b) ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

3.13. For the lines ℓ in the previous exercise

a) give parametric equations for ℓ,

b) describe $\mathrm{D}(\ell)$.

3.14. Consider a line ℓ. Show that

c) if $\mathbf{v}(v_1, v_2)$ is a direction vector for ℓ then $\mathbf{n}(v_2, -v_1)$ is a normal vector for ℓ,

d) if $\mathbf{n}(n_1, n_2)$ is a normal vector for ℓ then $\mathbf{v}(n_2, -n_1)$ is a direction vector for ℓ.

3.15. Consider the points $A(1,2)$, $B(-2,3)$ and $C(4,7)$. Determine the medians of the triangle ABC.

3.16. Let $M_1(1,2)$, $M_2(3,4)$ and $M_3(5,-1)$ be the midpoints of the sides of a triangle. Determine Cartesian equations and parametric equations for the lines containing the sides of the triangle.

3.17. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

a) the length of the altitude from A,

b) the line containing the altitude from A.

3.18. Determine the circumcenter and the orthocenter of the triangle with vertices $A(1,2)$, $B(3,-2)$, $C(5,6)$.

3.19. Determine the angle between the lines $\ell_1 : y = 2x+1$ and $\ell_2 : y = -x+2$.

3.20. Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle $\measuredangle A$.

3.21. Let A' be the orthogonal reflection of $A(10,10)$ in the line $\ell : 3x+4y-20=0$. Determine the coordinates of A'.

3.22. Determine Cartesian equations for the lines passing through $A(-2,5)$ which intersect the coordinate axes in congruent segments.

3.23. Determine Cartesian equations for the lines situated at distance 4 from the line $12x-5y-15=0$.

3.24. Determine the values k for which the distance from the point $(2,3)$ to the line $8x+15y+k=0$ equals 5.

3.25. Consider the points $A(3,-1)$, $B(9,1)$ and $C(-5,5)$. For each pair of these three points, determine the line which is equidistant from them.

~~3.26.~~ The point $A(3,-2)$ is the vertex of a square and $M(1,1)$ is the intersection point of its diagonals. Determine Cartesian equations for the sides of the square.

3.27. Determine a point on the line $5x-4y-4=0$ which is equidistant to the points $A(1,0)$ and $B(-2,1)$.

3.28. The point $A(2,0)$ is the vertex of an equilateral triangle. The side opposite to A lies on the line $x+y-1=0$. Determine Cartesian equations for the lines containing the other two sides.

3.29. Determine an equation for each plane passing through $P(3,5,-7)$ and intersecting the coordinate axes in congruent segments.

3.30. Let $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$ be vertices of a tetrahedron. Determine a Cartesian equation of the plane containing $[AB]$ and the midpoint of $[CD]$.

3.31. Show that a parallelepiped with faces in the planes $2x+y-2z+6=0$, $2x-2y+z-8=0$ and $x+2y+2z+1=0$ is rectangular.

3.32. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.

3.33. Determine the distance between the planes $x-2y-2z+7=0$ and $2x-4y-4z+17=0$.

3.34. Solve Exercise 2.16 using normal vectors.

3.35. Let $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,-5)$ be vertices of a triangle. Determine the equation of the internal angle bisector of $\measuredangle A$.

3.36. Determine the angles between the plane $\pi_1 : x-\sqrt{2}y+z-1=0$ and the plane $\pi_2 : x+\sqrt{2}y-z+3=0$.

3.37. Determine the values a and c for which the line $3x-2y+z+3=0 \cap 4x-3y+4z+1=0$ is perpendicular to the plane $ax+8y+cz+2=0$.

3.38. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x+4y-3z+7=0$.

3.39. Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $2x-3y+z-4=0$.

3.40. Consider the point $A(1,3,5)$ and the line $\ell : 2x+y+z-1=0 \cap 3x+y+2z-3=0$.

a) Determine the orthogonal projection of A on ℓ.

b) Determine the orthogonal reflection of A in ℓ.

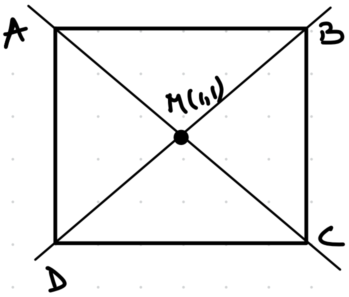
3.41. Determine the planes which pass through $P(0,2,0)$ and $Q(-1,0,0)$ and which form an angle of $60°$ with the z-axis.

3.42. Determine the orthogonal projection of the line $\ell : 2x-y-1=0 \cap x+y-z+1=0$ on the plane $\pi : x+2y-z=0$.

3.43. Determine the coordinates of a point A on the line $\ell : \frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{1}$ which is at distance $\sqrt{3}$ from the plane $x+y+z+3=0$.

3.44. The vertices of a tetrahedron are $A(-1,-3,1)$, $B(5,3,8)$, $C(-1,-3,5)$ and $D(2,1,-4)$. Determine the height of the tetrahedron relative to the face ABC.

3.26. The point $A(3,-2)$ is the vertex of a square and $M(1,1)$ is the intersection point of its diagonals. Determine Cartesian equations for the sides of the square.



$$M:\ x_M = \frac{x_A + x_C}{2} \Rightarrow x_C = 2x_M - x_A \Rightarrow x_C = 2-3 = -1 \Rightarrow C(-1,4)$$
$$y_M = \frac{y_A + y_C}{2} \qquad y_C = 2y_M - y_A \qquad y_C = 2+2 = 4$$

$$x_N = \frac{x_A + x_B}{2}$$

Let $d \perp AC \Rightarrow m_d \cdot m_{AC} = -1 \Rightarrow m_d = -\frac{1}{m_{AC}}$

$$d:\ y - y_M = m_d (x - x_M)$$
$$y - 1 = \frac{2}{3}(x-1)$$
$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{4+2}{-1-3} = \frac{6}{-4} = \frac{-3}{2} \Rightarrow m_d = \frac{2}{3}$$

$$AM = MC = MB = MD = \sqrt{(x_M - x_A)^2 + (y_M - y_A)^2} = \sqrt{(1-3)^2 + (1+2)^2} = \sqrt{4+9} = \sqrt{13}$$

for B, D: $\sqrt{(x - x_M)^2 + (y - y_M)^2} = \sqrt{13}$

$$\begin{cases} (x-1)^2 + (y-1)^2 = 13 \\ y = \frac{2}{3}x + \frac{1}{3} \end{cases} \Rightarrow x^2 - 2x + 1 + \frac{4}{9}(x-1)^2 = 13$$
$$x^2 - 2x + 1 + \frac{4}{9}x^2 - 4x + \frac{4}{9} = 13$$
$$\cancel{6x^2 - 12x + 6 = 26}$$
$$\cancel{6(x^2 - 2x + 1) = 26}$$
$$\cancel{6(x-1)^2 = 26}$$
$$(x-1)^2 = 9 \Rightarrow x - 1 = \pm 3$$

I $x - 1 = 3$
$$x = 4 \Rightarrow y = \frac{2}{3}\cdot 4 + \frac{1}{3} = \frac{9}{3} = 3$$

II $x - 1 = -3$
$$x = -2 \Rightarrow y = \frac{-4+1}{3} = -1$$

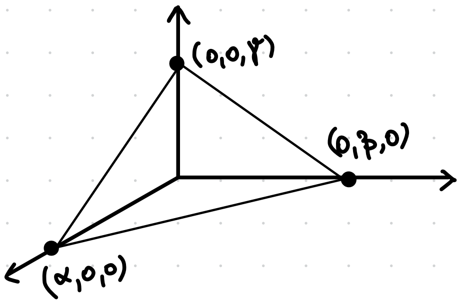
We choose $B(4,3)$, $D(-2,-1)$

$$\vec{AB} = B - A = (4,3) - (3,-2) = (1,5) \Rightarrow \vec{AB} = i + 5j$$
$$\vec{BC} = (-5,1) = -5i + j$$
$$\vec{CD} = (-1,-5) = -i - 5j$$
$$DA = (5,-1) = 5i - j$$

* write the cartesian equations

$$AB:\ \frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A}$$
$$\frac{x-3}{4-3} = \frac{y+2}{3+2}$$
$$x - 3 = \frac{y+2}{5} \Rightarrow 5x - 15 - y - 2 = 0 \Rightarrow 5x - y - 17 = 0$$

3.29 Determine an equation for each plane passing through $P(3,5,-7)$ and intersecting the coordinate axes in congruent segments.



$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$$

$$|\alpha|=|\beta|=|\gamma|=m$$

$$\alpha=\pm m \quad \beta=\pm m \quad \gamma=\pm m$$

I $\quad \alpha=\beta=\gamma=m \qquad x+y+z=m$

$P(3,5,-7) \Rightarrow 3+5-7=m$

$1=m$

$x+y+z=1$

II = $\quad \alpha=\beta=-\gamma=m \qquad x+y-z=m$

$3+5+7=m$

$15=m$

$$\begin{vmatrix} x & y & z & 1 \\ \alpha & 0 & 0 & 1 \\ 0 & \beta & 0 & 1 \\ 0 & 0 & \gamma & 1 \end{vmatrix}=0 \text{ eq plane}$$

III $\quad x-y-z=m$

$3-5+7=m$

$5=m$

V $\quad -x+y+z=m$

$\vdots$

IV $\quad x-y+z=m$

$3-5-7=m$

$-9=m$

3.30. Let $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$ be vertices of a tetrahedron. Determine a Cartesian equation of the plane containing $[AB]$ and the midpoint of $[CD]$.

$M \in CD$

$M(3,-2,6)$

$\vec{AM}\,(3-2,\ -2-1,\ 6-0)=\vec{AM}(1,-3,6)$

$\vec{AB}(-1,2,5)$

$$\begin{cases} x=2+1\lambda-\alpha \\ y=1-3\lambda+2\alpha \\ z=6\lambda+5\alpha \end{cases}, \text{ parametric eq}$$

$$(ABC)=\overbrace{\begin{vmatrix} x & y & z & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 5 & 1 \\ 6 & 3 & 4 & 1 \end{vmatrix}}^{F_1(x,y,z)=0}=0$$

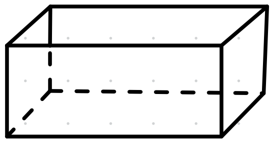
$$(ABD)=\underbrace{\begin{vmatrix} x & y & z & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 5 & 1 \\ 0 & -7 & 8 & 1 \end{vmatrix}}_{F_2(x,y,z)=0}=0$$

$\Rightarrow \pi_\lambda \quad F_1(x,y,z)+\lambda F_2(x,y,z)=0$

Cart. eq.

$$\begin{vmatrix} x-2 & y-1 & z \\ 1 & -3 & 6 \\ -1 & 2 & 5 \end{vmatrix}=0 \Rightarrow -27x-11y-z+65=0$$

$\boxed{3.31}$ Show that a parallelepiped with faces in the planes $2x + y - 2z + 6 = 0$, $2x - 2y + z - 8 = 0$ and $x + 2y + 2z + 1 = 0$ is rectangular.



$2x + y - 2z + 6 = 0$
$2x - 2y + z - 8 = 0$
$x + 2y + 2z + 1 = 0$

two planes are $\perp$ iff their norms are 0

$w_1(2,1,-2)$
$n_2(2,-2,1)$
$w_3(1,2,2)$

$u_1 \cdot u_2 = 4 - 2 - 2 = 0 \Rightarrow \vec{u_1} \perp \vec{u_2}$
$u_2 \cdot u_3 = 2 - 4 + 2 = 0 \Rightarrow \vec{u_2} \perp \vec{u_3}$
$u_1 \cdot u_3 = 2 + 2 - 4 = 0 \Rightarrow \vec{u_1} \perp \vec{u_3}$

$\boxed{3.32}$ Determine a Cartesian equation of the plane π if $A(1, -1, 3)$ is the orthogonal projection of the origin on π.

$\vec{OA} \perp \pi$

$\pi: A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

$\vec{OA}(1, -1, 3)$

$1 \cdot (x - 1) - 1(y + 1) + 3(z - 3) = 0$
$x - y + 3z - 1 - 1 - 9 = 0$
$x - y + 3z - 11 = 0$

$\boxed{3.35}$ Let $A(1, 2, -7)$, $B(2, 2, -7)$ and $C(3, 4, -5)$ be vertices of a triangle. Determine the equation of the internal angle bisector of $\measuredangle A$.

$m_{AB} = \frac{2-2}{2-1} = 0$

$AB: y = 2$
$y - y_A = m(x - x_A)$
$y = x + 1$

34.

$$d(P, \pi) = \frac{|Ax_p + By_p + Cz_p + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$\pi: Ax + By + Cz + D = 0$