Seminar 2

1. Prove using the ε-definition that $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and ★ $\lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$.

2. Study if the sequence (x_n) is bounded, monotone, and convergent, for each of the following:

   ~~(a)~~ $x_n = \sqrt{n+1} - \sqrt{n}$.

   ~~(b)~~ $x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$.

   (c) ★$x_n = \frac{\sin(n)}{n}$.

   ~~(d)~~ $x_n = \frac{2^n}{n!}$.

3. Find the limit for each of the following sequences:

   ~~(a)~~ $\sqrt{n}\big(\sqrt{n+1} - \sqrt{n}\big)$.

   ~~(b)~~ $\big(a_1^n + a_2^n + \ldots + a_k^n\big)^{\frac{1}{n}}$, with $a_i > 0$.

   ~~(c)~~ $\sqrt[n]{n}$.

   (d) ★ $\sqrt[n]{1+2+\ldots+n}$. $= \left(\frac{n(n+1)}{2}\right)^{\frac{1}{n}} = \underset{\downarrow\, 1}{\left(\frac{1}{2}\right)^{\frac{1}{n}}} \cdot [n(n+1)]^{\frac{1}{n}}$

   ~~(e)~~ $\frac{2^n + (-1)^n}{3^n}$.

   ~~(f)~~ $\frac{(an+1)^2}{4n^2-2n+1}$, $a \in \mathbb{R}$.

~~4.~~ Consider the sequence (e_n) given by

$$e_n = \big(1 + \frac{1}{n}\big)^n.$$

Prove that (e_n) is increasing and bounded – its limit is denoted by e (Euler's number).

5. Find the limit for each of the following sequences:

   ~~(a)~~ $\big(\frac{2n+1}{2n-1}\big)^n$.

   ~~(b)~~ $n\big(\ln(n+2) - \ln(n+1)\big)$.

   (c) ★$\big(\frac{\ln(n+1)}{\ln n}\big)^n$.

6. ★ Prove that the sequence (x_n) given by $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n$ is decreasing and bounded, hence convergent – its limit is denoted by γ (Euler's constant).

7. (Stolz-Cesàro lemma) Let $(a_n), (b_n)$ be two sequences such that (i) $a_n \to 0$ and $b_n \to 0$ with (b_n) decreasing; or (ii) $b_n \to \infty$ with (b_n) increasing.

   If

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell, \quad \text{then } \lim_{n\to\infty} \frac{a_n}{b_n} = \ell.$$

   (a) Let (x_n) be a convergent sequence. What can you say about the sequence (a_n) of averages

$$a_n = \frac{x_1 + x_2 + \ldots + x_n}{n}?$$

   Give an example where the averages converge, even though the sequence does not.

   (b) Compare $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ with n and $\ln n$ by taking the ratio, respectively.

   (c) Let (x_n) be a sequence such that $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell$. Prove that $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = \ell$.

1. Prove using the ε-definition that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

$$\boxed{x_n \to \ell : \quad \forall\, \varepsilon > 0,\ \exists\, N_\varepsilon \in \mathbb{N} \text{ s.t. } |x_n - \ell| < \varepsilon,\ \forall\, n \geq N_\varepsilon}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Let $\varepsilon > 0$, we want to find an index $N_\varepsilon \in \mathbb{N}$ s.t. $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon\ \forall\, n \geq N_\varepsilon$

you're given an ε

find an index for which this is ok

$$\frac{1}{\sqrt{n}} < \varepsilon \iff 1 < \varepsilon \cdot \sqrt{n} \iff \frac{1}{\varepsilon} < \sqrt{n} \quad \Big|\, ()^2$$

$$\frac{1}{\varepsilon^2} < n$$

Take $N_\varepsilon = \left[\frac{1}{\varepsilon^2}\right] + 1 > \frac{1}{\varepsilon^2}$

$$\boxed{n \geq N_\varepsilon > \frac{1}{\varepsilon^2} \Rightarrow n > \frac{1}{\varepsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon}$$

2. Study if the sequence (x_n) is bounded, monotone, and convergent, for each of the following:

(a) $x_n = \sqrt{n+1} - \sqrt{n}$.

(b) $x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$.

(c) ★$x_n = \frac{\sin(n)}{n}$.

(d) $x_n = \frac{2^n}{n!}$.

(a) $$x_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \in (0, 1]$$

$n = 0 \Rightarrow x_0 = \sqrt{0+1} - \sqrt{0} = 1$

$n = 1 \Rightarrow x_1 = \sqrt{2} - 1$

$$x_{n+1} \overset{?}{<} x_n \Rightarrow \frac{1}{\sqrt{n+2} + \sqrt{n+1}} < \frac{1}{\sqrt{n+1} + \sqrt{n}} \quad \Big|\, ()^{-1}$$

$$\sqrt{n+2} + \cancel{\sqrt{n+1}} > \cancel{\sqrt{n+1}} + \sqrt{n}$$

$$\sqrt{n+2} > \sqrt{n} \quad \Big|\, ()^2$$

$$n+2 > n, \quad n \in \mathbb{N} \to \text{true}$$

$\Rightarrow$ bounded, decreasing

convergent ✓ $\Rightarrow x_n \to 0$ as $n \to \infty$

(b) $$x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \ldots + \cancel{\frac{1}{n}} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1} \in (0, 1), \quad n \geq 1$$

$$x_{n+1} - x_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0 \Rightarrow x_{n+1} > x_n \Rightarrow (x_n) \text{ is increasing}$$

(x_n) convergent $\to \lim_{n\to\infty} x_n = 1$

(d) $x_n = \frac{2^n}{n!}$

$$\frac{x_{n+1}}{x_n} = \frac{\cancel{2^{n+1}}^{\,2}}{\cancel{(n+1)!}_{\,(n+1)}} \cdot \frac{\cancel{n!}}{\cancel{2^n}} = \frac{2}{n+1} < 1 \quad \text{for } n > 1$$

$$\left.\begin{array}{l} x_0 = 1 \\ x_1 = 2 \\ \text{for } n \geq 2 \Rightarrow (x_n) \downarrow \end{array}\right\} \Rightarrow x_n \downarrow \qquad \left.\begin{array}{l} x_n > 0 \ \forall n \\ x_n \leq 2 \end{array}\right\} \Rightarrow x_n \text{ convergent}$$

$$\boxed{\text{if } \frac{x_{n+1}}{x_n} \longrightarrow \ell \ \begin{cases} \text{if } \ell < 1 \Rightarrow x_n \to 0 \\ \text{if } \ell > 1 \Rightarrow x_n \to \infty \end{cases}}$$

$\Rightarrow$ behaves like a geometric progression $\Rightarrow q = \frac{x_{n+1}}{x_n}$

!

$$\Rightarrow \frac{x_{n+1}}{x_n} < 1 \Rightarrow x_n \to 0$$

3. Find the limit for each of the following sequences:

(a) $\sqrt{n}(\sqrt{n+1} - \sqrt{n})$.

(b) $(a_1^n + a_2^n + \ldots + a_k^n)^{\frac{1}{n}}$, with $a_i > 0$.

(c) $\sqrt[n]{n}$.

(d) ★ $\sqrt[n]{1+2+\ldots+n}$.

(e) $\frac{2^n+(-1)^n}{3^n}$.

(f) $\frac{(an+1)^2}{4n^2-2n+1}$, $a \in \mathbb{R}$.

(a) $\sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$

$$\lim_{n\to\infty} \sqrt{n}\,{}^{\sqrt{n+1}+\sqrt{n})}\!\left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(\sqrt{1+\underbrace{\tfrac{1}{n}}_{\to 0}}+1\right)} = \frac{1}{2}$$

(b) $\left(a_1^n + a_2^n + \ldots + a_k^n\right)^{\frac{1}{n}}$ with $a_i > 0$

$$\text{* helper: } \lim_{n\to\infty} \left(2^n + 3^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left\{3^n\left[\left(\tfrac{2}{3}\right)^n + 1\right]\right\}^{\frac{1}{n}} = 3 \cdot \lim_{n\to\infty} \left[\underbrace{\left(\tfrac{2}{3}\right)^n}_{\to 0} + 1\right]^{\frac{1}{n}} = 3 \cdot 1^0 = 3$$

$$\lim_{n\to\infty} \left(a_1^n + a_2^n + \ldots + a_k^n\right)^{\frac{1}{n}}$$

Let $a_m = \max\{a_1, a_2, \ldots, a_k\}$, $m \in \{1, 2, \ldots, k\}$

$$\lim_{n\to\infty} \left\{a_m^n \left[\left(\tfrac{a_1}{a_m}\right)^n + \left(\tfrac{a_2}{a_m}\right)^n + \ldots + \left(\tfrac{a_k}{a_m}\right)^n\right]\right\}^{\frac{1}{n}} =$$

$$= \lim_{n\to\infty} \left(a_m^n\right)^{\frac{1}{n}} \cdot \left[\ \epsilon\ \right]^{\frac{1}{n}} = \qquad 1 \leq \epsilon \leq k \ \Big| \ ()^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} a_m^{\frac{n}{n}} \cdot \epsilon^{\frac{1}{n}} = \qquad 1 \leq \epsilon^{\frac{1}{n}} \leq k^{\frac{1}{n}} \ \Big| \lim$$

$$= a_m \cdot \epsilon^0 = a_m \qquad \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ 1 & & 1 \end{array}$$

sandwich Th

$$\Rightarrow \lim \epsilon^{\frac{1}{n}} = 1$$

(c) $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}}$

$$\boxed{f^g = e^{g \ln f}}$$

$= \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^{\lim_{n\to\infty} \frac{\ln n}{n}} = e^0 = 1$

(e) $\lim_{n\to\infty} \frac{2^n + (-1)^n}{3^n} = \lim_{n\to\infty} \left[\underbrace{\left(\frac{2}{3}\right)^n}_{\to 0} + \frac{(-1)^n}{3^n} \right] = \lim_{n\to\infty} \left(-\frac{1}{3}\right)^n = 0$

(f) $\lim_{n\to\infty} \frac{(an+1)^2}{4n^2 - 2n + 1} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(a^2 + \frac{2a}{n} + \frac{1}{n^2}\right)}{\cancel{n^2}\left(4 - \frac{2}{n} + \frac{1}{n^2}\right)} = \frac{a^2}{4}, \ a \in \mathbb{R}^*$

for $a = 0$: $\lim_{n\to\infty} \frac{1}{4n^2 - 2n + 1} = 0$

4. Consider the sequence (e_n) given by

$$\boxed{e_n = (1 + \frac{1}{n})^n.}$$

Prove that (e_n) is increasing and bounded – its limit is denoted by e (Euler's number).

$(x+1)^n = x^n + \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} + \dots + \binom{n}{n-1} x + 1$

$\binom{n}{1} \downarrow$ n choose 1, C_n^1

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n(n-1)\dots(n-k+1)}{k!}$

$e_n = 1 + \cancel{n} \cdot \frac{1}{\cancel{n}} + \binom{n}{2} \cdot \frac{1}{n^2} + \dots + \binom{n}{k} \frac{1}{n^k} + \dots + \binom{n}{n-1} \cdot \frac{1}{n^{n-1}} + \frac{1}{n^n} =$

$= 1 + 1 + \frac{n(n-1)}{2n^2} + \frac{n(n-1)(n-2)}{2\cdot 3\cdot n^3} + \dots + \frac{n(n-1)(n-2)\dots(n-k+1)}{k!\,n^k} + \dots + \frac{1}{n^n} =$

$= 1 + 1 + \frac{1\cdot\left(1-\frac{1}{n}\right)}{2} + \dots + \frac{1\left(1-\frac{1}{n}\right)\dots\left(1-\frac{k-1}{n}\right)}{k!} + \dots + \frac{1}{n^n}$

$\wedge \qquad \wedge \qquad \wedge$

$e_{n+1} = 1 + 1 + \frac{1\cdot\left(1-\frac{1}{n+1}\right)}{2} + \dots + \frac{1\left(1-\frac{1}{n+1}\right)\dots\left(1-\frac{k-1}{n+1}\right)}{k!} + \dots + \} + \frac{1}{(n+1)^{n+1}}$

$\boxed{e_{n+1} > e_n} \Rightarrow (e_n)$ increasing

$$\boxed{1 + q + \dots + q^{n-1} = \frac{1-q^n}{1-q}}$$

$e_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{k!} + \dots + \frac{1}{n!}$ $\qquad \frac{1}{k!} = \frac{1}{2\cdot 3\cdot\dots\cdot k} < \frac{1}{2^{k-1}}$

$e_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} = 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 1 + 2\underbrace{\left(1 - \frac{1}{2^n}\right)}_{<1} < 1 + 2 = 3$

$\boxed{e_n < 3} \Rightarrow \exists \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

5. Find the limit for each of the following sequences:

(a) $\left(\frac{2n+1}{2n-1}\right)^n$. (b) $n\big(\ln(n+2)-\ln(n+1)\big)$.

(a) $\lim\limits_{n\to\infty}\left(\frac{2n+1}{2n-1}\right)^n = (1^\infty) = \lim\limits_{n\to\infty}\left(1+\frac{2n+1}{2n-1}-1\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{2}{2n-1}\right)^n = \lim\limits_{n\to\infty}\left[\left(1+\frac{2}{2n-1}\right)^{\frac{2n-1}{2}}\right]^{\frac{2n}{2n-1}} = e$

(b) $\lim\limits_{n\to\infty} n\left(\ln(n+2)-\ln(n+1)\right) = (\infty\cdot 0) = \lim\limits_{n\to\infty} n\cdot\ln\frac{n+2}{n+1} = \ln\lim\limits_{n\to\infty}\left(\frac{n+2}{n+1}\right)^n = \ln\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n =$

$= \ln\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{(n+1)\cdot\frac{n}{n+1}} = \ln e^{\lim\limits_{n\to\infty}\frac{n}{n+1}} = \lim\limits_{n\to\infty}\frac{n}{n+1} = 1$

7. (Stolz-Cesàro lemma) Let $(a_n),(b_n)$ be two sequences such that (i) $a_n\to 0$ and $b_n\to 0$ with (b_n) decreasing; or (ii) $b_n\to\infty$ with (b_n) increasing.

If

$$\boxed{\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\ell, \quad \text{then} \quad \lim_{n\to\infty}\frac{a_n}{b_n}=\ell.}$$

(a) Let (x_n) be a convergent sequence. What can you say about the sequence (a_n) of averages

$$a_n = \frac{x_1+x_2+\ldots+x_n}{n}?$$

Give an example where the averages converge, even though the sequence does not.

(b) Compare $1+\frac{1}{2}+\ldots+\frac{1}{n}$ with n and $\ln n$ by taking the ratio, respectively.

(c) Let (x_n) be a sequence such that $\lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n}=\ell$. Prove that $\lim\limits_{n\to\infty}\sqrt[n]{x_n}=\ell$.

(a) $(x_n)\to x$

$\lim\limits_{n\to\infty}\frac{x_1+x_2+\ldots+x_n}{n} \overset{S-C}{=} \lim\limits_{n\to\infty}\frac{x_{n+1}}{1} = x$

$\exists \lim\limits_{n\to\infty}\frac{x_1+\ldots+x_n}{n}$, but $\nexists \lim\limits_{n\to\infty} x_n$ Ex: $x_n=(-1)^n$: $\frac{x_1+\ldots+x_n}{n} \begin{cases} 0 \\ \frac{-1}{n}\to 0\end{cases} \Rightarrow \lim\limits_{n\to\infty}\frac{x_1+\ldots+x_n}{n}=0$

$\nexists \lim\limits_{n\to\infty}(-1)^n$

(b) $\lim\limits_{n\to\infty}\frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{n} \overset{S-C}{=} \lim\limits_{n\to\infty}\frac{\frac{1}{n+1}}{1} = 0 \to$ increases slower than n

$\lim\limits_{n\to\infty}\frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{\ln n} \overset{S-C}{=} \lim\limits_{n\to\infty}\frac{\frac{1}{n+1}}{\ln\frac{n+1}{n}} = \lim\limits_{n\to\infty}\frac{1}{\underbrace{(n+1)\ln\frac{n+1}{n}}_{\to \text{prev. ex.}}} = 1 \to$ increases the same as n

$1+\frac{1}{2}+\ldots+\frac{1}{n} = O(\ln n)$

(c) $\lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n}=\ell \overset{?}{\Rightarrow} \lim\sqrt[n]{x_n}=\ell$

$\sqrt[n]{x_n} = x_n^{\frac{1}{n}} = e^{\frac{1}{n}\ln x_n} \longrightarrow e^{\ln \ell} = \ell$

$\lim\limits_{n\to\infty}\frac{\ln x_n}{n} \overset{S-C}{=} \lim\limits_{n\to\infty}\ln\frac{x_{n+1}}{x_n} = \ln e = 1$

8. Find the limit for each of the following sequences: S-C

   (a) $\frac{n}{\sqrt[n]{n!}}$. (b) ★ $\frac{n^n}{1+2^2+3^3+\ldots+n^n}$. (c) $\frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}}$, $p \in \mathbb{N}$.

9. (Banach fixed point theorem) Let $f : [a,b] \to [a,b]$ be a contraction, meaning that there exists $\alpha \in (0,1)$ such that

$$|f(x) - f(y)| \leq \alpha \, |x - y| \,, \quad \forall x, y \in [a,b].$$

   Let an arbitrary $x_1 \in [a,b]$ and consider the sequence (x_n) given by

$$x_{n+1} = f(x_n), \quad \forall n \in \mathbb{N}.$$

   Prove that the sequence (x_n) is Cauchy and that its limit x^* is a fixed point, i.e. $f(x^*) = x^*$.

10. Study the convergence and find the limit of the following sequences:

   (a) $x_{n+1} = \sqrt{2 + x_n}$, $x_1 = 0$. (b) ★$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$, $x_1 = 1$ and $a > 1$.

   use 9 as a fact without proof

A sequence is said to be convergent if it approaches some limit (D'Angelo and West 2000, p. 259). Every bounded monotonic sequence converges. Every unbounded sequence diverges.

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.

1. Prove using ε-definition $\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$

Let $\varepsilon > 0$, we want to find an index $N_\varepsilon \in \mathbb{N}$ s.t. $\left|\frac{m+1}{2m+3} - \frac{1}{\varepsilon}\right| < \varepsilon \ \forall m \geq N_\varepsilon$

$$\frac{m+1}{2m+1} \overset{?}{>} \frac{1}{2}$$

$$2(m+1) \overset{?}{>} 2m+1$$

$$\cancel{2m}+2 \overset{?}{>} \cancel{2m}+1$$

$$2 > 1 \Rightarrow \text{true } \forall m \in \mathbb{N} \Rightarrow \left|\frac{m+1}{2m+1} - \frac{1}{2}\right| = \frac{m+1}{2m+1} - \frac{1}{2}$$

$$\Rightarrow \frac{^{2)}m+1}{2m+1} - \frac{^{2m+1)}1}{2} < \varepsilon$$

$$\frac{\cancel{2m}+2-\cancel{2m}-1}{2(2m+1)} < \varepsilon$$

$$\frac{1}{2(2m+1)} < \varepsilon \quad \Big| \cdot 2(2m+1)$$

$$1 < (4m+2)\cdot\varepsilon$$

$$1 < 4m\varepsilon + 2\varepsilon$$

$$4m\varepsilon > -2\varepsilon + 1$$

$$m > \frac{-2\varepsilon+1}{4\varepsilon}, \quad \varepsilon \neq 0$$

$$m > \frac{1}{4\varepsilon} - \frac{1}{2}$$

Take $N_\varepsilon = \left[\frac{1}{4\varepsilon} - \frac{1}{2}\right] + 1 > \frac{1}{4\varepsilon} - \frac{1}{2} \Rightarrow$ $m \geq N_\varepsilon > \frac{1}{4\varepsilon} - \frac{1}{2} \Rightarrow m > \frac{1}{4\varepsilon} - \frac{1}{2} \Rightarrow \frac{n+1}{2n+3} > \varepsilon$

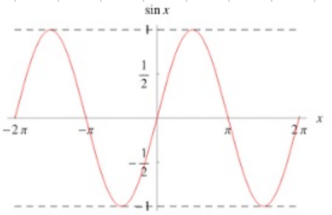
2. Study if the sequence is bounded, monotone and convergent:

(c) $x_n = \frac{\sin(n)}{n}$

$$-1 \leq \sin n \leq 1 \quad \Big| : m, \ m \neq 0, \ m > 0$$

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}, \quad m \text{ is an index } m = \{1, 2, \ldots\}$$

$$\Rightarrow -1 \leq \frac{\sin n}{n} \leq 1 \Rightarrow (x_n) \text{ bounded}$$



$\sin n$ – is not monotone $\Rightarrow \frac{\sin n}{n}$ is not increasing or decreasing $\Rightarrow (x_n)$ not mon.

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \quad \Big| \cdot \lim_{n\to\infty}$$

$$0 \leq \lim_{n\to\infty}(x_n) \leq 0 \Rightarrow (x_n) \text{ convergent}$$

3. Find the limit for the sequence:

(d) $\sqrt[n]{1+2+\ldots+n} = \sqrt[n]{\frac{n(n+1)}{2}}$

We proved that if $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell \Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{x_n} = \ell$. S-C. th. Let $(x_n) = 1+2+\ldots+n = \frac{n(n+1)}{2} \Rightarrow$

$$\lim_{n\to\infty} \frac{\frac{(n+1)(n+2)}{\cancel{2}}}{\frac{n(\cancel{n+1})}{\cancel{2}}} = \lim_{n\to\infty} \frac{n+2}{n} = 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = \ell$$

5. Find the limit.

(c) $x_n = \left(\frac{\ln(n+1)}{\ln(n)}\right)^n$

$$\lim_{n\to\infty}\left(\frac{\ln(n+1)}{\ln(n)}\right)^n = \lim_{n\to\infty}\left(\frac{\ln\left(\left(1+\frac{1}{n}\right)\cdot n\right)}{\ln n}\right)^n = \lim_{n\to\infty}\left(\frac{\ln n + \ln\left(1+\frac{1}{n}\right)}{\ln n}\right)^n = \lim_{n\to\infty}\left(1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right)^n - (1^\infty) = \lim_{n\to\infty}\left[\left(1+\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n}\right)^{\frac{\ln n}{\ln\left(1+\frac{1}{n}\right)}}\right]^{\frac{n\cdot\ln\left(1+\frac{1}{n}\right)}{\ln n}}$$

$$= e^{\lim\limits_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)^n}{\ln n}} = e^{\lim\limits_{n\to\infty}\frac{\ln e}{\ln n}} = e^{\lim\limits_{n\to\infty}\frac{1}{\ln n}} = e^0 = 1$$

with $\left(1+\frac{1}{n}\right)^n \to e$

6. ★ Prove that the sequence (x_n) given by $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n$ is decreasing and bounded, hence convergent – its limit is denoted by γ (Euler's constant).

$x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n$

If x_n is decreasing then $x_{n+1} - x_n < 0$

$$x_{n+1} - x_n = \cancel{1} + \cancel{\tfrac{1}{2}} + \ldots + \cancel{\tfrac{1}{n}} + \frac{1}{n+1} - \cancel{1} - \cancel{\tfrac{1}{2}} - \ldots - \cancel{\tfrac{1}{n}} - \ln(n+1) + \ln n = \underbrace{\frac{1}{n+1}}_{<1} + \underbrace{\ln\left(\frac{n}{n+1}\right)}_{<0} < 0 \Rightarrow (x_n) \text{ decreasing}$$

with $\frac{n}{n+1} < 1$

Since the sequence is decreasing $x_n \le x_1 \ \forall n \in \mathbb{N}$

$x_1 = 1 - \ln 1 = 1 \Rightarrow x_n < 1 \ \forall n \in \mathbb{N}$

$$x_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n$$

$$x_n = \sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\,dx$$

Let $f(x) = \frac{1}{x}$, $\int_1^n f(x)\,dx = \ln(n)$

$$x_n = f(1) + f(2) + \ldots + f(n) - \int_1^n f(x)\,dx$$

$$= f(1) - \int_1^2 f(x)\,dx + f(2) - \int_2^3 f(x)\,dx + \ldots + f(n-1) - \int_{n-1}^n f(x)\,dx + f(n)$$

$$= \underbrace{\int_1^2 [f(1) - f(x)]\,dx}_{\ge 0} + \underbrace{\int_2^3 [f(2) - f(x)]\,dx}_{\ge 0} + \ldots + \underbrace{\int_{n-1}^n [f(n-1) - f(x)]\,dx}_{\ge 0} + f(n) \ge f(n), \ \forall n \ge 1$$

$\Rightarrow x_n \ge f(n) \ \forall n \ge 1 \Rightarrow (x_n)$ bounded

$f(1) \ge (x_n) \ge f(n)$

8. Find the limit for:

$$\lim_{n\to\infty} \frac{n^n}{1+2^2+3^3+\dots+n^n} \overset{S-C}{=\!=} \lim_{n\to\infty} \frac{(n+1)^{n+1}-n^n}{\sum_{k=1}^{n+1} k^k - \sum_{k=1}^{n} k^k} = \lim_{n\to\infty} \frac{(n+1)^{n+1}-n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} 1-\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} =$$

$$= 1 - \lim_{n\to\infty} \underbrace{\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]}_{=e}^{\frac{-1}{n+1}\cdot n} \cdot \frac{1}{n+1} \to 0 = 1-e^{\lim_{n\to\infty}\frac{-n}{n+1}} \cdot 0 = 1-e^{-1}\cdot 0 = 1$$

10.
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) = \frac{1}{2}x_n - \frac{a}{2x_n} = \frac{1}{2x_n}\left(x_n^2 - a\right)$$

$$x_n^2 - a = \frac{1}{4}\left(x_{n-1} + \frac{a}{x_{n-1}}\right)^2 - a = \frac{1}{4}\left(x_{n-1}^2 + 2a + \frac{a^2}{x_{n-1}^2}\right) - a = \frac{x_{n-1}^2}{4} - \frac{a}{2} + \frac{a^2}{4x_{n-1}^2} = \frac{1}{4}\underbrace{\left(x_{n-1} - \frac{a}{x_{n-1}}\right)^2}_{\geq 0} \geq 0$$

$\Rightarrow x_n - x_{n+1} \geq 0 \Rightarrow x_n \geq x_{n+1} \Rightarrow$ the sequence is bounded from below

$x_n^2 - a \geq 0 \Rightarrow x_n^2 \geq a$

$\Rightarrow (x_n)$ is monotone and bounded $\Rightarrow$ the sequence converges

$$l = \lim_{n\to\infty} x_n$$

$$l = \frac{1}{2}\left(l + \frac{a}{l}\right)$$

$$\cancel{\tfrac{1}{2}} l = \frac{a}{\cancel{2} l} \quad \Big|\cdot l$$

$$l^2 = a \Rightarrow l = \sqrt{a}$$

Seminar 2

1. Prove using the ε-definition that $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and ★ $\lim\limits_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$.

2. Study if the sequence (x_n) is bounded, monotone, and convergent, for each of the following:

   (a) $x_n = \sqrt{n+1} - \sqrt{n}$.

   (b) $x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$.

   (c) ★$x_n = \frac{\sin(n)}{n}$.

   (d) $x_n = \frac{2^n}{n!}$.

3. Find the limit for each of the following sequences:

   (a) $\sqrt{n}\big(\sqrt{n+1} - \sqrt{n}\big)$.

   (b) $\big(a_1^n + a_2^n + \ldots + a_k^n\big)^{\frac{1}{n}}$, with $a_i > 0$.

   (c) $\sqrt[n]{n}$.

   (d) ★ $\sqrt[n]{1+2+\ldots+n}$. $= \left(\frac{n(n+1)}{2}\right)^{\frac{1}{n}} = \underset{\downarrow\, 1}{\left(\frac{1}{2}\right)^{\frac{1}{n}}} \cdot \left[n(n+1)\right]^{\frac{1}{n}}$

   (e) $\frac{2^n + (-1)^n}{3^n}$.

   (f) $\frac{(an+1)^2}{4n^2-2n+1}$, $a \in \mathbb{R}$.

4. Consider the sequence (e_n) given by

$$e_n = \big(1 + \frac{1}{n}\big)^n.$$

   Prove that (e_n) is increasing and bounded – its limit is denoted by e (Euler's number).

5. Find the limit for each of the following sequences:

   (a) $\big(\frac{2n+1}{2n-1}\big)^n$.

   (b) $n\big(\ln(n+2) - \ln(n+1)\big)$.

   (c) ★$\big(\frac{\ln(n+1)}{\ln n}\big)^n$.

6. ★ Prove that the sequence (x_n) given by $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n$ is decreasing and bounded, hence convergent – its limit is denoted by γ (Euler's constant).

7. (Stolz-Cesàro lemma) Let $(a_n), (b_n)$ be two sequences such that (i) $a_n \to 0$ and $b_n \to 0$ with (b_n) decreasing; or (ii) $b_n \to \infty$ with (b_n) increasing.

   If

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell, \quad \text{then} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \ell.$$

   (a) Let (x_n) be a convergent sequence. What can you say about the sequence (a_n) of averages

$$a_n = \frac{x_1 + x_2 + \ldots + x_n}{n}?$$

   Give an example where the averages converge, even though the sequence does not.

   (b) Compare $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ with n and $\ln n$ by taking the ratio, respectively.

   (c) Let (x_n) be a sequence such that $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell$. Prove that $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = \ell$.

1. Prove using the ε-definition that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and ★ $\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$.

$x_n \to \ell : \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}$ s.t. $|x_n - \ell| < \varepsilon, \forall n \geq N_\varepsilon$

(a) Let $\varepsilon > 0$, we want to find the index $N_\varepsilon \in \mathbb{N}$ s.t. $\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon \; \forall n \geq N_\varepsilon$

$\frac{1}{\sqrt{n}} < \varepsilon \iff 1 < \varepsilon\sqrt{n} \iff \frac{1}{\varepsilon} < \sqrt{n} \; |()^2$

$n > \frac{1}{\varepsilon^2}$

Take $N_\varepsilon = \left[\frac{1}{\varepsilon^2}\right] + 1 > \frac{1}{\varepsilon^2} \Rightarrow n \geq N_\varepsilon > \frac{1}{\varepsilon^2} \Rightarrow n > \frac{1}{\varepsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon \Rightarrow \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

(b) Let $\varepsilon > 0$, we want to find $N_\varepsilon \in \mathbb{N}$ s.t. $\left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon \; \forall n \geq N_\varepsilon$

$\frac{{}^{2)}n+1}{2n+3} - \frac{{}^{2n+3)}1}{2} < \varepsilon$

$\frac{\cancel{2n}+1 - \cancel{2n} - 3}{4n+6} < \varepsilon$

$\frac{-\cancel{2}}{\cancel{2}(2n+3)} < \varepsilon$

$-\frac{1}{2n+3} < \varepsilon \; | \cdot (2n+3)$

$-1 < (2n+3)\varepsilon$

$-1 < 2n\varepsilon + 3\varepsilon$

$2n\varepsilon > 3\varepsilon + 1$

$n > \frac{3\varepsilon + 1}{2\varepsilon}, \quad \varepsilon \neq 0$

$n > \frac{3}{2} + \frac{1}{2\varepsilon}$

Take $N_\varepsilon = \left[\frac{3}{2} + \frac{1}{2\varepsilon}\right] + 1 > \frac{3}{2} + \frac{1}{2\varepsilon} \Rightarrow n \geq \left[\frac{3}{2} + \frac{1}{2\varepsilon}\right] + 1 > \frac{3}{2} + \frac{1}{2\varepsilon}$

2. Study if the sequence (x_n) is bounded, monotone, and convergent, for each of the following:

(a) $x_n = \sqrt{n+1} - \sqrt{n}$.

(b) $x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)}$.

(c) ★$x_n = \frac{\sin(n)}{n}$.

(d) $x_n = \frac{2^n}{n!}$.

(a) $x_n = \overset{\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1} - \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \in (0, 1]$

$n = 0 \Rightarrow x_0 = \sqrt{0+1} - \sqrt{0} = 1$

$n = 1 \Rightarrow x_1 = \sqrt{2} - 1$

$x_{n+1} \overset{?}{<} x_n$

$\sqrt{n+2} - \sqrt{n+1} < \sqrt{n+1} - \sqrt{n}$

$\frac{1}{\sqrt{n+1} + \sqrt{n+2}} < \frac{1}{\sqrt{n} + \sqrt{n+1}} \; |()^{-1}$

$$\cancel{\sqrt{n+1}} + \sqrt{n+2} > \sqrt{n} + \cancel{\sqrt{n+1}}$$

$$\sqrt{n+2} > \sqrt{n} \quad |(\,)^2$$

$$n+2 > n \Rightarrow \text{true } \forall n \in \mathbb{N}$$

$\Rightarrow x_n$ is bounded and decreasing $\xrightarrow{\text{Weierstrass}}$ x_n - converges to 0 as $n \to \infty$

(b) $x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} \overset{n+1}{=} 1 - \frac{1}{n+1} = \frac{n}{n+1} \in (0,1)$, $n \geq 1$

$x_{n+1} - x_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0 \Rightarrow x_{n+1} > x_n \Rightarrow x_n$ increasing & bounded $\Rightarrow x_n$ convergent

(c) $x_n = \frac{\sin n}{n}$

$$-1 \leq \sin n \leq 1 \quad |\cdot \tfrac{1}{n} > 0$$

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \Rightarrow x_n \in \left[-\frac{1}{n}, \frac{1}{n}\right] \to \text{bounded}$$

$\left.\begin{array}{l} \lim\limits_{n\to\infty} \frac{-1}{n} = 0 \\ \lim\limits_{n\to\infty} \frac{1}{n} = 0 \end{array}\right\} \Rightarrow \lim\limits_{n\to\infty} \frac{\sin n}{n} = 0 \Rightarrow$ convergent even if it is NOT monotone

(d) $x_n = \frac{2^n}{n!}$

$\frac{x_{n+1}}{x_n} = \frac{\cancel{2^{n+1}}^{\,2}}{\cancel{(n+1)!}_{\,n+1}} \cdot \frac{\cancel{n!}}{\cancel{2^n}} = \frac{2}{n+1} < 1$ for $n > 1 \Rightarrow x_n$ decreasing $\forall n > 1$

$x_0 = 1$
$x_1 = 2$ $\Rightarrow x_n \leq 2$

$\Rightarrow x_n$ convergent
$x_n \to 0$

3. Find the limit for each of the following sequences:

(a) $\sqrt{n}(\sqrt{n+1} - \sqrt{n})$.

(b) $\left(a_1^n + a_2^n + \ldots + a_k^n\right)^{\frac{1}{n}}$, with $a_i > 0$.

(c) $\sqrt[n]{n}$.

(d) ★ $\sqrt[n]{1 + 2 + \ldots + n}$.

(e) $\frac{2^n + (-1)^n}{3^n}$.

(f) $\frac{(an+1)^2}{4n^2 - 2n + 1}$, $a \in \mathbb{R}$.

(a) $\lim\limits_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)^{\sqrt{n+1}+\sqrt{n}} = (\infty \cdot 0) = \lim\limits_{n\to\infty} \sqrt{n} \cdot \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}} = \left(\frac{\infty}{\infty}\right) = \lim\limits_{n\to\infty} \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(\sqrt{1+\underbrace{\left(\frac{1}{n}\right)}_{0}} + 1\right)} = \frac{1}{2}$

(b) $\lim\limits_{n\to\infty} \left(a_1^n + a_2^n + \ldots + a_k^n\right)^{\frac{1}{n}}$

Let $a_{max} = \max\{a_1, a_2, \ldots, a_k\}$

$$\lim_{n\to\infty} \left[a_{max}^n \underbrace{\left(\left(\frac{a_1}{a_{max}}\right)^n + \left(\frac{a_2}{a_{max}}\right)^n + \ldots + \left(\frac{a_k}{a_{max}}\right)^n\right)}_{\epsilon}\right]^{\frac{1}{n}} = \lim_{n\to\infty} a_{max} \cdot \epsilon^{\frac{1}{n}} = a_{max} \cdot \epsilon^0 = a_{max}$$

(c) $\lim\limits_{n\to\infty} \sqrt[n]{n} = \lim\limits_{n\to\infty} n^{\frac{1}{n}} = e^{\lim\limits_{n\to\infty} \frac{1}{n} \ln n} = e^{\lim\limits_{n\to\infty} \frac{\ln n}{n}} = e^{\lim \frac{\frac{1}{n}}{1}} = e^{\frac{1}{n}} = e^0 = 1$

(d) $\lim_{n\to\infty} \sqrt[n]{1+2+\dots+n} = \lim_{n\to\infty} \left(\frac{n(n+1)}{2}\right)^{\frac{1}{n}}$

Stoltz - Cesaro theorem: if $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = l$

$$\frac{x_{n+1}}{x_n} = \frac{\cancel{(n+1)}(n+2)}{\cancel{2}} \cdot \frac{\cancel{2}}{n\cancel{(n+1)}} = \frac{n+2}{n} = 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{\frac{n(n+1)}{2}} = 1$$

(e) $$\lim_{n\to\infty} \frac{2^n + (-1)^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n + \left(\frac{-1}{3}\right)^n = 0$$

(f) $$\lim_{n\to\infty} \frac{(an+1)^2}{4n^2-2n+1} = \lim_{n\to\infty} \frac{a^2n^2 + 2an + 1}{4n^2 - 2n + 1} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(a^2 + \frac{2a}{n} + \frac{1}{n^2}\right)}{\cancel{n^2}\left(4 - \frac{2}{n} + \frac{1}{n^2}\right)} = \frac{a^2}{4}$$

5. Find the limit for each of the following sequences:

(a) $\left(\frac{2n+1}{2n-1}\right)^n$. (b) $n\big(\ln(n+2) - \ln(n+1)\big)$. (c) ★$\left(\frac{\ln(n+1)}{\ln n}\right)^n$.

(a) $$\lim_{n\to\infty} \left(\frac{2n+1}{2n-1}\right)^n = (1^\infty) = \lim_{n\to\infty} \left[\left(1 + \frac{2}{2n-1}\right)^{\frac{2n-1}{2}}\right]^{\frac{2n}{2n-1}} = e$$

(b) $$\lim_{n\to\infty} n\big(\ln(n+2) - \ln(n+1)\big) = \lim_{n\to\infty} n \cdot \ln\frac{n+2}{n+1} = \ln \lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right)^{(n+1)\cdot\frac{n}{n+1}} = \ln e = 1$$

(c) $$\lim_{n\to\infty} \left(\frac{\ln(n+1)}{\ln(n)}\right)^n = \lim_{n\to\infty} \left(\frac{\ln n + \ln(1+\frac{1}{n})}{\ln n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{\ln(1+\frac{1}{n})}{\ln n}\right)^{\frac{\ln n}{\ln(1+\frac{1}{n})} \cdot \frac{\ln(1+\frac{1}{n})}{\ln n} \cdot n} =$$

$$= e^{\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})^n}{\ln n}} = e^{\lim_{n\to\infty} \frac{1}{\ln n}} = e^0 = 1$$

6. ★ Prove that the sequence (x_n) given by $x_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n$ is decreasing and bounded, hence convergent – its limit is denoted by γ (Euler's constant).

x_n decreasing (1) $\Leftrightarrow x_{n+1} - x_n < 0 \Rightarrow \cancel{1} + \cancel{\frac{1}{2}} + \dots + \cancel{\frac{1}{n}} + \frac{1}{n+1} - \ln(n+1) - \left(\cancel{1} + \cancel{\frac{1}{2}} + \dots + \cancel{\frac{1}{n}} - \ln n\right) =$

$$= \frac{1}{n+1} - \ln(n+1) + \ln n = \frac{1}{n+1} + \ln\frac{n}{n+1} = \frac{1}{n+1} + \underbrace{\ln\left(1 + \frac{-1}{n+1}\right)}_{<0} < 0$$

$x_n \le x_1 \ \forall n \in \mathbb{N}$

$x_1 = 1 - \ln 1 = 1 \Rightarrow x_n \le 1$

Let $f(x) = \frac{1}{x} \Rightarrow \int_1^n f(x)\,dx = \ln n$

$$x_n = f(1) + f(2) + \dots + f(n) - \int_1^n f(x)\,dx =$$

$$= \underbrace{f(1) - \int_1^2 f(x)\,dx}_{\ge 0} + \underbrace{f(2) - \int_2^3 f(x)\,dx}_{\ge 0} + \dots + \underbrace{f(n-1) - \int_{n-1}^n f(x)\,dx}_{\ge 0} + f(n) \ge f(n) \quad \forall n \ge 1$$

$\Rightarrow x_n \ge f(n) \Rightarrow (x_n)$ bounded

$f(1) \ge (x_n) \ge f(n)$ (2) $\xRightarrow{(1),(2)}$ Weierstrass convergent

7. (Stolz-Cesàro lemma) Let $(a_n), (b_n)$ be two sequences such that (i) $a_n \to 0$ and $b_n \to 0$ with (b_n) decreasing; or (ii) $b_n \to \infty$ with (b_n) increasing.

If

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \ell, \quad \text{then} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \ell.$$

(a) Let (x_n) be a convergent sequence. What can you say about the sequence (a_n) of averages

$$a_n = \frac{x_1 + x_2 + \ldots + x_n}{n}?$$

Give an example where the averages converge, even though the sequence does not.

(b) Compare $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ with n and $\ln n$ by taking the ratio, respectively.

(c) Let (x_n) be a sequence such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell$. Prove that $\lim_{n\to\infty} \sqrt[n]{x_n} = \ell$.

(a) $(x_n) \to x$

$$\lim_{n\to\infty} \frac{x_1+x_2+\ldots+x_n}{n} \overset{S-C}{=\!=} \lim_{n\to\infty} \frac{\cancel{x_1}+\cancel{x_2}+\ldots+\cancel{x_n}+x_{n+1}-\cancel{x_1}-\cancel{x_2}-\ldots-\cancel{x_n}}{\cancel{n}+1-\cancel{n}} = \lim_{n\to\infty} x_n = x$$

(b)
$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{1} = 0 \to \text{increases slower than } n$$

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\ldots+\frac{1}{n}}{\ln n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\ln\frac{n+1}{n}} = \lim_{n\to\infty} \frac{1}{\ln\left(\frac{n+1}{n}\right)^{n+1}} = 1 \to \text{increases the same}$$

(c)
$$\lim_{n\to\infty} \sqrt[n]{x_n} = \ell = e^{\lim\limits_{n\to\infty} \frac{\ln x_n}{n}} = e^{\lim\limits_{n\to\infty} \ln\frac{x_{n+1}}{x_n}} = e^{\ln \ell} = \ell$$

$$\sqrt[n]{x_n} = x_n^{\frac{1}{n}} = e^{\frac{1}{n}\ln x_n}$$

8. Find the limit for each of the following sequences:

(a) $\frac{n}{\sqrt[n]{n!}}$.

(b) $\bigstar \frac{n^n}{1+2^2+3^3+\ldots+n^n}$.

(c) $\frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}}$, $p \in \mathbb{N}$. $= \frac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}} = 1$

(a)
$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty} \frac{1}{\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}} = \lim_{n\to\infty} \frac{1}{e^{\frac{(n+1)!}{n+1}} - e^{\frac{n!}{n}}} = \lim_{n\to\infty} \frac{1}{e^{\frac{n!}{n}}\left(e^{\frac{(n+1)!}{n+1} - \frac{n!}{n}} - 1\right)} = \lim_{n\to\infty} \frac{1}{e^{\frac{n!}{n}}\left(e^{(n-1)!\,(n-1)} - 1\right)} = \lim_{n\to\infty} \frac{1}{e^{\frac{n!}{n}}\left(e^{(n-1)!\,(n-1)} - 1\right)} \longrightarrow 0$$

$$\sqrt[n]{n!} = e^{\frac{n!}{n}}$$

if $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \ell \Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{x_n} = \ell$

(b)
$$\lim_{n\to\infty} \frac{n^n}{1+2^2+3^3+\ldots+n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1} - n^n}{\sum\limits_{k=1}^{n+1} k^k - \sum\limits_{k=1}^{n} k^k} = \lim_{n\to\infty} \frac{(n+1)^{n+1} - n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} 1 - \left(\frac{n}{n+1}\right)^n \overset{\to 0}{\left(\frac{1}{n+1}\right)} = 1 - \lim_{n\to\infty} \left[\left(1 + \frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-n}{n+1}} =$$

$$= 1 - e^{\lim\limits_{n\to\infty} \frac{-1}{n+1}} \cdot 0 = 1 - \frac{0}{e} = 1$$

8. Find the limit for each of the following sequences:

   (a) $\frac{n}{\sqrt[n]{n!}}$.
   (b) ★ $\frac{n^n}{1+2^2+3^3+\ldots+n^n}$.
   (c) $\frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}}$, $p \in \mathbb{N}$.

9. (Banach fixed point theorem) Let $f : [a, b] \to [a, b]$ be a contraction, meaning that there exists $\alpha \in (0, 1)$ such that

$$|f(x) - f(y)| \leq \alpha \, |x - y| \,, \quad \forall x, y \in [a, b].$$

   Let an arbitrary $x_1 \in [a, b]$ and consider the sequence (x_n) given by

$$x_{n+1} = f(x_n), \quad \forall n \in \mathbb{N}.$$

   Prove that the sequence (x_n) is Cauchy and that its limit x^* is a fixed point, i.e. $f(x^*) = x^*$.

10. Study the convergence and find the limit of the following sequences:

   (a) $x_{n+1} = \sqrt{2 + x_n}$, $x_1 = 0$.
   (b) ★$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$, $x_1 = 1$ and $a > 1$.

Homework questions are marked with ★.
Solutions should be handed in at the beginning of next week's lecture.

10. Study the convergence and find the limit of the following sequences:

(a) $x_{n+1} = \sqrt{2 + x_n}$, $x_1 = 0$.

(b) ★$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$, $x_1 = 1$ and $a > 1$.

(a) $x_{n+1} = \sqrt{2+x_n} \qquad x_1 = 0$

$x_{n+1} - x_n = \sqrt{2+x_n} - x_n > 0 \Rightarrow$ increasing

$x_2 = \sqrt{2}$

$x_3 = \sqrt{2+\sqrt{2}}$

$x_4 = \sqrt{2+\sqrt{2+\sqrt{2}}}$

$l = \lim_{n \to \infty} x_n$

$l = \sqrt{2+l} \quad |(\ )^2$

$l^2 = 2 + l$

$l^2 - l - 2 = 0 \Rightarrow \begin{cases} l_1 = -1 < 0 \Rightarrow \text{false} \\ l_2 = 2 \end{cases}$

(b) $l = \frac{1}{2}(l + \frac{a}{l}) \quad |\cdot l$

$l^2 = \frac{1}{2}(l^2 + a)$

$\cancel{\frac{1}{2}} l^2 = \cancel{\frac{1}{2}} a$

$l = \sqrt{a}$