3. Find the limit for the sequence:

(d) $\sqrt[n]{1+2+\ldots+n} = \sqrt[n]{\frac{n(n+1)}{2}}$

S-C. th.

We proved that if $\lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n} = \ell \Rightarrow \lim\limits_{n\to\infty}\sqrt[n]{x_n} = \ell$. Let $(x_n) = 1+2+\ldots+n = \frac{n(n+1)}{2} \Rightarrow$

$$\lim_{n\to\infty}\frac{\frac{(n+1)(n+2)}{\cancel{2}}}{\frac{n(\cancel{n+1})}{\cancel{2}}} = \lim_{n\to\infty}\frac{n+2}{n} = 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{x_n} = \ell$$

5. Find the limit.

(c) $x_n = \left(\frac{\ln(n+1)}{\ln(n)}\right)^n$

$$\lim_{n\to\infty}\left(\frac{\ln(n+1)}{\ln(n)}\right)^n = \lim_{n\to\infty}\left(\frac{\ln\left(1+\frac{1}{n}\right)\cdot n}{\ln n}\right)^n = \lim_{n\to\infty}\left(\frac{\ln n+\ln\left(1+\frac{1}{n}\right)}{\ln n}\right)^n = \lim_{n\to\infty}\left(1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right)^n = (1^\infty) = \lim_{n\to\infty}\left[\left(1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right)^{\frac{\ln n}{\ln\left(1+\frac{1}{n}\right)}}\right]^{\frac{n\cdot\ln\left(1+\frac{1}{n}\right)}{\ln n}}$$

$$= e^{\lim\limits_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)^n}{\ln n}} = e^{\lim\limits_{n\to\infty}\frac{\ln e}{\ln n}} = e^{\lim\limits_{n\to\infty}\frac{1}{\ln n}} = e^0 = 1$$

$\left(1+\frac{1}{n}\right)^n \to e$

6. ★ Prove that the sequence (x_n) given by $x_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n$ is decreasing and bounded, hence convergent – its limit is denoted by γ (Euler's constant).

$x_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n$

If x_n is decreasing then $x_{n+1}-x_n < 0$

$$x_{n+1}-x_n = \cancel{1}+\cancel{\tfrac{1}{2}}+\ldots+\cancel{\tfrac{1}{n}}+\frac{1}{n+1}-\cancel{1}-\cancel{\tfrac{1}{2}}-\ldots-\cancel{\tfrac{1}{n}}-\ln(n+1)+\ln n = \underbrace{\frac{1}{n+1}}_{<1}+\underbrace{\ln\Big(\underbrace{\frac{n}{n+1}}_{<1}\Big)}_{<0} < 0 \Rightarrow (x_n) \text{ decreasing}$$

Since the sequence is decreasing $x_n \le x_1 \ \forall n \in \mathbb{N}$

$x_1 = 1-\ln 1 = 1 \Rightarrow x_n < 1 \ \forall n \in \mathbb{N}$

$$x_n = \sum_{k=1}^{n}\frac{1}{k}-\ln n$$

$$x_n = \sum_{k=1}^{n}\frac{1}{k}-\int_1^n\frac{1}{x}\,dx$$

Let $f(x)=\frac{1}{x}$, $\int_1^n f(x)\,dx = \ln(n)$

$$x_n = f(1)+f(2)+\ldots+f(n)-\int_1^n f(x)\,dx$$

$$= f(1)-\int_1^2 f(x)\,dx+f(2)-\int_2^3 f(x)\,dx+\ldots+f(n-1)-\int_{n-1}^n f(x)\,dx+f(n)$$

$$= \underbrace{\int_1^2[f(1)-f(x)]\,dx}_{\ge 0}+\underbrace{\int_2^3[f(2)-f(x)]\,dx}_{\ge 0}+\ldots+\underbrace{\int_{n-1}^n[f(n-1)-f(x)]\,dx}_{\ge 0}+f(n) \ge f(n),\ \forall n \ge 1$$

$\Rightarrow x_n \ge f(n) \ \forall n \ge 1 \Rightarrow (x_n)$ bounded

$f(1) \ge (x_n) \ge f(n)$

8. Find the limit for:

$$\lim_{n\to\infty} \frac{n^n}{1+2^2+3^3+\dots+n^n} \overset{S-C}{=\!=} \lim_{n\to\infty} \frac{(n+1)^{n+1}-n^n}{\sum_{k=1}^{n+1} k^k - \sum_{k=1}^{n} k^k} = \lim_{n\to\infty} \frac{(n+1)^{n+1}-n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} 1-\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} =$$

$$= 1 - \lim_{n\to\infty} \underbrace{\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]}_{=e}^{\frac{-1}{n+1}\cdot n} \cdot \frac{1}{n+1} \xrightarrow{} 0 = 1 - e^{\lim\limits_{n\to\infty}\frac{-n}{n+1}} \cdot 0 = 1 - e^{-1}\cdot 0 = 1$$

10. $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) = \frac{1}{2}x_n - \frac{a}{2x_n} = \frac{1}{2x_n}\left(x_n^2 - a\right)$$

$$x_n^2 - a = \frac{1}{4}\left(x_{n-1} + \frac{a}{x_{n-1}}\right)^2 - a = \frac{1}{4}\left(x_{n-1}^2 + 2a + \frac{a^2}{x_{n-1}^2}\right) - a = \frac{x_{n-1}^2}{4} - \frac{a}{2} + \frac{a^2}{4x_{n-1}^2} = \frac{1}{4}\underbrace{\left(x_{n-1} - \frac{a}{x_{n-1}}\right)^2}_{\geq 0} \geq 0$$

$\Rightarrow x_n - x_{n+1} \geq 0 \Rightarrow x_n \geq x_{n+1} \Rightarrow$ the sequence is bounded from below

$x_n^2 - a \geq 0 \Rightarrow x_n^2 \geq a$

$\Rightarrow (x_n)$ is monotone and bounded $\Rightarrow$ the sequence converges

$$l = \lim_{n\to\infty} x_n$$

$$l = \frac{1}{2}\left(l + \frac{a}{l}\right)$$

$$\cancel{\frac{1}{2}} l = \frac{a}{\cancel{2} l} \quad \Big| \cdot l$$

$$l^2 = a \Rightarrow l = \sqrt{a}$$