Seminar 1

1. Which ones of the usual symbols of addition, subtraction, multiplication and division define an operation (composition law) on the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$?

2. Let $A = \{a_1, a_2, a_3\}$. Determine the number of:
(i) operations on A;
(ii) commutative operations on A;
(iii) operations on A with identity element.
Generalization for a set A with n elements ($n \in \mathbb{N}^*$).

3. Decide which ones of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are groups together with the usual addition or multiplication.

4. Let "$*$" be the operation defined on $\mathbb{R}$ by $x * y = x + y + xy$. Prove that:
(i) $(\mathbb{R}, *)$ is a commutative monoid.
(ii) The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

5. Let "$*$" be the operation defined on $\mathbb{N}$ by $x * y = \text{g.c.d.}(x, y)$.
(i) Prove that $(\mathbb{N}, *)$ is a commutative monoid.
(ii) Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ ($n \in \mathbb{N}^*$) is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid.
(iii) Fill in the table of the operation "$*$" on D_6.

6. Determine the finite stable subsets of $(\mathbb{Z}, \cdot)$.

7. Let $(G, \cdot)$ be a group. Show that:
(i) G is abelian $\Longleftrightarrow \forall x, y \in G$, $(xy)^2 = x^2y^2$.
(ii) If $x^2 = 1$ for every $x \in G$, then G is abelian.

8. Let "$\cdot$" be an operation on a set A and let $X, Y \subseteq A$. Define an operation "$*$" on the power set $\mathcal{P}(A)$ by

$$X * Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

Prove that:
(i) If $(A, \cdot)$ is a monoid, then $(\mathcal{P}(A), *)$ is a monoid.
(ii) If $(A, \cdot)$ is a group, then in general $(\mathcal{P}(A), *)$ is not a group.

Seminar 2

1. Let r, s, t, v be the homogeneous relations defined on the set $M = \{2, 3, 4, 5, 6\}$ by

$$x \, r \, y \Longleftrightarrow x < y$$

$$x \, s \, y \Longleftrightarrow x | y$$

$$x \, t \, y \Longleftrightarrow g.c.d.(x, y) = 1$$

$$x \, v \, y \Longleftrightarrow x \equiv y \pmod 3 .$$

Write the graphs R, S, T, V of the given relations.

2. Let A and B be sets with n and m elements respectively $(m, n \in \mathbb{N}^*)$. Determine the number of:
(i) relations having the domain A and the codomain B;
(ii) homogeneous relations on A.

3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

4. Which ones of the properties of reflexivity, transitivity and symmetry hold for the following homogeneous relations: the strict inequality relations on $\mathbb{R}$, the divisibility relation on $\mathbb{N}$ and on $\mathbb{Z}$, the perpendicularity relation of lines in space, the parallelism relation of lines in space, the congruence of triangles in a plane, the similarity of triangles in a plane?

5. Let $M = \{1, 2, 3, 4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$, $R_2 = \Delta_M \cup \{(1, 2), (1, 3)\}$, $\pi_1 = \{\{1\}, \{2\}, \{3, 4\}\}$, $\pi_2 = \{\{1\}, \{1, 2\}, \{3, 4\}\}$.
(i) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.
(ii) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

6. Define on $\mathbb{C}$ the relations r and s by:

$$z_1 \, r \, z_2 \Longleftrightarrow |z_1| = |z_2| ; \qquad z_1 \, s \, z_2 \Longleftrightarrow arg \, z_1 = arg \, z_2 \text{ or } z_1 = z_2 = 0 .$$

Prove that r and s are equivalence relations on $\mathbb{C}$ and determine the quotient sets (partitions) $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometric interpretation).

7. Let $n \in \mathbb{N}$. Consider the relation ρ_n on $\mathbb{Z}$, called the *congruence modulo* n, defined by:

$$x \, \rho_n \, y \Longleftrightarrow n | (x - y) .$$

Prove that ρ_n is an equivalence relation on $\mathbb{Z}$ and determine the quotient set (partition) $\mathbb{Z}/\rho_n$. Discuss the cases $n = 0$ and $n = 1$.

8. Determine all equivalence relations and all partitions on the set $M = \{1, 2, 3\}$.

9. Let $M = \{0, 1, 2, 3\}$ and let $h = (\mathbb{Z}, M, H)$ be a relation, where

$$H = \{(x, y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : x = 4z + y\} .$$

Is h a function?

10. Consider the following homogeneous relations on $\mathbb{N}$, defined by:

$$m \, r \, n \Longleftrightarrow \exists a \in \mathbb{N} : m = 2^a n ,$$

$$m \, s \, n \Longleftrightarrow (m = n \text{ or } m = n^2 \text{ or } n = m^2) .$$

Are r and s equivalence relations?

Seminar 3

1. Let M be a non-empty set and let $S_M = \{f : M \to M \mid f \text{ is bijective}\}$. Show that $(S_M, \circ)$ is a group, called the *symmetric group* of M.

2. Let M be a non-empty set and let $(R, +, \cdot)$ be a ring. Define on $R^M = \{f \mid f : M \to R\}$ two operations by: $\forall f, g \in R^M$,

$$f + g : M \to R\,, \quad (f+g)(x) = f(x) + g(x)\,, \quad \forall x \in M\,,$$

$$f \cdot g : M \to R\,, \quad (f \cdot g)(x) = f(x) \cdot g(x)\,, \quad \forall x \in M\,.$$

Show that $(R^M, +, \cdot)$ is a ring. If R is commutative or has identity, does R^M have the same property?

3. Prove that $H = \{z \in \mathbb{C} \mid |z| = 1\}$ is a subgroup of $(\mathbb{C}^*, \cdot)$, but not of $(\mathbb{C}, +)$.

4. Let $U_n = \{z \in \mathbb{C} \mid z^n = 1\} \quad (n \in \mathbb{N}^*)$ be the *set of n-th roots of unity*. Prove that U_n is a subgroup of $(\mathbb{C}^*, \cdot)$.

5. Let $n \in \mathbb{N}$, $n \geq 2$. Prove that:
(i) $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) \neq 0\}$ is a stable subset of the monoid $(M_n(\mathbb{C}), \cdot)$;
(ii) $(GL_n(\mathbb{C}), \cdot)$ is a group, called the *general linear group of rank n*;
(iii) $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) = 1\}$ is a subgroup of the group $(GL_n(\mathbb{C}), \cdot)$.

6. Show that the following sets are subrings of the corresponding rings:
(i) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ in $(\mathbb{C}, +, \cdot)$.
(ii) $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$ in $(M_2(\mathbb{R}), +, \cdot)$.

7. (i) Let $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = |z|$. Show that f is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

(ii) Let $g : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $g(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that g is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}), \cdot)$.

8. Let $n \in \mathbb{N}$, $n \geq 2$. Prove that the groups $(\mathbb{Z}_n, +)$ of residue classes modulo n and $(U_n, \cdot)$ of n-th roots of unity are isomorphic.

9. Let $n \in \mathbb{N}$, $n \geq 2$. Consider the ring $(\mathbb{Z}_n, +, \cdot)$ and let $\widehat{a} \in \mathbb{Z}_n^*$.
(i) Prove that $\widehat{a}$ is invertible $\Longleftrightarrow (a, n) = 1$.
(ii) Deduce that $(\mathbb{Z}_n, +, \cdot)$ is a field $\Longleftrightarrow n$ is prime.

10. Let $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$. Show that $(\mathcal{M}, +, \cdot)$ is a field isomorphic to $(\mathbb{C}, +, \cdot)$.

Seminar 4

1. Let K be a field. Show that $K[X]$ is a K-vector space, where the addition is the usual addition of polynomials and the scalar multiplication is defined as follows: $\forall\ k \in K$, $\forall f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$,

$$k \cdot f = (ka_0) + (ka_1)X + \cdots + (ka_n)X^n\,.$$

2. Let K be a field and $m, n \in \mathbb{N}$, $m, n \geq 2$. Show that $M_{m,n}(K)$ is a K-vector space, with the usual addition and scalar multiplication of matrices.

3. Let K be a field, $A \neq \emptyset$ and denote $K^A = \{f \mid f : A \to K\}$. Show that K^A is a K-vector space, where the addition and the scalar multiplication are defined as follows: $\forall f, g \in K^A$, $\forall k \in K$, $f + g \in K^A$, $kf \in K^A$,

$$(f + g)(x) = f(x) + g(x)\,, \quad (k \cdot f)(x) = k \cdot f(x)\,, \forall x \in A\,.$$

4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define the operations: $x \perp y = xy$ and $k \top x = x^k$, $\forall k \in \mathbb{R}$ and $\forall x, y \in V$. Prove that V is a vector space over $\mathbb{R}$.

5. Let K be a field and let $V = K \times K$. Decide whether V is a K-vector space with respect to the following addition and scalar multiplication:
(i) $(x_1, y_1)+(x_2, y_2) = (x_1+x_2, y_1+2y_2)$ and $k\cdot(x_1, y_1) = (kx_1, ky_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.
(ii) $(x_1, y_1)+(x_2, y_2) = (x_1+x_2, y_1+y_2)$ and $k\cdot(x_1, y_1) = (kx_1, y_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

6. Let p be a prime number and let V be a vector space over the field $\mathbb{Z}_p$.
(i) Prove that $\underbrace{x + \cdots + x}_{p \text{ times}} = 0$, $\forall x \in V$.
(ii) Is there a scalar multiplication endowing $(\mathbb{Z}, +)$ with a structure of a vector space over $\mathbb{Z}_p$?

7. Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;
(iv) $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(v) $E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;
(vi) $F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$?

8. Which ones of the following sets are subspaces:
(i) $[-1, 1]$ of the real vector space $\mathbb{R}$;
(ii) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ of the real vector space $\mathbb{R}^2$;
(iii) $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Q} \right\}$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;
(iv) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

9. Which ones of the following sets are subspaces of the K-vector space $K[X]$:
(i) $K_n[X] = \{f \in K[X] \mid \text{degree}(f) \leq n\}$ $(n \in \mathbb{N})$;
(ii) $K'_n[X] = \{f \in K[X] \mid \text{degree}(f) = n\}$ $(n \in \mathbb{N})$.

10. Show that the set of all solutions of a homogeneous system of two equations and two unknowns with real coefficients is a subspace of the real vector space $\mathbb{R}^2$.

Seminar 5

1. Determine the following generated subspaces:
(i) $< 1, X, X^2 >$ in the real vector space $\mathbb{R}[X]$.
(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in the real vector space $M_2(\mathbb{R})$.

2. Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces with a minimal number of generators.

3. Consider the following vectors in the real vector space $\mathbb{R}^3$:

$$a = (-2, 1, 3), b = (3, -2, -1), c = (1, -1, 2), d = (-5, 3, 4), e = (-9, 5, 10).$$

Show that $\langle a, b \rangle = \langle c, d, e \rangle$.

4. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that S and T are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

5. Let S and T be the set of all even functions and of all odd functions in $\mathbb{R}^{\mathbb{R}}$ respectively. Prove that S and T are subspaces of the real vector space $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{R}} = S \oplus T$.

6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x + y, x - y),$$
$$g(x, y) = (2x - y, 4x - 2y),$$
$$h(x, y, z) = (x - y, y - z, z - x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

7. Which ones of the following functions are endomorphisms of the real vector space $\mathbb{R}^2$:
(i) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (ax + by, cx + dy)$, where $a, b, c, d \in \mathbb{R}$;
(ii) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (a + x, b + y)$, where $a, b \in \mathbb{R}$?

8. Let $a \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x \cos a - y \sin a, x \sin a + y \cos a)\,.$$

Prove that $f \in End_{\mathbb{R}}(\mathbb{R}^2)$.

9. Determine the kernel and the image of the endomorphisms from Exercise **6.**

10. Let V be a vector space over K and $f \in End_K(V)$. Show that the set

$$S = \{x \in V \mid f(x) = x\}$$

of fixed points of f is a subspace of V.

Seminar 6

1. Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:
(i) v_1, v_2, v_3 are linearly dependent and determine a dependence relationship.
(ii) v_1, v_2 are linearly independent.

2. Prove that the following vectors are linearly independent:
(i) $v_1 = (1, 0, 2)$, $v_2 = (-1, 2, 1)$, $v_3 = (3, 1, 1)$ in ${}_{\mathbb{R}}\mathbb{R}^3$.
(ii) $v_1 = (1, 2, 3, 4)$, $v_2 = (2, 3, 4, 1)$, $v_3 = (3, 4, 1, 2)$, $v_4 = (4, 1, 2, 3)$ in ${}_{\mathbb{R}}\mathbb{R}^4$.

3. Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly independent.

4. Let $v_1 = (1, -2, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (0, a, 1, 2)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^4$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly dependent.

5. Let $v_1 = (1, 1, 0)$, $v_2 = (-1, 0, 2)$, $v_3 = (1, 1, 1)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$.
(i) Show that the list (v_1, v_2, v_3) is a basis of the real vector space $\mathbb{R}^3$.
(ii) Express the vectors of the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$ as a linear combination of the vectors v_1, v_2 and v_3.
(iii) Determine the coordinates of $u = (1, -1, 2)$ in each of the two bases.

6. Let $n \in \mathbb{N}^*$. Show that the vectors

$$v_1 = (1, \dots, 1, 1), v_2 = (1, \dots, 1, 2), v_3 = (1, \dots, 1, 2, 3), \dots, v_n = (1, 2, \dots, n-1, n)$$

form a basis of the real vector space $\mathbb{R}^n$ and write the coordinates of a vector $(x_1, \dots, x_n)$ in this basis.

7. Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the lists (E_1, E_2, E_3, E_4) and (A_1, A_2, A_3, A_4) are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in each of the two bases.

8. Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X - a, (X - a)^2)$ $(a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial $f = a_0 + a_1 X + a_2 X^2 \in \mathbb{R}_2[X]$ in each basis.

9. Determine the number of bases of the vector space $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$.

10. Determine the number of elements of the general linear group $(GL_3(\mathbb{Z}_2), \cdot)$ of invertible 3×3-matrices over $\mathbb{Z}_2$.

Seminar 7

1. Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

2. Let K be a field and $S = \{(x_1, \dots, x_n) \in K^n \mid x_1 + \cdots + x_n = 0\}$.
(i) Prove that S is a subspace of the canonical vector space K^n over K.
(ii) Determine a basis and the dimension of S.

3. Determine a basis and the dimensions of the vector spaces $\mathbb{C}$ over $\mathbb{C}$ and $\mathbb{C}$ over $\mathbb{R}$. Prove that the set $\{1, i\}$ is linearly dependent in the vector space $\mathbb{C}$ over $\mathbb{C}$ and linearly independent in the vector space $\mathbb{C}$ over $\mathbb{R}$.

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (y, -x)$. Prove that f is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

5. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x, y, z) = (-y + 5z, x, y - 5z)$. Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

6. Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

7. Determine a complement for the following subspaces:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ in the real vector space $\mathbb{R}^3$;
(ii) $B = \{aX + bX^3 \mid a, b \in \mathbb{R}\}$ in the real vector space $\mathbb{R}_3[X]$.

8. Let V be a vector space over K and let S, T and U be subspaces of V such that $dim(S \cap U) = dim(T \cap U)$ and $dim(S + U) = dim(T + U)$. Prove that if $S \subseteq T$, then $S = T$.

9. Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\},$$

$$T =< (0, 1, 1), (1, 1, 0) >$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

10. Determine the dimensions of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $M_2(\mathbb{R})$, where

$$S = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle, \qquad T = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle.$$

Seminar 8

1. Let $A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Show that A is invertible, determine A^{-1} and solve the linear system $AX = B$.

2. Using the Kronecker-Capelli theorem, decide if the following linear systems are compatible and then solve the compatible ones:

$$(i) \begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases} \qquad (ii) \begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

$$(iii) \begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$$

3. Using the Rouché theorem, decide if the systems from **2.** are compatible and then solve the compatible ones.

4. Decide when the following linear system is compatible determinate and in that case solve it by using Cramer's method:

$$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases} \quad (a, b, c \in \mathbb{R}).$$

Solve the following linear systems by the Gauss and Gauss-Jordan methods:

$$\textbf{5}. \quad (i) \begin{cases} 2x + 2y + 3z = 3 \\ x - y \quad = 1 \\ -x + 2y + z = 2 \end{cases} \qquad (ii) \begin{cases} 2x + 5y + z = 7 \\ x + 2y - z = 3 \\ x + y - 4z = 2 \end{cases} \qquad (iii) \begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$$

$$\textbf{6}. \quad \begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases} \qquad (\lambda \in \mathbb{R})$$

$$\textbf{7}. \quad \begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases} \qquad (a \in \mathbb{R})$$

8. Determine the positive solutions of the following non-linear system:

$$\begin{cases} xyz = 1 \\ x^3y^2z^2 = 27 \\ \frac{z}{xy} = 81 \end{cases}$$

Seminar 9

Compute by applying elementary operations the ranks of the matrices:

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Compute by applying elementary operations the inverses of the matrices:

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

6. Let K be a field, let $B = (e_1, e_2, e_3, e_4)$ be a basis and let $X = (v_1, v_2, v_3)$ be a list in the canonical K-vector space K^4, where

$$v_1 = 3e_1 + 2e_2 - 5e_3 + 4e_4\,,$$

$$v_2 = 3e_1 - e_2 + 3e_3 - 3e_4\,,$$

$$v_3 = 3e_1 + 5e_2 - 13e_3 + 11e_4\,.$$

Write the matrix of the list X in the basis B, determine an echelon form for it and deduce that X is linearly dependent.

For the following exercises, for a list X of vectors in a canonical vector space $\mathbb{R}^n$, use that dim $< X >$ is equal to the rank of an echelon form C of the matrix consisting of the components of the vectors of X, and a basis of $< X >$ is given by the non-zero rows of C.

7. In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1, 0, 4)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 5, -36)$ and $v_4 = (2, 10, -72)$. Determine *dim* $< X >$ and a basis of $< X >$.

8. In the real vector space $\mathbb{R}^4$ consider the list $X = (v_1, v_2, v_3)$, where $v_1 = (1, 0, 4, 3)$, $v_2 = (0, 2, 3, 1)$ and $v_3 = (0, 4, 6, 2)$. Determine *dim* $< X >$ and a basis of $< X >$.

9. Determine the dimension of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S = < (1, 0, 4), (2, 1, 0), (1, 1, -4) >\,,$$

$$T = < (-3, -2, 4), (5, 2, 4), (-2, 0, -8) > \ .$$

10. Determine the dimension of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^4$ and a basis for the first three of them, where

$$S = < (1, 2, -1, -2), (3, 1, 1, 1), (-1, 0, 1, -1) >\,,$$

$$T = < (2, 5, -6, -5), (-1, 2, -7, -3) > \ .$$

Seminar 10

1. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by

$$f(x, y, z) = (x + y, y - z, 2x + y + z).$$

Determine the matrix $[f]_E$, where $E = (e_1, e_2, e_3)$ is the canonical basis for $\mathbb{R}^3$.

2. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1, 1, 0), (0, 1, 1), (1, 0, 1))$ of $\mathbb{R}^3$, $B' = (v'_1, v'_2) = ((1, 1), (1, -2))$ of $\mathbb{R}^2$ and let $E' = (e'_1, e'_2)$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

3. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^4)$ be defined by

$$f(e_1) = (1, 2, 3, 4), f(e_2) = (4, 3, 2, 1), f(e_3) = (-2, 1, 4, 1)$$

on the elements of the canonical basis of $\mathbb{R}^3$. Determine:
(*i*) $f(v)$ for every $v \in \mathbb{R}^3$.
(*ii*) the matrix of f in the canonical bases.
(*iii*) a basis and the diemnsion of $Ker\, f$ and $Im\, f$.

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

(*i*) Show that $v = (1, 4, 1, -1) \in Ker\, f$ and $v' = (2, -2, 4, 2) \in Im\, f$.
(*ii*) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(*iii*) Define f.

5. Consider the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ and its bases $E = (1, X, X^2)$ and $B = (1, X - 1, X^2 + 1)$. Consider $\varphi \in End_{\mathbb{R}}(\mathbb{R}_2[X])$ defined by

$$\varphi(a_0 + a_1 X + a_2 X^2) = (a_0 + a_1) + (a_1 + a_2)X + (a_0 + a_2)X^2.$$

Determine the matrices $[\varphi]_E$ and $[\varphi]_B$.

6. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1, 2), (1, 3))$ and $B' = (v'_1, v'_2) = ((1, 0), (2, 1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f + g]_B$ and $[f \circ g]_{B'}$.

7. Consider the endomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$f(x, y) = (x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha) \quad (\alpha \in \mathbb{R}).$$

Write its matrix in the canonical basis of $\mathbb{R}^2$ and show that f is an automorphism.

8. Let V be a vector space of dimension 2 over the field $K = \mathbb{Z}_2$. Determine $|V|$, $|End_K(V)|$ and $|Aut_K(V)|$.
[*Hint: use the isomorphism between* $End_K(V)$ *and* $M_n(K)$, *where* $dim_K(V) = n$.]

Seminar 11

1. In the real vector space $\mathbb{R}^3$ consider the bases $B = (v_1, v_2, v_3) = ((1,0,1),(0,1,1),(1,1,1))$ and $B' = (v_1', v_2', v_3') = ((1,1,0),(-1,0,0),(0,0,1))$. Determine the matrices of change of basis $T_{BB'}$ and $T_{B'B}$, and compute the coordinates of the vector $u = (2,0,-1)$ in both bases.

2. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

3. In the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$ consider the bases $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)\,(a \in \mathbb{R})$ and $B' = (1, X-b, (X-b)^2)\,(b \in \mathbb{R})$. Determine the matrices of change of bases T_{EB}, T_{BE} and $T_{BB'}$.

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^2)$ be defined by $f(x,y) = (3x+3y, 2x+4y)$.
(i) Determine the eigenvalues and the eigenvectors of f.
(ii) Write a basis B of $\mathbb{R}^2$ consisting of eigenvectors of f and $[f]_B$.

Compute the eigenvalues and the eigenvectors of the (endomorphisms having) matrices:

5. $\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$. **6.** $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

7. $\begin{pmatrix} x & 0 & y \\ 0 & x & 0 \\ y & 0 & x \end{pmatrix}$ $(x, y \in \mathbb{R}^*)$. **8.** $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$ $(x \in \mathbb{R})$.

9. Let $A \in M_2(\mathbb{R})$ and let λ_1, λ_2 be the eigenvalues of A in $\mathbb{C}$. Prove that:
(i) $\lambda_1 + \lambda_2 = Tr\,(A)$ and $\lambda_1 \cdot \lambda_2 = det\,(A)$, where $Tr\,(A)$ denotes the trace of A, that is, the sum of the elements of the principal diagonal. Generalization.
(ii) A has all the eigenvalues in $\mathbb{R} \Longleftrightarrow (Tr\,(A))^2 - 4 \cdot det\,(A) \geq 0$.
(iii) Show that A is a root of its characteristic polynomial.

10. Let $A \in M_2(\mathbb{R})$ be such that $det(A + iI_2) = 0$. Show that $det(A + 2I_2) = 5$.

Seminar 12

1. (*i*) Which of the following received words contain detectable errors when using the (3,2)-parity check code: 110, 010, 001, 111, 101, 000?

(*ii*) Decode the following words using the (3,1)-repeating code to correct errors: 111, 011, 101, 010, 000, 001. Which of them contain detectable errors?

2. Are $1+X^3+X^4+X^6+X^7$ and $X+X^2+X^3+X^6$ code words in the polynomial (8,4)-code generated by $p=1+X^2+X^3+X^4 \in \mathbb{Z}_2[X]$?

3. Write down all the words in the (6,3)-code generated by $p=1+X^2+X^3 \in \mathbb{Z}_2[X]$.

4. A code is defined by the generator matrix $G=\left(\frac{P}{I_3}\right) \in M_{5,3}(\mathbb{Z}_2)$, where:

$$P=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Write down the parity check matrix and all the code words.

5. Determine the minimum Hamming distance between the code words of the code with generator matrix $G=\left(\frac{P}{I_4}\right) \in M_{9,4}(\mathbb{Z}_2)$, where:

$$P=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Discuss the error-detecting and error-correcting capabilities of this code, and write down the parity check matrix.

6. Encode the following messages using the generator matrix of the (9,4)-code of Exercise **5.**: 1101, 0111, 0000, 1000.

Determine the generator matrix and the parity check matrix for:

7. The (4,1)-code generated by $p=1+X+X^2+X^3 \in \mathbb{Z}_2[X]$.

8. The (7,3)-code generated by $p=1+X^2+X^3+X^4 \in \mathbb{Z}_2[X]$.

Seminar 13

1. Consider a $(63, 56)$-code.
(i) What is the number of digits in the message before coding?
(ii) What is the number of check digits?
(iii) What is the information rate?
(iv) How many different syndromes are there?

2. Using the parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and the syndromes and coset leaders

Syndrome	000	001	010	011
Coset leader	000000	001000	010000	000010

Syndrome	100	101	110	111
Coset leader	100000	000110	000100	000001

decode the following words: 101110, 011000, 001011, 111111, 110011.

3. A (7,4)-code is defined by the equations $u_1 = u_4 + u_5 + u_7$, $u_2 = u_4 + u_6 + u_7$, $u_3 = u_4 + u_5 + u_6$, where u_4, u_5, u_6, u_7 are the message digits and u_1, u_2, u_3 are the check digits. Write its generator matrix and parity check matrix. Decode the received words 0000111 and 0001111.

4. Find the syndromes of all the received words in the (3,2)-parity check code and in the (3,1)-repeating code.

5. Construct a table of coset leaders and syndromes for the (7,4)-code with parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

6. Determine the parity check matrix and all syndromes and coset leaders of the (5,3)-code with generator matrix $G = \left(\frac{P}{I_3}\right) \in M_{5,3}(\mathbb{Z}_2)$, where:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

7. Construct a table of coset leaders and syndromes for the (3,1)-code generated by $p = 1 + X + X^2 \in \mathbb{Z}_2[X]$.

8. Construct a table of coset leaders and syndromes for the (7,3)-code generated by $p = 1 + X^2 + X^3 + X^4 \in \mathbb{Z}_2[X]$.