Recapitulare

V - vectorspace
B, B' basis of V
basechange from B' to B: $T_{B',B} = [id]_{B,B'} = [id]^{-1}_{B',B} = T^{-1}_{B',B}$

$\forall\, v \in V$

$$[v]_{B'} = [id]_{B,B'} \cdot [v]_B$$

More general formula:

$$[f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B$$

↳ expects a vector in B and "spits out" a vector in B'

$f \in Hom_{\mathbb{K}}(V, V')$
B_1, B_2 basis of V
B'_1, B'_2 basis of V'

$$[f]_{B_2,B'_2} = [id]_{B'_1,B'_2}\, [f]_{B_1,B'_1}\, [id]_{B_2,B_1}$$

* a base change acts on the

$f: V \to V'$ $\quad B = (v_1, \dots, v_n)$ - basis of V

$$\boxed{[f]_{B,B'} = \left([f(v_1)]_{B'}\; [f(v_2)]_{B'} \dots [f(v_n)]_{B'} \right)}$$

3. In the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ consider the bases $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)\,(a \in \mathbb{R})$ and $B' = (1, X-b, (X-b)^2)\,(b \in \mathbb{R})$. Determine the matrices of change of bases T_{EB}, T_{BE} and $T_{BB'}$.

$$T_{EB} = [id]_{B,E} = \left([1]_E, [x-a]_E, [(x-a)^2]_E \right)$$

$$[1]_E = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$[x-a]_E = \begin{pmatrix} -a \\ 1 \\ 0 \end{pmatrix} \qquad \Rightarrow T_{EB} = \begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}$$

$$[(x-a)^2]_E = \begin{pmatrix} a^2 \\ -2a \\ 1 \end{pmatrix}$$

$$T_{BE} = T_{EB}^{-1}$$

$$\left(\begin{array}{ccc|ccc} 1 & -a & a^2 & 1 & 0 & 0 \\ 0 & 1 & -2a & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_1 \leftarrow L_1 - a^2 L_3}{\overset{L_2 \leftarrow L_2 + 2aL_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & -a & 0 & 1 & 0 & -a^2 \\ 0 & 1 & 0 & 0 & 1 & 2a \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{\sim}{L_1 \leftarrow L_1 + aL_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & a & -a^2 \\ 0 & 1 & 0 & 0 & 1 & 2a \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

* check it by multiplying

$$T_{BE} = \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_{B,B'} = [id]_{B,B'} = [id]_{E,B} \, [id]_{B'E}$$

$$[id]_{B',E} = \left([1]_E, [x-b]_E, [(x-b)^2]_E \right) = \begin{pmatrix} 1 & -b & b^2 \\ 0 & 1 & -2b \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow T_{B,B'} = \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -b & b^2 \\ 0 & 1 & -2b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a-b & (a-b)^2 \\ 0 & 1 & 2(a-b) \\ 0 & 0 & 1 \end{pmatrix}$$

alternative

$$[id]_{B',B} = \left([1]_B, [x-b]_B, [(x-b)^2]_B \right)$$

$$[1]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x-b = x-a+a-b$$

$$[x-b]_B = \begin{pmatrix} a-b \\ 1 \\ 0 \end{pmatrix}$$

$$(x-b)^2 = (x-a+a-b)^2 = (x-a)^2 + 2(a-b)\cdot(x-a) + (a-b)^2$$

$$[(x-b)^2] = \begin{pmatrix} (a-b)^2 \\ 2(a-b) \\ 1 \end{pmatrix}$$

8. $\begin{pmatrix}\cos x & -\sin x \\ \sin x & \cos x\end{pmatrix}$ $(x \in \mathbb{R})$. * rotation matrix

$$P_A = \begin{vmatrix} \cos x - X & -\sin x \\ \sin x & \cos x - X \end{vmatrix} = (\cos x - X)^2 - \sin^2 x = X^2 - 2X\cos x + 1$$

$$\lambda_{1,2} = \frac{2\cos x \pm \sqrt{4\cos^2 x - 4}}{2}$$

$P_A = \det(A - X I_n)$

$$\Delta = 4\cos^2 x - 4 \le 0 \Rightarrow$$

for $x \neq k\pi$, $x \neq k + \frac{\pi}{2} + k\pi$ $\Rightarrow$ we have complex eigenval.

For $x = k\pi \Rightarrow \lambda = \cos(k\pi) = (-1)^k$

$$S(\lambda) = \left\{ v = (a,b) \;\middle|\; (A - \lambda I_2)\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \right\}$$

* eigenspace of λ

If k-even $\Rightarrow$ $\lambda = 1$ $\quad S(\lambda) = \{ (a,b) \mid \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \cdot \begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$

For $x = \frac{\pi}{2} + k\pi \Rightarrow$ $\lambda = \cos x = 0$

$$S(\lambda) = \{ v = (a,b) \mid \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} \cdot \begin{pmatrix}a\\b\end{pmatrix} \} \Rightarrow \{ v = (a,b) \mid b = 0,\ a = 0 \}$$

5. $\begin{pmatrix}3 & 1 & 0\\ -4 & -1 & 0\\ -4 & -8 & -2\end{pmatrix} = A$

$$P_A = \begin{vmatrix} 3 - X & 1 & 0 \\ -4 & -1 - X & 0 \\ -4 & -8 & -2 - X \end{vmatrix} = \det(A - X I_n)$$

* divizorii termenului liber

$$(-2 - X) \cdot \begin{vmatrix} 3 - X & 1 \\ -4 & -1 - X \end{vmatrix} = -(X + 2) \cdot (-3 - 3X + X + X^2 + 4) =$$

$$= -(X+2) \cdot (X^2 - 2X + 1) = -(X+2)(X-1)^2 \Rightarrow \lambda_1 = -2,\ \lambda_2 = 1$$

$$S(-2) = \left\{ v=(x,y,z) \;\middle|\; \begin{pmatrix} 5 & 1 & 0 \\ -4 & 1 & 0 \\ -4 & -8 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\left\{ v=(x,y,z) \;\middle|\; \begin{array}{l} 5x+y=0 \\ -4x+y=0 \\ -4x-8y=0 \end{array} \right\} = \{ (0,0,z) \mid z \in \mathbb{R} \} = \underline{\langle (0,0,1) \rangle}$$

Scrie asta !!

* Vectorii proprii sunt $(0,0,z) \mid z \in \mathbb{R}^*$

Hint dimensiunea subspatiului propriu $<$ multiplicitatea

$$S(1) = \left\{ v=(x,y,z) \;\middle|\; \begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ -4 & -8 & -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\left\{ x=(x,y,z) \;\middle|\; \begin{array}{l} 2x+y=0 \\ -4x-2y=0 \\ -4x-8y-3z=0 \end{array} \right\} \Rightarrow$$

$$y=-2y \Rightarrow -4x+16x-3z=0$$
$$z=4x$$

$$\Rightarrow S(1) = \{ (x,-2x,4x) \mid x \in \mathbb{R} \} = \langle (1,-2,4) \rangle$$