Seminar 8

8.2.

$$(ii)\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

$$\bar{A} = \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 & 5 \end{array}\right)$$

rank $A = 2$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0 \Rightarrow \text{rank } A = 2$$

rank $\bar{A} = ?$

$$\begin{vmatrix} 1 & -2 & 1 \\ 1 & -2 & -1 \\ 1 & -2 & -5 \end{vmatrix} = 0 \Rightarrow \text{rank } \bar{A} = 2$$

$\Rightarrow$ rank A = rank $\bar{A}$ $\Rightarrow$ comp.

eq 1,2 - prin

x_1, x_4 - prin

x_2, x_3 - sec

$$\begin{cases} x_1 + x_4 = 1 + 2\alpha - \beta \\ x_1 - x_4 = -1 + 2\alpha - \beta \end{cases} \quad (+)$$

$$2x_1 = 4\alpha - 2\beta \Rightarrow x_1 = 2\alpha - \beta$$

$$x_4 = 1 + 2\alpha - \beta - (2\alpha - \beta) = 1$$

$\Rightarrow S = (2\alpha - \beta, \alpha, \beta, 1)$

8.4. Decide when the following linear system is compatible determinate and in that case solve it by using Cramer's method:

$$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases} \quad (a, b, c \in \mathbb{R}).$$

$$\bar{A} = \left(\begin{array}{ccc|c} b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{array}\right) \Rightarrow \det A = -2abc \Rightarrow \text{system is comp. iff } abc \neq 0 \Rightarrow a, b, c \neq 0$$

$$x = \frac{\det A_x}{\det A}$$

$$\det A_x = \begin{vmatrix} c & a & 0 \\ b & 0 & a \\ a & c & b \end{vmatrix} = a^3 - ac^2 - ab^2 = a(a^2 - b^2 - c^2)$$

$$x = \frac{\cancel{a}(a^2 - b^2 - c^2)}{-2\cancel{a}bc} = \frac{-a^2 + b^2 + c^2}{2bc}$$

$$\begin{cases} x = \dfrac{-a^2 + b^2 + c^2}{2bc} \\ y = \dfrac{a^2 - b^2 + c^2}{2ac} \\ z = \dfrac{a^2 + b^2 - c^2}{2ab} \end{cases}$$

8.5

$$(iii)\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$$

$$A=\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3-2\cdot L_1 \\ L_4\leftarrow L_4-L_1}}{\overset{L_2\leftarrow L_2-L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & -3 & 0 & -3 \\ 0 & -1 & 0 & 1 \end{array}\right) \underset{L_4\leftarrow L_4(-1)}{\overset{L_2\leftrightarrow L_4}{\sim}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & -2 & 0 & -2 \\ 0 & -3 & 0 & -3 \end{array}\right) \underset{L_4\leftarrow L_4+3L_2}{\overset{L_3\leftarrow L_3+2L_2}{\sim}} \left(\begin{array}{ccc|c} \boxed{\begin{matrix}1 & 1 \\ 0 & 1\end{matrix}} & & 1 & 3 \\ & & 0 & -1 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & -6 \end{array}\right)$$

$$\Rightarrow \begin{cases} x+y+z=3 \\ y=-1 \\ z=\alpha \end{cases} \Rightarrow \begin{cases} x+y=3-\alpha \\ y=-1 \end{cases} \Rightarrow x=4-\alpha$$

8.7. $$\begin{cases} ax+y+z=1 \\ x+ay+z=a \\ x+y+az=a^2 \end{cases} \quad (a\in\mathbb{R})$$

$$A=\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & a & a^2 \\ 1 & a & 1 & a \\ a & 1 & 1 & 1 \end{array}\right) \underset{L_3\leftarrow L_3-aL_1}{\overset{L_2\leftarrow L_2-L_1}{\sim}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & a & a^2 \\ 0 & a-1 & 1-a & a-a^2 \\ 0 & 1-a & 1-a^2 & 1-a^3 \end{array}\right) \overset{L_3\leftarrow L_3+L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & a & a^2 \\ 0 & a-1 & 1-a & a-a^2 \\ 0 & 0 & 2-a-a^2 & 1-a^3+a-a^2 \end{array}\right)$$

$$\Rightarrow \begin{cases} x+y+az=a^2 \\ -(1-a)y+(1-a)z=a(1-a) \;\big|:(1-a),\ a\neq 1 \Rightarrow -y+z=a \\ (2-a-a^2)z=1-a^3+a-a^2 \Rightarrow z=\dfrac{-a^3-a^2+a+1}{-a^2-a+2}=\dfrac{-a^3-a^2+2a-a+1}{-a^2-a+2}= \end{cases}$$

$$= a + \frac{1-a}{-a^2+a-2a+2} = a + \frac{1-a}{-(1-a)a-2(1-a)} =$$

$$= a + \frac{\cancel{1-a}}{\cancel{(1-a)}(a+2)} \overset{a+2}{=} a + \frac{1}{a+2} = \frac{a^2+2a+1}{a+2} = \frac{(a+1)^2}{a+2}$$

8. Determine the positive solutions of the following non-linear system:

$$\begin{cases} xyz = 1 \Rightarrow z = \frac{1}{xy} \\ x^3y^2z^2 = 27 \Rightarrow z^2 = \frac{27}{x} \cdot \frac{1}{x^2 \cdot y^2} \Rightarrow \frac{27}{x} = 1 \Rightarrow x = 3^3 \\ \frac{z}{xy} = 81 \Rightarrow z = 81xy \end{cases}$$

$$\Rightarrow \frac{1}{xy} = 81xy \quad | xy$$

$$\frac{1}{81} = x^2y^2 \quad | \sqrt{\ }$$

$$\left.\begin{array}{l} \frac{1}{3^2} = xy \\ x = 3^3 \end{array}\right\} \Rightarrow \frac{1}{3^2} = 3^3 y \Rightarrow y = \frac{1}{3^5} \Rightarrow z = \frac{1}{3^3 \cdot \frac{1}{3^5}} = 3^2$$

Seminar 3

3.2.
$$\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix} \underset{L_3 \leftarrow L_3 + L_1}{\overset{L_2 \leftarrow L_2 + 2L_1}{\sim}} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & -2 & 9 & 3 \end{pmatrix} \overset{L_3 \leftarrow L_3 + 2L_2}{\sim} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 15 & 5 \end{pmatrix}$$

3.4.
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 - 2L_2}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \overset{L_1 \leftarrow 3L_1 + 2L_2}{\sim} \left(\begin{array}{ccc|ccc} 3 & 0 & -6 & -1 & 2 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \underset{L_1 \text{ -"-}}{\overset{L_2 \leftarrow 3L_2 + 2L_3}{\sim}}$$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 0 & -1 & 2 & 2 \\ 0 & -3 & 0 & -2 & -1 & 2 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \begin{array}{l} 1:3 \\ 1:-3 \\ 1:9 \end{array}$$

6. Let K be a field, let $B = (e_1, e_2, e_3, e_4)$ be a basis and let $X = (v_1, v_2, v_3)$ be a list in the canonical K-vector space K^4, where

$$v_1 = 3e_1 + 2e_2 - 5e_3 + 4e_4\,,$$

$$v_2 = 3e_1 - e_2 + 3e_3 - 3e_4\,,$$

$$v_3 = 3e_1 + 5e_2 - 13e_3 + 11e_4\,.$$

$$\begin{pmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \end{pmatrix}$$

$$\Rightarrow X = \begin{pmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \end{pmatrix} \underset{L_3 \leftarrow L_2 - L_1}{\overset{L_2 \leftarrow L_2 - L_1}{\sim}} \begin{pmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 3 & -8 & 7 \end{pmatrix} \overset{L_3 \leftarrow L_3 + L_2}{\sim}$$

$$\sim \begin{pmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow v_1, v_2, v_3 \text{ – lineary dependent}$$

* see theory

7. In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1,0,4)$, $v_2 = (2,1,0)$, $v_3 = (1,5,-36)$ and $v_4 = (2,10,-72)$. Determine $dim < X >$ and a basis of $< X >$.

$$X = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 5 & -36 \\ 2 & 10 & -72 \end{pmatrix} \underset{\substack{L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_3 - 2L_1}}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 5 & -40 \\ 0 & 10 & -80 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 10L_2}{\overset{L_3 \leftarrow L_3 - 5L_2}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank } X = 2 \Rightarrow \dim \langle X \rangle = 2 \Rightarrow \langle X \rangle = \langle v_1, v_2 \rangle$$

9. Determine the dimension of the subspaces S, T, $S+T$ and $S \cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S =< (1,0,4), (2,1,0), (1,1,-4) > ,$$

$$T =< (-3,-2,4), (5,2,4), (-2,0,-8) > \ .$$

$$S = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \dim T = 2$$

$$\langle T \rangle = \langle (1,0,4), (2,1,0) \rangle$$

$$T = \begin{pmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \overset{L_2 \leftarrow L_2 + L_1}{\sim} \begin{pmatrix} -3 & -2 & 4 \\ 2 & 0 & 8 \\ -2 & 0 & -8 \end{pmatrix}$$

$$\sim \begin{pmatrix} -3 & -2 & 4 \\ 2 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim T = 2, \ \langle T \rangle = \langle (-3,-2,4), (5,2,4) \rangle$$

$$S+T = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \\ -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \underset{L_5 \leftarrow L_5 + L_4}{\overset{L_3 \leftarrow L_3 + L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ \cancel{2} & \cancel{1} & \cancel{0} \\ -3 & -2 & 4 \\ -2 & 0 & 8 \\ -2 & 0 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \\ -3 & -2 & 4 \\ 2 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \\ -3 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\dim(S+T) = 3$$

$$\langle S+T \rangle = \langle (1,0,4), (2,1,0), (-3,-2,4) \rangle$$

$$\dim(S\cap T) = \dim S + \dim T - \dim(S+T) = 1$$

Seminar 10

10 **1.** Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by

$$f(x, y, z) = (x + y, y - z, 2x + y + z).$$

Determine the matrix $[f]_E$, where $E = (e_1, e_2, e_3)$ is the canonical basis for $\mathbb{R}^3$.

$$[f]_E = [f(e_1)\ f(e_2)\ f(e_3)]$$

$$\left.\begin{array}{l} f(e_1) = f(1,0,0) = (1,0,2) \\ f(e_2) = f(0,1,0) = (1,1,1) \\ f(e_3) = f(0,0,1) = (0,-1,1) \end{array}\right\} \Rightarrow [f]_E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

exam **2.** Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1, 1, 0), (0, 1, 1), (1, 0, 1))$ of $\mathbb{R}^3$, $B' = (v_1', v_2') = ((1, 1), (1, -2))$ of $\mathbb{R}^2$ and let $E' = (e_1', e_2')$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

$$v_1 = (1,1,0) \Rightarrow f(v_1) = f(1,1,0) = (1,-1)$$
$$v_2 = (0,1,1) \Rightarrow f(v_2) = f(0,1,1) = (1,0)$$
$$v_3 = (1,0,1) \Rightarrow f(v_3) = f(1,0,1) = (0,-1)$$

$$(1,-1) \underset{BE}{=} a\cdot e_1' + b\cdot e_2' = a\cdot(1,0) + b\cdot(0,1) = (a,b) = \begin{cases} a=1 \\ b=-1 \end{cases} \Rightarrow [f(v_1)]_{E'} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(1,0) = a\cdot(1,0) + b\cdot(0,1) \Rightarrow \begin{array}{l} a=1 \\ b=0 \end{array}$$

$$(0,-1) = a\cdot(1,0) + b(0,1) \Rightarrow \begin{array}{l} a=0 \\ b=-1 \end{array}$$

$$\Rightarrow [f]_{BE'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

B':

$$(1,-1) = a\cdot(1,1) + b(1,-2) = (a+b, a-2b) \Rightarrow \begin{cases} a+b=1 \\ a-2b=-1 \end{cases} \Rightarrow \begin{array}{l} a=\frac{1}{3} \\ b=\frac{2}{3} \end{array} \Rightarrow [f(v_1)]_{B'} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$(1,0) \Rightarrow \left.\begin{cases} a+b=1 \\ a-2b=0 \Rightarrow a=2b \end{cases}\right\} \Rightarrow [f(v_2)]_{B'} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$(0,-1) \Rightarrow \begin{cases} a+b=0 \\ a-2b=-1 \end{cases} \Rightarrow \begin{array}{l} a=\frac{-1}{3} \\ b=\frac{1}{3} \end{array} \Rightarrow [f(v_3)]_{B'} = \begin{pmatrix} \frac{-1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$[f]_{BB'} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

3. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^4)$ be defined by

$$f(e_1) = (1,2,3,4), f(e_2) = (4,3,2,1), f(e_3) = (-2,1,4,1)$$

on the elements of the canonical basis of $\mathbb{R}^3$. Determine:

(*i*) $f(v)$ for every $v \in \mathbb{R}^3$.

(*ii*) the matrix of f in the canonical bases.

(*iii*) a basis and the diemnsion of $Ker\, f$ and $Im\, f$.

$f(1,0,0) = (1,2,3,4)$
$f(0,1,0) = (4,3,2,1)$
$f(0,0,1) = (-2,1,4,1)$

i) let $v = (a,b,c) \Rightarrow f(v) = f(a\cdot e_1 + be_2 + c\cdot e_3) \Leftrightarrow a\cdot f(e_1) + b\cdot f(e_2) + c\cdot f(e_3) =$
$= a(1,2,3,4) + b(4,3,2,1) + c(-2,1,4,1) =$
$= (a+4\cdot b-2c,\ 2a+3b+c,\ 3a+2b+4c,\ 4a+b+c) \in \mathbb{R}^4$

ii) $[f]_{e,e'} = \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 1 \end{pmatrix}$

$Im f = \{ f(v) \mid v \in \mathbb{R}^3 \} = \{ w \in \mathbb{R}^4 \mid \exists v = (x,y,z) \text{ s.t. } f(v) = w \} = \{ w \in \mathbb{R}^4 \mid \exists v = (x,y,z)$
s.t. $[f]_{e,e'} \cdot [v]_{e} = [w]_{e'} \}$

Imf

$$\begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 4 & -2 & a \\ 2 & 3 & 1 & b \\ 3 & 2 & 4 & c \\ 4 & 1 & 1 & d \end{array}\right) \xrightarrow[\substack{L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - 4L_1}]{L_2 \leftarrow L_2 - 2L_1} \left(\begin{array}{ccc|c} 1 & 4 & -2 & a \\ 0 & -5 & 5 & b-2a \\ 0 & -10 & 10 & c-3a \\ 0 & -15 & 9 & d-4a \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 4 & -2 & a \\ 0 & -5 & 5 & b-2a \\ 0 & 0 & 0 & a-2b+c \\ 0 & 0 & -6 & 2a-3b+d \end{array}\right)$$

compatible iff $a - 2b + c = 0 \Rightarrow Imf = \{ (2b-c, b, c, d) \} =$
$= \langle (2,1,0,0), (-1,0,1,0), (0,0,0,1) \rangle \Rightarrow \dim Imf = 3$

$$\mathrm{Im} f = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow ((-1,0,1,0),(0,1,2,0),(0,0,0,1))$$

basis of $\mathrm{Im} f$

Ker f:
$$\left(\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 4 & 0 \\ 4 & 1 & 1 & 0 \end{array}\right) \ldots$$

exam **4.** Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

(i) Show that $v = (1,4,1,-1) \in Ker\, f$ and $v' = (2,-2,4,2) \in Im\, f$.
(ii) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(iii) Define f.

i) $v \in Ker f \Leftrightarrow (f(v)=0) \Leftrightarrow [f(v)]_E = 0 \Leftrightarrow [f]_E \cdot [v]_E = 0$

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+4-3-2 \\ -1+4+1-4 \\ 2+4-5-1 \\ 1+8-4-5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$v' \in \mathrm{Im} f \Leftrightarrow \exists\, u \in \mathbb{R}^4$ s.t. $f(u) = v' \Leftrightarrow \exists\, u = (x,y,z,t)$ s.t. $[f(u)]_E = [v']_E$

$\Leftrightarrow [f]_E \cdot [u]_E = [v']_E$

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix}$$

$$\begin{cases} x + y - 3z + 2t = 2 \\ -x + y + z + 4t = -2 \\ 2x + y - 5z + t = 4 \\ x + 2y - 4z + 5t = 2 \end{cases} \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \begin{array}{l} \\ L_2 \leftarrow L_2 + L_1 \\ L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \end{array}\right) \underset{L_4 \leftarrow L_4 + L_2}{\overset{L_3 \leftarrow L_3 - 2L_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{compatible}$$

$\Rightarrow \exists\ u$ s.t. $f(u) = v'$, $v' \in \operatorname{Im} f$

ii) $\operatorname{Ker} f = \{ v = (x,y,z,t) \mid f(v) = 0 \} = \{ v = (x,y,z,t) \mid [f]_e \cdot [v] = 0 \}$

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 2 & 1 & -5 & 1 & 0 \\ 1 & 2 & -4 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x + y = 3\alpha - 2\beta \Rightarrow x = 2\alpha + \beta \\ y = \alpha - 3\beta \end{cases}$$

$z = \alpha$
$t = \beta$

$\Rightarrow \operatorname{Ker} f = \{ (2\alpha + \beta, \alpha - 3\beta, \alpha, \beta) \mid \alpha, \beta \in \mathbb{R} \} = \langle (2,1,1,0), (1,-3,0,1) \rangle$

linearly independent $\Rightarrow$ basis of f

$\dim \operatorname{Ker} f = 2 \Rightarrow \operatorname{null} f = 2$

$\operatorname{Im} f = \{ v = (a,b,c,d) \mid \exists\ u \in \mathbb{R}^4 \text{ s.t. } [f(u)]_e = [v]_e \} = \{ v = (a,b,c,d) \mid [f]_e \cdot [u]_e = [v]_e \}$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 2 & -2 & 6 & a+b \end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c} \boxed{\begin{matrix}1 & 1\\ 0 & 1\end{matrix}} & & -3 & 2 & a \\ & & -1 & 3 & d-a \\ 0 & 0 & 0 & 0 & c+d-3a \\ 0 & 0 & 0 & 0 & 3a+b-2d \end{array}\right) \Rightarrow \text{compatible iff } \begin{cases} -3a+c+d=0 \\ 3a+b-2d=0 \end{cases}$$

$$\Rightarrow \begin{cases} -3a = -\alpha-\beta \\ 3a+b = 2\beta \end{cases} \quad (+)$$

$$b = -\alpha+\beta$$

$$a = \frac{\alpha+\beta}{3}$$

$$\Rightarrow u = \left(\frac{\alpha+\beta}{3}, -\alpha+\beta, \alpha, \beta\right) \Rightarrow \operatorname{Im} f = \langle (\tfrac{1}{3}, -1, 1, 0), (\tfrac{1}{3}, 1, 0, 1) \rangle$$

linearly indep $\Rightarrow$ basis $\Rightarrow \dim \operatorname{Im} f = 2$

iii)

$$[f(x,y,z,t)]_E = [f]_E \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+y-3z+2t \\ -x+y+z+4t \\ 2x+y-5z+t \\ x+2y-4z+5t \end{pmatrix}$$

$$\Rightarrow f(x,y,z,t) = (x+y-3z+t, -x+y+z+4t, 2x+y-5z+t, x+2y-4z+5t)$$

5. Consider the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ and its bases $E = (1, X, X^2)$ and $B = (\underbrace{1}_{b_1}, \underbrace{X-1}_{b_2}, \underbrace{X^2+1}_{b_3})$. Consider $\varphi \in End_{\mathbb{R}}(\mathbb{R}_2[X])$ defined by

$$\varphi(a_0 + a_1X + a_2X^2) = (a_0 + a_1) + (a_1 + a_2)X + (a_0 + a_2)X^2.$$

Determine the matrices $[\varphi]_E$ and $[\varphi]_B$.

$$\varphi(a_0+a_1X+a_2X^2) = (a_0+a_1) + (a_1+a_2)X + (a_0+a_2)X^2$$

$$[\varphi]_E = [\varphi(1)\ \varphi(X)\ \varphi(X^2)]$$

$$\left.\begin{array}{l} \varphi(1) = 1+X^2 \\ \varphi(X) = 1+X \\ \varphi(X^2) = X+X^2 \end{array}\right. \Rightarrow [f]_E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$[\varphi]_B = [\varphi(b_1)\ \varphi(b_2)\ \varphi(b_3)]$$

$$\left.\begin{array}{l} \varphi(b_1) = \varphi(1) = 1+X^2 \\ \varphi(b_2) = \varphi(X-1) = \varphi(-1+1\cdot X+0\cdot X^2) = 0+X-X^2 = X-X^2 \\ \varphi(b_3) = \varphi(X^2+1) = 1+X+2X^2 \end{array}\right\} \Rightarrow [f]_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

6. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$.

$v_1 = (1,2) \qquad v_1' = (1,0)$
$v_2 = (1,3) \qquad v_2' = (2,1)$

$[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \quad \det[f]_B \neq 0 \Rightarrow [2f]_B = 2 \cdot [f]_B = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$

$$\left.\begin{array}{l} f(v_1) = (1,-1) \Rightarrow f(1,2) = (1,-1) \\ f(v_2) = (2,-1) \quad\;\; f(1,3) = (2,-1) \end{array}\right\} \Rightarrow \begin{cases} a_1 + 2b_1 = 1 \\ a_2 + 2b_2 = -1 \\ a_1 + 3b_1 = 2 \\ a_2 + 3b_2 = -1 \end{cases} \begin{array}{l} \Rightarrow b_1 = 1 \Rightarrow a_1 = -1 \\ \Rightarrow b_2 = 0 \Rightarrow a_2 = -1 \end{array}$$

$f(x,y) = (y-x, -x)$

$$\left.\begin{array}{l} g(v_1') = (-7,5) \Rightarrow \\ g(v_2') = (-13,7) \end{array}\right. \begin{cases} a_1x + b_1y = -7 \\ a_2x + b_2y = 5 \\ a_1x + b_2y = -13 \\ a_2x + b_2y = 7 \end{cases} \begin{array}{l} \Rightarrow a_1 = -7 \\ \quad\; b_1 = 1 \\ \Rightarrow a_2 = 5 \\ \quad\; b_2 = -3 \end{array} \Rightarrow g(x,y) = (y-7x, 5x-3y)$$

$(f+g)(x,y) = f(x,y) + g(x,y) = (y-x,-x) + (y-7x, 5x-3y)$

$$\left.\begin{array}{l} (f+g)(v_1) = (2-1,-1) + (-5,-1) = (-4,-2) \\ (f+g)(v_2) = (-2,-5) \end{array}\right\} \Rightarrow [f+g]_B = \begin{pmatrix} -4 & -2 \\ -2 & -5 \end{pmatrix}$$

7. Consider the endomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$f(x,y) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha) \quad (\alpha \in \mathbb{R}).$$

Write its matrix in the canonical basis of $\mathbb{R}^2$ and show that f is an automorphism.

$\mathcal{E} = ((1,0),(0,1))$

$$\left.\begin{array}{l} f(e_1) = (\cos\alpha, \sin\alpha) \\ f(e_2) = (-\sin\alpha, \cos\alpha) \end{array}\right\} \Rightarrow [f]_{\mathcal{E}} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$\det[f]_{\mathcal{E}} = \cos^2\alpha + \sin^2\alpha = 1 \neq 0 \Rightarrow$ automorphism

? **8.** Let V be a vector space of dimension 2 over the field $K = \mathbb{Z}_2$. Determine $|V|$, $|End_K(V)|$ and $|Aut_K(V)|$. ?

[*Hint: use the isomorphism between $End_K(V)$ and $M_n(K)$, where $dim_K(V) = n$.*]

Seminar 11

11. **1.** In the real vector space $\mathbb{R}^3$ consider the bases $B = (v_1, v_2, v_3) = ((1,0,1),(0,1,1),(1,1,1))$ and $B' = (v_1', v_2', v_3') = ((1,1,0),(-1,0,0),(0,0,1))$. Determine the matrices of change of basis $T_{BB'}$ and $T_{B'B}$, and compute the coordinates of the vector $u = (2,0,-1)$ in both bases.

$v_1 = (1,0,1) \quad v_2 = (0,1,1) \quad v_3 = (1,1,1)$

$v_1' = (1,1,0) \quad v_2' = (-1,0,0) \quad v_3' = (0,0,1)$

$v_1 = a_1 \cdot v_1' + a_2 \cdot v_2' + a_3 \cdot v_3' = (a_1 - a_2, a_1, a_3) = (1,0,1) \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = -1 \\ a_3 = 1 \end{cases}$

$v_2 = (a_1 - a_2, a_1, a_3) = (0,1,1) \Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_3 = 1 \end{cases}$

$v_3 = (a_1 - a_2, a_1, a_3) = (1,1,1) \Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 0 \\ a_3 = 1 \end{cases}$

$\Rightarrow [id]_{B',B} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$[id]_{B,B'} = [id]_{B',B}^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & -1 & -1 \end{pmatrix}$

$[id]_{B',B}^{-1} = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right) \overset{L_2 \leftarrow L_2 + L_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 - 2L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right)$

$\underset{L_1 \leftarrow L_1 - L_3}{\overset{L_2 \leftarrow L_2 - L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right) \overset{L_1 \leftarrow L_1 - L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right)$

$[u]_{B'} = ?$

$(2,0,-1) = (a_1 - a_2, a_1, a_3) \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = -2 \\ a_3 = -1 \end{cases}$

$[u]_B = [id]_{B',B} \cdot [u]_{B'} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2-1 \\ -2 \\ -2-1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -3 \end{pmatrix}$

exam *

2. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

$$[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \Rightarrow [2f]_B = 2[f]_B = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'}$$

$[id]_{B,B'} = ?$

$$f(v_1) = a(1,0) + b(2,1) = (a+2b, b) = (1,2)$$
$$f(v_2) = c(1,0) + d(2,1) = (c+2d, d) = (1,3)$$

$$\Rightarrow \begin{cases} a = -3 \\ b = 2 \end{cases} \qquad \begin{cases} c = 1-6 = -5 \\ d = 3 \end{cases}$$

$$\Rightarrow [id(v_1)]_{B'} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \qquad [id(v_2)]_{B'} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

$$\Rightarrow [id]_{B,B'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$[id]_{B',B} = \left(\begin{array}{cc|cc} 2 & 3 & 0 & 1 \\ -3 & -5 & 1 & 0 \end{array}\right) \overset{L_2 \leftarrow L_1 \cdot 3 + L_2 \cdot 2}{\sim} \left(\begin{array}{cc|cc} 2 & 3 & 0 & 1 \\ 0 & 1 & -2 & -3 \end{array}\right) \overset{L_1 \leftarrow L_1 - 3L_2}{\sim}$$

$$\left(\begin{array}{cc|cc} 2 & 0 & 6 & 10 \\ 0 & 1 & -2 & -3 \end{array}\right) \overset{L_1 \leftarrow L_1 : 2}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -3 \end{array}\right)$$

$$\underbrace{\begin{matrix} 3 & 5 \\ -2 & -3 \end{matrix}}_{[id]_{B'B}}$$

$$[f+g]_B = [f]_B + \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -4 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} -20 & -32 \\ 13 & -20 \end{pmatrix} = \begin{pmatrix} -19 & -30 \\ 12 & -19 \end{pmatrix}$$

$$[f\circ g]_{B'} = [f]_{B'}\cdot[g]_{B'} = [id]_{B,B'}\cdot[f]_B\cdot[id]_{B,B'}\cdot[g]_{B'} = \begin{pmatrix}8 & 13\\ -5 & -8\end{pmatrix}\begin{pmatrix}-7 & -13\\ 5 & 7\end{pmatrix} = \begin{pmatrix}9 & 13\\ -5 & 9\end{pmatrix}$$

3. In the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ consider the bases $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)\,(a \in \mathbb{R})$ and $B' = (1, X-b, (X-b)^2)\,(b \in \mathbb{R})$. Determine the matrices of change of bases T_{EB}, T_{BE} and $T_{BB'}$.

$$T_{EB} = [id]_{B,E} = \left([1]_E \;\; [x-a]_E \;\; [(x-a)^2]_E\right)$$

$$[1]_E = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$[x-a]_E = \begin{pmatrix}-a\\1\\0\end{pmatrix} \qquad T_{EB} = \begin{pmatrix}1 & -a & a^2\\ 0 & 1 & -2a\\ 0 & 0 & 1\end{pmatrix}$$

$$[(x-a)^2]_E = \begin{pmatrix}a^2\\-2a\\1\end{pmatrix}$$

$$[id]_{E,B} = [id]_{B,E}^{-1} = \left(\begin{array}{ccc|ccc}1 & -a & a^2 & 1 & 0 & 0\\ 0 & 1 & -2a & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right) \underset{L_1 \leftarrow L_1 - a^2 L_3}{\overset{L_2 \leftarrow L_2 + L_3 \cdot 2a}{\sim}} \left(\begin{array}{ccc|ccc}1 & -a & 0 & 1 & 0 & -a^2\\ 0 & 1 & 0 & 0 & 1 & 2a\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right)$$

$$\overset{L_1 \leftarrow L_1 + aL_2}{\sim} \left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & a & a^2\\ 0 & 1 & 0 & 0 & 1 & 2a\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right) \Rightarrow [id]_{E,B} = \begin{pmatrix}1 & a & a^2\\ 0 & 1 & 2a\\ 0 & 0 & 1\end{pmatrix}$$

$$T_{BB'} = [id]_{B',B} = \left([1]_B \;\; [x-b]_B \;\; [(x-b)^2]_B\right)$$

$$[1]_B = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$[x-b]_B = \begin{pmatrix}a-b\\1\\0\end{pmatrix} \quad \Rightarrow [id]_{B',B} = \begin{pmatrix}1 & a-b & (a-b)^2\\ 0 & 1 & 2(a-b)\\ 0 & 0 & 1\end{pmatrix}$$

$$[(x-b)^2]_B = \begin{pmatrix}(a-b)^2\\ 2a-2b\\ 1\end{pmatrix}$$

x^2-2b+b^2

$x^2-2ax+a^2-2ax+a^2$

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^2)$ be defined by $f(x, y) = (3x + 3y, 2x + 4y)$.
(i) Determine the eigenvalues and the eigenvectors of f.
(ii) Write a basis B of $\mathbb{R}^2$ consisting of eigenvectors of f and $[f]_B$.

i) $[f]_E = \left([f(e_1)]_E, [f(e_2)]_E\right)$

$f(e_1) = f(1,0) = (3,2)$

$f(e_2) = f(0,1) = (3,4)$

$\Rightarrow [f]_{\mathcal{E}} = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$

$P_A(x) = \det(A - x\mathcal{I}_n) = \begin{vmatrix} 3-x & 3 \\ 2 & 4-x \end{vmatrix} = (3-x)(4-x) - 6 = x^2 + 12 - 7x - 6 = x^2 - 7x + 6$

$\Delta = 49 - 24 = 25 \Rightarrow \lambda_{1,2} = \frac{7 \pm 5}{2} < \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 6 \end{matrix}$

$S(\lambda_1) = \{ v \in V \mid f(v) = \lambda v \} = \{ v \in V \mid [f]_{\mathcal{E}} \cdot [v]_{\mathcal{E}} = \lambda [v]_{\mathcal{E}} \}$

$\lambda = \lambda_1 = 6 \Rightarrow S(\lambda_1) = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} 3x+3y \\ 2x+4y \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \end{pmatrix}$

$\left.\begin{matrix} 2y = 6x - 3x \\ 2y = 2x \end{matrix}\right\} \Rightarrow y = x \Rightarrow S(\lambda_1) = \{ (x,x) \mid x \in \mathbb{R} \} = \langle (1,1) \rangle$

$\lambda_2 = 1 \Rightarrow S(\lambda_2) \Rightarrow \begin{pmatrix} 3x+3y \\ 2x+4y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 3y = -2x \\ 2x = -3y \end{cases} \Rightarrow y = \frac{-2}{3} x$

$\Rightarrow S(\lambda_2) = \{ (x, \frac{-2}{3} x) \mid x \in \mathbb{R} \} = \langle (1, \frac{-2}{3}) \rangle = \langle (3,-2) \rangle$

ii) $\dim(\mathbb{R}^2) = 2 \quad \lambda_1 \neq \lambda_2 \Rightarrow B = \langle (1,1), (3,-2) \rangle$

$[f]_B = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$

11.5. $\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$.

$P_A(x) = \det(A - x\mathcal{I}_n) = \begin{vmatrix} 3-x & 1 & 0 \\ -4 & -1-x & 0 \\ -4 & -8 & -2-x \end{vmatrix} = -(2+x) \cdot \begin{vmatrix} 3-x & 1 \\ -4 & -(1+x) \end{vmatrix} =$

$= -(2+x) \cdot [-(3-x)(1+x) + 4] = (2+x)[3 + 2x - x^2 - 4] = (2+x)(-x^2 + 2x - 1) = -(2+x)(x-1)^2 \Rightarrow \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = -2 \end{matrix}$

$S(\lambda_1) = \{ v \in \mathbb{R}^3 \mid [f]_{\mathcal{E}} \cdot [v]_{\mathcal{E}} = (0,0,0) \}$

$\begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ -4 & -8 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + y \\ -4x - 2y \\ -4x - 8y - 3z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \left.\begin{matrix} 2x + y = 0 \\ -4x - 2y = 0 \end{matrix}\right\} \Rightarrow 2x = -y \\ -4x - 8y - 3z = 0 \end{cases}$

$-4x + 16x - 3z = 0 \Rightarrow z = 4x$

$\Rightarrow S(\lambda_1) = \{ (x, -2x, 4x) \mid x \in \mathbb{R} \} = \langle (1,-2,4) \rangle$

$S(\lambda_2) = \{ v \in \mathbb{R}^3 \mid \begin{pmatrix} 5 & 1 & 0 \\ -4 & 1 & 0 \\ -4 & -8 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \} \Rightarrow \begin{cases} 5x + y = 0 \\ -4x + y = 0 \\ -4x - 8y = 0 \end{cases} \Rightarrow y = -\frac{1}{2} x$

$\Rightarrow S(\lambda_1) = \langle (-2,1,0), (0,0,1) \rangle$

11. **7.** $\begin{pmatrix} x & 0 & y \\ 0 & x & 0 \\ y & 0 & x \end{pmatrix}$ $(x, y \in \mathbb{R}^*)$

$$P_A(X) = \det(A - XI_n) = \begin{vmatrix} x-X & 0 & y \\ 0 & x-X & 0 \\ y & 0 & x-X \end{vmatrix} = (x-X)^3 - y^2(x-X) = (x-X)[(x-X)^2 - y^2] =$$
$$= (x-X)(x-X+y)(x-X-y) = 0$$

$\Rightarrow \lambda_1 = x \quad \lambda_2 = x-y \quad \lambda_3 = x+y$

$S(\lambda_1)$: $\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} yc = 0 \\ ya = 0, \ y \neq 0 \end{cases} \Rightarrow a = c = 0 \Rightarrow$

$S(\lambda_1) = \{(0,b,0) \mid b \in \mathbb{R}\} = \langle (0,1,0) \rangle$

$S(\lambda_2)$: $\begin{pmatrix} y & 0 & y \\ 0 & y & 0 \\ y & 0 & y \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} (a+c)y = 0 \\ by = 0 \\ (a+c)y = 0 \end{cases} \Rightarrow \begin{matrix} a = -c \\ b = 0 \end{matrix} \Rightarrow S(\lambda_2) = \langle (-1,0,1) \rangle$

$S(\lambda_3)$: $\begin{pmatrix} -y & 0 & y \\ 0 & -y & 0 \\ y & 0 & -y \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} (-a+c)y = 0 \\ -yb = 0 \\ (a-c)y = 0 \end{cases} \Rightarrow \begin{cases} b = 0 \\ a = c \end{cases} \Rightarrow S(\lambda_3) = \langle (1,0,1) \rangle$

* **9.** Let $A \in M_2(\mathbb{R})$ and let λ_1, λ_2 be the eigenvalues of A in $\mathbb{C}$. Prove that:

(i) $\lambda_1 + \lambda_2 = Tr\,(A)$ and $\lambda_1 \cdot \lambda_2 = det\,(A)$, where $Tr\,(A)$ denotes the trace of A, that is, the sum of the elements of the principal diagonal. Generalization.

(ii) A has all the eigenvalues in $\mathbb{R} \iff (Tr\,(A))^2 - 4 \cdot det\,(A) \geq 0$.

(iii) Show that A is a root of its characteristic polynomial.

i) $P_A(X) = \det(A - XI_n) = \begin{pmatrix} a-X & b \\ c & d-X \end{pmatrix} = (a-X)(d-X) - bc = ad - (a+d)X + X^2 - bc$
$= X^2 - Tr(A) + \det(A)$

x_1, x_2 roots of $P_A(X) \Rightarrow x_1 + x_2 = Tr(A)$, $x_1 \cdot x_2 = \det A$

ii) $X^2 - Tr(A) \cdot X + \det A = 0$

$\Delta \geq 0 \Leftrightarrow (Tr(A))^2 - 4\det(A) \geq 0$

iii) $A^2 + Tr(A) \cdot A + \det(A) I_2 = O_2$

Seminar 12

2. $1\,0\,0\,1\,\underbrace{1\,0\,1\,1}$ $\Rightarrow$ $f = x^7 + x^6 + x^4 + x^3 + 1$

$$\begin{array}{r|l} x^7 + x^6 + x^4 + x^3 + 1 & x^4 + x^3 + x^2 + 1 \\ \cline{2-2} x^7 + x^6 + x^5 + x^3 & x^3 + x \\ \hline x^5 + x^4 + 1 & \\ x^5 + x^4 + x^3 + x & \\ \hline x^3 + x + 1 = r_m & \end{array}$$

$f \not| \, p \Rightarrow f$ not a code word

3. $(6,3)$ - code $\Rightarrow n = 6 \quad k = 3 \Rightarrow$ we have $2^3 = 8$ words

0 0 0	1 0 0
0 0 1	1 0 1
0 1 0	1 1 0
0 1 1	1 1 1

$m = 000 \Rightarrow m \cdot x^{n-k} = 0$

$r = m \cdot x^{n-k} \pmod p) \Rightarrow r = 0$

$v = r + m \cdot x^{n-k} \Rightarrow v = 0 \Rightarrow 000000$

$m = 001 \Rightarrow m = x^2 \to m \cdot x^{n-k} = x^5 \Rightarrow$

$$\begin{array}{r|l} x^5 & x^3 + x^2 + 1 \\ \cline{2-2} x^5 + x^4 + x^2 & x^2 + x + 1 \\ \hline x^4 + x^2 & \\ x^4 + x^3 + x & \\ \hline x^3 + x^2 + x & \\ x^3 + x^2 + 1 & \\ \hline x + 1 = r & \end{array}$$

$\Rightarrow v = 1 + x + x^5 = 110001$

$m = 011 \Rightarrow m = x + x^2 \to m \cdot x^{n-k} = x^4 + x^5 \Rightarrow$

$$\begin{array}{r|l} x^5 + x^4 & x^3 + x^2 + 1 \\ \cline{2-2} x^5 + x^4 + x^2 & x^2 \\ \hline x^2 & \end{array}$$

$m = 101 \Rightarrow m = 1 + x^2 \Rightarrow m \cdot x^{n-k} = x^3 + x^5 \Rightarrow$

$$\begin{array}{r|l} x^5 + x^3 & x^3 + x^2 + 1 \\ \cline{2-2} x^5 + x^4 + x^2 & x^2 + x \\ \hline x^4 + x^3 + x^2 & \\ x^4 + x^3 + x & \\ \hline x^2 + x & \end{array}$$

$\Rightarrow v = x + x^2 + x^3 + x^5 \Rightarrow 011101$

4. $G = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow n = 5 \quad k = 3$ - gen. matrix

$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$ parity check matrix

$U = (u_1, u_2, u_3, u_4, u_5)$

$$H \cdot u = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = \begin{cases} u_1 + u_5 = 0 \Rightarrow u_1 = -u_5 \\ u_2 + u_3 + u_4 + u_5 = 0 \Rightarrow u_5 = u_2 + u_3 + u_4 \end{cases}$$

$$\Rightarrow u = (u_2+u_3+u_4,\ u_2,\ u_3,\ u_4,\ u_2+u_3+u_4)$$

$\Rightarrow$ all pos. cont.

$\nexists$ a 0 col $\Rightarrow d(\mathcal{C}) > 1$
no identical col $\Rightarrow d(\mathcal{C}) > 2$
..... add 3 col $\Rightarrow d(\mathcal{C}) \leq 3$

min # of columns that add up to 0

5. Determine the minimum Hamming distance between the code words of the code with generator matrix $G = \left(\frac{P}{I_4}\right) \in M_{9,4}(\mathbb{Z}_2)$, where:

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Discuss the error-detecting and error-correcting capabilities of this code, and write down the parity check matrix.

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \underline{0} & 0 & 0 & 1 & \underline{1} & 0 & 0 & \underline{1} \end{pmatrix} \Rightarrow C_2 + C_6 + C_9 = 0 \Rightarrow d(\mathcal{C}) = 3$$

$\Rightarrow$ we can detect $d(\mathcal{C}) - 1$ errors and correct $\left\lfloor \frac{d(\mathcal{C})-1}{2} \right\rfloor = 1$ errors

6. Encode the following messages using the generator matrix of the (9,4)-code of Exercise **5.**: 1101, 0111, 0000, 1000.

$$[\gamma(m)]_{E'} = G \cdot [m]_E$$

$$G = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for 1101 $\Rightarrow$ $G \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

for 0111 $\Rightarrow$ $G \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Determine the generator matrix and the parity check matrix for:

↳ $G\left(\frac{?}{I_n}\right)$ ↳ $H = (I_n | P)$ $d(\mathcal{C})$

7. The (4,1)-code generated by $p = 1 + X + X^2 + X^3 \in \mathbb{Z}_2[X]$.

* encode the canonical message

$(4,1)$-code $\Rightarrow n = 4$ $k = 1$

$e_1 = 1 \Rightarrow m = 1 \Rightarrow m \cdot X^{n-k} = X^3 \Rightarrow r = X^2+X+1 \Rightarrow v = X^3+X^2+X+1 \Rightarrow 1111$

$$\Rightarrow G = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \left(\frac{P}{I_k}\right) \Rightarrow P = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$H = (I_{n-k} \mid P) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

8. The (7,3)-code generated by $p = 1 + X^2 + X^3 + X^4 \in \mathbb{Z}_2[X]$.

(7,3)-code $\Rightarrow n = 7 \quad k = 3$

$e_1 = 100$
$e_2 = 010$
$e_3 = 001$

$\Rightarrow m = 100 = 1 \Rightarrow m \cdot X^{n-k} = X^4 \Rightarrow X^4 : (X^4+X^3+X^2+1) = 1 \Rightarrow r = X^3+X^2+1$

$\Rightarrow v = X^4+X^3+X^2+1 \Rightarrow 1011\underbrace{100}$

$m = 010 = X \Rightarrow m \cdot X^{n-k} = X^5 \Rightarrow$

$$\begin{array}{l|l} X^5 & X^4+X^3+X^2+1 \\ \cline{2-2} X^5+X^4+X^3+X & X+1 \\ \cline{1-1} X^4+X^3+X & \\ X^4+X^3+X^2+1 & \\ \cline{1-1} X^2+X+1 = R_m & \end{array}$$

$\Rightarrow v = F_m + R_m = X^5+X^2+X+1 \Rightarrow 1110\underbrace{010}$

$m = 001 = X^2 \Rightarrow v = X^6+X^3+X^2+X \Rightarrow 0111\underbrace{001}$

$$G = \left(\frac{P}{I_K}\right) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$\to P$

$d(\mathcal{C}) = 4 \Rightarrow$ detect 3 errors and correct $\lfloor \frac{4-1}{2} \rfloor = 1$

Seminar 13

3. (7,4)-code $\Rightarrow n = 7 \quad k = 4$

check digs:
$u_1 = \underbrace{u_4 + u_5 + u_7}$
$u_2 = u_4 + u_6 + u_7$
$u_3 = u_4 + u_5 + u_6$

message digs.

$$\Rightarrow G = \Big(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_{n-k}} \;\; \underbrace{\begin{matrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}}_{P} \Big) \Rightarrow H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$G \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 1+1 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow 0000000$$

$$G \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \text{coset leader } 0001000$$

4. Parity check (3,2)

$G = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $\quad H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

$G \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+c \\ b+c \end{pmatrix}$

0 0	0 0 0
0 1	0 1 0
1 0	1 0 0
1 1	0 0 1

(3,1) - repeating code

$G = (1\ 1\ 1) \quad H = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

?

5. Construct a table of coset leaders and syndromes for the (7,4)-code with parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

syndr.	coset lead.
0 0 0	0000000
0 0 1	0010000
0 1 0	0100000
0 1 1	0000001
1 0 0	1000000
1 0 1	0000010
1 1 0	0000100
1 1 1	0001000

6. Determine the parity check matrix and all syndromes and coset leaders of the (5,3)-code with generator matrix $G = \begin{pmatrix} P \\ I_3 \end{pmatrix} \in M_{5,3}(\mathbb{Z}_2)$, where:

$n \; k$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$G = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow H = (I_{n-k} \mid P) = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

0 0	00000
0 1	01000
1 0	10000
1 1	00010

7. Construct a table of coset leaders and syndromes for the (3,1)-code generated by $p = 1 + X + X^2 \in \mathbb{Z}_2[X]$.

$m = 1 \Rightarrow f = 1 \qquad f_m = f \cdot X^{n-k} = 1 \cdot X^2 = X^2$

$$\begin{array}{l|l} X^2 & X^2+X+1 \\ \hline X^2+X+1 & 1 \\ \hline X+1 = R_m & \end{array}$$

$\Rightarrow g_m = X^2+X+1 \rightsquigarrow v = 11\underbrace{1}_{m}$

$H = \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right)$

S	
00	000
01	010
10	100
11	001

Past exams

2013: $\underset{\text{II}}{3)}$ $\mathcal{E} = (e_1, e_2, e_3, e_4)$ $\qquad v_1 = (1, 0, 0, 0)$

$B = (v_1, v_2, v_3, v_4) \qquad v_2 = (1, 1, 0, 0)$

$v_3 = (1, 1, 1, 0)$

$v_4 = (1, 1, 1, 1)$

a) $u = (1, 2, 2, 1)$

$[u]_B = ([u_1]_B\ [u_2]_B\ [u_3]_B\ [u_4]_B) \Rightarrow$

$\Rightarrow c_1 \cdot v_1 + c_2 \cdot v_2 + c_3 \cdot v_3 + c_4 \cdot v_4 = (1, 2, 2, 1)$

$\Rightarrow (c_1+c_2+c_3+c_4,\ c_2+c_3+c_4,\ c_3+c_4,\ c_4) = (1, 2, 2, 1)$

$\Rightarrow c_4 = 1 \Rightarrow c_3 = 1 \Rightarrow c_2 = 0 \Rightarrow c_1 = -1$

$\Rightarrow [u]_B = (-1, 0, 1, 1)$

$[u]_{\mathcal{E}} = (1, 2, 2, 1)$

$\exists\ w = (x, y, z, t)$ s.t.

$[id]_{\mathcal{E},B} = ?$

$e_1 = a_1 \cdot v_1 + a_2 v_2 + a_3 \cdot v_3 + a_4 \cdot v_4 = (a_1+a_2+a_3+a_4,\ a_2+a_3+a_4,\ a_3+a_4,\ a_4) = (1, 0, 0, 0) \Rightarrow \begin{cases} a_1 = 1 \\ a_2 = 0 \\ a_3 = 0 \\ a_4 = 0 \end{cases}$

$e_2 = (a_1+a_2+a_3+a_4,\ a_2+a_3+a_4,\ a_3+a_4,\ a_4) = (0, 1, 0, 0) = \begin{cases} a_1 = -1 \\ a_2 = 1 \\ a_3 = 0 \\ a_4 = 0 \end{cases}$

$e_3 = (0, 0, 1, 0) \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = -1 \\ a_3 = 1 \\ a_4 = 0 \end{cases}$

$$e_4 = (0,0,0,1) \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = -1 \\ a_4 = 1 \end{cases}$$

$$\Rightarrow [id]_{\mathcal{E},B} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[id]_{\mathcal{E},B} \cdot [w]_{\mathcal{E}} = [w]_B$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} x - y = 0 \Rightarrow x = 4 \\ y - z = 2 \Rightarrow y = 4 \\ z - t = 0 \Rightarrow z = 2 \\ t = 2 \end{cases} \Rightarrow w = (4,4,2,2)$$

b) $$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} x - y = 2 \Rightarrow x = 10 \\ y - z = 7 \Rightarrow y = 8 \\ z - t = 0 \Rightarrow z = 1 \\ t = 1 \end{cases} \Rightarrow w = (10, 8, 1, 1)$$

4) a) $$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \Rightarrow A^{-1} = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 - 2L_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -4 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\overset{L_3 \leftarrow L_3 + 4L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 9 & -2 & 4 & 1 \end{array}\right) :9 \sim \quad \ldots$$

b) $f(x,y,z) = (x-y, x+y-2z, -5y+5z)$

$$[f]_{\mathcal{E}} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -2 \\ 0 & -5 & 5 \end{pmatrix} \overset{L_2 \leftarrow L_2 - L_1}{\sim} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -5 & 5 \end{pmatrix} \overset{L_3 \leftarrow L_3 + 5L_2}{\sim} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim [f]_{\mathcal{E}} = 2$$

$\Rightarrow \langle (1,-1,0), (0,1,-1) \rangle$ basis of f

$$\operatorname{Im} f = \{ v = (a,b,c) \mid \exists\, u \in \mathbb{R}^3 \text{ s.t. } [f(u)]_{\mathcal{E}} = [v]_{\mathcal{E}} \} = \{ v = (a,b,c) \mid [f]_{\mathcal{E}} \cdot [u]_{\mathcal{E}} = [v]_{\mathcal{E}} \}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 1 & 1 & -2 & b \\ 0 & -5 & 5 & c \end{array}\right) \overset{L_2 \leftarrow L_2 - L_1}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 0 & 2 & -2 & b-a \\ 0 & -5 & 5 & c \end{array}\right) \overset{L_3 \leftarrow 2 \cdot L_3 + 5 \cdot L_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & a \\ 0 & 2 & -2 & b-a \\ 0 & 0 & 0 & 2c-5(b-a) \end{array}\right)$$

compatible iff $2c-5(b-a)=0 \Rightarrow 2c=5(b-a)$

$$c=\frac{5}{2}(b-a)$$

$$\Rightarrow u=\left(a,\ b-a,\ \frac{5}{2}(b-a)\right)=\langle(1,-1,-\tfrac{5}{2}),(0,1,\tfrac{5}{2})\rangle$$

$\mathrm{Ker} f=\{v=(a,b,c)\mid f(v)=0\}$

$$\left(\begin{array}{ccc|c}1&-1&0&0\\1&1&-2&0\\0&-5&5&0\end{array}\right)\sim\left(\begin{array}{ccc|c}1&-1&0&0\\0&1&-1&0\\0&0&0&0\end{array}\right)\Rightarrow \dim \mathrm{Ker} f=2 \Rightarrow$$

$$\begin{cases}a-b=0\Rightarrow a=\alpha\\ b=\alpha\\ c=\alpha\end{cases}\Rightarrow \mathrm{Ker} f=\langle(1,1,1)\rangle$$

5. $(5,2)$-code $\Rightarrow n=5\ k=2$

$$G=\begin{pmatrix}1&1\\1&0\\0&1\\1&0\\0&1\end{pmatrix}\Rightarrow H=(I_{n-k}\mid P)=\begin{pmatrix}1&0&0&1&1\\0&1&0&1&0\\0&0&1&0&1\end{pmatrix}$$

000	00000
001	00100
010	01000
011	01100
100	10000
101	00001
110	00010
111	00110

$$H\cdot\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0+1\\1+1\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}\Rightarrow e=10000$$

$$\Rightarrow v+e=\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\\1\\0\end{pmatrix}\Rightarrow m=10$$

$$H\cdot\begin{pmatrix}1\\0\\1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow 00000=e\Rightarrow v+e=v\Rightarrow m=01$$

4 II. $f(x,y,z)=(x-z,\ x+y-2z,\ -5y+z)$

$$\cancel{\begin{pmatrix}1&0&-1\\1&1&-2\\0&-5&1\end{pmatrix}\sim\begin{pmatrix}1&0&-1\\0&1&-1\\0&-5&1\end{pmatrix}\sim}\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&-4\end{pmatrix}$$

$$P_A(x)=\det(A-xI_n)=\begin{vmatrix}1-x&0&-1\\1&1-x&-2\\0&-5&1-x\end{vmatrix}=(1-x)^3+5-10(1-x)=$$