Sq. 16 18 19 20 3 22 23 24

affine morphism $\varphi : \mathbb{E}^n \longrightarrow \mathbb{E}^n$

$\varphi(P) = A \cdot P + b$
$A \in \mathcal{M}_n(\mathbb{R}), \quad b \in \mathbb{R}^n$

φ isometry $\Leftrightarrow \forall P, Q \in \mathbb{E}^n$:

$\operatorname{dist}(\varphi(P), \varphi(Q)) = \operatorname{dist}(P, Q) \Leftrightarrow A \in O(n) = \{M \in \mathcal{M}_n(\mathbb{R}) \mid M^{-1} = M^T\} \Leftrightarrow A \cdot A^T = I_n$

- $\det A = 1 \Rightarrow$ direct isometry
- $\det A = -1 \Rightarrow$ indirect isometry

$n = 2$, $\varphi : \mathbb{E}^n \to \mathbb{E}^n$ isometry

→ direct isometry:
- identity
- translation $T_{\vec{v}}$
- rotation around point C with angle θ ($Rot_{C,\theta}$)

→ indirect isometry:
- reflection w to a line ℓ Ref_ℓ
- glide - reflection $T_{\vec{v}} \circ Ref_\ell$
  $\vec{v} \in D(\ell)$

If φ rotation $\Rightarrow$ $\boxed{Tr\, A = 2 \cdot \cos\theta}$

$Fix(\varphi) = \{P \in \mathbb{E}^n \mid \varphi(P) = P\}$ → set of fixed points, the ones that remain unchanged

5.16 F isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a translation with vector $(-2, 5)$. Find F^{-1}

$F = Rot_{-\frac{\pi}{3}} \circ T_{(-2,5)}$

$\boxed{F^{-1} = T_{(-2,5)}^{-1} \circ Rot_{-\frac{\pi}{3}}^{-1}}$

$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

$F^{-1} = T_{(2,-5)} \circ Rot_{\frac{\pi}{3}}$

* when you compose two affine morphisms

$f(P) = AP + b$ $\quad \left(\begin{array}{c|c} A & b \\ 0 0 .. 0 & 1 \end{array}\right)$

$\hat{f}(\hat{P}) = \left(\begin{array}{c|c} A & b \\ 0 0 \dots & 1 \end{array}\right) \begin{pmatrix} P \\ 1 \end{pmatrix}$

$[T_{(2,-5)}] = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad [Rot_{\frac{\pi}{3}}] = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 0 \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$[F^{-1}] = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 2 \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & -5 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$F^{-1}(P) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \cdot P + \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

exam

$\boxed{5.8}$ $\varphi : \mathbb{E}^2 \to \mathbb{E}^2$

$$\varphi(P) = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \cdot P + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Show that φ is a rotation, find its center and angle

$$[A] = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \qquad A^T = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$A \cdot A^T = \begin{pmatrix} \frac{9}{25} + \frac{16}{25} & \frac{12}{25} - \frac{12}{25} \\ \frac{12}{25} - \frac{12}{25} & \frac{16}{25} + \frac{9}{25} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$\det A = \frac{9}{25} + \frac{16}{25} = 1 \Rightarrow A \in SO(2) \Rightarrow \varphi$ is a direct isometry

$\mathrm{fix}(\varphi) = \{ P \in \mathbb{E}^n \mid \varphi(P) = P \}$

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 1 \\ \frac{4}{5} & \frac{3}{5} & -2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \Rightarrow \begin{cases} \frac{3x}{5} - \frac{4y}{5} + 1 = x \\ \frac{4x}{5} + \frac{3y}{5} - 2 = y \end{cases} \Leftrightarrow \begin{cases} 3x - 4y + 5 = 5x \\ 4x + 3y - 10 = 5y \end{cases} \Leftrightarrow \begin{cases} -2x - 4y + 5 = 0 \\ 4x - 2y - 10 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -2x - 4y + 5 = 0 \Rightarrow -2x - 8x + 20 + 5 = 0 \Rightarrow -10x = -25 \Rightarrow x = \frac{25}{10} = \frac{5}{2} \\ 2x - y - 5 = 0 \Rightarrow y = 2x - 5 \Rightarrow y = 2 \cdot \frac{5}{2} - 5 = 0 \end{cases}$$

or $\varphi(P) = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \mathrm{fix}(\varphi) = \{ (\frac{5}{2}, 0) \} \Rightarrow f =$ rotation

$\mathrm{Tr} A = 2 \cdot \cos\theta \Rightarrow \theta = \cos^{-1} \frac{\mathrm{Tr} A}{2}$

$\cos\theta = \frac{\mathrm{Tr} A}{2} = \frac{6}{5 \cdot 2} = \frac{3}{5} \Rightarrow \theta = \cos^{-1} \frac{3}{5}$

rotation
* Isometry
* ortogonal
* direct / indirect Isometry

$n = 3$ $\quad \varphi : \mathbb{E}^n \to \mathbb{E}^n$ isometry

$\longrightarrow$ direct:
- identity
- translation $T_{\vec{v}}$
- rotation around axis ℓ with an angle θ
- glide rotation $T_{\vec{v}} \circ Rot_{\ell,\theta}$ $\quad \vec{v} \in D(\ell)$

$\to$ indirect
- reflection with a plane Ref_{π}
- glide-reflection $T_{\vec{v}} \circ Ref_{\pi}$, $\vec{v} \in D(\pi)$
- rotation-reflection $Rot_{\ell,\theta} \circ Ref_{\pi}$, $\ell \perp \pi$

$\boxed{5.19}$ Verify that the matrix $A = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix}$ belongs to $SO(3)$.

Find the axis and angle of rotation

$Tr(A) = 2\cos\theta + 1$

$$Rot_{\ell,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A \cdot A^T = \frac{1}{9}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = I \Rightarrow A \in SO(3)$$

$$\frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} -4x + 2y - 2z = 0 \quad | : 2 \\ -2x - y - z = 0 \end{cases} \quad (-)$$

$2y = 0 \Rightarrow y = 0 \Rightarrow -2x = z$

$Fix\begin{pmatrix} x \\ y \end{pmatrix} = \{(\alpha, 0, -2\alpha) \mid \alpha \in \mathbb{R}\}$, where is a line $\Rightarrow$ A is a rotation

$Tr(A) = \frac{1}{3}(-1 - 2 + 2) = \frac{-1}{3}$

$2\cos\theta + 1 = \frac{-1}{3}$

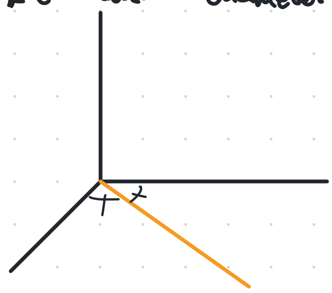
$\cos\theta = \frac{-2}{3}$

Euler – Rodrigues formula

$Rot_{\ell,\theta}(P) = \cos\theta \cdot \vec{P} + \sin\theta \cdot (\vec{v} \times \vec{P}) + (1 - \sin\theta) \langle \vec{v}, \vec{P} \rangle \vec{v}$, where $\vec{v} \in D(\ell)$ $\quad |\vec{v}| = 1$

5.22 Write down the matrix form of a rotation around the axis $\mathbb{R}\vec{v}$ where

where $\vec{v} = (1,1,0)$. Use this matrix to give a parametrization of a cylinder with axis $\mathbb{R}\vec{v}$ and diameter $\sqrt{2}$



$$\vec{w} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$Rot_{\pi,\vec{w}}(P) = \cos\theta \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sin\theta \cdot \begin{vmatrix} i & j & k \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ x & y & z \end{vmatrix} + (1-\sin\theta)$$

$$\cdot \left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \dots = M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$