Chapter 5

Proposition 5.4. A map $\phi : \mathbb{A}^n \to \mathbb{A}^m$ is an affine morphism if and only if there exists $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ and $b \in \mathrm{Mat}_{m\times 1}(\mathbb{R})$ such that

$$[\phi(P)]_{\mathcal{K}'} = A \cdot [P]_{\mathcal{K}} + b \tag{5.3}$$

relative to some reference frame $\mathcal{K}$ of $\mathbb{A}^n$ and some reference frame $\mathcal{K}'$ of $\mathbb{A}^m$.

$$P' = P - \frac{a_1p_1 + \cdots + a_np_n + a_{n+1}}{a_1v_1 + \cdots + a_nv_n}\mathbf{v} = P - \frac{\mathbf{a}^T \cdot P + a_{n+1}}{\mathbf{a}^T \cdot \mathbf{v}}\mathbf{v}.$$

projection on the hyperplane H parallel to v

$$\mathrm{Pr}_{H,\mathbf{v}}(P) = \left(I_n - \frac{\mathbf{v}\cdot\mathbf{a}^T}{\mathbf{v}^T\cdot\mathbf{a}}\right)\cdot P - \frac{a_{n+1}}{\mathbf{v}^T\cdot\mathbf{a}}\mathbf{v} = \left(I_n - \frac{v \otimes a}{\langle v, a\rangle}\right)\cdot P - \frac{a_{n+1}}{\langle v, a\rangle}\cdot v$$

$$\mathrm{Pr}_\ell^\perp(P) = \frac{\mathbf{a}\otimes\mathbf{a}}{|\mathbf{a}|^2}P + \left(I_n - \frac{\mathbf{a}\otimes\mathbf{a}}{|\mathbf{a}|^2}\right)Q$$

→ projection on a line ℓ parallel to a hyperplan

Cap 5: 1, 2, 4, 5, 11, 12, T. 13, 14, 16, 18, 19, T. 20.3, 22, 23, 24

5.1. Consider an orthonormal coordinate system $\mathcal{K}$ of $\mathbb{E}^n$ where $n = 2$ or 3. Starting from the matrix form of the projections and reflections described in this Chapter, deduce the matrices of

a) the orthogonal projections on the coordinate axes and on the coordinate hyperplanes of $\mathcal{K}$.

b) the orthogonal reflections in coordinate axes and in coordinate hyperplanes of $\mathcal{K}$.

a) $\left.\begin{array}{l} (xOy): z=0 \\ (yOz): x=0 \\ (xOz): y=0 \end{array}\right\} \Rightarrow$ $\quad Ox: \begin{cases} y=0 \\ z=0 \end{cases}$ $\quad Oy: \begin{cases} x=0 \\ z=0 \end{cases}$ $\quad Oz: \begin{cases} x=0 \\ y=0 \end{cases}$

$$\mathrm{Pr}_{H,v}(P) = \left(I_n - \frac{a \otimes v}{\langle a, v\rangle}\right)\cdot P - \frac{a_{n+1}}{\langle a, v\rangle}\cdot v$$

$H: a_1x_1 + a_2x_2 + \ldots + a_nx_n + a_{n+1} = 0$

$a = (0, 0, 1)$

$$\mathrm{Pr}_{xOy,a}(P) = \left(I_n - \frac{a \otimes a}{\langle a, a\rangle}\right)\cdot P - \frac{a_{n+1}}{\langle a, a\rangle}\cdot v$$

$$a \otimes a = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

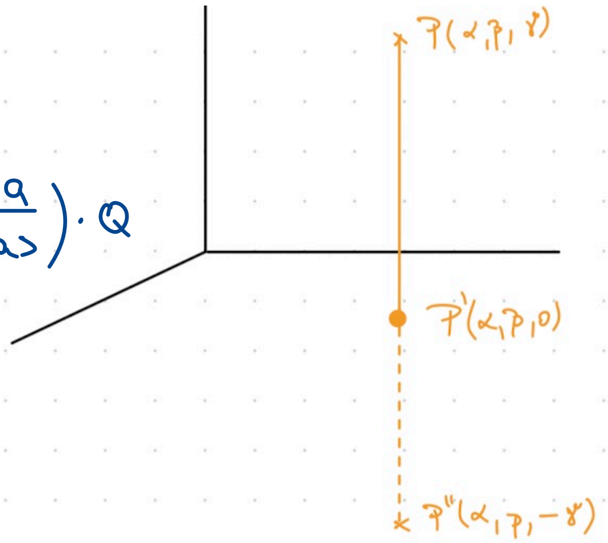
$$a_{n+1} = 0 \Rightarrow \mathrm{Pr}_{xOy,a}(P) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\cdot P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\cdot P$$

$$Pr_{\ell,w}(P) = \frac{w\otimes a}{\langle w,a\rangle}\cdot P + \left(\mathcal{I}_n - \frac{w\otimes a}{\langle w,a\rangle}\right)\cdot Q$$

if we choose $Q=(0,0,0)$

$$\Rightarrow Pr^{\perp}_{Ox}(P) = \frac{a\otimes a}{\langle a,a\rangle}\cdot P = \frac{(1,0,0)\otimes(1,0,0)}{1}\cdot P = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\cdot P$$

$$Pr^{\perp}_{Oy}P = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}P$$



$$Ref_{\mathcal{J}\ell,w}(P) = \left(-\mathcal{I}_n + 2\cdot\frac{w\otimes a}{\langle w,a\rangle}\right)\cdot P - \left(\mathcal{I}_n - \frac{w\otimes a}{\langle w,a\rangle}\right)\cdot Q$$

$$Ref_{\mathcal{J}H,v}(P) = \left(\mathcal{I}_n - 2\cdot\frac{v\otimes a}{\langle w,a\rangle}\right)\cdot P - 2\cdot\frac{a_{n+1}}{\langle a,v\rangle}\cdot v$$

$$Ref_{\mathcal{J}xOy}(P) = \left(\mathcal{I}_n - 2\cdot\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right)\cdot P = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\cdot P$$

5.2. Consider the vector $\mathbf{v}(2,1,1)\in\mathbb{V}^3$.

a) Give the matrix form for the parallel projection on the plane $\pi : z = 0$ parallel to $\mathbf{v}$.

b) Give the matrix form for the parallel reflection in the plane $\pi : z = 0$ parallel to $\mathbf{v}$.

a) $\pi : z=0 \quad \parallel \quad (2,1,1)$

$$Pr_{\pi,v}(P) = \left(\mathcal{I}_n - \frac{a\otimes v}{\langle a,v\rangle}\right)\cdot P - \frac{a_{n+1}}{\langle a,v\rangle}\cdot v$$

$\pi : z=0$ is $xOy \Rightarrow a:(0,0,1)$

$$Pr_{\pi,v}(P) = \left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&0\\2&1&1\end{pmatrix}\right)\cdot P = \begin{pmatrix}1&0&0\\0&1&0\\-2&-1&0\end{pmatrix}\cdot P$$

$\langle a,v\rangle = \langle(0,0,1),(2,1,1)\rangle = 1$

$$a\otimes v = \begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}2&1&1\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\2&1&1\end{pmatrix}$$

b) $$Ref_{\mathcal{J}\pi,v}(P) = \left(\mathcal{I}_n - 2\cdot\frac{a\otimes v}{\langle a,v\rangle}\right)\cdot P - 2\cdot\frac{a_{n+1}}{\langle a,v\rangle}\cdot v$$

$$= \left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0\\0&0&0\\4&2&2\end{pmatrix}\right)\cdot P = \begin{pmatrix}1&0&0\\0&1&0\\-4&-2&-1\end{pmatrix}\cdot P$$

5.4. Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $2x-3y+z-4=0$ by determining the matrix form of the reflection.

$\vec{v}: (2,-3,1)$

$$Ref^{\perp}_{\pi}(P) = \left(\mathcal{I}_n - 2\cdot\frac{a\otimes v}{\langle a,v\rangle}\right)\cdot P - 2\cdot\frac{a_{n+1}}{\langle a,v\rangle}\cdot v$$

$$a\otimes v = \begin{pmatrix}2\\-3\\1\end{pmatrix}\begin{pmatrix}2 & -3 & 1\end{pmatrix} = \begin{pmatrix}4 & -6 & 2\\-6 & 9 & -3\\2 & -3 & 1\end{pmatrix}$$

$$\langle a,v\rangle = \langle(2,-3,1),(2,-3,1)\rangle = 4+9+1 = 14$$

$$=\left(\mathcal{I}_n - \frac{1}{7}\cdot\begin{pmatrix}4 & -6 & 2\\-6 & 9 & -3\\2 & -3 & 1\end{pmatrix}\right)\cdot P - \cancel{2}\cdot\frac{-4}{\cancel{14}}\cdot\vec{v}$$

$$=\left(\mathcal{I}_n - \frac{1}{7}\cdot\begin{pmatrix}4 & -6 & 2\\-6 & 9 & -3\\2 & -3 & 1\end{pmatrix}\right)\cdot P + \frac{4}{7}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix}$$

5.5. Consider an orthonormal coordinate system $\mathcal{K}$ of $\mathbb{E}^2$. Starting from the matrix form of the projections and reflections described in this Section 5.1, show that

$$\mathrm{Pr}^{\perp}_{\mathbf{a}}(\mathbf{b}) = \frac{\langle \mathbf{a},\mathbf{b}\rangle}{\langle \mathbf{a},\mathbf{a}\rangle}\mathbf{a}$$

Compare this to the projections described in Section 3.1.

if $a(a_1,a_2)$ then $a^{\perp}(a_2,-a_1) \Rightarrow H: a_2x - a_1y = 0$

$$a^{\perp}\otimes a^{\perp} = \begin{pmatrix}a_2^2 & -a_1a_2\\-a_1a_2 & a_1^2\end{pmatrix}$$

$$\Rightarrow Pr^{\perp}_{a}(b) = \left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \frac{1}{a_1^2+a_2^2}\begin{pmatrix}a_2^2 & -a_1a_2\\-a_1a_2 & a_1^2\end{pmatrix}\right)$$

Proposition 5.29. A matrix A is in $SO(2)$ if and only if A has the form

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$$

5.11. In $\mathbb{E}^2$, for the lines/hyperplanes

$$\pi: ax+by+c=0,\quad \ell: \frac{x-x_0}{v_1} = \frac{y-y_0}{v_2}$$

with $\pi \nparallel \ell$, deduce the matrix forms of $\mathrm{Pr}_{\pi,\ell}$ and $\mathrm{Ref}_{\pi,\ell}$.

$$n_{\pi} = \begin{pmatrix}a\\b\end{pmatrix}$$

$$\ell: v_2x - x_0v_2 - v_1y - y_0v_1 = 0$$

$$d_{\ell}: \begin{pmatrix}v_2\\-v_1\end{pmatrix}$$

$$Pr_{\pi,\ell}(P) = \left(\mathcal{I}_n - \frac{1}{v_1^2+v_2^2}\begin{pmatrix}v_1^2 & v_1v_2\\v_1v_2 & v_2^2\end{pmatrix}\right)\cdot P$$

5.12. Let H be a hyperplane and let $\mathbf{v}$ be a vector which is not parallel to H. Use the deduced matrix forms to show that

a) $\Pr_{H,\mathbf{v}} \circ \Pr_{H,\mathbf{v}} = \Pr_{H,\mathbf{v}}$ and

b) $\text{Ref}_{H,\mathbf{v}} \circ \text{Ref}_{H,\mathbf{v}} = \text{Id}$.

b) $\text{Ref}_{H,v}(P) = \left(I_n - 2\frac{v \otimes v}{\langle v,v\rangle}\right)\cdot P - 2\frac{a_{n+1}}{\langle v,v\rangle}\cdot v$

5.14. The vertices of a triangle are $A(1,1)$, $B(4,1)$ and $C(2,3)$. Determine the image of the triangle ABC under a rotation by $90°$ around C followed by an orthogonal reflection relative to the line AB.

$\text{Ref}_{AB} \circ \text{Rot}_{C,\frac{\pi}{2}}(ABC)$

$\text{Rot}_{C,\theta}(P) = \overline{T_{(x_c,y_c)}} \circ \overline{\text{Rot}_\theta} \circ \overline{T_{(-x_c,-y_c)}} \cdot (P)$

$$\text{Rot}_{C,\frac{\pi}{2}}(ABC) = \left(\begin{array}{cc|c} 1 & 0 & x_c \\ 0 & 1 & y_c \\ \hline 0 & 0 & 1 \end{array}\right)\begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\begin{array}{cc|c} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ \hline 0 & 0 & 1 \end{array}\right)\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{Ref}^{\perp}_{AB}(P) = \left(2\cdot\frac{AB \otimes AB}{\langle AB, AB\rangle} - I_n\right)\cdot P - \left(I_2 - \frac{AB \otimes AB}{\langle AB, AB\rangle}\right)\cdot A =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Definition 5.24. An *isometry* is a map $\phi : \mathbb{E}^n \to \mathbb{E}^n$ which preserves distances, i.e.

$$d(\phi(P), \phi(Q)) = d(P, Q)$$

$A \in O(n) = \{M \in \mathcal{M}_n(\mathbb{R}) \mid M^{-1} = M^T\} \Leftrightarrow A \cdot A^T = I_n$ (check if the dist were preserved)

Theorem 5.33 (Chasles). A direct isometry of the plane $\mathbb{E}^2$ is either

a) the identity, or

b) a translation, or

c) a rotation.

$\det A = 1$

Theorem 5.34. An indirect isometry of the plane $\mathbb{E}^2$ fixes a line ℓ and is either

a) a reflection in ℓ, or

$\det A = -1$

b) the composition of a reflection in ℓ with a translation parallel to ℓ, in which case it is called a *glide-reflection*.

If φ rotation $\Rightarrow$ $\boxed{\operatorname{Tr} A = 2 \cdot \cos\theta}$

Set of all fixed points that remain unchanged $\operatorname{Fix}(\varphi) = \{P \in \mathbb{E}^n \mid \varphi(P) = P\}$

5.16. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2, 5)$. Determine the inverse transformation, T^{-1}.

$\Rightarrow$

$$T = Rot_{-\frac{\pi}{3}} \circ T_{(-2,5)}$$

* when composing 2 affine morphisms ?

$$T^{-1} = T_{(-2,5)} \cdot Rot_{\frac{\pi}{3}}$$

$$\boxed{f(P) = AP + b \Rightarrow \left(\begin{array}{c|c} A & b \\ 00\ldots0 & 1 \end{array}\right)}$$

$$\left[\hat{T}_{(2,-5)}\right] = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left[Rot_{\frac{\pi}{3}}\right] = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 0 \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left[\hat{T}^{-1}\right] = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 2 \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & -5 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

* **5.18.** Determine the cosine of the angle of the rotation f given in the previous exercise and find the inverse rotation, f^{-1}.

$$f(P) = \underbrace{\frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}}_{A} \cdot P + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$f: \mathbb{E}^2 \to \mathbb{E}^2$

* show that it's a rotation (direct isometry)

* find it's center and rotation angle

a) direct isometry iff $\det A = 1$ and $A \cdot A^T = I_n$

$$A = \begin{pmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \qquad A^T = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$A \cdot A^T = \begin{pmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} \frac{9}{25} + \frac{16}{25} & 0 \\ 0 & \frac{16}{25} + \frac{9}{25} \end{pmatrix} = I_n$$

$\det A = \frac{9}{25} + \frac{16}{25} = 1 \Rightarrow$ <u>rotation</u> $\Rightarrow \theta \in SO(2)$

b) Let $P \in \mathbb{E}^2$

$fix(P) = \{ P \in \mathbb{E}^2 \mid \varphi(P) = P \}$

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 1 \\ \frac{4}{5} & \frac{3}{5} & -2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \frac{3}{5}x - \frac{4}{5}y + 1 = x \\ \frac{4}{5}x + \frac{3}{5}y - 2 = y \end{cases} \Rightarrow \begin{cases} -2x - 4y + 5 = 0 \\ 4x - 2y - 10 = 0 \quad |:2 \end{cases}$$

$(+)$

$-5y = 0 \Rightarrow y = 0 \Rightarrow x = \frac{5}{2}$

$\Rightarrow fix(P) = \{ (\frac{5}{2}, 0) \}$

$Tr A = 2 \cdot \cos\theta$

$\cos\theta = \frac{Tr A}{2} = \frac{6}{5 \cdot 2} = \frac{3}{5} \Rightarrow \theta = \cos^{-1}\frac{3}{5}$

5.19. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to $SO(3)$. Moreover, determine the axis of rotation and the rotation angle.

$Tr A = 2\cos\theta + 1$

$$Rot_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A \cdot A^{-1} = \frac{1}{9}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix} = I_4 \Rightarrow A \in SO(3)$$

$\det A = 1$

Let $P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$A \cdot P = P$

$$\frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} -4x + 2y - 2z = 0 \\ -2x - 5y - z = 0 \\ -2x + y - z = 0 \end{cases}$$

$$-6y = 0 \Rightarrow y = 0$$

$$\Rightarrow -2x = z$$

$$\text{Fix}\begin{pmatrix} x \\ y \end{pmatrix} = \{(\alpha, 0, -2\alpha) \mid \alpha \in \mathbb{R}\} \Rightarrow \text{line} \quad D(\ell) = \langle (1, 0, -2) \rangle \text{ rotation}$$

$$\text{Tr} A = \frac{1}{3}(-1 - 2 + 2) = \frac{-1}{3}$$

$$2\cos\theta + 1 = \frac{-1}{3}$$

$$\cos\theta = \frac{-4}{3} \cdot \frac{1}{2} = \frac{-2}{3}$$

$$\text{Rot}_{\ell,\theta}(P) = \cos\theta \cdot P + \sin\theta \cdot (\vec{v} \times P) + (1 - \sin\theta)\langle \vec{v}, P \rangle \vec{v}$$

5.22. Using Euler-Rodrigues formula, write down the matrix form of a rotation around the axis $\mathbb{R}\mathbf{v}$ where $\mathbf{v} = (1, 1, 0)$. Use this matrix form to give a parametrization of a cylinder with axis $\mathbb{R}\mathbf{v}$ and diameter $\sqrt{2}$.

$$\|v\| = \sqrt{2} \rightarrow \quad \vec{w} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\text{Rot}_{\pi, \vec{w}}(P) = \cos\theta \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sin\theta \begin{vmatrix} i & j & k \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ x & y & z \end{vmatrix} + (1 - \sin\theta) \cdot \left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$= -\begin{pmatrix} x \\ y \\ z \end{pmatrix} + ?$$

Chapter 6 : 1, 3, 4, 5, 6, T. 7, 8, 9, 10 | 2, 18, 20, 26, 27, 28

Definition 6.1. A *quadratic curve* (or *conic*) in $\mathbb{E}^2$ is a curve described by a quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

for some $a, b, c, d, e, f \in \mathbb{R}$.

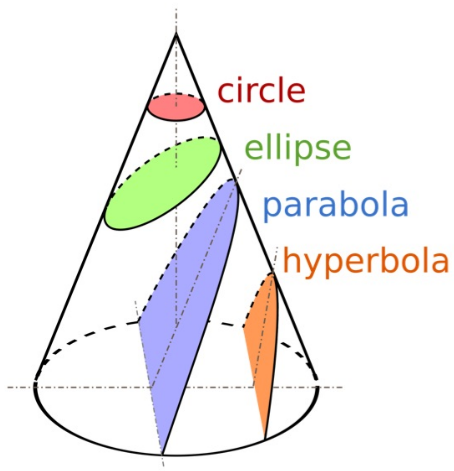


Figure 6.1: Conic sections[1]

Circle

$(x-x_0)^2 + (y-y_0)^2 = R^2 \rightarrow$ ecuatia cercului

• locus of points where the dist. to a fixed point $R(x_0, y_0)$ is $r>0$ (radius)

$$\mathcal{C}(R, r) = \begin{cases} x = x_0 + r \cdot \cos t \\ y = y_0 + r \cdot \sin t \end{cases} \quad t \in [0, 2\pi] \longleftarrow \text{parametric equation}$$

6.1. Find the equation of the circle:

a) of diameter $[A, B]$, with $A(1,2)$ and $B(-3,-1)$,

b) with center $I(2,-3)$ and radius $R = 7$,

c) with center $I(-1,2)$ and passing through $A(2,6)$,

d) centered at the origin and tangent to $\ell : 3x - 4y + 20 = 0$,

e) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $\ell : 3x - y - 2 = 0$,

f) passing through $A(1,1)$, $B(1,-1)$ and $C(2,0)$,

g) tangent to both $\ell_1 : 2x + y - 5 = 0$ and $\ell_2 : 2x + y + 15 = 0$ if one tangency point is $M(3,-1)$.

a) O – mij $[AB]$

$O(-1, \frac{1}{2})$

$AB = \sqrt{16+9} = \sqrt{25} = 5 \Rightarrow R = \frac{5}{2}$

$(x+1)^2 + (y-\frac{1}{2})^2 = \frac{25}{4}$

b) $I(2,3)$ $R = 7$

$(x-2)^2 + (y-3)^2 = 49$

c) $I(-1,2)$ $A \in \mathcal{C}$ $A(2,6)$

$AI = \sqrt{9+16} = 5$

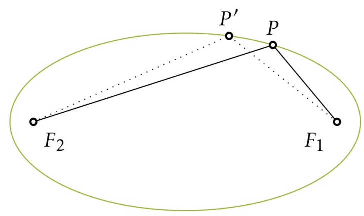
$(x+1)^2 + (y-2)^2 = 25$

d) $d(O,\ell)=\dfrac{|3\cdot 0-4\cdot 0+20|}{\sqrt{9+16}}=\dfrac{20}{5}=4$

$x^2+y^2=16$

e) $\begin{cases} 3x_0-y_0-2=0 \\ (3-x_0)^2+(1-y_0)^2=R^2 \\ (-1-x_0)^2+(3-y_0)^2=R^2 \end{cases}$ ... $C: (x-2)^2+(y-4)^2=10$

Elipse



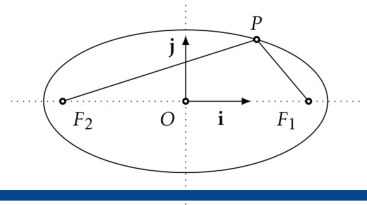
Definition 6.2. An *ellipse* is the geometric locus of points in $\mathbb{E}^2$ for which the sum of the distances from two given points, the *focal points*, is constant.

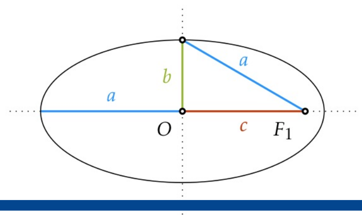
Proposition 6.3. Let F_1 and F_2 be two points in $\mathbb{E}^2$ and let a be a positive real scalar. Choose the coordinate system $Oxy=(O,\mathbf{i},\mathbf{j})$ such that F_1 and F_2 are on the Ox axis, such that $\overrightarrow{F_2F_1}$ has the same direction as $\mathbf{i}$ and such that O is the midpoint of $[F_1F_2]$. With these choices, the ellipse with focal points F_1 and F_2 for which the sum of distances from the focal points is $2a$ has an equation of the form

$$\mathcal{E}_{a,b}:\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{6.1}$$

for some positive scalar $b\in\mathbb{R}$. We denote this ellipse by $\mathcal{E}_{a,b}$.

if $F_1(c,0), F_2(-c,0)$
$c=\sqrt{a^2-b^2}$





6.3. Determine the foci (focal points) of the Ellipse $9x^2+25y^2-225=0$

$\mathcal{E}: 9x^2+25y^2-225=0$

$9x^2+25y^2=225 \quad |:225$

$\frac{9}{225}x^2+\frac{25}{225}y^2=1$

$\frac{x^2}{5^2}+\frac{y^2}{3^2}=1, \ a=5, b=3$

$\Rightarrow$ Foci: $c=\sqrt{25-9}=\sqrt{16}=4 \Rightarrow \begin{cases} F_1(4,0) \\ F_2(-4,0) \end{cases}$

6.4. Determine the intersection of the line $\ell : x + 2y - 7 = 0$ and the ellipse $\mathcal{E} : x^2 + 3y^2 - 25 = 0$.

$$y = \frac{-1}{2}x + \frac{7}{2} \qquad \frac{x^2}{5^2} + \frac{y^2}{\left(\frac{5}{\sqrt{3}}\right)^2} = 1$$

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\ell : y = kx + m$$

$$\mathcal{E} \cap \ell : \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ y = kx + m \end{cases} \Rightarrow \frac{x^2}{a^2} + \frac{k^2x^2 + 2kmx + m^2}{b^2} = 1$$

$$x^2\left(\frac{1}{a^2} + \frac{k^2}{b^2}\right) + \left(\frac{2km}{b^2}\right)x + \frac{m^2}{b^2} - 1 = 0$$

$$\delta = \frac{4}{a^2b^2}\left(-m^2 + b^2 + a^2k^2\right)$$

$$\Rightarrow \text{if } a^2k^2 + b^2 - m^2 = \begin{cases} < 0 & \text{no } \cap \\ = 0 & \text{one point} \\ > 0 & \text{2 points} \end{cases}$$

$$5^2 \cdot \frac{1}{4} + \left(\frac{5}{\sqrt{3}}\right)^2 - \frac{7}{2} = \frac{25}{4} + \frac{25}{3} - \frac{14}{4} = \frac{11}{4} + \frac{25}{3} = \frac{33 + 100}{12} = \frac{133}{12}$$

$$\boxed{T_{(x_0,y_0)}\mathcal{E}_{a,b} : \frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1.}$$ tangent line to an ellipse

If $a^2k^2 + b^2 - m^2 = 0 \Rightarrow$ the tangents are $y = kx \pm \sqrt{b^2 + a^2k^2}$

6.5. Determine the position of the line $\ell : 2x + y - 10 = 0$ relative to the ellipse $\mathcal{E} : \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$.

$$y = -2x + 10 \qquad \frac{x^2}{9} + \frac{y^2}{4} = 1$$

$\Rightarrow a^2k^2 + b^2 - m^2 = 9 \cdot 4 + 4 - 100 = -60 < 0 \Rightarrow$ no intersection between ℓ and $\mathcal{E}$

6.6. Determine an equation of a line which is orthogonal to $\ell : 2x - 2y - 13 = 0$ and tangent to the ellipse $\mathcal{E} : x^2 + 4y^2 - 20 = 0$.

$$y = x - \frac{13}{2}$$

$$\mathcal{E} : \frac{x^2}{20} + \frac{y^2}{5} = 1 \Rightarrow a = 2\sqrt{5} \qquad b = \sqrt{5}$$

$d \perp \ell$ and d tg $\mathcal{E}$

$$\Rightarrow \begin{rcases} m_d \cdot m_\ell = -1 \\ m_\ell = 1 \end{rcases} \Rightarrow m_d = -1$$

$$\frac{x^2}{20} + \frac{y^2}{5} = 1$$

let $d: y = m_d x + n$ where $n = \pm\sqrt{b^2 + a^2 \cdot m_d^2}$

$n = \pm\sqrt{5 + 20 \cdot 1} = \pm 5$

$y = -x \pm 5$

6.8. Consider the family of ellipses $\mathcal{E}_a : \frac{x^2}{a^2} + \frac{y^2}{16} = 1$. For what value $a \in \mathbb{R}$ is $\mathcal{E}_a$ tangent to the line $\ell : x - y + 5 = 0$?

$y = x + 5$

if ℓ tg $\mathcal{E}$ $\Rightarrow$ $b^2 = m^2 - a^2 k^2$ $\qquad a^2 = \frac{m^2 - b^2}{k^2} = \frac{25 - 16}{1} = 9 \Rightarrow a = 3$

$m = \pm\sqrt{b^2 + a^2 k^2}$

6.9. Consider the family of lines $\ell_c : \sqrt{5}x - y + c = 0$. For what values $c \in \mathbb{R}$ is ℓ_c tangent to the ellipse $\mathcal{E} : x^2 + \frac{y^2}{4} = 1$?

ℓ_c tg $\mathcal{E}$ iff $b^2 = m^2 - a^2 \cdot k^2$

$\ell_c: y = \sqrt{5}x + c$

$\mathcal{E}: \frac{x^2}{1} + \frac{y^2}{4} = 1$ $\qquad \Rightarrow 4 = c^2 - 5 \Rightarrow c^2 = 9 \Rightarrow c = \pm 3$

6.10. Determine the common tangents to the ellipses

$$\mathcal{E}_1: \frac{x^2}{45} + \frac{y^2}{9} = 1 \quad \text{and} \quad \frac{x^2}{9} + \frac{y^2}{18} = 1 : \mathcal{E}_2$$

let $\ell: y = k \cdot x + m$ be the tangent

if $\mathcal{E}_1$ tg ℓ : $m^2 = 9 + 45k^2$

$\mathcal{E}_2$ tg ℓ : $m^2 = 18 + 9k^2$ $\quad (-)$

$0 = -9 + 36k^2$

$9 = 36k^2 \Rightarrow k^2 = \frac{9}{36} \Rightarrow k = \pm\frac{3}{6} = \pm\frac{1}{2}$

$\Rightarrow m^2 = 18 + 9 \cdot \frac{1}{4} = \frac{81}{4} \Rightarrow m = \pm\frac{9}{2}$

$\Rightarrow \ell: y = \pm\frac{1}{2}x \pm \frac{9}{2}$

6.2. For a circle C of radius R:

a) Use the parametrization $x \mapsto (x, \pm\sqrt{R^2 - x^2})$ to deduce a parametrization of tangent lines to C.

b) Use the parametrization $\theta \mapsto (R\cos(\theta), R\sin(\theta))$ to deduce a parametrization of tangent lines to C.

c) Compare these to the equation of the tangent line $xx_0 + yy_0 = R^2$ where $(x_0, y_0) \in C$.

a) $f(t) = \sqrt{R^2 - t^2}$ $\qquad T_{(x(t), y(t))}(C): \frac{x - x(t)}{x'(t)} = \frac{y - y(t)}{y'(t)}$

$f'(t) = \frac{-2t}{2\sqrt{R^2 - t^2}} = \frac{-t}{\sqrt{R^2 - t^2}}$

$$T_{(t,f(t))}G_f: \quad y = \sqrt{R^2-t^2} - \frac{t}{\sqrt{R^2-t^2}}(x-t)$$

$$\frac{x-t}{1} = \frac{y-\sqrt{R^2-t^2}}{\frac{-t}{\sqrt{R^2-t^2}}}$$

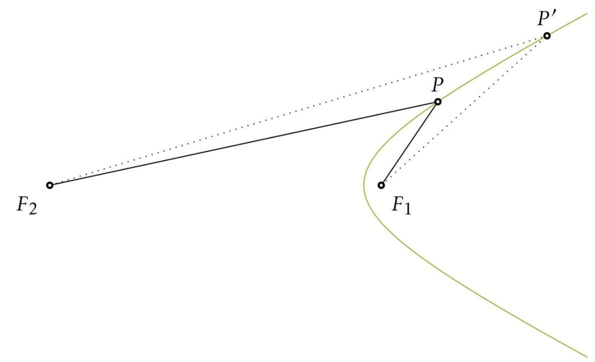
$$y\sqrt{R^2-t^2} = R^2 - tx$$

b) $$(C): \begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases} \Rightarrow \begin{cases} x'(\theta) = -R\sin\theta \\ y'(\theta) = R\cos\theta \end{cases}$$

$$T_{(x(\theta),y(\theta))}C: \frac{x-R\cos\theta}{-\cancel{R}\sin\theta} = \frac{y-R\sin\theta}{\cancel{R}\cos\theta}$$

$$R(x\cos\theta + y\sin\theta) = R^2$$

Hyperbola



Definition 6.4. A *hyperbola* is the geometric locus of points in $\mathbb{E}^2$ for which the difference of the distances from two given points, the *focal points*, is constant.

$$\boxed{\mathcal{H}_{a,b}: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

if $2c$ is the distance between the two focal points, then

$$\boxed{b^2 = c^2 - a^2}$$

$$\boxed{\begin{cases} \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \\ y = kx + m \end{cases} \Leftrightarrow \begin{cases} \frac{x^2}{a^2} - \frac{(kx+m)^2}{b^2} = 1 \\ y = kx + m \end{cases}.}$$

tangent iff $\boxed{y = kx \pm \sqrt{a^2k^2 - b^2}.}$

for $m = \pm\sqrt{a^2k^2 - b^2}$

The *tangent line to $\mathcal{H}_{a,b}$ at the point* $(x_0, y_0) \in \mathcal{H}_{a,b}$ has an equation of the form

$$T_{(x_0,y_0)}\mathcal{H}_{a,b} : \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1.$$

6.18. Determine the tangents to the hyperbola $\mathcal{H} : x^2 - y^2 = 16$ which contain the point $M(-1,7)$.

$$\frac{x^2}{16} - \frac{y^2}{16} = 1$$

Let ℓ_1, ℓ_2 tg $\mathcal{H}$ $\qquad y = kx + m$

$$\ell_k : y = kx \pm \sqrt{a^2k^2 - b^2}$$

$$7 = -k \pm \sqrt{16k^2 - 16}$$

$$7 + k = \pm 4\sqrt{k^2 - 1} \quad \Big| ()^2$$

$$49 + 14k + k^2 = 16k^2 - 16$$

$$65 + 14k - 15k^2 = 0$$

$$\Delta = 196 + 60 \cdot 65 = 4096 = 2^{12}$$

$$k_{1,2} = \frac{-14 \pm 64}{30} = \left\langle \begin{array}{l} \frac{50}{30} = \frac{5}{3} \\ \frac{-78}{30} = \frac{-26}{10} = \frac{-13}{5} \end{array} \right.$$

Sau

$$\left. \begin{array}{l} T_{(x_0,y_0)} : \frac{xx_0}{16} - \frac{yy_0}{16} = 1 \\ M \in T_{(x_0,y_0)} \end{array} \right\} \Rightarrow T_{(x_0,y_0)} : \left\{ \begin{array}{l} -x_0 - 7y_0 = 16 \\ x_0^2 - y_0^2 = 16 \end{array} \right.$$

$$\Rightarrow (-7y_0 - 16)^2 - y_0^2 = 16$$

$$49y_0^2 + 14 \cdot 16 y_0 + 240 - y_0^2 - 16 = 0 \quad \Big| :16$$

$$3y_0^2 + 14y_0 + 15 = 0$$

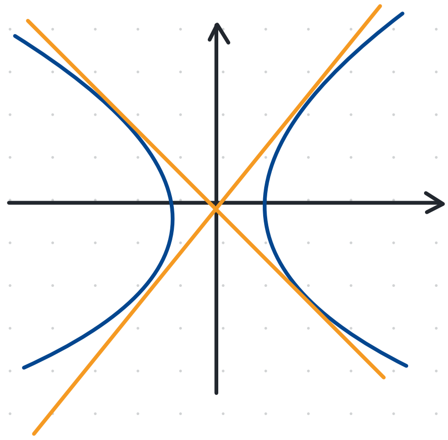
$$\Delta = 14^2 - 4 \cdot 3 \cdot 15 = 16 > 0$$

$$y_{01,02} = \frac{-14 \pm 4}{6} \left\langle \begin{array}{l} -3 \\ \frac{-5}{3} \end{array} \right. \Rightarrow \begin{array}{l} x_{01} = 5 \\ x_{02} = \frac{-13}{3} \end{array}$$

6.20. Find the area of the triangle determined by the asymptotes of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the line $\ell : 9x + 2y - 24 = 0$.

$$y = \frac{-9}{2}x + 12 \qquad\qquad \frac{x^2}{4} - \frac{y^2}{9} = 1$$

$$\boxed{\text{Asymptotes: } y = \pm\frac{b}{a}x}$$



$$\Rightarrow y = \pm\frac{3}{2}x$$

I) $\frac{3}{2}x = \frac{-9}{2}x + 12$

$6x = 12 \Rightarrow \underline{x=2} \Rightarrow y = 3 \qquad A(2,3)$

II) $\frac{-3}{2}x = \frac{-9}{2}x + 12$

$3x = 12 \Rightarrow \underline{x=4} \Rightarrow y = -6 \qquad B(4,-6)$

$$\mathcal{A}_{\Delta} = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 4 & -6 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}\cdot|-12-12| = 12$$

6.21. Find an equation for the tangent lines to:

a) the hyperbola $\mathcal{H}: \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$, orthogonal to the line $\ell: 4x + 3y - 7 = 0$; $\qquad m_\ell = \frac{-4}{3} \Rightarrow m_d = \frac{3}{4}$

a) $\ell_k: \; y = kx \pm\sqrt{k^2a^2 - b^2}$

$k = \frac{3}{4}$

$\Rightarrow y = \frac{3}{4}x \pm\sqrt{\frac{9}{\cancel{16}_{4}}\cdot\cancel{20}^{5} - \frac{4}{5}}$

$y = \frac{3}{4}x \pm\sqrt{\frac{45-20}{4}}$

$y = \frac{3}{4}x \pm \frac{5}{2}$

6.22. Find an equation for the tangent lines to:

a) the hyperbola $\mathcal{H}: \frac{x^2}{3} - \frac{y^2}{5} - 1 = 0$, passing through $P(1,-5)$;

$\ell_k: \; y = kx \pm\sqrt{k^2a^2 - b^2}$

$P(1,-5) \in \ell_k$

$\Rightarrow -5 = k \pm\sqrt{3k^2 - 5}$

$-(k+5) = \pm\sqrt{3k^2-5} \quad |()^2$

$k^2 + 10k + 25 = 3k^2 - 5$

$-2k^2 + 10k + 30 = 0 \quad |:(-2)$

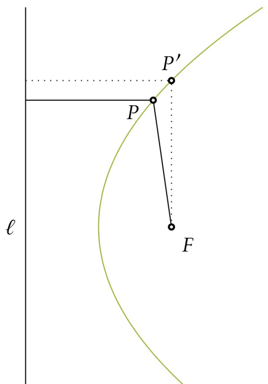
$k^2 - 5k - 15 = 0$

$\Delta = 25 + 60 = 85$

$k_{1,2} = \frac{5 \pm\sqrt{85}}{2}$

Parabola

$$\boxed{\mathcal{P}_p : y^2 = 2px}$$



tangent lines:

$$\boxed{y = kx + \frac{p}{2k}.}$$

6.26. For which value k is the line $y = kx + 2$ tangent to the parabola $\mathcal{P} : y^2 = 4x$?

$\mathcal{P}_p : y^2 = 2px$

$y^2 = 4x \Rightarrow p = 2$

tan: $y = kx + \frac{p}{2k}$

$y = kx + \frac{2}{2k} = kx + 2 \Rightarrow \frac{1}{k} = 2 \Rightarrow k = \frac{1}{2}$

6.27. Consider the parabola $\mathcal{P} : y^2 = 16x$. Determine the tangents to $\mathcal{P}$ which are

a) parallel to the line $\ell : 3x - 2y + 30 = 0$;

b) perpendicular to the line $\ell : 4x + 2y + 7 = 0$.

a) $\mathcal{P}_p : y^2 = 16x \Rightarrow p = 8$

$y = kx + \frac{p}{2k}$

$y \parallel \ell \Rightarrow k = \frac{3}{2}$

$y = \frac{3}{2}x + \frac{8}{\cancel{2} \cdot \frac{3}{\cancel{2}}} \Rightarrow y = \frac{3}{2}x + \frac{8}{3}$

b) $y \perp \ell : m_\ell = -2 \Rightarrow m_d = \frac{1}{2}$

$y = \frac{1}{2}x + \frac{8}{2 \cdot \frac{1}{2}} = \frac{1}{2}x + 8$

6.28. Determine the tangents to the parabola $\mathcal{P} : y^2 = 16x$ which contain the point $P(-2, 2)$.

$\mathcal{P}_p : y^2 = 16x \Rightarrow p = 8$

tg: $y = kx + \frac{p}{2k}$

$P \in tg \Rightarrow 2 = -2k + \frac{8}{2k} \quad | \cdot k$

$2k^2 + 2k - 4 = 0 \quad | : 2$

$k^2 + k - 2 = 0$

$\Delta = 1 + 8 = 9$

$k_{1,2} = \frac{-1 \pm 3}{2} \begin{cases} k_1 = -2 \\ k_2 = 1 \end{cases}$

$\Rightarrow tg_1 : y = x + 4$

$tg_2 : y = -2x - 2$

Chapter 7

2, 4, 6, 7, 8, 9, 10 (a, b)

A hyperquadric in $\mathbb{E}^2$ is a curve given by an equation of the form

$$\mathcal{C} : q_{11}x^2 + 2q_{12}xy + q_{22}y^2 + b_1x + b_2y + c = 0. \tag{7.7}$$

From the discussion in Section 7.2, we may apply a rotation and a translation to change the coordinate system such that Equation (7.7) becomes

$$\mathcal{C} : \lambda_1 x^2 + \lambda_2 y^2 = k \quad \text{or} \quad \mathcal{C} : \lambda_1 x^2 + v_2 y = k. \tag{7.8}$$

[Conclusion] Starting with an equation of the form (7.7), we may use rotations, translations and reflections in order to recognize that we obtain either

- degenerate cases: two lines, double lines, points, or
- non-degenerate cases:

$$\boxed{\mathcal{E}_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \mathcal{H}_{a,b} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \mathcal{P}_p : y^2 = 2px.}$$

Proposition 7.5. The type of curve described by Equation (7.15) is given with the following table.

$\widehat{D}$	D	T	curve C
$\widehat{D} = 0$	$D > 0$	-	A point
	$D = 0$	-	Two lines or the empty set
	$D < 0$	-	Two lines
$\widehat{D} \neq 0$	$D > 0$	$DT < 0$	An ellipse
	$D > 0$	$DT > 0$	The empty set
	$D = 0$	-	A parabola
	$D < 0$	-	A hyperbola

Table 7.1: Classification in dimension 2 via invariants.

$$\widehat{D} = \det(\widehat{Q}), \quad D = \det(Q) \quad \text{and} \quad T = \operatorname{tr}(Q).$$

$Q:\ a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$

$$M_Q = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

7.2. For each of the following matrices A, write down a quadratic equation with associated matrix A and find the matrix $M \in SO(2)$ which diagonalizes A.

a) $\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$

b) $\begin{bmatrix} 5 & -13 \\ -15 & 5 \end{bmatrix}$

c) $\begin{bmatrix} 7 & -2 \\ -2 & 5/3 \end{bmatrix}$

a) $P_A(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 6-\lambda & 2 \\ 2 & 9-\lambda \end{vmatrix} = 0$

$\Rightarrow (6-\lambda)(9-\lambda) - 4 = 0$

$54 - 15\lambda + \lambda^2 - 4 = 0$

$\Delta = 225 - 200 = 25 \Rightarrow \lambda_{1,2} = \frac{15 \pm 5}{2} \begin{cases} \lambda_1 = 5 \\ \lambda_2 = 10 \end{cases}$

$S(\lambda_2) = \{(x,y) \mid A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}\}$

$= \{(x,y) \mid \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$

$\Rightarrow \begin{cases} -4x + 2y = 0 \\ 2x - y = 0 \end{cases} \Rightarrow S(\lambda_2) = \langle (1,2) \rangle$

$S(\lambda_1) = \langle (-2,1) \rangle$

$M \cdot A \cdot M^T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$

choose randomly

$Q: \quad (x \ y) \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (x \ y) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3 = 0$

$\begin{pmatrix} 6x + 2y \\ 2x + 9y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + x + 2y + 3 = 0$

$6x^2 + 2xy + 2xy + 9y^2 + x + 2y + 3 = 0$

$\left(6x^2 + 2 \cdot \frac{2}{\sqrt{6}} \cdot \sqrt{6}\, xy + \frac{4}{6} y^2\right) + 9y^2 - \frac{4}{6} y^2 + x + 2y + 3 = 0$

$\left(\sqrt{6}x + \frac{2}{\sqrt{6}} y\right)^2 + \frac{50}{6} y^2 + 2y + 3 + x = 0$

$\left(\sqrt{6}x + \frac{2}{\sqrt{6}} y\right)^2 + \left(\left(\frac{5}{\sqrt{3}} y\right)^2 + 2 \cdot \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{5} y + \frac{3}{25}\right) + 3 - \frac{3}{25} + x = 0$

$\left(\sqrt{6}x + \frac{2}{\sqrt{6}} y\right)^2 + \cancel{\left(\frac{5}{\sqrt{3}} y + \frac{\sqrt{3}}{5}\right)^2} + \cancel{\frac{72}{25}} + x = 0$

let $\sqrt{6}x + \frac{2}{\sqrt{6}} y = x'$ $\Rightarrow x = \dfrac{x' - \frac{2}{\sqrt{6}} y}{\sqrt{6}}$

$\cancel{\frac{5}{\sqrt{3}} y + \frac{\sqrt{3}}{5}} = y'$

y

b) $\begin{bmatrix} 5 & -13 \\ -15 & 5 \end{bmatrix}$

$$(x\ y)\begin{pmatrix} 5 & -13 \\ -15 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (x\ y)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 5x & -15y \\ -13x & +5y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + x + 2y = 0$$

$$5x^2 - 28\ xy + 5y^2 + x + 2y = 0$$

$$\left(5x^2 - 2\cdot\sqrt{5}\cdot\frac{14}{\sqrt{5}}xy + \frac{196}{5}y^2\right) - \frac{196}{5}y^2 + \overset{5}{\cancel{5}}y^2 + x + 2y = 0$$

$$\left(\sqrt{5}x - \frac{14}{\sqrt{5}}y\right)^2 - \frac{171}{5}y^2 + 2y + x = 0$$

Let $x' = \sqrt{5}x - \frac{14}{5}y \Rightarrow x = \frac{x' - \frac{14}{\sqrt{5}}y}{\sqrt{5}}$

$y' = y$

$$x'^2 - \frac{171}{5}y'^2 + 2y' + \frac{x' - \frac{14}{\sqrt{5}}y'}{\sqrt{5}} = 0$$

$$x'^2 + \frac{x'}{\sqrt{5}} - \frac{171}{5}y'^2 + \frac{10}{5}y' - \frac{14}{5}y' = 0$$

$$x'^2 + \frac{x'}{\sqrt{5}} - \frac{171}{5}y'^2 - \frac{4}{5}y' = 0 \quad |\cdot 5$$

$$5x'^2 + \sqrt{5}x' - 171y'^2 - 4y' = 0$$

$$\left((\sqrt{5}x')^2 + 2\cdot\sqrt{5}\cdot\frac{1}{2}x' + \frac{1}{4}\right) - \left((\sqrt{171}y')^2 + 2\cdot\sqrt{171}\cdot\frac{2}{\sqrt{171}}y' + \frac{4}{171}\right) - \frac{1}{4} + \frac{4}{171} = 0$$

Let $\left(\sqrt{5}x' + \frac{1}{2}\right)^2 = x''$

$\left(\sqrt{171}y + \frac{2}{\sqrt{171}}\right)^2 = y''$

$$x''^2 - y''^2 = \frac{171}{684} - \frac{16}{684}$$

$$x''^2 - y''^2 = \frac{155}{684} \quad |\cdot\frac{684}{155}$$

Hyperbola

$$\begin{vmatrix} 5-\lambda & -13 \\ -15 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 195 = 25 - 10\lambda + \lambda^2 - 195$$

$$\lambda^2 - 10\lambda - 170 = 0$$

$$\Delta = 100 + 680 = 780$$

780	2
390	2
195	5
39	13
3	3

$$\Rightarrow \lambda_{1,2} = \frac{10 \pm 2\sqrt{195}}{2} \begin{cases} 5+\sqrt{195} \\ 5-\sqrt{195} \end{cases}$$

$$S(\lambda_1) = \left\{ (x,y) \,\middle|\, \begin{vmatrix} \sqrt{195} & -13 \\ -15 & \sqrt{195} \end{vmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

7.4. For each of the following equations write down the associated matrix and bring the equation in canonical form.

a) $-x^2 + xy - y^2 = 0$, $\qquad ax^2 + 2b\,xy + cy^2 = 0 \qquad \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

b) $6xy + x - y = 0$.

a) $A = \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix}$

$$\det(A - \lambda I_n) = 0 = \begin{vmatrix} -1-\lambda & \frac{1}{2} \\ \frac{1}{2} & -1-\lambda \end{vmatrix} = (1+\lambda)^2 - \frac{1}{4} = 1 + 2\lambda + \lambda^2 - \frac{1}{4} = 0$$

$$= 4\lambda^2 + 8\lambda + 3 = 0$$

$$\Delta = 64 - 48 = 16$$

$$\lambda_{1,2} = \frac{-8 \pm 4}{8} \begin{cases} \lambda_1 = \frac{-12}{8} = \frac{-3}{2} \\ \lambda_1 = \frac{-4}{8} = \frac{-1}{2} \end{cases}$$

$$S(\lambda_1) = \left\{ (x,y) \,\middle|\, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$\Rightarrow (x,y) = (1,-1) = v_1$$

$$S(\lambda_2) = \left\{ (x,y) \,\middle|\, \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$\Rightarrow (x,y) = (1,1) = v_2$$

normalize the vectors

$$\Rightarrow \frac{v_1}{\|v_1\|} = \frac{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$u = \frac{1}{\sqrt{2}}(x+y)$$

$$v = \frac{1}{\sqrt{2}}(x-y)$$

$$\Rightarrow \lambda_1 u + \lambda_2 v = 0$$

$$\frac{3}{2} \cdot \frac{1}{2}(x+y)^2 + \frac{1}{4}(x-y)^2 =$$

$$3x^2 + 6xy + 3y^2 + x^2 - 2xy + y^2 = 0$$

$$4x^2 + 4xy + 4y^2 = 0$$

$$x^2 + xy + y^2 = 0$$

b) $6xy + x - y = 0$

$$Q: \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

$$\det(Q - \lambda I_2) = \begin{vmatrix} -\lambda & 3 \\ 3 & -\lambda \end{vmatrix} = \lambda^2 - 9 = 0 \Rightarrow \lambda_{1,2} = \pm 3$$

$$S(\lambda_1) = \left\{ (x,y) \middle| \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$3x + 3y = 0 \Rightarrow x = -y$$

$$\Rightarrow S(\lambda_1) = \langle (1,-1) \rangle$$

$$S(\lambda_2) = \left\{ (x,y) \middle| \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$-3x + 3y = 0 \Rightarrow S(\lambda_2) = \langle (1,1) \rangle$$

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$u = \frac{1}{\sqrt{2}}(x+y)$$
$$v = \frac{1}{\sqrt{2}}(x-y)$$

$$\Rightarrow \lambda_1 u^2 + \lambda_2 v^2 = 0$$

$$3 \cdot \frac{1}{2}(x+y)^2 - 3 \cdot \frac{1}{2}(x-y)^2 = 0$$

$$\frac{3}{2}(\cancel{x^2} + 2xy + \cancel{y^2}) - \frac{3}{2}(\cancel{x^2} - 2xy + \cancel{y^2}) = 0$$

$$\frac{3}{\cancel{2}}\cancel{2}xy + \frac{3}{\cancel{2}}\cancel{2}xy = 0$$

$$6xy = 0$$

7.6. Consider the rotation R_{90° of $\mathbb{E}^2$ around the origin and the translation $T_{\mathbf{v}}$ of $\mathbb{E}^2$ with vector $\mathbf{v}(1,0)$.

a) Give the algebraic form of the isometries R_{90°, $T_{\mathbf{v}}$ and $T_{\mathbf{v}} \circ R_{90^\circ}$.

b) Determine the equations of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the parabola $\mathcal{P} : y^2 - 8x = 0$ after transforming them with R_{90° and with $T_{\mathbf{v}} \circ R_{90^\circ}$ respectively.

a) $Rot_{90^\circ} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \cdot P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot P = \begin{pmatrix} -y_0 \\ x_0 \end{pmatrix}$

$T_v = P + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_0+1 \\ y_0 \end{pmatrix}$

$T_v \circ Rot_{90^\circ} = T_v \cdot \begin{pmatrix} -y_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} -y_0+1 \\ x_0 \end{pmatrix}$

b) $\left.\begin{array}{l} \mathcal{H}: \frac{x^2}{4} - \frac{y^2}{9} = 1 \\ Rot_{90^\circ} = \begin{pmatrix} -y_0 \\ x_0 \end{pmatrix} \end{array}\right\} \Rightarrow \frac{y_0^2}{4} - \frac{x_0^2}{9} = 1$

$T_v \circ Rot_{90^\circ} \Rightarrow \frac{(-y_0+1)^2}{4} - \frac{x_0^2}{9} = 1$

$\frac{y_0^2 - 2y_0 + 1}{4} - \frac{x_0^2}{9} = 1$

$9y_0^2 - 18y_0 + 9 - 4x_0^2 - 36 = 0$

$-4x_0^2 + 9y_0^2 - 18y_0 - 27 = 0$

7.7. Find the canonical equation for each of the following cases

a) $5x^2 + 4xy + 8y^2 - 32x - 56y + 80 = 0$,

b) $8y^2 + 6xy - 12x - 26y + 11 = 0$,

c) $x^2 - 4xy + y^2 - 6x + 2y + 1 = 0$.

a) $5x^2 + 4xy + 9y^2 - 32x - 56y + 80 = 0$

$(4x^2 + 4xy + y^2) + (x^2 - 32x + 256) + \left(8y^2 - 56y + \frac{196}{3}\right) - 256 - \frac{196}{3} + 80 = 0$

$x'^2 + y'^2 + z'^2 - \frac{964}{3} + \frac{240}{3} = 0$

$x'^2 + y'^2 + z'^2 = \frac{724}{3}$

b) $8y^2 + 6xy - 12x - 26y + 11 = 0$

$(9y^2 + 6xy + x^2) - y^2 - 26y - x^2 - 12x + 11 = 0$

$(3y+x)^2 - (y + \sqrt{13})^2 - (x + \sqrt{6})^2 + 6 + 13 + 11 = 0$

$x'^2 - y'^2 - z'^2 + 30 = 0$

7.9. Discuss the type of the curve

$$x^2 + \lambda xy + y^2 - 6x - 16 = 0$$

in terms of $\lambda \in \mathbb{R}$.

$$Q: \quad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$\Delta = B^2 - 4AC = \lambda^2 - 4$$

if $\Delta > 0 \Rightarrow$ hyperbola

$\Delta < 0 \Rightarrow$ elipse

$\Delta = 0 \Rightarrow$ parabola

$\widehat{D}$	D	T	curve C
	$D > 0$	-	A point
$\widehat{D} = 0$	$D = 0$	-	Two lines or the empty set
	$D < 0$	-	Two lines
	$D > 0$	$DT < 0$	An ellipse
$\widehat{D} \neq 0$	$D > 0$	$DT > 0$	The empty set
	$D = 0$	-	A parabola
	$D < 0$	-	A hyperbola

Table 7.1: Classification in dimension 2 via invariants.

where $\hat{D} = \begin{pmatrix} A & \frac{B}{2} & F \\ \frac{B}{2} & C & E \\ D & E & F \end{pmatrix}$

$$\hat{D} = \begin{bmatrix} x & y & 1 \end{bmatrix} M^T \widehat{Q} M \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0.$$

* **7.10.** Using the classification of quadrics, decide what surfaces are described by the following equations.

a) $x^2 + 2y^2 + z^2 + xy + yz + zx = 1$,

b) $xy + yz + zx = 1$,

$$Q: \quad a x^2 + 2b xy + c y^2 + 2dx + 2ey + f = 0$$

$$M_a = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad Q: \begin{pmatrix} x & y & 1 \end{pmatrix} \hat{M_a} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

$$\hat{M_a} = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$$

isometry. $\hat{D} = \det(\hat{M_a})$

$D = \det(M_a)$

$T = \operatorname{tr}(M_a)$

a) $(x+y+z)^2 + y^2 - xy - yz - zx = 1$

$$(x+y+z)^2 + \left(y - \frac{x}{2}\right)^2 - \left(\frac{x^2}{4} + 2\frac{x}{2}z + z^2\right) + \left(z^2 - 2 \cdot z \cdot \frac{y}{2} + \frac{y^2}{4}\right) - \frac{y^2}{4} = 1$$

$$(x+y+z)^2 + \left(y - \frac{x}{2}\right)^2 - \left(\frac{x}{2} + z\right)^2 + \left(z - \frac{y}{2}\right)^2 - \frac{y^2}{4} = 1$$

?

Chapter 8

3, 4, 5, ? 10, 11, 14, 17, 18, 21, 24

$$\boxed{\mathcal{S} : q_{11}x^2 + q_{22}y^2 + q_{33}z^2 + 2q_{12}xy + 2q_{13}xz + 2q_{23}yz + a_1x + a_2y + a_3z + c = 0.}$$

Elipsoid

$$\boxed{\mathcal{E}_{a,b,c} : \underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}_{\varphi(x,y,z)} = 1}$$

equation

parametrization

$$\begin{cases} x(\theta_1,\theta_2) = a\cos(\theta_1)\cos(\theta_2) \\ y(\theta_1,\theta_2) = b\sin(\theta_1)\cos(\theta_2) \\ z(\theta_1,\theta_2) = c\sin(\theta_2) \end{cases} \quad \theta_1 \in [0,2\pi[\quad \theta_2 \in [-\frac{\pi}{2},\frac{\pi}{2}[$$

$$\boxed{T_p\mathcal{E}_{a,b,c} : \frac{x_px}{a^2} + \frac{y_py}{b^2} + \frac{z_pz}{c^2} = 1.}$$

tangent planes

Eliptic cone

An *elliptic cone* is a surface which (in some coordinate system) satisfies an equation of the form

$$\boxed{\mathcal{C}_{a,b,c} : \underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}}_{\varphi(x,y,z)} = 0} \quad \text{so} \quad \mathcal{E}_{a,b,c} = \varphi^{-1}(0)$$

for some positive constants $a,b,c \in \mathbb{R}$.

parametrization

$$\begin{cases} x(\theta,h) = ha\cos(\theta) \\ y(\theta,h) = hb\sin(\theta) \\ z(\theta,h) = hc \end{cases} \quad \theta \in [0,2\pi[\quad h \in \mathbb{R}.$$

$$\boxed{T_p\mathcal{C}_{a,b,c} : \frac{x_px}{a^2} + \frac{y_py}{b^2} - \frac{z_pz}{c^2} = 0.}$$

Hyperboloid of one sheet

$$\boxed{\mathcal{H}^1_{a,b,c} : \underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}}_{\varphi(x,y,z)} = 1} \quad \text{so} \quad \mathcal{H}^1_{a,b,c} = \varphi^{-1}(1)$$

$$T_p\mathcal{H}^1_{a,b,c} : \frac{x_px}{a^2} + \frac{y_py}{b^2} - \frac{z_pz}{c^2} = 1.$$

$$\sigma_1(\theta_1,\theta_2) = \begin{bmatrix} a\sqrt{1+\theta_2^2}\cos(\theta_1) \\ b\sqrt{1+\theta_2^2}\sin(\theta_1) \\ c\theta_2 \end{bmatrix} \quad \text{and} \quad \sigma_2(\theta_1,\theta_2) = \begin{bmatrix} a\cosh(\theta_2)\cos(\theta_1) \\ b\cosh(\theta_2)\sin(\theta_1) \\ c\sinh(\theta_2) \end{bmatrix}$$

Hyperboloid of two sheets

$$\mathcal{H}^2_{a,b,c} : \underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}}_{\varphi(x,y,z)} = -1$$

$$T_p\mathcal{H}^2_{a,b,c} : \frac{x_p x}{a^2} + \frac{y_p y}{b^2} - \frac{z_p z}{c^2} = -1.$$

$$\sigma_1(\theta_1,\theta_2) = \begin{bmatrix} a\sqrt{\theta_2^2 - 1}\cos(\theta_1) \\ b\sqrt{\theta_2^2 - 1}\sin(\theta_1) \\ c\theta_2 \end{bmatrix} \quad \text{and} \quad \sigma_2(\theta_1,\theta_2) = \begin{bmatrix} a\sinh(\theta_2)\cos(\theta_1) \\ b\sinh(\theta_2)\sin(\theta_1) \\ \varepsilon c\cosh(\theta_2) \end{bmatrix}$$

Eliptic paraboloid

$$\mathcal{P}^e_{a,b} : \underbrace{\frac{x^2}{a} + \frac{y^2}{b} - 2z}_{\varphi(x,y,z)} = 0$$

$$T_p\mathcal{P}^e_{a,b} : \frac{x_p x}{a} + \frac{y_p y}{b} - z_p - z = 0.$$

$$\sigma(\theta_1,\theta_2) = \begin{bmatrix} \sqrt{a}\theta_2\cos(\theta_1) \\ \sqrt{b}\theta_2\sin(\theta_1) \\ \theta_2/2 \end{bmatrix} \quad \theta_1 \in [0,2\pi[\quad \theta_2 \in [0,\infty[$$

Hyperbolic paraboloid

$$\mathcal{P}^h_{a,b} : \underbrace{\frac{x^2}{a} - \frac{y^2}{b} - 2z}_{\varphi(x,y,z)} = 0$$

$$T_p\mathcal{P}^h_{a,b} : \frac{x_p x}{a} - \frac{y_p y}{b} - z_p - z = 0.$$

$$\sigma_2(\theta_1,\theta_2) = \begin{bmatrix} \sqrt{a}\theta_1 \\ \sqrt{b}\theta_2 \\ \frac{1}{2}(\theta_1^2 - \theta_2^2) \end{bmatrix} \quad \theta_1,\theta_2 \in \mathbb{R}$$

8.3. Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,\sqrt{3},3} : \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \quad \text{with the line} \quad \ell : x = y = z.$$

Write down the equations of the tangent planes in the intersection points.

$$\left.\begin{aligned} &\mathcal{E}_{2,\sqrt{3},3} : \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \\ &\ell : x = y = z \end{aligned}\right\} \Rightarrow \frac{x^2}{4} + \frac{x^2}{3} + \frac{x^2}{9} = 1 \quad \Big| \cdot 36$$

$$9x^2 + 12x^2 + 4x^2 = 36$$

$$25x^2 = 36 \Rightarrow x = \pm\frac{6}{5}$$

$$\frac{x \cdot x_0}{4} + \frac{y y_0}{3} + \frac{z z_0}{9} = 1$$

$$\frac{x \cdot \frac{6}{5}}{4} + \frac{y \cdot \frac{6}{5}}{3} + \frac{z \cdot \frac{6}{5}}{9} = 1$$

$$\frac{6x}{20} + \frac{6y}{15} + \frac{6z}{45} = 1 \Big| \cdot \frac{5}{6}$$

$$\frac{x}{4} + \frac{y}{3} + \frac{z}{9} = \frac{5}{6}$$

8.4. Determine the tangent planes to the ellipsoid

$$\mathcal{E}_{2,3,2\sqrt{2}} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{8} = 1$$

which are parallel to the plane $\pi : 3x - 2y + 5z + 1 = 0$.

$$n_\pi = (3, -2, 5)$$

$$T_\mathcal{E} \parallel \pi \Rightarrow n_\pi \parallel n_T : \frac{x_0}{3 \cdot 4} = \frac{y_0}{9 \cdot (-2)} = \frac{z_0}{8 \cdot 5} = t$$

$$x_0 = 12t$$
$$y_0 = -18t$$
$$z_0 = 40t$$

$$\frac{144t^2}{4} + \frac{324t^2}{9} + \frac{1600t^2}{8} = 1$$

$$36t^2 + 36t^2 + 200t^2 = 1$$

$$t^2 = \frac{1}{272} \Rightarrow t = \pm\frac{1}{4\sqrt{17}}$$

8.5. Determine the points P of the ellipsoid

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

for which the tangent space $T_P\mathcal{E}$ intersects the coordinate axis in congruent segments.

$\cap$

$$\frac{x x_0}{a^2} + \frac{y y_0}{b^2} + \frac{z z_0}{c^2} = 1$$

Ox:

$$\frac{x x_0}{a^2} = 1 \Rightarrow x = \frac{a^2}{x_0} \quad \text{idem} \quad y = \frac{b^2}{y_0} \quad z = \frac{c^2}{z_0}$$

congruent seg: $\frac{a^2}{|x_0|} = \frac{b^2}{|y_0|} = \frac{c^2}{|z_0|} = k$

$$|x_0| = \frac{a^2}{k} \quad |y_0| = \frac{b^2}{k} \quad |z_0| = \frac{c^2}{k}$$

$$\frac{a^2 + b^2 + c^2}{k^2} = 1 \Rightarrow k = \sqrt{a^2 + b^2 + c^2}$$

8.11. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S} : \begin{cases} x = 4\cos(s)\cos(t) \\ y = 4\sin(s)\cos(t) \\ z = 2\sin(t) \end{cases} \quad s \in [0, 2\pi[\quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}[$$

a) Give an equation of $\mathcal{S}$.

b) Find the parameters of the point $P(3, \sqrt{3}, 1)$.

* c) Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.

* d) Give an equation of $T_P\mathcal{S}$.

a) $x^2+y^2 = 16(\cos^2 s \cos^2 t + \sin^2 s \cos^2 t) = 16\cos^2 t$

$x^2+y^2+4z^2 = 16(\sin^2 t + \cos^2 t) = 16 \quad |:16$

$\frac{x^2}{4^2}+\frac{y^2}{4^2}+\frac{z^2}{2^2} = 1 \Rightarrow$ elipsoid

b)

$$S: \begin{cases} 3 = 4\cot\cos s \\ \sqrt{3} = 4\cot\sin s \\ 1 = 2\cdot\sin t \Rightarrow t = \frac{\pi}{6} \quad \Rightarrow s = \frac{\pi}{6} \end{cases}$$

c) $P(x_0 y_0 z_0): \frac{xx_0}{16} + \frac{yy_0}{16} + \frac{zz_0}{4} = 1$?

8.14. Determine the tangent plane of the hyperboloid

$$\mathcal{H}^1_{2,3,1} : \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$$

in the point $M(2,3,1)$. Show that the tangent plane intersects the surface in two lines.

$\pi: \frac{xx_0}{4} + \frac{yy_0}{9} - \frac{zz_0}{1} = 1$

$M \in \pi: \frac{1}{2}x_0 + \frac{1}{3}y_0 - z = 1$

$\Rightarrow z = \frac{1}{2}x + \frac{1}{3}y - 1$

$\cancel{\frac{x^2}{4}} + \cancel{\frac{y^2}{9}} - \left(\cancel{\frac{x^2}{4}} + \cancel{\frac{y^2}{9}} + 1 + \frac{xy}{3} - x - \frac{2y}{3}\right) = 1$

$-\frac{xy}{3} + x + \frac{2y}{3} = 2 \quad |\cdot 3$

$-xy + 3x + 2y = 6$

$x(3-y) - 2(3-y) = 0$

$(x-2)(3-y) = 0$

$$\Rightarrow \begin{cases} x-2=0 & P_1 \\ 3-y=0 & P_2 \end{cases}$$

8.17. Determine the intersection of the paraboloid

$$\mathcal{P}^h_{2,\frac{1}{2}} : x^2 - 4y^2 = 4z \quad \text{with the line} \quad \ell = \begin{bmatrix}2\\0\\3\end{bmatrix} + \langle\begin{bmatrix}2\\1\\-2\end{bmatrix}\rangle.$$

Write down the equations of the tangent planes in the intersection points.

$$\ell = \begin{cases} x = 2+2\lambda \\ y = \lambda \\ z = 3-2\lambda \end{cases}$$

$4 + 8\lambda + \cancel{4\lambda^2} - \cancel{4\lambda^2} = 12 - 8\lambda$

$-8 = -16\lambda \Rightarrow \lambda = \frac{1}{2} \Rightarrow M(3, \frac{1}{2}, 2)$

$$T_p\mathcal{P}^h_{a,b} : \frac{x_p x}{a} - \frac{y_p y}{b} - z_p - z = 0.$$

$$\mathcal{P}^h_{2,\frac{1}{2}} = \frac{x_p x}{2} - \frac{y_p y}{\frac{1}{2}} - z_p - z = 0$$

$$\frac{3x}{2} - y - z = 2 \;|\cdot 2$$

$$3x - 2y - 2z - 4 = 0$$

8.18. Determine the tangent plane of

a) the elliptic paraboloid $\frac{x^2}{5} + \frac{y^2}{3} = z$ and of

b) the hyperbolic paraboloid $x^2 - \frac{y^2}{4} = z$

which are parallel to the plane $x - 3y + 2z - 1 = 0$.

a) $$T_p\mathcal{P}^e_{ab} : \frac{x x_0}{5} + \frac{y y_0}{3} - \frac{1}{2} z - \frac{1}{2} z_0 = 0$$

$$T_p\mathcal{P}^e_{ab} \parallel e \Rightarrow \frac{\frac{x_0}{5}}{1} = \frac{\frac{y_0}{3}}{-3} = \frac{-\frac{1}{2}}{2} = -\frac{1}{4}$$

$$x_0 = -\frac{5}{4} \quad y_0 = \frac{9}{4}$$

$$\frac{\frac{\cancel{25}\,5}{16}}{\cancel{5}} + \frac{\frac{\cancel{81}\,27}{16}}{\cancel{3}} = z_0$$

$$\frac{32}{16} = 2 = z_0$$

$$T_p\mathcal{P}^e_{ab} : \frac{-x}{4} + \frac{3y}{4} - \frac{1}{2} z - 1 = 0 \;|\cdot 4$$

$$-x + 3y - 2z - 2 = 0$$

b) $$T_p\mathcal{P}^h_{ab} : \frac{x x_0}{1} - \frac{y y_0}{4} - \frac{1}{2} z - \frac{1}{2} z_0 = 0$$

$$\frac{x_0}{1} = \frac{\frac{y_0}{4}}{-3} = \frac{-\frac{1}{2}}{2} = -\frac{1}{4}$$

$$x_0 = \frac{-1}{4} \Rightarrow y_0 = 3$$

$$x^2 - \frac{y^2}{4} = z$$

$$\frac{1}{16} - \frac{9}{4} = z \;|\cdot 16$$

$$1-36=16z$$

$$z=\frac{-35}{16}$$

$$T_P P^h_{a,b}: \frac{x x_0}{1}-\frac{y y_0}{4}-\frac{1}{2}z-\frac{1}{2}z_0=0$$

$$-\frac{1}{4}x-\frac{3}{4}y-\frac{1}{2}z+\frac{35}{32}=0 \mid \cdot 4$$

$$-x-3y-z+\frac{35}{8}=0$$

8.21. Use a parametrization of a parabola and a rotation matrix to deduce a parametrization of an elliptic paraboloid of revolution.

8.24. Which of the following is a hyperboliod?

a) $\mathcal{S}: 2xz+2xy+2yz=1$

b) $\mathcal{S}: 5x^2+3y^2+xz=1$

c) $\mathcal{S}: 2xy+2yz+y+z=2$

b) $5x^2+2\cdot\frac{\sqrt{5}}{2\sqrt{5}}xz+\frac{1}{20}z^2+3y^2-\frac{1}{20}z^2=1$

$$\underbrace{\left(\sqrt{5}x+\frac{1}{2\sqrt{5}}z\right)}^2+3y^2-\frac{1}{20}z^2=1$$

$$x'^2+3y^2-\frac{1}{20}z^2=1$$

B. Name and group:

Problem		1	2	3	4	Total
Points	1	2	2	[illegible]	[illegible]	[illegible]
Obtained	1					

Justify your answers.

P 1. Give the equation of an ellipse containing the points (2, 0) and (0, 3) and having focal points on one of the coordinate axis equally distanced from the origin.

P 2. Determine equations for the tangent lines to the hyperbola

$$x^2 - y^2 = 2$$

which contain the point $M(2, 2)$.

P 3. Consider the surface

$$\mathcal{S} : y^2 + 2xy - 2xz + z - 2 = 0$$

given with respect to some coordinate system.

a) Determine the intersection of $\mathcal{S}$ with the coordinate axis Oy.

b) Bring the equation in canonical form. What type of quadric is it? Why?

P 4. Consider the matrix

$$B = \frac{1}{11}\begin{bmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{bmatrix}.$$

a) Explain why it corresponds to a rotation.

b) Determine the cosine of the rotation angle.

c) Determine the axis of rotation.

oints: Each subitem in P3 and P4 is worth 1 Point. You receive 2 Point for solving Problem 1 a
espectively. This is a total of 9 Points to which we add 1 Point in order to obtain the final gra
his exam.

ime: 1h:20min

1) $\mathcal{E}: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1$

$(0,3), (2,0) \in \mathcal{E}$

$\dfrac{4}{a^2} = 1 \Rightarrow a^2 = 4$ $\qquad \Rightarrow \mathcal{E}: \dfrac{x^2}{4} + \dfrac{y^2}{3} = 0$

$\dfrac{9}{b^2} = 1 \Rightarrow b^2 = 9$

2) $\mathcal{H}: x^2 - y^2 = 2 \quad | : 2$

$\dfrac{x^2}{2} - \dfrac{y^2}{2} = 1$

t tg $\mathcal{H}: \dfrac{x x_0}{2} - \dfrac{y y_0}{2} = 1$

$M(2,2) \qquad x - y = 1 \Rightarrow y = x - 1$

$$\frac{x_0^2}{2} - \frac{(x_0-1)^2}{2} = 1 \quad | \cdot 2$$

$$x_0^2 - x_0^2 + 2x_0 - 1 = 2$$

$$2x_0 = 3 \Rightarrow x_0 = \frac{3}{2}$$

$$y_0 = \frac{3}{2} - 1 = \frac{1}{2}$$

$$t: \frac{3}{4}x - \frac{1}{4}y = 1 \quad | \cdot 4$$

$$3x - y = 4$$

3) **P 3.** Consider the surface

$$S: y^2 + 2xy - 2xz + z - 2 = 0$$

given with respect to some coordinate system.

~~a)~~ Determine the intersection of S with the coordinate axis Oy.

~~b)~~ Bring the equation in canonical form. What type of quadric is it? Why?

a) $S \cap Oy$

$$Oy: \begin{cases} x = 0 \\ z = 0 \end{cases}$$

$$S \cap Oy: \begin{cases} x = 0 \\ z = 0 \\ y^2 + 2xy - 2xz + z - 2 = 0 \end{cases}$$

$$y^2 - 2 = 0 \qquad y = \pm\sqrt{2}$$

$$P_1: y = \sqrt{2} \qquad P_2 = -\sqrt{2}$$

b) $S: y^2 + 2xy - 2xz + z - 2 = 0$

$$(y^2 + 2xy + x^2) - (x^2 + 2xz + z^2) + z^2 + z - 2 = 0$$

$$(y+x)^2 - (x+z)^2 + \left(z^2 + 2\cdot\frac{1}{2}z + \frac{1}{4}\right) - \frac{1}{4} - 2 = 0$$

$$(y+x)^2 - (x+z)^2 + \left(z+\frac{1}{2}\right)^2 = \frac{9}{4}$$

* puun – la capăt

Let $x + y = x'$

$$z + \frac{1}{2} = y'$$

$$x + z = z'$$

$$x'^2 + y'^2 - z'^2 = \frac{9}{4} \quad | \cdot \frac{4}{9}$$

$$\frac{x'^2}{\frac{9}{4}} + \frac{y'^2}{\frac{9}{4}} - \frac{z'^2}{\frac{9}{4}} = 1$$

Hyperboloid in one sheet

P 4. Consider the matrix

$$B = \frac{1}{11}\begin{bmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{bmatrix}.$$

a) Explain why it corresponds to a rotation.

b) Determine the cosine of the rotation angle.

c) Determine the axis of rotation.

a)
$$B = \frac{1}{11}\begin{bmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{bmatrix}$$

$$\frac{1}{121}\begin{bmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{bmatrix}\begin{bmatrix} -2 & -6 & 9 \\ 6 & 7 & 6 \\ -9 & 6 & 2 \end{bmatrix} = \frac{1}{121}\begin{bmatrix} 4+36+81 & 12+42-54 & 0 \\ 0 & 121 & 0 \\ 0 & 0 & 121 \end{bmatrix} = I_n$$

$$\det B = \frac{1}{11}\begin{vmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{vmatrix} = \frac{1}{11}(-28+324+324+567+42+72) = ?$$

$$\mathrm{Fix}(B) = \{(x,y,z) \mid B\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\}$$

$$\frac{1}{11}\begin{bmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{cases} -2x + 6y - 9z = 11x \\ -6x + 7y + 6z = 11y \\ 9x + 6y + 2z = 11z \end{cases}$$

$$\Rightarrow \begin{cases} -13x + 6y - 9z = 0 \\ -6x - 4y + 6z = 0 \quad |:(-2) \\ 9x + 6y - 9z = 0 \quad |:(3) \end{cases}$$

$$\Rightarrow \begin{cases} -13x + 6y - 9z = 0 \\ 3x + 2y - 3z = 0 \\ 3x + 2y - 3z = 0 \end{cases}$$

$$\begin{cases} -13x + 6y - 9z = 0 \\ 3x + 2y - 3z = 0 \quad | \cdot (-3) \end{cases} \quad (+)$$

$$X = 0$$

$$\Rightarrow \quad 2y = 3z \Rightarrow \quad y = \frac{3}{2}z$$

$$\Rightarrow F = \langle (0, \tfrac{3}{2}, 1) \rangle$$

$$\cos\theta = \frac{\operatorname{Tr} A - 1}{2} \quad \text{in 3D!}$$

$$\cos\theta = \frac{\frac{7}{11} - \frac{11}{11}}{2} = \frac{\frac{7-11}{11}}{2} = \frac{-4}{22} = \frac{-2}{11} \quad \Rightarrow \theta = \arccos\frac{-2}{11}$$

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1) $\mathcal{E}: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$(2,0), (0,3) \in \mathcal{E}$

$$\left.\begin{aligned} \frac{4}{a^2} = 1 \Rightarrow a^2 = 4 \\ \frac{9}{b^2} = 1 \Rightarrow b^2 = 9 \end{aligned}\right\} \Rightarrow \mathcal{E}: \frac{x^2}{4} + \frac{y^2}{9} = 1$$

2) $x^2 - y^2 = 2 \quad | : 2$

$$\frac{x^2}{2} - \frac{y^2}{2} = 1$$

tg: $\dfrac{xx_0}{2} - \dfrac{yy_0}{2} = 1$

$M \in$ tg $\quad x - y = 1 \Rightarrow x = y + 1$

$$\frac{y^2 + 2y + 1}{2} - \frac{y^2}{2} = 1$$

$$2y + 1 = 2 \Rightarrow y = \frac{3}{2} \Rightarrow x = \frac{5}{2}$$

tg: $\dfrac{5}{4}x - \dfrac{3}{4}y = 1 \quad | \cdot 4$

$$5x - 3y = 4$$

3) $S \quad y^2+2xy-2xz+z-2=0$

a) $S \cap Oy$

$Oy: \begin{cases} x=0 \\ z=0 \end{cases}$

$\Rightarrow y^2-2=0 \Rightarrow y=\pm\sqrt{2}$

b) $(y^2+2xy+x^2)-(x^2+2xz+z^2)+z^2-z-2=0$

$\underbrace{(y+x)^2}_{x'}-\underbrace{(x+z)^2}_{z'}+\underbrace{\left(z-\frac{1}{2}\right)^2}_{y'}-\frac{1}{4}-\frac{4}{2}=0$

$\Rightarrow x'^2+y'^2-z'^2=\frac{9}{4} \quad \Big| \cdot \frac{4}{9}$

$\frac{x'^2}{\frac{9}{4}}+\frac{y'^2}{\frac{9}{4}}-\frac{z'^2}{\frac{9}{4}}=1 \Rightarrow$ Hyperb. 1 shot

4)

$B=\frac{1}{11}\begin{bmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{bmatrix}$

a) $\left.\begin{matrix} B\cdot B^{-1}=I_n \\ \det B=1 \end{matrix}\right\} \Rightarrow \in SO_3$

b) $\cos\theta=\frac{\operatorname{Tr}B-1}{2}$

c) $Fix\ F=\{(x\ y\ z) \mid B\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}\}$

$\begin{cases} -2x+6y-9z=11x \\ -6x+7y+6z=11y \\ 9x+6y+2z=11z \end{cases}$