Revision

$r = (A, B, R)$ — relation

domain (A), codomain (B), graph (R)

- a triple
- A, B - sets
- $R \subseteq A \times B$ → its meant to show us smt

if $A = B$ → homogenous relation

- reflexivity: $\forall\, x \in A : x\, r\, x \quad ((x,x) \in R)$
- symmetry: $\forall\, x, y \in A : [\text{if } x\, r\, y \Rightarrow y\, r\, x]$
- transitivity: $\forall\, x, y, z \in A : [\text{if } x\, r\, y \text{ and } y\, r\, z \Rightarrow x\, r\, z]$

→ r is an equivalence relation (* ex: ppl born in the same place)

Question from lecture

on $\mathbb{Z}$, $m > 1$

$$x \equiv y \pmod{m} \Leftrightarrow m \mid (x-y)$$

$$\Rightarrow \forall\, x, y : x \equiv y \pmod{1}$$

if $m = 0 \Rightarrow x \equiv y \pmod{0} \Leftrightarrow 0 \mid (x-y) \Rightarrow$ the graph of this relation is $\emptyset$

* concentrate on definitions

2.1) r, s, t, v hom. relations defined on $M = \{2, 3, 4, 5, 6\}$ by:

$x\, r\, y \Leftrightarrow x < y$

$x\, s\, y \Leftrightarrow x \mid y$

$x\, t\, y \Leftrightarrow \gcd(x,y) = 1$

$x\, v\, y \Leftrightarrow x \equiv y \pmod{3}$

* plural graf - grafuri (praf - prafe)

Write the graphs r, s, t, v of these relations.

$R = \{(2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}$

$S = \{(2,4), (2,6), (3,6), (2,2), (3,3), (4,4), (5,5), (6,6)\}$ → reflexive, but not symmetric + transitive (no contradiction)

$T = \{(2,3), (2,5), (3,2), (3,5), (3,4), (4,3), (4,5), (5,2), (5,3), (5,4), (5,6), (6,5)\}$

$V = \{(3,6), (2,5), (6,3), (5,2), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

Seminar 2

1. Let r, s, t, v be the homogeneous relations defined on the set $M = \{2, 3, 4, 5, 6\}$ by

$$x\, r\, y \Longleftrightarrow x < y$$

$$x\, s\, y \Longleftrightarrow x | y$$

$$x\, t\, y \Longleftrightarrow g.c.d.(x, y) = 1$$

$$x\, v\, y \Longleftrightarrow x \equiv y \pmod 3\,.$$

Write the graphs R, S, T, V of the given relations.

2. Let A and B be sets with n and m elements respectively ($m, n \in \mathbb{N}^*$). Determine the number of:
(i) relations having the domain A and the codomain B;
(ii) homogeneous relations on A.

3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

4. Which ones of the properties of reflexivity, transitivity and symmetry hold for the following homogeneous relations: the strict inequality relations on $\mathbb{R}$, the divisibility relation on $\mathbb{N}$ and on $\mathbb{Z}$, the perpendicularity relation of lines in space, the parallelism relation of lines in space, the congruence of triangles in a plane, the similarity of triangles in a plane?

5. Let $M = \{1, 2, 3, 4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$, $R_2 = \Delta_M \cup \{(1, 2), (1, 3)\}$, $\pi_1 = \{\{1\}, \{2\}, \{3, 4\}\}$, $\pi_2 = \{\{1\}, \{1, 2\}, \{3, 4\}\}$.
(i) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.
(ii) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

6. Define on $\mathbb{C}$ the relations r and s by:

$$z_1\, r\, z_2 \Longleftrightarrow |z_1| = |z_2|\,; \qquad z_1\, s\, z_2 \Longleftrightarrow arg\, z_1 = arg\, z_2 \text{ or } z_1 = z_2 = 0\,.$$

Prove that r and s are equivalence relations on $\mathbb{C}$ and determine the quotient sets (partitions) $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometric interpretation).

7. Let $n \in \mathbb{N}$. Consider the relation ρ_n on $\mathbb{Z}$, called the *congruence modulo* n, defined by:

$$x\, \rho_n\, y \Longleftrightarrow n | (x - y)\,.$$

Prove that ρ_n is an equivalence relation on $\mathbb{Z}$ and determine the quotient set (partition) $\mathbb{Z}/\rho_n$. Discuss the cases $n = 0$ and $n = 1$.

8. Determine all equivalence relations and all partitions on the set $M = \{1, 2, 3\}$.

9. Let $M = \{0, 1, 2, 3\}$ and let $h = (\mathbb{Z}, M, H)$ be a relation, where

$$H = \{(x, y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : x = 4z + y\}\,.$$

Is h a function?

10. Consider the following homogeneous relations on $\mathbb{N}$, defined by:

$$m\, r\, n \Longleftrightarrow \exists a \in \mathbb{N} : m = 2^a n\,,$$

$$m\, s\, n \Longleftrightarrow (m = n \text{ or } m = n^2 \text{ or } n = m^2)\,.$$

Are r and s equivalence relations?

2.3) Give examples of relations that have one of the three properties, but not the others

$M = \{1, 2, 3\}$ $\qquad$ $x \, r \, y \Rightarrow x > y$

$x \, r \, y = \{(1,1), (1,2), (2,2), (3,3)\} \rightarrow$ reflexive + transitive

$t, \cancel{s}, \cancel{r}$ – transitive

$M = \mathbb{Z}, \quad x \, \rho \, y \Leftrightarrow \gcd(x,y) = 1$

$M = \mathbb{Z}, \quad x \, \rho \, y \Leftrightarrow x < y$

$r, \cancel{s}, t$

$M = \{2,3,4\} \quad x \, \rho \, y \Leftrightarrow x \bmod y = y \bmod x$

$R = \{(2,2), (3,3), (4,4)$

$M = \{1, 2, 3\}$

$x \, r \, y = \{(1,1), (1,2), (2,2), (3,3), (2,3)\} \rightarrow$ transitive

$M = \mathbb{Z}^* \quad x v y \Leftrightarrow x = -y \rightarrow$ symmetric

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Symmetric but not others

$M = \{1, 2, 3\} \Rightarrow R = \{(1,2), (2,1)\}$ (for it to be transitive I'd need $(1,1)$)

transitivity

$M = \{1, 2, 3\} \Rightarrow R = \{(1,2), (2,3), (1,3)\}$

$R = \{(1,2)\}$ cannot be disproved

Th) M set. There is a bijection: between all equivalences and partitions of M

$$\mathcal{E}(M) \xleftrightarrow{\sim} \mathcal{P}(M)$$

$\mathcal{E}(M)$ = the equivalences on M; $\mathcal{P}(M)$ = partitions on M (2 subsets cannot overlap & all subsets give the initial set)

$$r \longmapsto \underbrace{M/r}_{\text{the quotient set mod } r}$$ (read as: M modulo r)

$$r_P \longleftarrow P$$

$$M/r = \{ \underbrace{r\langle x\rangle}_{\substack{\text{class modulo } r \\ = \hat{x}}} \mid x \in M \}$$

$$r\langle x\rangle = \{ y \in M \mid x\, r\, y \}$$

$$x\, r_P\, y \iff \exists A \in P : A \ni x, y$$

2.5) $M = \{1,2,3,4\}$, r_1, r_2 hom. relations on M, $\pi_1, \pi_2 \subseteq \mathcal{P}(M)$ ⟶ set of all subsets of M

$R_1 = \underbrace{\Delta_M}_{\{(1,1),(2,2),(3,3),(4,4)\}} \cup \{(1,2),(2,3),(3,1),(2,1),(3,2),(1,3)\}$

5. Let $M = \{1,2,3,4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.
(*i*) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.
(*ii*) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

$R_2 = \Delta_M \cup \{(1,2),(1,3)\}$

* happening only between equivalences & partitions

$\pi_1 = \{\{1\},\{2\},\{3,4\}\}$

$\pi_2 = \{\underline{\{1\},\{1,2\}},\{3,4\}\}$

(i) Are r_1 and r_2 equivalences on M? If so, write their associated partition}

(ii) Are π_1 and π_2 partitions of M? If so, write the graphs of their associated equivalences.

(i) r_2 – is not symmetric ⟹ is not equivalence, $(1,2) \in R_2$, but $(2,1) \notin R_2$

r_1 is equivalent ⟹ $\tilde{\pi} = \{\{1,2,3\}\ \{4\}\}$ ⟶ $M/r_1 = \{r_1\langle x\rangle \mid x \in M\}$ $\quad r_1\langle 1\rangle = \{1,2,3\}$, $r_1\langle 4\rangle = \{4\}$ ⟹ $M/r_1 = \{\{1,2,3\},\{4\}\}$

⟶ the class of 4 in relation to π_1

(ii) $\pi_1 = \{\underbrace{\{1\}}_{r\langle 1\rangle}, \underbrace{\{2\}}_{r\langle 2\rangle}, \underbrace{\{3,4\}}_{r\langle 3\rangle\, r\langle 4\rangle}\}$ ⟹ $R_{\pi_1} = \{(1,1),(2,2),(3,3),(3,4),(4,4),(4,3)\}$

π_2 is NOT because $\{1\} \cap \{1,2\} = \{1\} \neq \emptyset$

$\forall\ A, B \in \Pi_1,\quad A \cap B = \emptyset$

$\{1\} \cap \{2\} \cap \{3,4\} = M$ $\Rightarrow \Pi_1 =$ partition

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$r = (A, B, R)$ relation

$$\boxed{r \text{ function} \iff \forall\ x \in A : \ |r\langle x\rangle| = 1}$$

$f : A \to B$

$Gr_f = \{(x, f(x)) \mid x \in A\}$

9. Let $M = \{0,1,2,3\}$ and let $h = (\mathbb{Z}, M, H)$ be a relation, where

$$H = \{(x,y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : \underbrace{x = 4z + y}_{\text{disguised mod 4}}\}.$$

Is h a function?

for $x \in \mathbb{Z}$, $\exists!\, y \in \{0,1,2,3\}$: $x \equiv y \pmod 4$

which is the same as $4 \mid (x-y) \iff \exists\, z \in \mathbb{Z}$

$x - y = 4z$

$r = (A, B, R)$

$A = \{1,2,3,4\}$

$B = \{0,1,5\}$

$R = \{(1,0), (4,1), (2,5), (3,5), (4,5)\}$

NOT a function: $|r\langle 4\rangle| = 2 \neq 1$