Seminar 12

1. Use Lagrange multipliers to find the extrema of the following functions subject to constraints:

   (a) $x^2 + y^2$ subject to $x - y + 1 = 0$.

   (b) $(x + y)^2$ subject to $x^2 + y^2 = 1$.

   (c) ★ $x^2 - y^2$ subject to $x^2 + y^2 = 1$.

   (d) $x + 2y + 3z$ subject to $x^2 + y^2 + z^2 = 1$.

   (e) $2x^2 + y^2 + 3z^2$ subject to $x^2 + y^2 + z^2 = 1$.

   (f) ★ $x^3 + y^3 + z^3$ subject to $x^2 + y^2 + z^2 = 1$.

2. Find the minimum value of $\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ subject to the following constraints:

   (a) $x_1 + x_2 + x_3 = 3$.

   (b) $x_1 + x_2 + x_3 = 3$ and $x_1 + 2x_2 + 3x_3 = 12$.

3. Compute the following integrals:

   (a) $\iint\limits_R \cos x \sin y \, \mathrm{d}x \, \mathrm{d}y$, where $R = [0, \pi/2] \times [0, \pi/2]$.

   (b) $\iint\limits_R \frac{1}{(x+y)^2} \, \mathrm{d}x \, \mathrm{d}y$ and $\iint\limits_R y e^{xy} \, \mathrm{d}x \, \mathrm{d}y$, where $R = [1, 2] \times [0, 1]$.

   ★ (c) $\iint\limits_R \min\{x, y\} \, \mathrm{d}x \, \mathrm{d}y$, where $R = [0, 1] \times [0, 1]$.

4. Let $D \subseteq \mathbb{R}^2$ be the subset bounded by the parabola $y = x^2$ and the lines $x = 2$ and $y = 0$.

   (a) Express D as a simple set first w.r.t. the y-axis and then w.r.t. the x-axis.

   (b) Compute $\iint\limits_D xy \, dx \, dy$ in two ways.

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.

Lagrange multipliers:

min / max $f(x)$ subject to $g(x) = c$

$\mathcal{L}(x, \lambda) = f(x) + \lambda(g(x) - c)$

$\nabla \mathcal{L} = 0$, $\nabla_x \mathcal{L} = 0 = \frac{\partial \mathcal{L}}{\partial x}$ (all partial derivatives are zero)

1. Use Lagrange multipliers to find the extrema of the following functions subject to constraints:

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(e) $2x^2 + y^2 + 3z^2$ subject to $x^2 + y^2 + z^2 = 1$.

(f) ★ $x^3 + y^3 + z^3$ subject to $x^2 + y^2 + z^2 = 1$.

(a) $f(x,y) = x^2 + y^2$ $\quad g(x,y) = x - y + 1$

$\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y) = x^2 + y^2 + \lambda(x - y + 1)$: Lagrange functional

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 2x + \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= 2y - \lambda = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &x + y = 0 \Rightarrow y = -x \\ &\lambda = 2y \end{aligned}$$

$\frac{\partial \mathcal{L}}{\partial \lambda} = x - y + 1 = 0 \Rightarrow 2x + 1 = 0 \Rightarrow \boxed{x = \frac{-1}{2}, \quad y = \frac{1}{2}, \quad \lambda = 1}$

$f(-\frac{1}{2}, \frac{1}{2}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Another solution: $x - y + 1 = 0 \Rightarrow y = x + 1$ $\quad f(x,y) = x^2 + y^2 = x^2 + (x+1)^2 = 2x^2 + 2x + 1$

vector $x = \frac{-1}{2}$ (min)

(b) $f(x,y) = (x+y)^2$ $\quad g(x,y) = x^2 + y^2 - 1 = 0$

$= x^2 + 2xy + y^2$

$\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda g(x,y) = x^2 + 2xy + y^2 + \lambda(x^2 + y^2 - 1)$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 2x + 2y + 2\lambda x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= 2x + 2y + 2\lambda y = 0 \end{aligned}\right\} \Rightarrow \lambda(x - y) = 0 \Rightarrow \lambda = 0 \text{ or } x = y$$

(i) $\lambda = 0 \Rightarrow x + y = 0$, $y = -x$ $\quad 2x^2 = 1$, $x^2 = \frac{1}{2}$ $\quad x = \pm \frac{\sqrt{2}}{2}$

$y = \mp \frac{\sqrt{2}}{2}$

$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 1 = 0$

(ii) $x = y \Rightarrow (2 + \lambda)x = 0$ $\quad \lambda = -2$

$2x^2 = 1$ $\quad x = \pm \frac{\sqrt{2}}{2} = y$

(i) $f(x,y) = (x+y)^2 = 0 \to$ min

(ii) $f(x,y) = (x+y)^2 = 4x^2 = 2 \to$ max

(d) $f(x,y,z) = x + 2y + 3z \qquad g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$

$L(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z) = x + 2y + 3z + \lambda(x^2 + y^2 + z^2 - 1)$

$\frac{\partial L}{\partial x} = 1 + 2\lambda x = 0 \Rightarrow x = \frac{-1}{2\lambda}$

$\frac{\partial L}{\partial y} = 2 + 2\lambda y = 0 \Rightarrow y = \frac{-1}{\lambda}$

$\frac{\partial L}{\partial z} = 3 + 2\lambda z = 0 \Rightarrow z = \frac{-3}{2\lambda}$

$\frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1 = 0 \Rightarrow \frac{1}{4\lambda^2} + \frac{\overset{4}{\cancel{1}}}{\lambda^2} + \frac{9}{4\lambda^2} = 1$

$\Rightarrow 14 = 4\lambda^2 \Rightarrow \lambda = \pm\sqrt{\frac{7}{2}}$

$\lambda = \sqrt{\frac{7}{2}} \quad f(x,y,z) \to \min$

$\lambda = -\sqrt{\frac{7}{2}} \quad f(x,y,z) \to \max$

(e) $f(x,y,z) = 2x^2 + y^2 + 3z^2 \qquad g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$

$L(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z) = 2x^2 + y^2 + 3z^2 + \lambda(x^2 + y^2 + z^2 - 1)$

$\frac{\partial L}{\partial x} = 4x + 2\lambda x = 0, \quad (2+\lambda)x = 0$

$\frac{\partial L}{\partial y} = 2y + 2\lambda y = 0 \quad (1+\lambda)y = 0$

$\frac{\partial L}{\partial z} = 6z + 2\lambda z = 0 \quad (3+\lambda)z = 0$

$\frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1 = 0$ Note that we cannot have $x = y = z = 0$

We have 3 cases:

(i) $\lambda = -2, \quad y = z = 0, \quad x^2 = 1 \Rightarrow x = \pm 1 \quad f(x,y,z) = 2x^2 = 2$

(ii) $\lambda = -1 \quad x = z = 0, \quad y^2 = 1 \Rightarrow y = \pm 1 \quad f(x,y,z) = y^2 = 1 \longrightarrow \min$

(iii) $\lambda = -3 \quad x = y = 0, \quad z^2 = 1 \Rightarrow z = \pm 1 \quad f(x,y,z) = 3z^2 = 3 \longrightarrow \max$

2. Find the minimum value of $\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ subject to the following constraints:

(a) $x_1 + x_2 + x_3 = 3$. (b) $x_1 + x_2 + x_3 = 3$ and $x_1 + 2x_2 + 3x_3 = 12$.

(a) $f(x_1,x_2,x_3) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \qquad g(x_1,x_2,x_3) = x_1 + x_2 + x_3 - 3 = 0$

$L(x_1,x_2,x_3,\lambda) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \lambda(x_1 + x_2 + x_3 - 3)$

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= x_1 + \lambda = 0 \\ \frac{\partial L}{\partial x_2} &= x_2 + \lambda = 0 \\ \frac{\partial \lambda}{\partial x_3} &= x_3 + \lambda = 0 \end{aligned}\right\} \Rightarrow x_1 = x_2 = x_3 = -\lambda = 1$$

$$\frac{\partial L}{\partial \lambda} = x_1+x_2+x_3-3=0 \Rightarrow \quad 3(-\lambda-1)=0 \Rightarrow \lambda=-1$$

$$f(x_1,x_2,x_3)=\frac{3}{2}$$

(b) $L(x_1,x_2,x_3,\lambda,\mu)$ * because we have 2 constrains: $x_1+x_2+x_3=3$

$x_1+2x_2+3x_3=12$

$$L(x_1,x_2,x_3,\lambda,\mu) = f(x_1,x_2,x_3) + \lambda g_1(x_1,x_2,x_3) + \mu g(x_1,x_2,x_3)$$
$$= \frac{1}{2}(x_1^2+x_2^2+x_3^2) + \lambda(x_1+x_2+x_3-3) + \mu(x_1+2x_2+3x_3-12)$$

$$\left.\begin{array}{l}(1)\ \frac{\partial L}{\partial x_1} = x_1+\lambda+\mu=0 \\ (2)\ \frac{\partial L}{\partial x_2} = x_2+\lambda+2\mu=0 \\ (3)\ \frac{\partial L}{\partial x_3} = x_3+\lambda+3\mu=0\end{array}\right\} \overset{(4)}{\Longrightarrow} 3+3\lambda+6\mu=0 \Rightarrow \lambda+2\mu=-1$$

$$\lambda=-1-2\mu$$
$$2\mu=-1-\lambda$$

(4) $\frac{\partial L}{\partial \lambda} = x_1+x_2+x_3-3=0 \Rightarrow x_1+x_2+x_3=3$

(5) $\frac{\partial L}{\partial \mu} = x_1+2x_2+3x_3-12=0$

$$(1)+2\cdot(2)+3\cdot(3) = 12+6\lambda+14\mu=0$$
$$12+6\lambda-7-7\lambda=0$$
$$\lambda=5 \Rightarrow \mu=-3$$

$$\Rightarrow x_1=-2,\quad x_2=1,\quad x_3=4$$

$$f(-2,1,4)=\frac{1}{2}(4+1+16)=\frac{21}{2} > \frac{3}{2}$$

* the more constrains I put, the "worse" the min might be, since I'm moving in a smaller space

3. Compute the following integrals:

(a) $\iint\limits_R \cos x \sin y \, dx\, dy$, where $R=[0,\pi/2]\times[0,\pi/2]$.

(b) $\iint\limits_R \frac{1}{(x+y)^2}\, dx\, dy$ and $\iint\limits_R ye^{xy}\, dx\, dy$, where $R=[1,2]\times[0,1]$.

(c) $\iint\limits_R \min\{x,y\}\, dx\, dy$, where $R=[0,1]\times[0,1]$.

(a) $$\int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}} \cos x \,\underline{\sin y}\, dx\right)dy = \int_0^{\frac{\pi}{2}} \sin y\left(\int_0^{\frac{\pi}{2}}\cos x\, dx\right)dy = \left(\int_0^{\frac{\pi}{2}}\sin y\, dy\right)\cdot\left(\int_0^{\frac{\pi}{2}}\cos x\, dx\right) = 1\cdot 1 = 1$$

will be a constant
⇒ it can be pulled out of the integral

* separable function (product of $f(x)$ and $g(y)$)
↳ do them separatly and then multiply

(6.1) $\int_0^1 \left(\int_1^2 \frac{1}{(x+y)^2}\,dx\right)dy =$

$\int_1^2 \frac{1}{(x+y)^2}\,dx = -\frac{1}{x+y}\Big|_1^2 = \frac{-1}{y+2} + \frac{1}{y+1}$

$\int_0^1 \left(\frac{-1}{y+2} + \frac{1}{y+1}\right)dy = -\ln(y+2)\Big|_0^1 + \ln(y+1)\Big|_0^1 = -\ln 3 + \ln 2 + \ln 2 = \ln\frac{4}{3}$

(6.2) $\int_0^1 \left(\int_1^2 y e^{xy}\,dx\right)dy = \int_0^1 e^{xy}\Big|_1^2 dy = \int_0^1 (e^{2y} - e^{y})\,dy = \frac{e^{2y}}{2}\Big|_0^1 - e^{y}\Big|_0^1 = \frac{e^2}{2} - \frac{1}{2} - e + 1 = \frac{e^2 - 2e + 1}{2}$

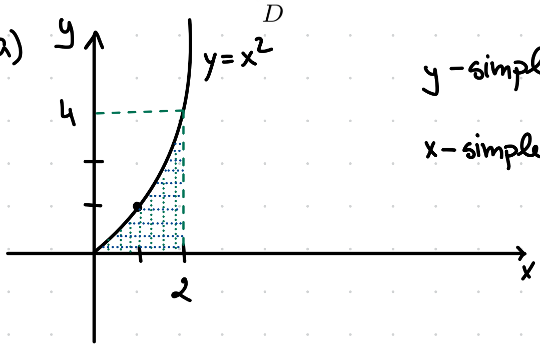
$\int y e^{xy}\,dx = e^{xy}$ like $\int 2e^{2x} = e^{2x}$

4. Let $D \subseteq \mathbb{R}^2$ be the subset bounded by the parabola $y = x^2$ and the lines $x = 2$ and $y = 0$.

(a) Express D as a simple set first w.r.t. the y-axis and then w.r.t. the x-axis.

(b) Compute $\iint_D xy\,dx\,dy$ in two ways.

(a)



y-simple: $D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le x^2\}$

x-simple: $D = \{(x,y) \mid 0 \le y \le 4,\ \sqrt{y} \le x \le 2\}$

$y \le x^2$ $\sqrt{y} \le x$

(b) $\iint_D xy\,dx\,dy = \int_0^2 \left(\int_0^{x^2} xy\,dy\right)dx = \int_0^2 x\frac{y^2}{2}\Big|_{y=0}^{y=x^2} = \int_0^2 \frac{x^5}{2}\,dx = \frac{x^6}{12}\Big|_0^2 = \frac{64}{12} = \frac{16}{3}$

$= \int_0^4 \left(\int_{\sqrt{y}}^2 xy\,dx\right)dy = \int_0^4 y\cdot\frac{x^2}{2}\Big|_{\sqrt{y}}^2 dy = \int_0^4 y\left(2 - \frac{y}{2}\right)dy = \int_0^4 \left(2y - \frac{y^2}{2}\right)dy = y^2\Big|_0^4 - \frac{y^3}{6}\Big|_0^4$

$= 16 - 0 - \frac{64}{6} + 0 = \frac{32}{6} = \frac{16}{3}$