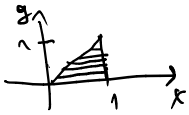
# Seminar 13

Sunday, January 14, 2024

1. By changing the order of integration, evaluate the following:

(a) $\displaystyle\int_0^1\int_y^1 \sin(x^2)\,\mathrm{d}x\,\mathrm{d}y.$ (b) $\displaystyle\int_0^1\int_x^1 e^{y^2}\,\mathrm{d}y\,\mathrm{d}x.$ (c) $\bigstar \displaystyle\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin(y)}{y}\,\mathrm{d}y\,\mathrm{d}x.$

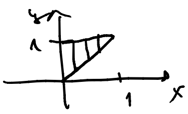
(a) $0\le y\le 1$, $y\le x\le 1$



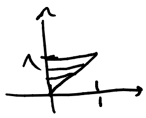
$0\le x\le 1$, $0\le y\le x$

$$\int_0^1\int_0^x \sin(x^2)\,dy\,dx = \int_0^1 \sin(x^2)\cdot x\,dx = -\frac12\cos(x^2)\Big|_0^1 = -\frac12(\cos 1-1) = \frac12(1-\cos 1).$$

(b) $0\le x\le 1$, $x\le y\le 1$



$0\le y\le 1$, $0\le x\le y$



$$\int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 e^{y^2}\cdot y\,dy = \frac12 e^{y^2}\Big|_0^1 = \frac12(e-1).$$

2. Compute the following integrals by doing a change of variables:

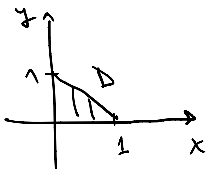
(a) $\displaystyle\iint_D e^{\frac{x-y}{x+y}}\,\mathrm{d}x\,\mathrm{d}y$, where $D=\{(x,y)\in\mathbb{R}^2 \mid x\ge 0,\ y\ge 0, x+y\le 1\}$.

(b) $\displaystyle\iint_D \left(\frac{x-y}{x+y+2}\right)^2\mathrm{d}x\,\mathrm{d}y$, where $D=\{(x,y)\in\mathbb{R}^2 \mid |x|+|y|\le 1\}$.

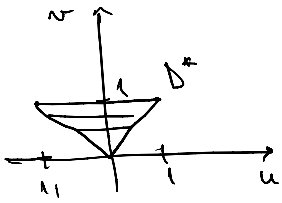
(c) $\displaystyle\iint_D \frac{y^2}{x}\,\mathrm{d}x\,\mathrm{d}y$, where $D$ is the region between the parabolas $x=1-y^2$ and $x=3(1-y^2)$.

(d) $\bigstar\displaystyle\iint_D xy\,\mathrm{d}x\,\mathrm{d}y$, where $D$ is the parallelogram with vertices $(0,0)$, $(2,2)$, $(1,2)$, $(3,4)$.

(a)



let $u=x-y$, $v=x+y$, $x=\frac{u+v}{2}$, $y=\frac{v-u}{2}$

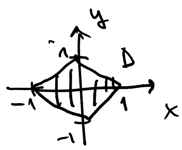


$v\in[0,1]$, $u\in[-v,v]$

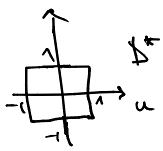
$$J=\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix} = \begin{bmatrix}\frac12 & \frac12\\ -\frac12 & \frac12\end{bmatrix},\ \det J=\frac12.$$

$$\iint_D e^{\frac{x-y}{x+y}}dx\,dy = \iint_{D^*} e^{\frac uv}\cdot\frac12\,du\,dv = \frac12\int_0^1\left(\int_{-v}^{v} e^{\frac uv}du\right)dv = \frac12\int_0^1\left(v e^{\frac uv}\right)\Big|_{-v}^{v}dv = \frac12\int_0^1\left(ve-\frac ve\right)dv = \frac e4-\frac1{4e}.$$

(b)



$u=x-y$, $v=x+y$, $u\in[-1,1]$, $v\in[-1,1]$

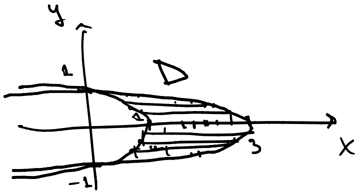


$$\iint_D\left(\frac{x-y}{x+y+2}\right)^2dx\,dy = \iint_{D^*}\left(\frac{u}{v+2}\right)^2\cdot\frac12\,du\,dv$$

$$=\frac12\int_{-1}^1\int_{-1}^1\frac{u^2}{(v+2)^2}du\,dv = \frac12\int_{-1}^1 u^2du\cdot\int_{-1}^1\frac{1}{(v+2)^2}dv$$

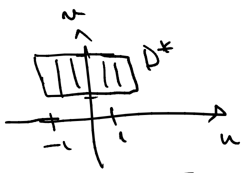
$$=\frac12\cdot\frac23\cdot\frac{-1}{v+2}\Big|_{-1}^1 = \frac13\cdot\frac23=\frac29.$$

(c)



$y=u\in[-1,1]$

$x=v(1-u^2)$, $v\in[1,3]$.



$$\frac{\partial x}{\partial u}=-2uv,\ \frac{\partial x}{\partial v}=1-u^2,\ \frac{\partial y}{\partial u}=1,\ \frac{\partial y}{\partial v}=0,\ J=\begin{bmatrix}-2uv & 1-u^2\\ 1 & 0\end{bmatrix},\ \det J=u^2-1$$

$|\det J|=1-u^2.$

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$$|\det J|=1-u^2.$$

$$\iint_D \frac{y^2}{x}\,dx\,dy=\iint_{D^*}\frac{u^2}{v(1-u^2)}\cdot(1-u^2)\,du\,dv=\int_1^3\int_{-1}^1\frac{u^2}{v}\,du\,dv=\int_1^3\frac{2}{3v}\,dv=\frac{2}{3}\ln 3.$$

3. Compute the following integrals using polar coordinates:

(a) $\iint_D \sqrt{x^2+y^2}\,dx\,dy$, where $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 4\}$.

(b) $\iint_D \sin(x^2+y^2)\,dx\,dy$, where $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le a^2,\ x\le 0\}$.

(c) $\iint_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy$, where $D=\{(x,y)\in\mathbb{R}^2 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}$.

(d) ★ $\iint_D \ln(x^2+y^2)\,dx\,dy$, where $D$ is the region in the first quadrant between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$, with $0<a<b$.

$$x=r\cos\theta,\ y=r\sin\theta,\ x^2+y^2=r^2.$$

$$J=\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix}=\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}.$$

$$|\det J|=r.$$

(a) $r\in[0,2],\ \theta\in[0,2\pi]$, $\displaystyle\iint_D\sqrt{x^2+y^2}\,dx\,dy=\int_0^{2\pi}\int_0^2 r\cdot r\,dr\,d\theta=2\pi\cdot\frac{r^3}{3}\Big|_0^2=\frac{16\pi}{3}.$

(b) $r\in[0,a],\ \theta\in[\frac{\pi}{2},\frac{3\pi}{2}]$, $\displaystyle\iint_D\sin(x^2+y^2)\,dx\,dy=\int_{\pi/2}^{3\pi/2}\int_0^a\sin(r^2)\cdot r\,dr\,d\theta=\pi\cdot\frac{-1}{2}\cos(r^2)\Big|_0^a=\frac{\pi}{2}(1-\cos a^2).$

(c) $\frac{x}{a}=r\cos\theta,\ x=ar\cos\theta.$ $\quad J=\begin{bmatrix}a\cos\theta & -ar\sin\theta\\ b\sin\theta & br\cos\theta\end{bmatrix},\ |\det J|=abr.$

$\frac{y}{b}=r\sin\theta,\ y=br\sin\theta.$

$r\in[0,1],\ \theta\in[0,2\pi].$ $\displaystyle\iint_D\sqrt{1-x^2-y^2}\,dx\,dy=\int_0^{2\pi}\int_0^1\sqrt{1-r^2}\cdot abr\,dr\,d\theta=2\pi\cdot ab\cdot\int_0^1 r\sqrt{1-r^2}\,dr$

$$=2\pi ab\ \frac{-(1-r^2)^{3/2}}{3/2}\Big|_0^1=\frac{4\pi ab}{3}.$$

4. Find the area of the region bounded by the curve defined through the equation:

(a) $(x^2+y^2)^2=a^2(x^2-y^2)$, $x>0$, $a>0$. (b) $(x^2+y^2)^2=2a^2xy$, $a>0$.

Hint: use polar coordinates.

I skipped this question about lemniscates in the seminar. It can be considered as an extra question for students to try for themselves.

(a) Polar coordinates: $r^4=a^2r^2(\cos^2\theta-\sin^2\theta)=a^2r^2\cos 2\theta\to r^2=a^2\cos 2\theta\Rightarrow\theta\in[-\frac{\pi}{4},\frac{\pi}{4}].$

$r=a\sqrt{\cos 2\theta}.$

$$\text{Area}=\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta=\int_{-\pi/4}^{\pi/4}\frac{a^2\cos 2\theta}{2}\,d\theta=\frac{a^2}{2}.$$

(b) $r^4=2a^2r^2\cos\theta\sin\theta=a^2r^2\sin 2\theta\Rightarrow r^2=a^2\sin 2\theta\Rightarrow\theta\in[0,\frac{\pi}{2}]\cup[\pi,\frac{3\pi}{2}]$, $r=a\sqrt{\sin 2\theta}.$

$$\text{Area}=2\int_0^{\pi/2}\int_0^{a\sqrt{\sin 2\theta}} r\,dr\,d\theta=\cancel{2}\int_0^{\frac{\pi}{2}}\frac{a^2\sin 2\theta}{\cancel{2}}\,d\theta=a^2.$$

5. Compute $\iiint_D \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, where $D=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\le az\}$.

Spherical coordinates: $x=(r\sin\varphi)\cos\theta,\ y=(r\sin\varphi)\sin\theta,\ z=r\cos\varphi,\ x^2+y^2+z^2=r^2.$

Domain: $r^2\le ar\cos\varphi\Rightarrow r\le a\cos\varphi\to\varphi\in[0,\frac{\pi}{2}],\ r\in[0,a\cos\varphi],\ \theta\in[0,2\pi].$

$2\pi \quad \pi/2 \quad a\cos\varphi \qquad \pi/2$

Domain : $r^2 \leq a r \cos\varphi \Rightarrow r \leq a\cos\varphi \to \varphi \in [0, \frac{\pi}{2}], r \in [0, a\cos\varphi], \theta \in [0, 2\pi]$.

$$\iiint_D \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a\cos\varphi} r \cdot r^2 \sin\varphi \, dr\, d\varphi\, d\theta = 2\pi \int_0^{\pi/2} \sin\varphi \cdot \frac{a^4\cos^4\varphi}{4} d\varphi, \quad u = \cos\varphi,\ du = -\sin\varphi\, d\varphi.$$

$$= -\frac{a^4\pi}{2} \cdot \frac{u^5}{5}\Big|_1^0 = \frac{a^4\pi}{10}.$$

6. Show that the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is given by $\frac{4\pi}{3}abc$.

$u = \frac{x}{a}, x = au$

$v = \frac{y}{b}, y = bv$

$w = \frac{z}{c}, z = cw$

$J = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, $\det J = abc$, $D^* = \{(u,v,w) \mid u^2+v^2+z^2 \leq 1\} = B(0,1)$.

$$\text{Volume} = \iiint_{B(0,1)} 1 \cdot abc\, du\, dv\, dw = abc \cdot \text{Vol}(B(0,1)) = abc \cdot \frac{4\pi}{3}.$$