(b) ★ $\underbrace{\sum_{n\geq 1} \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}\cdot \frac{1}{n^2}}_{x_n}$.

ratio test: $\frac{x_{n+1}}{x_n} = \frac{1\cdot 3\cdot \ldots \cancel{(2n+1)}}{\cancel{2\cdot 4\cdot \ldots}\, 2(n+1)}\cdot\frac{1}{(n+1)^2}\cdot\frac{\cancel{2\cdot 4\cdot \ldots\cdot 2n}}{\cancel{1\cdot 3\cdot \ldots (2n-1)}}\cdot n^2 = \frac{2n+1}{2n+2}\cdot\frac{n^2}{(n+1)^2} = 1$

R-D Test: $\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right) = L$

$L > 1 \Rightarrow \sum x_n$ conv

$L < 1 \Rightarrow \sum x_n$ div

$\lim_{n\to\infty} n\left(\frac{2(n+1)\cdot (n+1)^2}{(2n+1)\,n^2} - 1\right) = \lim_{n\to\infty} n\left(\frac{2(n+1)(n^2+2n+1)}{2n^3+n^2}-1\right) = \lim_{n\to\infty} n\left(\frac{2n^3+4n^2+2n+n^2+2n+1}{2n^3+n^2}-1\right) = \lim_{n\to\infty} n\left(\cancel{1}+\frac{4n^2+4n+1}{n^2(2n+1)}\cancel{-1}\right)$

$= \lim_{n\to\infty} \cancel{n}\cdot\frac{(2n+1)^{\cancel{2}}}{\cancel{(2n+1)}\cdot n^{\cancel{2}}} = \lim_{n\to\infty}\frac{2n+1}{n} = 2 > 1 \Rightarrow \sum_{n\geq 1} x_n$ converges

(c) ★ $\sum_{n\geq 1}\frac{nx^n}{2^n}$. — find the radius of convergence

$S(x) = \sum_{n\geq 1} a_n (x-c)^n$ $\qquad \lim_{n\to\infty}\sqrt[n]{a_n} = L \in [0,\infty]$, where $a_n = \frac{n}{2^n}$

root test: $\lim_{n\to\infty}\sqrt[n]{\frac{n}{2^n}} = \frac{1}{2}\lim_{n\to\infty} n^{\frac{1}{n}} = \frac{1}{2}\lim_{n\to\infty} e^{\frac{1}{n}\ln n} =$

$= \frac{1}{2}\cdot e^{\lim_{n\to\infty}\frac{\ln n}{n}} = \frac{1}{2}\cdot e^0 = \frac{1}{2} \in (0,\infty)$

$R = \begin{cases} \frac{1}{L}, & \text{if } L\in(0,\infty)\\ 0, & \text{if } L=\infty \\ \infty, & \text{if } L=0\end{cases}$ $\Rightarrow R = \frac{1}{L} = \frac{1}{\frac{1}{2}} = 2$ $\Rightarrow$ the convergence set C contains $(-2,2)$ and possibly $\{-2,2\}$

for $x=2 \Rightarrow \sum_{k=1}^{\infty}\frac{n\cdot \cancel{2^n}}{\cancel{2^n}} = \sum_{n=1}^{\infty} n \rightarrow$ divergent

$x=-2 \Rightarrow \sum_{n=1}^{\infty}\frac{n\cdot(-2)^n}{2^n} = \sum_{n=1}^{\infty}(-1)^n\cdot n \rightarrow$ divergent

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$a_n = \frac{nx^n}{2^n}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+1)\cancel{x^{n+1}}}{\cancel{2^{n+1}}}}{\frac{n\cancel{x^n}}{\cancel{2^n}}} = \frac{(n+1)x}{2n} = \frac{x}{2}\cdot\frac{n+1}{n}$

$\Longrightarrow \lim_{n\to\infty}\frac{x}{2}\cdot\frac{n+1}{n} = \frac{x}{2} \Rightarrow$ based on the ratio test we have that the series is absolutely convergent for $\left|\frac{x}{2}\right|<1$ and divergent for $\left|\frac{x}{2}\right|>1$

for $\left|\frac{x}{2}\right| = 1$ the test is inconclusive $\Rightarrow$ we verify case by case

a) $\frac{x}{2} = 1 \Rightarrow x=2$

$a_n = \frac{n\cancel{2^n}}{\cancel{2^n}} = n$ which is divergent

b) $\frac{x}{2} = -1 \Rightarrow x=-2 \Rightarrow$ $a_n = \frac{n(-1)^n\cancel{2^n}}{\cancel{2^n}} = (-1)^n\cdot n \rightarrow$ divergent