Mathematical Analysis
1st Year

Babeș-Bolyai University
Bachelor in Computer Science

# Seminar 6

1. Recall that the Taylor series for sin and cos are given, for any $x \in \mathbb{R}$, by:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots.$$

   (a) Prove that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, for any $x \in \mathbb{R}$.

   (b) Deduce that $x - \frac{x^3}{6} < \sin x < x$, $\forall x > 0$ and $1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$, $\forall x \in \mathbb{R}$.

   (c) Where does Euler's formula $e^{ix} = \cos x + i \sin x$ come from?

2. (a) For $\alpha \in \mathbb{R}$ and $|x| < 1$, prove the generalized binomial expansion (binomial series)

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{k} x^k, \quad \binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}.$$

   (b) Find the first four terms in the binomial series of $\sqrt{1+x}$ and $1/\sqrt{1+x}$.

3. Find the MacLaurin series and its radius of convergence for the following functions:

   $x_0 = 0$

   (a) $a^x$, $a > 0$.

   (b) $(1+x)\ln(1+x)$.

   (c) $\sin^2(x)$.

   (d) $\arctan x$.

4. For each function $f : \mathbb{R} \to \mathbb{R}$ given below check that $f'(0) = 0$ and find the first $n \in \mathbb{N}$ such that $f^{(n)}(0) \neq 0$. Then, deduce whether 0 is a local extremum point of $f$ or not; in the affirmative, specify if 0 is a global extremum point or just a local one.

   (a) $f(x) = e^x + e^{-x} - x^2$.

   (b) $f(x) = \cos(x^2)$.

   (c) $f(x) = 6 \sin x - 6x + x^3$.

No homework this week. Prepare for the midterm.

1. (a) $(\sin x)' = \cos x$

$$(\sin x)' = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right)' = \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n+1)!} x^{2n+1}\right)' = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)' = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x$$

$(\cos x)' = -\sin x$

$$(\cos x)' = \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right)' = \sum_{n=1}^{\infty} (-1)^n \frac{\cancel{2n} \cdot x^{2n-1}}{\cancel{2n}!} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} \cdot x^{2n-1} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} \cdot x^{2n-1} = -\sin x$$

(b) $x - \frac{x^3}{6} < \sin x < x \quad \forall x > 0$

Let $f : [0, \infty) \to \mathbb{R}$ $\quad f(x) = \sin x - x$

$f'(x) = \cos x - 1$

$f''(x) = -\sin x \quad < 0$

$\Rightarrow \underset{x \geq 0}{f'(x)} \searrow \Rightarrow f'(x) \leq f'(0) = 0 \Rightarrow \underset{x \geq 0}{f \searrow} \Big\} \Rightarrow f(x) \leq f(0) = 0$

$\Rightarrow \sin x - x \leq 0 \quad \forall x \in [0, \infty)$

$\Leftrightarrow \sin x \leq x \quad \forall x \in [0, \infty)$

Let $g : [0, \infty) \to \mathbb{R}$ $\quad g(x) = \sin x + \frac{x^3}{6} - x$

$g'(x) = \cos x + \frac{x^2}{2} - 1$

$g''(x) = -\sin x + x \quad \geq 0 \Rightarrow \underset{x \geq 0}{g' \nearrow} \Big\} \Rightarrow g'(x) \geq g'(0) = 0 \quad \Rightarrow g(x) \geq g(0) = 0$

(c) $e^{ix} = \cos x + i \sin x \quad$ Euler's formula $\quad \boxed{e^{i\pi} = -1}$

$$e^t = \sum_{n=0}^{\infty} \frac{(e^t)^{(n)}|_{t=0}}{n!} \cdot t^n = \sum_{n=0}^{\infty} \frac{e^0 \cdot t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \quad \Rightarrow e^{i \cdot x} = \sum_{n=0}^{\infty} \frac{(i \cdot x)^n}{n!} = \frac{x^0}{1} + \frac{i \cdot x}{1} - \frac{x^2}{2!} - \frac{i x^3}{3!} + \frac{x^4}{4!} - \dots$$

$t \to i \cdot x$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) = \cos x + i \sin x$$

2. (a) $\alpha \in \mathbb{R} \quad |x| < 1 \qquad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$

$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$

Let $x_0 = 0 \qquad f(x) = (1-x)^{\alpha}$

$f'(x) = \alpha(1+x)^{\alpha-1} \qquad f'(0) = \alpha$

$f''(x) = \alpha(\alpha-1)(1+x)$ $\qquad$ $f'(0) = \alpha(\alpha-1)$

$\vdots$

$f^{(n)}(0) = \alpha(\alpha-1)\dots(\alpha-n+1) = \cancel{\binom{\alpha}{n}}$

$$\boxed{(1+x)^{\alpha} = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k}$$ $\longrightarrow$ generalization for Newton's binomial

(b) find the first 4 terms of $\sqrt{1+x}$ and $\frac{1}{\sqrt{1+x}}$

$\sqrt{1+x} = (1+x)^{\frac{1}{2}} \Rightarrow \alpha = \frac{1}{2}$

$T_4(x) = f(0) + f'(0)\cdot x + f''(0)\cdot x^2 + f'''(0)\cdot x^3 =$

$= 1 + \alpha\cdot x + \frac{\alpha\cdot(\alpha-1)}{2!}\cdot x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\cdot x^3 =$

$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2}\cdot x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^3}{6} =$

$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \approx \sqrt{1+x}$ for $x \approx 0$

$\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} \Rightarrow \alpha = -\frac{1}{2} \Rightarrow T_4(x) = 1 - \frac{x}{2} + \frac{3}{8}x^2 - \dots$

3. (a) $a^x$ $\quad a > 0$ $\qquad$ $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f'(x) = a^x \ln a$ $\qquad$ $f'(0) = \ln a$

$f''(x) = a^x \ln^2 a$ $\qquad$ $f''(0) = \ln^2 a$

$\vdots$

$f^{(n)}(x) = a^x \ln^{(n)} a$ $\qquad$ $f^{(n)}(0) = \ln^n a$

$a^x = \sum_{n=0}^{\infty} \frac{\ln^n a}{n!} x^n$

Ratio test $\frac{\cancel{\ln^{n+1} a}\cdot \cancel{x^{n+1}}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{\cancel{\ln^n a}\,\cancel{x^n}} = \left| \frac{\ln a \cdot x}{n+1} \right|$

(b) $(1+x)\ln(1+x)$ — ignore it now & multiply it at the end

I. $f'(x) = \ln(1+x) + 1$ $\qquad$ $f'(0) = 1$

II $f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2}$ $\qquad f''(0) = -1$

$f'''(x) = \frac{-2}{(1+x)^3}$ $\qquad f'''(0) = 2$

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$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ $\qquad f^{(n)}(0) = (-1)^{n-1}\cdot(n-1)!$

$$\ln(1+x) = 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cancel{(n-1)!}}{\cancel{n!}} \cdot x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$(x+1)\cdot\ln(1+x) = (x+1)\cdot\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) = x(x+1) - \frac{x^2(x+1)}{2} + \frac{x^3(x+1)}{3} - \dots = x + x^2 - \frac{1}{2}x^2 - \frac{x^3}{2} + \frac{x^3}{3} + \frac{x^4}{3} - \dots =$$

$$= x + \frac{x^2}{2} + \frac{-x^3}{6} + \dots$$

Finish ↑

$\sum a_n(x-x_0)^n$

$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\sqrt[n]{|a_n|}$

$R = \frac{1}{L}$ $\Rightarrow$ Abs Convergence in $(x_0 - R, x_0 + R)$

(a) $\sum \underbrace{\frac{\ln^n a}{n!}}_{a_n} x^n$

$R = ?$

$L = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\frac{\ln a}{n+1} \longrightarrow 0 \Rightarrow$ $\boxed{R = \infty \Rightarrow \text{Convergent EVERYWHERE}}$

(b) $\sum \underbrace{\frac{(-1)^{n-1}}{n}}_{a_n} x^n$

$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1 \Rightarrow R = 1 \Rightarrow$ the series is conv. on $[-1, 1]$

(c) $f(x) = \sin^2 x = \frac{1-\cos 2x}{2}$ $\qquad \cos 2x = 1 - 2\sin^2 x$

$$= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}\underbrace{\frac{(-1)^n}{(2n)!}}_{a_n}\cdot(2x)^{2n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}}{\underbrace{\cancel{(2n+2)!}}_{(2n+1)(2n+2)}} \cdot \frac{\cancel{2n!}}{(-1)^n}\right| = \left|\frac{-1}{(2n+1)(2n+2)}\right| \longrightarrow 0 \Rightarrow R = 0$$

(d) $f(x) = \arctan x$

$$f'(x) = \frac{1}{1+x^2} \qquad f'(0) = 1$$

$$f''(x) = \frac{-2x}{(1+x^2)^2} \qquad f''(0) = 0$$

$$f'''(x) = \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4} \qquad f'''(0) = -2$$

$$\frac{1}{1-t} = 1 + t + t^2 + \dots = \sum_{n=0}^{\infty} t^n \qquad |t| < 1$$

$$\frac{1}{1+t} = 1 - t + t^2 - \dots = \sum_{n=0}^{\infty} (-t)^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \Big| \text{ integrate}$$

$$\Rightarrow \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n}{2n+1}}_{a_n} \cdot x^{2n+1}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^n}{2n+1} \cdot \frac{2n+3}{(-1)^{n+1}}\right| = \frac{2n+3}{2n+1} \longrightarrow 1 \Rightarrow R = 1 \quad \text{and series conv } (-1, 1)$$

3. Find the MacLaurin series and its radius of convergence for the following functions:

$x_0 = 0$

(a) $a^x$, $a > 0$.
(b) $(1 + x)\ln(1 + x)$.
(c) $\sin^2(x)$.
(d) $\arctan x$.

4. For each function $f : \mathbb{R} \to \mathbb{R}$ given below check that $f'(0) = 0$ and find the first $n \in \mathbb{N}$ such that $f^{(n)}(0) \neq 0$. Then, deduce whether 0 is a local extremum point of $f$ or not; in the affirmative, specify if 0 is a global extremum point or just a local one.

(a) $f(x) = e^x + e^{-x} - x^2$.
(b) $f(x) = \cos(x^2)$.
(c) $f(x) = 6\sin x - 6x + x^3$.

3. (a)

$f(x) = a^x$

$f'(x) = a^x \ln a \qquad f'(0) = \ln a$

$f''(x) = a^x \ln^2 a \qquad f''(x) = \ln^2 a$

$\vdots$

$f^{(n)}(x) = a^x \ln^n a \qquad f^{(n)}(x) = \ln^n a$

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\cdot(x-x_0)^n = \sum_{n\geq 0}^{\infty} \underbrace{\frac{\ln^n a}{n!}} x^n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\cancel{\ln^{n+1}} a}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{\cancel{\ln^n a}} \right| = \lim_{n\to\infty} \frac{\ln a}{n+1} \to 0 \Rightarrow R = \infty \Rightarrow \text{conv. everywhere}$$

(b)

$f(x) = (1+x)\ln(1+x)$

$f'(x) = \ln(1+x) + \cancel{(1+x)}\cdot\frac{1}{\cancel{1+x}} = \ln(1+x) + 1 \qquad f'(0) = 1$

$f''(x) = \frac{1}{1+x} \qquad f''(0) = 1$

$f'''(x) = \frac{-1}{(1+x)^2} \qquad f'''(0) = -1$

$f^{IV}(x) = \frac{2}{(1+x)^3} \qquad f^{IV}(0) = 2$

$f^{V}(x) = \frac{-2\cdot 3}{(1+x)^4} \qquad f^{V}(0) = -3!$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^n\cdot n!}{(1+x)^{n-1}} \qquad f^{(n)}(0) = (-1)^n\cdot n!$

$$T(x) = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n\cdot \cancel{n!}}{(1+x)^{n-1}}\cdot\frac{1}{\cancel{n!}}} x^n$$

Matho :

$f(x) = (1+x) \cdot \ln(x+1)$

$$1 - \frac{1}{2} + \frac{1}{3} - \dots = \ln 2$$

$g(x) = \ln(x+1)$

$g'(x) = \frac{1}{x+1}$

$g''(x) = \frac{-1}{(x+1)^2}$

$g'''(x) = \frac{2}{(x+1)^3}$

$$\frac{(-1)^n \cdot (n-1)!}{(x+1)^n}$$

$$T(x) = \sum_{n=0}^{\infty} (1+x) \cdot \frac{(-1)^n \cancel{(n-1)!}}{\cancel{n!}} \cdot x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} x^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \cdot x^{n+1}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^n} \right| = \frac{n}{n+1} \longrightarrow 1 \Rightarrow R = 1 \Rightarrow \text{con } (-1, 1)$$