$a x'' + b x' + c = 0, \qquad x = x(t)$

$a r^2 + b r + c = 0$

I. $r_1 \pm r_2 \in \mathbb{R}$

$x(t) = c_1 \cdot e^{r_1 t} + c_2 \cdot e^{r_2 t}$

II. $r = r_1 = r_2 \in \mathbb{R}$ * we expect 2 linearly independent solutions.

$(r - r_1)(r - r_2) = (r - r_1)^2$

$$\left.\begin{aligned} x_1(t) &= e^{rt} \\ x_2(t) &= t \cdot e^{rt} \end{aligned}\right\} \text{linearly independent solutions}$$

$X(t) = c_1 \cdot e^{rt} + c_2 t e^{rt}$

$x_2(t) = t \cdot x_1(t)$

$x_2'(t) = x_1(t) + t x_1'(t)$

$x_2''(t) = x_1'(t) + x_1'(t) + t \cdot x_1''(t) =$

$= 2 x_1'(t) + t x_1''(t)$

$a\left(2 x_1'(t) + t x_1''(t)\right) + b \cdot \left(x_1(t) + t x_1'(t)\right) + c \cdot t \cdot x_1(t) = 0$

$t\underbrace{\left(a \cdot x_1''(t) + b \cdot x_1' + c \cdot x_1\right)}_{=0} + 2a x_1'(t) + b x_1(t) = 0$

(since $x_1(t)$ is a solution of the system)

$\Rightarrow \left(2a x_1'(t) + b x_2(t)\right)' = 0$

$\left((r - r_1)^2\right)' = 0 \Rightarrow 2(r - r_1) = 0$

III $r_1, r_2 \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{cases} r_1 = \alpha + i\beta \\ r_2 = \alpha - i\beta \end{cases}$$

$\Rightarrow x_1(t) = e^{\alpha t}(c_1 \cdot \sin \beta t + c_2 \cdot \cos \beta t)$

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1. a) $x' + 6x = 0$

$r + 6 = 0 \Rightarrow r = -6 \Rightarrow x(t) = c_1 \cdot e^{-6t}, t \in \mathbb{R}$

b) $x'' + 4x' + 4x = 0$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \Rightarrow r_{1,2} = -2$

$x_1(t) = e^{-2t}$

$x_2(t) = t e^{-2t}$

$\Rightarrow x(t) = c_1 e^{-2t} + c_2 \cdot t \cdot e^{-2t}$

c) $x^{(4)} - x = 0$ $\qquad$ $e^{it} = \cos t + i \sin t$

$(r^4 - 1) = 0$

$(r^2-1)(r^2+1) = 0$

$r_{1,2} = \pm 1 \Rightarrow \begin{cases} x_1(t) = e^{t} \\ x_2(t) = e^{-t} \end{cases}$

$r_{3,4} = \pm i = 0 \pm 1 \cdot i \Rightarrow \begin{cases} x_3(t) = e^{0 \cdot t}(\sin t) \\ x_4(t) = e^{0 t}(\cos t) \end{cases}$

$\Rightarrow x(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t) + c_4 x_4(t)$

2. Find the linear homogenous ode with constant coefficients and of <u>minimal</u> order s.t. the given functions are solutions.

a) $e^{-3t}, e^{5t}$

the order is 2 $\Rightarrow$ $ax'' + bx' + cx = 0$

$ar^2 + br + c = 0$

$\left.\begin{matrix} r_1 = -3 \\ r_2 = 5 \end{matrix}\right\} \Rightarrow \underbrace{(r - r_1)(r - r_2)}_{\text{char. poly}} = 0$

$\Rightarrow (r+3)(r-5) = 0$

$r^2 - 2r - 15 = 0$

$x'' - 2x' - 15x = 0$

b) $2 \cdot e^{-3t} + 3e^{5t}$

$ax'' + bx' + cx = 0$

$ar^2 + br + c = 0$

if this is a solution $\nLeftarrow$ $e^{-3t}, e^{5t}$ are lin. indep. $\Rightarrow$ $e^{-3t}, e^{5t}$ are solutions

c) $ax'' + bx' + cx = d$

$x(t) = x_{\text{homogen}}(t) + x_{\text{particular}}(t)$

3. Find solutions (if exists) of

a) $\begin{cases} x''(t) + x(t) = 0 \\ x(0) = x(\pi) = 0 \end{cases}$

pol. charact.: $r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i$

$\Rightarrow x(t) = c_1 \sin t + c_2 \cos t$

but $x(0)=0$

$x(0) = c_1 \cdot 0 + c_2 \cos 0 \Rightarrow c_2 = 0$

$x(\pi) = c_1 \cdot 0 + c_2 \cdot \cos \pi \Rightarrow c_2 = 0$

$\Rightarrow \quad x(t) = c_1 \sin t, \quad t \in \mathbb{R} \qquad$ * homogeneous solution

b) $\begin{cases} x''(t) + x(t) = 1 \\ x(0) = x(\pi) = 0 \end{cases}$ * particular equation

$x(t) = c_1 \sin t + c_2 \cos t + x_{particular}(t)$

we take $x(t) = 1$

$$\frac{x''(t) = 0}{x''(t) + x(t) = 1} \quad (1)$$

$\Rightarrow x(t) = c_1 \sin t + c_2 \cos t + 1$

$\begin{aligned} x(0) &= c_1 \cdot 0 + c_2 + 1 = c_2 + 1 = 0 \\ x(\pi) &= c_1 \cdot 0 - c_2 + 1 = -c_2 + 1 = 0 \end{aligned} \Bigg\} \Rightarrow \begin{cases} c_2 = -1 \\ c_2 = 1 \end{cases}$ impossible $\Rightarrow \nexists$ sol.

4. Find $\lambda$ such that $x'' + \lambda \cdot x = 0$ has a $2\pi$ periodic solution.

pol ch. $\quad r^2 + \lambda = 0$

1. $\lambda < 0 \Rightarrow r^2 = -\lambda = |\lambda|$

$r^2 = |\lambda| \Rightarrow r_{1,2} = \pm \sqrt{|\lambda|} \in \mathbb{R}$

$x(t) = c_1 e^{\sqrt{|\lambda|} t} + c_2 e^{-\sqrt{|\lambda|} t}$ — non-periodic

2. $\lambda > 0 \Rightarrow r^2 = -\lambda \Rightarrow r_{1,2} = \pm i \sqrt{\lambda}$

$x(t) = c_1 \sin \sqrt{\lambda} t + c_2 \cos \sqrt{\lambda} t$

$x(t) = \sin \sqrt{\lambda} t$

$x(2\pi + t) = x(t) \;?$

$x(2\pi + t) = \sin(2\pi + \sqrt{t})t = \sin(\sqrt{\lambda} \cdot 2\pi + \sqrt{\lambda} t)$

$\sin(\sqrt{\lambda}\, 2\pi + \sqrt{\lambda} t) = \sin(\sqrt{\lambda} t)$

$\sqrt{\lambda}\, \cancel{2\pi} = k \cancel{2\pi}, \; k \in \mathbb{N}$

$\lambda = k^2$

5. Find the solutions of

$\begin{cases} x'' + \pi^2 x = 0 \\ x(0) = 0 \\ x'(0) = \mu \end{cases}$

$x^2 + \pi^2 = 0$

$x^2 = -\pi^2 \Rightarrow x_{1,2} = \pm i \cdot \pi$

$x(t) = c_1 \sin \pi t + c_2 \cos \pi t$

$\Rightarrow \quad x(0) = c_1 \cdot \sin \pi \cdot 0 + c_2 \cos \pi \cdot 0 = c_2 = 0$

$x'(0) = c_1 \pi \cdot \cos \pi t - c_2 \cdot \pi \cdot \sin \pi t =$

$= c_1 \pi \cdot 1 - c_2 \cdot \pi \cdot 0 = c_1 \pi = \mu \Rightarrow \quad c_1 = \frac{\mu}{\pi}$

$x(t) = \frac{\mu}{\pi} \sin \pi t$