Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză

Curs: Dynamical Systems

Primăvara 2019

Linear difference equations. Discrete Dynamical Systems[1]

1. a) Find all the sequences $(x_k)_{k\geq 0}$ that satisfy $x_{k+3} = k3^k + 5k - 2$, $k \geq 0$.
b) Find all the sequences $(x_k)_{k\in\mathbb{Z}}$ that satisfy $x_{k+3} = k3^k + 5k - 2$, $k \in \mathbb{Z}$.

2. Let $\eta \in \mathbb{R}$ be a fixed parameter. Find the solution $(x_k)_{k\geq 0}$ of the initial value problem

$$x_{k+1} = 2x_k, \quad x_0 = \eta.$$

Check the solution you obtained. What is the long term behavior of this sequence? Discuss with respect to η.

3. Let $\lambda \in \mathbb{R}^*$ and $\eta \in \mathbb{R}$ be fixed parameters. Find the solution $(x_k)_{k\geq 0}$ of the initial value problem

$$x_{k+1} = \lambda x_k, \quad x_0 = \eta.$$

Check the solution you obtained. What is the long term behavior of this sequence? Discuss with respect to λ and η.

4. Find the solution $(x_k)_{k\in\mathbb{Z}}$ of the initial value problem

$$x_{k+2} + x_{k+1} + x_k = 0, \quad x_0 = 0, \quad x_1 = 1.$$

Check the solution you obtained. What is the long term behavior of this sequence?

5. Find solutions of the form $x_k = a\,3^k$ of the difference equation $x_{k+1} = 2x_k + 3^k$, $k \geq 0$. Here we look for $a \in \mathbb{R}$.

6. Find solutions of the form $x_k = ak + b$ of the difference equation $x_{k+1} = 2x_k - k$, $k \geq 0$. Here we look for $a, b \in \mathbb{R}$.

For each of the following difference equations find a particular solution of the indicated form. If no form is indicated (and the equation is nonhomogeneous), try a constant solution. Find its general solution using the fundamental theorems for linear difference equations and the characteristic equation method.

7. $x_{k+1} = \frac{1}{3}\, x_k$; **8.** $x_{k+1} = \frac{1}{2}\, x_k + 2$; **9.** $x_{k+1} = 2\, x_k + \frac{1}{2}$; **10.** $x_{k+1} = -3x_k$;

11. $x_{k+1} = 4\, x_k + 3^{k+1},\ \ x_k^p = a \cdot 3^k$; **12.** $x_{k+1} = 1/3\, x_k + 2^k,\ \ x_k^p = a \cdot 2^k$;

13. $x_{k+2} - 6x_{k+1} + 9x_k = 0$; **14.** $x_{k+2} + x_{k+1} + x_k = 0$.

15. Find the general solution of the linear planar system

$x_{k+1} = 1/2\, x_k + y_k, \quad y_{k+1} = -1/5\, y_k$

in two ways. One way must be by reducing it to a second order difference equation in x.

16. Find the general solution of the linear planar system

$x_{k+1} = -x_k - y_k, \quad y_{k+1} = -x_k + y_k$

by reducing it to a second order difference equation.

17. Write the Euler numerical formula with stepsize $h > 0$ for the IVP $x' = -120x,\ x(0) = 1$ and solve the difference equation you obtained in two cases: $h = 0.1$ and $h = 0.001$. Discuss on the long term behavior of the solution of the differential equation and, also, of the solution of the difference equation.

18. Write the Euler numerical formula with stepsize $h > 0$ for the IVP $x' = 2x(1 - x),\ x(0) = 1$.

19. Find the fixed points of the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x - \frac{1}{4}(x^2 - 2)$ and study their stability.

20*. Represent the stair-step diagram of the function from the previous exercise and try to discuss the long term behavior of the solution of $x_{k+1} = x_k - \frac{1}{4}(x_k^2 - 2)$, $k \geq 0$ with respect to an arbitrary $x_0 \in \mathbb{R}$.

21*. Prove that the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = m + \varepsilon \sin x$, where $m > 0$ and $0 < \varepsilon < 1$, has a unique fixed point which is an attractor. The equation $x = m + \varepsilon \sin x$ is known as *Kepler equation* and arises in the study of planetary motion.

22. Study the stability of the fixed point $(0,0)$ of the following planar linear difference system:

a) $x_{k+1} = 3/5\, x_k + 1/5\, y_k, \quad y_{k+1} = 1/5\, x_k + 3/5\, y_k;$

b) $x_{k+1} = 1/2\, x_k + y_k, \quad y_{k+1} = -1/5\, y_k;$

c) $x_{k+1} = -x_k - y_k, \quad y_{k+1} = -x_k + y_k.$

23. Find the fixed points and the 2-periodic points of the maps $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = -x^3$ and $g(x) = x^2 - 1$.

24. In order to study the stability of the fixed point 0 of the maps $f, g, h : \mathbb{R} \to \mathbb{R}$, $f(x) = x + x^3$, $g(x) = x - x^3$, $h(x) = x + x^2$, note that the linearization method can not be applied. Study the stability of the fixed point 0 using the stair-step diagram.

25. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$.

(a) Find the fixed points and the 2-periodic points of f and study their stability using the linearization method.

(b) Represent the graph of f and find geometrically the fixed points of f. Also, depict the 2-periodic orbit using the stair-step diagram.

(c) Find directly $f^k(0)$ for any $k \geq 0$. Depict this orbit using the stair-step diagram.

d) Let $\eta = 2$, and, respectively, $\eta = -1/4$. Using the stair-step diagram describe the long-term behavior of the orbit that starts at η (in other notation, of the sequence defined by $x_{k+1} = x_k^2 - 1, \ x_0 = \eta$).

26. Using the stair-step diagram, estimate the basin of attraction for each of the fixed points (if there is any which is an attractor) of the map

$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{x^2 + 5}{2x}\ .$$

27. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x(1 - x)$.

a) Find its fixed points and study their stability.

b) Let $I_1 = (-\infty, 0)$, $I_2 = (0, 1)$ and $I_3 = (1, \infty)$. Find $f(I_1)$, $f(I_2)$ and $f(I_3)$.

c) Find the orbits corresponding to the initial states $\eta = 0$ and, respectively, $\eta = 1$.

d) Using the stair-step diagram, describe the long-term behavior of the orbits corresponding to the initial states: $\eta = 1/8$, $\eta = 7/8$, $\eta = -1/8$ and, respectively, $\eta = 9/8$.

e) Estimate the basin of attraction of the attractor fixed point of f.

28. Find the expression of the Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k,\ x_0 = 0,\ x_1 = 1.$$

29*. Find the solution of

$$x_{k+2} - 6x_{k+1} + 9x_k = 12k, \quad x_0 = 0, \quad x_1 = 0.$$

Hint: look for $a, b \in \mathbb{R}$ such that $(x_k)_p = ak + b$ is a particular solution of the difference equation.

30. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (-x^2 + y/5,\ x)$.

(a) Find the fixed points of f and study their stability.

(b) In case that you found an attracting fixed point, write the consequence of this fact for the sequence $(f^k(\eta))_{k\geq 0}$ where $\eta \in \mathbb{R}^2$ is properly chosen. As usual, f^k denotes the k-th iterate of f.

31. We consider the map

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{50}x(100 - x).$$

(a) Find its fixed points and study their stability.

(b) Using the stair-step diagram estimate the basin of attraction of the attractor fixed point.

(c) If $(x_k)_{k\geq 0}$ represent the number of fish in some lake at month k and

$$x_{k+1} = \frac{1}{50}x_k(100 - x_k), \quad x_0 = \eta$$

try to predict the fate of the fish in the case $\eta = 80$ and also in the case $\eta = 10$.

32. We consider the IVP $\quad y' = -200y, \quad y(0) = 1,$
where the unknown is the function $y(t)$.

a) Find the solution and its limit as $t \to \infty$.

b) Write the Euler's numerical formula with constant step-size h.

c) For $h = 0.001$, and, respectively, $h = 0.01$ find the solution $(y_k)_{k\geq 0}$ of the difference equation found at b) and decide if it satisfies $\lim\limits_{k\to\infty} y_k = 0$.

d) Find a range of values for the step-size h such that the solution $(y_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} y_k = 0$.

33. (a) Write the Euler's numerical formula with stepsize $h = 0.01$ to approximate the solution of the IVP $y' = y, \quad y(0) = 1$.

(b) Using (a) find a rational approximation of the Euler's constant e.

34. Let $g : I \to \mathbb{R}$ be a C^1 map such that $g'(x) \neq 0$ for all x in the interval I. Assume that there exists $r \in I$ such that $g(r) = 0$. Prove that for $\eta \in I$ sufficiently close to r the unique solution $(x_k)_{k\geq 0}$ of the IVP

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}, \quad x_0 = \eta$$

satisfies

$$\lim_{k\to\infty} x_k = r.$$

35. Let $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ be such that $a_{12} \neq 0$. Show that the roots of the characteristic equation corresponding to the second-order difference equation obtained by reducing the linear system

$$x_{k+1} = a_{11}x_k + a_{12}y_k, \quad y_{k+1} = a_{21}x_k + a_{22}y_k$$

are the eigenvalues of the matrix associated to this system.

36. We consider the map

$$T : [0.1] \to \mathbb{R}, \quad T(x) = 1 - |2x - 1|.$$

a) Represent the graph of T. Find its fixed points.

b) Let $n \in \mathbb{N}$, $n \geq 2$. Find the orbit of the initial state $\frac{3}{2^n}$.

c) Find the 2-periodic points of T.

d) Represent the graphs of T^2 and T^3. How many fixed points they have?

e) The map T has a 2-periodic orbit? Or a 3-periodic orbit? T has a 2019-periodic orbit?

37. We consider the IVP $y' = 1 + xy^2,\ y(0) = 0$. Write the Euler numerical formula on the interval $[0, 1]$ with step-size $h = 0.02$. Specify the initial values and the number of steps necessary to find the approximate value of $\varphi(0.5)$ and, respectively, of $\varphi(1)$. Here with φ is denoted the exact solution of the given IVP.

38. We consider the difference equation

$$x_{k+2} + x_k = \cos\frac{k\pi}{2}\ .$$

a) Find a solution of the form $(x_k)_p = ak\cos\frac{k\pi}{2}$, with $a \in \mathbb{R}$. (Hint: we recall that $\cos(t + \pi) = -\cos t$ for any $t \in \mathbb{R}$)

b) Find its general solution.

c) Find the solution with $x_0 = x_1 = 0$ and describe its long-term behavior (is it periodic? is it bounded? is it oscillatory around 0?).

39. We consider the linear difference system

$$x_{k+1} = \frac{3}{5}x_k + \frac{1}{5}y_k, \quad y_{k+1} = \frac{1}{5}x_k + \frac{3}{5}y_k.$$

a) Study the stability of this system.

b) Find its general solution.

40. Find the linear homogeneous difference equation with constant coefficients, of minimal order, which has as solutions the two sequences

$$1,\ \frac{1}{2}\ ,\ \frac{1}{2^2}\ ,\ \frac{1}{2^3}\ ,\ \frac{1}{2^4}\ ,\ \frac{1}{2^5}\ ,\ \ldots$$

and

$$1,\ -\frac{1}{2}\ ,\ \frac{1}{2^2}\ ,\ -\frac{1}{2^3}\ ,\ \frac{1}{2^4}\ ,\ -\frac{1}{2^5}\ ,\ \ldots\ .$$

41. We consider the scalar difference equation

$$x_{k+1} = x_k + \lambda x_k(2 - x_k),$$

whose unknown is the sequence $(x_k)_{k\geq 0}$, and where $\lambda \in (0,1)$ is a parameter. Find its constant solutions (fixed points) and study their stability. Discuss with respect to the parameter λ.

42. We consider the IVP $x' = -10^3 x, \quad x(0) = 1$.

a) Find the solution and its limit as $t \to \infty$.

b) Write the Euler's numerical formula with constant step-size h.

c) Find a range of values for the step-size h such that the solution $(x_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} x_k = 0$.

43. We consider the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x - \frac{1}{4}(x^2 - 2)$ and, given $x_0 \in \mathbb{R}$, consider the sequence $(x_k)_{k\geq 0}$ satisfying the recurrence

$$x_{k+1} = f(x_k)\ .$$

a) Find the fixed points of f, and study their stability.

b) Find $(x_k)_{k\geq 0}$ when $x_0 = \sqrt{2}$.

c) There exists an $x_0 \in \mathbb{R} \setminus \{\sqrt{2}\}$ such that $\lim\limits_{k\to\infty} x_k = \sqrt{2}$?

d) There exists an $x_0 \in \mathbb{R}$ such that $\lim\limits_{k\to\infty} x_k = 2$?