Seminar 8

1. Let $A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Show that A is invertible, determine A^{-1} and solve the linear system $AX = B$.

2. Using the Kronecker-Capelli theorem, decide if the following linear systems are compatible and then solve the compatible ones:

$$(i) \begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases} \qquad (ii) \begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

$$(iii) \begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$$

3. Using the Rouché theorem, decide if the systems from **2.** are compatible and then solve the compatible ones.

4. Decide when the following linear system is compatible determinate and in that case solve it by using Cramer's method:

$$\begin{cases} ay + bx = c \\ cx + az = b \\ bz + cy = a \end{cases} \quad (a, b, c \in \mathbb{R}).$$

Solve the following linear systems by the Gauss and Gauss-Jordan methods:

$$\mathbf{5}. \quad (i) \begin{cases} 2x + 2y + 3z = 3 \\ x - y \quad = 1 \\ -x + 2y + z = 2 \end{cases} \qquad (ii) \begin{cases} 2x + 5y + z = 7 \\ x + 2y - z = 3 \\ x + y - 4z = 2 \end{cases} \qquad (iii) \begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$$

$$\mathbf{6}. \quad \begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases} \qquad (\lambda \in \mathbb{R})$$

$$\mathbf{7}. \quad \begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases} \qquad (a \in \mathbb{R})$$

8. Determine the positive solutions of the following non-linear system:

$$\begin{cases} xyz = 1 \\ x^3y^2z^2 = 27 \\ \frac{z}{xy} = 81 \end{cases}$$

Kronecker - Capelli

System is compatible iff $\operatorname{rank} M_S = \operatorname{rank} \overline{M_S}$

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n \end{cases} \quad \overset{(*)}{\Leftrightarrow} \quad x_1 \underbrace{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}}_{v_1} + x_2 \cdot \underbrace{\begin{pmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{pmatrix}}_{v_2} + \dots + x_n \cdot \underbrace{\begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nn} \end{pmatrix}}_{v_n} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

S - compatible $\Leftrightarrow \exists\ \alpha_1, \dots x_n$ s.t. $(*)$ is true $\Leftrightarrow v = \alpha_1 v_1 + \dots + x_n \cdot v_n \Leftrightarrow$

$\Leftrightarrow v \in \langle v_1, \dots, v_n \rangle$

Rouché

S system is compatible $\Leftrightarrow$ all the characteristic minors are 0.

principal minor	column of free terms
row of the matrix	

– characteristic minor

10. Consider the following homogeneous relations on $\mathbb{N}$, defined by:

$$m\,r\,n \iff \exists a \in \mathbb{N} : m = 2^a n\,,$$

$$m\,s\,n \iff (m = n \text{ or } m = n^2 \text{ or } n = m^2)\,.$$

Are r and s equivalence relations?

5II. 10

Suppose $m\ s\ n$
$n\ s\ k$

	$n = k$	$n = k^2$	$m^2 = k$
$m = n$	$m = k$	$m = k^2$	$m^2 = k$
$m = n^2$	$m = k^2$	...	
$n = m^2$			

contradiction:

$k = 2 \quad n = 4 \quad m = 16$

$16 = 4^2 \Rightarrow m\ s\ n$

$4 = 2^2 \Rightarrow n\ s\ k$

but $\left.\begin{array}{l} 16 \neq 2 \\ 16 \neq 2^2 \\ 16^2 \neq 2 \end{array}\right\} \Rightarrow m \not s\ k$

$\Rightarrow$ msn not an equivalence relation

Ex) 8.2.3 Using Kronecker Capelli - Rouché decide if the system is comp. & if so, find the solution

$$\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases}$$

$$M_s = \left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 2 & 1 & -2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 3 \end{array}\right)$$

rank M_s = rank $\overline{M_s}$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = 1 - 6 - 4 - 2 - 6 - 2 = -19 \neq 0$$

$\Rightarrow$ there are no characteristic minors $\Rightarrow$ system is compatible

$$\begin{cases} \text{eq } 1,2,3 \text{ principle equations} \\ \text{no secondary equations} \\ x_1, x_2, x_3 - \text{principal unknowns} \\ x_4 - \text{secondary unknown} \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 5 + 2\alpha \\ 2\alpha_1 + x_2 - 2x_3 = 1 - \alpha \\ 2x_1 - 3x_2 + x_3 = 3 - 2\alpha \end{cases}$$

$$x_1 = \frac{\Delta x_1}{\Delta} \qquad x_2 = \frac{\Delta x_2}{\Delta} \qquad x_3 = \frac{\Delta x_3}{\Delta}$$

$$\Delta x_1 = \begin{vmatrix} 5+2\alpha & 1 & 1 \\ 1-\alpha & 1 & -2 \\ 3-2\alpha & -3 & 1 \end{vmatrix} = 5+2\alpha - 3(1-\alpha) - 2(3-2\alpha) - (3-2\alpha) - 6(5+2\alpha) - (1-\alpha) =$$

$$= 5 + \underline{2\alpha} - 3 + \underline{3\alpha} - 6 + \underline{4\alpha} - 3 + \underline{2\alpha} - 30 - \underline{12\alpha} - 1 + \underline{\alpha} =$$

$$= -38$$

$$x_1 = \frac{-38}{-19} = 2$$

$$\Delta x_2 = \begin{vmatrix} 1 & 5+2\alpha & 1 \\ 2 & 1-\alpha & -2 \\ 2 & 3-2\alpha & 1 \end{vmatrix} = 1 - \underline{\alpha} + 6 - \underline{4\alpha} - 20 - \underline{8\alpha} - 2 + \underline{2\alpha} + 6 - \underline{4\alpha} - 10 - \underline{4\alpha} =$$

$$= -19\alpha - 19$$

$$x_2 = \frac{-19\alpha - 19}{-19} = 1 + \alpha$$

$$\Delta x_3 = \begin{vmatrix} 1 & 1 & 5+2\alpha \\ 2 & 1 & 1-\alpha \\ 2 & -3 & 3-2\alpha \end{vmatrix} = 3 - 2\alpha - 6(5+2\alpha) + 2 - 2\alpha - 10 - 4\alpha + 3 - 3\alpha - 6 + 4\alpha =$$

$$= -19\alpha - 38$$

$$\Delta x_3 = \frac{-19\alpha - 38}{-19} = \alpha + 2$$

8.5. Solve the following linear systems using Gauss or Gauss-Jordan Method

$$(i) = \begin{cases} 2x + 2y + 3z = 3 \\ x - y = 1 \\ -x + 2y + z = 2 \end{cases}$$

$$A = \left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right) \xrightarrow{\text{row echalon}}_{\text{form}} \underset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ -1 & 2 & 1 & 2 \end{array}\right) \underset{L_3 \leftarrow L_3 + L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right)$$

pivot

* No operations on colums!! All operations must be done on Rows

Go 1 row lower and to the right

$$\underset{L_2 \leftrightarrow L_3}{\sim}\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 4 & 3 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 4L_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right)$$

0's should be s. increasing by each row

Stop and revert to a system:

$$\begin{cases} x - y = -1 & \Rightarrow x = -1 + (-8) = -7 \\ y + z = 3 & \Rightarrow y = 3 - 11 = -8 \\ -z = -11 & \Rightarrow z = 11 \end{cases}$$

If you are applying Gauss-Jordan we continue the transformations in the opposite order

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right) \underset{L_2 \leftarrow L_2 + L_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & -1 & -11 \end{array}\right) \underset{L_1 \leftarrow L_1 + L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & -1 & -11 \end{array}\right) \Rightarrow \begin{array}{l} x = -7 \\ y = -8 \\ z = 11 \end{array}$$

pivot

(ii) $$\begin{cases} 2x + 5y + z = 7 \\ x + 2y - z = 3 \\ x + y - 4z = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right) \underset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -4 & 2 \end{array}\right) \underset{\substack{L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - L_1}}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \end{array}\right)$$

principal unknowns

$$\underset{L_3 \leftarrow L_2 + L_3}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \Rightarrow \begin{cases} x + 2y = 3 + \alpha \\ y = 1 - 3\alpha \\ z = \alpha \end{cases} \Rightarrow x = 3 + \alpha - 2y = 3 + \alpha - 2 + 6\alpha = 1 + 7\alpha$$

comes from a redundant equations

$$(x, y, z) = (1 + 7\alpha,\ 1 - 3\alpha,\ \alpha)$$

8. 6. $\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases}$

$\lambda \in \mathbb{R}$

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \overset{L_2 \leftrightarrow L_1}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 3 & -3 & 7 & \lambda-2 \end{array}\right) \overset{L_3 \leftarrow L_3 + L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & \lambda-5 \end{array}\right)$$

$\Rightarrow \lambda = 5$

$$\begin{cases} x + 2y = 2 + \alpha - 4\beta & \mid \cdot 3 \\ \quad -3y = -3 - 3\alpha + 7\beta & \mid \cdot 2 \end{cases}$$

$$\begin{cases} 3x + 6y = 6 + 3\alpha - 12\beta \\ \quad -6y = -6 - 6\alpha + 14\beta \end{cases} \quad (+)$$

$3x = -3\alpha + 2\beta$

$x = -\alpha + \frac{2}{3}\beta$

$y = 1 + \alpha - \frac{7}{3}\beta$

7. $\begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases} \qquad (a \in \mathbb{R})$

$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ a & 1 & 1 & 1 \\ 1 & 1 & a & a^2 \end{array}\right) \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - aL_1}{\sim}} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a^2 & 1-a & 1-a^2 \\ 0 & 1-a & a-1 & a(a-1) \end{array}\right)$$

$$\underset{\sim}{L_2 \leftrightarrow L_3}\left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a(a-1) \\ 0 & 1-a^2 & 1-a & 1-a^2 \end{array}\right) \underset{\sim}{L_3 \leftarrow L_3-(1+a)L_2} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a(a-1) \\ 0 & 0 & \begin{matrix}1-a-\\ a^2+1\end{matrix} & \underbrace{\begin{matrix}1-a^2-\\ a(a^2-1)\end{matrix}} \end{array}\right)$$

$1-a-(a+1)(a-1)$

$(a-1)(-1-a-1) = (a-1)(-a-2)$

$-(a^2-1)-a(a^2-1)$

$-(a^2-1)(a+1)$

if $a=1 \Rightarrow x + \quad z = 1$

$$\left.\begin{array}{l} x = 1-z \\ \text{rank } A = 1 \end{array}\right\} \Rightarrow (\alpha, \beta, 1-\alpha)$$

$$a \neq 1 \Rightarrow \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & -(a-1) & a-1 & a(a-1) \\ 0 & 0 & -(a-1)(a+2) & -(a^2-1)(a+1) \end{array}\right)$$

$$\underset{\sim}{L_2 \leftarrow L_3 + L_2(a+2)} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & -(a-1) & 0 & -(a^2-1)(a+1)+a(a-1)(a+2) \\ 0 & 0 & -(a-1)(a+2) & -(a^2-1)(a+1) \end{array}\right)$$