Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză

Curs: Dynamical Systems

Primăvara 2019

Continuous Dynamical Systems[1]

$\boxed{\textbf{1.}}$ Represent the phase portrait of the scalar dynamical system $\dot{x} = 1 - x^2$. Find $\varphi(t, 1)$ and justify. Specify the properties of the functions $\varphi(t, 2)$ and, respectively, $\varphi(t, 0.5)$.

$\boxed{\textbf{2.}}$ Let $0 < c < 1$ be a parameter and consider the scalar dynamical system $\dot{x} = x(1 - x) - cx$.

a) Find its equilibria and study their stability using the linearization method.

b) Represent its phase portrait.

c) When $x(t) > 0$ is considered to be the density of fish in a lake, and $0 < c < 1$ to be the rate of fishing, try to predict the fate of the fish from the lake interpreting the theoretical result obtained at a) and b). ⋄

3. Let $c > 1/4$ be a parameter and consider the scalar dynamical system $\dot{x} = x(1 - x) - c$.

a) Represent its phase portrait.

b) When $x(t) \geq 0$ is considered to be the density of fish in a lake, and $c > 1/4$ to be the rate of fishing, try to predict the fate of the fish from the lake interpreting the theoretical result obtained at a). ⋄

4. Represent the phase portrait of the scalar dynamical system $\dot{x} = 2 - x^2$. Study the stability of the equilibrium points using the linearization method. Find $\varphi(t, \sqrt{2})$ and specify the properties of the functions $\varphi(t, -1.5)$, $\varphi(t, 0)$ and $\varphi(t, 2)$. ⋄

5. Represent the phase portrait of the scalar dynamical system $\dot{x} = 2x - x^2$. Study the stability of the equilibrium points using the linearization method. Find $\varphi(t, 2)$, $\varphi(t, 0)$ and study the properties of the functions $\varphi(t, -2)$, $\varphi(t, 1)$ and $\varphi(t, 3)$.

If there is an attractor, specify its basin of attraction. $\diamond$

6. Represent the phase portrait, specify the orbits and emphasize the properties of the solution $\varphi(\cdot,\eta)$ for any $\eta\in\mathbb{R}$ for the following equations. Specify the stability of the equilibria.

(a) $\dot x = x - x^2$, (b) $\dot x = x - x^3 + 1$, (c) $\dot x = x - x^3 + 0.2$, (d) $\dot x = -x - x^3 + 1$, (e) $\dot x = 2\sin x$, (f) $\dot x = 1-2\sin x$, (g) $\dot x = 1-\sin x$, (h) $\dot x = 2-\sin x$, (i) $\dot x = \tanh x$. $\diamond$

7. Represent the phase portrait of $\dot x = \lambda - x^2$. Discuss with respect to the parameter $\lambda\in\mathbb{R}$. $\diamond$

8. We consider the following linear planar system

(a) $\dot x = -6x,\quad \dot y = -3y$, (b) $\dot x = -6y,\quad \dot y = -3x$, (c) $\dot x = 6y,\quad \dot y = -3x$, (d) $\dot x = x,\quad \dot y = -3y$, (e) $\dot x = 2y,\quad \dot y = -3x$, (f) $\dot x = -2y,\quad \dot y = 2x$.

i) Find its flow.

ii) Specify the type and stability of this linear system.

iii) Find a first integral. There is a global first integral?

iv) Represent its phase portrait. $\diamond$

9. Specify the type and stability of the linear systems

(a) $\dot x = 4x - 5y,\quad \dot y = x - 2y$, (b) $\dot x = x + y,\quad \dot y = -2x + 4y$, (c) $x' = x + y,\quad y' = x - 4y$. $\diamond$

10. (i) For what values of the real parameter a, the system $\dot x = ax - 5y$, $\dot y = x - 2y$ is a center?. In that cases find the general solution of the system.

(ii) For what values of the real parameter a, the system from (i) has a line filled with equilibrium points? $\diamond$

11. There are uncoupled linear systems which are centers? $\diamond$

12. Find all the equilibrium points of the nonlinear planar system $\dot x = x(1-x)$, $\dot y = (y+1)(y-2)$. $\diamond$

13*. Represent the phase portrait of the following uncoupled (product) systems

(a) $\dot{x} = x(1-x)$, $\dot{y} = 0$, (b) $\dot{x} = x(1-x)$, $\dot{y} = y$, (c) $\dot{x} = x(1-x)$, $\dot{y} = (y+1)(y-2)$. $\diamond$

14. Find the equilibrium points and decide whether they are or not hyperbolic, for the nonlinear planar system $\ddot{\theta} + \dot{\theta} + \theta^3 = 0$. $\diamond$

15. We consider the following nonlinear planar systems

$\dot{x} = -x + xy, \quad \dot{y} = -2y + 3y^2$.

a) Find its equilibrium points and study their stability using the linearization method.

b) Find $\varphi(t, 0, 2/3)$, $\varphi(t, 4, 0)$ and $\varphi(t, 1, 2/3)$.

c) Represent in the phase plane the orbits corresponding to the initial values $(0, 2/3)$, $(4, 0)$ and $(1, 2/3)$. $\diamond$

16. Consider the following planar system

$$\dot{x} = 2x - x^2, \quad \dot{y} = y - xy^2.$$

a) Find the equilibrium points and study their stability.

b) Find $\varphi(t, 2, 1/2)$, $\varphi(t, 2, 0)$ and $\varphi(t, 0, 2)$. $\diamond$

17. Find the polar coordinates of the following points of cartesian coordinates. Represent all these points in the plane.

$(0, 1,$ $(1, 0)$, $(2, 0)$, $(0, 3)$, $(-3, 0)$, $(0, -2)$, $(1, 1)$, $(1, 1/2)$, $(-2, 1)$, $(-6, -3)$, $(\eta_1 \cos t - \eta_2 \sin t,\ \eta_1 \sin t + \eta_2 \cos t)$ where $t \in \mathbb{R}$ and $(\eta_1, \eta_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $(\eta_1 e^t \cos t - \eta_2 e^t \sin t,\ \eta_1 e^t \sin t + \eta_2 e^t \cos t)$ where $t \in \mathbb{R}$ and $(\eta_1, \eta_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. $\diamond$

18. We consider the linear planar system $\dot{x} = -x + y$, $\dot{y} = -x - y$. Specify its type and stability. Pass to polar coordinates. Represent its phase portrait. $\diamond$

19. Consider the following planar system

$$\dot{x} = x - 2\,x\,y, \quad \dot{y} = x - y.$$

Find the equilibrium points and study their stability. $\diamond$

20. Consider the following planar system

$$\dot{x} = -y(x^2 + y^2), \quad \dot{y} = x(x^2 + y^2).$$

a) Does this system have other equilibria besides $(0, 0)$? Justify.

b) Decide whether the equilibrium point $(0, 0)$ is hyperbolic or not.

c) Verify that $\varphi(t, 1, 0) = (\cos t, \sin t)$, $\varphi(t, 2, 0) = (2\cos 4t, 2\sin 4t)$ for all $t \in \mathbb{R}$. Find $\varphi(t, 3, 0)$. Represent the corresponding orbits.

d) Pass to polar coordinates and represent the phase portrait. Deduce that all the solutions of the system are periodic. $\diamond$

21. We consider the planar system

$\dot{x} = -y + x(1 - x^2 - y^2)$, $\dot{y} = x + y(1 - x^2 - y^2)$.

a) Study the stability of the equilibrium point $(0, 0)$ using the linearization method. There are other equilibrium points?

b) Check that $\varphi(t, 1, 0) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$. Represent the corresponding orbit. $\diamond$

22. Consider the following planar system

$$\dot{x} = -y - x(x^2 + y^2), \quad \dot{y} = x - y(x^2 + y^2).$$

a) Does this system have other equilibria besides $(0, 0)$? Justify.

b)Prove that the equilibrium point $(0, 0)$ is not hyperbolic.

c) Use the system obtained by passing to polar coordinates to determine the shape of the orbits. $\diamond$

23. Denote by $x(t)$ the density of a trout population at time t. Describe its evolution in each of the following cases.

(a) $\dot{x} = 100x$.

(b) $\dot{x} = 100x - x^2$.

Assume that someone is fishing, first with a constant rate $k > 0$, that is,

(c) $\dot{x} = 100x - x^2 - k$,

and another one, in another basin, with rate proportional to the density, $kx(t)$, where $k > 0$, that is,

(d) $\dot{x} = 100x - x^2 - kx$.

Discuss with respect to k. $\diamond$

24. For each $k > 0$ we consider the differential equation

$$\dot{x} = -k\,(x - 21),$$

which is the model of Newton for cooling processes, here $x(t)$ being the temperature of a cup of tea at time t.

(a) Find its flow. (b) An experiment revealed the following fact. A cup of tea with initial temperature of $49°C$ has a temperature of $37°C$ after 10 minutes. Find the initial temperature of a cup of tea such that after 20 minutes the tea has $37°C$. $\diamond$

25. Find a global first integral of the following nonlinear planar system $\ddot{\theta} + \omega^2 \sin\theta = 0$ (where $\omega > 0$). $\diamond$

26. Find a first integral in $(0,\infty) \times (0,\infty)$ of the following nonlinear planar system $\dot{x} = N_1 x - xy, \quad \dot{y} = -N_2 y + xy$ (where $N_1, N_2 > 0$). $\diamond$

27. Find a first integral in $(0,\infty) \times (0,\infty)$ of the following nonlinear planar system $\dot{x} = x - xy, \quad \dot{y} = -0.3y + 0.3xy$. $\diamond$

28. a) Give an example of a coupled linear planar system which has a node at the origin.

b) Give an example of a coupled linear planar system which has a saddle at the origin. $\diamond$

29. Find the the flow of each of the systems. Decide wether, for every $\eta \in \mathbb{R}^2$ we have $\lim_{t\to\infty} \varphi(t,\eta) = 0$ or we have that $\varphi(t,\eta)$ is bounded for $t \in (0,\infty)$.

a) $x' = -2x, \;\; y' = -3y$

b) $x' = -2x, \;\; y' = 3y$

c) $x' = -3x, \;\; y' = x - 3y$

d) $x' = y, \;\; y' = \omega^2 x$ (here $\omega > 0$ is a parameter)

e) $x' = -x - y, \;\; y' = x - y$

f) $x' = -5x - 9y, \;\; y' = 2x + y$. $\diamond$