Seminar 1

1. Find the lower and the upper bounds, then sup, inf, max, min for each of the following:

   (a) $[-3, 2) \cup \{3\}$.
   (b) $(-1, 1] \cup (2, \infty)$.
   (c) $(-5, 5) \cap \mathbb{Z}$.
   ★(d) $\emptyset$.

2. Find the sup, inf, max, min for each of the following sets:

   (a) $\{x \in \mathbb{Q} \mid x^2 < 3\}$.
   (b) $\{x^2 - 4x + 3 \mid x \in \mathbb{R}\}$.
   (c) $\{\frac{n}{n+1} \mid n \in \mathbb{N}\}$.
   (d) $\{2^{-k} + 3^{-m} \mid k, m \in \mathbb{N}\}$.

3. Suppose that S is nonempty and bounded above. Show that the set $-S := \{-x \mid x \in S\}$ is bounded below and $\inf(-S) = -\sup(S)$.

4. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two functions defined on a nonempty set D. Prove that

$$★ \inf_{x\in D} \big(f(x) + g(x)\big) \geq \inf_{x\in D} f(x) + \inf_{x\in D} g(x) \quad \text{and} \quad \sup_{x\in D} \big(f(x) + g(x)\big) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x).$$

   Give examples where the above inequalities are strict.

5. ★ Let $a, b \in \mathbb{R}$ with $a > 0$. If S is nonempty and bounded above, prove that

$$\sup_{x\in S}(ax + b) = a \sup(S) + b.$$

6. Which of the following sets are neighborhoods of 0?

$$[-1, 1] \cup \{2\}; \quad (-1, 1) \cap \mathbb{Q}; \quad \bigcap_{n=1}^{\infty} [-\frac{1}{n}, \frac{1}{n}].$$

7. Let $x \in \mathbb{R}$ and $U, V \in \mathcal{V}(x)$. Prove that $U \cap V \in \mathcal{V}(x)$.

HW ★8. Let $a, b \in \mathbb{R}$. Prove that there exist neighborhoods $U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t. $U \cap V = \emptyset$.
contra exemplu

9. Find the interior and the closure for each of the following sets:

   (a) $(1, 2]$.
   (b) $[-3, 2) \cup \{3\}$.
   ★(c) $(-1, 1] \cup (2, \infty)$.
   ★(d) $(-5, 5) \cap \mathbb{Z}$.

10. ★ Let $A = (0, 1) \cap \mathbb{Q}$. Show that $\inf A = 0$, $\sup A = 1$, $\mathrm{int}A = \emptyset$ and $\mathrm{cl}A = [0, 1]$.

Homework questions are marked with ★.
Solutions should be uploaded on Teams before the next lecture.

 deadline Thu 12AM

1. Find the lower and the upper bounds, then sup, inf, max, min for each of the following:

(a) $[-3,2) \cup \{3\}$.

(b) $(-1,1] \cup (2,\infty)$.

(c) $(-5,5) \cap \mathbb{Z}$.

(d) $\emptyset$.

(a) $A = [-3,2) \cup \{3\}$

$lb(A) = (-\infty,-3]$, $\inf(A) = -3$, $\max = 3$

$ub(A) = [3,+\infty)$, $\sup(A) = 3$, $\min = -3$

(b) $(-1,1] \cup (2,\infty)$ $\to$ set bounded from below, unbounded from above

$lb(A) = (-\infty,-1]$ $\inf(A) = -1$ $\nexists \min(A), \max(A)$

$ub(A) = \emptyset$ (in $\mathbb{R}$) $\sup(A) = \infty$

(c) $A = (-5,5) \cap \mathbb{Z} = \{-4,-3,-2,\dots,3,4\}$ discrete set (there's a finite nr. of them)

$lb(A) = (-\infty,-4]$ $\inf(A) = -4$ $\min(A) = -4$

$ub(A) = [4,+\infty)$ $\sup(A) = 4$ $\max(A) = 4$

(d) $A = \emptyset$ $x \in lb(A)$ iff $x \le a, \forall a \in A$

$lb(\emptyset) = ?$

Let $x \in \mathbb{R}$. Then $x \le a, \forall a \in A$.

If this is false, then $\underline{\exists a \in \emptyset}$ s.t. $x > a$ $\leftarrow$ contradiction

$\Rightarrow$ the statement is true

$lb(\emptyset) = \mathbb{R}$ $\boxed{\inf(A) = \infty}$ $\nexists \min(\emptyset), \max(\emptyset)$

$up(\emptyset) = \mathbb{R}$ $\boxed{\sup(A) = -\infty}$

2. Find the sup, inf, max, min for each of the following sets:

(a) $\{x \in \mathbb{Q} \mid x^2 < 3\}$.

(b) $\{x^2 - 4x + 3 \mid x \in \mathbb{R}\}$.

(c) $\{\frac{n}{n+1} \mid n \in \mathbb{N}\}$.

(d) $\{2^{-k} + 3^{-m} \mid k, m \in \mathbb{N}\}$.

(a) $\{x \in \mathbb{Q} \mid x^2 < 3\} \Rightarrow A = (-\sqrt{3}, \sqrt{3}) \cap \mathbb{Q}$

$\sup(A) = \sqrt{3}$ $\min(A) = \emptyset$

$\inf(A) = -\sqrt{3}$ $\max(A) = \emptyset$

(b) $\{x^2 - 4x + 3 \mid x \in \mathbb{R}\}$ $\sup(A) = \emptyset$ $\max(A) = \emptyset$

$x^2 - 3x - x + 3 = (x-3)(x-1)$ $\inf(A) = -1$ $\min(A) = -1$

$\min(A) = f(2) = 4 - 8 + 3 = -1 \Rightarrow A = [-1,\infty)$

(c) $A = \left\{ \frac{m}{m+1} \mid m \in \mathbb{N} \right\} = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \dots \right\}$

$\sup(A) = 1$ $\qquad$ $\max(A) = \emptyset$ $\qquad$ $1-\varepsilon$, $\underset{\frac{m}{m+1}}{\uparrow}$, 1 vecinity

$\inf(A) = 0$ $\qquad$ $\min(A) = 0$

(d) $A = \left\{ 2^{-k} + 3^{-m} \mid k, m \in \mathbb{N} \right\} = \left\{ \left(\frac{1}{2}\right)^k + \left(\frac{1}{3}\right)^m \mid k, m \in \mathbb{N} \right\}$

$\inf(A) = 0$ $\qquad$ $\nexists \min(A)$

$\sup(A) = 2 = \max(A)$ (k, m must be the smallest possible)

3. Suppose that S is nonempty and bounded above. Show that the set $-S := \{-x \mid x \in S\}$ is bounded below and $\inf(-S) = -\sup(S)$.

S bounded above $\Rightarrow \exists \sup(S)$

- $\sup(S) \geq a, \forall a \in S$
- if $u \geq a, \forall a \in S$, then $\sup(S) \leq u$

Prove that $\inf(-S) \overset{(*)}{=} -\sup(S)$, $-S := \{-x \mid x \in S\}$

$-\sup(S) \leq -a, \forall a \in S \Rightarrow -S$ is bounded below by $-\sup(S)$

$\Rightarrow -\sup(S) \in lb(-S)$

- Let $u \in lb(-S)$, $u \leq -a, \forall a \in S$, $-u \geq a \; \forall a \in S \Rightarrow -u \geq \sup(S) \Rightarrow$

$\Rightarrow \boxed{u \leq -\sup(S)}$

$\Rightarrow (*) \quad \inf(-S) = -\sup(S)$

4. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two functions defined on a nonempty set D. Prove that

$$\inf_{x \in D} \left(f(x) + g(x)\right) \geq \inf_{x \in D} f(x) + \inf_{x \in D} g(x) \quad \text{and} \quad \sup_{x \in D} \left(f(x) + g(x)\right) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x).$$

Give examples where the above inequalities are strict.

$f, g : D \to \mathbb{R}$, $\sup\limits_{x \in D} \big(\underbrace{f(x) + g(x)}_{h(x)}\big) \leq \sup\limits_{x \in D} f(x) + \sup\limits_{x \in D} g(x)$

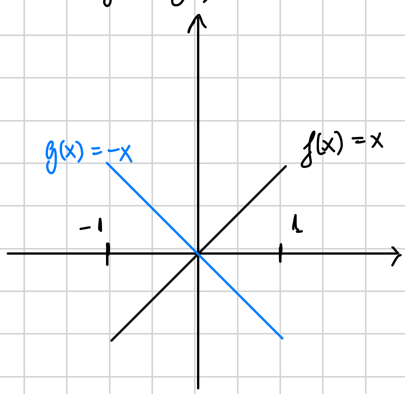
Let $F = \sup\limits_{x \in D} (f(x))$ and $G = \sup\limits_{x \in D} (g(x))$ and $h(x) = f(x) + g(x)$

$f(x) \leq F$

$g(x) \leq G$ $\quad (+)$

$h(x) \leq F + G \Rightarrow \sup\limits_{x \in D} h(x) \leq F + G$

$\sup\limits_{x \in D} h(x) < \sup\limits_{x \in D} f(x) + \sup\limits_{x \in D} g(x)$



$\Rightarrow \sup f(x) = 1$

$\sup g(x) = 1$

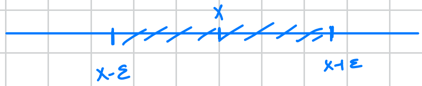
$\Rightarrow \sup h = 0 \Rightarrow$

$\sup h(x) < \sup f(x) + \sup g(x)$

6. Which of the following sets are neighborhoods of 0?

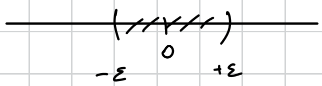
$$[-1,1]\cup\{2\};\quad (-1,1)\cap\mathbb{Q};\quad \bigcap_{n=1}^{\infty}[-\frac{1}{n},\frac{1}{n}].$$

$V \in \mathcal{V}(x)$ iff $\exists\, \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq V$



- $[-1,1] \cup \{2\} \in \mathcal{V}(0)$
- $(-1,1) \cap \mathbb{Q} \notin \mathcal{V}(0)$

$A \subseteq B$ $\forall a \in A$, $a \in B$



$(-\varepsilon, \varepsilon) \overset{?}{\not\subseteq} (-1,1) \cap \mathbb{Q}$

$\uparrow$

continuous irrational numbers

- $\bigcap_{n=1}^{\infty} [-\frac{1}{n}, \frac{1}{n}] \longrightarrow$

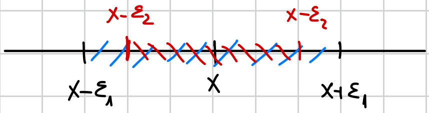
$\bigcap_{n=1}^{\infty} [-\frac{1}{n}, \frac{1}{n}] = \{0\} \notin \mathcal{V}(0)$

7. Let $x \in \mathbb{R}$ and $U, V \in \mathcal{V}(x)$. Prove that $U \cap V \in \mathcal{V}(x)$.

$U, V \in \mathcal{V}(x) \overset{?}{\Rightarrow} U \cap V \in \mathcal{V}(x)$

$\exists\, \varepsilon_1 > 0$ s.t. $(x-\varepsilon_1, x+\varepsilon_1) \subseteq U$

$\exists\, \varepsilon_2 > 0$ s.t. $(x-\varepsilon_2, x+\varepsilon_2) \subseteq V$



Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. Then

$(x-\varepsilon, x+\varepsilon) \subseteq U \cap V$

Hence $U \cap V \in \mathcal{V}(x)$

9. Find the interior and the closure for each of the following sets:

(a) $(1,2]$.

(b) $[-3,2)\cup\{3\}$.

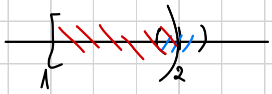
(c) $(-1,1]\cup(2,\infty)$.

(d) $(-5,5)\cap\mathbb{Z}$.

$\text{int}(A) = \{x \in \mathbb{R} \mid \exists\, V \in \mathcal{V}(x),\ V \subseteq A\}$

$\text{cl}(A) = \{x \in \mathbb{R} \mid \forall\, V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}$

(a) $\text{int}(A) = (1,2)$ $\quad$ $\text{cl}(A) = [1,2]$



(b) $[-3,2) \cup \{3\}$

$\text{int}(A) = (-3,2)$ $\quad$ $\text{cl}(A) = [-3,2] \cup \{3\}$

(c)

5. ★ Let $a, b \in \mathbb{R}$ with $a > 0$. If S is nonempty and bounded above, prove that

$$\sup_{x\in S}(ax+b) = a\sup(S) + b.$$

$\exists\ ub(S)$, $S \neq \emptyset$ $\quad a,b \in \mathbb{R}\quad a>0 \overset{?}{\Rightarrow} \sup\limits_{x\in S}(ax+b) = a\sup(S)+b$

Let $y := \sup(S) \Rightarrow y \geq a\ \forall a\in S \Rightarrow y \in ub(S)$

~~if u is an upper bound for A, then $y \leq u$~~

$x \leq y \quad \forall x \in S$

$x \leq y \quad |\cdot a \neq 0$

$ax \leq ay \quad |+b$

$ax+b \leq ay+b$

$ax+b \leq a\sup(S)+b$

Let u be any upperbound

$y \leq u \quad |\cdot a$

$ay \leq au \quad |+b$

$ay+b \leq au+b$

$ax+b \leq a\sup(S)+b \leq a\cdot u+b$

$\sup(ax+b) = a\sup(S)+b$

Let $f(x) = ax+b \Rightarrow f'(x) = \underline{a}$

$$\sup_{x\in D}\big(f(x)+g(x)\big) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x).$$

based on this relation proved during seminar:

let us take $\left.\begin{array}{ll} f(x) = ax, & a\in\mathbb{R} \\ g(x) = b, & b\in\mathbb{R} \end{array}\right\} \Rightarrow \sup(ax+b) \leq \sup(ax) + \sup(b)$

$\sup f(x) = a\cdot\sup(S)$ since a is a constant and $x\in S$

$\sup g(x) = b$ since b is a constant

$\Rightarrow \sup(ax+b) \leq a\sup(S)+b$

5. ★ Let $a, b \in \mathbb{R}$ with $a > 0$. If S is nonempty and bounded above, prove that

$$\sup_{x \in S}(ax + b) = a \sup(S) + b.$$

I. Let $y := \sup(S) \Rightarrow y \geq s \ \forall s \in S \Rightarrow y \in ub(S)$

$x \leq y \quad \forall x \in S$

$x \leq y \ | \cdot a \neq 0$

$ax \leq ay \ | + b$

$ax + b \leq ay + b$

$ax + b \leq a \sup(S) + b \quad (1) \Rightarrow a \cdot \sup(S) + b$ is an upperbound for $\{ax + b \mid x \in S\}$

Let $u \in ub(ax + b)$ (any upperbound of the set $ax + b$)

$ax + b \leq u \ | - b$

$ax \leq u - b \ | : a \neq 0$

$x \leq \frac{u - b}{a}$ - bounded above

$\sup(S) \leq \frac{u-b}{a} \Rightarrow \quad a \sup(S) + b \leq u \quad (2)$

Using (1) and (2) $\Rightarrow$ $a \sup(S) + b$ is the least upperbound for $\{ax + b \mid x \in S\} \Rightarrow$

$\Rightarrow \sup(ax + b) = a \sup(S) + b$

II.
$$\sup_{x \in D}\big(f(x) + g(x)\big) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x).$$

based on this relation proved during seminar:

let us take $\left.\begin{array}{ll} f(x) = ax & , a \in \mathbb{R} \\ g(x) = b & , b \in \mathbb{R} \end{array}\right\} \Rightarrow \sup(ax + b) \leq \sup(ax) + \sup(b)$

$\sup f(x) = a \cdot \sup(S)$ since a is a constant and $x \in S$

$\sup g(x) = b$ since b is a constant

$\Rightarrow \sup(ax + b) \leq a \sup(S) + b$

★8. Let $a, b \in \mathbb{R}$. Prove that there exist neighborhoods $U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t. $U \cap V = \emptyset$.

contra exemplu

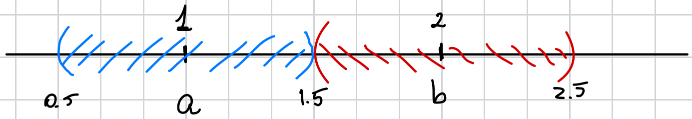
Assume that $\nexists\ U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t. $U \cap V = \emptyset$

$\exists\ \varepsilon > 0$ s.t. $(a-\varepsilon, a+\varepsilon) \subseteq U$

$\exists\ \varepsilon > 0$ s.t. $(b-\varepsilon, b+\varepsilon) \subseteq V$

Let us take $a = 1$, $b = 2$, $\varepsilon_1 = 0.5$

$\varepsilon_1 = 0.5$



Given if $\left.\begin{array}{l}(\frac{1}{2}, \frac{3}{2}) \in \mathcal{V}(1) \\ (\frac{3}{2}, \frac{5}{2}) \in \mathcal{V}_2\end{array}\right\} \Rightarrow (\frac{1}{2}, \frac{3}{2}) \cap (\frac{3}{2}, \frac{5}{2}) = \emptyset$

$\Rightarrow$ For any $\varepsilon > \frac{a+b}{2}$ $U \cap V \neq \emptyset$, but for any $\varepsilon \leq \frac{a+b}{2} \Rightarrow U \cap V = \emptyset$

10. ★ Let $A = (0,1) \cap \mathbb{Q}$. Show that $\inf A = 0$, $\sup A = 1$, $\text{int}A = \emptyset$ and $\text{cl}A = [0,1]$.

$A = (0,1) \cap \mathbb{Q}$

$lb(A) = (-\infty, 0]$

$ub(A) = [1, +\infty)$

1) $x = \inf(A) = 0 \leq a\ \forall a \in A$, $x \in lb(A)$

$x \geq u,\ \forall u \in lb(A)$

2) $y = \sup(A) = 1 \geq a\ \forall a \in A$, $y \in ub(A)$

$y \leq u,\ \forall u \in ub(A)$

3) $\text{int}\, A = \emptyset$

$\text{int}(A) := \{x \in \mathbb{R} \mid \exists\ V \in \mathcal{V}(x) \text{ s.t. } V \subseteq A\}$

$A = (0,1) \cap \mathbb{Q} \Rightarrow \nexists\ V \in \mathcal{V}(x), x \in A$, s.t. $V \subseteq A$

4) $cl(A) = [0,1]$

$cl(A) := \{x \in \mathbb{R} \mid \forall\ V \in \mathcal{V}(x), V \cap A \neq 0\}$

$A = (0,1) \cap \mathbb{Q} \Rightarrow [0,1] \cap (0,1) \cap \mathbb{Q} \neq \emptyset$

Redo for midterm prep

Seminar 1

1. Find the lower and the upper bounds, then sup, inf, max, min for each of the following:
   (a) $[-3, 2) \cup \{3\}$.
   (b) $(-1, 1] \cup (2, \infty)$.
   (c) $(-5, 5) \cap \mathbb{Z}$.
   (d) $\emptyset$.

2. Find the sup, inf, max, min for each of the following sets:
   (a) $\{x \in \mathbb{Q} \mid x^2 < 3\}$.
   (b) $\{x^2 - 4x + 3 \mid x \in \mathbb{R}\}$.
   (c) $\{\frac{n}{n+1} \mid n \in \mathbb{N}\}$.
   (d) $\{2^{-k} + 3^{-m} \mid k, m \in \mathbb{N}\}$.

3. Suppose that S is nonempty and bounded above. Show that the set $-S := \{-x \mid x \in S\}$ is bounded below and $\inf(-S) = -\sup(S)$.

4. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two functions defined on a nonempty set D. Prove that
$$\inf_{x\in D} \big(f(x) + g(x)\big) \geq \inf_{x\in D} f(x) + \inf_{x\in D} g(x) \quad \text{and} \quad \sup_{x\in D} \big(f(x) + g(x)\big) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x).$$
   Give examples where the above inequalities are strict.

5. ★ Let $a, b \in \mathbb{R}$ with $a > 0$. If S is nonempty and bounded above, prove that
$$\sup_{x\in S}(ax + b) = a \sup(S) + b.$$

6. Which of the following sets are neighborhoods of 0?
$$[-1, 1] \cup \{2\}; \quad (-1, 1) \cap \mathbb{Q}; \quad \bigcap_{n=1}^{\infty} [-\frac{1}{n}, \frac{1}{n}].$$

7. Let $x \in \mathbb{R}$ and $U, V \in \mathcal{V}(x)$. Prove that $U \cap V \in \mathcal{V}(x)$.

8. Let $a, b \in \mathbb{R}$. Prove that there exist neighborhoods $U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t. $U \cap V = \emptyset$.

9. Find the interior and the closure for each of the following sets:
   (a) $(1, 2]$.
   (b) $[-3, 2) \cup \{3\}$.
   (c) $(-1, 1] \cup (2, \infty)$.
   (d) $(-5, 5) \cap \mathbb{Z}$.

10. ★ Let $A = (0, 1) \cap \mathbb{Q}$. Show that $\inf A = 0$, $\sup A = 1$, $\mathrm{int}A = \emptyset$ and $\mathrm{cl}A = [0, 1]$.

Homework questions are marked with ★.
Solutions should be uploaded on Teams before the next lecture.

1. Find the lower and the upper bounds, then sup, inf, max, min for each of the following:

(a) $[-3, 2) \cup \{3\}$.

(b) $(-1, 1] \cup (2, \infty)$.

(c) $(-5, 5) \cap \mathbb{Z}$.

(d) $\emptyset$.

(a) $A = [-3,2) \cup \{3\}$

$lb(A) = (-\infty, -3] \Rightarrow \inf(A) = -3$ $\qquad \max(A) = 3$

$ub(A) = [3, +\infty) \Rightarrow \sup(A) = 3$ $\qquad \min(A) = -3$

(b) $A = (-1, 1] \cup (2, \infty)$

$lb(A) = (-\infty, -1] \Rightarrow \inf(A) = -1 \Rightarrow \nexists \min(A)$

$ub(A) = \emptyset$ $\qquad$ ~~$\nexists \sup(A)$~~ $\qquad \Rightarrow \nexists \max(A)$

$\sup(A) = +\infty$

(c) $A = (-5,5) \cap \mathbb{Z} = \{-4, -3, -2, \dots, 3, 4\}$

$lb(A) = (-\infty, -4]$ $\Rightarrow$ $\inf(A) = -4$ $\Rightarrow$ $\min(A) = -4$

$ub(A) = [4, +\infty)$ $\qquad \sup(A) = 4$ $\qquad \max(A) = 4$

(d) $A = \emptyset$

$lb(A) = (-\infty, \infty)$ $\qquad \inf(A) = -\infty$

$ub(A) = (-\infty, \infty)$ $\qquad \sup(A) = \infty$ $\qquad \min(A) = \max(A) = \emptyset$

2. Find the sup, inf, max, min for each of the following sets:

(a) $\{x \in \mathbb{Q} \mid x^2 < 3\}$.

(b) $\{x^2 - 4x + 3 \mid x \in \mathbb{R}\}$.

(c) $\{\frac{n}{n+1} \mid n \in \mathbb{N}\}$.

(d) $\{2^{-k} + 3^{-m} \mid k, m \in \mathbb{N}\}$.

(a) $\{x \in \mathbb{Q} \mid x^2 < 3\} \Rightarrow A = (-\sqrt{3}, \sqrt{3}) \cap \mathbb{Q}$

$\sup(A) = \sqrt{3}$ $\qquad \max(A) = \min(A) = \emptyset$

$\inf(A) = -\sqrt{3}$

(b) $\{x^2 - 4x + 3 \mid x \in \mathbb{R}\} = \{x^2 - 3x - x + 3 \mid x \in \mathbb{R}\} = \{(x-1)(x-3) \mid x \in \mathbb{R}\}$

let $f(x) = (x-1)(x-3)$ $\qquad f'(x) = 2x - 4$

x	$-\infty$		2		$+\infty$
$f'(x)$		$-\ -\ -\ -\ -$	0	$+\ +\ +\ +\ +$	
$f(x)$	$+\infty$	$\searrow$		$\nearrow$	$+\infty$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (x^2 - 4x + 3) = +\infty$

$f(2) = 1 \cdot (-1) = -1 \Rightarrow \min(A) = f(2) = -1 \Rightarrow A = [-1, +\infty) \Rightarrow \sup(A) = \emptyset$ $\qquad \max(A) = \emptyset$

$\inf(A) = -1$ $\qquad \min(A) = -1$

(c) $\left\{ \frac{n}{n+1} \;\middle|\; n \in \mathbb{N} \right\} = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\}$

$\inf(A) = 0$ $\quad \min(A) = 0$

$\sup(A) = 1$ $\quad \nexists \max(A)$

$1-\varepsilon, \frac{n}{n+1}, 1$ vecinity

(d) $\left\{ 2^{-k} + 3^{-m} \;\middle|\; k, m \in \mathbb{N} \right\} = \left\{ \left(\frac{1}{2}\right)^k + \left(\frac{1}{3}\right)^m \;\middle|\; k, m \in \mathbb{N} \right\}$

$\inf(A) = 0$ $\quad \nexists \min(A)$

$\sup(A) = 2$ $\quad \max(A) = 2$?

3. Suppose that S is nonempty and bounded above. Show that the set $-S := \{-x \mid x \in S\}$ is bounded below and $\inf(-S) = -\sup(S)$.

$S \neq \emptyset$, $\exists$ up$(S) \neq 0 \Rightarrow \exists \sup(A)$

$\begin{cases} \sup(S) \geq a \;\; \forall a \in S \\ \text{if } u \in up(S),\; u \geq a \; \forall a \in S \Rightarrow \sup(S) \leq u \end{cases}$

$\inf(-S) \overset{?}{=} -\sup(S)$

$\sup(S) \geq a \quad \mid \cdot(-1)$

$-\sup(S) \leq -a \;\; \forall a \in S \Rightarrow -S$ is bounded below by $-\sup(S)$

$\Rightarrow -\sup(S) \in lb(-S)$ (1)

let $l \in lb(S)$, $l \leq -a \; \forall a \in S$

$-l \geq a \;\; \forall a \in S \Rightarrow -l \in up(S) \Rightarrow -l \geq \sup(S) \quad \mid \cdot(-1)$

$l \leq -\sup(S)$ (2) $\overset{(1),(2)}{\Rightarrow} -\sup(S) = \inf(-S)$

4. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two functions defined on a nonempty set D. Prove that

(a) $\inf\limits_{x\in D} \big(f(x) + g(x)\big) \geq \inf\limits_{x\in D} f(x) + \inf\limits_{x\in D} g(x)$ and (b) $\sup\limits_{x\in D} \big(f(x) + g(x)\big) \leq \sup\limits_{x\in D} f(x) + \sup\limits_{x\in D} g(x)$.

Give examples where the above inequalities are strict.

$f, g : D \to \mathbb{R}$

(b) $\sup\limits_{x\in D} (f(x) + g(x)) \leq \sup\limits_{x\in D} f(x) + \sup\limits_{x\in D} g(x)$

Let $F = \sup\limits_{x\in D} (f(x))$ $\quad G = \sup\limits_{x\in D} (g(x))$ $\quad h(x) = g(x) + f(x)$

$F = \sup f(x) \Rightarrow f(x) \leq F \;\; \forall x \in D$

$g(x) \leq G \;\; \forall x \in D$ $\quad (+)$

$h(x) \leq F + G \;\; \forall x \in D \Rightarrow \sup\limits_{x\in D} h(x) \leq F + G$

$\Rightarrow \sup_{x\in D} (f(x)+g(x)) \leq \sup_{x\in D} f(x) + \sup_{x\in D} g(x)$

ex: $\left.\begin{array}{l} f(x) = -x \\ g(x) = x \\ D=[-1,1] \end{array}\right\} \Rightarrow \begin{array}{l} \sup f(x) = 1 \\ \sup g(x) = 1 \\ \text{but } \sup h(x) = 0 \end{array} \Rightarrow \sup(f(x)+g(x)) < \sup f(x) + \sup g(x)$

(a) Let $\inf f(x) = F$ and $\inf g(x) = G$ $\quad h(x) = f(x)+g(x)$

$$\begin{array}{l} \inf_{x\in D} f(x) \leq f(x) \\ \inf_{x\in D} g(x) \leq g(x) \\ \hline F+G \leq f(x)+g(x) \end{array} \quad (+)$$

$F+G \leq h(x) \quad \forall x\in D \Rightarrow \inf_{x\in D} h(x) \geq F+G$

$\inf(f(x)+g(x)) \geq \inf f(x) + \inf g(x), \forall x\in D$

ex: $\left.\begin{array}{l} f(x) = x \\ g(x) = -x \\ D=[-2,2] \end{array}\right\} \Rightarrow \begin{array}{l} \inf f(x) = -2 \\ \inf g(x) = -2 \\ \inf h(x) = 0 \end{array} \Rightarrow \inf_{x\in D}(f(x)+g(x)) > \inf_{x\in D} f(x) + \inf_{x\in D} g(x)$

5. ★ Let $a, b \in \mathbb{R}$ with $a > 0$. If S is nonempty and bounded above, prove that

$$\sup_{x\in S}(ax+b) = a\sup(S) + b.$$

Let $y := \sup(S) \Rightarrow y \geq s \ \forall s \in S \Rightarrow y \in ub(S)$

$x \leq y \quad \forall x \in S$

$ax \leq ay$

$ax+b \leq ay+b$

$ax+b \leq a\sup(S)+b \Rightarrow a\sup(S)+b \in up(ax+b)$

Let $u \in up(ax+b)$

$ax+b \leq u \ |-b$

$ax \leq u-b \ |:a$

$x \leq \frac{u-b}{a}$ - bounded above

$\Rightarrow \sup(S) \leq \frac{u-b}{a} \Rightarrow a\sup(S)+b \leq u$

$\Rightarrow \sup(ax+b) = a\sup(S)+b$

6. Which of the following sets are neighborhoods of 0?

$$[-1,1]\cup\{2\};\quad (-1,1)\cap\mathbb{Q};\quad \bigcap_{n=1}^{\infty}[-\frac{1}{n},\frac{1}{n}] = \{0\} \notin \mathcal{V}(0)$$

$\in \mathcal{V}(0)$ $\qquad$ $\notin \mathcal{V}(0)$

7. Let $x\in\mathbb{R}$ and $U,V\in\mathcal{V}(x)$. Prove that $U\cap V\in\mathcal{V}(x)$.

$U, V \in \mathcal{V}(x)$

$\exists\ \varepsilon_1 > 0$ st. $(x-\varepsilon_1, x+\varepsilon_1) \subseteq U$

$\varepsilon_2 > 0$ st. $(x-\varepsilon_2, x+\varepsilon_2) \subseteq V$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$

$\underbrace{(x-\varepsilon, x+\varepsilon)}_{\in \mathcal{V}(x)} \subseteq U\cap V$ $\qquad$ Hence $U\cap V \in \mathcal{V}(x)$

8. Let $a,b\in\mathbb{R}$. Prove that there exist neighborhoods $U\in\mathcal{V}(a)$ and $V\in\mathcal{V}(b)$ s.t. $U\cap V=\emptyset$.

$\exists\ \varepsilon > 0$ st. $(a+\varepsilon, a-\varepsilon) \subseteq U$

$\varepsilon > 0$ $\qquad (b+\varepsilon, b-\varepsilon) \subseteq V$

For $a=1$ $\quad b=2$ $\quad \varepsilon = 0.5$

$(\frac{1}{2},\frac{3}{2}) \subseteq U$ $\qquad (\frac{1}{2},\frac{3}{2}) \cap (\frac{3}{2},\frac{5}{2}) = \emptyset$

$(\frac{3}{2},\frac{5}{2}) \subseteq V$

9. Find the interior and the closure for each of the following sets:

(a) $(1,2]$.

(b) $[-3,2)\cup\{3\}$.

(c) $(-1,1]\cup(2,\infty)$.

(d) $(-5,5)\cap\mathbb{Z}$.

(a) $\operatorname{int}(A) = (1,2)$

$\operatorname{cl}(A) = [1,2]$

(b) $\operatorname{Int}(B) = (-3,2)$

$\operatorname{cl}(B) = [-3,2]\cup\{3\}$

(c) $\operatorname{int}(C) = (-1,1)\cup(2,\infty)$

$\operatorname{cl}(C) = [-1,1]\cup[2,\infty)$

(d) $\operatorname{int}(D) = \emptyset$

$\operatorname{cl}(D) = [-5,5]$

10. ★ Let $A=(0,1)\cap\mathbb{Q}$. Show that $\inf A=0$, $\sup A=1$, $\operatorname{int}A=\emptyset$ and $\operatorname{cl}A=[0,1]$.

$lb(A) = (-\infty, 0] \Rightarrow \inf(A) = 0$

$ub(A) = [1, +\infty) \qquad \sup(A) = 1$

$\operatorname{int} A = \emptyset$

$\operatorname{int} A := \{x\in\mathbb{R} \mid \exists V \in \mathcal{V}(x) \text{ s.t. } V \subseteq A\}$

$A = (0,1)\cap\mathbb{Q} \Rightarrow \nexists\ V \in \mathcal{V}(x)$ s.t. $V \subseteq A$

$\operatorname{cl} A = [0,1]$

$\operatorname{cl} A := \{x\in\mathbb{R} \mid \forall\ V\in\mathcal{V}(x),\ V\cap A \neq \emptyset\}$

$((0,1)\cap\mathbb{Q}) \cap [0,1] \neq \emptyset$