Seminar 9

1. Study the limits of the following functions when $(x, y) \to (0, 0)$:

   (a) $\dfrac{x^2 - y^2}{x^2 + y^2}$. (b) $\dfrac{x + y}{x^2 + y^2}$ (c) $\dfrac{x^3 + y^3}{x^2 + y^2}$. (d) $\dfrac{\sin x - \sin y}{x - y}$.

2. Compute the partial derivatives (and specify where they exist) for the following functions:

   (a) $f(x, y) = e^{-(x^2+y^2)}$.
   (b) $f(x, y) = \cos x \cos y - \sin x \sin y$.
   (c) $f(x, y) = \|(x, y)\| = \sqrt{x^2 + y^2}$.
   (d) $f(x, y, z) = x^2yz + ye^z$.

3. Let $f : \mathbb{R}^2 \to R, f(x, y) = xy$. Using the definition, prove that $Df(x_0, y_0) = (y_0, x_0)$.

4. Prove that

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

   is continuous and has partial derivatives, but it is not differentiable in the origin.

5. Find the gradient of the function f at the point a for the following:

   (a) $f(x, y) = e^{-x} \sin(x + 2y)$, $a = (0, \frac{\pi}{4})$.
   (b) $f(x, y) = \arctan(\frac{y}{x})$, $a = (1, 1)$.
   (c) $f(x, y, z) = e^{xyz}$, $a = (0, 0, 0)$.
   (d) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $a = (1, 1, 1)$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x, y) = f(x^2 + y^2),\ \forall\, (x, y) \in \mathbb{R}^2.$$

   Prove that

$$y\frac{\partial g}{\partial x}(x, y) = x\frac{\partial g}{\partial y}(x, y).$$

1. Study the limits of the following functions when $(x, y) \to (0, 0)$:

(a) $\dfrac{x^2 - y^2}{x^2 + y^2}$. (b) $\dfrac{x + y}{x^2 + y^2}$ (c) $\dfrac{x^3 + y^3}{x^2 + y^2}$. (d) $\dfrac{\sin x - \sin y}{x - y}$.

(a) let $y = mx \Rightarrow$

$$\frac{x^2-y^2}{x^2+y^2} = \frac{x^2-m^2x^2}{x^2+m^2x^2} = \frac{\cancel{x^2}(1-m^2)}{\cancel{x^2}(1+m^2)} = \frac{1-m^2}{1+m^2} \text{ (depends on } m) \Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

or take $y = 0 \Rightarrow \lim_{x\to 0} \frac{x^2}{x^2} = 1$ (left / right)

$x = 0 \Rightarrow \lim_{y \to 0} \frac{-y^2}{y^2} = -1$ (up / down) $\Bigg\} \Rightarrow \nexists \lim_{(x,y)\to(0,0)}$

* take examples of moving on axes / lines $\Rightarrow$ the limit might exist

(b) $\dfrac{x+y}{x^2+y^2}$

let $y = 0 \Rightarrow \lim_{x\to 0} \frac{x}{x^2} = \lim_{x \to 0} \frac{1}{x}$ doesn't exist $\Longrightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$

(c) $\dfrac{x^3+y^3}{x^2+y^2}$

$y=0 \Rightarrow \lim_{x\to 0} \frac{x^3}{x^2} = \lim_{x\to 0} x = 0$

$x = 0 \Rightarrow$ —— \|\| —— $\lim_{x\to 0} y = 0$ $\Bigg\} \Rightarrow$ we could have the limit $= 0$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \le \underbrace{|x|+|y|}_{\dots} \to 0$$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| = \frac{|x^3+y^3|}{x^2+y^2} \le \frac{|x|^3}{x^2+y^2} + \frac{|y|^3}{x^2+y^2}$$

$$\le |x| \cdot \underbrace{\frac{x^2}{x^2+y^2}}_{\le 1} + |y| \cdot \underbrace{\frac{y^2}{x^2+y^2}}_{\le 1}$$

$$\le |x| + |y|$$

$\frac{\sin t}{t} \to 1$ as $t \to 0$

(d) $$\frac{\sin x - \sin y}{x-y} = \frac{\cos\frac{x+y}{2} \sin\frac{x-y}{2}}{\frac{x-y}{2}} = \cos 0 = 1$$

2. Compute the partial derivatives (and specify where they exist) for the following functions:

(a) $f(x, y) = e^{-(x^2+y^2)}$.

(b) $f(x, y) = \cos x \cos y - \sin x \sin y$.

(c) $f(x, y) = \|(x, y)\| = \sqrt{x^2 + y^2}$.

(d) $f(x, y, z) = x^2yz + ye^z$.

(a) $$\frac{\partial f}{\partial x} = e^{-(x^2+y^2)} \frac{d}{dx}\left[-(x^2+y^2)\right] = e^{-(x^2+y^2)} \cdot (-2x)$$ (y is a constant)

$$\frac{\partial f}{\partial y} = e^{-(x^2+y^2)} \cdot (-2y)$$ defined $\forall (x,y) \in \mathbb{R}^2$

(b) $f(x,y) = \cos x \cos y - \sin x \sin y = \cos(x+y)$

$$\frac{\partial f}{\partial x} = -\sin(x+y) \qquad \frac{\partial f}{\partial y} = -\sin(x+y)$$

- $\frac{\partial f}{\partial x} = -\sin x \cos y - \cos x \sin y = -\sin(x+y)$

  $\frac{\partial f}{\partial y} = -\cos x \cdot \sin y - \cos y \sin x = -\sin(x+y)$

(c) $f(x,y) = \|(x,y)\| = \sqrt{x^2+y^2}$

$\frac{\partial f}{\partial x} = \left(\sqrt{x^2+y^2}\right)'_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$ Defined for $\forall (x,y) \neq (0,0)$

(d) $f(x,y,z) = x^2yz + ye^z$

$\frac{\partial f}{\partial x} = 2xyz$ $\qquad \frac{\partial f}{\partial y} = x^2z + e^z$ $\qquad \frac{\partial f}{\partial z} = x^2 y + ye^z$ $\qquad$ Defined $\forall\, x,y,z \in \mathbb{R}^3$

3. Let $f : \mathbb{R}^2 \to R, f(x,y) = xy$. Using the definition, prove that $Df(x_0,y_0) = (y_0,x_0)$.

$$\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^n,\, x_0 \in \mathbb{R}^n}} \frac{f(x) - f(x_0) - Df(x_0)\cdot(x-x_0)}{\|x-x_0\|} = 0$$

$$L = \lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \overbrace{Df(x_0,y_0)}^{(y_0,\,x_0)}\cdot(x-x_0,y-y_0)}{\|(x-x_0,y-y_0)\|} = 0$$

$$\begin{aligned} f(x,y) - f(x_0,y_0) - (y_0,x_0)\cdot(x-x_0,y-y_0) &= xy - x_0y_0 - y_0(x-x_0) - x_0(y-y_0) \\ &= xy - x_0y_0 - y_0x + y_0x_0 - x_0y + x_0y_0 \\ &= xy - xy_0 - x_0y + x_0y_0 \\ &= x(y-y_0) - x_0(y-y_0) \\ &= (x-x_0)(y-y_0) \end{aligned}$$

$\|(x-x_0, y-y_0)\| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$

$$L = \lim_{(x,y)\to(0,0)} \frac{(x-x_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \lim_{(u,v)\to(0,0)} \frac{uv}{\sqrt{u^2+v^2}} = 0$$

$$u^2+v^2 \geq 2\cdot|uv| \Rightarrow \frac{1}{\sqrt{u^2+v^2}} \leq \frac{1}{\sqrt{2}\cdot\sqrt{|uv|}} \Rightarrow 0 \leq \frac{|uv|}{\sqrt{u^2+v^2}} \leq \frac{\sqrt{uv}}{\sqrt{2}} \longrightarrow 0$$

4. Prove that

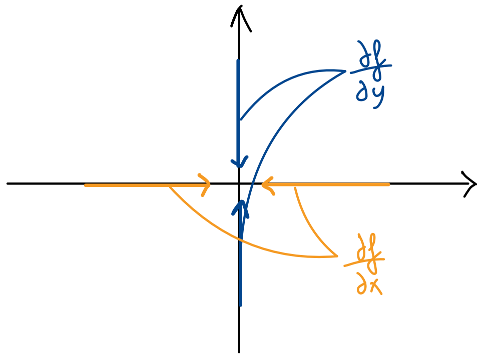
$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

is continuous and has partial derivatives, but it is not differentiable in the origin.

- Continuity : $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 = f(0,0) \Rightarrow$ cont. at $(0,0)$

$$0 \leq \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{|xy|}}{\sqrt{2}} \longrightarrow 0$$

Partial derivatives at $(0,0)$:



$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

Assume that f is diff. at $(0,0)$ $\Rightarrow$ $Df(0,0) = \nabla f(0,0) = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right) = (0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \overbrace{f(0,0)}^{0} - \overbrace{(0,0)\cdot(x,y)}^{=0}}{\|xy\|} = \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\|xy\|} = \lim_{(x,y)\to(0,0)} \frac{\frac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

let $y = mx$

$$\lim_{x\to 0} \frac{m x^2}{(1+m^2)x^2} = \frac{m}{1+m^2} \text{ dep. on } m \Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \Rightarrow f \text{ is } \underline{\text{not}} \text{ diff at } (0,0)$$

5. Find the gradient of the function f at the point a for the following:

(a) $f(x,y) = e^{-x}\sin(x+2y)$, $a = (0, \frac{\pi}{4})$.

(b) $f(x,y) = \arctan(\frac{y}{x})$, $a = (1,1)$.

(c) $f(x,y,z) = e^{xyz}$, $a = (0,0,0)$.

(d) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $a = (1,1,1)$

(a) $$\frac{\partial f}{\partial x} = -e^{-x}\sin(x+2y) + e^{-x}\cos(x+2y), \quad \frac{\partial f}{\partial x}(0,\tfrac{\pi}{4}) = -\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = -1$$

$$\frac{\partial f}{\partial y} = 2\cdot e^{-x}\cos(x+2y), \quad \frac{\partial f}{\partial x}(0,\tfrac{\pi}{4}) = 2\cdot\cos\frac{\pi}{2} = 0 \Rightarrow \nabla f(0,\tfrac{\pi}{4}) = (-1,0)$$

(b) $$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2} = \frac{\cancel{x^2}}{x^2+y^2}\cdot\frac{-y}{\cancel{x^2}} = \frac{-y}{x^2+y^2}, \quad \frac{\partial f}{\partial x}(1,1) = \frac{-1}{2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x^2}{x^2+y^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} \qquad \frac{\partial f}{\partial y}(1,1) = \frac{1}{2}$$

$\Rightarrow \nabla f(1,1) = \left(-\frac{1}{2}, \frac{1}{2}\right)$

(c) $$\frac{\partial f}{\partial x} = e^{xyz}\cdot yz$$

$$\frac{\partial f}{\partial y} = e^{xyz}\cdot xz \qquad \nabla f(0,0,0) = (0,0,0)$$

$$\frac{\partial f}{\partial z} = e^{xyz}\cdot xy$$

(d) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ $\quad a = (1,1,1)$

$$\frac{\partial f}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla f(1,1,1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$