**11.** Let $SABCD$ be a pyramid with apex $S$ and base the parallelogram $ABCD$. Show that

$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}$$

where $O$ is the center of the parallelogram.

**12.** Give the coordinates of the vertices of the parallelepiped whose faces lie in the coordinate planes and in the planes $x = 1$, $y = 3$ and $z = -2$.

**13.** In $\mathbb{E}^3$ consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.

**14.** Which of the following sets of vectors form a basis?

a) $\mathbf{v}(1,2)$, $\mathbf{w}(3,4)$; ✓

b) $\mathbf{u}(-1,2,1)$, $\mathbf{v}(2,1,1)$, $\mathbf{w}(1,0,-1)$;

$$\begin{vmatrix} -1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = 3-(-3) = 6 \neq 0 \Rightarrow \text{lin indep.}$$

c) $\mathbf{u}(-1,2,1)$, $\mathbf{v}(2,1,1)$, $\mathbf{w}(0,5,3)$; ✗

$$\begin{vmatrix} -1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = -1 \cdot \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -(2-1) + (1+2-1+4) = -1+6 = 5 \neq 0$$

d) $\mathbf{v}_1(-1,2,1,0)$, $\mathbf{v}_2(2,1,1,0)$, $\mathbf{v}_3(1,0-1,1)$, $\mathbf{v}_4(1,0,0,1)$;

**15.** With respect to the basis $\mathcal{B} = (\mathbf{i},\mathbf{j},\mathbf{k})$ consider the vectors $\mathbf{u} = \mathbf{i}+\mathbf{j}$, $\mathbf{v} = \mathbf{j}+\mathbf{k}$ and $\mathbf{w} = \mathbf{i}+\mathbf{k}$. Check that $\mathcal{B}' = (\mathbf{u},\mathbf{v},\mathbf{w})$ is a basis and give the base change matrix $M_{\mathcal{B}',\mathcal{B}}$.

**16.** Consider the two coordinate systems $\mathcal{K} = (O,\mathbf{i},\mathbf{j})$ and $\mathcal{K}' = (O',\mathbf{i}',\mathbf{j}')$ given in Example 1.20. Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.

**17.** Consider the tetrahedron $ABCD$ and the coordinate systems

$$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}), \quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}), \quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

a) the coordinates of the vertices of the tetrahedron in the three coordinate systems,

b) the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,

c) the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.

**18.** Consider the two coordinate systems $\mathcal{K} = (O,\mathbf{i},\mathbf{j},\mathbf{k})$ and $\mathcal{K}' = (O',\mathbf{i}',\mathbf{j}',\mathbf{k}')$ given in Example 1.21. Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.

**16.** Consider the two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}')$ given in Example 1.20. Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad [O']_K = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

in the system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously obtained coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$ and $[C]_{\mathcal{K}}$.

$K = (0, i, j)$ $\qquad \vec{i}' = -2\vec{i} + \vec{j} \qquad \vec{j}' = \vec{i} + 2\vec{j}$

$K' = (0', i', j')$

$M_{K'K} = (M_{KK'})^{-1}$

$$M_{KK'} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow \left(\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 5 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} \end{array}\right)$$

$$\Rightarrow M_{K'K} = \frac{1}{5} \cdot \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\Rightarrow [A]_{K'} = M_{K'K}\left([A]_K - [O']_K\right) = \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \end{pmatrix}\right) = \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 15 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$[B]_{K'} = M_{K'K}\left([B]_K - [O']_K\right) = \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\left(\begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \end{pmatrix}\right) = \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 15 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$[C]_{K'} = M_{K'K}\left([C]_K - [O']_K\right) = \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\left(\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \end{pmatrix}\right) = \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 10 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

**18.** Consider the two coordinate systems $\mathcal{K} = (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ and $\mathcal{K}' = (O', \mathbf{i}', \mathbf{j}', \mathbf{k}')$ given in Example 1.21. Determine the base change matrix from $\mathcal{K}$ to $\mathcal{K}'$ and the coordinates of the points

$$[A]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \quad [B]_{\mathcal{K}} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}, \quad [C]_{\mathcal{K}} = \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}, \quad [D]_{\mathcal{K}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \qquad [O']_K = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

in the coordinate system $\mathcal{K}'$. Further, determine the base change matrix from $\mathcal{K}'$ to $\mathcal{K}$ and use it with the previously determined coordinates to calculate $[A]_{\mathcal{K}}$, $[B]_{\mathcal{K}}$, $[C]_{\mathcal{K}}$ and $[D]_{\mathcal{K}}$.

$\vec{i}' = -i - 2j \qquad \vec{j}' = -2i + j \qquad \vec{k}' = j + 2k$

$$M_{KK'} = \begin{pmatrix} -1 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$M_{K'K} = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & -\frac{2}{5} & \frac{2}{10} \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right) \Rightarrow M_{K'K} = \frac{1}{10} \cdot \begin{pmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{pmatrix}$$

$$[A]_{K'} = M_{K'K} \cdot \left([A]_K - [O']_K\right) = \frac{1}{10} \cdot \begin{pmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}\right) = \frac{1}{10} \cdot \begin{pmatrix} -2 & -4 & 2 \\ -4 & 2 & -1 \\ 0 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ -2 \end{pmatrix}$$

$$= \frac{1}{10}\cdot\begin{pmatrix}6\\12\\-10\end{pmatrix} = \frac{1}{5}\cdot\begin{pmatrix}3\\6\\-5\end{pmatrix}$$

$$[B]_{K'} = \mathcal{M}_{K'K}\cdot\left([B]_K - [O']_K\right) = \frac{1}{10}\cdot\begin{pmatrix}-2&-4&2\\-4&2&-1\\0&0&5\end{pmatrix}\cdot\left[\begin{pmatrix}1\\5\\1\end{pmatrix}-\begin{pmatrix}4\\5\\-1\end{pmatrix}\right] = \frac{1}{10}\cdot\begin{pmatrix}-2&-4&2\\-4&2&-1\\0&0&5\end{pmatrix}\cdot\begin{pmatrix}-3\\0\\2\end{pmatrix}$$

$$= \frac{1}{10}\cdot\begin{pmatrix}10\\10\\10\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$[C]_{K'} = \frac{1}{10}\cdot\begin{pmatrix}-2&-4&2\\-4&2&-1\\0&0&5\end{pmatrix}\cdot\left[\begin{pmatrix}-3\\7\\1\end{pmatrix}-\begin{pmatrix}4\\5\\-1\end{pmatrix}\right] = \frac{1}{10}\cdot\begin{pmatrix}-2&-4&2\\-4&2&-1\\0&0&5\end{pmatrix}\cdot\begin{pmatrix}-7\\2\\2\end{pmatrix} =$$

$$= \frac{1}{10}\cdot\begin{pmatrix}10\\30\\10\end{pmatrix} = \begin{pmatrix}1\\3\\1\end{pmatrix}$$

$$[D]_{K'} = \frac{1}{10}\cdot\begin{pmatrix}-2&-4&2\\-4&2&-1\\0&0&5\end{pmatrix}\cdot\begin{pmatrix}-4\\-2\\2\end{pmatrix} = \frac{1}{10}\cdot\begin{pmatrix}20\\10\\10\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}$$

**17.** Consider the tetrahedron $ABCD$ and the coordinate systems

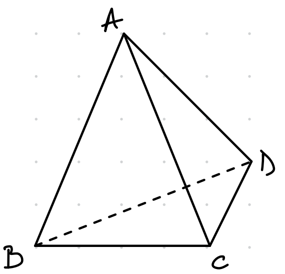
$$\mathcal{K}_A = (A, \underset{i}{\overrightarrow{AB}}, \underset{j}{\overrightarrow{AC}}, \underset{k}{\overrightarrow{AD}}),\quad \mathcal{K}'_A = (A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AC}),\quad \mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD}).$$

Determine

a) the coordinates of the vertices of the tetrahedron in the three coordinate systems,

b) the base change matrix from $\mathcal{K}_A$ to $\mathcal{K}'_A$,

c) the base change matrix from $\mathcal{K}_B$ to $\mathcal{K}_A$.




$[A]_{K_A} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ $\quad [A]_{K'_A} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ $\quad [A]_{K_B} = [\overrightarrow{BA}]_{K_B} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$[B]_{K_A} = [\overrightarrow{AB}]_{K_A} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\quad [B]_{K'_A} = [\overrightarrow{AB}]_{K'_A} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\quad [B]_{K_B} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$[C]_{K_A} = [\overrightarrow{AC}]_{K_A} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad [C]_{K'_A} = [\overrightarrow{AC}]_{K'_A} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ $\quad [C]_{K_B} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$[D]_{K_A} = [\overrightarrow{AD}]_{K_A} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ $\quad [D]_{K'_A} = [\overrightarrow{AD}]_{K'_A} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad [D]_{K_B} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

b)

$$\mathcal{M}_{K'_A, K_A} = \left([\overrightarrow{AB}]_{K_A}\ [\overrightarrow{AC}]_{K_A}\ [\overrightarrow{AD}]_{K_A}\right) = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

c)

$$\mathcal{M}_{K_A, K_B} = \begin{pmatrix}-1&-1&-1\\0&1&0\\0&0&1\end{pmatrix}$$

$$[\overrightarrow{BA}]_{K_A} = -[\overrightarrow{AB}]_{K_A} = \begin{pmatrix}-1\\0\\0\end{pmatrix}$$

$$[\overrightarrow{BC}]_{K_A} = [\overrightarrow{AC} - \overrightarrow{AB}]_{K_A} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$[\overrightarrow{BD}]_{K_A} = [\overrightarrow{AD} - \overrightarrow{AB}]_{K_A} = \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

**15.** With respect to the basis $\mathcal{B} = (\mathbf{i}, \mathbf{j}, \mathbf{k})$ consider the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + \mathbf{k}$. Check that $\mathcal{B}' = (\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a basis and give the base change matrix $M_{\mathcal{B}',\mathcal{B}}$.

$$u = i + j \qquad v = j + k \qquad w = i + k$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = (-1) \cdot \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \Rightarrow \text{Rank}(A) = 3 \Rightarrow u, v, w \text{ lin. indep.}$$

$$M_{B,B'} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow M_{B',B} = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \cancel{0}\,{-1} & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

$$\Rightarrow M_{B,B} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\Rightarrow M_{B',B} = \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$