7.5 Exercises

7.1. For each of the equations in Table 7.2, discuss the geometric locus of points satisfying them.

7.2. For each of the following matrices A, write down a quadratic equation with associated matrix A and find the matrix $M \in \mathrm{SO}(2)$ which diagonalizes A.

a) $\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$

b) $\begin{bmatrix} 5 & -13 \\ -15 & 5 \end{bmatrix}$

c) $\begin{bmatrix} 7 & -2 \\ -2 & 5/3 \end{bmatrix}$

7.3. Check the calculations in Examples 7.2, 7.3 and 7.4.

7.4. For each of the following equations write down the associated matrix and bring the equation in canonical form.

a) $-x^2 + xy - y^2 = 0$,

b) $6xy + x - y = 0$.

7.5. In each of the following cases, decide the type of the quadratic curve based on the parameter $a \in \mathbb{R}$.

a) $x^2 - 4xy + y^2 = a$,

b) $x^2 + 4xy + y^2 = a$.

7.6. Consider the rotation R_{90° of $\mathbb{E}^2$ around the origin and the translation $T_{\mathbf{v}}$ of $\mathbb{E}^2$ with vector $\mathbf{v}(1,0)$.

a) Give the algebraic form of the isometries R_{90°, $T_{\mathbf{v}}$ and $T_{\mathbf{v}} \circ R_{90^\circ}$.

b) Determine the equations of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the parabola $\mathcal{P} : y^2 - 8x = 0$ after transforming them with R_{90° and with $T_{\mathbf{v}} \circ R_{90^\circ}$ respectively.

7.7. Find the canonical equation for each of the following cases

a) $5x^2 + 4xy + 8y^2 - 32x - 56y + 80 = 0$,

b) $8y^2 + 6xy - 12x - 26y + 11 = 0$,

c) $x^2 - 4xy + y^2 - 6x + 2y + 1 = 0$.

7.8. For each of the conics in the previous exercise, indicate the affine change of coordinates which brings the equation in canonical form.

7.9. Discuss the type of the curve

$$x^2 + \lambda xy + y^2 - 6x - 16 = 0$$

in terms of $\lambda \in \mathbb{R}$.

7.10. Using the classification of quadrics, decide what surfaces are described by the following equations.

a) $x^2 + 2y^2 + z^2 + xy + yz + zx = 1$,

b) $xy + yz + zx = 1$,

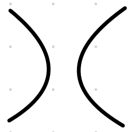
c) $x^2 + xy + yz + zx = 1$,

d) $xy + yz + zx = 0$.

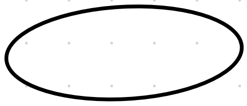
4.1

$x^2+y^2+1=0$ – imaginary eclipse

$x^2-y^2-1=0$ – hyperbola

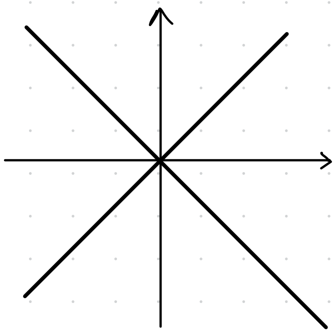


$x^2+y^2-1=0$ – elipse



$x^2+y^2=0$ – two complex lines

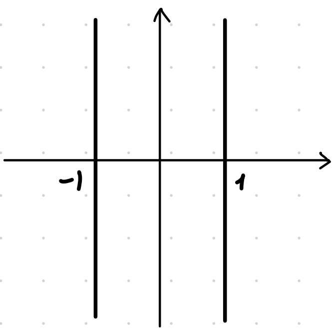
$x^2-y^2=0$ – two real lines



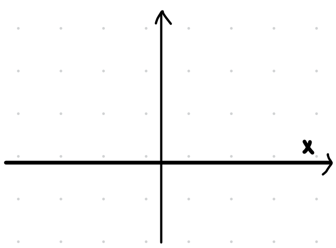
$x^2+1=0$ – two complex lines

$x^2-1=0$ – two real lines

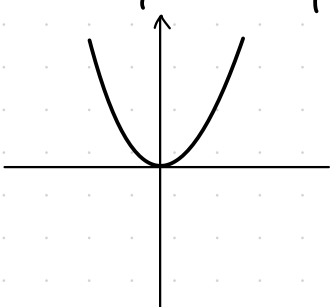
$x=\pm 1$



$x^2=0$ – a real double line



$x^2-y=0$ – parabola



7.2

$$\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} = A$$

$$Q: \ (x\ y) A \begin{pmatrix} x \\ y \end{pmatrix} + (x\ y)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 3 = 0$$

$$\begin{pmatrix} 6x + 2y \\ 2x + 9y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + x + 2y + 3 = 0$$

$$6x^2 + 2xy + 2xy + 9y^2 + x + 2y + 3 = 0$$

$$6x^2 + 4xy + 9y^2 + x + 2y + 3 = 0$$

$$\left(6x^2 + 2\sqrt{6}\,xy \cdot \frac{2}{\sqrt{6}} + \frac{4}{6}y^2\right) + 9y^2 - \frac{2}{3}y^2 + x + 2y + 3 = 0$$

$$\left(\sqrt{6}x + \frac{2}{\sqrt{6}}y\right)^2 + 9y^2 - \frac{2}{3}y^2 + x + 2y + 3 = 0$$

$$x' = \sqrt{6}x + \frac{2}{\sqrt{6}}y \qquad x = \frac{x' - \frac{2}{\sqrt{6}}y}{\sqrt{6}} = \frac{x' - \frac{2}{\sqrt{6}}y'}{\sqrt{6}}$$

$$y' = y$$

$$x'^2 + \frac{25}{3}y'^2 + \frac{x' - \frac{2}{\sqrt{6}}y'}{\sqrt{6}} + 2y' + 3 = 0$$

$$x'^2 + \frac{25}{3}y'^2 + \frac{\sqrt{6}}{6}x' - \frac{2}{6}y' + 2y' + 3 = 0$$

$$6x'^2 + 50y'^2 + \sqrt{6}x' - 2y' + 12y' + 18 = 0$$

$$6x'^2 + 2\sqrt{6}x' \cdot \frac{1}{2} + \frac{1}{4} + \left((5\sqrt{2}y')^2 - 2 \cdot 5\sqrt{2} \cdot \frac{1}{\sqrt{2}}y' + \frac{1}{2}\right) - \frac{1}{4} - \frac{1}{2} + 18 = 0$$

$$(\sqrt{6}x' + \tfrac{1}{2})^2 + (5\sqrt{2}y' - \tfrac{1}{2})^2 + 18 - \frac{3}{4} = 0$$

$$x''^2 + y''^2 + 69 = 0$$

$$\det(A - \lambda I_2) = \begin{vmatrix} 6-\lambda & 2 \\ 2 & 9-\lambda \end{vmatrix} = (6-\lambda)(9-\lambda) - 4 = 54 - 15\lambda + \lambda^2 - 4 = \lambda^2 - 15\lambda + 50$$

$$\lambda_1 = 10$$
$$\lambda_2 = 5$$

$$S(\lambda_2) = \left\{ (x,y) \,\middle|\, \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$x + 2y = 0$$
$$2x + 4y = 0$$

$$S(\lambda_2) = \{(-2y, y) \mid y \in \mathbb{R}\} = \langle(-2,1)\rangle$$

$$S(\lambda_1) = \left\{ (x,y) \,\middle|\, (A - \lambda I_2)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \right\}$$

$$S(10) = \left\{ (x,y) \,\middle|\, \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \right\}$$

$$2y = 4x$$
$$y = 2x$$

$$S(10) = \{(x, 2x) \mid x \in \mathbb{R}\} = \langle(1,2)\rangle$$

$$v_1 = \frac{1}{\sqrt{5}}(1,2) \qquad v_2 = \frac{1}{\sqrt{5}}(-2,1)$$

$$\frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix} = M$$

$$M \cdot M^t = \mathcal{I}_2$$

$$M \cdot A \cdot M^T = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}\begin{pmatrix}6 & 2\\ 2 & 9\end{pmatrix} \cdot \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$$

$$\cancel{= \frac{1}{5}\begin{pmatrix}2 & -16\\ 14 & 13\end{pmatrix}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix} =}$$

$$\cancel{= \frac{1}{5}}\Big(30$$

$$= \frac{1}{5}\begin{pmatrix}50 & 0\\ 0 & 25\end{pmatrix} = \begin{pmatrix}10 & 0\\ 0 & 5\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

4.4 a) $-x^2 + xy + y^2 = a$

$$Q: \begin{pmatrix}-1 & \frac{1}{2}\\ \frac{1}{2} & -1\end{pmatrix}$$

$$\det(Q - \lambda \mathcal{I}_2) = 0$$

$$\begin{vmatrix}-1-\lambda & \frac{1}{2}\\ \frac{1}{2} & -1-\lambda\end{vmatrix} = (1+\lambda)^2 - \frac{1}{4} = 1 + 2\lambda + \lambda^2 - \frac{1}{4}$$

$$\lambda_{1,2} = \frac{-2 \pm 1}{2}$$

$$\lambda_1 = -\frac{1}{2} \qquad \lambda_2 = \frac{-3}{2}$$

$$\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}1\\ 1\end{pmatrix} = v_1$$

$$\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 0$$

$$\frac{1}{2}x = -\frac{1}{2}y \Rightarrow \begin{pmatrix}1\\ -1\end{pmatrix} = v_2$$

$$M = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$u = \frac{1}{\sqrt{2}}(x+y)$$
$$v = \frac{1}{\sqrt{2}}(x-y)$$

$$\frac{3}{2}u^2 + \frac{1}{2}v^2 = 0$$

$$\frac{3}{2} \cdot \frac{1}{2}(x+y)^2 + (x-y)^2 = 0$$

$$x^2 + y^2 + xy = 0$$

b) $6xy + x - y = 0$

$Q = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$

$\lambda^2 - 9 = 0 \Rightarrow \lambda_{1,2} = \pm 3$

$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow u = \frac{1}{\sqrt{2}}(x+y)$

$v = \frac{1}{\sqrt{2}}(x \cdot y)$

$3u^2 - 3v^2 = 0$