2. ★ Find the sum for each of the following series:

(a) $\sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right)$. (b) $\sum_{n\geq 1} \frac{n+1}{3^n}$. (c) $\sum_{n\geq 1} \frac{n}{n^4+n^2+1}$.

(a) $$\sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right) = \sum_{n\geq 2} \ln \frac{(n+1)(n-1)}{n^2} = \sum_{n\geq 2} \left[\ln(n+1)+\ln(n-1)-2\ln n\right] = \sum_{n\geq 2}\left(\ln\frac{n-1}{n} - \ln\frac{n}{n+1}\right) \Rightarrow \lim_{n\to\infty}\left(\ln\frac{1}{2} - \cancel{\ln\frac{2}{3}} + \cancel{\ln\frac{2}{3}} - \dots + \cancel{\ln\frac{n-1}{n}} - \ln\frac{n}{n+1}\right)$$

$$= \lim_{n\to\infty} \ln\frac{1}{2} - \ln\frac{n}{n+1} = \lim_{n\to\infty} \ln\frac{n+1}{2n} = \ln\frac{1}{2} = \ln 1 - \ln 2 = 0 - \ln 2 = -\ln 2$$

(b) $$\sum_{n\geq 1} \frac{n+1}{3^n} \Rightarrow \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+2}{\cancel{3^{n+1}}_{3}} \cdot \frac{\cancel{3^n}}{n+1} = \lim_{n\to\infty} \frac{n+2}{3(n+1)} = \frac{1}{3} < 1 \Rightarrow \lim_{n\to\infty} x_n = 0$$

$$S = \sum_{n\geq 1} \frac{n}{3^n} + \sum_{n\geq 1} \frac{1}{3^n} = 3\cdot \sum_{n\geq 1} \frac{n}{3^{n+1}} + \sum_{n\geq 1} \frac{1}{3^n} = 3\cdot S + \frac{1}{1-\frac{1}{3}} - 1 = 3\cdot S + \frac{3}{2} - 1 = 3\cdot S + \frac{1}{2}$$

$$S = 3S + \frac{1}{2} \quad |\cdot 2 \Rightarrow 2S = 6S + 1$$

$$-1 = 4S \Rightarrow S = -\frac{1}{4}$$

(c) $$\sum_{n\geq 1} \frac{n}{n^4+n^2+1} = \sum_{n\geq 1} \frac{n}{n^4+2n^2+1-n^2} = \sum_{n\geq 1} \frac{n}{(n^2+1)^2-n^2} = \sum_{n\geq 1} \frac{n}{(n^2-n+1)(n^2+n+1)} = \frac{1}{2}\sum_{n\geq 1} \frac{2n}{(n^2-n+1)(n^2+n+1)} = \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right)$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \dots + \cancel{\frac{1}{n^2-n+1}} - \frac{1}{n^2+n+1}\right) = \frac{1}{2}\lim_{n\to\infty}\left(1^{(n^2+n+1)} - \frac{1}{n^2+n+1}\right) = \frac{1}{2}\lim_{n\to\infty} \frac{n^2+n+\cancel{1}-\cancel{1}}{n^2+n+1} = \frac{1}{2}$$

4. ★ Study if the following series are convergent or divergent:

(a) $\sum_{n\geq 1} \frac{x^n}{n^p}$, $x>0, p\in\mathbb{N}$. (b) $\sum_{n\geq 2} \frac{1}{(\ln n)^{\ln n}}$. (c) $\sum_{n\geq 1} (\sqrt[n]{n}-1)$.

(a) $\sum_{n\geq 1} \frac{x^n}{n^p}$, $x>0$, $p\in\mathbb{N}$

$$x_n = \frac{x^n}{n^p} \qquad x_{n+1} = \frac{x^{n+1}}{(n+1)^p}$$

ratio test $$\frac{x_{n+1}}{x_n} = \frac{\cancel{x^{n+1}}}{(n+1)^p} \cdot \frac{n^p}{\cancel{x^n}} = x\cdot \underbrace{\left(\frac{n}{n+1}\right)^p}_{\to 1} \longrightarrow x, \text{ x is fixed} \Rightarrow \sum_{n\geq 1} \frac{x^n}{n^p} \text{ converges to } x$$

(b) $\sum_{n\geq 2} \frac{1}{(\ln n)^{\ln n}}$

Cauchy condensation method $\sum_{n\geq 1} x_n$ "like" $\sum 2^n x_{2^n}$

$$x_n = \frac{1}{(\ln n)^{\ln n}} \Rightarrow \sum_{n\geq 1} x_n = \sum_{n\geq 1} 2^n \cdot \frac{1}{(\ln 2^n)^{\ln 2^n}} = \sum_{n\geq 1} \frac{2^n}{\big(\underbrace{(n\cdot \ln 2)^{\ln 2}}_{\to\infty}\big)^n} = \sum_{n\geq 1}\left(\frac{2}{(n\cdot\ln 2)^{\ln 2}}\right)^n$$

we apply the root test

Let $\sum_{n=1}^{\infty} x_n$. $\quad \lim_{n\to\infty} \sqrt[n]{x_n} = l$

- if $l<1 \Rightarrow \sum_{n=1}^{\infty} x_n$ convergent
- if $l>1 \Rightarrow \sum_{n=1}^{\infty} x_n$ divergent

$$\lim_{n\to\infty} \sqrt[\cancel{n}]{\left(\frac{2}{(n\ln 2)^{\ln 2}}\right)^{\cancel{n}}} = \lim_{n\to\infty} \frac{2}{(n\ln 2)^{\ln 2}} = \frac{2}{\ln^2 2}\cdot \lim_{n\to\infty}\frac{1}{n} = 0 \Rightarrow \text{convergent}$$

(c) $$\sum_{n\geq 1} \sqrt[n]{n} - 1 = \sum_{n\geq 1}\left(n^{\frac{1}{n}} - 1\right) = \sum_{n\geq 1}\left(e^{\frac{1}{n}\ln n} - 1\right)$$

$$\lim_{n\to\infty} \frac{\ln n}{n} \overset{\text{l'H}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty}\frac{1}{n} = 0 \Rightarrow \text{we can apply } \lim_{x\to 0}\frac{e^x-1}{x} = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{e^{\frac{1}{n}\ln n}-1}{\frac{\ln n}{n}} = 1$$

$$\Rightarrow e^{\frac{\ln n}{n}} - 1 \approx \frac{\ln n}{n}$$

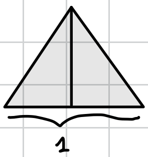
$$\Rightarrow S = \sum_{n\geq 1} \left(e^{\frac{\ln n}{n}} - 1\right) \approx \sum_{n \geq 1} \frac{\ln n}{n}$$

Comparison test: $\ln n > 1$, $n > 3 \mid \cdot \frac{1}{n}$

$$\left.\begin{array}{l} \frac{\ln n}{n} > \frac{1}{n} \\ \sum_{n\geq 1} \frac{1}{n} \text{ is divergent} \end{array}\right\} \Rightarrow \sum_{n\geq 1} \frac{\ln n}{n} \text{ diverges} \Rightarrow \sum_{n \geq 1} \sqrt[n]{n} - 1 \text{ diverges}$$

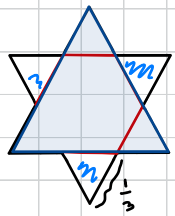
5. ★ Start with an equilateral triangle of side 1. For each side, remove the middle third and add there another equilateral triangle. Repeat this process at each iteration (see the figure). How many sides are there at iteration n? What is the limit of the perimeter and the area?

iteration 1



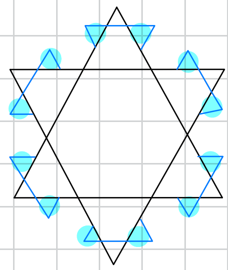
$$p = 1 \cdot 3 = 3$$
$$A = \frac{\sqrt{3}}{4} \cdot a^2 = \frac{\sqrt{3}}{4}$$

iteration 2

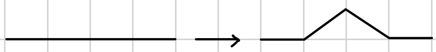


$$p = \frac{1}{3} \cdot 12 = 4$$

iteration 3



* after each split, the side goes into 4 sides



$$P_1 = \frac{4}{3} \cdot P_0 = \frac{4}{3} \cdot 3 = 4$$
$$P_2 = \frac{4}{3} \cdot P_1 = \frac{4}{3} \cdot 4 = \frac{4^2}{3}$$
$$P_3 = \frac{4}{3} \cdot P_2 = \frac{4}{3} \cdot \frac{4^2}{3} = \frac{4^3}{3^2}$$
$$\vdots$$
$$P_n = \frac{4^n}{3^{n-1}}$$

* nr. sides increases 4 times after each iteration

sides : $3 \rightarrow 4 \cdot 3 \rightarrow 4^2 \cdot 3 \rightarrow \dots$

area : let l be the original length of the triangle

$$A_1 = \frac{l^2\sqrt{3}}{4}$$
$$A_2 = 3 \cdot \frac{\left(\frac{l}{3}\right)^2 \sqrt{3}}{4}$$
$$A_3 = 3 \cdot 4 \cdot \frac{\left(\frac{l}{3^2}\right)^2 \sqrt{3}}{4}$$
$$A_4 = 3 \cdot 4^2 \cdot \frac{\left(\frac{l}{3^3}\right)^2 \sqrt{3}}{4}$$

$$A = \frac{\ell^2\sqrt{3}}{4}\left(1 + 3\cdot\left(\frac{1}{3}\right)^2 + 3\cdot 4\cdot\left(\frac{1}{3^2}\right)^2 + 3\cdot 4^2\cdot\left(\frac{1}{3^3}\right)^2 + \dots + 3\cdot 4^{n-1}\cdot\left(\frac{1}{3^n}\right)^2\right)$$

$$= \frac{\ell^2\sqrt{3}}{4^2}\left(4 + 3\cdot 4\cdot\left(\frac{1}{3}\right)^2 + \cancel{3\cdot 4}^2\left(\frac{1}{3^2}\right)^2 + \dots + 3\cdot 4^n\cdot\left(\frac{1}{3^n}\right)^2\right)$$

$$= \frac{\ell^2\sqrt{3}}{4^2}\left(4 + \underbrace{3\cdot\frac{4}{9} + 3\cdot\left(\frac{4}{9}\right)^2 + 3\cdot\left(\frac{4}{9}\right)^3 + \dots + 3\cdot\left(\frac{4}{9}\right)^n}_{\text{geometric progression}}\right) \qquad q = \frac{4}{9}$$

$$\sum_{k=1}^{n} 3\cdot\left(\frac{4}{9}\right)^k = 3\left(\frac{4}{9}\right)^1\cdot\frac{\left(\frac{4}{9}\right)^n - 1}{\frac{4}{9} - 1} = \frac{4}{\cancel{3}}\cdot\frac{\cancel{9}^{3}\left(\left(\frac{4}{9}\right)^n - 1\right)}{-5} = \frac{12}{-5}\cdot\left(\left(\frac{4}{9}\right)^n - 1\right)$$

$$= \frac{\sqrt{3}}{4^2}\left[4 - \frac{12}{5}\left(\left(\frac{4}{9}\right)^n - 1\right)\right]$$