Chapter 5 : 16, 18, 19, 20.3, 22, 23, 24

$\varphi : \mathcal{E}^n \to \mathcal{E}^n$ affine morphism

$f(p) = A \cdot p + b$, $A \in M_n(\mathbb{R})$, $b \in \mathbb{R}^n$

f isometry $\Leftrightarrow \forall\ X, Y \in \mathcal{E}^n$

$\quad \text{dist}(f(X), f(Y)) = \text{dist}(X, Y)$

$(\text{lin } f)(p) = A \cdot p$

$\Leftrightarrow A \in \underbrace{O(n)}_{\text{orthogonal matrices}} = \{ M \in M_n(\mathbb{R}) \mid M^{-1} = M^T \}$

$\Leftrightarrow \boxed{A \cdot A^T = I_n}$ $\Rightarrow$ check if the distances are preserved

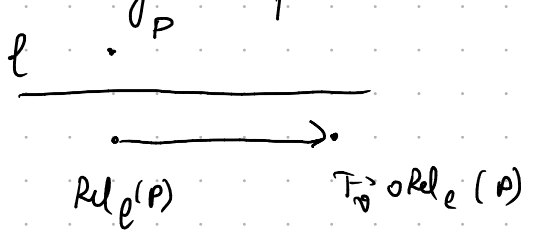
$A \in SO(n)$ (special orthogonal)

I. $\det A = 1 \Rightarrow f$ direct isometry

II. $\det A = -1 \Rightarrow f$ indirect isometry

$n = 2$

- direct isometry:
  - identity
  - translation by a vector $\vec{v}$ — $T_{\vec{v}}$
  - rotation around a point Q by an angle θ — $Rot_{Q,\theta}$
- indirect isometry:
  - reflection wrt a line ℓ — Ref_ℓ
  - glide reflection wrt a line ℓ w/ a vector $\vec{v} \in D(\ell)$ — $T_{\vec{v}} \circ Ref_\ell$



$$\mathrm{Fix}(f) = \{ P \in \mathbb{E}^n \mid f(P) = P \}$$

5.16. F isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a translation by the vector $(-2,5)$

Let the inverse F^{-1}.

$$\Rightarrow F = \mathrm{Rot}_{-\frac{\pi}{3}} \circ T_{(-2,5)} \qquad (f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

$$\begin{pmatrix} \cos -\frac{\pi}{3} & -\sin -\frac{\pi}{3} & 0 \\ \sin -\frac{\pi}{3} & \cos -\frac{\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$= \begin{pmatrix} \cos -\frac{\pi}{3} & -\sin -\frac{\pi}{3} & -2\cos -\frac{\pi}{3} - 5\sin -\frac{\pi}{3} \\ \sin -\frac{\pi}{3} & \cos -\frac{\pi}{3} & -2\sin -\frac{\pi}{3} + 5\cos -\frac{\pi}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & -1 + \frac{5\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{5}{2} + \sqrt{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow F^T = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -1 + \frac{5\sqrt{3}}{2} & \frac{5}{2} + \sqrt{3} & 1 \end{pmatrix}$$

$$[R^{-1}] = \mathrm{Rot}_{\frac{\pi}{3}} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T^{-1}] = T_{(2,-5)} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T^{-1}] \cdot [R^{-1}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

S.R. $f: \mathcal{E}^2 \to \mathcal{E}^2$ $\qquad$ $Tr\, A = 2\cos\theta$

$$f(P) = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \cdot P + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

- Show that f is a rotation (direct isometry)
- Find its center and the rotation angle

$$M \cdot M^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix} = I_2 \Rightarrow \text{direct isometry.}$$

$$tr\, A = \frac{9}{5} = 2\cdot\cos\theta \Rightarrow \frac{9}{10} = \cos\theta \Rightarrow \theta = \arccos\frac{9}{10}$$

$$\det A = \frac{9+16}{25} = 1 \to A \in SO(2)$$

$\Rightarrow f$ direct isometry.

Let $P \in \mathcal{E}^2$, $P \in Fix(f) \Leftrightarrow f(P) = P$

$$P\binom{x}{y}$$

$$f(P) = AP + B = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3x-4y}{5}+1 \\ \frac{4x+3y}{5}-2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow x = \frac{5}{2},\ y = 0 \Rightarrow P = \frac{1}{5}\begin{pmatrix} 2 \\ 0 \end{pmatrix} \Rightarrow f \text{ is a rotation.}$$

$$Tr(f) = \frac{6}{5} = 2\cos\theta \Rightarrow \frac{3}{5} = \cos\theta \Rightarrow \theta = \arccos\frac{3}{5}$$

$n=3$

direct:

- identity
- rotation around an axis l by an angle θ $Rot_{l,\theta}$
- glide rotation

$T_{\vec{v}} \circ Rot_{l,\theta}$

indirect:

- reflection with respect to a plane π Ref_{π}
- glide reflection

$T_{\vec{v}} \circ Ref_{\pi}$

$\vec{v} \in D(\pi)$

- rotation reflection

$Rot_{l,\theta} \circ Ref_{\pi}$, where $l \perp \pi$

If T rotation: $Tr A = 2\cos\theta + 1$

FINAL EXAM 18,15

Verify that the matrix

$$A = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \in SO(3)$$

Det. the axis and the angle of rotation

$$\begin{pmatrix} -1 & 2 & 2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$= I_3 \Rightarrow A \in O(3)$

$\det A = \frac{1}{27}(4 + 4 + \dots) = \frac{27}{27} = 1 \quad A \in SO(3)$

$\Rightarrow f$ direct isometry

Let $P\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ s.t. $A \cdot P = P$.

$$\Rightarrow \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{1}{3}\begin{pmatrix} -x + 2y - 2z \\ -2x - 2y - z \\ -2x + y + 2z \end{pmatrix} = \begin{matrix} x \\ y \\ z \end{matrix}$$

$$\Rightarrow \begin{cases} -4x + 2y - 2z = 0 \\ -2x - 5y - z = 0 \\ -2x + y - z = 0 \end{cases}$$

$$\Rightarrow -5y = y \Rightarrow y = 0$$

$$-4x - 2z = 0 \quad \Big| \cdot -\frac{1}{2}$$

$$2x + z = 0 \Rightarrow z = -2x$$

$$\Rightarrow S(\alpha, 0, -2\alpha)$$

$$\mathrm{Fix}(A) = \{(x, 0, -2x) \mid x \in \mathbb{R}\}$$

$$D(\ell) = \langle (1, 0, -2) \rangle \Rightarrow A \text{ rotation}$$

$$\mathrm{Tr}\, A = 2\cos\theta + 1$$

$$\Rightarrow -\frac{1}{3} = 2\cos\theta + \frac{3}{3}$$

$$-\frac{4}{3} = 2\cos\theta \Rightarrow -\frac{2}{3} = \cos\theta$$

22. Using Euler Rodriguez formula write a rotation around the axis $\mathbb{R}$ where $\vec{v}(1,1,0)$ and use this matrix for the parametrization of a cylinder of axis $\mathbb{R}\vec{v}$ and diameter r_2

Euler Rodriguez:

$$Rot_{\ell,\theta}(p) = \cos\theta \cdot p - \sin\theta(\vec{v} \times p) + (1 - \sin\theta)\langle \vec{v}, p\rangle \vec{v}$$

plug in p and v $\Rightarrow$ write result as matrix = matrix of rotation.