3.5 Exercises

3.1. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 60°$. Determine the length of the diagonals in the parallelogram spanned by the vectors $\mathbf{a} = 2\mathbf{m} + \mathbf{n}$ and $\mathbf{b} = \mathbf{m} - 2\mathbf{n}$.

3.2. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 120°$. Determine the angle between the vectors $\mathbf{a} = 2\mathbf{m} + 4\mathbf{n}$ and $\mathbf{b} = \mathbf{m} - \mathbf{n}$.

3.3. You are given two vectors $\mathbf{a}(2,1,0)$ and $\mathbf{b}(0,-2,1)$ with respect to an orthonormal basis. Determine the angles between the diagonals of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

3.4. Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis. Consider the vectors $\mathbf{q} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{p} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p},\mathbf{q})$ is $5/12$.

3.5. Using the scalar product, prove the Cauchy-Bunyakovsky-Schwarz inequality, i.e. show that for any $a_1,\dots,a_n,b_1,\dots,b_n \in \mathbb{R}$ we have

$$(a_1b_1 + \cdots + a_nb_n)^2 \le (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2).$$

3.6. Let ABC be a triangle. Show that

$$\overrightarrow{AB}^2 + \overrightarrow{AC}^2 - \overrightarrow{BC}^2 = 2\overrightarrow{AB}\cdot\overrightarrow{AC}$$

and deduce the law of cosines in a triangle.

3.7. Let $ABCD$ be a tetrahedron. Show that

$$\cos(\measuredangle(\overrightarrow{AB},\overrightarrow{CD})) = \frac{AD^2 + BC^2 - AC^2 - BD^2}{2\cdot AB\cdot CD}.$$

This is a $3D$-version of the law of cosine.

3.8. Let $ABCD$ be a rectangle. Show that for any point O

$$\overrightarrow{OA}\cdot\overrightarrow{OC} = \overrightarrow{OB}\cdot\overrightarrow{OD} \quad\text{and}\quad \overrightarrow{OA}^2 + \overrightarrow{OC}^2 = \overrightarrow{OB}^2 + \overrightarrow{OD}^2.$$

3.9. Consider the vector $\mathbf{v}$ which is perpendicular on $\mathbf{a}(4,-2,-3)$ and on $\mathbf{b}(0,1,3)$. If $\mathbf{v}$ describes an acute angle with Ox and $|\mathbf{v}| = 26$ determine the components of $\mathbf{v}$.

3.10. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

3.11. Let $\mathbf{v} \in \mathbb{V}^n$ be a vector. Show that the set $\mathbf{v}^\perp$ is an $(n-1)$-dimensional vector subspace of $\mathbb{V}^n$. Deduce that there is a basis $\mathbf{v},\mathbf{v}_2,\mathbf{v}_3,\dots,\mathbf{v}_n$ of $\mathbb{V}^n$ with $\mathbf{v}_2,\dots,\mathbf{v}_{n-1}$ a basis of $\mathbf{v}^\perp$. (*Hint.* use Steinitz Theorem - Algebra, Lecture 6).

3.12. Determine a Cartesian equations for the line ℓ in the following cases:

a) ℓ contains the point $A(-2,3)$ and has an angle of $60°$ with the Ox-axis,

b) ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

3.13. For the lines ℓ in the previous exercise

a) give parametric equations for ℓ,

b) describe $\mathrm{D}(\ell)$.

3.14. Consider a line ℓ. Show that

c) if $\mathbf{v}(v_1,v_2)$ is a direction vector for ℓ then $\mathbf{n}(v_2,-v_1)$ is a normal vector for ℓ,

d) if $\mathbf{n}(n_1,n_2)$ is a normal vector for ℓ then $\mathbf{v}(n_2,-n_1)$ is a direction vector for ℓ.

3.15. Consider the points $A(1,2)$, $B(-2,3)$ and $C(4,7)$. Determine the medians of the triangle ABC.

3.16. Let $M_1(1,2)$, $M_2(3,4)$ and $M_3(5,-1)$ be the midpoints of the sides of a triangle. Determine Cartesian equations and parametric equations for the lines containing the sides of the triangle.

3.17. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

a) the length of the altitude from A,

b) the line containing the altitude from A.

$\boxed{\textbf{3.18.}}$ Determine the circumcenter and the orthocenter of the triangle with vertices $A(1,2)$, $B(3,-2)$, $C(5,6)$.

3.19. Determine the angle between the lines $\ell_1 : y = 2x+1$ and $\ell_2 : y = -x+2$.

$\boxed{\textbf{3.20.}}$ Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle $\measuredangle A$.

3.21. Let A' be the orthogonal reflection of $A(10,10)$ in the line $\ell : 3x+4y-20=0$. Determine the coordinates of A'.

3.22. Determine Cartesian equations for the lines passing through $A(-2,5)$ which intersect the coordinate axes in congruent segments.

3.23. Determine Cartesian equations for the lines situated at distance 4 from the line $12x-5y-15=0$.

3.24. Determine the values k for which the distance from the point $(2,3)$ to the line $8x+15y+k=0$ equals 5.

3.25. Consider the points $A(3,-1)$, $B(9,1)$ and $C(-5,5)$. For each pair of these three points, determine the line which is equidistant from them.

3.26. The point $A(3,-2)$ is the vertex of a square and $M(1,1)$ is the intersection point of its diagonals. Determine Cartesian equations for the sides of the square.

3.27. Determine a point on the line $5x-4y-4=0$ which is equidistant to the points $A(1,0)$ and $B(-2,1)$.

3.28. The point $A(2,0)$ is the vertex of an equilateral triangle. The side opposite to A lies on the line $x+y-1=0$. Determine Cartesian equations for the lines containing the other two sides.

3.29. Determine an equation for each plane passing through $P(3,5,-7)$ and intersecting the coordinate axes in congruent segments.

3.30. Let $A(2,1,0)$, $B(1,3,5)$, $C(6,3,4)$, $D(0,-7,8)$ be vertices of a tetrahedron. Determine a Cartesian equation of the plane containing $[AB]$ and the midpoint of $[CD]$.

3.31. Show that a parallelepiped with faces in the planes $2x+y-2z+6=0$, $2x-2y+z-8=0$ and $x+2y+2z+1=0$ is rectangular.

3.32. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.

3.33. Determine the distance between the planes $x-2y-2z+7=0$ and $2x-4y-4z+17=0$.

3.34. Solve Exercise 2.16 using normal vectors.

3.35. Let $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,-5)$ be vertices of a triangle. Determine the equation of the internal angle bisector of $\measuredangle A$.

3.36. Determine the angles between the plane $\pi_1 : x-\sqrt{2}y+z-1=0$ and the plane $\pi_2 : x+\sqrt{2}y-z+3=0$.

3.37. Determine the values a and c for which the line $3x-2y+z+3=0 \cap 4x-3y+4z+1=0$ is perpendicular to the plane $ax+8y+cz+2=0$.

3.38. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x+4y-3z+7=0$.

3.39. Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $2x-3y+z-4=0$.

3.40. Consider the point $A(1,3,5)$ and the line $\ell : 2x+y+z-1=0 \cap 3x+y+2z-3=0$.

a) Determine the orthogonal projection of A on ℓ.

b) Determine the orthogonal reflection of A in ℓ.

3.41. Determine the planes which pass through $P(0,2,0)$ and $Q(-1,0,0)$ and which form an angle of $60°$ with the z-axis.

3.42. Determine the orthogonal projection of the line $\ell : 2x-y-1=0 \cap x+y-z+1=0$ on the plane $\pi : x+2y-z=0$.

3.43. Determine the coordinates of a point A on the line $\ell : \frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{1}$ which is at distance $\sqrt{3}$ from the plane $x+y+z+3=0$.

3.44. The vertices of a tetrahedron are $A(-1,-3,1)$, $B(5,3,8)$, $C(-1,-3,5)$ and $D(2,1,-4)$. Determine the height of the tetrahedron relative to the face ABC.

3.1. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 60°$. Determine the length of the diagonals in the parallelogram spanned by the vectors $\mathbf{a} = 2\mathbf{m} + \mathbf{n}$ and $\mathbf{b} = \mathbf{m} - 2\mathbf{n}$.

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos(\widehat{a,b})$

$\cos(\widehat{a,b}) = \frac{\langle a,b\rangle}{\|a\|\,\|b\|}$

$\|u\| = \sqrt{\langle u,u\rangle} \Leftrightarrow \|u\|^2 = \langle u,u\rangle$

m, n – unit vectors

$\measuredangle(m,n) = 60°$

$\vec{a} = 2\vec{m} + \vec{n}$

$\vec{b} = \vec{m} - 2\vec{n}$

diagonals : $\vec{a}+\vec{b}$; $\vec{a}-\vec{b}$

$\vec{a}+\vec{b} = 3\vec{m} - \vec{n}$

$\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b})(\vec{a}+\vec{b}) = (3\vec{m}-\vec{n})(3\vec{m}-\vec{n}) = 9 - 3\vec{m}\vec{n} - 3\vec{m}\vec{n} + 1 = 10 - 6\vec{m}\vec{n}$

$\vec{m}\cdot\vec{n} = \|m\|\|n\|\cdot\cos(m,n) = 1\cdot 1\cdot \cos 60° = \frac{1}{2}$

$\Rightarrow \|\vec{a}+\vec{b}\|^2 = 10 - 6\cdot\frac{1}{2} = 10 - 3 = 7 \Rightarrow \|\vec{a}+\vec{b}\| = \sqrt{7}$

$\vec{a} - \vec{b} = \vec{m} + 3\vec{n}$

$\|\vec{a}-\vec{b}\|^2 = (\vec{a}-\vec{b})(\vec{a}-\vec{b}) = (\vec{m}+3\vec{n})(\vec{m}+3\vec{n}) = 1 + 6\vec{m}\vec{n} + 9 = 10 + 6\vec{m}\vec{n} = 10 + 6\cdot\frac{1}{2} = 13$

3.2. Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors such that $\measuredangle(\mathbf{m},\mathbf{n}) = 120°$. Determine the angle between the vectors $\mathbf{a} = 2\mathbf{m} + 4\mathbf{n}$ and $\mathbf{b} = \mathbf{m} - \mathbf{n}$.

$m, n \rightarrow$ unit vect.

$\measuredangle(m,n) = 120°$

$a = 2m + 4n$

$b = m - n$ $\Rightarrow \measuredangle(a,b) = ?$

$\cos(\widehat{a,b}) = \frac{\vec{a}\cdot\vec{b}}{\|a\|\cdot\|b\|}$

$\|\vec{a}\|^2 = 4 + 16mn + 16 = 20 + 16mn$

$\|\vec{b}\|^2 = 1 - 2mn + 1 = 2 - 2mn$

$\vec{m}\vec{n} = \|m\|\cdot\|n\|\cdot\cos 120° = -\frac{1}{2}$ $\Rightarrow \|a\|^2 = 20 - 8 = 12 \Rightarrow \|a\| = 2\sqrt{3}$

$\|b\|^2 = 2 + 1 = 3 \Rightarrow \|b\| = \sqrt{3}$

$\Rightarrow \cos(\widehat{a,b}) = \frac{(2m+4n)(m-n)}{2\sqrt{3}\cdot\sqrt{3}} = \frac{2 - 2mn + 4mn - 4}{6} = \frac{-2-1}{6} = \frac{-1}{2}$

$\boxed{3.3.}$ You are given two vectors $\mathbf{a}(2,1,0)$ and $\mathbf{b}(0,-2,1)$ with respect to an orthonormal basis. Determine the angles between the diagonals of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

$(a+b)(2,-1,1)$

$(a-b)(2,3,-1)$

$\cos(a+b, a-b) = \frac{0}{\dots} = 0 \Rightarrow \vec{a}\cdot\vec{b} = 90°$

* if dot product is 0, then the vectors are $\perp$

3.4 Let $(\mathbf{i},\mathbf{j},\mathbf{k})$ be an orthonormal basis. Consider the vectors $\mathbf{q} = 3\mathbf{i}+\mathbf{j}$ and $\mathbf{p} = \mathbf{i}+2\mathbf{j}+\lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p},\mathbf{q})$ is 5/12.

$$\|q\| = \sqrt{3^2+1^2+0^2} = \sqrt{10}$$

$$\|p\| = \sqrt{1^2+2^2+\lambda^2} = \sqrt{5+\lambda^2}$$

$$\vec{p}\cdot\vec{q} = 3\cdot 1 + 1\cdot 2 + 0\cdot\lambda = 5$$

$$\cos(p,q) = \frac{p\cdot q}{\|p\|\cdot\|q\|} = \frac{5}{\sqrt{5+\lambda^2}\cdot\sqrt{10}} = \frac{5}{12} \Rightarrow \sqrt{(5+\lambda^2)\cdot 10} = 12$$

$$(5+\lambda^2)\cdot 10 = 144$$

$$50+\lambda^2\cdot 10 = 144$$

$$\lambda^2\cdot 10 = 94$$

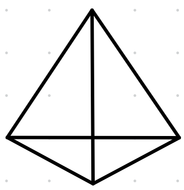
$$\lambda^2 = \frac{94}{10}$$

$$\lambda = \pm\sqrt{\frac{94}{10}} = \pm\sqrt{\frac{47}{5}}$$

3.7. Let $ABCD$ be a tetrahedron. Show that

$$\cos(\measuredangle(\overrightarrow{AB},\overrightarrow{CD})) = \frac{AD^2+BC^2-AC^2-BD^2}{2\cdot AB\cdot CD}.$$

This is a $3D$-version of the law of cosine.



$$\vec{AB} = \vec{AC}+\vec{CB}$$

$$\vec{CD} = \vec{CA}+\vec{AD}$$

$$\vec{AB}\cdot\vec{CD} = (\vec{AC}+\vec{CB})(\vec{CA}+\vec{AD}) = -\|AC\|^2 + \vec{AC}\,\vec{AD} + \vec{CB}\cdot\vec{CA} + \vec{CB}\cdot\vec{AD} =$$

$$\vec{AB} = \vec{AD}+\vec{DB}$$

$$\vec{CD} = \vec{CB}+\vec{BD}$$

$$\vec{AB}\cdot\vec{CD} = \vec{AB}\cdot\vec{CB} + \vec{AB}\cdot\vec{BD} + \vec{DB}\cdot\vec{CB} - \|BD\|^2 \quad (+)$$

$$2\vec{AB}\cdot\vec{CD} = -\|AC\|^2 + \vec{AC}\cdot\vec{AD} + \vec{CB}\cdot\vec{CA} + \vec{CB}\cdot\vec{AD} + \vec{AB}\cdot\vec{CB} + \vec{AB}\cdot\vec{BD} + \vec{DB}\cdot\vec{CB} - \|BD\|^2 =$$

$$= -\|AC\|^2 + \vec{AB}(\vec{AC}+\vec{CB}+\vec{BD}) + \vec{BC}(\vec{AC}+\vec{DA}+\vec{BD}) - \|BD\|^2 =$$

$$= -\|AC\|^2 + \|AD\|^2 + \|BC\|^2 - \|BD\|^2$$

$$\cos(\widehat{AB, CD}) = \frac{AD^2+BC^2-AC^2-BD^2}{2AB\cdot CD}$$

$$\vec{x}^2-\vec{y}^2 = \langle\vec{x}-\vec{y},\vec{x}+\vec{y}\rangle$$

$$(\vec{x}\pm\vec{y})^2 = \vec{x}^2 \pm 2\langle x,y\rangle + \vec{y}^2$$

* another method ↑ ask mara

3.9. Consider the vector $\mathbf{v}$ which is perpendicular on $\mathbf{a}(4,-2,-3)$ and on $\mathbf{b}(0,1,3)$. If $\mathbf{v}$ describes an acute angle with Ox and $|\mathbf{v}| = 26$ determine the components of $\mathbf{v}$.

$$v\perp a \Rightarrow \measuredangle(v,a) = 90^\circ \Rightarrow \cos(\widehat{v,a}) = 0$$

$$v\perp b \Rightarrow \measuredangle(v,b) = 90^\circ \Rightarrow \cos(\widehat{v,a}) = 0$$

$$|v| = 26 \Rightarrow \sqrt{x^2+y^2+z^2} = 26 \Rightarrow x^2+y^2+z^2 = 676$$

$$v\cdot a = \|v\|\cdot\|a\|\cdot\cos(\widehat{v,a}) = 0$$

$$v\cdot a = 4x-2y-3z$$

$$v\cdot b = 0$$

$$v\,b = y+3z$$

$$\Rightarrow \begin{cases} 4x-2y-3z=0 \\ y+3z=0 \end{cases} \quad (+)$$

$$4x-y=0 \Rightarrow 4x=y$$

$$y-2y-3z=0 \Rightarrow -y=3z \Rightarrow 3z=-4x \Rightarrow z=\frac{-4}{3}x$$

$$x^2 + 16x^2 + \frac{16}{9}x^2 = 676$$

$$x^2\left(\frac{9}{1} + \frac{9}{16} + \frac{16}{9}\right) = 676$$

$$x^2 \cdot \frac{9 + 9\cdot 16 + 16}{9} = 676$$

$$\frac{169}{9}x^2 = 676$$

$$169x^2 = 6\cdot 26^2$$

$$x^2 = \frac{6\cdot \cancel{26}^{2^2}}{\cancel{13}^{2}}$$

$$x^2 = 16 \Rightarrow x = 6$$

$$y = 24$$

$$z = -8$$

$\boxed{3.10}$ In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

$$Pr_{v_1'}(v_2) = \frac{\langle v_2, v_1'\rangle}{\langle v_1', v_1'\rangle} \cdot v_1'$$

$$v_1' = v_1 = (0,1,0)$$

$$v_2' = v_2 - Pr_{v_1'}(v_2)$$

$$Pr_{v_1'} = \frac{\langle v_2, v_1'\rangle}{\langle v_1', v_1'\rangle} \cdot v_1' = \frac{2\cdot 0 + 1\cdot 1 + 0\cdot 0}{0\cdot 0 + 1\cdot 1 + 0\cdot 0} \cdot (0,1,0) = (0,1,0)$$

$$\left.v_2' = v_2 - Pr_{v_1'}(v_2) = (2,1,0) - (0,1,0) = (2,0,0)\right\} \Rightarrow v_2' = 2\cdot(1,0,0)$$

*normalization process

$$* \; v_2'' = \frac{v_2'}{|v_2'|} = \frac{(2,0,0)}{2} = (1,0,0)$$

$$v_3' = v_3 - Pr_{v_1'}(v_3) - Pr_{v_2'}(v_3) =$$

$$Pr_{v_1'}(v_3) = \frac{\langle v_3, v_1'\rangle}{\langle v_1', v_1'\rangle} \cdot v_1' = \frac{-1\cdot 0 + 0\cdot 1 + 1\,0}{1} \cdot (0,1,0) = (0,0,0)$$

$$Pr_{v_2'}(v_3) = \frac{\langle v_3, v_2'\rangle}{\langle v_2', v_2'\rangle} \cdot v_2' = \frac{1\cdot 0 -}{\quad}$$

?