Seminar 10

1. For $f : \mathbb{R}^2 \to R, f(x, y) = x^2 + xy$ find:
   (a) the gradient of f and the direction of steepest descent at the point $(1, 0)$.
   (b) the directional derivative at the point $(1, 0)$ in the direction of $\vec{i} + \vec{j}$.
   (c) the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 0, 1)$.

2. Find the equation of the tangent line to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at an arbitrary point (x_0, y_0).

3. Let $f : \mathbb{R}^n \to \mathbb{R}, f(x) = \frac{1}{2}\|x\|^2$. Find the gradient of f. Find the directional derivative $D_v f(x)$ in two ways: using the definition and using the gradient.

4. Let $D = \mathrm{diag}(d_1, \ldots, d_n)$ be a diagonal $n \times n$ matrix and consider the quadratic function $f : \mathbb{R}^n \to R$, $f(x) = \frac{1}{2}x^T Dx$. Prove that $\nabla f(x) = Dx$ and $H(x) = D$.

5. For each of the following functions, compute $\dfrac{\mathrm{d}f}{\mathrm{d}t}$ directly and using the chain rule:
   (a) $f(x, y) = \ln(x^2 + y^2)$,
   $x = t$, $y = t^2$.
   (b) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$,
   $x = \cos t$, $y = \sin t$, $z = t > 0$.

6. Let $f : \mathbb{R}^2 \to R$ and $(x, y) = (g_1(u, v), g_2(u, v)) = g(u, v)$, $f(x, y) = (f \circ g)(u, v)$. Prove that

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

Homework questions will be given in a separate document.

1.

(a) $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x), \frac{\partial f}{\partial y}(y) \right)$ — gradient

$f : \mathbb{R}^2 \to \mathbb{R} \qquad f(x,y) = x^2 + xy$

$\frac{\partial f}{\partial x}(x,y) = 2x + y$

$\frac{\partial f}{\partial y}(x,y) = x$

$\nabla f(1,0) = (2,1)$

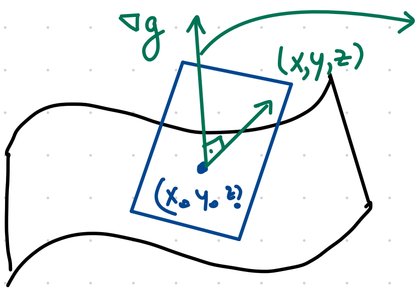
direction of the steepest descent : $-\nabla f(1,0) = (-2,-1)$

(b) $D_v f(x,y) = \nabla f(x,y) \cdot v = (2,1) \cdot \underbrace{(1,1)}_{\vec{i}+\vec{j}} = 2+1 = 3$

(c) $z = f(x,y)$ at $(1,0,1)$

$\Rightarrow z = x^2 + xy \Rightarrow \underbrace{x^2 + xy - z}_{g(x,y,z)} = 0$

$\boxed{\text{Gradient } \perp \text{ level set } !}$



norm of the plane

* gradient will be perpendicular on any vector of the plane

$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) = (2x+y, x, -1)$

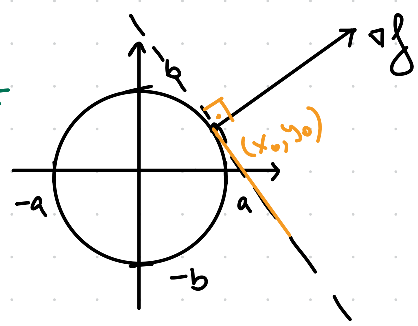
$\nabla g(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$

$\nabla g(1,0,1) = (2,1,-1)$

$(2,1,-1) \cdot (x-1, y, z-1) = 0$

$2(x-1) + y - z + 1 = 0 \Rightarrow 2x + y - z - 1 = 0 \Rightarrow \boxed{z = 2x + y - 1}$

2. $\underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2}}_{f(x,y)} = 1$ tangent line at (x_0, y_0)



elipse = level set = $\{(x,y) \mid f(x,y) = 1\}$ — all points in the plane that satisfy

$\nabla f(x,y) \perp$ level set

$$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0) = 0$$

$$\frac{\partial f}{\partial x} = \frac{2x}{a^2} \qquad \frac{\partial f}{\partial y} = \frac{2y}{b^2}$$

$$\Rightarrow \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) = 0 \quad |:2$$

$$\frac{x_0 \cdot x - x_0^2}{a^2} + \frac{y_0 \cdot y - y_0^2}{b^2} = 0$$

$$\frac{x_0 \cdot x}{a^2} + \frac{y_0 \cdot y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = f(x_0, y_0) = 1$$

3. $f : \mathbb{R}^n \to \mathbb{R}$

$$f(x) = \frac{1}{2}\|x\|^2 = \frac{1}{2} x \cdot x = \frac{1}{2}(x_1, x_2, \ldots, x_n) \cdot (x_1, x_2, \ldots, x_n) =$$

$$= \frac{1}{2}(x_1^2 + x_2^2 + \ldots + x_n^2)$$

$$\nabla f(x) = ?$$

$$\frac{\partial f}{\partial x_1} = \frac{1}{2} 2x_1 = x_1$$

$$\vdots$$

$$\frac{\partial f}{\partial x_i} = x_i$$

$$\Rightarrow \nabla f(x) = (x_1, x_2, \ldots, x_n) = x$$

$D_v f(x) = \nabla f(x) \cdot v = x \cdot v$ (direction of derivative)

$$\lim_{h \to 0} \frac{f(x + hv) - f(x)}{h} \longleftarrow \text{definition}$$

$$\hookrightarrow f(x + hv) = \frac{1}{2}\|x + hv\|^2 = \frac{1}{2}(x + hv) \cdot (x + hv) =$$

$$= \frac{1}{2}(x \cdot x + 2hv + h^2 v \cdot v)$$

$$\Rightarrow f(x + hv) - f(x) = \frac{1}{2}(\cancel{x \cdot x} + 2hv + h^2 v \cdot v - \cancel{x \cdot x}) =$$

$$= h(x \cdot v) + \frac{h^2}{2}(v \cdot v)$$

$$\Rightarrow \lim_{h \to 0} \frac{\cancel{h}(x \cdot v + \frac{h}{2}(v \cdot v))}{\cancel{h}} = x \cdot v$$

4. $D = \mathrm{diag}(d_1, \dots, d_n)$

$f : \mathbb{R}^n \to \mathbb{R} \quad f(x) = \frac{1}{2} x^T D x$

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$\nabla f(x) = Dx \qquad H(x) = D$

$x \cdot y = x^T y$ where $x, y \in \mathbb{R}^n$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad D = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{bmatrix}$$

$$\Rightarrow Dx = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{bmatrix}$$

$$f(x) = \frac{1}{2} x^T D x = \frac{1}{2} [x_1 \; x_2 \dots x_n] \begin{bmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{bmatrix} = \frac{1}{2}(x_1^2 d_1 + x_2^2 d_2 + \dots + x_n^2 d_n)$$

linear function

(a) $\nabla f(x) \overset{?}{=} Dx$

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} &= d_1 x_1 \\ \frac{\partial f}{\partial x_2} &= d_2 x_2 \\ &\vdots \\ \frac{\partial f}{\partial x_n} &= d_n x_n \end{aligned}\right\} \Rightarrow \nabla f(x) = (d_1 x_1, d_2 x_2, \dots, d_n x_n) = Dx$$

(b) Hessian matrix

$$H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x_1^2} &= d_1 \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} &= 0 = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad j \neq i \\ \frac{\partial^2 f}{\partial x_1 x_2} &= 0 \\ \frac{\partial^2 f}{\partial x_2^2} &= d_2 \end{aligned}\right\} \Rightarrow \begin{aligned} \frac{\partial^2 f}{\partial x_i^2} &= d_i, \; i = \overline{1,n} \\ \frac{\partial^2 f}{\partial x_i \partial x_j} &= 0, \; i \neq j \end{aligned}$$

$\Rightarrow H(x) = D$

constant

* Chain rule

5. (a) $f(x,y) = \ln(x^2+y^2)$

$x = t \quad y = t^2$

$$f(t) = \ln(t^2+t^4) \Rightarrow \frac{df}{dt} = \frac{2t+4t^3}{t^2+t^4} = \frac{2+4t^2}{t+t^3}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$\left.\begin{aligned} &\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{dx}{dt} = 1 \\ &\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} \quad \frac{dy}{dt} = 2t \end{aligned}\right\} \Rightarrow \frac{\partial f}{\partial t} = \frac{2x}{x^2+y^2} + \frac{2y}{x^2+y^2}\cdot 2t =$$

$$= \frac{2t}{t^2+t^4} + \frac{4t^3}{t^2+t^4} = \frac{2+4t^2}{t+t^3}$$

(b) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$x = \cos t, \quad y = \sin t, \quad z = t > 0$

$$f(t) = \sqrt{\cos^2 t + \sin^2 t + t^2} = \sqrt{t^2+1} \Rightarrow \frac{df}{dt} = \frac{t}{\sqrt{t^2+1}}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad \frac{dx}{dt} = -\sin t$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad \frac{dy}{dt} = \cos t$$

$$\frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}, \quad \frac{dz}{dt} = 1$$

$$\Rightarrow \frac{\partial f}{\partial t} = \cancel{\frac{\cos t}{\sqrt{\cos^2 t + \sin^2 t + t^2}}(-\sin t)} + \cancel{\frac{\sin t}{\sqrt{\cos^2 t + \sin^2 t + t^2}}(\cos t)}$$

$$+ \frac{t}{\sqrt{\cos^2 t + \sin^2 t + t^2}} = \frac{t}{\sqrt{1+t^2}}$$

6. $f: \mathbb{R}^2 \to \mathbb{R} \qquad (x,y) = (g_1(u,v), g_2(u,v)) = g(u,v)$

$$(f\circ g)(u,v) = f(x,y)$$

Prove: $\dfrac{\partial f}{\partial u} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial u}$ and $\dfrac{\partial f}{\partial v} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial v}$

Chain rule: $D(f\circ g)(x) = Df(g(x))\ Dg(x)$

$$Df(u,v) = \nabla f(u,v) = \left(\frac{\partial f}{\partial x}(u,v), \frac{\partial f}{\partial y}(u,v)\right)_{1\times 2}$$

$$Dg(u,v) = \begin{bmatrix} \nabla g_1(u,v) \\ \nabla g_2(u,v) \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial u} = \frac{\partial x}{\partial u} (u,v) & \frac{\partial g_1}{\partial v}(u,v) \\ \frac{\partial g_2}{\partial u}(u,v) & \frac{\partial g_2}{\partial v}(u,v) \end{bmatrix}_{2\times 2}$$

$$D(f\circ g)(u,v) = \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}, \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}\right) =$$

$$= \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$$