1.4 Exercises

~~1.1.~~ Let $A_0, \ldots, A_n$ be the vertices of a polygon. Determine $\overrightarrow{A_0A_1} + \overrightarrow{A_1A_2} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_0}$.

~~1.2.~~ In each of the following cases, decide if the indicated vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ can be represented with the vertices of a triangle: * triangle exists iff $\vec{u}+\vec{v}+\vec{w}=\vec{0}$

a) $\mathbf{u}(7,3)$, $\mathbf{v}(2,8)$, $\mathbf{w}(-5,5)$. $\vec{u}(7,3)+\vec{v}(2,8)+\vec{w}(-5,5)=(0,0)$

b) $\mathbf{u}(1,2,-1)$, $\mathbf{v}(2,-1,0)$, $\mathbf{w}(1,-3,1)$.

~~1.3.~~ Let $ABCD$ be a quadrilateral. Let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that

$$\overrightarrow{MN} + \overrightarrow{PQ} = 0.$$

Deduce that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram.

~~1.4.~~ Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AC]$ and let F be the midpoint of $[BD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right) = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{CB}\right).$$

~~1.5.~~ Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AB]$ and let F be the midpoint of $[CD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{BC}\right).$$

~~1.6.~~ Let A', B' and C' be midpoints of the sides of a triangle ABC. Show that for any point O we have

$$\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

$$\left.\begin{aligned} \overrightarrow{OA'} &= \tfrac{1}{2}(\overrightarrow{OC}+\overrightarrow{OB}) \\ \overrightarrow{OB'} &= \tfrac{1}{2}(\overrightarrow{OA}+\overrightarrow{OC}) \\ \overrightarrow{OC'} &= \tfrac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB}) \end{aligned}\right\} \Rightarrow \overrightarrow{OA'}+\overrightarrow{OB'}+\overrightarrow{OC'} = \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$$

~~1.7.~~ Show that the medians in a triangle intersect in one point and deduce the ratio in which the common intersection point divides the medians.

~~1.8.~~ In each of the following cases, decide if the given points are collinear:
determine slope

a) $P(3,-5)$, $Q(-1,2)$, $R(-5,9)$.

b) $A(11,2)$, $B(1,-3)$, $C(31,13)$.

c) $P(1,0,-1)$, $Q(0,-1,2)$, $R(-1,-2,5)$.

d) $A(-1,-1,-4)$, $B(1,1,0)$, $C(2,2,2)$.

~~1.9.~~ Let $ABCD$ be a tetrahedron. Determine the sums

a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$, $=\overrightarrow{AD}$

b) $\overrightarrow{AD} + \overrightarrow{BC} + \overrightarrow{DB}$, $=\overrightarrow{AC}$

c) $\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{BC} + \overrightarrow{DA}$. $=\vec{0}$

~~1.10.~~ Let $ABCD$ be a tetrahedron. Show that

$$\overrightarrow{AD} + \overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{AC}.$$

$\overrightarrow{AD}+\overrightarrow{DB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$

~~1.11~~. Let $SABCD$ be a pyramid with apex S and base the parallelogram $ABCD$. Show that

$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}$$

where O is the center of the parallelogram.

1.12. Give the coordinates of the vertices of the parallelepiped whose faces lie in the coordinate planes and in the planes $x = 1$, $y = 3$ and $z = -2$.

1.13. In $\mathbb{E}^3$ consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.

1.14. Which of the following sets of vectors form a basis?

a) $\mathbf{v}(1,2)$, $\mathbf{w}(3,4)$;

b) $\mathbf{u}(-1,2,1)$, $\mathbf{v}(2,1,1)$, $\mathbf{w}(1,0,-1)$;

c) $\mathbf{u}(-1,2,1)$, $\mathbf{v}(2,1,1)$, $\mathbf{w}(0,5,3)$;

d) $\mathbf{v}_1(-1,2,1,0)$, $\mathbf{v}_2(2,1,1,0)$, $\mathbf{v}_3(1,0-1,1)$, $\mathbf{v}_4(1,0,0,1)$;

1.15. With respect to the basis $\mathcal{B} = (\mathbf{i},\mathbf{j},\mathbf{k})$ consider the vectors $\mathbf{u} = \mathbf{i}+\mathbf{j}$, $\mathbf{v} = \mathbf{j}+\mathbf{k}$ and $\mathbf{w} = \mathbf{i}+\mathbf{k}$. Check that $\mathcal{B}' = (\mathbf{u},\mathbf{v},\mathbf{w})$ is a basis and give the base change matrix $M_{\mathcal{B}',\mathcal{B}}$.

1.16. Consider the two coordinate systems $\mathcal{K} = (O,\mathbf{i},\mathbf{j})$ and $\mathcal{K}' = (O',\mathbf{i}',\mathbf{j}')$ given in Example 1.20. Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix}1\\2\end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix}2\\4\end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix}3\\1\end{bmatrix}$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.

1.17. Consider the tetrahedron $ABCD$ and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

a) the coordinates of the vertices of the tetrahedron in the three coordinate systems,

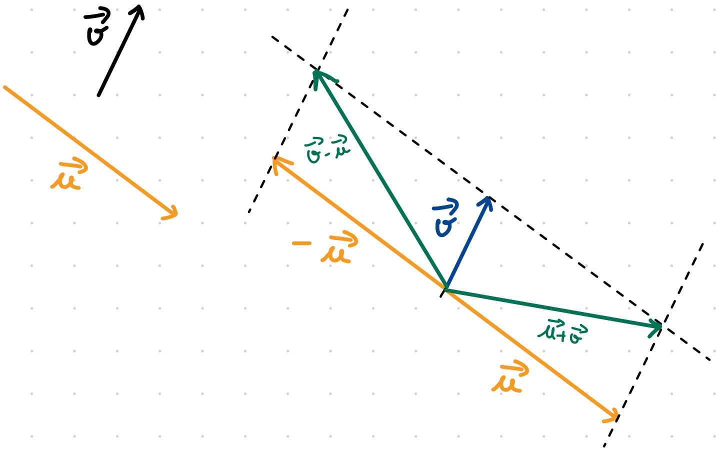
b) the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,

c) the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.

1.18. Consider the two coordinate systems $\mathcal{K} = (O,\mathbf{i},\mathbf{j},\mathbf{k})$ and $\mathcal{K}' = (O',\mathbf{i}',\mathbf{j}',\mathbf{k}')$ given in Example 1.21. Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix}1\\4\\-1\end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix}1\\5\\1\end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix}-3\\7\\1\end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix}0\\3\\1\end{bmatrix}$$

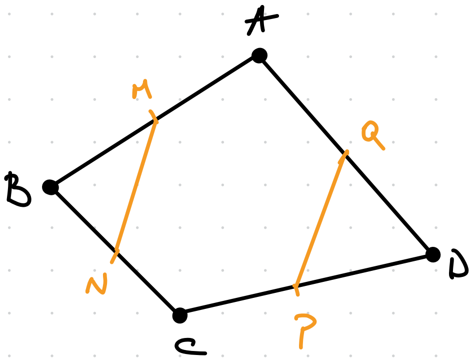
in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.



1.3. Let $ABCD$ be a quadrilateral. Let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that

$$\overrightarrow{MN} + \overrightarrow{PQ} = 0.$$

Deduce that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram.

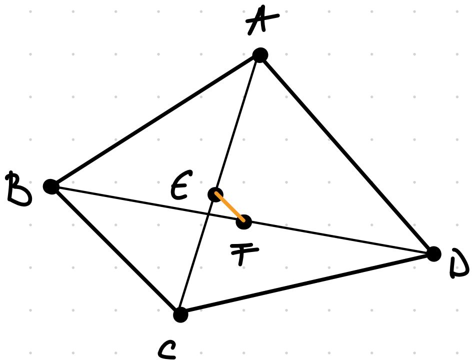


$$\overrightarrow{MN} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC})$$
$$\overrightarrow{PQ} = \frac{1}{2}(\overrightarrow{CD} + \overrightarrow{DA})$$

$$\Rightarrow \overrightarrow{MN} + \overrightarrow{PQ} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}) = \vec{0}$$

1.4. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AC]$ and let F be the midpoint of $[BD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right) = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{CB}\right).$$

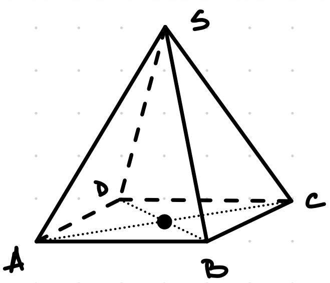


$$\begin{aligned}
\overrightarrow{EF} &= \overrightarrow{EC} + \overrightarrow{CD} + \overrightarrow{DF} \\
&= \frac{1}{2}\overrightarrow{AC} + \overrightarrow{CD} + \frac{1}{2}\overrightarrow{DB} \\
&= \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{CD}) + \frac{1}{2}(\overrightarrow{CD} + \overrightarrow{DB}) \\
&= \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}\overrightarrow{CB}
\end{aligned}$$

1.11. Let $SABCD$ be a pyramid with apex S and base the parallelogram $ABCD$. Show that

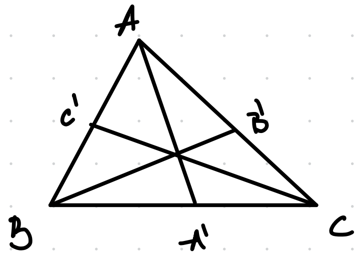
$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}$$

where O is the center of the parallelogram.



$$\overrightarrow{SO} = \overrightarrow{SA} + \overrightarrow{AO}$$
$$\vdots$$
$$(+)$$
$$4\overrightarrow{SO} = \overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} + \underbrace{\overrightarrow{AO} + \overrightarrow{CO}}_{\vec{0}} + \underbrace{\overrightarrow{BO} + \overrightarrow{DO}}_{\vec{0}}$$

? **1.7.** Show that the medians in a triangle intersect in one point and deduce the ratio in which the common intersection point divides the medians.



$$\vec{AA'} = \vec{r_{A'}} - \vec{r_A} = \frac{1}{2}(\vec{r_B} - \vec{r_C}) - \vec{r_A}$$

$$\vec{AG} = \vec{r_G} - \vec{r_A} = \frac{1}{3}(\vec{r_A} + \vec{r_B} + \vec{r_C}) - \vec{r_A}$$

$$\vec{AG} = \alpha \cdot \vec{AA'}$$

$$\Rightarrow -\frac{2}{3}\vec{r_A} + \frac{1}{3}\vec{r_B} + \frac{1}{3} r_C = \frac{\alpha}{2}\vec{r_B} - \frac{\alpha}{2}\vec{r_C} - \alpha\vec{r_A}$$

$\Rightarrow \alpha = \frac{2}{3}$ s.t. A, G, A' - collinear

A, B, C - col iff $\exists\, \alpha$ s.t. $\vec{AB} = \alpha \vec{BC}$

1.8. In each of the following cases, decide if the given points are collinear:

determine slope

a) $P(3,-5)$, $Q(-1,2)$, $R(-5,9)$.

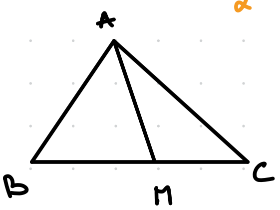
b) $A(11,2)$, $B(1,-3)$, $C(31,13)$.

c) $P(1,0,-1)$, $Q(0,-1,2)$, $R(-1,-2,5)$.

d) $A(-1,-1,-4)$, $B(1,1,0)$, $C(2,2,2)$.

a) $\vec{PQ} = \vec{r_Q} - \vec{r_P} = (-1,2) - (3,5) = (-4,7)$
$\vec{QR} = (-4,7)$
$\Rightarrow \vec{AB} = 1 \cdot \vec{BC} \Rightarrow$ colinear

* vector equation of a line: $\vec{r_T} = \lambda \vec{r_A} + (1-\lambda)\vec{r_B}$



$$G \in AM \Rightarrow \vec{r_G} = \lambda \vec{r_A} + (1-\lambda)\vec{r_M} = \lambda \vec{r_A} + \frac{1-\lambda}{2}\vec{r_B} + \frac{1-\lambda}{2}\vec{r_C}$$

$$G \in BN \Rightarrow \vec{r_G} = \mu \vec{r_B} + \frac{1-\mu}{2}\vec{r_A} + \frac{1-\mu}{2} r_C$$

$$\Rightarrow \begin{cases} \mu = \frac{1-\lambda}{2} \\ \lambda = \frac{1-\mu}{2} \Rightarrow 2\lambda = 1-\lambda \Rightarrow \lambda = \frac{1}{3} = \mu \\ \frac{1-\lambda}{2} = \frac{1-\mu}{2} \Rightarrow \lambda = \mu \end{cases}$$

$$\Rightarrow \vec{r_G} = \frac{1}{3}(\vec{r_A} + \vec{r_B} + \vec{r_C})$$

$$GP = \frac{1}{6}\vec{r_A} + \frac{1}{6}\vec{r_B} - \frac{1}{3}\vec{r_C}$$

$CP = \frac{1}{2}\vec{r_A} + \frac{1}{2}\vec{r_B} - \vec{r_C} = 3 \cdot GP \Rightarrow G \in CP \Rightarrow AM, BN, CP$ congruent

Help Nechita ♡

* un spațiu affine este un spațiu $\hat{}$ în unde se pot înmulți vectori între ei și scalari

2.20. Consider the lines $\ell_1 : x = 1 + t, y = 1 + 2t, z = 3 + t, t \in \mathbb{R}$ and $\ell_2 : x = 3 + s, y = 2s, z = -2 + s, s \in \mathbb{R}$. Show that ℓ_1 and ℓ_2 are parallel and find the equation of the plane determined by the two lines.

$$\ell_1 : \begin{cases} x = 1+t \\ y = 1+2t \\ z = 3+t \end{cases} \qquad \ell_2 : \begin{cases} x = 3+s \\ y = 2s \\ z = -2+s \end{cases}$$

vect. $(1,2,1)$ este vectorul director pentru ambele linii $\Rightarrow$ sunt paralele

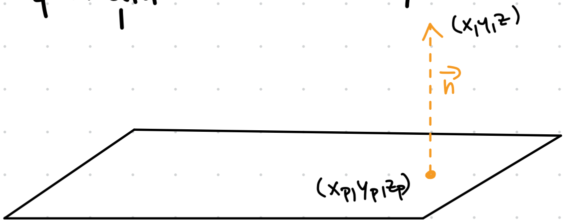
$$ax + by + cz + d = 0$$

$$\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$v_1 \to$ vect. director
$v_2 \to$ unesc două puncte
} cross product și ec. planului

3.42. Determine the orthogonal projection of the line $\ell : 2x - y - 1 = 0 \cap x + y - z + 1 = 0$ on the plane $\pi : x + 2y - z = 0$.

* proiecția unui vector pe altul



$$(x - x_p, y - y_p, z - z_p) = \alpha \vec{n}$$

$$\begin{cases} x - x_p = \alpha n_1 \\ y - y_p = \alpha n_2 \\ z - z_p = \alpha n_3 \end{cases}$$

*proiectezi două puncte de pe dreapta și faci dreapta dintre ele

3.43. Determine the coordinates of a point A on the line $\ell : \frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{1}$ which is at distance $\sqrt{3}$ from the plane $x + y + z + 3 = 0$.

$$d(P, \pi)$$

$$\frac{\text{coord } A}{\sqrt{a^2+b^2+c^2}}$$

Distance from a Point to a Plane The distance from (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is

$$\text{Distance} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Pintea

2.27. In each of the following, find a Cartesian equation of the plane in $\mathbb{A}^3$ passing through Q and parallel to the lines ℓ and ℓ':

a) $Q(1,-1,-2)$, $\ell : x - y = 1, x + z = 5$, $\ell' : x = 1, z = 2$

b) $Q(0,1,3)$, $\ell : x + y = -5, x - y + 2z = 0$, $\ell : 2x - 2y = 1, x - y + 2z = 1$

a) $Q(1,-1,-2)$

$$\ell : \begin{cases} x - y = 1 \\ x + z = 5 \end{cases} \qquad \ell' : \begin{cases} x = 1 \\ z = 2 \end{cases}$$

$Q \in \pi \qquad \pi \parallel \ell, \ell'$

$$\Leftrightarrow \ell : \begin{cases} y = x - 1 \\ z = 5 - x \end{cases} \Rightarrow \ell : \begin{cases} x = 1 \cdot t \\ y = 1 \cdot t - 1 \\ z = 5 + (-1) t \end{cases}, t \in \mathbb{R} \Rightarrow D(\ell) = \langle (1,1,-1) \rangle$$

$$\ell': \begin{cases} x = 1+0\lambda \\ y = \lambda \\ z = 2+0\cdot\lambda \end{cases}, \lambda \in \mathbb{R} \Rightarrow D(\ell') = \langle(0,1,0)\rangle$$

$$\tilde{\pi}: \begin{vmatrix} x-1 & y+1 & z+2 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} x-1 & z+2 \\ 1 & -1 \end{vmatrix} = -x+1-z-2 = -x-z-1=0 \Rightarrow \boxed{x+z=-1}$$

2.30. In each of the following, find Cartesian equations for the line ℓ in $\mathbb{A}^3$ passing through Q, contained in the plane π and intersecting the line ℓ'

a) $Q = (1,1,0)$, $\pi : 2x - y + z - 1 = 0$, $\ell' : x = 2 - t, y = 2 + t, z = t$

b) $Q = (-1,-1,-1)$, $\pi : x + y + z + 3 = 0$, $\ell' : x - 2z + 4 = 0, 2y - z = 0$

b) $Q = (-1,-1,-1)$

$\pi: x+y+z+3=0$

$$\ell' = \begin{cases} x-2z+4=0 \Rightarrow x = 2z-4 = 4y-4 \\ 2y-z=0 \Rightarrow z = 2y \end{cases} \Leftrightarrow \ell' = \begin{cases} x = 4t-4 \\ y = t \\ z = 2t \end{cases} \Rightarrow D(\ell') = \langle(4,1,2)\rangle$$

$\ell \in \pi$ and $Q \in \ell$

$$\pi \cap \ell' = \begin{cases} x+y+z+3=0 \\ x = 4y-4 \\ z = 2y \end{cases} \Rightarrow 4y-4+y+2y+3=0 \Rightarrow 7y-1=0 \Rightarrow y = \frac{1}{7} \Rightarrow z = \frac{2}{7} \text{ and } x = \frac{4}{7} - 4 = \frac{4-28}{7} = \frac{-24}{7}$$

$-4+2t+\frac{1}{2}t+t+3=0 \quad |\cdot 2$

$-8+4t+t+2t+6=0$

$7t = 2$

$t = \frac{2}{7} = z$

$$\overrightarrow{QR}: \frac{x-x_R}{x_Q-x_R} = \frac{y-y_R}{y_Q-y_R} = \frac{z-z_R}{z_Q-z_R}$$

$$\frac{x+\frac{24}{7}}{-1+\frac{24}{7}} = \frac{y-\frac{1}{7}}{-1-\frac{1}{7}} = \frac{z-\frac{2}{7}}{-1-\frac{2}{7}}$$

$$\frac{7x+24}{-17} = \frac{7y-1}{-8} = \frac{7z-2}{-9}$$

2.26. Determine the values a and d for which the line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$ is contained in the plane $ax + y - 2z + d = 0$.

$\ell \in \pi$

$$\ell: \begin{cases} \frac{x-2}{3} = \frac{y+1}{2} \\ \frac{y+1}{2} = \frac{z-3}{-2} \end{cases} \Rightarrow \ell: \begin{cases} 2x-4 = 3y+3 \\ y+1 = 3-z \end{cases} \Rightarrow \ell: \begin{cases} 2x = 3y+7 \\ y = 2-z \end{cases}$$

$$\Rightarrow \ell: \begin{cases} x = \frac{6-3t+7}{2} = \frac{13-3t}{2} \\ y = 2-t \\ z = t \end{cases}$$

$\pi: \quad a\cdot\frac{13-3t}{2} + 2-t-2t+d = 0 \quad |\cdot 2$

$13a-3at+4-2t-4t+2d=0$

$t(-3a-6)+13a+2d+4=0$

3.20. Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle $\measuredangle A$.

$$AB: \frac{x-1}{5-1} = \frac{y+2}{4+2} \Rightarrow 3x-2y-7=0$$

$$AC: \frac{x-1}{-2-1} = \frac{y+2}{2} \Leftrightarrow \frac{x-1}{-3} = \frac{y+2}{2} \Leftrightarrow 2x-2 = -3y-6 \Leftrightarrow 2x+3y+4=0$$

$M(x,y) \in$ bisector iff $dist(M, AB) = dist(M, AC)$

$$\frac{|3x-2y-7|}{\sqrt{3^2+(-2)^2}} = \frac{|2x-3y+4|}{\sqrt{13}} \iff |3x-2y-7| = \pm(2x+3y+4)$$

a) $3x-2y-7 = 2x+3y+4$

$x-5y-11=0$

b) $3x-2y-7 = -2x-3y-4$

$5x+y-3=0$

2.22. Determine an equation of the plane containing $P(2,0,3)$ and the line $\ell : x = -1+t, y = t, z = -4+2t, t \in \mathbb{R}$.

$$\ell : \frac{x+1}{1} = \frac{y}{1} = \frac{z+4}{2} \iff \begin{cases} x+1 = y \\ y = \frac{z+4}{2} \end{cases} \iff \begin{cases} x-y+1=0 \\ 2y-z-4=0 \end{cases}$$

$\pi_\lambda : x-y+1+\lambda(2y-z-4)=0$

$$\left.\begin{array}{l} x-y+2\lambda y-\lambda z-\lambda\cdot 4+1=0 \\ P \in \pi_\lambda \quad P(2,0,3) \end{array}\right\} \Rightarrow \begin{array}{l} 2-0+2\lambda\cdot 0-\lambda\cdot 3-\lambda\cdot 4+1=0 \\ 3-7\lambda=0 \Rightarrow \lambda=\frac{3}{7} \end{array}$$

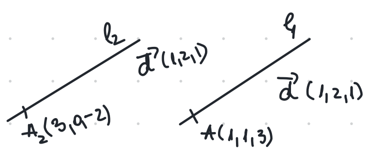
$\Rightarrow \pi : x-y+1+\frac{6}{7}y-\frac{3}{7}z-\frac{12}{7}=0$

$7x-y-3z-5=0$

2.20. Consider the lines $\ell_1 : x = 1+t, y = 1+2t, z = 3+t, t \in \mathbb{R}$ and $\ell_2 : x = 3+s, y = 2s, z = -2+s, s \in \mathbb{R}$. Show that ℓ_1 and ℓ_2 are parallel and find the equation of the plane determined by the two lines.

$D(\ell_1) = \langle(1,2,1)\rangle \qquad D(\ell_2) = \langle(1,2,1)\rangle$ same director space $\Rightarrow \ell_1 \parallel \ell_2$

$(1,1,3) \in \ell_1 \setminus \ell_2 \Rightarrow \ell_1 \neq \ell_2$



$\overrightarrow{A_1A_2}(2,-1,-5)$

$$\pi : \begin{vmatrix} x-1 & y-1 & z-3 \\ 2 & -1 & -5 \\ 1 & 2 & 1 \end{vmatrix} = 0 = -x+1+4z-12-5y+5+z-3+10x-10-2y+2 =$$

$= 9x-7y+5z-17=0$

$\mathcal{K}(O, \mathcal{B})$ reference system where O - origin & $\mathcal{B}$ - basis

$M_{B'B}(f) = ([f(v_1)]_{B'} \ldots [f(v_n)]_{B'}) = [id]_{BB'}$

$[v]_{B'} = M_{BB'} \cdot [v]_B$

$[P]_{K'}$ in terms of $[P]_K \Rightarrow [P]_{K'} = M_{B'B}[\overrightarrow{OP}-\overrightarrow{OO'}]_B = M_{B'B}([P]_K-[O']_K) = M_{BB'}[P]_K+[O]_{K'}$

1.16. Consider the two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ given in Example 1.20. Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.

$K = (O, \vec{i}, \vec{j})$ $\qquad K' = (O', \vec{i}', \vec{j}')$ $\qquad [O']_K = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

$$\left.\begin{array}{l} \vec{i}' = -2\vec{i} + \vec{j} \\ \vec{j}' = \vec{i} + 2\vec{j} \end{array}\right\} \Rightarrow M_{KK'} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow M_{K'K} = M_{KK'}^{-1} = \frac{1}{5}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow [A]_{K'} = M_{K'K}[\vec{O'A}]_K = M_{K'K}[A]_K - M_{K'K}[O']_K = \frac{1}{5}\left(\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ -1 \end{bmatrix}\right) = \frac{1}{5}\left(\begin{bmatrix} 0 \\ 5 \end{bmatrix} - \begin{bmatrix} -15 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$[B]_{K'} = \frac{1}{5}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -5 \\ 5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 15 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$[C]_{K'} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

reverse:

$$[A]_K = M_{KK'}[\vec{O'A}]_{K'} = M_{KK'}\left([A]_{K'} - [O']_{K'}\right) = M_{KK'}\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$M_{KK'} \cdot [O']_{K'} = [O']_K = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

1.17. Consider the tetrahedron $ABCD$ and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

a) the coordinates of the vertices of the tetrahedron in the three coordinate systems,

b) the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,

c) the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.

a) $A = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \overrightarrow{AB} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \overrightarrow{AC} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad \overrightarrow{AD} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

b) $M_{K'_A K_A} = \left([\overrightarrow{AB}]_{K'_A} \;\; [\overrightarrow{AC}]_{K'_A} \;\; [\overrightarrow{AD}]_{K'_A}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\vec{v}_T = \vec{v}_A + \lambda \cdot \vec{v}$$

$[T]_K = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad [A]_K = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$ $\quad [\vec{v}]_K = \begin{pmatrix} x_{\vec{v}} \\ y_{\vec{v}} \end{pmatrix}$

$$\ell: \begin{cases} x = x_A + \lambda x_{\vec{v}} \\ y = y_A + \lambda y_{\vec{v}} \end{cases}$$

$$Ax + By + C = 0$$

$\boxed{A_T = \lambda \vec{v} + \mu \vec{w}}$ → vector eq. of plane

$$\begin{vmatrix} x - x_A & y - y_A & z - z_A \\ x_{\vec{v}} & y_{\vec{v}} & z_{\vec{v}} \\ x_{\vec{w}} & y_{\vec{w}} & z_{\vec{w}} \end{vmatrix} = 0 \quad \text{plane eq.}$$

2.5 Exercises

a) $A(1,2) \in \ell \quad \ell \parallel a(3,-1)$

$\Rightarrow \ell : \begin{cases} x = 1 + 3\cdot\lambda \\ y = 2 - \lambda \end{cases} \Rightarrow \lambda = 2 - y$

$\Rightarrow x - 1 = 6 - 3y \Rightarrow D(\ell) = \langle(3,-1)\rangle$

b) $O \in \ell \quad \ell \parallel (4,5)$

$\begin{cases} x = \lambda\cdot 4 \\ y = \lambda\cdot 5 \Rightarrow \lambda = \frac{y}{5} \end{cases} \Rightarrow x = \frac{4}{5}y \Rightarrow 4y = 5x$

~~2.1.~~ Determine parametric equations for the line $\ell \subseteq \mathbb{A}^2$ in the following cases:

a) ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,

b) ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,

c) ℓ contains the point $M(1,7)$ and is parallel to Oy,

d) ℓ contains the points $M(2,4)$ and $N(2,-5)$.

2.5.2. $\vec{v} = (-5\sqrt{2}, 7)$

$S \cap T = \begin{cases} x - y = 0 \\ x + y = 1 \end{cases}$

$x = \frac{1}{2} \Rightarrow y = \frac{-1}{2}$

$\ell : \begin{cases} x = \frac{1}{2} - 5\sqrt{2}\lambda \\ y = \frac{-1}{2} + 7\lambda \Rightarrow \lambda = \frac{y + \frac{1}{2}}{7} \end{cases}$

~~2.2.~~ For the lines ℓ in the previous exercise

a) determine a Cartesian equation for ℓ,

b) describe all direction vectors for ℓ.

2.3. With the assumptions in Exercise 1.16, give parametric equations and Cartesian equations for the lines AB, AC, BC both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

2.4. Find a Cartesian equation of the line ℓ in $\mathbb{A}^2$ containing the points $P = S \cap S'$ and $Q = T \cap T'$ where

$$S : x + 5y - 8 = 0, \quad S' : 3x + 6 = 0, \quad T : 5x - \frac{1}{2}y = 1, \quad T' : x - y = 5.$$

~~2.5.~~ Deterimine an equation for the line in $\mathbb{A}^2$ parallel to $\mathbf{v}$ and passing through $S \cap T$ in each of the following cases:

1. $\mathbf{v} = (2,4)$, $S : 3x - 2y - 7 = 0$, $T : 2x + 3y = 0$,
2. $\mathbf{v} = (-5\sqrt{2}, 7)$, $S : x - y = 0$, $T : x + y = 1$.

$D(\ell) = D(v)$

$S \cap T = \begin{cases} 3x - 2y - 7 = 0 \ (\cdot 2) \\ 2x + 3y = 0 \ (\cdot -3) \end{cases}$ (+)

$-13y = 14 \Rightarrow y = \frac{14}{13} \Rightarrow x = \frac{3\cdot\frac{\cancel{14}^{7}}{13}}{\cancel{2}} = \frac{21}{13} \Rightarrow \left(\frac{21}{13}, \frac{14}{13}\right)$

$\ell : \begin{cases} x = \frac{21}{13} + 2\lambda \\ y = \frac{14}{13} + 4\lambda \end{cases} \Rightarrow 2x - y - \frac{56}{13} = 0$

2.6. Let ABC be a triangle in the affine space $\mathbb{A}^n$. Consider the points C' and B' on the sides AB and AC respectively, such that

$$\overrightarrow{AC'} = \lambda \overrightarrow{BC'} \quad \text{and} \quad \overrightarrow{AB'} = \mu \overrightarrow{CB'}.$$

The lines BB' and CC' meet in the point M. For a fixed but arbitrary point $O \in \mathbb{A}^n$, show that

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} - \lambda \overrightarrow{OB} - \mu \overrightarrow{OC}}{1 - \lambda - \mu}.$$

Deduce a formula for $\overrightarrow{OG}$ where G is the centroid of the triangle.

2.7. In $\mathbb{A}^n$, consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\overrightarrow{OA} + n\frac{m-1}{m-n}\overrightarrow{OA'}$$

where $M = AB' \cap A'B$ and $N = AA' \cap BB'$ and where $\overrightarrow{OB} = m\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\overrightarrow{OA'}$.

2.8. Show that the midpoints of the diagonals of a complete quadrilateral are collinear.

~~2.9.~~ Determine parametric equations for the plane π in the following cases:

a) π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1(3,-1,1)$ and $\mathbf{a}_2(0,3,1)$,

b) π contains the points $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$,

c) π contains the point $A(1,2,1)$ and is parallel to $\mathbf{i}$ and $\mathbf{j}$,

d) π contains the point $M(1,7,1)$ and is parallel coordinate plane Oyz,

e) π contains the points $M_1(5,3,4)$ and $M_2(1,0,1)$, and is parallel to the vector $\mathbf{a}(1,3,-3)$,

f) π contains the point $A(1,5,7)$ and the coordinate axis Ox.

~~2.10.~~ Determine Cartesian equations for the plane π in the following cases:

a) $\pi : x = 2+3u-4v,\ y = 4-v,\ z = 2+3u$;

b) $\pi : x = u+v,\ y = u-v,\ z = 5+6u-4v$.

~~2.11.~~ Determine parametric equations for the plane π in the following cases:

a) $3x-6y+z=0$; $\Rightarrow z = 6y-3x$ $\quad \ell: \begin{cases} x = u \\ y = v \\ z = 6v-3u \end{cases} \Rightarrow \begin{vmatrix} x & y & z \\ 1 & 0 & -3 \\ 0 & 1 & 6 \end{vmatrix} = 0 \Rightarrow D(\pi) = \langle (1,0,-3), (0,1,6) \rangle$

b) $2x-y-z-3=0$.

2.12. With the assumptions in Exercise 1.18, give parametric equations and Cartesian equations for the line AB and the plane ACD both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

2.13. Show that the points $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ and $D(4,2,3)$ are coplanar.

~~2.14.~~ Determine the relative positions of the planes in the following cases

a) $\pi_1 : x+2y+3z-1=0$, $\pi_2 : x+2y-3z-1=0$.

b) $\pi_1 : x+2y+3z-1=0$, $\pi_2 : 2x+y+3z-2=0$, $\pi_3 : x+2y+3z+2=0$.

2.15. Show that the planes

$$\pi_1 : 3x+y+z-1=0, \quad \pi_2 : 2x+y+3z+2=0, \quad \pi_3 : -x+2y+z+4=0$$

have a point in common.

2.16. Show that the pairwise intersection of the planes

$$\pi_1 : 3x+y+z-5=0, \quad \pi_2 : 2x+y+3z+2=0, \quad \pi_3 : 5x+2y+4z+1=0$$

are parallel lines.

2.17. Determine parametric equations for the line ℓ in the following cases:

$\boxed{\textbf{2.10.}}$ Determine Cartesian equations for the plane π in the following cases:

~~a)~~ $\pi : x = 2 + 3u - 4v,\ y = 4 - v,\ z = 2 + 3u$;

~~b)~~ $\pi : x = u + v,\ y = u - v,\ z = 5 + 6u - 4v$.

a) $\pi : \begin{cases} x = 2 + 3u - 4v \\ y = 4 \quad\ \ - v \\ z = 2 + 3u \end{cases}$ eq. $\begin{vmatrix} x-2 & y-4 & z-4 \\ 3 & 0 & 3 \\ -4 & -1 & 0 \end{vmatrix} = 0$

$-3z + 12 - 12y + 48 + 3x - 6 = 0$

$3x - 12y - 3z + 54 = 0 \quad | : 3$

$\pi : \ x - 4y - z + 16 = 0$

b) $\pi : \begin{vmatrix} x & y & z-5 \\ 1 & 1 & 6 \\ 1 & -1 & -4 \end{vmatrix} = 0$

$-4x - z + 5 + 6y - z + 5 + 6x + 4y = 0$

$2x + 10y - 2z + 10 = 0 \quad | : 2$

$x + 5y - z + 5 = 0 \Rightarrow D(\pi) = \langle u, v \rangle = \langle (1,1,6), (1,-1,-4) \rangle$

$\boxed{\textbf{2.14.}}$ Determine the relative positions of the planes in the following cases

a) $\pi_1 : x + 2y + 3z - 1 = 0$, $\pi_2 : x + 2y - 3z - 1 = 0$.

b) $\pi_1 : x + 2y + 3z - 1 = 0$, $\pi_2 : 2x + y + 3z - 2 = 0$, $\pi_3 : x + 2y + 3z + 2 = 0$.

a) $\begin{cases} x + 2y + 3z - 1 = 0 \\ x + 2y - 3z - 1 = 0 \end{cases} \Rightarrow 3z = 0 \Rightarrow z = 0$ $\quad (+)$

$2x + 4y - 2 = 0 \quad | : 2$

$x + 2y - 1 = 0 \Rightarrow$ they intersect on this line

b) $\begin{cases} x + 2y + 3z - 1 = 0 \\ 2x + y + 3z - 2 = 0 \\ x + 2y + 3z + 2 = 0 \end{cases}$ $\cap$ $\cap$

$(1) - (3)$

$-3 = 0$ false $\Rightarrow (1) \cancel{\cap} (3)$ they have the same $D(\pi) \Rightarrow$ parallel

$\boxed{\textbf{2.18.}}$ Give Cartesian equations for the lines ℓ in the previous exercise.

a) ℓ contains the point $M_0(2, 0, 3)$ and is parallel to the vector $\mathbf{a}(3, -2, -2)$,

b) ℓ contains the point $A(1, 2, 3)$ and is parallel to the Oz-axis,

c) ℓ contains the points $M_1(1, 2, 3)$ and $M_2(4, 4, 4)$.

a) $M_0(2, 0, 3) \in \ell \qquad \ell \parallel (3, -2, -2)$

$\ell : \begin{cases} x = 2 + 3\lambda \Rightarrow x = 2 - \frac{3}{2}y \\ y = -2\lambda \Rightarrow \lambda = \frac{-y}{2} \\ z = 3 - 2\lambda \Rightarrow z = 3 + y \end{cases}$ $\Rightarrow \begin{cases} 2x = 4 - 3y \\ z = 3 + y \end{cases}$ $y = t$ $\Rightarrow \begin{cases} x = \frac{4 - 3t}{2} \\ z = 3 + t \\ y = t \end{cases}$

$\boxed{\textbf{2.19.}}$ Determine parametric equations for the line contained in the planes $x + y + 2z - 3 = 0$ and $x - y + z - 1 = 0$.

$\ell : \begin{cases} x + y + 2z - 3 = 0 \\ x - y + z - 1 = 0 \end{cases}$ $\Rightarrow 4 - 3z + y + 2z - 3 = 0$

$y - z + 1 = 0 \Rightarrow y = 1 - z$

$(+)$

$x + 3z - 4 = 0$

$x = 4 - 3z$

$\Rightarrow \ell = \begin{cases} x = 4 - 3t \\ y = 1 - t \\ z = t \end{cases}$

$\boxed{\textbf{2.26.}}$ Determine the values a and d for which the line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$ is contained in the plane π: $ax + y - 2z + d = 0$.

$\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2} \in \pi$

$2x - 4 = 3y + 3 = -3z + 9$

$\ell : \begin{cases} 2x - 4 = 3y + 3 \Rightarrow x = \frac{6 - 3z + 3 + 4}{2} = \frac{13 - 3z}{2} \\ 3y + 3 = -3z + 9 \quad | : 3 \\ y + 1 = 3 - z \Rightarrow y = 2 - z \end{cases}$

$\Rightarrow a \cdot \frac{13 - 3t}{2} + 2 - t - 2t + d = 0 \quad | \cdot 2$

$13a - 3at + 4 - 2t - 4t + 2d = 0$

$\underbrace{-3(a+2)}_{= 0 \Rightarrow a = -2} t + \underbrace{4 + 2d}_{= 0 \Rightarrow d = -2} = 0$

a) ℓ contains the point $M_0(2,0,3)$ and is parallel to the vector $\mathbf{a}(3,-2,-2)$,

b) ℓ contains the point $A(1,2,3)$ and is parallel to the Oz-axis,

c) ℓ contains the points $M_1(1,2,3)$ and $M_2(4,4,4)$.

2.18. Give Cartesian equations for the lines ℓ in the previous exercise.

2.19. Determine parametric equations for the line contained in the planes $x+y+2z-3=0$ and $x-y+z-1=0$.

2.20. Consider the lines $\ell_1 : x = 1+t, y = 1+2t, z = 3+t, t \in \mathbb{R}$ and $\ell_2 : x = 3+s, y = 2s, z = -2+s, s \in \mathbb{R}$. Show that ℓ_1 and ℓ_2 are parallel and find the equation of the plane determined by the two lines.

2.21. Determine parametric equations of the line passing through $P(5,0,-2)$ and parallel to the planes $\pi_1 : x-4y+2z=0$ and $\pi_2 : 2x+3y-z+1=0$.

2.22. Determine an equation of the plane containing $P(2,0,3)$ and the line $\ell : x = -1+t, y = t, z = -4+2t, t \in \mathbb{R}$.

2.23. For the points $A(2,1,-1)$ and $B(-3,0,2)$, determine an equation of the bundle of planes passing through A and B.

2.24. Determine the relative positions of the lines $x = -3t, y = 2+3t, z = 1, t \in \mathbb{R}$ and $x = 1+5s, y = 1+13s, z = 1+10s, s \in \mathbb{R}$.

2.25. Determine the parameter m for which the line $x = -1+3t, y = 2+mt, z = -3-2t$ doesn't intersect the plane $x+3y+3z-2=0$.

2.26. Determine the values a and d for which the line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$ is contained in the plane $ax+y-2z+d=0$.

2.27. In each of the following, find a Cartesian equation of the plane in $\mathbb{A}^3$ passing through Q and parallel to the lines ℓ and ℓ':

a) $Q(1,-1,-2)$, $\ell : x-y=1, x+z=5$, $\ell' : x=1, z=2$

b) $Q(0,1,3)$, $\ell : x+y=-5, x-y+2z=0$, $\ell : 2x-2y=1, x-y+2z=1$

2.28. In each of the following, find the relative positions of the line ℓ and the plane π in $\mathbb{A}^3$, and, if they are incident, determine the point of intersection.

a) $\ell : x = 1+t, y = 2-2t, z = 1-4t$, $\pi : 2x-y+z-1=0$

b) $\ell : x = 2-t, y = 1+2t, z = -1+3t$, $\pi : 2x+2y-z+1=0$

2.29. In each of the following, find a Cartesian equation for the plane in $\mathbb{A}^3$ containing the point Q and the line ℓ.

a) $Q = (3,3,1)$, $\ell : x = 2+3t, y = 5+t, z = 1+7t$

b) $Q = (2,1,0)$, $\ell : x - y + 1 = 0, 3x + 5z - 7 = 0$

$\boxed{\cancel{\textbf{2.30}}}$**.** In each of the following, find Cartesian equations for the line ℓ in $\mathbb{A}^3$ passing through Q, contained in the plane π and intersecting the line ℓ'

a) $Q = (1,1,0)$, $\pi : 2x - y + z - 1 = 0$, $\ell' : x = 2 - t, y = 2 + t, z = t$

b) $Q = (-1,-1,-1)$, $\pi : x + y + z + 3 = 0$, $\ell' : x - 2z + 4 = 0, 2y - z = 0$

2.31. In each of the following, find Cartesian equations for the line ℓ in $\mathbb{A}^3$ passing through Q and coplanar to the lines ℓ' and ℓ''. Furthermore, establish whether ℓ meets or is parallel to ℓ' and ℓ''

a) $Q = (1,1,2)$, $\ell' : 3x - 5y + z = -1, 2x - 3z = -9$, $\ell'' : x + 5y = 3, 2x + 2y - 7z = -7$

b) $Q = (2,0,-2)$, $\ell' : -x + 3y = 2, x + y + z = -1$, $\ell'' : x = 2 - t, y = 3 + 5t, z = -t$

2.32. In each of the following, find the value of the real parameter k for which the lines ℓ and ℓ' are coplanar. Find a Cartesian equation for the plane that contains them, and find the point of intersection whenever they meet

a) $\ell : x = k + t, y = 1 + 2t, z = -1 + kt$, $\ell' : x = 2 - 2t, y = 3 + 3t, z = 1 - t$

b) $\ell : x = 3 - t, y = 1 + 2t, z = k + t$, $\ell' : x = 1 + t, y = 1 + 2t, z = 1 + 3t$

2.33. Find a Cartesian equation for the plane π in $\mathbb{A}^3$ which contains the line of intersection of the two planes

$$x + y = 3 \quad \text{and} \quad 2y + 3z = 4$$

and is parallel to the vector $\mathbf{v} = (3,-1,2)$.

2.34. In the affine space $\mathbb{A}^4$ consider

$$\text{the plane} \quad \alpha = \left\langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix} \right\rangle + \begin{bmatrix}2\\4\\1\\2\end{bmatrix} \quad \text{and the line} \quad \beta = \left\langle \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} \right\rangle + \begin{bmatrix}2\\3\\-1\\1\end{bmatrix}.$$

Determine $\alpha \cap \beta$.

2.35. In $\mathbb{A}^4$ consider the affine subspaces

$$\alpha = \begin{bmatrix}2\\1\\2\\1\end{bmatrix} \quad \beta = \left\langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \right\rangle + \begin{bmatrix}1\\3\\0\\0\end{bmatrix} \quad \gamma = \left\langle \begin{bmatrix}2\\1\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-2\end{bmatrix} \right\rangle + \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad \delta = \left\langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\rangle.$$

punct dreapta plan hyperplane

Which of the following is true?

$\alpha \in \gamma \Leftrightarrow \begin{pmatrix}2\\1\\2\\1\end{pmatrix} \in \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}2\\1\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\2\\-2\end{pmatrix} \right\rangle \Leftrightarrow \exists\, \alpha, \beta \in \mathbb{R}$ a.î.

$\begin{pmatrix}2\\1\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \alpha \begin{pmatrix}2\\1\\3\\-1\end{pmatrix} + \beta \begin{pmatrix}1\\0\\2\\-2\end{pmatrix}$

a) $\alpha \in \beta$

b) $\alpha \in \gamma$

c) $\alpha \in \delta$

d) $\beta \parallel \gamma \Leftrightarrow D(\beta) \subseteq D(\gamma)$

$\hookrightarrow \dim \beta = 1$

$\dim \gamma = 1$

$\Leftrightarrow \left\langle \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix}2\\1\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\2\\-1\end{pmatrix} \right\rangle$

$\Leftrightarrow$

e) $\beta \parallel \delta$

f) $\gamma \parallel \delta$

g) $\beta \subseteq \gamma$

h) $\gamma \subseteq \delta$

i) $\beta \subseteq \delta$

2.36. Consider the following affine subspaces of $\mathbb{A}^4$

$$Y : \begin{cases} x_1 + x_3 - 2 &= 0 \\ 2x_1 - x_2 + x_3 + 3x_4 - 1 &= 0 \end{cases}$$

$$Z : \begin{cases} x_1 + x_2 + 2x_3 - 3x_4 &= 1 \\ x_2 + x_3 - 3x_4 &= -1 \\ x_1 - x_2 + 3x_4 &= 3 \end{cases}$$

a) Determine the dimensions of Y and Z.

b) Find parametric equations for each of the two affine subspaces.

c) Is $Y \parallel Z$?

2.37. In Section 2.2.2 we deduce a linear equation for a plane in $\mathbb{A}^3$ via a determinant. What is the analogue of this description for lines? I.e. deduce Cartesian equations for lines starting from linear dependence of vectors (both in $\mathbb{A}^2$ and $\mathbb{A}^3$).

2.38. Consider the affine space $\mathbb{A}^3$. Show that if a line ℓ doesn't intersect a plane π then $\ell \parallel \pi$ in the sense of the Definition 2.14. Moreover, give an example in $\mathbb{A}^4$ of a line and a plane which do not intersect and which are not parallel.

2.39. Consider the affine space $\mathbb{A}^4$. Describe the relative positions of two planes.

2.40. In $\mathbb{A}^3$ discuss the relative positions of a plane and a line in terms of their Cartesian equations.

2.41. In $\mathbb{A}^3$ discuss the relative positions two lines in terms of their Cartesian equations.

3.5 Exercises

$\boxed{\cancel{3.1.}}$ Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 60°$. Determine the length of the diagonals in the parallelogram spanned by the vectors $\mathbf{a} = 2\mathbf{m}+\mathbf{n}$ and $\mathbf{b} = \mathbf{m}-2\mathbf{n}$.

$\boxed{\cancel{3.2.}}$ Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 120°$. Determine the angle between the vectors $\mathbf{a} = 2\mathbf{m}+4\mathbf{n}$ and $\mathbf{b} = \mathbf{m}-\mathbf{n}$. $\cos(\widehat{a,b}) = \frac{\vec a \cdot \vec b}{\|a\|\,\|b\|} = \frac{\langle a,b\rangle}{\|a\|\,\|b\|} = \frac{(2m+4n)(m-n)}{2\sqrt3 \cdot \sqrt3} = \frac{-1}{2}$

? $\boxed{3.3.}$ You are given two vectors $\mathbf{a}(2,1,0)$ and $\mathbf{b}(0,-2,1)$ with respect to an orthonormal basis. Determine the angles between the diagonals of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. $\langle a+b, a-b\rangle = \langle (2,-1,1),(2,3,-1)\rangle = \sqrt6 \cdot \sqrt{14} \cdot \cos\alpha$

$\boxed{3.4.}$ Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis. Consider the vectors $\mathbf{q} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{p} = \mathbf{i}+2\mathbf{j}+\lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p},\mathbf{q})$ is $5/12$. $\|q\| = \sqrt{10}$ $\|p\| = \sqrt{5+\lambda^2}$ $\vec p \cdot \vec q = 5 \Rightarrow \cos\alpha = \frac{5}{\sqrt{5+\lambda^2}\cdot\sqrt{10}} \Rightarrow \lambda = \pm\sqrt{\frac{47}{5}}$

3.5. Using the scalar product, prove the Cauchy-Bunyakovsky-Schwarz inequality, i.e. show that for any $a_1,\dots,a_n,b_1,\dots,b_n \in \mathbb{R}$ we have

$$(a_1b_1+\cdots+a_nb_n)^2 \le (a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2).$$

3.6. Let ABC be a triangle. Show that

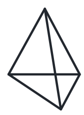
$$\overrightarrow{AB}^2 + \overrightarrow{AC}^2 - \overrightarrow{BC}^2 = 2\overrightarrow{AB}\cdot\overrightarrow{AC}$$

and deduce the law of cosines in a triangle.

ask Mara for a simpler method

$\boxed{3.7.}$ Let $ABCD$ be a tetrahedron. Show that

$$\cos(\measuredangle(\overrightarrow{AB},\overrightarrow{CD})) = \frac{AD^2+BC^2-AC^2-BD^2}{2\cdot AB\cdot CD}.$$



use points as midterms

$\vec{AB} = \vec{AC} + \vec{CB}$ ..

This is a $3D$-version of the law of cosine.

3.8. Let $ABCD$ be a rectangle. Show that for any point O

$$\overrightarrow{OA}\cdot\overrightarrow{OC} = \overrightarrow{OB}\cdot\overrightarrow{OD} \quad \text{and} \quad \overrightarrow{OA}^2+\overrightarrow{OC}^2 = \overrightarrow{OB}^2+\overrightarrow{OD}^2.$$

$\boxed{\cancel{3.9.}}$ Consider the vector $\mathbf{v}$ which is perpendicular on $\mathbf{a}(4,-2,-3)$ and on $\mathbf{b}(0,1,3)$. If $\mathbf{v}$ describes an acute angle with Ox and $|\mathbf{v}| = 26$ determine the components of $\mathbf{v}$.

$\boxed{\cancel{3.10.}}$ In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

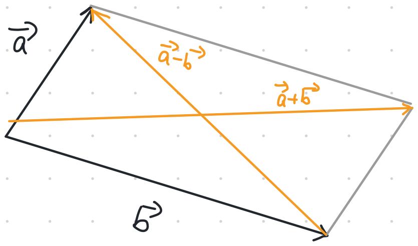
3.11. Let $\mathbf{v} \in \mathbb{V}^n$ be a vector. Show that the set $\mathbf{v}^\perp$ is an $(n-1)$-dimensional vector subspace of $\mathbb{V}^n$. Deduce that there is a basis $\mathbf{v},\mathbf{v}_2,\mathbf{v}_3,\dots,\mathbf{v}_n$ of $\mathbb{V}^n$ with $\mathbf{v}_2,\dots,\mathbf{v}_{n-1}$ a basis of $\mathbf{v}^\perp$. (*Hint.* use Steinitz Theorem - Algebra, Lecture 6).

$\boxed{\cancel{3.12.}}$ Determine a Cartesian equations for the line ℓ in the following cases:

a) ℓ contains the point $A(-2,3)$ and has an angle of $60°$ with the Ox-axis,

$\vec{a}\cdot\vec{b} = \langle a,b\rangle = \|a\|\cdot\|b\|\cdot\cos\alpha$

3.1. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 60°$. Determine the length of the diagonals in the parallelogram spanned by the vectors $\mathbf{a} = 2\mathbf{m}+\mathbf{n}$ and $\mathbf{b} = \mathbf{m}-2\mathbf{n}$.



$\vec{a}+\vec{b} = 3m-n$

$\|\vec{a}+\vec{b}\|^2 = (3m-n)^2 = 9+1+2\cdot 3mn = 10+6mn$

$\vec{m}\cdot\vec{n} = \|m\|\cdot\|n\|\cdot\cos\alpha = \cos 60° = \frac{1}{2}$

$\Rightarrow \|\vec{a}+\vec{b}\| = 10+6\cdot\frac{1}{2} = 13$

$\vec{a}-\vec{b} = m+3n$

$\|\vec{a}-\vec{b}\| = 13$

3.9. Consider the vector $\mathbf{v}$ which is perpendicular on $\mathbf{a}(4,-2,-3)$ and on $\mathbf{b}(0,1,3)$. If $\mathbf{v}$ describes an acute angle with Ox and $|\mathbf{v}| = 26$ determine the components of $\mathbf{v}$.

$v\perp a$, $v\perp b$ $\quad |v|=26 \quad \sphericalangle(v,Ox) < 90°$

$\cos(\widehat{v,a}) = 0 = \cos(\widehat{v,b})$

let $v=(x,y,z)$

$\vec{v}\cdot\vec{a} = 4x-2y-3z=0$

$\vec{v}\cdot\vec{b} = y+3z=0$

$\Rightarrow \vec{v}: \begin{cases} 4x-2y-3z=0 \\ y+3z=0 \Rightarrow z=\frac{-y}{3} \end{cases}$ $(+)$

$4x-y=0 \Rightarrow y=4x$

$\Rightarrow v: \begin{cases} x=t \\ y=4t \\ z=-\frac{4}{3}t \end{cases}$

$|v| = t^2+16t^2+\frac{16}{9}t^2 = 676$

$t^2\left(1+16+\frac{16}{9}\right) = 676 \Rightarrow x^2=16 \Rightarrow x=4$

$y=24$

$z=-8$

3.10. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

$v_1(0,1,0)$

$v_2=(2,1,0)$

$v_3=(-1,0,1)$

$Pr_{v_1'}(v_2) = \frac{\langle v_2, v_1'\rangle}{\langle v_1', v_1'\rangle}\cdot v_1'$

$$\boxed{\mathbf{p} = \frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{a}\|^2}\,\mathbf{a}}$$

direction vector

normalised

let $v_1' = v_1 = (0,1,0)$

$P(v_2) = \frac{v_1\cdot v_2}{\|v_1\|^2}\cdot v_1 = \frac{1}{1}\cdot(0,1,0) = (0,1,0)$

$P(v_3) = \frac{v_1\cdot v_3}{\|v_1\|^2}\cdot v_1 = \frac{0}{1}\cdot(0,1,0) = (0,0,0)$

$\Rightarrow v_2' = v_2 - P(v_2) = (2,1,0)-(0,1,0) = (2,0,0)$

$v_3' = (-1,0,1)-(0,0,0) = (-1,0,1)$

$e_1' = e_1$

$e_2' = e_2 - \frac{e_1'\cdot e_2}{\|e_1'\|^2}\cdot e_1'$

$e_3' = e_3 - \frac{e_1'\cdot e_3}{\|e_1'\|^2}\cdot e_1' - \frac{e_2'\cdot e_3}{\|e_2'\|^2}\cdot e_2'$

$\vdots$

$B' = \{e_1', \dots, e_n'\}$

3.12. Determine a Cartesian equations for the line ℓ in the following cases:

a) ℓ contains the point $A(-2,3)$ and has an angle of $60°$ with the Ox-axis,

a) $A(-2,3)\in\ell \quad \cos(\widehat{Ox,\ell}) = \frac{1}{2}$

$\alpha = 60° \Rightarrow \tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \ell: \ y = \frac{1}{\sqrt{3}}x + c \quad |\cdot\sqrt{3}$

$\sqrt{3}y = x+\sqrt{3}c$

let $\ell: ax+by+c=0$

$\ell: \begin{cases} \cos(\widehat{Ox,\ell}) = \frac{1}{2} \Rightarrow \langle Ox, \ell\rangle = \|Ox\|\|\ell\|\cdot\cos(\widehat{Ox,\ell}) \Rightarrow (0,1)\cdot(a,b) = \sqrt{a^2+b^2}\cdot\frac{1}{2} \Rightarrow 2b = \sqrt{a^2+b^2} \quad |()^2 \\ A\in\ell \Rightarrow \end{cases}$

$4b^2 = a^2+b^2 \Rightarrow a^2 = 3b^2 \Rightarrow a = b\sqrt{3}$

$\left.\begin{array}{l} -2a+3b+c=0 \\ a=b\sqrt{3} \end{array}\right\} \Rightarrow b(3-2\sqrt{3})+c=0$

$3\sqrt{3} = -2+\sqrt{3}c$

b) ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

$B(1,7) \in \ell$

$\ell \cdot n = 0$ $\qquad \ell: \; ax+by+c=0$

$(a,b)\cdot(4,3)=0 \Rightarrow 4a+3b=0 \Rightarrow a=\frac{-3}{4}b$

$-\frac{3}{4}bx+by+c=0 \quad |\cdot 4$

$-3bx+4by+4c=0$

$B\in\ell \Rightarrow -3b+28b+4c=0$

$25b=-4c$

$y-y_1=m(x-x_1)$

$m=\frac{-4}{3}$

$y-7=\frac{-4}{3}(x-1) \quad |\cdot 3$

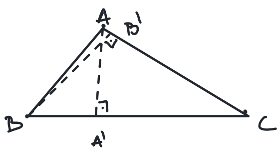
$3y-21=-4x+4$

$4x+3y-25=0$

$\boxed{3.18.}$ Determine the circumcenter and the orthocenter of the triangle with vertices $A(1,2)$, $B(3,-2)$, $C(5,6)$.

mediatoare $\qquad$ height

$A(1,2) \quad B(3,-2) \quad C(5,6)$



$\vec{AB}(3-1,-2-2)=\vec{AB}(2,-4)$

$\vec{BC}(5-3,6+2)=\vec{BC}(2,8)$

$BC: \begin{cases} x=3+2\lambda & |\cdot(-4) \\ y=-2+8\lambda \end{cases}$ (+)

$y-4x=-14 \Rightarrow n_{BC}=\langle(-4,1)\rangle$

$AA' \perp BC \Rightarrow \langle AA', BC\rangle=0 \Rightarrow -4\cdot a+b=0$

$AA' \begin{cases} A\in AA' \\ -4a+b=0 \end{cases} - \begin{cases} x=1-4\lambda \\ y=2+\lambda \;|\cdot 4 \end{cases}$ (+)

$4y+x=9 \Rightarrow AA': x+4y-9=0$

$\vec{AC}(4,4)$ $\qquad AC: \begin{cases} x=5+4\lambda \\ y=6+4\lambda \end{cases}$ (-)

$x-y=-1 \Rightarrow AC: x-y+1=0 \Rightarrow n_{AC}=\langle(1,-1)\rangle$

$BB' \perp AC \Rightarrow BB': \begin{cases} x=3+\lambda \\ y=-2-\lambda \end{cases}$ (+)

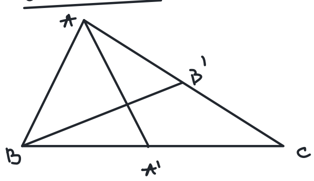
$x+y=1 \Rightarrow BB': x+y-1=0$

$AA'\cap BB' = \begin{cases} x+4y-9=0 \\ x+y-1=0 \end{cases}$ (-)

$3y-8=0 \Rightarrow y=\frac{8}{3} \Rightarrow x=1-\frac{8}{3}=\frac{-5}{3}$

$\Rightarrow H\left(\frac{-5}{3},\frac{8}{3}\right)$

circum center



$A'\left(\frac{x_B+x_C}{2},\frac{y_B+y_C}{2}\right)=A'(4,2)$

$B'(3,4)$

$\cancel{\vec{BB'}=(0,6)}$

$\cancel{\vec{AA'}=(3,0)}$

$BC\perp AA': \begin{cases} x=4+4\lambda \\ y=2-\lambda \end{cases}$ (+)

$x+4y=12$

$AC\perp BB': \begin{cases} x=3+\lambda \\ y=4-\lambda \end{cases}$ (+)

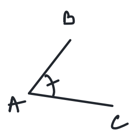
$x+y=7$

$\Rightarrow AA'\cap BB'=\{O\left(\frac{16}{3},\frac{5}{3}\right)\}$

$\boxed{3.20.}$ Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle $\measuredangle A$.

$AB: \frac{x-1}{5-1}=\frac{y+2}{4+2} \Rightarrow 3x-2y-7=0$

$AC: \frac{x-1}{-2-1}=\frac{y+2}{2} \Rightarrow 2x-2=-3y-6 \Rightarrow 2x+3y+4=0$



let $M\in\ell$ st $dist(M,AB)=dist(M,AC)$

$\frac{|3x-2y-7|}{\sqrt{3^2+2^2}}=\frac{|2x+3y+4|}{\sqrt{2^2+3^2}}$

$\Rightarrow |3x-2y-7|=\pm(2x+3y+4)$

$3x-2y-7=2x+3y+4$

$x-5y-11=0$

$3x-2y-7=-2x-3y-4$

$5x+y-3=0$

b) ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

3.13. For the lines ℓ in the previous exercise

a) give parametric equations for ℓ,

b) describe $\mathrm{D}(\ell)$.

3.14. Consider a line ℓ. Show that

c) if $\mathbf{v}(v_1,v_2)$ is a direction vector for ℓ then $\mathbf{n}(v_2,-v_1)$ is a normal vector for ℓ,

d) if $\mathbf{n}(n_1,n_2)$ is a normal vector for ℓ then $\mathbf{v}(n_2,-n_1)$ is a direction vector for ℓ.

3.15. Consider the points $A(1,2)$, $B(-2,3)$ and $C(4,7)$. Determine the medians of the triangle ABC.

3.16. Let $M_1(1,2)$, $M_2(3,4)$ and $M_3(5,-1)$ be the midpoints of the sides of a triangle. Determine Cartesian equations and parametric equations for the lines containing the sides of the triangle.

3.17. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

a) the length of the altitude from A,

b) the line containing the altitude from A.

3.18. Determine the circumcenter and the orthocenter of the triangle with vertices $A(1,2)$, $B(3,-2)$, $C(5,6)$.

3.19. Determine the angle between the lines $\ell_1 : y = 2x+1$ and $\ell_2 : y = -x+2$.

3.20. Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle $\measuredangle A$.

3.21. Let A' be the orthogonal reflection of $A(10,10)$ in the line $\ell : 3x+4y-20 = 0$. Determine the coordinates of A'.

3.22. Determine Cartesian equations for the lines passing through $A(-2,5)$ which intersect the coordinate axes in congruent segments.

3.23. Determine Cartesian equations for the lines situated at distance 4 from the line $12x-5y-15 = 0$.

3.24. Determine the values k for which the distance from the point $(2,3)$ to the line $8x+15y+k = 0$ equals 5.

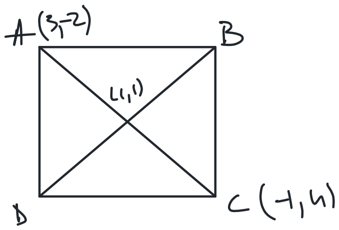
3.25. Consider the points $A(3,-1)$, $B(9,1)$ and $C(-5,5)$. For each pair of these three points, determine the line which is equidistant from them.

3.26. The point $A(3,-2)$ is the vertex of a square and $M(1,1)$ is the intersection point of its diagonals. Determine Cartesian equations for the sides of the square.

3.27. Determine a point on the line $5x-4y-4 = 0$ which is equidistant to the points $A(1,0)$ and $B(-2,1)$.

3.26. The point $A(3,-2)$ is the vertex of a square and $M(1,1)$ is the intersection point of its diagonals. Determine Cartesian equations for the sides of the square.

M – mid $AC \Rightarrow \left.\begin{matrix} x_C = 2-3 = -1 \\ y_C = 2+2 = 4 \end{matrix}\right\} \Rightarrow C = (-1,4)$



$$m_{AC} = \frac{1+2}{1-3} = \frac{3}{-2} = -\frac{3}{2}$$

$AC \perp BD \Rightarrow m_{AC} \cdot m_{BD} = -1 \Rightarrow m_{BD} = \frac{2}{3}$

$BD: \; y - y_M = \frac{2}{3}(x - x_M)$

$3y - 3 = 2x - 2 \Rightarrow 2x - 3y + 1 = 0 \Rightarrow x = \frac{3y-1}{2}$

$\|AC\| = \sqrt{16+36} = \sqrt{52} = 2\sqrt{13} \Rightarrow \|BD\| = 2\sqrt{13}$

$(y-y_M)^2 + (x-x_M)^2 = 13$

$y^2 - 2y + x^2 - 2x = 11$

$y^2 - 2y + \frac{9y^2 - 6y + 1}{4} - 3y + 1 = 11 \quad | \cdot 4$

$4y^2 - 8y + 9y^2 - 6y + 1 - 12y + 4 = 44$

$13y^2 - 26y - 39 = 0 \quad | : 13$

$y^2 - 2y - 3 = 0$

$\Delta = 4 + 12 = 16 \Rightarrow y_{1,2} = \frac{2 \pm 4}{2} \; \begin{matrix} y_1 = 3 \Rightarrow x = 4 \\ y_2 = -1 \Rightarrow x = -2 \end{matrix}$

3.29. Determine an equation for each plane passing through $P(3,5,-7)$ and intersecting the coordinate axes in congruent segments.

π: plane $\quad P(3,5,-7) \in \pi$

$$\begin{vmatrix} x & y & z & 1 \\ \alpha & 0 & 0 & 1 \\ 0 & \beta & 0 & 1 \\ 0 & 0 & \gamma & 1 \end{vmatrix} = 0 \qquad \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$$

$\Rightarrow |\alpha| = |\beta| = |\gamma| = m$

take cases

3.30. Let $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$ be vertices of a tetrahedron. Determine a Cartesian equation of the plane containing $[AB]$ and the midpoint of $[CD]$.

$\pi : \begin{cases} AB \in \pi \\ M \in \pi \end{cases}$

$M(3,-2,6) \in \pi$

$\vec{AB} = (-1,2,5)$

we need 2 vect. and 1 point

$\vec{AM}(1,-3,6)$

$$\Rightarrow \begin{vmatrix} x-x_A & y-y_A & z-z_A \\ x_{AB} & y_{AB} & z_{AB} \\ x_{AM} & y_{AM} & z_{AM} \end{vmatrix} = \begin{vmatrix} x-2 & y-1 & z \\ -1 & 2 & 5 \\ 1 & -3 & 6 \end{vmatrix} = 12x - 24 + 3z + 5y$$

$-5 - 2z + 15x - 30 + 6y - 6 = 27x + 11y + z - 65 = 0$

3.31. Show that a parallelepiped with faces in the planes $\underbrace{2x + y - 2z + 6 = 0}_{\pi_1}$, $\underbrace{2x - 2y + z - 8 = 0}_{\pi_2}$ and $\underbrace{x + 2y + 2z + 1 = 0}_{\pi_3}$ is rectangular.

$n_{\pi_1}(2,1,-2)$

$n_{\pi_2}(2,-2,1)$

$n_{\pi_3}(1,2,2)$

$\left.\begin{matrix} \Rightarrow n_1 \cdot n_2 = 4 - 2 - 2 = 0 \\ n_2 \cdot n_3 = 2 - 4 + 2 = 0 \\ n_1 \cdot n_3 = 2 + 2 - 4 = 0 \end{matrix}\right\} \Rightarrow$ all $\perp$ to one another

3.28. The point $A(2,0)$ is the vertex of an equilateral triangle. The side opposite to A lies on the line $x+y-1=0$. Determine Cartesian equations for the lines containing the other two sides.

3.29. Determine an equation for each plane passing through $P(3,5,-7)$ and intersecting the coordinate axes in congruent segments.

3.30. Let $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$ be vertices of a tetrahedron. Determine a Cartesian equation of the plane containing $[AB]$ and the midpoint of $[CD]$.

3.31. Show that a parallelepiped with faces in the planes $2x+y-2z+6=0$, $2x-2y+z-8=0$ and $x+2y+2z+1=0$ is rectangular.

3.32. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.

3.33. Determine the distance between the planes $x-2y-2z+7=0$ and $2x-4y-4z+17=0$.

3.34. Solve Exercise 2.16 using normal vectors. cross prod of norm. vectors

3.35. Let $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,-5)$ be vertices of a triangle. Determine the equation of the internal angle bisector of $\measuredangle A$.

3.36. Determine the angles between the plane $\pi_1 : x-\sqrt{2}y+z-1=0$ and the plane $\pi_2 : x+\sqrt{2}y-z+3=0$. dot prod.

* **3.37.** Determine the values a and c for which the line $3x-2y+z+3=0 \cap 4x-3y+4z+1=0$ is perpendicular to the plane $ax+8y+cz+2=0$.

3.38. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x+4y-3z+7=0$.

3.39. Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $2x-3y+z-4=0$.

3.40. Consider the point $A(1,3,5)$ and the line $\ell : 2x+y+z-1=0 \cap 3x+y+2z-3=0$.

a) Determine the orthogonal projection of A on ℓ.

$$\ell : \begin{cases} 2x+y+z-1=0 \Rightarrow z=-2x-y+1 \\ 3x+y+2z-3=0 \end{cases} \quad \begin{matrix} 2y+2-y+1 \\ y+3 \end{matrix} \Rightarrow \ell : \begin{cases} x=-t-1 \\ y=t \\ z=t+3 \end{cases} \Rightarrow \vec{v}(-1,1,1)$$

$$\Rightarrow 3x+y-4x-2y+2-3=0$$

$$-x-y-1=0 \Rightarrow x+y+1=0 \Rightarrow x=-y-1$$

b) Determine the orthogonal reflection of A in ℓ.

3.41. Determine the planes which pass through $P(0,2,0)$ and $Q(-1,0,0)$ and which form an angle of $60°$ with the z-axis.

3.42. Determine the orthogonal projection of the line $\ell : 2x-y-1=0 \cap x+y-z+1=0$ on the plane $\pi : x+2y-z=0$.

3.43. Determine the coordinates of a point A on the line $\ell : \frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{1}$ which is at distance $\sqrt{3}$ from the plane $x+y+z+3=0$.

$$3x-3=2y=6z+6 \Rightarrow 3x-3-2y-6z-6=0 \Rightarrow 3x-2y-6-9=0$$

3.44. The vertices of a tetrahedron are $A(-1,-3,1)$, $B(5,3,8)$, $C(-1,-3,5)$ and $D(2,1,-4)$. Determine the height of the tetrahedron relative to the face ABC.

$$\sqrt{3} = \frac{|x_1+y_1+z_1+3|}{\sqrt{3}}$$

$$\Rightarrow |x_1+y_1+z_1+3| = 3$$

I $x_1+y_1+z_1=0 \Rightarrow x_1=-y_1-z_1$

II $x_1+y_1+z_1=-6$

$$3x-2y-6z-9=0$$

$$-3y_1-3z_1-2y_1-6z_1-9=0$$

$$-9z_1-5y_1-9=0 \Rightarrow$$

$$\ell : \frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{1} = \lambda$$

$$\ell : \begin{cases} x=2\lambda+1 \\ y=3\lambda \\ z=\lambda-1 \end{cases}$$

$$\Rightarrow 3 = |2\lambda+\cancel{1}+3\lambda+\lambda-\cancel{1}+3|$$

$6\lambda+3=3$ $\quad$ $6\lambda+3=-3$

$6\lambda=0 \Rightarrow \lambda_1=0$ $\quad$ $6\lambda=-6$, $\lambda_2=-1$

3.32. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.

π plane $\quad A(1,-1,3)$ orthogonal proj. of origin

$\Rightarrow \vec{OA} \perp \pi$

let $\pi: ax + by + cz + d = 0$

$\vec{OA}$ – normal vect. of $\pi \Rightarrow x - y + 3z + d = 0$

$A \in \pi: 1 + 1 + 9 + d = 0 \Rightarrow d = -11 \Rightarrow x - y + 3z - 11 = 0$

3.33. Determine the distance between the planes $x - 2y - 2z + 7 = 0$ and $2x - 4y - 4z + 17 = 0$.

$1 - 2y - 2z + 7 = 0$

$8 - 2y - 2z = 0$

$(1,2,2) \in \pi \qquad d(P, \pi_2) = \dfrac{|2x - 4y - 4z + 17|}{\sqrt{4+16+16}} = \dfrac{|2 - 8 - 8 + 17|}{\sqrt{36}} = \dfrac{3}{6} = \dfrac{1}{2}$

3.35. Let $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,-5)$ be vertices of a triangle. Determine the equation of the internal angle bisector of $\measuredangle A$.

$AB: x - 1 = 0$

$AC: \dfrac{x-1}{3-1} = \dfrac{y-2}{4-2} = \dfrac{z+7}{-5+7}$

$x - 1 = y - 2 = z + 7$

$x - y - z - 6 = 0$

$\vec{AB}(2,0,0)$

$\vec{AC}(2,2,2)$

let $M \in \ell$

$\text{dist}(M, AB) = \text{dist}(M, AC)$

$AB: \dfrac{x-1}{2-1} = \dfrac{y-2}{0} = \dfrac{z+7}{0}$

$\dfrac{|x_M - 1|}{\sqrt{1}} = \dfrac{|x_M - y_M - z_M - 6|}{\sqrt{3}}$

3.37. Determine the values a and c for which the line $3x - 2y + z + 3 = 0 \cap 4x - 3y + 4z + 1 = 0$ is perpendicular to the plane $ax + 8y + cz + 2 = 0$.

$\ell: \begin{cases} 3x - 2y + z + 3 = 0 & | \cdot (-4) \\ 4x - 3y + 4z + 1 = 0 & | \cdot (3) \end{cases}$

$-y + 8z - 9 = 0 \Rightarrow y = 8z - 9$

$3x - 16z - 18 + z + 3 = 0$

$3x - 15z - 15 = 0 \quad | : 3$

$x - 5z - 5 = 0$

$x = 5z + 5$

$\ell: \begin{cases} x = 5t + 5 \\ y = 8t - 9 \\ z = t \end{cases} \Rightarrow \ell = \langle (5t+5, 8t-9, t) \rangle \Rightarrow D(\ell) = \langle (10, -1, 1) \rangle$

$\ell \perp \pi$ iff $\langle D(\ell), n \rangle = 0$

$(10,-1,1) \cdot (a,8,c) = 0$

$10a - 8 + c = 0 \Rightarrow c = 8 - 10a$

$(10,-1,1) \parallel (a,8,1)$ iff

$8t - 9 = 8 \Rightarrow t = \dfrac{17}{8}$

$\left(5 \cdot \dfrac{17}{8} + 5, 8, \dfrac{17}{8}\right)$?

3.38. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x + 4y - 3z + 7 = 0$.

$d = \dfrac{a \cdot v}{\|a\|^2} \cdot a$

$v(1,4,-3)$

$\vec{OA}(2,11,-5)$

$d = \dfrac{2 + 44 + 15}{1 + 16 + 9} \cdot (1,4,-3) = \dfrac{61}{26}(1,4,-3)$

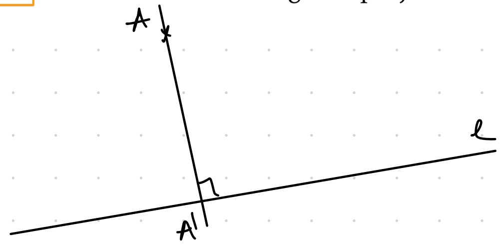
$\begin{cases} \dfrac{32}{7} + a = 12 \\ \dfrac{-48}{7} + b = -10 \\ \dfrac{16}{7} + c = 10 \end{cases}$

3.39. Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $2x - 3y + z - 4 = 0$.

$d = \dfrac{12 + 15}{4 + 9 + 1} (2,-3,1) = \dfrac{32}{14}(2,-3,1) = \dfrac{16}{7}(2,-3,1)$ projection

reflection $\Rightarrow$ P-mid. $\Rightarrow (6,-5,5) = \left(\dfrac{\frac{32}{7} + a}{2}, \dfrac{\frac{16}{7}\cdot(-3) + b}{2}, \dfrac{\frac{16}{7} + c}{2} \right)$

3.38. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x+4y-3z+7=0$.



$n_{\pi} = (1,4,-3)$

$\frac{x-1}{1} = \frac{y-11}{4} = \frac{z+5}{-3} = k$

$l: \begin{cases} x = k+2 \\ y = 4k+11 \\ z = -3k-5 \end{cases}$

$\Rightarrow k+1+4(4k+11)-3(-3k-5)+7=0$

$k+2+16k+44+9k+15+7=0$

$26k+68=0$

$k = \frac{68}{26}$

3.45. In $\mathbb{E}^4$, determine the angles between the hyperplanes:

$$H_1 : 3w - x - y + z + 2 = 0 \quad \text{and} \quad H_2 : -w + 2x + y - z + 2 = 0.$$

3.46. In $\mathbb{E}^4$, show that the lines

$$\ell_1 : \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \ell_2 : \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

are skew and determine the hyperplane containing them.

3.47. In $\mathbb{E}^4$, determine a point on the first coordinate axis which is equidistant from $P_1(1,-1,0,2)$ and $P_2(0,-2,0,1)$.

3.48. In $\mathbb{E}^4$, consider the line

$$\ell : \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{and the hyperplane} \quad H : \lambda w - x - y + \mu z + 2 = 0.$$

For what values λ and μ is H orthogonal to ℓ? For what values λ and μ is H parallel to ℓ?

3.19. Determine the angle between the lines $\ell_1 : y = 2x + 1$ and $\ell_2 : y = -x + 2$.

$\ell_1 : y = 2x + 1 \Rightarrow 2x - y + 1 = 0$

$\ell_2 : y = -x + 2 \Rightarrow -x - y + 2 = 0$

$2 \cdot (-1) + (-1)(-1) \quad \sqrt{2^2+1} \quad \sqrt{1^2+1^2}$

$\langle \ell_1, \ell_2 \rangle = |\ell_1| \, |\ell_2| \cos\alpha$

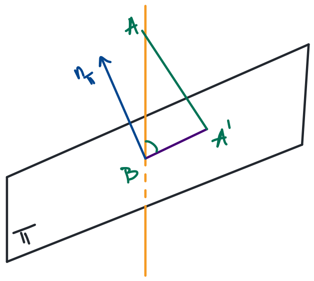
$-2 + 1 = \sqrt{5}\sqrt{2} \cos\alpha$

$-1 = \sqrt{10} \cos\alpha$

$-\frac{1}{\sqrt{10}} = \cos\alpha$

$\frac{-\sqrt{10}}{10}$

3.41. Determine the planes which pass through $P(0,2,0)$ and $Q(-1,0,0)$ and which form an angle of $60°$ with the z-axis.



$\ell \cap \tilde{\pi} = \{B\}$

$A' = pr_\pi(A)$

$m(\ell, \pi) = \frac{\pi}{2} - m(\vec{n_\pi}, \vec{v})$

unde $\vec{v} \in D(\ell)$

$m(\vec{n_\pi}, \vec{v}) = 30^\circ$

$\vec{n_\pi}(a,b,c) \Rightarrow m((a,b,c),(0,0,1)) = 30^\circ$

$\Rightarrow \cos 30^\circ = \frac{\langle (a,b,c),(0,0,1)\rangle}{\sqrt{a^2+b^2+c^2}}$

$\tilde{\pi}: ax+by+cz+d=0$

$P(0,2,0) \in \tilde{\pi} \Rightarrow 2b+d=0 \Rightarrow b=\frac{-a}{2}$

$Q(-1,0,0) \in \pi \Rightarrow -a+d=0 \Rightarrow d=a$

$\Rightarrow \frac{c^2}{a^2+b^2+c^2} = \frac{3}{4} \Rightarrow \frac{c^2}{a^2+\frac{a^2}{4}+c^2} = \frac{-3}{4} \Rightarrow 4c^2 = 3c^2 + \frac{15a^2}{4}$

$c^2 = \frac{15a^2}{4} \Rightarrow c = \frac{\sqrt{15}\,a}{2}$

$\tilde{\pi}: ax - \frac{a}{2}y \pm \frac{\sqrt{15}\,a}{2}z + a = 0 \quad |:a$

$x - \frac{1}{2}y \pm \frac{\sqrt{15}}{2} + 1 = 0$

$\boxed{\cancel{3.29}}$. Determine an equation for each plane passing through $P(3,5,-7)$ and intersecting the coordinate axes in congruent segments.

$\tilde{\pi} \cap Ox = (\pm\alpha, 0, 0)$

$\tilde{\pi} \cap Oy = (0, \pm\alpha, 0)$?

$\tilde{\pi} \cap Oz = (0, 0, \pm\alpha)$

Plan determinat de 3 puncte $A_i(x_i, y_i, z_i)$

$$\tilde{\pi}: \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

, dezvoltăm după prima linie

4.19. Consider two lines ℓ_1 and ℓ_2 in $\mathbb{E}^3$. Suppose that the common perpendicular line is

$$\ell : \begin{cases} x = 1 + t \\ y = 2 - t \\ z = t \end{cases},$$

that $P_1(1,0,1) \in \ell_1$ and that $P_2(-1,1,0) \in \ell_2$. Determine the two lines.

$\vec{b}(1,-1,1)$ vect. normal pt π_1 și π_2

$\pi_1: 1\cdot(x-1) + (-1)\cdot(y-0) + 1\cdot(z-1) = 0$

$x - y + z - 2 = 0$

$\pi_2: 1(x+1) + (-1)(y-1) + 1\cdot(z-0) = 0$

$x - y + z + 2 = 0$

Fie $A = \ell \cap \pi_1$, $B = \ell \cap \pi_2$

$$A = \begin{cases} x - y + z - 2 = 0 \\ x = 1+t \\ y = 2-t \\ z = t \end{cases} \Rightarrow 1+t-2+t+t-2=0,\quad 3t-3=0 \Rightarrow t=1 \quad \Rightarrow A(2,1,1)$$

$$\ell_1: \frac{x-2}{1-2} = \frac{y-1}{0-1} = \frac{z-1}{1-1}$$

$$\ell: \begin{cases} \dfrac{x-2}{-1} = \dfrac{y-1}{-1} \\ z = 1 \end{cases} \Rightarrow \begin{cases} x - y - 1 = 0 \\ z = 1 \end{cases}$$

analog ℓ_2

4.17. With respect to an orthonormal system consider the vectors $\mathbf{a}(8,4,1)$, $\mathbf{b}(2,2,1)$ and $\mathbf{c}(1,1,1)$. Determine a vector $\mathbf{d}$ satisfying the following properties

a) the angles $\measuredangle(\mathbf{d},\mathbf{a})$ and $\measuredangle(\mathbf{d},\mathbf{b})$ are equal, * dot prod

b) $\mathbf{d}$ is orthogonal to $\mathbf{c}$,

c) $(\mathbf{a},\mathbf{b},\mathbf{c})$ and $(\mathbf{a},\mathbf{b},\mathbf{d})$ have the same orientation.

$\vec{d}(x,y,z)$ $\quad x^2+y^2+z^2=1$ $\quad$ vrem doar directia

a) $\Rightarrow \dfrac{\vec{d}\cdot\vec{a}}{|a|} = \dfrac{\vec{d}\cdot\vec{b}}{|b|}$

$$\frac{8x+4y+z}{\sqrt{64+16+1}} = \frac{2x+2y+z}{\sqrt{4+4+1}}$$

$$3(8x+4y+z) = 9(2x+2y+z)$$

$$6x - 6y - 6z = 0 \quad |:6$$

$$\boxed{x - y - z = 0}$$

b) $\Rightarrow \vec{d} \perp \vec{c} \Rightarrow \vec{d}\cdot\vec{c} = 0 \Rightarrow x+y+z=0$

c) $[a,b,c]\cdot[a,b,d]$ Box product !! >0 pt aceeasi orientare

$$\begin{vmatrix} 8 & 4 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 16+2+4-2-8-8 = 4 > 0$$

$$\begin{vmatrix} 8 & 4 & 1 \\ 2 & 2 & 1 \\ x & y & z \end{vmatrix} > 0 \Rightarrow 16z+2y+4x-2x-8y-8z>0$$

$$2x - 6y + 8z > 0 \quad |:2$$

$$x - 3y + 4z > 0$$

$$\begin{cases} x - y - z = 0 \\ x + y + z = 0 \\ x - 3y + 4z > 0 \end{cases} \Rightarrow \begin{cases} x = y+z \Rightarrow x = 0 \\ y = -z \\ 3z+4z>0 \Rightarrow z>0 \\ z^2+z^2 = 1 \Rightarrow z = \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \vec{d}\left(0, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)$$

3.42. Determine the orthogonal projection of the line $\ell: 2x-y-1=0 \cap x+y-z+1=0$ on the plane $\pi: x+2y-z=0$.

$$\ell: \begin{cases} 2x-y-1=0 \\ x+y-z+1=0 \end{cases} \Rightarrow \begin{cases} y = 2x-1 \\ x+2x\cancel{-1}-z\cancel{+1}=0 \Rightarrow 3x = z \end{cases} \Rightarrow \ell: \begin{cases} x = t \\ y = 2t-1 \\ z = 3t \end{cases} \Rightarrow \vec{v}(1,2,3)$$

let 2 points A and $B \in \ell$: $\quad A(0,-1,0)$ and $B(1,1,3)$

~~$P_1 = \dfrac{b\cdot a}{\|a\|^2}\cdot a =$~~

$n_\pi (1,2,-1)$ $\quad \ell_1: \begin{cases} x = t+\lambda \\ y = 2t-1-2\lambda \\ z = 3t-\lambda \end{cases}$

$$P_\pi = \begin{cases} x = t+\lambda \\ y = 2t-1+2\lambda \\ z = 3t-\lambda \end{cases}$$

$x+2y-z=0$

$t+\lambda+4t-2+4\lambda-3t+\lambda=0$

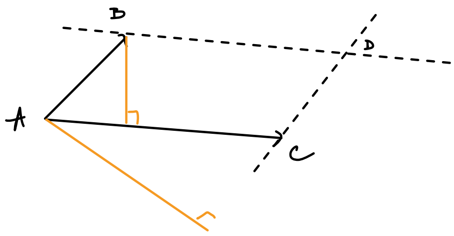
$2t+6\lambda-2=0 \mid :2$

$t+3\lambda-1=0 \Rightarrow \lambda = \frac{1-t}{3}$

4.3. Determine the distances between opposite sides of a parallelogram spanned by the vectors $\overrightarrow{AB}(6,0,1)$ and $\overrightarrow{AC}(1.5,2,1)$ if the coordinates of the vectors are given with respect to a right oriented orthonormal basis.

$d(B,AC) = \frac{|AC \times AB|}{\sqrt{36+1}} = \frac{\sqrt{673}}{2\sqrt{24}}$

$CC' = \frac{|\vec{AC}\times\vec{AB}|}{|\vec{AB}|}$



$d(C,AB)$

4.10. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis. Determine the matrices of the linear maps $\phi,\psi:\mathbb{V}^3\to\mathbb{V}^3$ defined by $\phi(\mathbf{v})=\mathbf{w}\times\mathbf{v}$ and $\psi(\mathbf{v})=\mathbf{v}\times\mathbf{u}$ where

~~a)~~ $\mathbf{w}=-\mathbf{i}+3\mathbf{j}+\mathbf{k}$,

b) $\mathbf{w}=\mathbf{i}+\mathbf{k}$,

c) $\mathbf{u}=2\mathbf{i}-\mathbf{j}$,

d) $\mathbf{u}=\mathbf{i}+\mathbf{j}+\mathbf{k}$.

a) $w = -i+3j+k$

$$w\times v = \begin{vmatrix} i & j & k \\ -1 & 3 & 1 \\ a & b & c \end{vmatrix} = 3ci - bk + aj - 3ak - bi + cj = (3c-b)i + (a+c)j + (-b-3a)k$$

$\phi(i) = \phi(1,0,0) = j-3k$

$\phi(j) = -i-k$

$\phi(k) = 3i+j$

$$\Rightarrow [\phi]_B = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}$$

4.13. The points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

$\vec{AB}(-1,-1,6)$ $\quad \vec{AC}(-2,0,2)$ $\quad \vec{AB}(1,-1,4)$

$$\begin{vmatrix} -1 & -1 & 6 \\ -2 & 0 & 2 \\ 1 & -1 & 4 \end{vmatrix} = 0+12-2-0-2-8=0 \Rightarrow \text{coplanar}$$

4.16. The volume of a tetrahedron $ABCD$ is 5. With respect to an orthonormal system $Oxyz$ the vertices are $A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$ and $D\in Oy$. Determine the coordinates of D.

Volume $\Rightarrow$ Box product!

$V = \frac{1}{6}[\vec{AB},\vec{AC},\vec{AD}]$

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ -2 & y-1 & 1 \end{vmatrix} = -2+0+\cancel{8}-\cancel{8}-4y+4 = -4y+2$$

$V = \frac{1}{6}(-4y+2) \Rightarrow -4y+2 = 30$

$y = \frac{28}{-4} = -7 \Rightarrow D(0,-7,0)$

4.5 Exercises

4.1. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis of $\mathbb{V}^3$. Consider the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Determine $\mathbf{a} \times \mathbf{b}$ in terms of the given basis vectors.

~~4.2.~~ With respect to a right oriented orthonormal basis of $\mathbb{V}^3$ consider the vectors $\mathbf{a}(3,-1,-2)$ and $\mathbf{b}(1,2,-1)$. Calculate

$$\mathbf{a} \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times \mathbf{b}, \quad (2\mathbf{a} + \mathbf{b}) \times (2\mathbf{a} - \mathbf{b}).$$

~~4.3.~~ Determine the distances between opposite sides of a parallelogram spanned by the vectors $\overrightarrow{AB}(6,0,1)$ and $\overrightarrow{AC}(1.5,2,1)$ if the coordinates of the vectors are given with respect to a right oriented orthonormal basis.

4.4. Consider the vectors $\mathbf{a}(2,3,-1)$ and $\mathbf{b}(1,-1,3)$ with respect to an orthonormal basis.

a) Determine the vector subspace $\langle \mathbf{a}, \mathbf{b} \rangle^{\perp}$. $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 9i - 2k - j - 3k - i - 6j = 8i - 7j - 5k = \langle (8,-7,-5) \rangle$

b) Determine the vector $\mathbf{p}$ which is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ and for which $\mathbf{p} \cdot (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 51$.

$\alpha(8,-7,-5)(2,-3,4) = 51 \Rightarrow 17\alpha = 51 \Rightarrow \alpha = 3$

4.5. Consider the points $A(1,2,0)$, $B(3,0,-3)$ and $C(5,2,6)$ with respect to an orthonormal coordinate system.

a) Determine the area of the triangle ABC.

b) Determine the distance from C to AB.

4.6. Let $ABCD$ be a quadrilateral in $\mathbb{E}^3$ and let E, F be the midpoints of $[AB]$ and $[CD]$ respectively. Denote by K, L, M and N the midpoints of the segments $[AF]$, $[CE]$, $[BF]$ and $[DE]$ respectively. Prove that $KLMN$ is a parallelogram.

4.7. Let ABC be a triangle and let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{CA}$. Show that

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}.$$

and deduce the law of sines in a triangle.

4.8. With respect to a right oriented orthonormal coordinate system consider the vectors $\mathbf{a}(2,-3,1)$, $\mathbf{b}(-3,1,2)$ and $\mathbf{c}(1,2,3)$. Calculate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

4.9. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\psi : \mathbb{V}^3 \to \mathbb{V}^3$ be the map $\phi(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to a right oriented orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w} \times \mathbf{v}$?

4.10. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be a right oriented orthonormal basis. Determine the matrices of the linear maps $\phi, \psi : \mathbb{V}^3 \to \mathbb{V}^3$ defined by $\phi(\mathbf{v}) = \mathbf{w} \times \mathbf{v}$ and $\psi(\mathbf{v}) = \mathbf{v} \times \mathbf{u}$ where

~~a)~~ $\mathbf{w} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$,

b) $\mathbf{w} = \mathbf{i} + \mathbf{k}$,

c) $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$,

d) $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

4.11. Prove the following identities:

a) the Grassmann identity,

b) the Jacobi identity,

c) the Lagrange identity,

d) the formula for the cross product of two cross products.

4.12. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be a right oriented orthonormal basis. Consider the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{k}$ and $\mathbf{c} = \mathbf{k}$. Determine if

a) $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a basis of $\mathbb{V}^3$,

b) if it is a basis, decide if it is left or right oriented.

~~4.13~~. The points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

4.14. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be an orthonormal basis and consider the vectors $\mathbf{u} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{k}$. Determine the matrix of the linear map $\phi : \mathbb{V}^3 \to \mathbb{R}$ defined by $\phi(\mathbf{v}) = [\mathbf{v}, \mathbf{u}, \mathbf{w}]$.

4.15. Determine the volume of the tetrahedron with vertices $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,-1)$ and $D(4,1,3)$ given with respect to an orthonormal system.

~~4.16~~. The volume of a tetrahedron $ABCD$ is 5. With respect to an orthonormal system $Oxyz$ the vertices are $A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$ and $D \in Oy$. Determine the coordinates of D.

~~4.17~~. With respect to an orthonormal system consider the vectors $\mathbf{a}(8,4,1)$, $\mathbf{b}(2,2,1)$ and $\mathbf{c}(1,1,1)$. Determine a vector $\mathbf{d}$ satisfying the following properties

a) the angles $\measuredangle(\mathbf{d}, \mathbf{a})$ and $\measuredangle(\mathbf{d}, \mathbf{b})$ are equal,

b) $\mathbf{d}$ is orthogonal to $\mathbf{c}$,

c) $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and $(\mathbf{a}, \mathbf{b}, \mathbf{d})$ have the same orientation.

4.18. For the tetrahedron $ABCD$ in Exercise 4.15, determine the common perpendicular of the sides AB and CD.

~~4.19~~. Consider two lines ℓ_1 and ℓ_2 in $\mathbb{E}^3$. Suppose that the common perpendicular line is

$$\ell : \begin{cases} x = 1 + t \\ y = 2 - t \\ z = t \end{cases},$$

that $P_1(1,0,1) \in \ell_1$ and that $P_2(-1,1,0) \in \ell_2$. Determine the two lines.

4.20. In $\mathbb{E}^4$ consider the affine subspaces

$$\ell : \begin{cases} w = -1 + 2t \\ x = 1 + t \\ y = 2 - t \\ z = -2 + t \end{cases} \quad \text{and} \quad \pi : \begin{cases} w + 2y - 1 = 0 \\ x - z + 2 = 0 \end{cases} .$$

Show that $\ell \parallel \pi$ and determine the distance between them.

4.21. In $\mathbb{E}^4$ consider the parallel lines

$$\ell_1 : \begin{cases} w = -1 + 2t \\ x = 1 + t \\ y = 2 - t \\ z = -2 + t \end{cases} \quad \text{and} \quad \begin{cases} w = 1 + 2t \\ x = t \\ y = 1 - t \\ z = 2 + t \end{cases} .$$

Determine the distance between them.

Partial 1

Part 1 $\quad K=(O, i, j) \quad K'=(O', i', j')$

$[O']_K = \begin{pmatrix}4\\0\end{pmatrix} \quad [i']_K = \begin{bmatrix}-1\\1\end{bmatrix} \quad [j']_K = \begin{bmatrix}0\\1\end{bmatrix}$

$[A]_K = \begin{pmatrix}2\\4\end{pmatrix} \quad [B]_K = \begin{pmatrix}4\\1\end{pmatrix} \quad [C]_K = \begin{pmatrix}4\\2\end{pmatrix}$

a) $\vec{BC}\,(0,1) = j \Rightarrow$ yes

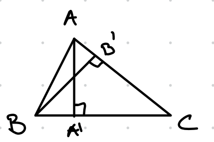
b) $\dfrac{x-2}{4-2} = \dfrac{y-4}{2-4}$

$\dfrac{x-2}{2} = \dfrac{y-4}{-2}$

$x-2 = -y+4 \Rightarrow x+y-6=0 : AC$

$d(B,AC) = \dfrac{|x+y-6|}{\sqrt{1+1}} = \dfrac{|4+1-6|}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

c)



$\vec{BC}\,(0,1) \qquad m_{BC} = 0$

$\vec{AA'} \perp \vec{BC}$

$A' \in BC \Rightarrow AA' : y - y_A = m(x - x_A)$

$y - 4 = 0 \Rightarrow \boxed{y=4}$

$\vec{AC}\,(2,-2) \qquad m_{AC} = \dfrac{-2}{2} = -1 \Rightarrow m_{BB'} = 1$

$y - y_B = x - x_B$

$y - 1 = x - 4 \Rightarrow y = x - 3$

$BB' \cap AA' = \begin{cases} y = 4 \\ y = x-3 \end{cases} \Rightarrow x = 4+3 = 7 \Rightarrow O(7,4)$

d)

$M_{KK'} = \begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}$

$[A]_{K'} = M_{KK'}\left([A]_K - [O']_K\right) = \begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}\left(\begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}4\\0\end{pmatrix}\right) =$

$= \begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}-2\\4\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$

$[C]_{K'} = \begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$

$[\vec{AC}]_{K'} = \begin{pmatrix}-2\\0\end{pmatrix}$

Part II $\quad (O, i, j, k)$

$[A]_K = \begin{pmatrix}1\\2\\0\end{pmatrix} \quad [B]_K = \begin{pmatrix}1\\2\\1\end{pmatrix} \quad [C]_K = \begin{pmatrix}0\\3\\2\end{pmatrix} \quad [D]_K = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$K'\,(D, \vec{BA}, \vec{DA}, \vec{DC})$

a) $d(A, BD)$

$\vec{BD}\,(0,-1,0)$

$\vec{AB}\,(0,0,1)$

$pr_{AB} BD : \dfrac{(0,0,1)(0,-1,0)}{1}(0,-1,0) = \dfrac{0}{1}(0,-1,0) = (0,0,0)$

$\|pr_{AB} BD\| = 0 \in BD$ or $AB \perp BD$

b) Plane BCD

$\vec{BC}(-1,1,1)$

$\vec{CD}(1,-2,-1)$

$$\vec{BC} \times \vec{CD} = \begin{vmatrix} i & j & k \\ -1 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = i + 2k + j - k + 2i + j = 3i + 2j + k \Rightarrow n_{BCD}(3,2,1)$$

$\vec{BA}(0,0,-1)$

$$Pr_{\vec{n}} \vec{BA} = \frac{(0,0,-1)(3,2,1)}{\|(3,2,1)\|^2} \cdot (3,2,1) = \frac{-1}{9+4+1}(3,2,1) = \frac{-1}{14}(3,2,1)$$

Projection point to plane: $P = \underset{\substack{\uparrow \\ \text{point on plane}}}{B} + Pr_{\vec{n}} \vec{BA}$

$\vec{BD}(0,-1,0)$

$$\vec{BC} \times \vec{BD} = \begin{vmatrix} i & j & k \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix} = k + i \Rightarrow n(1,0,1)$$

π: $3(x-1) + 2(y-2) + 1(z-1) = 0$

$3x + 2y + z - 8 = 0$

c) $(\vec{AB} \times \vec{AC}) \times k$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -1 & 1 & 2 \end{vmatrix} = -j + i \Rightarrow (1,-1,0)$$

$$\begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -i - j \Rightarrow \boxed{(-1,-1,0)}$$

d) K' left ‖ right oriented

* box product $(v_1 \times v_2) \cdot v_3 \begin{cases} >0 & \text{right} \\ <0 & \text{left} \\ =0 & \text{coplanar} \end{cases}$

$K'(D, \vec{DA}, \vec{DA}, \vec{DC})$

$$[A]_{K'} = [\vec{DA}]_{K'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{K'}$$

$$[B]_{K'} = [\vec{DA}]_{K'} + [\vec{AB}]_{K'} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}_{K'}$$

$$[C]_{K'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[D]_{K'} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[\vec{AB}]_{K'} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$[\vec{AC}]_{K'} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[\vec{AD}]_{K'} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = \begin{vmatrix} i & j & k \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix} \cdot \vec{AD} = (-k + j) \cdot \vec{AD} = (0,1,-1) \cdot (0,-1,0) = 0 - 1 + 0 = -1 < 0 \Rightarrow \text{left oriented}$$

e)
$$\begin{vmatrix} x-0 & y-1 & z-0 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 0 + z + 0 - 0 + y - 1 = y + z - 1 = 0$$

Partial II

I) $K=(O,i,j)$ $\qquad K'=(O',i',j')$

$[O']_K = \begin{bmatrix}0\\1\end{bmatrix} \qquad [i']_K = \begin{pmatrix}2\\1\end{pmatrix} \qquad [j']_K = \begin{pmatrix}-1\\2\end{pmatrix}$

$[A]_K = \begin{pmatrix}1\\3\end{pmatrix} \qquad [B]_K = \begin{pmatrix}-3\\1\end{pmatrix} \qquad [C]_K = \begin{pmatrix}-1\\1\end{pmatrix}$

a) $\vec{BA}(-4,2)$

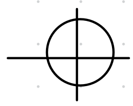
$\vec{CA}(-2,-2)$

$\vec{BA}\cdot\vec{CA} = 8-4=4 \neq 0$

b) l: $\quad C\in l \quad \measuredangle 45^\circ\ AB$

$l\cdot AB = \|l\|\cdot\|AB\|\cdot\cos\alpha$

$AB: \dfrac{x-1}{-3-1} = \dfrac{y-3}{1-3}$



$\dfrac{x-1}{-4} = \dfrac{y-3}{-2} \quad |\cdot(-4)$

$x-1 = 2y-6 \Rightarrow x-2y+5=0$

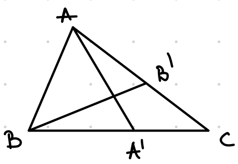
$\|AB\| = \sqrt{1+4} = \sqrt{5}$

$$\boxed{m_{AB} = \frac{1-3}{-3-1} = \frac{-2}{-4} = \frac{1}{2}}$$

$\Rightarrow m_l = \dfrac{-1}{\frac{1}{2}} = -2 \Rightarrow y-1 = -2(x+1)$

$l: 2x+y+1=0$

c) circumcenter



$A'(-2,1) \qquad B'(-1,2)$

$AA'(-3,-2) \qquad BB'(-4,1)$

$AA': \begin{cases} -3x-2y+c=0 \\ A\in AA' \end{cases}$

$-3\cdot(-2)-2\cdot(1)+c=0$

$6-2+c=0 \Rightarrow c=-4 \Rightarrow AA': -3x-2y+4=0$

$BB': \begin{cases} -4x+y+c=0 \\ B\in BB' \end{cases}$

$-4\cdot(-3)+1+c=0$

$12+1+c=0 \Rightarrow c=-13 \Rightarrow BB': -4x+y-13=0$

$AA'\cap BB' = \begin{cases} -3x-2y+4=0 \\ -4x+y-13=0 \quad |\cdot 2 \end{cases}$

$-11x-22=0 \Rightarrow x=-2 \Rightarrow$

$8-13+y=0 \Rightarrow y=5$ $\Big\} \Rightarrow O(-2,5)$

d) $M_{K'K} = \begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}$

$\left(\begin{array}{cc|cc}2 & -1 & 1 & 0\\1 & 2 & 0 & 1\end{array}\right) \sim \left(\begin{array}{cc|cc}1 & 2 & 0 & 1\\2 & -1 & 1 & 0\end{array}\right) \sim \left(\begin{array}{cc|cc}1 & 2 & 0 & 1\\0 & -7 & 1 & -2\end{array}\right) \sim \left(\begin{array}{cc|cc}1 & 2 & 0 & 1\\0 & 1 & -\frac{1}{7} & \frac{2}{7}\end{array}\right) \sim \left(\begin{array}{cc|cc}1 & 0 & \frac{2}{7} & \frac{3}{7}\\0 & 1 & -\frac{1}{7} & \frac{2}{7}\end{array}\right)$

$\Rightarrow M_{KK'} = \frac{1}{7}\begin{pmatrix}2 & 3\\-1 & 2\end{pmatrix}$

$$[A]_{k'} = M_{kk'} \cdot \left[[A]_k - [O']_k \right] = M_{kk'} \left(\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \frac{1}{7} \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

$\vdots$

II

b) ort. ref. A in π_{BCD}

$\vec{BC}(-1,1,1)$

$\vec{BD}(0,-1,0)$

$$n_\pi : \vec{BC} \times \vec{BD} = \begin{vmatrix} i & j & k \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix} = k + i$$

$n_\pi(1,0,1)$

$$e: \begin{cases} x + z + c = 0 \\ (1,1,-1) \in e \end{cases}$$

$$\left.\begin{array}{r} 1 - 1 + c = 0 \\ c = 0 \end{array}\right\} \Rightarrow x + z = 0$$

$$Pr_n \vec{BA} : \frac{(1,0,1)(0,0,-1)}{\|(0,0,-1)\|^2} \cdot (0,0,-1) = \frac{-1}{1}(0,0,-1) = (0,0,1)$$

$\vec{BA}(0,0,-1)$

$$Pr_\pi A = B + Pr_\pi \vec{BA} = (1,1,0) + (0,0,1) = (1,1,1) \; ?$$

c) $k'(B, \vec{BA}, \vec{AC}, \vec{CD})$

$[B]_{k'} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$[A]_{k'} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$[C]_{k'} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$[D]_{k'} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 + 0 + 0 - 0 - 0 - 0 = 1 > 0 \Rightarrow \text{right}$$