Lecture 3

- Let V be a vectorspace over K. Then $\forall k, k' \in K$ and $\forall v, v' \in V$:

(i) $k \cdot 0 = 0 \cdot v = 0$

$$k \cdot 0 + k \cdot v = k(0 + v) = k \cdot v \quad \Big| -(k \cdot v)$$
$$k \cdot 0 = 0$$
$$0 \cdot v + k \cdot v = (0 + k) \cdot v = k \cdot v \quad \Big| -k \cdot v$$
$$0 \cdot v = 0$$

(ii) $k(-v) = (-k)v = -kv$

$$kv + k(-v) = k(v - v) = k \cdot 0 = 0 \Rightarrow k(-v) = -kv$$
$$kv + (-k)v = (k - k)v = 0 \cdot v = 0 \Rightarrow (-k)v = -kv$$

(iii) $k(v - v') = kv - kv'$

$$k(v - v') + kv' = kv - kv' + kv' = k(v - v' + v') = kv \Rightarrow k(v - v') = kv - kv'$$

(iv) $(k - k')v = kv - k'v$

$$(k - k')v + k'v = (k - k' + k')v = kv \Rightarrow (k - k')v = kv - k'v$$

- Let V be a vector space over K and let $k \in K$ and $v \in V$. Then: $kv = 0 \Leftrightarrow k = 0$ or $v = 0$

$\boxed{\Longrightarrow}$ Assume that $kv = 0$. Suppose that $k \neq 0$, k is invertable in the k field so:

$$kv = 0 \Rightarrow kv = k \cdot 0 \Rightarrow k^{-1}(kv) = k^{-1}(k \cdot 0) \Rightarrow (k^{-1} \cdot k)v = (k^{-1} \cdot k)0 \Rightarrow v = 0$$

- Let V be a vectorspace over K and let $S \subseteq V$.

$$S \leq_K V \Leftrightarrow \begin{cases} S \neq \emptyset \quad (0 \in S) \\ \forall k_1, k_2 \in K, \forall v_1, v_2 \in S, \ k_1 v_1 + k_2 v_2 \in S \end{cases}$$

$\boxed{\Longrightarrow}$ take $k = 0$, $v_1 \in S \neq \emptyset$ $\Rightarrow 0 = 0 \cdot v_1 \in S$

let $k_1, k_2 \in K$, $v_1, v_2 \in S \Rightarrow k_1 v_1, k_2 v_2 \in S \Rightarrow k_1 v_1 + k_2 v_2 \in S$

Lecture 4

- **Thm** Let V be a vector space over K and $(S_i)_{i\in I}$ be a family of subspaces

$$\bigcap_{i\in I} S_i \in S(V)$$

$$S \leq_K V \iff \begin{cases} S \neq \emptyset \\ \forall v_1, v_2 \in S \quad v_1+v_2 \in S \\ \forall k \in K, \forall v \in S, \ k\cdot v \in S \end{cases} \iff \begin{cases} S \neq \emptyset \\ \forall k_1, k_2 \in K \\ \forall v_1, v_2 \in S \end{cases} k_1v_1 + k_2v_2 \in S$$

↳ $0 \in \bigcap_{i\in I} S_i \neq \emptyset$ because $0 \in S_i \ \forall i \in I \ (S_i \leq_K V)$

↳ Let $k_1, k_2 \in K$ and $v_1, v_2 \in \bigcap_{i\in I} S_i \Rightarrow \left.\begin{array}{l} v_1, v_2 \in S_i \ \forall i \in I \\ S_i \leq_K V, \forall i \in I \end{array}\right\} \Rightarrow$

$\Rightarrow k_1v_1 + k_2v_2 \in S_i, \forall i \in I \Rightarrow k_1v_1 + k_2v_2 \in \bigcap_{i\in I} S_i$

Hence $\bigcap_{i\in I} S_i \leq_K V$

- **Thm** Characterization of the generated subspace

Let V be a vector space over K and $\emptyset \neq X \subseteq V$. Then

$\langle X \rangle = \{k_1v_1 + \ldots + k_nv_n \mid k_i \in K, v_i \in X\}$ is set of all finite linear comb of vectors of X

$$L = \{k_1v_1 + \ldots + k_nv_n \mid k_i \in K, v_i \in X\}$$

(i) Let $v \in X$. Then $v = 1\cdot v \in L$, hence $L \neq \emptyset$. Now let $k, k' \in K$ and $v, v' \in L$

$v = \sum_{i=1}^{n} k_iv_i$ and $v' = \sum_{j=1}^{m} k'_jv'_j$ for some $k_1, \ldots, k_n, k'_1, \ldots, k'_m \in K$ and $v_1, \ldots, v_n, v'_1, \ldots, v'_m \in X$

Hence

$$kv + k'v' = k\sum_{i=1}^{n} k_iv_i + k'\sum_{j=1}^{m} k'_jv'_j = \sum_{i=1}^{n}(kk_i)v_i + \sum_{j=1}^{m}(k'k'_j)v'_j \in L \rightarrow \text{finite linear comb.}$$

or: $k_1, k_2 \in K$, $v_1, v_2 \in L \Rightarrow v_1, v_2$ are finite linear comb. of K

$k_1v_1 + k_2v_2$ also a finite lin. comb of vect. from $X \Rightarrow k_1v_1 + k_2v_2 \in L$

$\Rightarrow L \leq_K V$

(ii) $\begin{array}{l} n=1 \\ k_1 = 1 \end{array} \Rightarrow X \subseteq L$ ✓

(iii) Let $S \leq V$ s.t. $X \subseteq S$. Since $X \subseteq S$ and $S \leq V \Rightarrow$

$k_1v_1 + \ldots + k_nv_n \in S$ Hence $L \subseteq S$

Thus $\langle X \rangle = L$ by the remark from the beginning

Def Let V be v.s. over K and $S, T \leq V$. We define the sum of the subspaces S and T as
$$S+T = \{ s+t \mid s \in S, t \in T \}$$

Thm If $S \cap T = \{0\}$ then $S+T$ is the direct sum of the subspaces S and T

$$S+T = \langle S \cup T \rangle \text{ hence } S+T \leq V$$

$S+T = \langle S \cup T \rangle$ - generated set

$\boxed{\subseteq}$ Let $v \in S+T \Rightarrow v = s+t \Rightarrow v = 1 \cdot s + 1 \cdot t \in \langle S \cup T \rangle$

$\boxed{\supseteq}$ Let $v \in \langle S \cup T \rangle \Rightarrow v = k_1 v_1 + \dots + k_n v_n$ with $k_1, \dots, k_n \in K$, $v_1, \dots, v_n \in S \cup T$

$$v = \sum_{i=1} k_i \cdot v_i + \sum_{j=1} k_j \cdot v_j \in S \cup T$$

$I = \{ i \in \{1, \dots, n\} \mid v_i \in S \}$

$J = \{1, \dots, n\} \setminus I$

- $V = S \oplus T \Leftrightarrow \forall v \in V, \exists! s \in S, t \in T: v = s+t$

$\Rightarrow$ Suppose $V = S \oplus T \Rightarrow V = S+T$ and $S \cap T = \{0\} \Rightarrow$

$\Rightarrow \forall v \in V, v = s+t$ with $s \in S, t \in T$

for uniqueness, suppose that $v = s' + t' \Rightarrow s + t = s' + t' \Rightarrow$

$$\underbrace{s - s'}_{\in S} = \underbrace{t' - t}_{\in T} \in S \cap T = \{0\}$$

$\Rightarrow s = s'$ and $t = t'$

$\Leftarrow$ Suppose $\forall v \in V, \exists! s \in S, t \in T : v = s+t$

Let $v \in V \Rightarrow v = s+t$, with $s \in S, t \in T \Rightarrow V \subseteq S+T \Rightarrow V = S+T$

Suppose $S \cap T = \{u\} \Rightarrow \underset{\in S}{u} + \underset{\in T}{0} = \underset{\in S}{0} + \underset{\in T}{u} \xrightarrow{\text{uniqueness of writing}} u = 0$, $S \cap T = \{0\}$

$\Longrightarrow V = S \oplus T$

- **Thm** Let V and V' vector spaces over K and $f: V \to V'$

f is a K-linear map $\Leftrightarrow f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2)$ $\forall k_1, k_2 \in K, \forall v_1, v_2 \in V$

$\Longrightarrow$ Suppose that f is a K-linear map $\forall k_1, k_2 \in K, \forall v_1 v_2 \in V$

$$f(k_1 v_1 + k_2 v_2) = f(k_1 v_1) + f(k_2 v_2) = k_1 f(v_1) + k_2 f(v_2)$$

$\Longleftarrow$ Choose $k_1 = k_2 = 1$

$$\left.\begin{array}{l} f(v_1 + v_2) = f(v_1) + f(v_2) \\ k_2 = 0 \\ f(k_1 v_1 + 0 \cdot v_2) = f(k_1 v_1) = k_1 \cdot f(v_1) \end{array}\right\} \Rightarrow \text{the two cond. for a linear map}$$

- **Thm** (i) Let $f: V \to V'$ be an isomorphism of vector spaces over K
 $f^{-1}: V' \to V$ also isomorphic over K

 (ii) Let $f: V \to V'$ and $g: V' \to V''$ be K-linear maps. $g \circ f: V \to V''$ lin. map

(i) Since f is isomorphic, f is bijective and so is f^{-1}

let $k_1, k_2 \in K$, $v_1', v_2' \in V'$ we wat to prove that

$$f^{-1}(k_1 v_1' + k_2 v_2') = k_1 f^{-1}(v_1') + k_2 f^{-1}(v_2')$$

Let $v_1 = f^{-1}(v_1')$, $v_2 = f^{-1}(v_2') \Rightarrow f(v_1) = v_1'$ and $f(v_2) = v_2'$ hence

$$k_1 v_1' + k_2 v_2' = k_1 f(v_1) + k_2 f(v_2) = f(k_1 v_1 + k_2 v_2)$$

Thus we have $f^{-1}(k_1 v_1' + k_2 v_2') = k_1 v_1 + k_2 v_2 = k_1 f^{-1}(v_1') + k_2 f^{-1}(v_2') \to$ isomorphism of vect. over K

(ii) Let $k_1, k_2 \in K$ and $v_1, v_2 \in V$.

$$(g \circ f)(k_1 v_1 + k_2 v_2) = g(f(k_1 v_1 + k_2 v_2))$$
$$= g(k_1 f(v_1) + k_2 f(v_2)) = k_1 g(f(v_1)) + k_2 g(f(v_2)) = k_1 (g \circ f)(v_1) + k_2 (g \circ f)(v_2)$$

Hence $g \circ f$ is a K-linear map

- **Thm** Let $f: V \to V'$ be a K-linear map.

$$\boxed{\operatorname{Ker} f \leq V \qquad \text{and} \qquad \operatorname{Im} f \leq V'}$$

$\operatorname{Ker} f \leq_K V \iff \begin{cases} \operatorname{Ker} f \neq \emptyset \quad f(0) = 0' \Rightarrow 0 \in \operatorname{Ker} f \neq \emptyset \\ \text{Let } k_1, k_2 \in K \text{, and } v_1, v_2 \in \operatorname{Ker} f. \text{ Show that } k_1 v_1 + k_2 v_2 \in \operatorname{Ker} f \end{cases}$

$$f(k_1 v_1 + k_2 v_2) = k_1 \underbrace{f(v_1)}_{0'} + k_2 \underbrace{f(v_2)}_{0'} = 0'$$

thus $k_1 v_1 + k_2 v_2 \in \operatorname{Ker} f$ Hence $\operatorname{Ker} f \leq V$

$\operatorname{im} f \leq_K V \iff 0' = f(0) \in \operatorname{Im} f \neq \emptyset$

Let $k_1, k_2 \in K$ and $v_1', v_2' \in \operatorname{Im} f$. Show that $k_1 v_1' + k_2 v_2' \in \operatorname{Im} f$

$v_1' = f(v_1)$ and $v_2' = f(v_2)$ for some $v_1, v_2 \in V$

$$k_1 v_1' + k_2 v_2' = k_1 f(v_1) + k_2 f(v_2) = f(k_1 v_1 + k_2 v_2) \in \operatorname{Im} f$$

Hence $\operatorname{Im} f \leq_K V$

- **Thm** Let $f : V \to V'$ be a K-linear map:

$$\boxed{\operatorname{Ker} f = \{0\} \iff f \text{ is injective}}$$

$\Longrightarrow$ Assume that $\operatorname{Ker} f = \{0\}$. Let $v_1, v_2 \in V$ s.t. $f(v_1) = f(v_2)$.

$f(v_1 - v_2) = 0$ hence $v_1 - v_2 \in \operatorname{Ker} f = \{0\}$ and thus $v_1 = v_2 \Rightarrow$

f is injective

$\Longleftarrow$ Assume that f is injective. Clearly $\{0\} \subseteq \operatorname{Ker} f$. (1)

Let $v \in \operatorname{Ker} f \Rightarrow f(v) = 0' = f(0)$. By the injectivity of f

we deduce that $v = 0$ Thus $\operatorname{Ker} f \subseteq \{0\}$ (2)

$$\xRightarrow{(1),(2)} \operatorname{Ker} f = \{0\}$$

- **Thm** Let $f : V \to V'$ a K-linear map. $X \subseteq V$

$$f(\langle X \rangle) = \langle f(X) \rangle$$

if $X = \emptyset$ then:

$$f(\langle \emptyset \rangle) = f(\{0\}) = \{f(0)\} = \{0'\} = \langle \emptyset \rangle = \langle f(\emptyset) \rangle$$

how assume that $X \neq \emptyset$

$$\langle X \rangle = \{k_1 v_1 + \dots + k_n v_n \mid k_i \in K, v_i \in X\}$$

since f is a K linear map $\Rightarrow$

$$f(\langle X \rangle) = \{f(k_1 v_1 + \dots + k_n v_n)\} = \{k_1 f(v_1) + \dots + k_n f(v_n)\} = \langle f(X) \rangle$$

Lecture 5

Theorem

Let V be a vector space over K. Then the vectors $v_1, \ldots, v_n \in V$ are linearly dependent if and only if one of the vectors is a linear combination of the others, that is, $\exists j \in \{1, \ldots, n\}$ such that

$$v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i v_i$$

for some $\alpha_i \in K$, where $i \in \{1, \ldots, n\}$ and $i \neq j$.

Proof. $\boxed{\Longrightarrow}$ Assume that $v_1, \ldots, v_n \in V$ are linearly dependent. Then $\exists k_1, \ldots, k_n \in K$ not all zero, say $k_j \neq 0$, such that $k_1 v_1 + \cdots + k_n v_n = 0$. But this implies

$$-k_j v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} k_i v_i$$

and further,

$$v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} (-k_j^{-1} k_i) v_i \,.$$

Now choose $\alpha_i = -k_j^{-1} k_i$ for each $i \neq j$ to get the conclusion.

$\boxed{\Longleftarrow}$ Assume that $\exists j \in \{1, \ldots, n\}$ such that

$$v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i v_i$$

for some $\alpha_i \in K$, where $i \in \{1, \ldots, n\}$ and $i \neq j$. Then

$$(-1) v_j + \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i v_i = 0 \,.$$

Since there exists such a linear combination equal to zero and the scalars are not all zero, the vectors $v_1, \ldots, v_n$ are linearly dependent. $\square$

Theorem 2.6.5 *Let $n \in \mathbb{N}$, $n \geq 2$.*
(i) Two vectors in the canonical vector space K^n are linearly dependent $\Longleftrightarrow$ their components are respectively proportional.
(ii) n vectors in the canonical vector space K^n are linearly dependent $\Longleftrightarrow$ the determinant consisting of their components is zero.

Proof. (*i*) Let $v = (x_1, \ldots, x_n)$, $v' = (x'_1, \ldots, x'_n) \in K^n$. By Theorem 2.6.3, the vectors v and v' are linearly dependent if and only if one of them is a linear combination of the other, say $v' = kv$ for some $k \in K$. That is, $x'_i = kx_i$ for each $i \in \{1, \ldots, n\}$.

(*ii*) Let $v_1 = (x_{11}, x_{21}, \ldots, x_{n1})$, $\ldots$, $v_n = (x_{1n}, x_{2n}, \ldots, x_{nn}) \in K^n$. The vectors $v_1, \ldots, v_n$ are linearly dependent if and only if $\exists k_1, \ldots, k_n \in K$ not all zero such that

$$k_1 v_1 + \cdots + k_n v_n = 0\,.$$

But this is equivalent to

$$k_1(x_{11}, x_{21}, \ldots, x_{n1}) + \cdots + k_n(x_{1n}, x_{2n}, \ldots, x_{nn}) = (0, \ldots, 0)\,,$$

and further to

$$\begin{cases} k_1 x_{11} + k_2 x_{12} + \cdots + k_n x_{1n} = 0 \\ k_1 x_{21} + k_2 x_{22} + \cdots + k_n x_{2n} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ k_1 x_{n1} + k_2 x_{n2} + \cdots + k_n x_{nn} = 0\,. \end{cases}$$

We are interested in the existence of a non-zero solution for this homogeneous linear system. We will see later on that such a solution does exist if and only if the determinant of the system is zero. $\square$

Remark 2.7.3 We are going to see that a vector space may have more than one basis.

Let us give now a characterization theorem for a basis of a vector space.

Theorem 2.7.4 *Let V be a vector space over K. A list $B = (v_1, \ldots, v_n)$ of vectors in V is a basis of V if and only if every vector $v \in V$ can be uniquely written as a linear combination of the vectors $v_1, \ldots, v_n$, that is,*

$$v = k_1 v_1 + \cdots + k_n v_n$$

for some unique $k_1, \ldots, k_n \in K$.

Proof. $\boxed{\Longrightarrow}$ Assume that B is a basis of V. Hence B is linearly independent and $\langle B \rangle = V$. The second condition assures us that every vector $v \in V$ can be written as a linear combination of the vectors of B. Suppose now that $v = k_1v_1 + \cdots + k_nv_n$ and $v = k'_1v_1 + \cdots + k'_nv_n$ for some $k_1, \dots, k_n, k'_1, \dots, k'_n \in K$. It follows that

$$(k_1 - k'_1)v_1 + \cdots + (k_n - k'_n)v_n = 0\,.$$

By the linear independence of B, we must have $k_i = k'_i$ for each $i \in \{1, \dots, n\}$. Thus, we have proved the uniqueness of writing.

$\boxed{\Longleftarrow}$ Assume that every vector $v \in V$ can be uniquely written as a linear combination of the vectors of B. Then clearly, $V = \langle B\rangle$. For $k_1, \dots, k_n \in K$, we have by the uniqueness of writing

$$k_1v_1 + \cdots + k_nv_n = 0 \Longrightarrow k_1v_1 + \cdots + k_nv_n = 0 \cdot v_1 + \cdots + 0 \cdot v_n \Longrightarrow$$

$$\Longrightarrow k_1 = \cdots = k_n = 0\,,$$

hence B is linearly independent. Consequently, B is a basis of V. $\square$