' = [0,1]$

(ii) $A = \mathbb{Z} \Rightarrow A' = \emptyset$ (in $\mathbb{R}$) but in $\overline{\mathbb{R}} \Rightarrow A' = \{\pm\infty\}$

(iii) $A = \{0.1, 0.11, \dots\}$

$A' = \{\frac{1}{9}\}$

2. $f: \mathbb{R} \to \mathbb{R}$ discontinuous everywhere with $|f|$ continuous everywhere

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$$ $\to$ useful to get examples like this

Let $x_0 \in \mathbb{R}$ $\left.\begin{matrix} x_n \subseteq \mathbb{Q},\ x_n \to x_0 : f(x_n) = 1 \\ y_n \subseteq \mathbb{R}\setminus\mathbb{Q},\ y_n \to x_0 : f(y_n) = -1 \end{matrix}\right\} \Rightarrow \nexists \lim_{x \to x_0} f(x)$

$\Rightarrow |f(x)| = 1,\ x \in \mathbb{R}$ continuous

3. $f: [a,b] \to [a,b]$ continuous

x^* - fixed point s.t. $x^* = f(x^*)$

$\Rightarrow f(x^*) - x^* = 0$

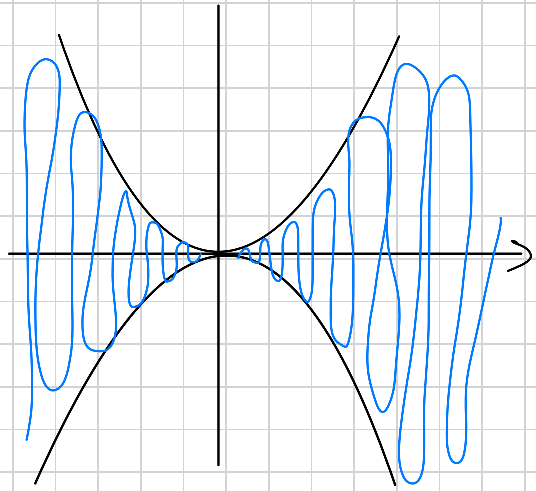
Let $g: \mathbb{R} \to \mathbb{R},\ g(x) = f(x) - x$

$$\left.\begin{matrix} \boxed{\begin{matrix} g(a) = f(a) - a \geq a - a = 0 \\ g(b) = f(b) - b \leq b - b = 0 \end{matrix}} \\ g \text{ cont.} \end{matrix}\right\} \xRightarrow[\text{property}]{\text{intermediate value}} \exists\ c \in [a,b] \text{ s.t. } g(c) = 0 \Leftrightarrow f(c) = c$$

4. $$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \underset{\downarrow\, 0}{x^2} \underbrace{\sin\frac{1}{x}}_{\in[-1,1]} = 0 = f(0) \Rightarrow f$ is continuous at 0 and f cont at $x \neq 0$

$\Rightarrow f$ cont. on $\mathbb{R}$



$$f(x) = x^2 \sin\frac{1}{x}$$

$$f'(x) = \begin{cases} \left(x^2 \sin\frac{1}{x}\right)' , & x \neq 0 \\ \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} , & x = 0 \end{cases}$$

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \underset{\downarrow\, 0}{x} \underbrace{\sin\frac{1}{x}}_{\in[-1,1]} = 0 = f'(0)$$

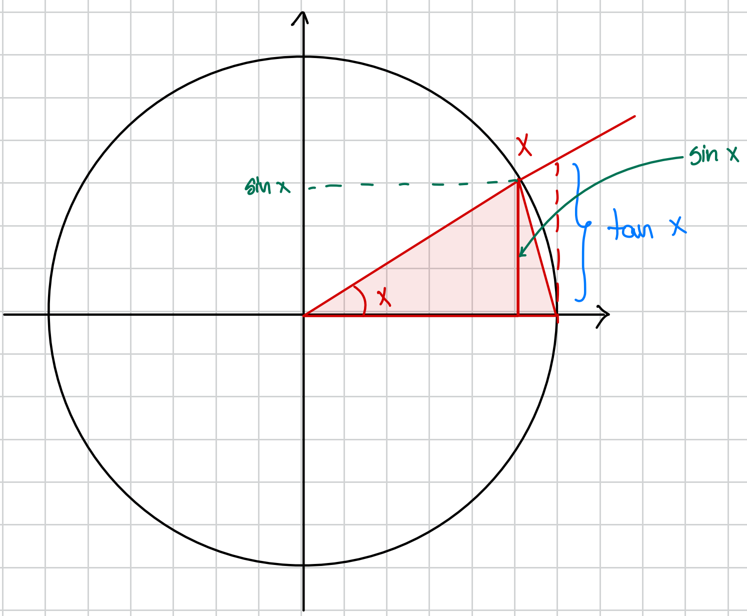
$$\left(x^2 \sin\frac{1}{x}\right)' = 2x \sin\frac{1}{x} + \cancel{x^2}\cos\left(\frac{1}{x}\right)\cdot\frac{-1}{\cancel{x^2}} = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$$

$$f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} , & x \neq 0 \\ 0 , & x = 0 \end{cases}$$

$\Rightarrow \nexists \lim\limits_{x\to 0} f'(x)$ because $\nexists \lim\limits_{x\to 0} \cos\frac{1}{x} \Rightarrow f'$ is NOT continious at 0

$\Rightarrow f$ is not differentiable at 0

5. $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$



$$\frac{\sin x}{2} < \text{Area Sector} = \frac{x}{2} < \frac{\tan x}{2}$$

$$\Rightarrow \sin x < x < \tan x$$

$$\underset{\searrow\, 1}{\cos x} < \underset{\downarrow\, 1}{\frac{\sin x}{x}} < \underset{\swarrow\, 1}{1}$$

$(\sin x)' = \cos x$ $\qquad\qquad$ $\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2}$

$$\lim_{x\to x_0} \frac{\sin x - \sin x_0}{x - x_0} \overset{?}{=} \cos x_0$$

$$\lim_{x\to x_0} \frac{\sin\frac{x-x_0}{2}\cos\frac{x+x_0}{2}}{\frac{x-x_0}{2}} = \cos x_0$$

$\lfloor\ \rfloor$ – floor

$\lceil\ \rceil$ – ceiling

6. (a) $\lim_{x\to\infty} \frac{\lfloor x\rfloor}{x}$

$x-1 < k \le \lfloor x\rfloor \le x < k+1,\ k\in\mathbb{Z} \ \Big|:x$

$$\frac{x-1}{x} < \frac{k}{x} \le \frac{\lfloor x\rfloor}{x} \le 1 < \frac{k+1}{x}$$

$\downarrow 1$

(b) $\lim_{n\to\infty} x\left(\ln(x+2) - \ln(x+1)\right) = \lim_{n\to\infty} x\ln\frac{x+2}{x+1} = \lim_{x\to\infty}\ln\left(1+\frac{1}{x+1}\right)^x = \ln e^{\lim_{n\to\infty}\frac{x}{x+1}} = \ln e^1 = 1$

(c) $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} = (1^\infty) = e^{\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}} = \left(\frac{0}{0}\right) \overset{L'H}{=} e^{\lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{2x}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

$$\lim_{x\to 0}\left[\left(1+\cos x - 1\right)^{\frac{1}{\cos x - 1}}\right]^{\frac{\cos x - 1}{x^2}} = e^{\lim_{x\to 0}\frac{\cos x - 1}{x^2}} = e^{\lim_{x\to 0}\frac{-\sin x}{2x}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

(d) $\lim_{\substack{x\to 0\\ x>0}} x^x = (0^0) = e^{\lim_{\substack{x\to 0\\ x>0}} x\ln x} = e^0 = 1$

$$\lim_{\substack{x\to 0\\ x>0}} x\ln x = (0\cdot(-\infty)) = \lim_{\substack{x\to 0\\ x>0}}\frac{\ln x}{\frac{1}{x}} \overset{L'H}{=} \lim_{\substack{x\to 0\\ x>0}}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{\substack{x\to 0\\ x>0}} -x = 0$$

(e) $\lim_{\substack{x\to 0\\ x>0}} (\sin x)^x = e^{x\ln\sin x} = e^0 = 1$

$$\lim_{x\to 0} x\ln(\sin x) = \frac{\ln(\sin x)}{\frac{1}{x}} = \lim_{\substack{x\to 0\\ x>0}}\frac{\frac{1}{\sin x}\cdot\cos x}{-\frac{1}{x^2}} = \lim_{\substack{x\to 0\\ x>0}}\frac{-x^2\cdot\cos x}{\sin x} = \lim_{\substack{x\to 0\\ x>0}}\frac{x}{\sin x}\cdot(-x\cdot\cos x) = \lim_{\substack{x\to 0\\ x>0}}(-x\cdot\cos x) = 0$$

(f) $\lim_{x\to\infty} x\left(\left(1+\frac{1}{x}\right)^x - e\right) = \lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^x - e}{\frac{1}{x}} = \lim_{x\to\infty}\frac{e^{x\ln\left(1+\frac{1}{x}\right)} - e}{\frac{1}{x}} = \lim_{x\to\infty}\frac{e^{x(\ln(x+1)-\ln x)} - e}{\frac{1}{x}}$

$$\left(e^{x(\ln(x+1)-\ln x)}\right)' = e^{x(\ln(x+1)-\ln x)}\cdot\left(x(\ln(x+1)-\ln x)\right)' = e^{x(\ln(x+1)-\ln x)}\cdot\left(\ln(x+1) - \ln x + x\cdot\left(\frac{1}{x+1}-\frac{1}{x}\right)\right)$$

$$= e^{x(\ln(x+1)-\ln x)}\cdot\left(\ln(x+1) - \ln x - \frac{1}{x+1}\right) = e^{\ln\left(\frac{x+1}{x}\right)^x}\cdot\left(\ln\frac{x+1}{x} - \frac{1}{x+1}\right) = \left(1+\frac{1}{x}\right)^x\cdot\left(\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right)$$

$$= \lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]}{-\frac{1}{x^2}} = e\cdot\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}}{-\frac{1}{x^2}} \overset{\left(\frac{0}{0}\right)}{=} e\cdot\lim_{x\to\infty}\frac{\frac{\cancel{x}}{x+1}\cdot\left(-\frac{1}{x^{\cancel{2}}}\right) + \frac{1}{(x+1)^2}}{\frac{2}{x^3}} = e\cdot\lim_{x\to\infty}\frac{-(x+1)+x}{x(x+1)^2}\cdot\frac{x^3}{2} = e\cdot\lim_{x\to\infty}\frac{-x^2}{2(x+1)^2} = e\cdot\left(-\frac{1}{2}\right)$$

$(\ln f)' = \frac{f'}{f}$, $\boxed{f' = f\cdot(\ln f)'}$, $\quad f = \left(1+\frac{1}{x}\right)^2$, $\ln f = x\ln\left(1+\frac{1}{x}\right)$

7.

$$f : (-1, \infty) \to \mathbb{R} \qquad f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$f'''(x) = -\left(-\frac{2(1+x)}{(1+x)^4} = \frac{2}{(1+x)^3}\right.$$

$$f^{IV}(x) = -2 \cdot \frac{3(1+x)^2}{(1+x)^4} = \frac{-3!}{(1+x)^4}$$

$$\vdots$$

$$f^{n}(x) = (-1)^{n-1} \cdot \frac{n!}{(1+x)^n}$$

$$f^{n+1}(x) = \left(f^{n}(n)\right)' = (-1)^{n-1}(n-1)!\left(-\frac{n(1+x)^{n-1}}{(1+x)^{2n}}\right) = (-1)^{n}\, n! \frac{1}{(1+x)^{n+1}}$$

(f) $\lim\limits_{x\to\infty} x\left(\left(1+\frac{1}{x}\right)^x - e\right)$.

$$\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^x - e}{\frac{1}{x}} = \lim_{x\to\infty}\frac{e^{x\ln\left(1+\frac{1}{x}\right)} - e}{\frac{1}{x}} = \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{x\to\infty}\frac{f'(x)}{\frac{-1}{x^2}}$$

$$\left(e^{x\ln\left(1+\frac{1}{x}\right)}\right)' = \left(e^{x(\ln(x+1)-\ln x)}\right)' = e^{x(\ln(x+1)-\ln x)}\cdot\left(x(\ln(x+1)-\ln x)\right)'$$

$$= e^{x(\ln(x+1)-\ln x)}\cdot\left[(\ln(x+1)-\ln x) + x\cdot\frac{1}{x+1} - \frac{x}{x}\right]$$

$$= e^{x(\ln(x+1)-\ln x)}\left(\ln\frac{x+1}{x} - \frac{1}{x+1}\right) = e^{x\ln\frac{x+1}{x}}\left(\ln\frac{x+1}{x} - \frac{1}{x+1}\right)$$

$$= \left(1+\frac{1}{x}\right)^x\left(\ln\frac{x+1}{x} - \frac{1}{x+1}\right)$$

$$\Rightarrow \lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^{x}_{\to e}\left(\ln\frac{x+1}{x} - \frac{1}{x+1}\right)}{\frac{-1}{x^2}} = e\cdot\lim_{x\to\infty}\frac{\ln\frac{x+1}{x} - \frac{1}{x+1}}{\frac{-1}{x^2}} = \left(\frac{0}{0}\right) \overset{L'H}{=} e\cdot\lim_{x\to\infty}\frac{\frac{\frac{x+1}{\cancel{x}}}{x+1}\cdot\frac{-1}{x^2} + \frac{1}{(x+1)^2}}{\frac{2\cancel{x}}{x^{\cancel{4}}}} =$$

$$= e\cdot\lim_{x\to\infty}\frac{(-\cancel{x}-1+\cancel{x})\,x^{\cancel{3}}}{2\cancel{x}(x+1)^2} = e\cdot\lim_{x\to\infty}\frac{-x^2}{2(x+1)^2} = -\frac{1}{2}e$$

$$(f') = f\cdot(\ln f)'$$

(d) $f:\mathbb{R}\to\mathbb{R},\ f(x) = e^{2x}x^3$.

$$f'(x) = e^{2x}\cdot 2x^3 + e^{2x}\cdot 3x^2 = e^{2x}(2x^3+3x^2) = e^{2x}(2^1x^3+3x^2)$$

$$f''(x) = e^{2x}\cdot 2(2x^3+3x^2) + e^{2x}(6x^2+6x) = e^{2x}(4x^3+12x^2+6x) = e^{2x}(2^2x^3+3\cdot 2^2x^2+6x)$$

$$f'''(x) = e^{2x}\cdot 2(4x^3+12x^2+6x) + e^{2x}(12x^2+24x+6) =$$

$$= e^{2x}(\underset{2^3}{8}x^3+36x^2+36x+6) = e^{2x}(2^3x^3+3^2\cdot 2^2\cdot x^2+6^2x+6)$$

$$f^{IV}(x) = e^{2x}\,2(2^3x^3+36x^2+36x+6) + e^{2x}(24x^2+72x+36)$$

$$= e^{2x}(2^4x^3+96x^2+144x+48) = e^{2x}(2^4x^3+\qquad+6^4x+6\cdot 4)$$

$$f^{V}(x) = e^{2x}(2^5x^3+2\cdot 96x^2+2\cdot 144x+2\cdot 48+\underbrace{3\cdot 2^4}_{48}x^2+2\cdot 96x+144)$$

$$= e^{2x}(2^5x^3+\underline{240x^2}+\underline{480x}+\underline{240})$$