1.4 Exercises

1. Let $A_0, \ldots, A_n$ be the vertices of a polygon. Determine $\overrightarrow{A_0A_1} + \overrightarrow{A_1A_2} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_0}$.

2. In each of the following cases, decide if the indicated vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ can be represented with the vertices of a triangle:

a) $\mathbf{u}(7,3)$, $\mathbf{v}(2,8)$, $\mathbf{w}(-5,5)$.

b) $\mathbf{u}(1,2,-1)$, $\mathbf{v}(2,-1,0)$, $\mathbf{w}(1,-3,1)$.

3. Let $ABCD$ be a quadrilateral. Let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that

$$\overrightarrow{MN} + \overrightarrow{PQ} = 0.$$

Deduce that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram.

4. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AC]$ and let F be the midpoint of $[BD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right) = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{CB}\right).$$

5. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AB]$ and let F be the midpoint of $[CD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{BC}\right).$$

6. Let A', B' and C' be midpoints of the sides of a triangle ABC. Show that for any point O we have

$$\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

7. Show that the medians in a triangle intersect in one point and deduce the ratio in which the common intersection point divides the medians.

8. In each of the following cases, decide if the given points are collinear:

* tablă a) $P(3,-5)$, $Q(-1,2)$, $R(-5,9)$.

b) $A(11,2)$, $B(1,-3)$, $C(31,13)$.

c) $P(1,0,-1)$, $Q(0,-1,2)$, $R(-1,-2,5)$.

d) $A(-1,-1,-4)$, $B(1,1,0)$, $C(2,2,2)$.

9. Let $ABCD$ be a tetrahedron. Determine the sums

a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$,

b) $\overrightarrow{AD} + \overrightarrow{BC} + \overrightarrow{DB}$,

c) $\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{BC} + \overrightarrow{DA}$.

10. Let $ABCD$ be a tetrahedron. Show that

$$\overrightarrow{AD} + \overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{AC}.$$

11. Let $SABCD$ be a pyramid with apex S and base the parallelogram $ABCD$. Show that

$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}$$

where O is the center of the parallelogram.

12. Give the coordinates of the vertices of the parallelepiped whose faces lie in the coordinate planes and in the planes $x = 1$, $y = 3$ and $z = -2$.

13. In $\mathbb{E}^3$ consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.

14. Which of the following sets of vectors form a basis?

a) $\mathbf{v}(1,2)$, $\mathbf{w}(3,4)$;

b) $\mathbf{u}(-1,2,1)$, $\mathbf{v}(2,1,1)$, $\mathbf{w}(1,0,-1)$;

c) $\mathbf{u}(-1,2,1)$, $\mathbf{v}(2,1,1)$, $\mathbf{w}(0,5,3)$;

d) $\mathbf{v}_1(-1,2,1,0)$, $\mathbf{v}_2(2,1,1,0)$, $\mathbf{v}_3(1,0-1,1)$, $\mathbf{v}_4(1,0,0,1)$;

15. With respect to the basis $\mathcal{B} = (\mathbf{i},\mathbf{j},\mathbf{k})$ consider the vectors $\mathbf{u} = \mathbf{i}+\mathbf{j}$, $\mathbf{v} = \mathbf{j}+\mathbf{k}$ and $\mathbf{w} = \mathbf{i}+\mathbf{k}$. Check that $\mathcal{B}' = (\mathbf{u},\mathbf{v},\mathbf{w})$ is a basis and give the base change matrix $\mathrm{M}_{\mathcal{B}',\mathcal{B}}$.

16. Consider the two coordinate systems $\mathcal{K} = (O,\mathbf{i},\mathbf{j})$ and $\mathcal{K}' = (O',\mathbf{i}',\mathbf{j}')$ given in Example 1.20. Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix}1\\2\end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix}2\\4\end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix}3\\1\end{bmatrix}$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.

17. Consider the tetrahedron $ABCD$ and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}_A' = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

a) the coordinates of the vertices of the tetrahedron in the three coordinate systems,

b) the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}_A'$,

c) the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.

18. Consider the two coordinate systems $\mathcal{K} = (O,\mathbf{i},\mathbf{j},\mathbf{k})$ and $\mathcal{K}' = (O',\mathbf{i}',\mathbf{j}',\mathbf{k}')$ given in Example 1.21. Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

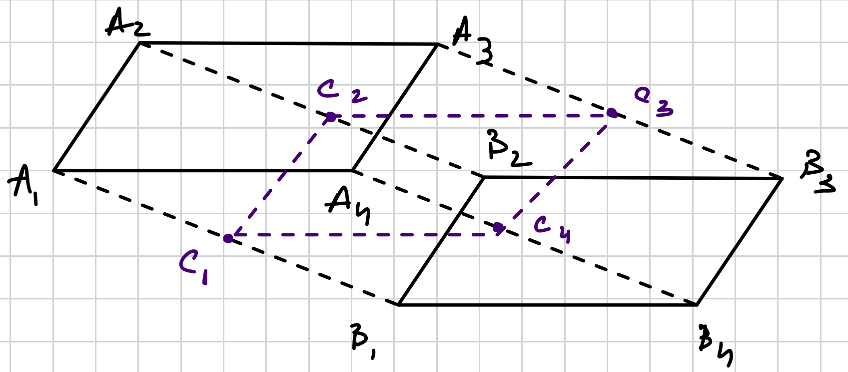
$$[A]_{\mathcal{K}} = \begin{bmatrix}1\\4\\-1\end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix}1\\5\\1\end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix}-3\\7\\1\end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix}0\\3\\1\end{bmatrix}$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.

1. Let $A_0, \ldots, A_n$ be the vertices of a polygon. Determine $\overrightarrow{A_0A_1} + \overrightarrow{A_1A_2} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_0}$.

$$\overrightarrow{A_0A_1} + \overrightarrow{A_1A_2} + \ldots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_0} =$$
$$= \overrightarrow{A_0A_2} + \ldots + \overrightarrow{A_nA_0} =$$
$$= \ldots$$
$$= \overrightarrow{A_0A_n} + \overrightarrow{A_nA_0} =$$
$$= \overrightarrow{A_0A_0} = 0$$

13. In $\mathbb{E}^3$ consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.



C_1, C_2, C_3, C_4 – midpoints of $[A_1B_1], [A_2B_2], [A_3B_3], [A_4B_4]$

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2C_2} + \overrightarrow{C_2C_1} + \overrightarrow{C_1A_1} = 0$$
$$\overrightarrow{B_1B_2} + \overrightarrow{B_2C_2} + \overrightarrow{C_2C_1} + \overrightarrow{C_1B_1} = 0 \quad (+)$$
$$\frac{1}{2}\left(\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2}\right) = \overrightarrow{C_1C_2} \quad (1)$$

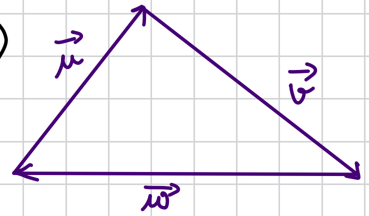
$$\overrightarrow{A_4A_3} + \overrightarrow{A_3C_3} + \overrightarrow{C_3C_4} + \overrightarrow{C_4A_4} = 0$$
$$\overrightarrow{B_4B_3} + \overrightarrow{B_3C_3} + \overrightarrow{C_3C_4} + \overrightarrow{C_4B_4} = 0 \quad (+)$$
$$\overrightarrow{C_4C_3} = \frac{1}{2}\left(\overrightarrow{A_4A_3} + \overrightarrow{B_4B_3}\right) \quad (2)$$

$(1), (2) \Rightarrow \begin{cases} \overrightarrow{A_1A_2} = \overrightarrow{A_4A_3} \\ \overrightarrow{B_1B_2} = \overrightarrow{B_4B_3} \end{cases} \Rightarrow \overrightarrow{C_1C_2} = \overrightarrow{C_4C_3} \Rightarrow C_1C_2C_3C_4$ is a parallelogram

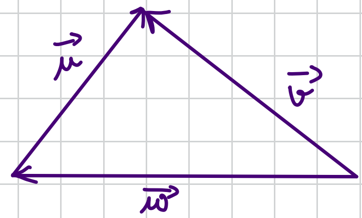
2. In each of the following cases, decide if the indicated vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ can be represented with the vertices of a triangle:

a) $\mathbf{u}(7,3)$, $\mathbf{v}(2,8)$, $\mathbf{w}(-5,5)$.

b) $\mathbf{u}(1,2,-1)$, $\mathbf{v}(2,-1,0)$, $\mathbf{w}(1,-3,1)$.

a) 

$\vec{u}+\vec{v}+\vec{w}=0$



$\vec{u}-\vec{v}+\vec{w}=0$

$ab=\sqrt{|a_x-b_x|^2+|a_y-b_y|^2}$

write each vector in terms of $\vec{i}$ and $\vec{j}$

$\vec{u}=7\vec{i}+3\vec{j}$

$\vec{v}=2\vec{i}+8\vec{j}$

$\vec{w}=-5\vec{i}+5\vec{j}$

$\vec{u}-\vec{v}+\vec{w}=7\vec{i}+3\vec{j}-2\vec{i}-8\vec{j}-5\vec{i}+5\vec{j}=\vec{0}\Rightarrow \vec{u},\vec{w},\vec{v}$ can be represented as vertices of a triangle

b) $u(1,2,-1)$
$v(2,-1,0)$
$w(1,-3,1)$

$\|u\|=\sqrt{1^2+2^2+(-1)^2}=\sqrt{6}$

$\|v\|=\sqrt{5}$

$\|u+v\|=\sqrt{3^2+1^2+(-1)^2}=\sqrt{11}\leq \|u\|+\|v\|$

$\vec{u}-\vec{v}+\vec{w}=0$

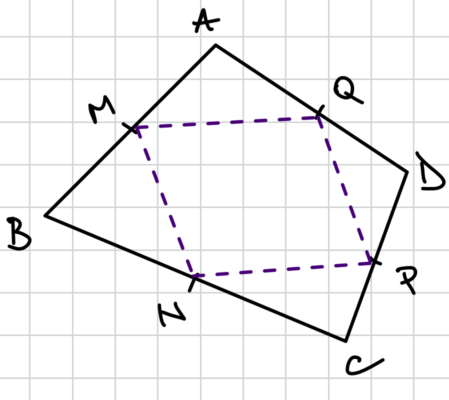
$\cancel{\vec{i}+2\vec{j}-\vec{k}-2\vec{i}+\vec{j}+\vec{i}-3\vec{j}+\vec{k}}=0$

$0=0\Rightarrow \vec{u}-\vec{v}+\vec{w}$ can be represented as a triangle

3. Let $ABCD$ be a quadrilateral. Let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that

$$\overrightarrow{MN}+\overrightarrow{PQ}=0.$$

$\overrightarrow{\overrightarrow{MN}+\overrightarrow{PQ}}=\vec{0}$



$\overrightarrow{MN}+\overrightarrow{PQ}=0$

$\overrightarrow{MB}+\overrightarrow{BN}+\overrightarrow{PD}+\overrightarrow{DQ}=0$

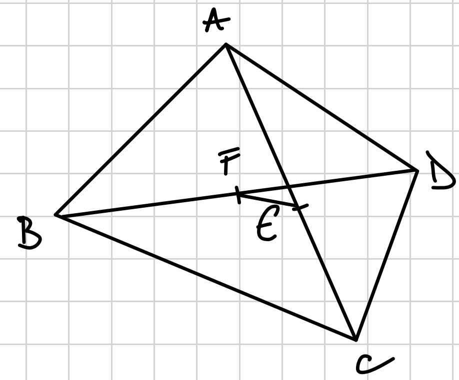
$\frac{1}{2}\overrightarrow{AB}+\frac{1}{2}\overrightarrow{BC}+\frac{1}{2}\overrightarrow{CD}+\frac{1}{2}\overrightarrow{DA}=0$

$\frac{1}{2}\overrightarrow{AC}+\frac{1}{2}\overrightarrow{CA}=\vec{0}$

$\frac{1}{2}(\overrightarrow{AC}+\overrightarrow{CA})=\vec{0}$ – true

4. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AC]$ and let F be the midpoint of $[BD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right) = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{CB}\right).$$

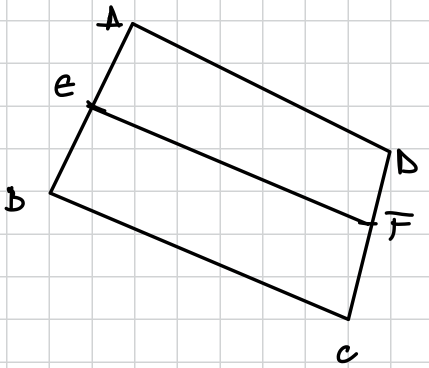


$\vec{EF} = \frac{1}{2}(\vec{AB}+\vec{CD})$

$\vec{EF} = \vec{EC} + \vec{CB} + \vec{BF} = \frac{1}{2}\vec{AC} + \vec{CB} + \frac{1}{2}\vec{BD} = \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{CB}$
$+ \frac{1}{2}\vec{BD}$
$= \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{CD}$

5. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AB]$ and let F be the midpoint of $[CD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{BC}\right).$$

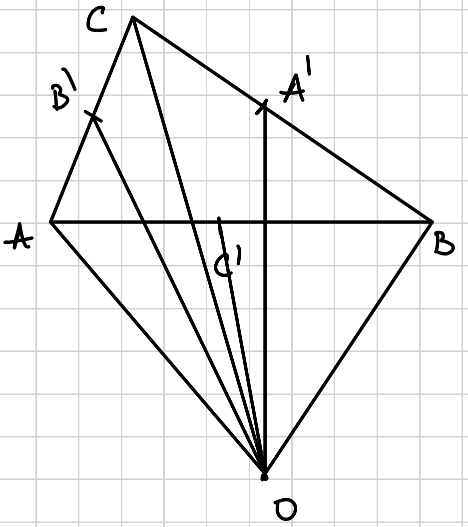


$\vec{EF} = \frac{1}{2}\vec{AB} + BC + \frac{1}{2}\vec{CD} = \frac{1}{2}\vec{AC} + \frac{1}{2}\vec{BD} =$

$= \frac{1}{2}\vec{AD} + \cancel{\frac{1}{2}\vec{DC}} + \frac{1}{2}\vec{BC} + \cancel{\frac{1}{2}\vec{CD}} = \frac{1}{2}\vec{AD} + \frac{1}{2}\vec{BC}$

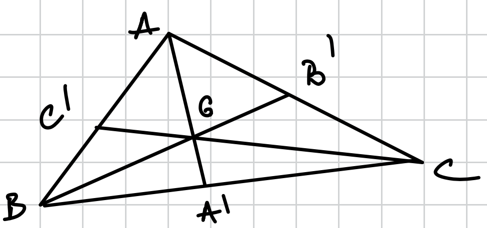
6. Let A', B' and C' be midpoints of the sides of a triangle ABC. Show that for any point O we have

$$\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$



$\vec{OA'} = \vec{OB} + \frac{1}{2}\vec{BC}$
$\vec{OB'} = \vec{OC} + \frac{1}{2}\vec{CA}$
$\vec{OC'} = \vec{OA} + \frac{1}{2}\vec{AB}$
(+)
$\vec{OA'} + \vec{OB'} + \vec{OC'} = \vec{OA} + \vec{OB} + \vec{OC} + \frac{1}{2}\underbrace{(\vec{AB} + \vec{BC} + \vec{CA})}_{\vec{0}}$

7. Show that the medians in a triangle intersect in one point and deduce the ratio in which the common intersection point divides the medians.



$\vec{AG} = \vec{AC} + \vec{CA'} + \vec{A'G} = \vec{AC} + \frac{1}{2}\vec{CB} + \vec{A'G} = \frac{1}{2}(\vec{AC} + \vec{AB}) + \vec{A'G}$
$BB' = \frac{1}{2}(\vec{BC} + \vec{BA})$
$CC' = \frac{1}{2}(\vec{CA} + \vec{CB})$

$\vec{OA} = \vec{a}$

$\vec{OB} = \vec{b}$

$\vdots$

$\vec{OG} = \frac{1}{3}\vec{a} + \frac{2}{3}\vec{a}$

$\left(\frac{1}{3} + \frac{2}{3}\right)\vec{OG} = \frac{1}{3}\vec{a} + \frac{2}{3}\vec{a}$

$\Rightarrow \vec{AG} = 2\vec{A'G}$