Seminar 11

1. In the real vector space $\mathbb{R}^3$ consider the bases $B = (v_1, v_2, v_3) = ((1,0,1),(0,1,1),(1,1,1))$ and $B' = (v_1', v_2', v_3') = ((1,1,0),(-1,0,0),(0,0,1))$. Determine the matrices of change of basis $T_{BB'}$ and $T_{B'B}$, and compute the coordinates of the vector $u = (2,0,-1)$ in both bases.

exam * **2.** In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

Seminar 14 **3.** In the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ consider the bases $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)\,(a \in \mathbb{R})$ and $B' = (1, X-b, (X-b)^2)\,(b \in \mathbb{R})$. Determine the matrices of change of bases T_{EB}, T_{BE} and $T_{BB'}$.

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^2)$ be defined by $f(x,y) = (3x+3y, 2x+4y)$.
(i) Determine the eigenvalues and the eigenvectors of f.
(ii) Write a basis B of $\mathbb{R}^2$ consisting of eigenvectors of f and $[f]_B$.

Compute the eigenvalues and the eigenvectors of the (endomorphisms having) matrices:

$$\textbf{5.}\ \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}. \qquad \textbf{6.}\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$\textbf{7.}\ \begin{pmatrix} x & 0 & y \\ 0 & x & 0 \\ y & 0 & x \end{pmatrix} (x, y \in \mathbb{R}^*). \qquad \textbf{8.}\ \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} (x \in \mathbb{R}).$$

9. Let $A \in M_2(\mathbb{R})$ and let λ_1, λ_2 be the eigenvalues of A in $\mathbb{C}$. Prove that:
(i) $\lambda_1 + \lambda_2 = Tr\,(A)$ and $\lambda_1 \cdot \lambda_2 = det\,(A)$, where $Tr\,(A)$ denotes the trace of A, that is, the sum of the elements of the principal diagonal. Generalization.
(ii) A has all the eigenvalues in $\mathbb{R} \iff (Tr\,(A))^2 - 4 \cdot det\,(A) \geq 0$.
(iii) Show that A is a root of its characteristic polynomial.

10. Let $A \in M_2(\mathbb{R})$ be such that $det(A + iI_2) = 0$. Show that $det(A + 2I_2) = 5$.

V, V' – K vect. spaces

B, B' basis of V, V'

$B = (v_1, \dots, v_n)$

$f \in Hom_K(V, V')$

$$\boxed{[f]_{B,B'} = \left([f(v_1)]_{B'} \dots [f(v_n)]_{B'}\right)}$$

$f_1, f_2 \in Hom_K(V, V')$, B, B' - basis of V, V'

$$[f_1 + f_2]_{B,B'} = [f_1]_{B,B'} + [f_2]_{B,B'}$$

$$\forall \alpha \in K : [\alpha f]_{B,B'} = \alpha [f]_{B,B'}$$

$f \in Hom_K(V, V')$

$g \in Hom_K(V', V'')$ $\Rightarrow$ $\boxed{g \circ f \in Hom_K(V, V'')}$ * theoritical question at the exam

B, B', B'' basis of V, V', V''

$$\boxed{[g \circ f]_{B,B''} = [g]_{B',B''} \cdot [f]_{B,B'}}$$

right hand side has to be looked at from right to left

Change of bases for a K-linear map:

V, V' - K v.s.

$f \in Hom_K(V, V')$

B_1, B_2 basis of V

B_1', B_2' basis of V'

$$\boxed{[f]_{B_2,B_2'} = [id]_{B_1',B_2'} \cdot [f]_{B_1,B_1'} \cdot [id]_{B_2,B_1}}$$

identity map

$[id]_{B_2,B_1} = T_{B_1,B_2}$ = base chage matrix from B_1 to B_2

$[id]_{B_1,B_2} = [id]^{-1}_{B_2,B_1}$ — NOT TRUE IN GENERAL !!
but true for id!

to convert vectors from a basis to another:

$\forall v \in V$:

$[v]_{B'} = [id]_{B,B'} \cdot [v]_B$

expects vectors from B and outputs vectors in B'

2. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

$$[2f]_B = 2 \cdot [f]_B = 2 \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [g]_B = [f]_B + [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'}$$

$B = (v_1, v_2) = ((1,2),(1,3))$

$B' = ((1,0),(2,1))$

$v_1 = (1,2) \Rightarrow f(v_1) = a \cdot (1,0) + b(2,1) = (a+2b, b)$

$v_2 = (1,3) \qquad f(v_2) = c(1,0) + d(2,1) = (c+2b, d)$

$$\begin{cases} 1 = a + 2b \\ 2 = b \end{cases} \qquad \begin{cases} 1 = c + 2b \\ d = 3 \end{cases}$$

$\Rightarrow a = 1 - 4 = -3 \qquad c = 1 - 6 = -5$

$b = 2 \qquad d = 3$

$\Rightarrow [id(v_1)]_{B'} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \qquad \Rightarrow [id(v_2)]_{B'} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$

$$[id]_{B,B'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$v_1' = (1,0) \Rightarrow id(v_1') = a\cdot(1,2) + b(1,3) = (a+b, 2a+3b)$$

$$v_2' = (2,1) \qquad id(v_2') = (c+d, 2c+3d)$$

$$\begin{cases} a+b = 1 \Rightarrow \frac{-1}{2} b = 1 \Rightarrow b = -2 \\ 2a+3b = 0 \Rightarrow 2a = -3b \Rightarrow a = \frac{-3}{2} b \Rightarrow a = 3 \end{cases}$$

$$\begin{cases} c+d = 2 \quad \cdot(-3) \\ 2c+3d = 1 \end{cases}$$

$$-c = -5 \Rightarrow c = 5$$

$$d = -3 \qquad \Rightarrow [id]_{B'B} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'}$$

$$\begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$[f+g]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix} = \begin{pmatrix} -19 & -30 \\ 12 & 19 \end{pmatrix}$$

$$\boxed{[g \circ f]_{B,B''} = [g]_{B',B''} \cdot [f]_{B,B'}}$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'}$$

$$[f]_{B'} = \left([f(v_1)]_{B'}, [f(v_2)]_{B'}\right) = [id]_{B,B'} \cdot [f]_B \cdot [id]_{B',B}$$

$$[id]_{B,B'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$[id]_{B',B} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} -56+65 & -13\cdot 8 + 13\cdot 7 \\ 35 - 40 & 65 - 56 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

3. In the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ consider the bases $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)$ $(a \in \mathbb{R})$ and $B' = (1, X-b, (X-b)^2)$ $(b \in \mathbb{R})$. Determine the matrices of change of bases T_{EB}, T_{BE} and $T_{BB'}$.

$\mathcal{E} = (1, x, x^2)$ basis of $\mathbb{R}_2[x]$

$B = (1, x-a, (x-a)^2)$ $a \in \mathbb{R}$

$B' = (1, x-b, (x-b)^2)$

$T_{\mathcal{E}B} = [id]_{B,\mathcal{E}}$

$v_1 = 1$ $\Rightarrow f(v_1) =$

$v_2 = x - a$

$v_3 = (x-a)^2$

* finish ex on your own

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^2)$ be defined by $f(x, y) = (3x + 3y, 2x + 4y)$.
(i) Determine the eigenvalues and the eigenvectors of f.
(ii) Write a basis B of $\mathbb{R}^2$ consisting of eigenvectors of f and $[f]_B$.

Step 1: write f in a convenient basis (usually $\mathcal{E}$)

$[f]_{\mathcal{E}} = ([f(e_1)]_{\mathcal{E}}, [f(e_2)]_{\mathcal{E}})$

$f(e_1) = f(1,0) = (3,2)$

$f(e_2) = f(0,1) = (3,4)$

$\Rightarrow [f]_{\mathcal{E}} = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$

Step 2: find the characteristic polinomial of $A = [f]_{\mathcal{E}}$

$P_A(x) = \det(A - xI_n)$

$$= \begin{vmatrix} 3-x & 3 \\ 2 & 4-x \end{vmatrix} = (3-x)(4-x)-6 = x^2-7x+6$$

Step 3 : the eigenvalue of A are the distinct roots of the poly

$$\lambda_{1,2} = \frac{7 \pm \sqrt{49-24}}{2} = \frac{7 \pm 5}{2} \Rightarrow \begin{cases} \lambda_1 = 6 \\ \lambda_2 = 1 \end{cases}$$

Step 4 : for every eigenvalue λ we have the eigenspace $\boxed{S(\lambda) = \{v \in V \mid f(v) = \lambda v\}}$

* 0 is never an eigenvector

(set of eigenvectors corresponding to λ and 0)

$$\Rightarrow S(\lambda) = \{ v \in V \mid [f]_\epsilon \cdot [v]_\epsilon = \lambda \cdot [v]_\epsilon \}$$

If $\lambda = \lambda_1 = 6 \Rightarrow \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{cases} 3x + 3y = 6x \\ 3x + 4y = 6y \end{cases} \Leftrightarrow \begin{cases} 3y = 3x \\ 2y = 2x \end{cases} \Leftrightarrow y = x \Rightarrow S(\lambda_1) = \{(x,x) \mid x \in \mathbb{R}\} \Rightarrow$$

$$\Rightarrow S(\lambda_1) = \langle (1,1) \rangle$$

If $\lambda = \lambda_2 = 1 \Rightarrow \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{cases} 3x + 3y = x \Rightarrow 3y = -2x \\ 2x + 4y = y \Rightarrow -3y = 2x \end{cases} \Leftrightarrow y = \frac{-2x}{3}$$

$$\Rightarrow S(\lambda_2) = \left\{ \left(x, -\frac{2x}{3}\right) \mid x \in \mathbb{R} \right\} = \left\langle \left(1, -\frac{2}{3}\right) \right\rangle = \langle (3,-2) \rangle$$

* Vectors from diff. eigenvector spaces are by default linearly indep.

$\Rightarrow$ we have a basis of eigenvectors for $\mathbb{R}^2$: $B = ((1,1),(3,-2))$

If f is a linear map and B is a basis of eigenvectors

$$[f]_B = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

* do these for exam