DATA STRUCTURES AND ALGORITHMS

LECTURE 6

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In Lecture 5...

- Doubly linked list
- Sorted list
- Circular list

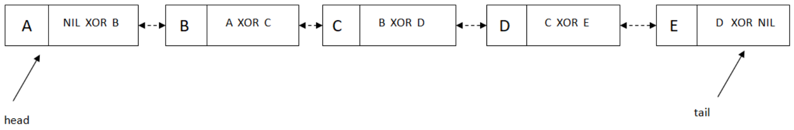
Today

- XOR linked list
- Skip list
- Linked list on array

XOR Linked List

- Doubly linked lists are better than singly linked lists because they offer better complexity for some operations. They are also more flexible, since they can be traversed in both directions.
- Their disadvantage is that they occupy more memory, because you have two links to memorize, instead of just one.
- A memory-efficient solution is to have a *XOR Linked List*, which is a doubly linked list (we can traverse it in both directions), where every node retains one single link, which is the XOR of the previous and the next node.

XOR Linked List - Example



- How do you traverse such a list?
  - We start from the head (but we can have a backward traversal starting from the tail in a similar manner), the node with A
  - The address from node A is directly the address of node B (NIL XOR B = B)
  - When we have the address of node B, its link is A XOR C. To get the address of node C, we have to XOR B's link with the address of A (it is the previous node we come from): A XOR C XOR A = A XOR A XOR C = NIL XOR C = C

XOR Linked List - Representation

- We need two structures to represent a XOR Linked List: one for a node and one for the list

```
XORNode:
    info: TElem
    link: ↑ XORNode
```

```
XORList:
    head: ↑ XORNode
    tail: ↑ XORNode
```

XOR Linked List - Traversal

```
**subalgorithm** printListForward(xorl) **is:**
*//pre: xorl is a XORList*
*//post: true (the content of the list was printed)*
   prevNode $\leftarrow$ NIL
   currentNode $\leftarrow$ xorl.head
   **while** currentNode $\neq$ NIL **execute**
      **write** [currentNode].info
      nextNode $\leftarrow$ prevNode XOR [currentNode].link
      prevNode $\leftarrow$ currentNode
      currentNode $\leftarrow$ nextNode
   **end-while**
**end-subalgorithm**
```

- Complexity: $\Theta(n)$

XOR Linked List - addToBeginning

- How can we add an element to the beginning of the list?

```
**subalgorithm** addToBeginning(xorl, elem) **is:**
//pre: xorl is a XORList
//post: a node with info elem was added to the beginning of the list
    newNode ← allocate()
    [newNode].info ← elem
    [newNode].link ← xorl.head
    if xorl.head = NIL then
        xorl.head ← newNode
        xorl.tail ← newNode
    else
        [xorl.head].link ← [xorl.head].link XOR newNode
        xorl.head ← newNode
    end-if
end-subalgorithm
```

- Complexity: $\Theta(1)$

Skip Lists

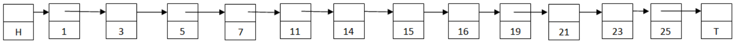
- Assume that we want to memorize a sequence of sorted elements. The elements can be stored in:
    - dynamic array
    - linked list (let's say doubly linked list)
- We know that the most frequently used operation will be the insertion of a new element, so we want to choose a data structure for which insertion has the best complexity. Which one should we choose?

Skip Lists

- We can divide the insertion operation into two steps: *finding where to insert* and *inserting the element*
    - For a dynamic array finding the position can be optimized (binary search $O(log_2 n)$), but the insertion is $O(n)$
    - For a linked list the insertion is optimal ($\Theta(1)$), but finding where to insert is $O(n)$

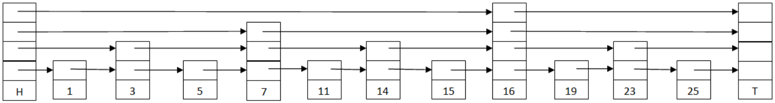
Skip List

- A skip list is a data structure that allows *fast search* in an ordered linked list.
- How can we do that?



- Starting from an ordered linked list, we add to every second node another pointer that skips over one element.
- We add to every fourth node another pointer that skips over 3 elements.
- etc.

Skip List

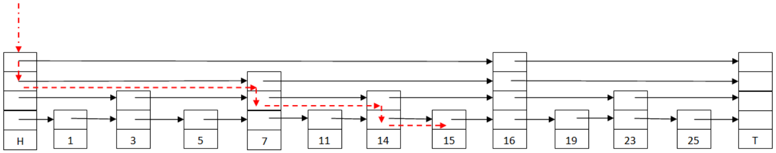


- H and T are two special nodes, representing *head* and *tail*. They cannot be deleted, they exist even in an empty list (we need them to keep the *height* of the list).

like a tree?

Skip List - Search

- Search for element 15.



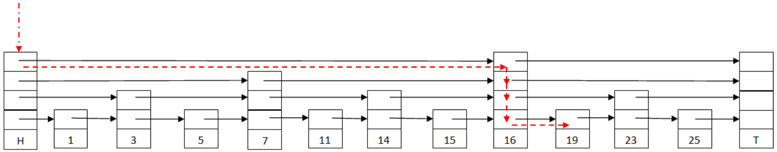
- Start from head and from highest level.
- If possible, go right.
- If cannot go right (next element is greater), go down a level.

Skip List

- Lowest level has all n elements.
- Next level has $\frac{n}{2}$ elements.
- Next level has $\frac{n}{4}$ elements.
- etc.
- $\Rightarrow$ there are approx $log_2 n$ levels.
- From each level, we check at most 2 nodes.
- Complexity of search: $O(log_2 n)$

Skip List - Insert

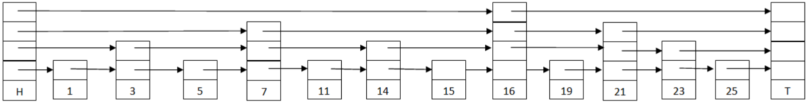
- Insert element 21.



- How *high* should the new node be?

Skip List - Insert

- *Height* of a new node is determined *randomly* with a method called *coin flip*. This method guarantees that approximately half of the nodes will be on level 2, a quarter of them on level 3, etc.



- Assume we randomly generate the height 3 for the node with 21.

Skip List

- Skip Lists are *probabilistic* data structures, since we decide randomly the height of a newly inserted node.
- There might be a worst case, where every node has height 1 (so it is just a linked list).
- However, the structure (i.e. heights of nodes) is independent of the order in which the elements are inserted, so there is no *bad sequence of insertion*.
- In practice, they function well.

Skip List representation ideas

- How would you represent a skip list?
- Since this is a linked list, we need a structure for a node. What do we have in a node?
- There are two approaches for defining a node:
    - Each node has an array of pointers (so that you can have several *next* pointers for different levels)
    - Each node has one single *next* pointer and it has a *down* pointer, pointing to the same node one level below.
- Another important issue is to decide whether we will have singly or doubly linked lists. Or maybe a combination: bottom level doubly linked list, higher levels singly linked.

Skip List implementation ideas

- It is possible to keep special *head* and *tail* nodes, which always have the maximum possible height.
- While insertion and deletion seems quite straightforward, for both operations, potentially, several linked lists need to be modified. In order to be able to do so, we, potentially, need to store somewhere the nodes that we pass during the search for the position. Even in this case, there might be nodes involved, which are not on this path.
- For example, in the previously used figure, if we want to remove 16, we will immediately find it, as we start from the *head* node on the highest level. However, when it is removed, pointers of nodes 7, 14 and 15 will need to be changed.

Disadvantages of skip lists

- One disadvantage of skip lists is that they use extra space to memorize the extra links, for N elements we need 2N nodes.
- In the basic version, skip lists are unidirectional (they are singly linked lists).
- Adjacent nodes in a skip list might be stored in totally different regions of the memory, so skip lists are less optimized for caching. This disadvantage can be reduced a little, if the skip list node contains an *array* of pointers (we can benefit from the cache at least when descending a level).

Linked Lists on Arrays

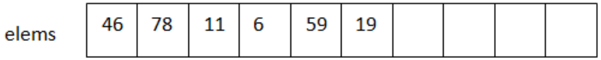
- What if we need a linked list, but we are working in a programming language that does not offer pointers (or references)?

- We can still implement linked data structures, without the explicit use of pointers or memory addresses, simulating them using arrays and array indexes.

Linked Lists on Arrays

- Usually, when we work with arrays, we store the elements in the array starting from the leftmost position and place them one after the other (no empty spaces in the middle of the array are allowed).

- The order of the elements is given by the order in which they are placed in the array.



- Order of the elements: 46, 78, 11, 6, 59, 19

Linked Lists on Arrays

- We can define a linked data structure on an array, if we consider that the order of the elements is not given by their relative positions in the array, but by an integer number associated with each element, which shows the index of the next element in the array (thus we have a singly linked list).

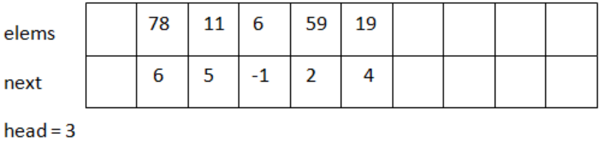
elems	46	78	11	6	59	19				
next	5	6	1	-1	2	4				

head = 3

- Order of the elements: 11, 46, 59, 78, 19, 6

Linked Lists on Arrays

- Now, if we want to delete the number 46 (which is actually the second element of the list), we do not have to move every other element to the left of the array, we just need to modify the links:



elems		78	11	6	59	19				
next		6	5	-1	2	4				

head = 3

- Order of the elements: 11, 59, 78, 19, 6

Linked Lists on Arrays

- If we want to insert a new element, for example 44, at the 3rd position in the list, we can put the element anywhere in the array, the important part is setting the links correctly:

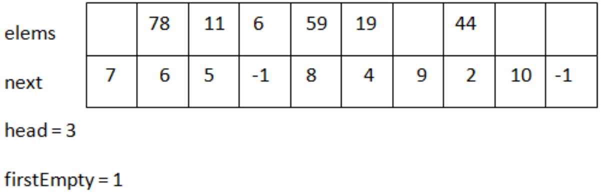
elems		78	11	6	59	19		44		
next		6	5	-1	8	4		2		

head = 3

- Order of the elements: 11, 59, 44, 78, 19, 6

Linked Lists on Arrays

- When a new element needs to be inserted, it can be put to any empty position in the array. However, finding an empty position has $O(n)$ complexity, which will make the complexity of any insert operation (anywhere in the list) $O(n)$. In order to avoid this, we will keep a linked list of the empty positions as well.



elems		78	11	6	59	19		44		
next	7	6	5	-1	8	4	9	2	10	-1

head = 3

firstEmpty = 1

Linked Lists on Arrays

- In a more formal way, we can simulate a singly linked list on an array with the following:
    - an array in which we will store the elements.
    - an array in which we will store the links (indexes to the next elements).
    - the capacity of the arrays (the two arrays have the same capacity, so we need only one value).
    - an index to tell where the *head* of the list is.
    - an index to tell where the first empty position in the array is.

SLL on Array - Representation

- The representation of a singly linked list on an array is the following:

```
SLLA:
    elems: TElem[]
    next: Integer[]
    cap: Integer
    head: Integer
    firstEmpty: Integer
```

SLLA - Operations

- We can implement for a SLLA any operation that we can implement for a SLL:
  - insert at the beginning, end, at a position, before/after a given value
  - delete from the beginning, end, from a position, a given element
  - search for an element
  - get an element from a position

SLLA - Search

```
function search (slla, elem) is:
//pre: slla is a SLLA, elem is a TElem
//post: return True is elem is in slla, False otherwise
   current ← slla.head
   while current $\neq$ -1 and slla.elems[current] $\neq$ elem execute
      current ← slla.next[current]
   end-while
   if current $\neq$ -1 then
      search ← True
   else
      search ← False
   end-if
end-function
```

- Complexity: $O(n)$

SLLA - Search

- From the *search* function we can see how we can go through the elements of a SLLA (and how similar this traversal is to the one done for a SLL):
  - We need a *current* element used for traversal, which is initialized to the index of the *head* of the list.
  - We stop the traversal when the value of *current* becomes -1
  - We go to the next element with the instruction: *current* $\leftarrow$ *slla.next[current]*.

SLLA - Init

```
**subalgorithm** init(slla) **is:**
//pre: true; post: slla is an empty SLLA
  slla.cap ← INIT_CAPACITY
  slla.elems ← @an array with slla.cap positions
  slla.next ← @an array with slla.cap positions
  slla.head ← -1
  for i ← 1, slla.cap-1 execute
    slla.next[i] ← i + 1
  end-for
  slla.next[slla.cap] ← -1
  slla.firstEmpty ← 1
end-subalgorithm
```

- Complexity: $\Theta(n)$ -where n is the initial capacity

SLLA - InsertFirst

```
subalgoritm insertFirst(slla, elem) is:
//pre: slla is an SLLA, elem is a TElem
//post: the element elem is added at the beginning of slla
  if slla.firstEmpty = -1 then
    newElems ← @an array with slla.cap * 2 positions
    newNext ← @an array with slla.cap * 2 positions
    for i ← 1, slla.cap execute
      newElems[i] ← slla.elems[i]
      newNext[i] ← slla.next[i]
    end-for
    for i ← slla.cap + 1, slla.cap*2 - 1 execute
      newNext[i] ← i + 1
    end-for
    newNext[slla.cap*2] ← -1
//continued on the next slide...
```

SLLA - InsertFirst

```
        //free slla.elems and slla.next if necessary
        slla.elems ← newElems
        slla.next ← newNext
        slla.firstEmpty ← slla.cap+1
        slla.cap ← slla.cap * 2
    end-if
    newPosition ← slla.firstEmpty
    slla.elems[newPosition] ← elem
    slla.firstEmpty ← slla.next[slla.firstEmpty]
    slla.next[newPosition] ← slla.head
    slla.head ← newPosition
end-subalgorithm
```

- Complexity: $\Theta(1)$ amortized (due to resize)

SLLA -InsertPosition

```
subalgorithm insertPosition(slla, elem, poz) is:
//pre: slla is an SLLA, elem is a TElem, poz is an integer number
//post: the element elem is inserted into slla at position pos
   if (pos < 1) then
      @error, invalid position
   end-if
   if slla.firstEmpty = -1 then
      @resize
   end-if
   if poz = 1 then
      insertFirst(slla, elem)
   else
      pozCurrent ← 1
      nodCurrent ← slla.head
//continued on the next slide...
```

SLLA - InsertPosition

```
while nodCurrent $\neq$ -1 and pozCurrent $<$ poz - 1 execute
    pozCurrent $\leftarrow$ pozCurrent + 1
    nodCurrent $\leftarrow$ slla.next[nodCurrent]
end-while
if nodCurrent $\neq$ -1 atunci
    newElem $\leftarrow$ slla.firstEmpty
    slla.firstEmpty $\leftarrow$ slla.next[firstEmpty]
    slla.elems[newElem] $\leftarrow$ elem
    slla.next[newElem] $\leftarrow$ slla.next[nodCurrent]
    slla.next[nodCurrent] $\leftarrow$ newElem
else
//continued on the next slide...
```

SLLA - InsertPosition

```
      @error, invalid position
    end-if
  end-if
end-subalgorithm
```

- Complexity: $O(n)$

SLLA - InsertPosition

- Observations regarding the *insertPosition* subalgorithm
    - Similar to the SLL, we iterate through the list until we find the element *after* which we insert (denoted in the code by *nodCurrent* - which is an index in the array).
    - We treat as a special case the situation when we insert at the first position (no node to insert after).
    - Since it is an operation which takes as parameter a position we need to check if it is a valid position
    - Since the elements are stored in an array, we need to see at every add operation if we still have space or if we need to do a resize. And if we do a resize, the extra positions have to be added in the list of empty positions.

SLLA - DeleteElement

```
subalgorithm deleteElement(slla, elem) is:
//pre: slla is a SLLA; elem is a TElem
//post: the element elem is deleted from SLLA
    nodC ← slla.head
    prevNode ← -1
    while nodC ≠ -1 and slla.elems[nodC] ≠ elem execute
        prevNode ← nodC
        nodC ← slla.next[nodC]
    end-while
    if nodC ≠ -1 then
        if nodC = slla.head then
            slla.head ← slla.next[slla.head]
        else
            slla.next[prevNode] ← slla.next[nodC]
        end-if
//continued on the next slide...
```

SLLA - DeleteElement

```
//add the nodC position to the list of empty spaces
        slla.next[nodC] ← slla.firstEmpty
        slla.firstEmpty ← nodC
    else
        @the element does not exist
    end-if
end-subalgorithm
```

- Complexity: $O(n)$

SLLA - Iterator

- How would you define an iterator for an SLLA?
- Iterator for a SSLA is a combination of an iterator for an array and of an iterator for a singly linked list:
- Since the elements are stored in an array, the *currentElement* will be an index from the array.
- But since we have a linked list, going to the next element will not be done by incrementing the *currentElement* by one; we have to follow the *next* links.
- Also, initialization will be done with the position of the head, not position 1.

Summary

- Today we have talked about:
  - XOR linked lists
  - Skip lists
  - How to define a linked list on an array
- Extra reading - Solution to the problem from the previous extra reading