Vector spaces

V K- vector space

- $(V, +)$ abelian group
  the parallelogram rule when adding vectors
- $(K, +, \cdot)$ field (most of the time $K = \mathbb{R}$, some times $K = \mathbb{Z}_2$)
- $\bullet : K \times V \longrightarrow V$
  external operation

  → $\forall\, x, y \in V$, $\forall\, \alpha \in K$: $\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y$

  → $\forall\, x \in V$, $\forall\, \alpha, \beta \in K$: $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$

  → $\forall\, x \in V$, $\forall\, \alpha, \beta \in K$: $(\alpha\beta)\, x = \alpha(\beta x)$

  → $\forall\, x \in V$, $1 \cdot x = x$

Definition

A *vector space over K* (or a *K-vector space*) is an abelian group $(V, +)$ together with a so-called *external operation* or *scalar multiplication*

$\cdot : V \times V \to V$

order matters!

we don't know what $v \cdot k$ is !!

$\cdot : K \times V \to V$
external

$$\cdot : K \times V \to V\,, \quad (k, v) \mapsto k \cdot v \quad (\text{or simply } kv)\,,$$

different operations that are noted the same way (for whatever reason)

satisfying the following axioms:

(L_1) $k \cdot (v_1 + v_2) = k \cdot v_1 + k \cdot v_2$; distributivity

(L_2) $(k_1 + k_2) \cdot v = k_1 \cdot v + k_2 \cdot v$; distributivity on the right

(L_3) $(k_1 \cdot k_2) \cdot v = k_1 \cdot (k_2 \cdot v)$; associativity

(L_4) $1 \cdot v = v$, neutral element

for every $k, k_1, k_2 \in K$ and every $v, v_1, v_2 \in V$. k- scalars, v-vectors

The elements of K are called *scalars* and the elements of V are called *vectors*.

Sometimes a vector space is also called a *linear space*.

Seminar 4

1. Let K be a field. Show that $K[X]$ is a K-vector space, where the addition is the usual addition of polynomials and the scalar multiplication is defined as follows: $\forall\ k \in K$, $\forall f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$,

$$k \cdot f = (ka_0) + (ka_1)X + \cdots + (ka_n)X^n\,.$$

2. Let K be a field and $m, n \in \mathbb{N}$, $m, n \geq 2$. Show that $M_{m,n}(K)$ is a K-vector space, with the usual addition and scalar multiplication of matrices.

3. Let K be a field, $A \neq \emptyset$ and denote $K^A = \{f \mid f : A \to K\}$. Show that K^A is a K-vector space, where the addition and the scalar multiplication are defined as follows: $\forall f, g \in K^A$, $\forall k \in K$, $f + g \in K^A$, $kf \in K^A$,

$$(f+g)(x) = f(x) + g(x)\,, \quad (k \cdot f)(x) = k \cdot f(x)\,, \forall x \in A\,.$$

~~4.~~ Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define the operations: $x \perp y = xy$ and $k \top x = x^k$, $\forall k \in \mathbb{R}$ and $\forall x, y \in V$. Prove that V is a vector space over $\mathbb{R}$.

5. Let K be a field and let $V = K \times K$. Decide whether V is a K-vector space with respect to the following addition and scalar multiplication:

(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + 2y_2)$ and $k \cdot (x_1, y_1) = (kx_1, ky_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

(ii) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k \cdot (x_1, y_1) = (kx_1, y_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

6. Let p be a prime number and let V be a vector space over the field $\mathbb{Z}_p$.

(i) Prove that $\underbrace{x + \cdots + x}_{p \text{ times}} = 0$, $\forall x \in V$.

(ii) Is there a scalar multiplication endowing $(\mathbb{Z}, +)$ with a structure of a vector space over $\mathbb{Z}_p$?

7. Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:

(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;
(iv) $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(v) $E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;
(vi) $F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$?

8. Which ones of the following sets are subspaces:

(i) $[-1, 1]$ of the real vector space $\mathbb{R}$;
(ii) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ of the real vector space $\mathbb{R}^2$;
(iii) $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Q} \right\}$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;
(iv) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

9. Which ones of the following sets are subspaces of the K-vector space $K[X]$:

(i) $K_n[X] = \{f \in K[X] \mid \text{degree}(f) \leq n\}$ $(n \in \mathbb{N})$;
(ii) $K'_n[X] = \{f \in K[X] \mid \text{degree}(f) = n\}$ $(n \in \mathbb{N})$.

10. Show that the set of all solutions of a homogeneous system of two equations and two unknowns with real coefficients is a subspace of the real vector space $\mathbb{R}^2$.

4.4.) $V = \{x \in \mathbb{R} \mid x > 0\}$. Define for $\forall x, y \in V$, $\forall k \in \mathbb{R}$

$x \perp y = xy$ $\longrightarrow$ on vectors

Prove that V is a vector space over $\mathbb{R}$

$k \top x = x^k$ $\longrightarrow$ external operation

- $(V, \perp)$ abelian group $\iff \begin{cases} \text{"}\perp\text{" associative} \\ \text{"}\perp\text{" commutative} \\ \exists\, e \text{ s.t. } e \perp x = x \perp e = x \\ \exists\, x' \text{ s.t. } x \perp x' = x' \perp x = e \end{cases}$

  - "$\perp$" associative $\iff \forall x, y, z \in \mathbb{R} \Rightarrow (x \perp y) \perp z = x \perp (y \perp z)$

$$xy \perp z = x \perp yz$$
$$xyz = xyz$$

  - "$\perp$" commutative $\iff \forall x, y \in \mathbb{R} \Rightarrow x \perp y = y \perp x$

$$xy = yx$$

  - "$\perp$" has a neutral element $\Rightarrow \exists\, e \in \mathbb{R}$ s.t. $\underbrace{x \perp e = e \perp x}_{\text{commutative}} = x \Rightarrow \underline{e = 1}$

  - "$\perp$" inversable $\iff \forall x \in \mathbb{R}\ \exists\ x^{-1} \in \mathbb{R}$ s.t. $x \perp x^{-1} = e$

$$xx^{-1} = 1 \quad | : x \neq 0$$
$$x^{-1} = \frac{1}{x} \in V$$

$\Rightarrow (V, \perp)$ - abelian group

- We check if "$\top$" is well defined:

Let $x \in V$, $k \in \mathbb{R}$

$x \top k = x^k$

$x > 0 \Rightarrow \forall k \in \mathbb{R}$, $x^k > 0 \Rightarrow x^k \in V$

$\Rightarrow$ "$\top$" is an external operation

- For V to be a vectorspace:

(1) $\forall x, y \in V$, $\alpha \in \mathbb{R}$:

$$\left.\begin{array}{l} \alpha \top (x \perp y) = \alpha \top (xy) = (xy)^\alpha = x^\alpha \cdot y^\alpha \\ (\alpha \top x) \perp (\alpha \top y) = x^\alpha \perp y^\alpha = x^\alpha \cdot y^\alpha \end{array}\right\} \Rightarrow \text{axyom } 1$$

(2) $\forall x \in V$, $\forall \alpha, \beta \in \mathbb{R}$

$(\alpha + \beta) \top x = (\alpha \top x) \perp (\beta \top y)$

$$\left.\begin{array}{l} (\alpha + \beta) \top x = x^{\alpha+\beta} \\ (\alpha \top x) \perp (\beta \top x) = x^\alpha \cdot x^\beta = x^{\alpha+\beta} \end{array}\right\} \text{true}$$

(3) $\forall x \in V,\ \forall \alpha, \beta \in \mathbb{R}$

$$(\alpha \cdot \beta) T x = \alpha T (\beta T x)$$

$$\left.\begin{aligned} (\alpha \cdot \beta) T x &= x^{\alpha \cdot \beta} \\ \alpha T (\beta T x) &= \alpha T x^{\beta} = (x^{\beta})^{\alpha} = x^{\beta \alpha} \end{aligned}\right\} \to \text{true}$$

(4) $\forall x \in V \Rightarrow 1 T x = x$

$x^1 = x \to$ true

Due to everything proven before $\Rightarrow V$ is a vectorspace over $\mathbb{R}$

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* linear = smt that behaves well with respect to the operation

V k - vector space

$S \subseteq V$

$S \leq_k V$ (S is a k-subspace of V)

$\Leftrightarrow$ (i) $S \neq \emptyset$

(ii) $\forall x, y \in S : x + y \in S$

(iii) $\forall k \in K,\ x \in S : kx \in S$

basically scalar

we get the inverse from here if $k = -1$

$\Updownarrow$

(i) $S = \emptyset$

(ii) $\forall x, y \in S,\ k_1, k_2 \in K : \underbrace{k_1 x + k_2 y \in S}_{\text{linear comedation}}$

4.7) Which ones of the following are subspaces of $\mathbb{R}^3$?

midterm

(i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{x = 0}_{\text{one eq.}}\}$ ✓ * condition needs to be a homogenous linear sistem

(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0$ or $z = 0\}$

(iii) $C = \{(x,y,z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$ ✗

(iv) $D = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0\}$ ✓ $\to$ plane through the origin

(v) $E = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 1\}$ ✗ not homogenous $\to$ plane, just not passing through the origin

(vi) $F = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}$ ✓

(i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\}$ a subspace of $\mathbb{R}^3$?

(1) $A \neq \emptyset$ because $(0,1,2) \in A$

(2) $\forall X, Y \in A : X + Y \in A$

$$\left.\begin{aligned} \text{Let } X &= (0, x_1, x_2) \\ Y &= (0, y_1, y_2) \end{aligned}\right\} \Rightarrow X + Y = (0, x_1, x_2) + (0, y_1, y_2) = (0, x_1 + y_1, x_2 + y_2) \in A$$

(3) $\forall k \in \mathbb{R},\ x \in A .\ x_1 = (0, y_1, z_1) \in A$

$k \cdot x = k \cdot (0, y_1, z_1) = (0, k \cdot y_1, k \cdot z_1) \in A$

$\xRightarrow{(1)\,(2)\,(3)}$ A a subspace of $\mathbb{R}^3$

(ii) (1) $B \neq 0$ $\quad (0,1,1) \in B$

(2) Let $X = (0, y_1, z_1) \in B$

$Y = (x_2, y_2, 0) \in B$

$(+)$

$X+Y = (x_2, y_1+y_2, z_1) \notin B \Rightarrow$ not a subspace

let $X = (0,1,1)$

$Y = (2,1,0)$

$(+)$

$(2,2,2) \Rightarrow$ count. ex.

(iii) $C = \{(x,y,z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$

(1) $C \neq \emptyset$ $\quad (0,1,1) \in C$

(2) Let $X, Y \in C$

$X = (x_1, y_1, z_1) \in C$

$Y = (x_2, y_2, z_2)$

$X+Y = (x_1+x_2, y_1+y_2, z_1+z_2) \in C$

(3) Let $k \in \mathbb{R}$ $\quad X \in A$

$$k \cdot X = k \cdot (x, y, z) = (k \cdot x, k \cdot y, k \cdot z)$$

$\left.\begin{array}{l} \left.\begin{array}{l} k \in \mathbb{R} \\ x \in \mathbb{Z} \end{array}\right\} \Rightarrow k \cdot x \notin \mathbb{Z} \end{array}\right\} \Rightarrow k \cdot X \notin C$

(iv) $D = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$

(1) $D \neq \emptyset$, $(1,-1,0) \in D$

(2) $\left.\begin{array}{l} (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2) \\ x_1+x_2+y_1+y_2+z_1+z_2 = 0 \end{array}\right\} \Rightarrow \in D$

(3) $\left.\begin{array}{l} k \cdot (x,y,z) = (kx, ky, kz) \\ k(x+y+z) = k \cdot 0 = 0 \end{array}\right\} \Rightarrow \in D$

(v) $E = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=1\}$

(1) $(1,-1,1) \in E$

(2) $(x_1+x_2, y_1+y_2, z_1+z_2) \Rightarrow x_1+y_1+z_1+z_2+y_2+x_2 = 2 \neq 1 \Rightarrow \notin E$

(vi) $x = y = z$

(1) $(1,1,1) \in F$

(2) $(x_1+x_2, y_1+y_2, z_1+z_2)$

$\left.\begin{array}{l} x_1 = y_1 = z_1 \\ x_2 = y_2 = z_2 \end{array}\right\} \Rightarrow$ sums e equal

(3) $\left.\begin{array}{l} k \cdot (x,y,z) = (kx, ky, kz) \\ x = y = z \end{array}\right\} \Rightarrow kx = ky = kz \Rightarrow k \cdot X \in E$

$\Rightarrow$ subspace

8. Which ones of the following sets are subspaces:
(*i*) $[-1,1]$ of the real vector space $\mathbb{R}$;
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(*iii*) $\left\{\begin{pmatrix} a & b\\ 0 & c\end{pmatrix} \,\middle|\, a,b,c\in\mathbb{Q}\right\}$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;
(*iv*) $\{f:\mathbb{R}\to\mathbb{R}\mid f \text{ continuous}\}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

(i) $[-1,1] \le_{\mathbb{R}} \mathbb{R}$

Let $x,y\in[-1,1]$

$x+y\in[-2,2] \Rightarrow$ not a subspace

(ii) $B=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1\}$ of $\mathbb{R}^2$

(1) $B\neq 0$, $(1,0)\in B$

(2) $X+Y=(x_1,y_1)+(x_2,y_2)=((x_1+x_2),(y_1+y_2))$

$$x_1^2+2x_1x_2+x_2^2+y_1^2+2y_1y_2+y_2^2 \overset{?}{\le} 1$$

$\le 1 \qquad \le 1$

$$\underbrace{x_1^2+y_1^2}_{\le 1}+\underbrace{x_2^2+y_2^2}_{\le 1}+2(x_1x_2+y_1y_2) \overset{?}{\le} 1$$ $\Rightarrow$ not a subspace

(iii) $T_2(\mathbb{Q})=\left\{\begin{pmatrix}a&b\\0&c\end{pmatrix} \,\middle|\, a,b,c\in\mathbb{Q}\right\}$ is a subspace of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$

because $\begin{pmatrix}1&1\\0&1\end{pmatrix}\in T_2(\mathbb{Q})$ but $\sqrt{2}\cdot\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}\sqrt{2}&\sqrt{2}\\0&\sqrt{2}\end{pmatrix}\notin T_2(\mathbb{Q})$

(ii) $D' = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1\} \Rightarrow$ a DISC $\Rightarrow$ NOT a subspace

$(1,0)\in D'$, but $(1,0)+(1,0)=(2,0)\notin D'$