

## Seminar 6

1. Recall that the Taylor series for sin and cos are given, for any  $x \in \mathbb{R}$ , by:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

- (a) Prove that  $(\sin x)' = \cos x$  and  $(\cos x)' = -\sin x$ , for any  $x \in \mathbb{R}$ .
- (b) Deduce that  $x \frac{x^3}{6} < \sin x < x$ ,  $\forall x > 0$  and  $1 \frac{x^2}{2} < \cos x < 1 \frac{x^2}{2} + \frac{x^4}{24}$ ,  $\forall x \in \mathbb{R}$ .
- (c) Where does Euler's formula  $e^{ix} = \cos x + i \sin x$  come from?
- (a) For  $\alpha \in \mathbb{R}$  and |x| < 1, prove the generalized binomial expansion (binomial series)

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} {\alpha \choose k} x^k, \quad {\alpha \choose k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}.$$

- (b) Find the first four terms in the binomial series of  $\sqrt{1+x}$  and  $1/\sqrt{1+x}$ .
- 3. Find the MacLaurin series and its radius of convergence for the following functions:

(a) 
$$a^x, a > 0.$$

(c) 
$$\sin^2(x)$$
.

(b) 
$$(1+x)\ln(1+x)$$
.

(d) 
$$\arctan x$$
.

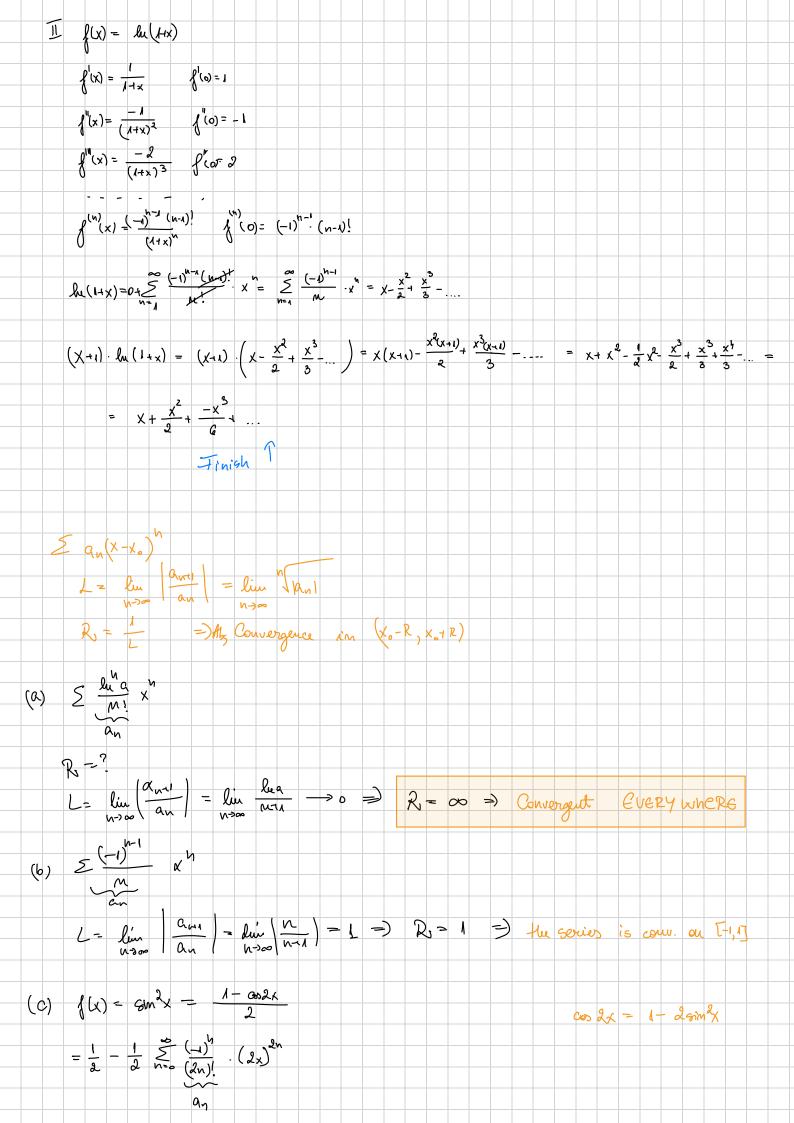
4. For each function  $f: \mathbb{R} \to \mathbb{R}$  given below check that f'(0) = 0 and find the first  $n \in \mathbb{N}$  such that  $f^{(n)}(0) \neq 0$ . Then, deduce whether 0 is a local extremum point of f or not; in the affirmative, specify if 0 is a global extremum point or just a local one.

(a) 
$$f(x) = e^x + e^{-x} - x^2$$
. (b)  $f(x) = \cos(x^2)$ .

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(c) 
$$f(x) = 6\sin x - 6x + x^3$$
.

$$\begin{cases} f(x) = x \cdot (x-1) \cdot (x-x) \\ f(x) = x \cdot (x-1)$$

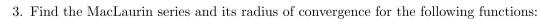


$$\frac{a_{n+1}}{a_{n}} = \frac{(-1)^{n+1}}{2^{n+1}} \cdot \frac{2^{n+1}}{2^{n+1}} = \frac{-1}{(2^{n+1})(2^{n+1})} \longrightarrow 0 \implies 2, = 0$$

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(c)  $\sin^2(x)$ .

(b)  $(1+x) \ln(1+x)$ .

(d)  $\arctan x$ .

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$$f(x) = e^x + e^{-x} - x^2$$
. (b)  $f(x) = \cos(x^2)$ .

(c)  $f(x) = 6\sin x - 6x + x^3$ .

3. (a) 
$$f(x) = a^{x}$$
  
 $f'(x) = a^{x} \ln a$   $f'(x) = \ln a$   
 $f''(x) = a^{x} \ln^{2} a$   $f''(x) = \ln^{2} a$ 

$$f^{(n)}(x) = a^{x} lu^{n} a \qquad f^{(n)}(x) = lu^{n} a$$

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n = \sum_{n=0}^{\infty} \frac{\ln^n a}{n!} x^n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\ln a}{\ln a} - \frac{\ln a}{\ln a}\right| = \lim_{n\to\infty} \frac{\ln a}{\ln a} \to 0 \Rightarrow R = \infty \Rightarrow conv. everywhere$$

$$\frac{\ln a}{\ln 1} \rightarrow 0 \Rightarrow R = \infty \Rightarrow com.$$
 everywha

$$\int_{|||} (x) = \frac{-1}{(1+x)^2}$$

$$\int_{0}^{1} (x) z \frac{-\lambda \cdot 5}{(1-x)^{4}}$$

$$\int_{(m)} (x) = \frac{(\mu + \kappa)^{m-1}}{(\mu + \kappa)^{m-1}}$$

$$\int_{0}^{1}(x) = \int_{0}^{1}(x + 1) dx = \int_{0}$$