Seminar 8

↳ a matrix is invertible $\Leftrightarrow \det A \neq 0$, $A^{-1} = \frac{1}{\det A} A^*$

Kronecker - Capelli : a system is compatible if $\text{Rank}(A) = \text{Rank}(\bar{A})$ where $\bar{A}$ is the matrix A with a column consisting of the free terms

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n \end{cases} \overset{(*)}{\Leftrightarrow} x_1 \cdot \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + \dots + x_n \cdot \begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

the system is compatible iff $\exists\, x_1, \dots x_n$ s.t. $(*)$ is true $\Leftrightarrow v = x_1 v_1 + \dots + x_n v_n$
$v \in \langle v_1, \dots, v_n \rangle$

Rouche : a system is compatible if all the characteristics determinants are zero

$$\begin{vmatrix} \text{principal minor} & \text{column of free terms} \\ \text{row of matrix} & \end{vmatrix} \text{ — characteristic minor}$$

Cramer $x_i = \frac{\det(A_i)}{\det A}$ are the solutions

Gauss - Jordan zeros under the main diagonal

Seminar 9

inverting a matrix using Gauss - Jordan $A \in \mathcal{M}_n(K)$

$$(A \mid I_n) \sim \dots \sim (I_n \mid A_n^{-1})$$

* use Gauss elimination to extract a basis out of a system of generators. place the generators as rows in a matrix, bring it to the echalon form. the rows will form a basis.

* $\dim \langle X \rangle = \text{rank}$ (echalon form of matrix)

basis $\langle X \rangle$ is given by the non - zero rows

Seminar 10

V, V' - K vector space
$B = (v_1, \dots, v_n)$ basis of V
$B' = (v_1', \dots, v_m')$ basis of V'
$f : V \to V'$

$$[f]_{B,B'} = \left([f(v_1)]_{B'} \dots [f(v_n)]_{B'} \right)$$

if $w = a_1 v_1 + a_2 v_2 + \dots + a_m v_m$,

$$[w]_{B'} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

$$[f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B$$

* applying a linear map is just multiplying a vect. with a matrix

$f(v) = 0 \Leftrightarrow [f(v)]_E = 0$

$v' \in \operatorname{Im} f \Leftrightarrow \exists\, u \in \mathbb{R}^4$ s.t. $f(u) = v' \Leftrightarrow \exists\, u = (x,y,z,t) \in \mathbb{R}^4$ s.t. $[f(u)]_E = [v']_E$

$\Leftrightarrow [f]_E \cdot [u]_E = [v]_E$

$$[f]_E \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to \text{a basis of } \ker(f)$$

Seminar 11

Let V, V' - K vector spaces ; B, B' basis of V, V' with $B = (v_1, \dots, v_n)$ $B' = (v_1', \dots, v_m')$

$f \in \operatorname{Hom}_K(V, V')$

$$\boxed{[f]_{B,B'} = \left([f(v_1)]_{B'} \dots [f(v_n)]_{B'}\right)}$$

$f_1, f_2 \in \operatorname{Hom}_K(V, V')$ B, B' - basis

$[f_1 + f_2]_{B,B'} = [f_1]_{B,B'} + [f_2]_{B,B'}$

$\forall\, \alpha \in K$: $[\alpha f]_{B,B'} = \alpha [f]_{B,B'}$

* $f \in \operatorname{Hom}_K(V, V')$
  $g \in \operatorname{Hom}_K(V', V'')$
  B, B', B'' basis of V, V', V''
  $g \circ f \in \operatorname{Hom}_K(V, V'')$

$\Rightarrow$ $\boxed{[g \circ f]_{B,B''} = [g]_{B',B''} \cdot [f]_{B,B'}}$

Change of bases for a K-linear map

B_1, B_2 basis of V

B_1', B_2' basis of V'

$$\boxed{[f]_{B_2, B_2'} = [id]_{B_1', B_2'} \cdot [f]_{B_1, B_1'} \cdot [id]_{B_2, B_1}}$$

→ Identity map

$[id]_{B_2, B_1} = T_{B_1 B_2}$ = base change matrix from B_1 to B_2

$$[id]_{B_2, B_1} = [id]^{-1}_{B_1, B_2}$$

to convert vectors from a basis to another: $[v]_{B'} = [id]_{B,B'} \cdot [v]_B$

$f(v) = \lambda \cdot v$ where λ is an eigenvalue and v is the eigenvector

$P_A(x) = (x-\lambda_1)(x-\lambda_2) = \lambda^2 - \lambda \cdot Tr(A) + det(A)$

Steps: 1) write f in a convenient basis (such as e)
2) find the characteristic polinomial of $A = [f]_e$
$P_A(x) = det(A - x I_n)$
3) the eigenvalues of A are the distinct roots of the polynomial
4) for every eigenvalue, there's an eigenspace
$S(\lambda) = \{ v \in V \mid f(v) = \lambda v \} = \{ v \in V \mid [f]_e \cdot [v]_e = \lambda \cdot [v]_e \}$

* vectors from different eigenspaces are by default linearly independent

Seminar 12

$P \in \mathbb{Z}_2[x]$ of degree $n-k$ → a generator of a polynomial code (n,k) whose words are polynomials of degree $< n$, divisible by P

for (n,k) polynomial code we have 2^k code words

parity check matrix $H = (I_{n-k} \mid P)$ $\quad u \in \mathcal{M}_{n,1}(\mathbb{Z}_2) \Leftrightarrow H \cdot u = 0$, $\mathcal{C} = Im\gamma \leq \mathbb{Z}_2^n$

hamming distance: u, v are the same length ⇒ the number of positions in which they differ

$w(v, v') = \#$ of 1's in $v - v'$

encoder $\gamma : \mathbb{Z}_2^k \to \mathbb{Z}_2^n$ $\quad [\gamma]_{e e'} = G = ([\gamma(e_1)]_e \dots [\gamma(e_n)]_e)$ – generator matrix

n-k bits	k
check digits	message

$G = \begin{pmatrix} P \\ \hline I_k \end{pmatrix}$ $\qquad [\gamma(m)]_{e'} = G \cdot [m]_e$

We can detect at most $d(\mathcal{C})-1$ errors and can correct $\left\lfloor \frac{d(\mathcal{C})-1}{2} \right\rfloor$ errors

$d(\mathcal{C}) = $ min # of colums in H that add up to 0

(n,k) polynomial code generated by $P \in \mathbb{Z}_2[x]$

$m = \overline{a_0 a_1 \dots a_{k-1}} \rightsquigarrow f_m = a_0 \cdot 1 + a_1 \cdot X + \dots + a_{k-1} \cdot X^{k-1}$

if $\deg P = n-k$ ⇒ the code is linear

steps: 1) encode m as a polynomial
2) multiply f_m by x^{n-k}
3) devide F_m by P
4) Compute $g_m = F_m + R_m$ (the multiplied poly. + the remainder)

5) convert back to a vector

* Proofs

1) Let $f: V \to V'$ be a K-linear map, B - basis of V, B' - basis of V' and $v \in V$.

Prove that $[f(v)]_{B'} = [f]_{BB'} \cdot [v]_B$

- Let $f \in End_K(V)$ a scalar λ is called an eigenvalue of f if there exists a non-zero vector $v \in V$ s.t. $\boxed{f(v) = \lambda \cdot v}$

ex: $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ - the eigenvalues of the endomorphism represented by the matrix A are 2 and 3

Th: Let V be a vector space over K, B - basis of V and $f \in End_K(V)$ with the matrix $[f]_B = A = (a_{ij}) \in \mathcal{M}_n(K)$, then $\lambda \in K$ is an eigenvalue of f iff

$$\boxed{\det(A - \lambda I_n) = 0}$$

Proof: $\lambda \in K$ - eigenvalue of f iff $\exists\, v \in V$ s.t. $f(v) = \lambda v$

consider $[v]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$f(v) = \lambda v \Leftrightarrow f(v) - \lambda v = 0 \Leftrightarrow (f - \lambda \cdot 1_V) \cdot (v) = 0 \Leftrightarrow [(f - \lambda 1_V) \cdot (v)]_B = [0]_B \Leftrightarrow$$

$$\Leftrightarrow [(f - \lambda \cdot 1_V)]_B \cdot [v]_B = [0]_B \Leftrightarrow ([f]_B - \lambda \cdot [1_V]_B) \cdot [v]_B = [0]_B \Leftrightarrow$$

$$(A - \lambda \cdot I_n) \cdot [v]_B = [0]_B \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow \lambda$ - eigenvalue $\Leftrightarrow$ system has a non-zero solution