Seminar 4

1. Study if the following series are convergent or divergent:

   ~~(a)~~ $\displaystyle\sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}$.

   (b) ★ $\displaystyle\sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}\cdot\frac{1}{n^2}$.

   ~~(c)~~ $\displaystyle\sum_{n\geq 1} a^{\ln n}$, $a>0$.

   ~~(d)~~ $\displaystyle\sum_{n\geq 1} \frac{a^n n!}{n^n}$ $a>0$.

2. Study the convergence and the absolute convergence of the following series:

   ~~(a)~~ $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$.

   ~~(b)~~ $\displaystyle\sum_{n\geq 1} (-1)^n \sin\frac{1}{n}$.

   ~~(c)~~ $\displaystyle\sum_{n\geq 1} \frac{\sin n}{n^2}$.

~~3.~~ Prove by differentiating the geometric series that, for $|x|<1$,

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}, \quad \sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}.$$

~~4.~~ Prove by integrating the geometric series that, for $|x|<1$,

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x), \quad \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = \ln(1+x).$$

~~5.~~ Prove that $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots = \arctan x$, for $x\in[-1,1]$.

6. Find the radius of convergence and the convergence set for each of the following series:

   (a) $\displaystyle\sum_{n\geq 1} \frac{(x-2)^n}{(n+1)3^n}$.

   (b) $\displaystyle\sum_{n\geq 1} \frac{(x-1)^n}{n^p}$, $p>0$.

   (c) ★ $\displaystyle\sum_{n\geq 1} \frac{nx^n}{2^n}$.

7. ★ [Python] Show numerically that $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = \ln 2$. Change the order of summation in this series – for example by first adding p positive terms, then q negative terms, and so on – and show numerically that the rearrangement gives a different sum (depending on p, q).

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.

(a) $\sum_{n\geq 1} \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}$.

R-D Test: $\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right)=L$

$L>1 \Rightarrow \sum x_n$ conv

$L<1 \Rightarrow \sum x_n$ div

ratio test: $\frac{x_{n+1}}{x_n} = \frac{\cancel{1\cdot 3\cdot 5\ldots}(2(n+1)-1)}{\cancel{2\cdot 4\ldots}2(n+1)}\cdot\frac{\cancel{2\cdot 4\ldots 2n}}{\cancel{1\cdot 3\cdot 5\ldots(2n-1)}} = \frac{(2n+1)}{2(n+1)} = \frac{2n+1}{2n+2} \to 1$

$\Rightarrow$ R-D test: $\lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}}-1\right) = \lim_{n\to\infty} n\cdot\left(\frac{2n+2}{2n+1}-1\right) = \lim_{n\to\infty} n\cdot\left(\cancel{1}+\frac{1}{2n+1}-\cancel{1}\right) = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2} < 1 \Rightarrow x_n$ diverges

(c) $\sum_{n\geq 1} a^{\ln n}$, $a>0$.

ratio test: $\frac{a^{\ln(n+1)}}{a^{\ln n}} = a^{\ln(n+1)-\ln n} = a^{\ln\frac{n+1}{n}} \longrightarrow a^0 = 1$

R-D: $\lim_{n\to\infty} n\left(\frac{a^{\ln n}}{a^{\ln(n+1)}}-1\right) = \lim_{n\to\infty} n\cdot\underbrace{\frac{a^{\ln\frac{n}{n+1}}-1}{\ln\frac{n}{n+1}}}_{\to\ln a}\cdot\ln\frac{n}{n+1} = \ln a\cdot\lim_{n\to\infty} n\ln\left(1+\frac{-1}{n+1}\right) = \ln a\cdot\ln\lim_{n\to\infty}\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-n}{n+1}}$

$= \ln a\cdot\ln e^{-1} = -\ln a = \ln\frac{1}{a}$

if $\ln\frac{1}{a}>1 \Leftrightarrow \left.\begin{array}{l}\frac{1}{a}>e\\ a>0\end{array}\right\}\Rightarrow a<\frac{1}{e} \Rightarrow x_n$ converges

$\ln\frac{1}{a}<1 \Leftrightarrow \left.\begin{array}{l}\frac{1}{a}<e\\ a>0\end{array}\right\}\Rightarrow a>\frac{1}{e} \Rightarrow x_n$ diverges

if $a=\frac{1}{e} \Rightarrow x_n = \sum_{n\geq 1}\left(\frac{1}{e}\right)^{\ln n} = \sum\frac{1}{n} \to\infty$, divergent

(d) $\sum_{n\geq 1}\frac{a^n n!}{n^n}$ $a>0$. * exam

root
$\lim_{n\to\infty}\sqrt[n]{x_n} = \lim_{n\to\infty}\frac{a}{n}\sqrt[n]{n!}$

ratio test: $\frac{\cancel{a^{n+1}}\,\cancel{(n+1)!}}{(n+1)^{n+1}}\cdot\frac{n^n}{\cancel{a^n}\,\cancel{n!}} = a\left(\frac{n}{n+1}\right)^n$

$\lim_{n\to\infty} a\cdot\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-n}{n+1}} = a\cdot e^{\lim_{n\to\infty}\frac{-n}{n+1}} = a\cdot e^{-1} = \frac{a}{e}$

$\boxed{\left(\frac{n+1}{n}\right)^n = e^{n\ln\frac{n+1}{n}}}$

if $\frac{a}{e}<1 \Rightarrow a<e \Rightarrow x_n$ converges

$a>e \Rightarrow x_n$ diverges

$a=e \Rightarrow$ R-D: $\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right) = \lim_{n\to\infty} n\cdot\left(\frac{1}{e}\cdot\underbrace{\left(\frac{n+1}{n}\right)^n}_{e}-1\right) = (0\cdot\infty) =$ l'H...

go back

if $\sum\underbrace{\frac{e^n n!}{n^n}}_{x_n}$ converges $\Rightarrow \frac{e^n n!}{n^n} \longrightarrow 0$ (1) * sterling aproximation

$$\left(1+\frac{1}{n}\right)^n \nearrow e \Rightarrow \left(1+\frac{1}{n}\right)^n < e$$

$$\frac{x_{n+1}}{x_n} = e\left(\frac{n}{n+1}\right)^n > 1 \Rightarrow x_{n+1} > x_n \Rightarrow \text{it's increasing (2)}$$

$$\Rightarrow \lim_{n\to\infty} x_n > 0 \xRightarrow{(1),(2)} \sum_{n\geq 1} x_n \text{ divergent}$$

(a) $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$. → altering series

Let $S_n = \displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$

$$S_n = \frac{1}{\sqrt{1\cdot 2}} - \frac{1}{\sqrt{2\cdot 3}} + \frac{1}{\sqrt{3\cdot 4}} - \dots + \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$$

Leibnitz test

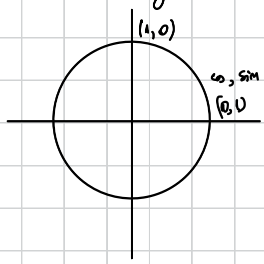
(x_n) if $(x_n) \to 0$, (x_n) decreasing $\Rightarrow \sum_{n\geq 1}(-1)^n x_n$ convergent $\left(\text{also } \sum (-1)^{n+1} x_n \text{ conv}\right)$

$$\left.\begin{array}{l} x_n = \frac{1}{\sqrt{n(n+1)}} \longrightarrow 0 \\ \sqrt{n(n+1)} - \text{increasing} \end{array}\right\} \Rightarrow x_n \text{ decreasing} \Rightarrow \sum_{n\geq 1}(-1)^{n+1}\cdot x_n \text{ converges}$$

$$\left.\begin{array}{l} |x_n| = \frac{1}{\sqrt{n(n+1)}} \\ y_n = \frac{1}{n} \end{array}\right| \to \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n(n+1)}} = 1 \in (0,1) \left.\begin{array}{l} \\ \text{since } \sum \frac{1}{n} \text{ diverges} \end{array}\right\} \Rightarrow \sum |x_n| \text{ diverges}$$

$\Rightarrow \displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$ is conditionally convergent

(b) $\displaystyle\sum_{n\geq 1}(-1)^n \sin\frac{1}{n}$.



$$\left.\begin{array}{l} x_n = \sin\frac{1}{n} \\ \frac{1}{n} \in (0,1] \end{array}\right\} \Rightarrow \sin\frac{1}{n} \in [0,1)$$

Leibnitz test: $x_n \to 0$, decreasing $\Rightarrow \sum(-1)^n x_n$ conv.

Comp. test: $y_n = \frac{1}{n} \Rightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} \cdot \frac{1}{n} = 1\cdot 0 = 0 \Rightarrow \lim \sin\frac{1}{n} \to 0$$

or

$$\left.\begin{array}{l} \frac{1}{n} \to 0 \\ \sin - \text{continuous} \end{array}\right\} \Rightarrow \sin\frac{1}{n} \to 0$$

$\frac{1}{n} \in [0,1)$

sin $\nearrow$ on $[0,1)$ $\quad (\sin x)' = \cos x > 0$, $x \in [0, \frac{\pi}{2}]$ $\Rightarrow$ $\sin x \nearrow$ on $[0, \frac{\pi}{2}]$

$\Rightarrow$ $\sin \frac{1}{n} \searrow$ on $[0,1]$ $\xRightarrow{(1)(2)}$ the series converges

$|x_n| = \sin \frac{1}{n}$

$\sum |x_n| = \sum \sin \frac{1}{n}$

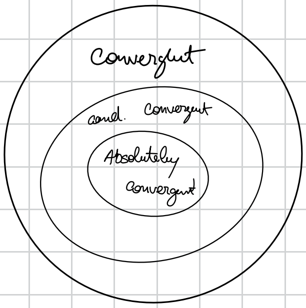
Comp test : $y_n = \frac{1}{n}$

$\lim\limits_{n\to\infty} \frac{|x_n|}{y_n} = \lim \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$ $\Rightarrow$ they have the same nature

$\sum \frac{1}{n} \to \infty$ $\quad \Rightarrow \sum |x_n|$ diverges

$\Rightarrow$ $\sum (-1)^n \sin \frac{1}{n}$ is conditionally convergent

(c) $\displaystyle\sum_{n\geq 1} \frac{\sin n}{n^2}$.



$\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2}$

Since $\sum \frac{1}{n^2}$ converges $\Rightarrow$ $\sum\limits_{n\geq 1} \left| \frac{\sin n}{n^2} \right|$ converges $\Rightarrow$ (x_n) is absolutely convergent (stronger than convergence)

$\Rightarrow$ $\sum \frac{\sin n}{n^2}$ is also convergent

3. Prove by differentiating the geometric series that, for $|x| < 1$,

$$(1) \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}, \quad (2) \sum_{n=2}^{\infty} n(n-1) x^n = \frac{2x^2}{(1-x)^3}.$$

(1) $\boxed{\sum\limits_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots = \frac{1}{1-x}}$ $= f(x)$ $\to$ differentiate term by term

converges absolutely when $|x| < 1$

$\sum |x|^n$ ratio test : $\frac{x_{n+1}}{x_n} = |x|$ $\quad$ <1, converges; $\quad >1$, diverges $\quad$ for this you can do all of the next ones

$f'(x) = \left(\frac{1}{1-x} \right)' = \sum\limits_{n=1}^{\infty} n x^{n-1}$

$= \frac{1}{(1-x)^2} = \sum\limits_{n=1}^{\infty} n x^{n-1} = (1 + 2x + 3x^2 + 4x^3 + \dots) \Big| \cdot x$

$\boxed{\frac{x}{(1-x)^2} = \sum\limits_{n=1}^{\infty} n x^n}$

(2) differentiate once more

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1+2x+3x^2+\dots$$

to get

$$\left((1-x)^{-2}\right)' = -2\cdot\frac{-1}{(1-x)^3} = \frac{2}{(1-x)^3}$$

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = 2+3\cdot 2x+\dots \quad \Big|\cdot x^2$$

$$\Rightarrow \boxed{\frac{2x^2}{(1-x)^3} = n(n-1)x^n}$$

4. Prove by integrating the geometric series that, for $|x| < 1$,

$$(1)\ \sum_{n=1}^{\infty}\frac{x^n}{n} = -\ln(1-x), \quad (2)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = \ln(1+x).$$

(1) $\displaystyle\sum_{n=0}^{\infty} t^n = 1+t+t^2+\dots = \frac{1}{1-t} = f(t)$

converges absolutely for $|t|<1$

integrating from 0 to x, for $|x|<1$

$$\int_0^x f(t)\,dt = \int_0^x \frac{1}{1-t}\,dt = -\ln(1-t)\Big|_0^x = -\ln(1-x)$$

$$= \sum_{n=0}^{\infty}\int_0^x t^n\,dt = \sum_{n=0}^{\infty}\frac{t^{n+1}}{n+1}\Big|_0^x = \sum_{n=0}^{x}\frac{x^{n+1}}{n+1} = -\ln(1-x)$$

$$= \sum_{n=1}^{\infty}\frac{x^n}{n} = -\ln(1-x) \quad \text{for } |x|<1$$

(2) $x \longmapsto -x$

$$\sum_{n=1}^{\infty}\frac{(-x)^n}{n} = -\ln(1+x)$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n} = -\ln(1+x) \quad \Big|\cdot(-1)$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = \ln(1+x)$$

5. Prove that $\displaystyle\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{2n+1} = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots = \arctan x$, for $x\in[-1,1]$.

replace t with $-t$

$$\sum_{n=0}^{\infty}(-t)^n = \frac{1}{1+t}, \quad |t|<1$$

$t \longmapsto t^2$: $\displaystyle\sum_{n=0}^{\infty}(-t^2)^n = \frac{1}{1+t^2}, \quad |t|<1$

$$\sum_{n=0}^{\infty}(-1)^n\cdot t^{2n} = \frac{1}{1+t^2}$$

Integrate $\int_0^x dt$ for $|x|<1$

$$\sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n} \, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \int_0^x \frac{1}{1+t^2} = \arctan x$$

$$\Longrightarrow \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

$\underline{x=1}$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Seminar 4

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   (b) ★ $\displaystyle\sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n} \cdot \frac{1}{n^2}$.

   (c) $\displaystyle\sum_{n\geq 1} a^{\ln n}$, $a>0$.

   (d) $\displaystyle\sum_{n\geq 1} \frac{a^n n!}{n^n}$ $a>0$.

2. Study the convergence and the absolute convergence of the following series:

   (a) $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$.

   (b) $\displaystyle\sum_{n\geq 1} (-1)^n \sin\frac{1}{n}$.

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3. Prove by differentiating the geometric series that, for $|x|<1$,

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5. Prove that $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots = \arctan x$, for $x\in[-1,1]$.

6. Find the radius of convergence and the convergence set for each of the following series:

   (a) $\displaystyle\sum_{n\geq 1} \frac{(x-2)^n}{(n+1)3^n}$.

   (b) $\displaystyle\sum_{n\geq 1} \frac{(x-1)^n}{n^p}$, $p>0$.

   (c) ★ $\displaystyle\sum_{n\geq 1} \frac{nx^n}{2^n}$.

7. ★ [Python] Show numerically that $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = \ln 2$. Change the order of summation in this series – for example by first adding p positive terms, then q negative terms, and so on – and show numerically that the rearrangement gives a different sum (depending on p, q).

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.

(a) $\sum_{n\geq1} \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}$.

(b) ★ $\sum_{n\geq1} \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\cdot\frac{1}{n^2}$.

(c) $\sum_{n\geq1} a^{\ln n}$, $a>0$.

(d) $\sum_{n\geq1} \frac{a^n n!}{n^n}$ $a>0$.

(b) $$\sum_{n\geq1}\frac{1\cdot3\cdot\ldots(2n-1)}{2\cdot4\cdot\ldots\cdot2n}\cdot\frac{1}{n^2}$$

$$\cancel{\frac{\cancel{1\cdot3\cdots(2n-1)}(2n+1)}{\cancel{2\cdot4\cdots2n}(2n+2)}\cdot\frac{1}{(n+1)^2}\cdot\frac{\cancel{2\cdot4\cdots2n}}{\cancel{1\cdot3\cdots(2n-1)}}}\cdot\frac{n^2}{1}=\frac{(2n+1)n^2}{(2n+2)(n+1)^2}\to1\Rightarrow\text{inconclusive}$$

$$\text{R-D: }\lim_{n\to\infty}n\left(\frac{x_n}{x_{n+1}}-1\right)=\lim_{n\to\infty}n\cdot\left(\frac{(2n+2)(n^2+2n+1)}{(2n+1)n^2}-1\right)=\lim_{n\to\infty}\cancel{n}\cdot\frac{\cancel{2n^3}+4n^2+2n+2n^2+4n+2-\cancel{2n^3}-n^2}{(2n+1)n^{\cancel{2}}}=\lim_{n\to\infty}\frac{5n^2+6n+2}{2n^2+n}$$

$$=\frac52>1\Rightarrow\text{convergent}$$

(c) $$\text{R-D: }\lim_{n\to\infty}n\left(\frac{a^{\ln n}}{a^{\ln(n+1)}}-1\right)=\lim_{n\to\infty}n\left(a^{\ln\frac{n}{n+1}}-1\right)=\cancel{\lim_{n\to\infty}\left(a^{n\ln\frac{n}{n+1}}-n\right)=\lim_{n\to\infty}a^{\left[\ln\left(1+\frac{-1}{n}\right)^{-n}\right]\frac{n}{-n}}=a}$$

$$=\lim_{n\to\infty}n\cdot\underbrace{\frac{a^{\ln\frac{n}{n+1}}-1}{\ln\frac{n}{n+1}}}_{\ln a}\cdot\ln\frac{n}{n+1}=\ln a\cdot\lim_{n\to\infty}\ln\left(1+\frac{-1}{n}\right)^n=\ln a\cdot\ln e^{-1}=-\ln a=\ln\frac1a$$

$$\ln\frac1a<1\iff\frac1a<e\Rightarrow a<e^{-1}$$

$$\ln\frac1a=1\Rightarrow a=e^{-1}=\frac1e\Rightarrow\sum_{n\geq1}\left(\frac1e\right)^{\ln n}=\sum\frac1n\to\infty$$

(d) $$\sum_{n\geq1}\frac{a^n n!}{n^n}$$

root: $\lim_{n\to\infty}\frac{a}{n}\sqrt[n]{n!}=$ ~~not~~

$$\frac{a^{\cancel{n+1}}\cancel{(n+1)!}}{(n+1)^{\cancel{n+1}}}\cdot\frac{n^n}{\cancel{a^n}\cdot\cancel{n!}}=a\left(\frac{n}{n+1}\right)^n=a\left(1+\frac{-1}{n}\right)^n\quad\Big|\ \lim\Rightarrow\frac{a}{e}$$

$\frac ae<1\Rightarrow a<e\Rightarrow$ conv

$\frac ae>1\Rightarrow$ div.

$$a=e\Rightarrow\text{R-D }\lim_{n\to\infty}n\left(\frac{x_n}{x_{n+1}}-1\right)=\lim_{n\to\infty}n\left(\frac{\cancel{a^n}\cancel{n!}}{n^n}\cdot\frac{(n+1)^{n+1}}{\underset{a}{\cancel{a^{n+1}}}\cancel{(n+1)!}}-1\right)=\lim_{n\to\infty}n\cdot\left(\underbrace{\frac1e\left(1+\frac1n\right)^n}_{e}-1\right)=(0\cdot\infty)=$$

2. Study the convergence and the absolute convergence of the following series:

(a) $\sum_{n\geq1}\frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$.

(b) $\sum_{n\geq1}(-1)^n\sin\frac1n$.

(c) $\sum_{n\geq1}\frac{\sin n}{n^2}$.

(a) if $(x_n)\to0$, (x_n) decreasing $\Rightarrow\sum_{n\geq1}(-1)^n\cdot x_n$ convergent

$$\left.\begin{array}{l}\frac{1}{\sqrt{n(n+1)}}\to0\\ \sqrt{n(n+1)}\text{ - increasing}\end{array}\right\}\Rightarrow\frac{1}{\sqrt{n(n+1)}}\text{ decreasing}\Rightarrow\text{convergent}$$

$$\left.\begin{array}{l}|x_n|=\frac{1}{\sqrt{n(n+1)}}\\ y_n=\frac1n\end{array}\right\}\Rightarrow\frac{x_n}{y_n}=\frac{n}{\sqrt{n(n+1)}}\to1\quad\frac{x_n}{y_n}\in(0,1)\Rightarrow\sum x_n\text{ diverges}$$

6. Find the radius of convergence and the convergence set for each of the following series:

(a) $\sum_{n\geq 1} \frac{(x-2)^n}{(n+1)3^n}$. (b) $\sum_{n\geq 1} \frac{(x-1)^n}{n^p}$, $p>0$. (c) ★ $\sum_{n\geq 1} \frac{nx^n}{2^n}$.

(a) $\sum_{n\geq 1} \underbrace{\frac{1}{(n+1)3^n}}_{a_n} \underbrace{(x-2)^n}_{(x-c)^n}$ – a power series centered at 2

$|x-2| < R \Rightarrow$ absolutely convergent

$|x-2| > R \Rightarrow$ divergent

$\cancel{L = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(n+1)3^n}} = \frac{1}{3} \lim_{n\to\infty} \frac{1}{\sqrt[n]{n+1}} = \frac{1}{3} \lim_{n\to\infty} \frac{1}{(1+n)^{\frac{1}{n}}} =}$

$$\lim_{n\to\infty} \frac{\frac{1}{(n+2)\cancel{3^{n+1}}3}}{\frac{1}{(n+1)\cancel{3^n}}} = \frac{1}{3} \lim_{n\to\infty} \frac{n+1}{n+2} = \frac{1}{3} \in (0,\infty) \Rightarrow R = \frac{1}{\frac{1}{3}} = 3$$

$\Rightarrow$ x_n convergent on $(2-3, 2+3) \to (-1, 5)$

$x=-1 \Rightarrow \sum_{n\geq 1} \frac{(-1)^n}{n+1} \to \ln 2 \Rightarrow$ conv.

$x=5 \Rightarrow \sum_{n\geq 1} \frac{1}{n+1} \to \infty$ div.

$\Rightarrow$ conv on $[-1, 5)$

(c) $\sum \frac{n x^n}{2^n} = \sum \underbrace{\frac{n}{2^n}}_{a_n} x^n$

$$\lim_{n\to\infty} \frac{n+1}{\cancel{2^{n+1}}\,2} \cdot \frac{\cancel{2^n}}{n} = \frac{1}{2} \in (0,\infty) \Rightarrow R = 2$$

conv on $(-2, 2)$

$x=-2 \Rightarrow \sum (-1)^n n \to \infty$

$x=2 \Rightarrow \sum n \to \infty$

$\Rightarrow$ conv on $(-2, 2)$