(b) ★ $\lim_{n\to\infty} \frac{\sqrt[n]{e}+2\sqrt[n]{e^2}+\cdots+n\sqrt[n]{e^n}}{n^2}. = 1$

$$\frac{\sqrt[n]{e}+2\sqrt[n]{e^2}+\dots+n\sqrt[n]{e^n}}{n^2} = \frac{1}{n^2}\left(e^{\frac1n}+2\cdot e^{\frac2n}+\dots+n\cdot e^{\frac nn}\right) = \sum_{k=1}^{n} \frac{1}{n^2}\left(k\cdot e^{\frac kn}\right) =$$

$$= \sum_{k=1}^{n} \frac{k}{n^2} e^{\frac kn} = \sum_{k=1}^{n} \frac1n\left(\frac kn e^{\frac kn}\right) = \sum_{k=1}^{n} \frac1n f\left(\frac kn\right) \longrightarrow \int_0^1 f(x)\,dx = \int_0^1 xe^x\,dx =$$

$$f(x) = x\cdot e^x$$

$$= xe^x\Big|_0^1 - e^x\Big|_0^1 = e^x(x-1)\Big|_0^1 = e(1-1) - e^0(0-1) = 1$$

(d) ★ $\lim_{n\to\infty} \sqrt[n]{\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\dots\sin\frac{(n-1)\pi}{2n}}. = \frac12$

$$p = \left(\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\dots\sin\frac{(n-1)\pi}{2n}\right)^{\frac1n} = e^{\frac1n \ln\left(\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\dots\sin\frac{(n-1)\pi}{2n}\right)} = e^{g}$$

$$g = \frac1n \ln\left(\sin\frac{\pi}{2n}\cdot\sin\frac{2\pi}{2n}\cdot\dots\cdot\sin\frac{(n-1)\pi}{2n}\right) = \frac1n\left[\ln\left(\sin\frac{\pi}{2n}\right)+\ln\left(\sin\frac{2\pi}{2n}\right)+\dots+\ln\left(\sin\frac{(n-1)\pi}{2n}\right)\right] =$$

$$= \frac1n\sum_{k=1}^{n-1}\ln\left(\sin\frac{k\pi}{2n}\right) = \sum_{k=1}^{n-1}\frac1n \underbrace{\ln\left(\sin\left(\frac kn\right)\frac\pi2\right)}_{f\left(\frac kn\right)} \longrightarrow \int_0^1 \ln\sin\frac{x\pi}{2}\,dx = \frac2\pi\int_0^{\frac\pi2}\ln\sin t\,dt = \frac2\pi\int_0^{\frac\pi2}\ln\cos t\,dt$$

let $t = \frac{x\pi}{2}$

$\frac{dt}{dx}=\frac\pi2$

$dx = \frac2\pi dt$

Let $I = \int_0^{\frac\pi2}\ln\sin t\,dt$

$$2I = \int_0^{\frac\pi2}\ln\sin t\,dt + \int_0^{\frac\pi2}\ln\cos t\,dt =$$

$$= \int_0^{\frac\pi2}\ln\sin t\cos t\,dt =$$

$$= \int_0^{\frac\pi2}(\ln\sin2t - \ln2)\,dt = \int_0^{\frac\pi2}\ln\sin2t\,dt - \frac{\pi\ln2}{2} =$$

$$= \frac12\int_0^{\pi} \ln\sin t\,dt - \frac{\pi\ln 2}{2} =$$

$$= \frac12\left(\int_0^{\frac\pi2}\ln\sin t\,dt + \int_{\frac\pi2}^{\pi}\ln\sin t\,dt\right) - \frac{\pi\ln2}{2} =$$

$$= \frac12\left(\int_0^{\frac\pi2}\ln\sin t\,dt + \int_0^{\frac\pi2}\ln\cos t\,dt\right) - \frac{\pi\ln2}{2} =$$

$$= \frac12\left(\cancel{\ln\sin\frac\pi2} - \cancel{\ln\sin0} + \cancel{\ln\cos\frac\pi2} - \cancel{\ln\cos0}\right) - \frac{\pi\ln2}{2}) \qquad I = -\frac{\pi\ln2}{2}$$

$$\Rightarrow e^{\frac2\pi I} = e^{-\ln2} = \frac{1}{e^{\ln2}} = \frac12$$

(d) $\star \int_0^\infty e^{-x}\sin x\,\mathrm{d}x. = \lim_{t\to\infty} \int_0^t e^{-x}\sin(x)\,dx = \lim_{t\to\infty} \frac{1}{2}\int_0^t e^{-x}(\sin(x) - \sin(-x))\,dx =$

$$= \lim_{t\to\infty} \frac{1}{2}\int_0^t [e^{-x}\sin x - e^{-x}\sin(-x)]dx$$

I $\frac{1}{2}\int_0^t e^{-x}\sin x\,dx = \frac{1}{2}\left(-e^{-x}\sin x - \int -e^{x}\cos x\,dx\right) = \frac{-1}{2}e^{-x}\sin(x) + \frac{1}{2}\int_0^t e^{-x}\cos(x)\,dx$

II $\frac{1}{2}\int_0^t e^{-x}\sin(-x)\,dx = -\frac{1}{2}\int_0^t e^{-x}\sin(x)$

$$\Rightarrow \int_0^\infty e^{-x}\sin(x)\,dx = \lim_{t\to\infty}\frac{1}{2}\int_0^t e^{-x}\sin(x) - e^{-x}\sin(-x)\,dx =$$

$$= \lim_{t\to\infty}\frac{1}{2}\left(-\frac{1}{2}\underbrace{e^{-t}}_{=0}\sin(t) + \frac{1}{2}\int_0^t e^{-x}\cos(x)\,dx + \frac{1}{2}\underbrace{e^{-t}}_{=0}\right) =$$

$$= \lim_{t\to\infty}\frac{1}{4}\int_0^t e^{-x}\cos(x)\,dx = \lim_{t\to\infty}\frac{1}{4}\left[\frac{1}{2}e^{-x}\cos x + \frac{1}{2}\int e^{-x}\sin(x)\,dx\right]\Big|_0^t =$$

$$= \lim_{t\to\infty}\frac{1}{8}\left[e^{-t}\cos(t) + \int_0^t e^{-x}\sin x\,dx\right] =$$

$$= \lim_{t\to\infty}\frac{1}{8}\left[e^{-t}\cos t + \underbrace{\frac{1}{8}\int_0^t e^{-x}\cos(x)\,dx}\right] =$$

$$\frac{1}{8}\int_0^t e^{-x}\cos x\,dx = \lim_{t\to\infty}\frac{1}{8}\left(e^{-t}\cos t + \frac{1}{8}\int_0^t e^{-x}\cos(x)\,dx\right) =$$

$$\frac{-1}{64}\int_0^t e^{-x}\cos x\,dx = \lim_{t\to\infty}\frac{1}{8}e^{-t}\cos t$$

$$\frac{7}{64}\int_0^t e^{-x}\cos x\,dx = \lim_{t\to\infty}\frac{1}{8}e^{-t}\cos t$$

$$\int_0^t e^{-x}\cos x\,dx = \frac{8}{7}\lim_{t\to\infty}\underbrace{e^{-t}\cos t}_{\to 0}$$

$$\int_0^t e^{-x}\cos x\,dx = 0 \quad \Rightarrow \int_0^\infty e^{-x}\sin x\,dx = 0$$