Seminar 11

1. Find the second-order Taylor polynomial for the following functions at the given points:

   (a) $f(x, y) = \sin(x + 2y)$ at $(0, 0)$.
   (b) $f(x, y) = e^{x+y}$ at $(0, 0)$ and $(1, -1)$.
   (c) $f(x, y) = \sin(x)\sin(y)$ at $(\pi/2, \pi/2)$.
   (d) $f(x, y) = e^{-(x^2+y^2)}$ at $(0, 0)$.

2. Compute the Hessian matrix and its eigenvalues for the following:

   (a) $f(x, y) = (y - 1)e^x + (x - 1)e^y$ at $(0, 0)$.
   (b) $f(x, y) = \sin(x)\cos(y)$ at $(\pi/2, 0)$.

3. Find and classify the critical points for each of the following functions:

   (a) $f(x, y) = x^3 - 3x + y^2$.
   (b) $f(x, y) = x^3 + y^3 - 6xy$.
   (c) $f(x, y) = x^4 + y^4 - 4(x - y)^2$.
   (d) $f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z$.

4. Let A be a symmetric $n \times n$ matrix and the quadratic function $f : \mathbb{R}^n \to R$, $f(x) = \frac{1}{2}x^T Ax$. Prove that $\nabla f(x) = Ax$ and $H(x) = A$. *Hint: use the Taylor expansion.*

5. Let A be an $m \times n$ matrix, b a vector in R^m and the least squares minimization problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|^2.$$

   Prove that the solution x^* of this problem satisfies (the so-called normal equations)

$$A^T Ax^* = A^T b.$$

6. ★[Python] Let A be a 2×2 matrix and let the quadratic function $f : \mathbb{R}^2 \to R$, $f(x) = \frac{1}{2}x^T Ax$.

   (a) Give a matrix A such that f has a unique minimum.
   (b) Give a matrix A such that f has a unique maximum.
   (c) Give a matrix A such that f has a unique saddle point.

   In each case plot the 3d surface, three contour lines and the gradient at three different points.

Homework questions are marked with ★.

$$T_2(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0) \cdot (x-x_0, y-y_0) + $$
$$+ \frac{1}{2}(x-x_0, y-y_0) H(x_0,y_0) \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

1. (a) $f(x,y) = \sin(x+2y)$ in $(0,0)$

$f(0,0) = 0$

$\nabla f(x_0,y_0) = (1,2)$

$$\frac{\partial f}{\partial x} = \cos(x+2y) \qquad \frac{\partial^2 f}{\partial x^2} = -\sin(x+2y) \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -2\sin(x+2y)$$

$$\frac{\partial f}{\partial y} = 2\cos(x+2y) \qquad \frac{\partial f}{\partial y^2} = -4\sin(x+2y)$$

$$H(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T_2(0,0) = 0 + (1,2)\cdot(x,y) = x+2y$$

or: $t = x+2y \Rightarrow \sin t = t - \frac{t^3}{3!} + \dots$ but $T_2 = t$

b) $f(x,y) = e^{x+y}$ in $(0,0), (1,-1)$

let $t = x+y \Rightarrow f(t) = e^t = \sum_{n=0}^{\infty} \frac{t^n}{t!} = 1 + t + \frac{t^2}{2!} + \dots \Rightarrow T_2 = 1 + t + \frac{t^2}{2} = 1 + x + y + \frac{(x+y)^2}{2}$

$f(0,0) = 1$

$f(1,-1) = 1$

$\nabla f(x,y) =$

$$\frac{\partial f}{\partial x} = e^{x+y} = \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial x^2} = e^{x+y} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\Rightarrow \nabla f(0,0) = (1,1) \qquad \Rightarrow H(x,y) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$\nabla f(1,-1) = (1,1)$$

$$\boxed{a \cdot b = a^T b}$$

$$T_2(0,0) = 1 + (1,1)\cdot(x,y) + \frac{1}{2}(x,y)\cdot\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1 + x + y + \frac{1}{2}(x,y)\cdot\begin{pmatrix} x+y \\ x+y \end{pmatrix} = 1 + x + y + \frac{1}{2}(x(x+y) + y(x+y))$$
$$= 1 + x + y + \frac{(x+y)^2}{2}$$

$$T_2(1,-1) = 1 + (1,1)(x+1, y-1) + \frac{1}{2}(x+1, y-1)\cdot\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x-1 \\ y+1 \end{pmatrix} =$$

$$= 1 + x\cancel{+1} + y\cancel{-1} + \frac{1}{2}(x+1, y-1)\cdot(x\cancel{-1}+y\cancel{+1}, x\cancel{-1}+y\cancel{+1}) = 1 + x + y + \frac{(x+y)^2}{2}$$

(c) $f(x,y) = \sin(x)\sin(y)$ at $(\pi/2, \pi/2)$. $= \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

$$f\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \sin^2\frac{\pi}{2} = 1$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots, \ \forall t \in \mathbb{R}$$

$$t = x-y \Rightarrow \cos(x-y) = 1 - \frac{(x-y)^2}{2} + \dots$$

$$t = x+y \Rightarrow \cos(x+y) = 1 - \frac{(x+y)^2}{2} + \dots$$

$$\Rightarrow f(x,y) = \frac{1}{2}\left[\cancel{1} - \frac{(x-y)^2}{2} - \cancel{1} + \frac{(x+y)^2}{2}\right] = \frac{1}{2}\cdot\frac{(x+y)^2 - (x-y)^2}{2} = \frac{4xy}{4} = x\cdot y$$

or : $f(x,y) = (\underbrace{x - \frac{x^3}{3!}}_{\text{only care up to the second order}} + \dots)(y - \frac{y^3}{3!} + \dots) = xy$

(d) $f(x,y) = e^{-(x^2+y^2)}$ at $(0,0)$.

$$T_2(x,y) = f(x_0,y_0) + \nabla f(x_0,y_0)\cdot(x-x_0, y-y_0) + \frac{1}{2}(x-x_0, y-y_0)H(x_0,y_0)\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

$$f(0,0) = e^0 = 1$$

$$\nabla f(x_0,y_0) = ?$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= e^{-(x^2+y^2)}\cdot(-2x) \\ \frac{\partial f}{\partial y} &= e^{-(x^2+y^2)}\cdot(-2y) \end{aligned}\right\} \Rightarrow \nabla f(x_0,y_0) = (0,0)$$

$$\frac{\partial^2 f}{\partial x^2} = e^{-(x^2+y^2)}\cdot 4x^2 + e^{-(x^2+y^2)}\cdot(-2) = e^{-(x^2+y^2)}(4x^2-2)$$

$$\frac{\partial^2 f}{\partial y^2} = e^{-(x^2+y^2)}(4y^2-2)$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{d}{dx}\left(e^{-(x^2+y^2)}\cdot(-2y)\right) = e^{-(x^2+y^2)}\cdot 4xy$$

$$H(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$T_2(x,y) = 1 + \frac{1}{2}(x,y)\cdot\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1 + \frac{1}{2}(x,y)\cdot(-2x,-2y) = 1 + \frac{1}{2}(-2x^2-2y^2) = 1 - x^2 - y^2$$

2. Compute the Hessian matrix and its eigenvalues for the following:

(a) $f(x,y) = (y-1)e^x + (x-1)e^y$ at $(0,0)$. (b) $f(x,y) = \sin(x)\cos(y)$ at $(\pi/2, 0)$.

(a) $f(x,y) = (y-1)e^x + (x-1)e^y$ at $(0,0)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{d}{dx}\left(\frac{\partial f}{\partial x}\right) = \frac{d}{dx}\left((y-1)e^x + e^y\right) = (y-1)e^x$$

$$\frac{\partial^2 f}{\partial y^2} = (x-1)e^y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{d}{dy}\left((y-1)e^x + e^y\right) = e^x + e^y = \frac{\partial^2 f}{\partial y \partial x}$$

$$H(0,0) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

eigenvalues: $A v = \lambda v$

$(A - \lambda I) \cdot v = 0$, $v \neq 0$

$\det(A - \lambda I) = 0$

$\det(H(0,0) - \lambda I) = 0$

$$\begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 - 4 = \lambda^2 + 2\cdot\lambda + 1 - 4 = \lambda^2 + 2\lambda - 3 = 0$$

$$\Delta = 16 \Rightarrow \lambda_{1,2} = \frac{-2 \pm 4}{2} < \begin{matrix} \lambda_1 = -3 \\ \lambda_2 = 1 \end{matrix}$$

(a) $f(x,y) = x^3 - 3x + y^2$.

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

$$\frac{\partial f}{\partial y} = 2y = 0 \Rightarrow y = 0$$

$\Rightarrow (1,0), (-1,0)$ critical points

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\Rightarrow H(1,0) = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \left.\begin{matrix} \lambda_1 = 6 > 0 \\ \lambda_2 = 2 > 0 \end{matrix}\right\} \Rightarrow (1,0) \text{ local max}$$

$$H(-1,0) = \begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \left.\begin{matrix} \lambda_1 = -6 < 0 \\ \lambda_2 = 2 > 0 \end{matrix}\right\} \Rightarrow (-1,0) \text{ saddle point}$$

$f(x+h) = f(x) + \underbrace{\nabla f(x) \cdot h}_{\langle \nabla f(x), h \rangle} + \frac{1}{2} h^T H(x) h + \dots$ — Taylor expansion

4. $f(x) = \frac{1}{2} x^T A x$

Method 1: $\frac{\partial f}{\partial x_1} = ?$ a lot of work, don't

$\langle a, b \rangle = a \cdot b = a^T b$

$(AB)^T = B^T A^T$

Method 2: $f(x+h) = \frac{1}{2} (x+h)^T A (x+h) = \frac{1}{2} x^T A x + \frac{1}{2} x^T A h + \frac{1}{2} h^T A x + \frac{1}{2} h^T A h.$

$x^T A h = \langle x, Ah \rangle = x \cdot (Ah) = \langle Ah, x \rangle = (Ah)^T x = h^T A^T x = h^T A x = \langle h, Ax \rangle$

$f(x+h) = f(x) + \boxed{Ax} \cdot h + \frac{1}{2} h^T \boxed{A} h$

$\nabla f(x) = Ax, \quad H(x) = A$

5. Let A be an $m \times n$ matrix, b a vector in R^m and the least squares minimization problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2.$$

Prove that the solution x^* of this problem satisfies (the so-called normal equations)

$$A^T A x^* = A^T b.$$

$f(x) = \|Ax - b\|^2 = \langle Ax - b, Ax - b \rangle$

$f(x) \longrightarrow \min \Rightarrow \nabla f(x) = 0$

$f(x) = \langle Ax, Ax \rangle - 2 \langle Ax, b \rangle + \langle b, b \rangle$

$\langle Ax, Ax \rangle = (Ax)^T A x = x^T (A^T A) x$

$\Rightarrow f(x) = x^T (A^T A) x - \underbrace{2 b^T A x}_{\langle x, A^T b \rangle = x \cdot (A^T b)} + \|b\|^2$

$\nabla f(x) = 2 A^T A x - 2 A^T b = 0$

$\Rightarrow A^T A x = A^T b$