affine variety: $\mathcal{A} = a + U = \{a + \vec{v} \mid \vec{v} \in U\}$

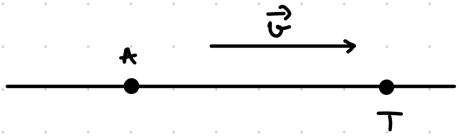
$a \in \mathbb{E}^n$ (if the point is the origin, then it's a subspace)

$U \leq V^n$ "a subspace that you can move around"

if $d=1 \Rightarrow a+U$ <u>line</u> $\rightarrow$ one dimensional object that requires 2 distinct geometric objects to describe

$d=2 \Rightarrow a+U$ <u>plane</u>

$d = \dim U$



we want to describe T

Fix an origin 0

$\vec{v_T} = \vec{v_A} + \vec{A_T}$

positional vector of T

$\exists \lambda \in \mathbb{R}: \vec{AT} = \lambda \cdot \vec{v} \Rightarrow \vec{v_T} = \vec{v_A} + \lambda \cdot \vec{v}$ - vector equation of ℓ

Fix the reference system

$\mathcal{K} = (0, B)$

$[T]_{\mathcal{K}} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad [A]_{\mathcal{K}} = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$

$[\vec{v}]_{\mathcal{K}} = \begin{pmatrix} x_{\vec{v}} \\ y_{\vec{v}} \end{pmatrix}$

$\ell: \begin{cases} x = x_A + \lambda x_{\vec{v}} \\ y = y_A + \lambda y_{\vec{v}} \end{cases}, \lambda \in \mathbb{R}$ — parametric equation

If $x_v, y_v \neq 0$: $\dfrac{x - x_A}{x_{\vec{v}}} = \dfrac{y - y_A}{y_{\vec{v}}}$ - symmetric form

If $x_{\vec{v}} = 0 \Rightarrow x = x_A$ (vertical line)

$y_{\vec{v}} = 0 \Rightarrow y = y_A$ (horizontal line)

Implicit form:

$y_{\vec{v}}(x - x_A) - x_{\vec{v}}(y - y_A) = 0$

$Ax + By + C = 0$

Explicit form:

if $A = 0, B \neq 0 \Rightarrow y = -\dfrac{C}{B}$

$B = 0, A \neq 0 \Rightarrow x = -\dfrac{C}{A}$

$A, B \neq 0 \Rightarrow y = -\dfrac{A}{C} x - \dfrac{C}{B}$

$y = mx + n$

$\ell: Ax + By + C = 0$

$\ell: \begin{cases} x = \lambda \\ y = -\dfrac{A}{B}\lambda - \dfrac{C}{B} \end{cases}$

$D(\ell) = \langle \vec{v} \rangle = \langle (x_{\vec{v}}, y_{\vec{v}}) \rangle$

2.5 Exercises

2.1. Determine parametric equations for the line $\ell \subseteq \mathbb{A}^2$ in the following cases:

a) ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,

b) ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,

c) ℓ contains the point $M(1,7)$ and is parallel to Oy,

d) ℓ contains the points $M(2,4)$ and $N(2,-5)$.

2.2. For the lines ℓ in the previous exercise

a) determine a Cartesian equation for ℓ,

b) describe all direction vectors for ℓ.

2.3. With the assumptions in Example 1.20, give parametric equations and Cartesian equations for the lines AB, AC, BC both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

2.4. Find a Cartesian equation of the line ℓ in $\mathbb{A}^2$ containing the points $P = S \cap S'$ and $Q = T \cap T'$ where

$$S : x + 5y - 8 = 0, \quad S' : 3x + 6 = 0, \quad T : 5x - \frac{1}{2}y = 1, \quad T' : x - y = 5.$$

2.5. Deterimine an equation for the line in $\mathbb{A}^2$ parallel to $\mathbf{v}$ and passing through $S \cap T$ in each of the following cases:

1. $\mathbf{v} = (2,4)$, $S : 3x - 2y - 7 = 0$, $T : 2x + 3y = 0$,
2. $\mathbf{v} = (-5\sqrt{2}, 7)$, $S : x - y = 0$, $T : x + y = 1$.

2.6. Let ABC be a triangle in the affine space $\mathbb{A}^n$. Consider the points C' and B' on the sides AB and AC respectively, such that

$$\overrightarrow{AC'} = \lambda \overrightarrow{BC'} \quad \text{and} \quad \overrightarrow{AB'} = \mu \overrightarrow{CB'}.$$

The lines BB' and CC' meet in the point M. For a fixed but arbitrary point $O \in \mathbb{A}^n$, show that

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} - \lambda \overrightarrow{OB} - \mu \overrightarrow{OC}}{1 - \lambda - \mu}.$$

Deduce a formula for $\overrightarrow{OG}$ where G is the centroid of the triangle.

2.7. In $\mathbb{A}^n$, consider the angle BOB' and the points $A \in [OB]$, $A' \in [OB']$. Show that

$$\overrightarrow{OM} = m\frac{1-n}{1-mn}\overrightarrow{OA} + n\frac{1-m}{1-mn}\overrightarrow{OA'}$$

$$\overrightarrow{ON} = m\frac{n-1}{n-m}\overrightarrow{OA} + n\frac{m-1}{m-n}\overrightarrow{OA'}$$

where $M = AB' \cap A'B$ and $N = AA' \cap BB'$ and where $\overrightarrow{OB} = m\overrightarrow{OA}$ and $\overrightarrow{OB'} = n\overrightarrow{OA'}$.

2.1. Determine parametric equations for the line $\ell \subseteq \mathbb{A}^2$ in the following cases:

a) ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,

b) ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,

c) ℓ contains the point $M(1,7)$ and is parallel to Oy,

d) ℓ contains the points $M(2,4)$ and $N(2,-5)$.

2.2. For the lines ℓ in the previous exercise

a) determine a Cartesian equation for ℓ,

b) describe all direction vectors for ℓ.

a) $\ell : \begin{cases} x = x_A + \lambda x_{\vec{a}} \\ y = y_A + \lambda y_{\vec{a}} \end{cases}, \lambda \in \mathbb{R}$ — parametric equation

$\ell \ni A(1,2), \quad \ell \parallel \vec{a}(3,-1)$

$\ell : \begin{cases} x = 1 + \lambda \cdot 3 \\ y = 2 + \lambda \cdot (-1) \end{cases} \Rightarrow \lambda = \frac{x-1}{3}, \quad \lambda = \frac{y-2}{(-1)} = -y+2$

$\Rightarrow \ell : 2 - y = \frac{x-1}{3} \Leftrightarrow y = \frac{1-x}{3} + 2 \Leftrightarrow \frac{x-1}{3} + y - 2 = 0$

$D(\ell) = \langle \vec{a} \rangle = \langle (3,-1) \rangle$

b) $\ell \ni O(0,0), \quad \ell \parallel \vec{b}(4,5)$

$\ell : \begin{cases} x = 4 \cdot \lambda \\ y = 5 \cdot \lambda \end{cases} \Rightarrow \lambda = \frac{x}{4}, \quad \lambda = \frac{y}{4}$

$\ell : \frac{x}{4} = \frac{y}{5} \Leftrightarrow 5x - 4y = 0 \Leftrightarrow x = \frac{4}{5} y \Leftrightarrow y = \frac{5}{4} x$

$D(\ell) = \langle \vec{b} \rangle = \langle (4,5) \rangle$

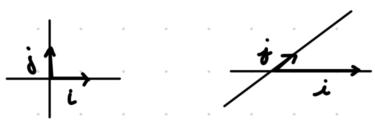
c) $\ell \ni M(1,7) \quad \ell \parallel Oy \Rightarrow \ell \parallel \vec{c}(0,1)$

$\ell : \begin{cases} x = 1 \\ y = 7 + \lambda \end{cases}$ → this doesn't tell us anything

$\Rightarrow \ell : x - 1 = 0 \Leftrightarrow \ell : x = 1$

$D(\ell) = \langle \vec{c} \rangle = \langle (0,1) \rangle = Oy$

* as long as you don't talk about dot product and perpendicularity, everything valid here is valid here too



d) $\ell \ni M(2,4), N(2,-5)$

$\vec{MN} = (2-2, -5-4) = (0,-9)$

$\ell : \begin{cases} x = 2 \\ y = 4 - 9\lambda \end{cases}$

$\ell : x - 2 = 0 \Leftrightarrow \ell : x = 2 \quad \Rightarrow D(\ell) = \langle \vec{MN} \rangle = \langle (0,-9) \rangle = \langle (0,1) \rangle$

* to prove that two vectors are parallel, you can use $D(\ell)$ and say it's the sam

2.5. Deterimine an equation for the line in $\mathbb{A}^2$ parallel to $\mathbf{v}$ and passing through $S \cap T$ in each of the following cases:

1. $\vec{\mathbf{v}} = (2,4)$, $S : 3x - 2y - 7 = 0$, $T : 2x + 3y = 0$,
2. (home*) $\vec{\mathbf{v}} = (-5\sqrt{2}, 7)$, $S : x - y = 0$, $T : x + y = 1$.

$S \cap T$ – solve the system

$$\begin{cases} 3x - 2y - 7 = 0 & \mid \cdot 3 \\ 2x + 3y = 0 & \mid \cdot 2 \end{cases} \Leftrightarrow \begin{cases} 9x - 6y - 21 = 0 \\ 4x + 6y = 0 \end{cases} \quad (+)$$

$$13x - 21 = 0$$

$$x = \frac{21}{13}$$

$$x = \frac{-3}{2}y$$

$$\frac{-9}{2}y - 2y - 7 = 0$$

$$-9y - 4y - 14 = 0$$

$$-13y - 14 = 0$$

$$y = \frac{-14}{13}$$

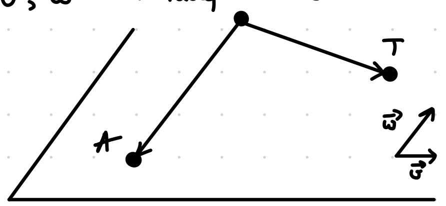
$$\Rightarrow S \cap T = \vec{w}\left(\frac{21}{13}, \frac{-14}{13}\right) \in \ell$$

$$\ell : \begin{cases} x = \frac{21}{13} + 2\lambda \\ y = \frac{-14}{13} + 4\lambda \end{cases} \Rightarrow \frac{x - \frac{21}{13}}{2} = \lambda, \quad \lambda = \frac{y + \frac{14}{13}}{4}$$

$$\Rightarrow \ell : 2x - \frac{42}{13} = y + \frac{14}{13} \Leftrightarrow y = 2x - \frac{56}{13} \Leftrightarrow 2x - y - \frac{56}{13} = 0$$

π plane, $A \in \pi$

$\vec{v}, \vec{w}$ lin. indep. vectors in $D(\pi)$ (they form a basis of $D(\pi)$)



$$\vec{r}_T = \vec{r}_A + \vec{AT}$$

$\vec{AT} \in D(\pi) \Rightarrow \exists \lambda, \mu \in \mathbb{R} : \vec{AT} = \lambda \vec{v} + \mu \vec{w}$

$$\vec{r}_T = \vec{r}_A + \lambda \vec{v} + \mu \cdot \vec{w} \rightarrow \text{vector eq of plane}$$

Fix $\mathcal{K} = (0, \mathcal{B})$

$$\Rightarrow \pi : \begin{cases} x = x_A + \lambda x_{\vec{v}} + \mu x_{\vec{w}} \\ y = y_A + \lambda y_{\vec{v}} + \mu y_{\vec{w}} \\ z = z_A + \lambda z_{\vec{v}} + \mu z_{\vec{w}} \end{cases} \rightarrow \text{parametric equation of plane}$$

* a line in 2D is more symmilar to a plane in 3D because they are d-1 of the dimention of the space

the symmetric form:

$$\begin{vmatrix} x-x_A & y-y_A & z-z_A \\ x_{\vec{v}} & y_{\vec{v}} & z_{\vec{v}} \\ x_{\vec{w}} & y_{\vec{w}} & z_{\vec{w}} \end{vmatrix} = 0$$

the implicit form (after opening up↑)

$$Ax + By + Cz + D = 0$$

2.10. Determine Cartesian equations for the plane π in the following cases:

a) $\pi : x = 2 + 3u - 4v,\ y = 4 - v,\ z = 2 + 3u;$

b) $\pi : x = u + v,\ y = u - v,\ z = 5 + 6u - 4v.$, write $D(\pi)$

a)
$$\begin{vmatrix} x-2 & y-4 & z-2 \\ 3 & 0 & 3 \\ -4 & -1 & 0 \end{vmatrix} = 0$$

$$= -3z + \cancel{6} - 12y + 48 + 3x - \cancel{6} =$$

$$= 3x - 12y - 3z + 48 = 0 \quad | :3$$

$$\pi : \quad x - 4y - z + 16 = 0$$

$$D(\pi) = \langle (3,0,3), (-4,-1,0) \rangle = \langle u, v \rangle$$

b)
$$\begin{vmatrix} x & y & z-5 \\ 1 & 1 & 6 \\ 1 & -1 & -4 \end{vmatrix} = 0$$

$$= -4x - z + 5 + 6y - z + 5 + 6x + 4y =$$

$$= 2x + 10y - 2z + 10 = 0 \quad | :2$$

$$\pi : \quad x + 5y - z + 5 = 0$$

$$D(\pi) = \langle (1,1,6), (1,-1,-4) \rangle = \langle u, v \rangle$$

$$A, B, C \in \pi \iff \begin{matrix} A \in \pi \\ \vec{AB}, \vec{AC} \in D(\pi) \end{matrix}$$

$$\iff \begin{vmatrix} x-x_A & y-y_A & z-z_A \\ x_B-x_A & y_B-y_A & z_B-z_A \\ x_C-x_A & y_C-y_A & z_C-z_A \end{vmatrix} = 0$$ symmetric form of a plane described by 3 points

$$\iff \begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0$$

2.8. Show that the midpoints of the diagonals of a complete quadrilateral are collinear.

2.9. Determine parametric equations for the plane π in the following cases:

a) π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1(3,-1,1)$ and $\mathbf{a}_2(0,3,1)$,

b) π contains the points $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$,

c) π contains the point $A(1,2,1)$ and is parallel to $\mathbf{i}$ and $\mathbf{j}$,

d) π contains the point $M(1,7,1)$ and is parallel coordinate plane Oyz,

e) π contains the points $M_1(5,3,4)$ and $M_2(1,0,1)$, and is parallel to the vector $\mathbf{a}(1,3,-3)$,

f) π contains the point $A(1,5,7)$ and the coordinate axis Ox.

2.10. Determine Cartesian equations for the plane π in the following cases:

a) $\pi : x = 2 + 3u - 4v,\ y = 4 - v,\ z = 2 + 3u;$

b) $\pi : x = u + v,\ y = u - v,\ z = 5 + 6u - 4v.$

2.11. Determine parametric equations for the plane π in the following cases:

a) $3x - 6y + z = 0;$

b) $2x - y - z - 3 = 0.$

2.12. With the assumptions in Example 1.21, give parametric equations and Cartesian equations for the line AB and the plane ACD both in the coordinate system $\mathcal{K}$ and in the coordinate system $\mathcal{K}'$.

2.13. Show that the points $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ and $D(4,2,3)$ are coplanar.

2.14. Determine the relative positions of the planes in the following cases

a) $\pi_1 : x + 2y + 3z - 1 = 0$, $\pi_2 : x + 2y - 3z - 1 = 0$.

b) $\pi_1 : x + 2y + 3z - 1 = 0$, $\pi_2 : 2x + y + 3z - 2 = 0$, $\pi_3 : x + 2y + 3z + 2 = 0$.

2.15. Show that the planes

$$\pi_1 : 3x + y + z - 1 = 0, \quad \pi_2 : 2x + y + 3z + 2 = 0, \quad \pi_3 : -x + 2y + z + 4 = 0$$

have a point in common.

2.16. Show that the pairwise intersection of the planes

$$\pi_1 : 3x + y + z - 5 = 0, \quad \pi_2 : 2x + y + 3z + 2 = 0, \quad \pi_3 : 5x + 2y + 4z + 1 = 0$$

are parallel lines.

2.17. Determine parametric equations for the line ℓ in the following cases:

2.11. Determine parametric equations for the plane π in the following cases:

a) $3x - 6y + z = 0$;

b) $2x - y - z - 3 = 0$.

> find $D(\pi)$

a) $3x - 6y + z = 0$

$z = 6y - 3x$

let $\begin{cases} x = u \\ y = v \\ z = 6v - 3u \end{cases} \Rightarrow \begin{vmatrix} x & y & z \\ 1 & 0 & -3 \\ 0 & 1 & 6 \end{vmatrix} = 0 \Rightarrow D(\pi) = \langle (1,0,-3), (0,1,6) \rangle$

b) $2x - y - z - 3 = 0$

$z = 2x - y - 3$

$\begin{cases} x = u \\ y = v \\ z = 2u - v - 3 \end{cases} \Rightarrow \begin{vmatrix} x & y & z+3 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{vmatrix} = 0 \Rightarrow D(\pi) = \langle (1,0,2), (0,1,-1) \rangle$

a) ℓ contains the point $M_0(2,0,3)$ and is parallel to the vector $\mathbf{a}(3,-2,-2)$,

b) ℓ contains the point $A(1,2,3)$ and is parallel to the Oz-axis,

c) ℓ contains the points $M_1(1,2,3)$ and $M_2(4,4,4)$.

2.18. Give Cartesian equations for the lines ℓ in the previous exercise.

2.19. Determine parametric equations for the line contained in the planes $x+y+2z-3=0$ and $x-y+z-1=0$.

2.20. Consider the lines $\ell_1 : x = 1+t, y = 1+2t, z = 3+t, t \in \mathbb{R}$ and $\ell_2 : x = 3+s, y = 2s, z = -2+s, s \in \mathbb{R}$. Show that ℓ_1 and ℓ_2 are parallel and find the equation of the plane determined by the two lines.

2.21. Determine parametric equations of the line passing through $P(5,0,-2)$ and parallel to the planes $\pi_1 : x-4y+2z=0$ and $\pi_2 : 2x+3y-z+1=0$.

2.22. Determine an equation of the plane containing $P(2,0,3)$ and the line $\ell : x = -1+t, y = t, z = -4+2t, t \in \mathbb{R}$.

2.23. For the points $A(2,1,-1)$ and $B(-3,0,2)$, determine an equation of the bundle of planes passing through A and B.

2.24. Determine the relative positions of the lines $x = -3t, y = 2+3t, z = 1, t \in \mathbb{R}$ and $x = 1+5s, y = 1+13s, z = 1+10s, s \in \mathbb{R}$.

2.25. Determine the parameter m for which the line $x = -1+3t, y = 2+mt, z = -3-2t$ doesn't intersect the plane $x+3y+3z-2=0$.

2.26. Determine the values a and d for which the line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$ is contained in the plane $ax+y-2z+d=0$.

2.27. In each of the following, find a Cartesian equation of the plane in $\mathbb{A}^3$ passing through Q and parallel to the lines ℓ and ℓ':

a) $Q(1,-1,-2)$, $\ell : x-y=1, x+z=5$, $\ell' : x=1, z=2$

b) $Q(0,1,3)$, $\ell : x+y=-5, x-y+2z=0$, $\ell : 2x-2y=1, x-y+2z=1$

2.28. In each of the following, find the relative positions of the line ℓ and the plane π in $\mathbb{A}^3$, and, if they are incident, determine the point of intersection.

a) $\ell : x = 1+t, y = 2-2t, z = 1-4t$, $\pi : 2x-y+z-1=0$

b) $\ell : x = 2-t, y = 1+2t, z = -1+3t$, $\pi : 2x+2y-z+1=0$

2.29. In each of the following, find a Cartesian equation for the plane in $\mathbb{A}^3$ containing the point Q and the line ℓ.

a) $Q = (3,3,1)$, $\ell : x = 2+3t, y = 5+t, z = 1+7t$

b) $Q = (2,1,0)$, $\ell : x - y + 1 = 0, 3x + 5z - 7 = 0$

2.30. In each of the following, find Cartesian equations for the line ℓ in $\mathbb{A}^3$ passing through Q, contained in the plane π and intersecting the line ℓ'

a) $Q = (1,1,0)$, $\pi : 2x - y + z - 1 = 0$, $\ell' : x = 2 - t, y = 2 + t, z = t$

b) $Q = (-1,-1,-1)$, $\pi : x + y + z + 3 = 0$, $\ell' : x - 2z + 4 = 0, 2y - z = 0$

2.31. In each of the following, find Cartesian equations for the line ℓ in $\mathbb{A}^3$ passing through Q and coplanar to the lines ℓ' and ℓ''. Furthermore, establish whether ℓ meets or is parallel to ℓ' and ℓ''

a) $Q = (1,1,2)$, $\ell' : 3x - 5y + z = -1, 2x - 3z = -9$, $\ell'' : x + 5y = 3, 2x + 2y - 7z = -7$

b) $Q = (2,0,-2)$, $\ell' : -x + 3y = 2, x + y + z = -1$, $\ell'' : x = 2 - t, y = 3 + 5t, z = -t$

2.32. In each of the following, find the value of the real parameter k for which the lines ℓ and ℓ' are coplanar. Find a Cartesian equation for the plane that contains them, and find the point of intersection whenever they meet

a) $\ell : x = k + t, y = 1 + 2t, z = -1 + kt$, $\ell' : x = 2 - 2t, y = 3 + 3t, z = 1 - t$

b) $\ell : x = 3 - t, y = 1 + 2t, z = k + t$, $\ell' : x = 1 + t, y = 1 + 2t, z = 1 + 3t$

2.33. Find a Cartesian equation for the plane π in $\mathbb{A}^3$ which contains the line of intersection of the two planes

$$x + y = 3 \quad \text{and} \quad 2y + 3z = 4$$

and is parallel to the vector $\mathbf{v} = (3,-1,2)$.

2.34. In the affine space $\mathbb{A}^4$ consider

$$\text{the plane} \quad \alpha = \langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix} \rangle + \begin{bmatrix}2\\4\\1\\2\end{bmatrix} \quad \text{and the line} \quad \beta = \langle \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} \rangle + \begin{bmatrix}2\\3\\-1\\1\end{bmatrix}.$$

Determine $\alpha \cap \beta$.

2.35. In $\mathbb{A}^4$ consider the affine subspaces

$$\alpha = \begin{bmatrix}2\\1\\2\\1\end{bmatrix} \quad \beta = \langle \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \rangle + \begin{bmatrix}1\\3\\0\\0\end{bmatrix} \quad \gamma = \langle \begin{bmatrix}2\\1\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-2\end{bmatrix} \rangle + \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad \delta = \langle \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \rangle.$$

Which of the following is true?

a) $\alpha \in \beta$

b) $\alpha \in \gamma$

c) $\alpha \in \delta$

d) $\beta \parallel \gamma$

e) $\beta \parallel \delta$

f) $\gamma \parallel \delta$

g) $\beta \subseteq \gamma$

h) $\gamma \subseteq \delta$

i) $\beta \subseteq \delta$

2.36. Consider the following affine subspaces of $\mathbb{A}^4$

$$Y : \begin{cases} x_1 + x_3 - 2 &= 0 \\ 2x_1 - x_2 + x_3 + 3x_4 - 1 &= 0 \end{cases}$$

$$Z : \begin{cases} x_1 + x_2 + 2x_3 - 3x_4 &= 1 \\ x_2 + x_3 - 3x_4 &= -1 \\ x_1 - x_2 + 3x_4 &= 3 \end{cases}$$

a) Determine the dimensions of Y and Z.

b) Find parametric equations for each of the two affine subspaces.

c) Is $Y \parallel Z$?

2.37. In Section 2.2.2 we deduce a linear equation for a plane in $\mathbb{A}^3$ via a determinant. What is the analogue of this description for lines? I.e. deduce Cartesian equations for lines starting from linear dependence of vectors (both in $\mathbb{A}^2$ and $\mathbb{A}^3$).

2.38. Consider the affine space $\mathbb{A}^3$. Show that if a line ℓ doesn't intersect a plane π then $\ell \parallel \pi$ in the sense of the Definition 2.14. Moreover, give an example in $\mathbb{A}^4$ of a line and a plane which do not intersect and which are not parallel.

2.39. Consider the affine space $\mathbb{A}^4$. Describe the relative positions of two planes.

2.40. In $\mathbb{A}^3$ discuss the relative positions of a plane and a line in terms of their Cartesian equations.

2.41. In $\mathbb{A}^3$ discuss the relative positions two lines in terms of their Cartesian equations.