Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză

Curs: Dynamical Systems

Primăvara 2024

Seminar 4

1. For each $k > 0$ we consider the differential equation

$$\dot{x} = -k\,(x - 21),$$

which is the model of Newton for cooling processes, here $x(t)$ being the temperature of a cup of tea at time t.

(a) Find its flow. (b) An experiment revealed the following fact. A cup of tea with initial temperature of $49°C$ has a temperature of $37°C$ after 10 minutes. Find the initial temperature of a cup of tea such that after 20 minutes the tea has $37°C$. $\diamond$

Theorem 1 *Let $f \in C^1(\mathbb{R})$ and $\eta^* \in \mathbb{R}$ be such that $f(\eta^*) = 0$.*
If $f'(\eta^*) < 0$ then η^* is an attractor equilibrium point of $\dot{x} = f(x)$.
If $f'(\eta^*) > 0$ then η^* is a repeller equilibrium point of $\dot{x} = f(x)$.

2. Let $0 < c < 1$ be a parameter and consider the scalar dynamical system
$\dot{x} = x(1 - x) - cx$.

a) Find its equilibria and study their stability using the linearization method.

b) Represent its phase portrait.

c) When $x(t) > 0$ is considered to be the density of fish in a lake, and $0 < c < 1$ to be the rate of fishing, try to predict the fate of the fish from the lake interpreting the theoretical result obtained at a) and b). $\diamond$

3. Represent the phase portrait of the scalar dynamical system
$\dot{x} = x - x^3$. Find $\varphi(t, -1)$ and $\varphi(t, 0$ and justify. Specify the properties of the functions $\varphi(t, -2)$, $\varphi(t, 3)$ and, respectively, $\varphi(t, -0.5)$.

4. Represent the phase portrait of the scalar dynamical systems
a) $\dot{x} = x - x^3 + 1$; b) $\dot{x} = -x^3$; c) $\dot{x} = x^3$; d) $\dot{x} = -x^2$. Try to use the linearization method.

1. For each $k > 0$ we consider the differential equation

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which is the model of Newton for cooling processes, here $x(t)$ being the temperature of a cup of tea at time t.

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Flow → mulțimea tuturor ecuațiilor posibile
$\varphi(\cdot, \mu)$ is the solution of the $\begin{cases} x' = f(x) \\ x(0) = \mu \end{cases}$

a) Let $\mu \in \mathbb{R}$ fixed

$\begin{cases} x' = -k(x-21) \\ x(0) = 0 \end{cases}$

$\frac{x'}{x-21} = -k$

$(\ln(x-21))' = -k \quad | \int$

$x - 21 = e^{-kt} \cdot c$

$x(t) = e^{-kt} \cdot c + 21$

$\mu = x(0) = 21 + c$

$c = \mu - 21$

$\Rightarrow x(t) = (\mu - 21) \cdot e^{-kt} + 21$

$\varphi(t, \mu) = (\mu - 21)\, e^{-kt} + 21$

x^* equilibrium point iff $(x^*)' = 0$

$0 = (x^*)' = f(x^*)$

We find the equilibrium points by solving $f(x) = 0$

$-k(x - 21) = 0 \Rightarrow x^* = 21$

b) $49°C \xrightarrow{10'} 37°C$

initial state $t=0$ $\qquad$ $t=10$

$\varphi(10, 49) = (49 - 21) \cdot e^{-k \cdot 10}$

$37 = 28\, e^{-10k} + 21$

$e^{-10k} = \frac{16}{28} = \frac{4}{7} \Rightarrow k = -\frac{1}{10} \ln \frac{4}{7}$

$x^\circ \xrightarrow{20'} 37°$

$\varphi(20, \mu) = 37$

$(\mu - 21) e^{-20k} + 21 = 37$

$(\mu - 21)(e^{-10k})^2 = 16$

$$(\mu-21)\frac{16}{49}=16$$

$$\mu-21=49$$

$$\mu=70 \Rightarrow t_0=70^\circ$$

2. Let $0<c<1$ be a parameter and consider the scalar dynamical system $\dot{x}=x(1-x)-cx$.

~~a)~~ Find its equilibria and study their stability using the linearization method.

~~b)~~ Represent its phase portrait.

c) When $x(t)>0$ is considered to be the density of fish in a lake, and $0<c<1$ to be the rate of fishing, try to predict the fate of the fish from the lake interpreting the theoretical result obtained at a) and b). $\diamond$

Pentru $x'=f(x)$, x^* eq points: $f(x^*)=0$

x^* < attractor / repeller

x^* is an attractor if $\exists\, r>0$ s.t. $|\varphi(t,x^*)-\varphi(t,\mu)|\to 0$, $t\to\infty$ $\forall\, \mu\in B(x^*,R)$

$$\begin{cases} x'=1-x \\ f(x)=1-x \end{cases}$$

$x^*=1$ eq. point

$\varphi(t,\mu)=-e^{-t}\cdot(1-\mu)+1 \Rightarrow \varphi(t,1)=1$

$|\varphi(t,\mu)-\varphi(t,x^*)|=|-e^{-t}(1-\mu)+1-1|=e^{-t}|1-\mu|\to 0$ $\quad r=\infty$

* exam

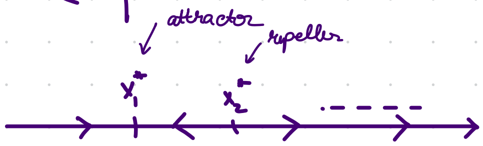
$x'=f(x)$

x^* eq. point

if $f'(x^*)<0 \Rightarrow x^*$ is an attractor

$f'(x^*)>0 \Rightarrow x^*$ is a repeller

Phase portrait



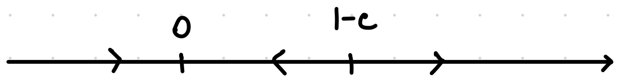
$x'=x(1-x)-cx$, $c\in(0,1)$

a) $f(x) = x(1-x) - cx = 0$

$x(1-x-c) = 0 \Rightarrow \begin{cases} x_1^* = 0 \\ x_2^* = 1-c \end{cases}$

$f'(x) = 1-2x-c$

$$\left.\begin{array}{l} f'(0) = 1-c \\ f'(1-c) = 1-2+2c-c = c-1 \\ c \in (0,1) \end{array}\right\} \Rightarrow \begin{array}{l} f'(0) > 0 \ - \text{attract.} \\ f'(1-c) < 0 - \text{repeller} \end{array}$$



3. Represent the phase portrait of the scalar dynamical system
$\dot{x} = x - x^3$. Find $\varphi(t,-1)$ and $\varphi(t, 0$ and justify. Specify the properties of the functions $\varphi(t,-2)$, $\varphi(t,3)$ and, respectively, $\varphi(t,-0.5)$.

$x' = x - x^3$

$\varphi(t,\mu) - \begin{cases} x' = f(x) \\ x(0) = \mu \end{cases}$

Orbits imaginea soluției

$x' = f(x)$

$\varphi(t,\mu)$ flow

$\Phi(\mu) = \{\varphi(t,\mu) \mid t \in \text{domain}\} \to$ orbit

if x_1^*, x_2^* eq points

$\Phi(\mu_1) = \Phi(\mu_2) \ \forall \ \mu_1, \mu_2 \in (-\infty, x_1^*)$

$\mu_1, \mu_2 \in (x_1^*, x_2^*)$

$\mu_1, \mu_2 \in (x_2^*, \infty)$

$\Phi(\mu_1) = (-\infty, x_1^*), \mu \in (-\infty, x_1^*)$

$\Phi(\mu_2) = (x_1^*, x_2^*), \mu \in (x_1^*, x_2^*)$

$\varphi(t,\mu) \to$ is increasing in μ $\quad \varphi(t,\mu_1) \le \varphi(t,\mu_2)$
$\forall \mu_1 \le \mu_2$

$x' = x - x^3$

$f(x) = x(1-x^2)$

$x(1-x^2) = 0 \Rightarrow \begin{cases} x_1^* = 0 \\ x_2^* = 1 \\ x_3^* = -1 \end{cases}$

$\gamma(-\infty, x_3^*) = (-\infty, -1)$

$\gamma(x_3^*, x_1^*) = (-1, 0)$

$\gamma(x_1^*, x_2^*) = (0, 1)$

$\gamma(x_2^*, +\infty) = (1, \infty)$

4. Represent the phase portrait of the scalar dynamical systems
a) $\dot{x} = x - x^3 + 1$; b) $\dot{x} = -x^3$; c) $\dot{x} = x^3$; d) $\dot{x} = -x^2$. Try to use the linearization method.

$\cancel{a)\ x' = x - x^3 + 1 = 0}$

b) $\left.\begin{array}{l} x' = -x^3 \\ f(x) = 0 \end{array}\right\} \Rightarrow x = 0 \Rightarrow x_1^* = 0$ eq. point

$f'(x) = -3x^2$

$f'(x_1^*) = f'(0) = 0$

$\left\{\begin{array}{l} x' = -x^3 \;\; | : x^3 \\ x_0 = \mu \end{array}\right. \Rightarrow \dfrac{x'}{x^3} = -1$

$\dfrac{x^{-2}}{-2} = -t$

$\dfrac{1}{x^2(t)} = 2t - 2c$

$x^2(t) = \dfrac{1}{2t - 2c}$

$x(t) = \dfrac{1}{\sqrt{2t - 2c}} \Rightarrow x(0) = \dfrac{1}{\sqrt{-2c}} \Rightarrow \left.\begin{array}{l} |c| = \dfrac{1}{2\mu^2} \\ c < 0 \end{array}\right\} \Rightarrow c = -\dfrac{1}{2\mu^2}$

$\varphi(t, \mu) = \dfrac{1}{\sqrt{2t + \frac{1}{\mu^2}}}$

$|\varphi(t, \mu) - \varphi(t, 0)|$

$\left| \dfrac{1}{\sqrt{2t + \frac{1}{\mu^2}}} - \dfrac{1}{\sqrt{2t}} \right| = \left| \dfrac{\sqrt{2t} - \sqrt{2t + \frac{1}{\mu^2}}}{\sqrt{2t} \cdot \sqrt{2t + \frac{1}{\mu^2}}} \right| \to 0 \Rightarrow$ attractor