$$x'(t) + \underline{a(t)}\, x(t) = \underline{b(t)}$$

$$x(t) = \underbrace{x_h(t)}_{\text{sol. of hom. eq}} + \underbrace{x_p(t)}_{\text{a particular solution}}$$

$x(t) \neq 0 \rightarrow$ nu se anulează nicăieri

$$x'(t) = -a(t)\cdot x(t) \quad | \, x(t)$$

$$\frac{dx}{dt} = -a(t)\cdot x(t)$$

$$\frac{dx}{dt} = -a\cdot x \quad |\cdot x$$

$$\frac{dx}{x}\cdot\frac{1}{dt} = -a(t)$$

$$\frac{dx}{x} = -a(t)\,dt$$

$$\int \frac{1}{x}\,dx = \int -a(t)\,dt$$

$$\ln x = \int -a(t)\,dt$$

$$\frac{x'(t)}{x(t)} = -a(t) \,\Big|\int_0^s$$

$$\int_0^s \frac{x'(t)}{x(t)}\,dt = -\int_0^s a(t)\,dt$$

$$\ln x(t)\Big|_0^s = -\int_0^s a(t)\,dt$$

$$\ln x(s) - \ln x(0) = -\int_0^s a(t)\,dt \quad \Big|\, e^{(\,)}$$

$$\boxed{x(s) = x(0)\cdot e^{-\int_0^s a(x)\,dx}}$$ solution of homogeneous equation

* Metoda de separare a variabilelor a lui La Grange

$$\boxed{x_p(t) = \gamma(t)\cdot x_h(t)}$$ a particular eq. solution

1) $$x'(t) + \underbrace{\frac{1}{t}}_{a(t)}\, x(t) = \underbrace{\frac{1}{t}\cdot e^{-2t+1}}_{b(t)}$$

$X = x_h + x_p$

hom. eq $\quad x'(t) + \frac{1}{t}x(t) = 0$

$$x'(t) = -\frac{1}{t}x(t)$$

$$\frac{x'(t)}{x(t)} = -\frac{1}{t} \quad \Big|\int_c^s$$

$$\int_c^s \frac{x'(t)}{x(t)}\,dt = -\int_c^s \frac{1}{t}\,dt$$

$$\ln x(t)\Big|_c^s = -\ln t\Big|_c^s$$

$$\ln x(s) - \ln x(c) = -\ln s + \ln c$$

$$\ln\frac{x(s)}{x(c)} = \ln\frac{c}{s} \quad \Big|\, e^{(\,)}$$

$$\frac{x(s)}{x(c)} = \frac{c}{s}$$

$$x(s) = \frac{c\cdot x(c)}{s}$$

let $c\cdot x(c) = c_2$

$$x'(s) + \frac{1}{s}\cdot\frac{c_2}{s} = 0$$

$$c_1 \cdot \frac{-1}{s^2} + \frac{c_2}{s^2} = 0$$

$$\Rightarrow x_h(s) = \frac{c_2}{s}$$

$$x_p = \varphi(t) \cdot x_n(t)$$

$$x_p = \varphi(t) \cdot \frac{1}{t}$$

$$\left(\frac{\varphi(t)}{t}\right)' = \left(\varphi(t) \cdot \frac{1}{t}\right)' = \varphi'(t) \cdot \frac{1}{t} - \varphi(t) \cdot \frac{1}{t^2}$$

$$\left(\frac{\varphi(t)}{t}\right)' - \frac{1}{t} \cdot \frac{1}{t}\varphi(t) = \frac{1}{t} \cdot e^{-2t+1}$$

$$\frac{\varphi'(t)}{t} - \cancel{\frac{\varphi(t)}{t^2}} + \cancel{\frac{\varphi(t)}{t^1}} = \frac{1}{t} \cdot e^{-2t+1}$$

$$\left.\begin{aligned} &\frac{\varphi'(t)}{t} = \frac{1}{t} e^{-2t+1} \\ &t \neq 0 \end{aligned}\right\} \Rightarrow \varphi(t) = -\frac{1}{2} e^{-2t+1} + c_3$$

$$x_p(t) = \frac{-1}{2} e^{-2t+1} \cdot \frac{1}{t}$$

$$x = x_n + x_p = \frac{c_2}{t} - \frac{1}{2t} e^{-2t+1}$$

sau $\quad x'(t) + \frac{1}{t} x(t) = \frac{1}{t} \cdot e^{-2t+1} \quad | \cdot t$

$$\underbrace{t x'(t) + x(t)} = e^{-2t+1}$$

$$(t \cdot x(t))' = e^{-2t+1}$$

$$t \cdot x(t) = \int e^{-2t+1} \, dt$$

$$t \cdot x(t) = \frac{-1}{2} e^{-2t+1} + c$$

$$x(t) = \frac{-1}{2t} e^{-2t+1} + \frac{c}{t}$$

2) Let $f : [0,\infty) \mapsto \mathbb{R}$ a smooth function. Assume $f'(t) \le 5 f(t)$, $\forall t$ and $f(0) = 2$. Show that $f(t) \le 2 \cdot e^{5t}$, $\forall t$.

$$f'(t) \le 5 f(t)$$

$$f'(t) - 5 f(t) \le 0 \quad | \cdot e^{-5t}$$

$$e^{-5t} f'(t) - 5 e^{-5t} f(t) \le 0$$

$(e^{-5t} f(t))' \le 0 \quad | \int \qquad$ and since the function is descending $\Rightarrow f(t) \le f(0)\ \forall t \in [0,\infty)$

$$e^{-5t} f(t) \le \frac{e^{-5 \cdot 0}}{1} \cdot \frac{f(0)}{2}$$

$$f(t) \le 2 e^{5t}$$

3) $x'(t) - x(t) = e^{t-1}$

$a(t) = -1$

$b(t) = e^{t-1}$

homogeneous eq: $x'(t) - x(t) = 0$

$x'(t) = x(t) \quad | : x(t) \neq 0$

$\frac{x'(t)}{x(t)} = 1 \quad | \int_c^s$

$\int_c^s \frac{x'(t)}{x(t)} dt = \int_c^s 1\, dt$

$\ln x(t) \Big|_c^s = t \Big|_c^s$

$\ln \frac{x(s)}{x(c)} = s - c$

$\frac{x(s)}{x(c)} = e^{s-c}$

$x(s) = x(c) \cdot e^{s-c}$

$x(s) = \underbrace{x c \cdot e^{-c}}_{c_1} \cdot e^s$

particular eq. $x_p(s) = \varphi(s) \cdot x_h(s) = \varphi(s) \cdot e^s$

$\Rightarrow (\varphi(s) \cdot e^s)' - \varphi(s) \cdot e^s \; e^{s-1}$

$\varphi'(s) \cdot e^s + \cancel{\varphi(s) \cdot e^s} - \cancel{\varphi(s) \cdot e^s} = e^{s-1}$

$\varphi'(s) \cdot e^s = e^{s-1}$

$\varphi'(s) = e^{-1} \quad | \int$

$\varphi(s) = \int \frac{1}{e} ds$

$\varphi(s) = \frac{1}{e} \cdot t$

$\Rightarrow x_p(s) = s \cdot e^{s-1}$

$x = x_h(s) + x_p(s) = c \cdot e^s + s \cdot e^{s-1}$

4) Let $w > 0$ and denote $\varphi(\cdot, w)$ the solution of $\begin{cases} x'' + x = \cos wt \\ x(0) = x'(0) = 0 \end{cases}$

a) For $w \neq 1$, find a solution of the form

$x(t) = a \cos wt + b \cdot \sin wt$

$x'(t) = -aw \sin wt + bw \cos wt$

$x''(t) = -a w^2 \cos t - b w^2 \sin wt$

$x''(t) + x(t) = -aw^2 \cos wt - bw^2 \sin wt + a \cos wt + b \sin wt$

$= a(1 - w^2) \cos wt + b(1 - w^2) \sin wt$

$$\Rightarrow \begin{cases} a(1-w^2) = 1 \Rightarrow a = \frac{1}{1-w^2} \\ b(1-w^2) = 0 \Rightarrow \begin{cases} b=0 \\ \text{sau} \\ 1-w^2=0 \end{cases} - \text{impos} \end{cases} \Rightarrow b=0$$

$$x(t) = \frac{1}{1-w^2}\cos wt$$

b) Find a solution of the form

$$x(t) = t(a\cdot\cos t + b\sin t)$$

for $x'' + x = \cos t$

$$x'(t) = a\cos t - at\sin t + b\sin t + bt\cos t$$

$$= \cos t(a+bt) + \sin t(-at+b)$$

$$x''(t) = -\sin t(a+bt) + \cos t\cdot b + \cos t(-at+b) - a\sin t =$$

$$= \cos t(2b-at) + \sin t(bt)$$

$$\cos t(2b-at) + \sin t(bt) + t(a\cdot\cos t + b\sin t) = \cos t$$

$$\cos t(2b - \cancel{at} + \cancel{at}) + \sin t(bt+bt) = \cos t$$

$$2b\cos t + 2bt\sin t = \cos t \qquad \text{ariana: } \begin{matrix} a=0 \\ b=\frac{1}{2} \end{matrix}$$

? * redo that

c) Find the general solution $\begin{cases} x'' + x = \cos wt \\ x(0) = x'(0) = 0 \end{cases}$

$$x(t) = x_H + x_P$$

x_H: $r^2 + 1 = 0$

$r = \pm i = 0 \pm 1\cdot i$

x_H: $c_1\cdot\sin 1\cdot t + c_2\cdot\cos 1\cdot t$

$x(t) = c_1\sin t + c_2\cos t + \frac{1}{2}\sin t$, $w=1$

$x(0) = c_2 \Rightarrow c_2 = 0$

$x'(t) = c_1\cos t + \frac{1}{2}\sin t + \frac{1}{2}\cos t$

$x'(0) = c_1 = 0$

$$\Rightarrow x(t) = \frac{t}{2}\sin t$$