8.7 Exercises

8.1. Determine the intersection of the ellipsoid

$$\mathcal{E}_{4,2\sqrt{3},2} : \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} - 1 = 0 \quad \text{with the line} \quad \ell = \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} + \langle \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

8.2. Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,3,4} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

with planes parallel to the coordinate planes. Treat the various cases separately.

8.3. Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,\sqrt{3},3} : \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \quad \text{with the line} \quad \ell : x = y = z.$$

Write down the equations of the tangent planes in the intersection points.

8.4. Determine the tangent planes to the ellipsoid

$$\mathcal{E}_{2,3,2\sqrt{2}} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{8} = 1$$

which are parallel to the plane $\pi : 3x - 2y + 5z + 1 = 0$.

8.5. Determine the points P of the ellipsoid

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

for which the tangent space $T_P\mathcal{E}$ intersects the coordinate axis in congruent segments.

8.6. Show that the line

$$\begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} + \langle \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \rangle \quad \text{is tangent to the quadric} \quad \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} - 1 = 0$$

and determine the tangency point.

8.7. Prove that the intersection of a quadric in $\mathbb{E}^3$ with a plane is either the empty set or a point or a line or two lines or an ellipse or a hyperbola or a parabola.

8.8. Prove that the intersection of an ellipsoid with a plane is either the empty set or a point or an ellipse.

8.9. Show that the ellipsoid $\mathcal{E}_{a,b,b}$ is the locus of points for which the sum of the distances to two given points is constant. Such a surface is called *ellipsoid of revolution*.

8.10. Use a parametrization of an ellipse and a rotation matrix to deduce a parametrization of an ellipsoid of revolution.

8.11. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}:\begin{cases} x = 4\cos(s)\cos(t) \\ y = 4\sin(s)\cos(t) \\ z = 2\sin(t) \end{cases} \qquad s \in [0, 2\pi[\quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}[$$

a) Give an equation of $\mathcal{S}$.

b) Find the parameters of the point $P(3, \sqrt{3}, 1)$.

c) Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.

d) Give an equation of $T_P\mathcal{S}$.

8.12. Prove that the intersection of an elliptic cone with a plane is either a point or a line or an ellipse or a hyperbola or a parabola.

8.13. Determine the intersection of the hyperboloid

$$\mathcal{H}^1_{4,3,1} : \frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{1} = 1 \quad \text{with the line} \quad \ell = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + \langle \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

8.14. Determine the tangent plane of the hyperboloid

$$\mathcal{H}^1_{2,3,1} : \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$$

in the point $M(2,3,1)$. Show that the tangent plane intersects the surface in two lines.

8.15. Determine the generators of the hyperboloid

$$\frac{x^2}{36} + \frac{y^2}{9} - \frac{x^2}{4} = 1$$

which are parallel to the plane $x + y + z = 0$.

8.16. Determine the intersection of the hyperboloid

$$\mathcal{H}^2_{2,1,3} : \frac{x^2}{4} + \frac{y^2}{1} - \frac{z^2}{9} = -1 \quad \text{with the line} \quad \ell = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} + \langle \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

8.17. Determine the intersection of the paraboloid

$$\mathcal{P}^h_{2,\frac{1}{2}} : x^2 - 4y^2 = 4z \quad \text{with the line} \quad \ell = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \langle \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \rangle.$$

Write down the equations of the tangent planes in the intersection points.

8.18. Determine the tangent plane of

a) the elliptic paraboloid $\frac{x^2}{5} + \frac{y^2}{3} = z$ and of

b) the hyperbolic paraboloid $x^2 - \frac{y^2}{4} = z$

which are parallel to the plane $x - 3y + 2z - 1 = 0$.

8.19. Determine the plane which contains the line

$$\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \langle \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \rangle \quad \text{and is tangent to the quadric} \quad x^2 + 2y^2 - z^2 + 1 = 0.$$

8.20. Show that the parabolid $\mathcal{P}^e_{p,p}$ is the locus of points for which the distance from a point equals the distance to a plane. Such a surface is called *elliptic paraboloid of revolution*.

8.21. Use a parametrization of a parabola and a rotation matrix to deduce a parametrization of an elliptic paraboloid of revolution.

8.22. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S} : \begin{cases} x = \sqrt{1+t^2}\cos(s) \\ y = \sqrt{1+t^2}\sin(s) \\ z = 2t \end{cases}$$

a) Give the equation of $\mathcal{S}$.

b) Find the parameters of the point $P(1,1,2)$.

c) Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.

d) Give the equation of $T_P\mathcal{S}$.

8.23. Determine the generators of the paraboloid

$$\frac{x^2}{16} - \frac{y^2}{4} = z$$

which are parallel to the plane $3x + 2y - 4z = 0$.

8.24. Which of the following is a hyperboliod?

a) $\mathcal{S} : 2xz + 2xy + 2yz = 1$

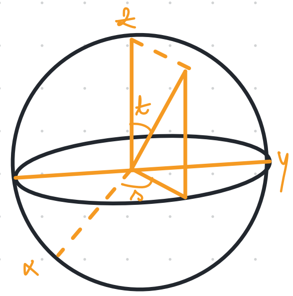
b) $\mathcal{S} : 5x^2 + 3y^2 + xz = 1$

c) $\mathcal{S} : 2xy + 2yz + y + z = 2$

Ex: Cap: 8 — 11, 14, 17, 18, 21, 24

8.11. For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}: \begin{cases} x = 4\cos(s)\cos(t) \\ y = 4\sin(s)\cos(t) \\ z = 2\sin(t) \end{cases} \qquad s \in [0, 2\pi[\quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}[$$



~~a)~~ Give an equation of $\mathcal{S}$.

~~b)~~ Find the parameters of the point $P(3, \sqrt{3}, 1)$.

c) Calculate a parametrization of the tangent plane $T_P\mathcal{S}$ using partial derivatives.

d) Give an equation of $T_P\mathcal{S}$.

a) $x^2 + y^2 = 16\cos^2(s)\cos^2(t) + 16\sin^2(s)\cos^2(t) = 16\cos^2(t)$

$x^2 + y^2 + 4z^2 = 16\cos^2(t) + 16\sin^2(t) = 16$

$\mathcal{S}: \ x^2 + y^2 + 4z^2 = 16 \Leftrightarrow \frac{x^2}{16} + \frac{y^2}{16} + \frac{z^2}{4} = 1$

b) $P(3, \sqrt{3}, 1)$

$3 = 4\cos(s)\cos(t)$
$\sqrt{3} = 4\sin(s)\cos(t)$
$1 = 2\sin(t) \Rightarrow t = \frac{\pi}{6}$

$$\begin{cases} 3 = \cancel{4}^{2}\cos(s) \cdot \frac{\sqrt{3}}{\cancel{2}} \\ \sqrt{3} = \cancel{4} \cdot \sin(s) \frac{\sqrt{3}}{\cancel{2}} \end{cases} = \begin{cases} 3 = 2\sqrt{3}\cos(s) \Rightarrow \cos(s) = \frac{\sqrt{3}}{2} \\ \sqrt{3} = 2\sqrt{3}\sin(s) \Rightarrow \sin(s) = \frac{1}{2} \Rightarrow s = \frac{\pi}{6} \end{cases}$$

c) $P(x_0, y_0, z_0) \Rightarrow T_P \mathcal{E} = \frac{x_0 x}{16} + \frac{y_0 y}{16} + \frac{z_0 z}{4} = 1$

8.14. Determine the tangent plane of the hyperboloid

$$\mathcal{H}^1_{2,3,1}: \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$$

in the point $M(2, 3, 1)$. Show that the tangent plane intersects the surface in two lines.

8.14. Det the tangent plane of the hyperboloid:

$H^1_{2,3,1}: \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1$ in the point $M(2,3,1)$. Show that the tangent plane intersects the surface in 2 lines.

$\pi: \ \frac{x x_0}{4} + \frac{y y_0}{9} - \frac{z z_0}{1} = 1$

$\Rightarrow \ \frac{x \cdot 2}{4} + \frac{y \cdot 3}{9} - \frac{z \cdot 1}{1} = 1$

$\pi: \ \frac{x}{2} + \frac{y}{3} - z = 1$

$$\pi \cap H: \begin{cases} \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = 1 \\ \frac{x}{2} + \frac{y}{3} - z = 1 \Rightarrow z = \frac{x}{2} + \frac{y}{3} - 1 \end{cases}$$

$$\Rightarrow \frac{x^2}{4}+\frac{y^2}{9}-\left(\frac{x}{2}+\frac{y}{3}-1\right)^2=1$$

$$\cancel{\frac{x^2}{4}}+\cancel{\frac{y^2}{9}}-\cancel{\frac{x^2}{4}}-\cancel{\frac{y^2}{9}}-1-\frac{xy}{3}+x+\frac{2y}{3}=1$$

$$x-\frac{xy}{3}+\frac{2y}{3}=2$$

$$\frac{y}{3}(2-x)+x-2=0$$

$$(x-2)\left(1-\frac{y}{3}\right)=0$$

$$(x-2)(3-y)=0$$

$l_1: \; x-2=0$

$l_2: \; 3-y=0$

8.17. Determine the intersection of the paraboloid

$$\mathcal{P}^h_{2,\frac{1}{2}} : x^2-4y^2=4z \quad \text{with the line} \quad \ell=\begin{bmatrix}2\\0\\3\end{bmatrix}+\left\langle\begin{bmatrix}2\\1\\-2\end{bmatrix}\right\rangle.$$

Write down the equations of the tangent planes in the intersection points.

$$\ell: \begin{cases} x=2+2\lambda \\ y=0+\lambda \\ z=3-2\lambda \end{cases} \qquad M\left(3,\frac{1}{2},2\right)$$

$$T_{\pi P} = xx_0-4yy_0=2(z+z_0)$$

$$4+8\lambda+4\lambda^2-4\lambda^2=12-8\lambda$$

$$16\lambda=8 \Rightarrow \lambda=\frac{1}{2}$$

$$M\left(3,\frac{1}{2},2\right) \Rightarrow T_{\pi P}= xx_0-4yy_0=2(z+z_0)$$

$$3x-2y-2z-4=0$$

8.18. Determine the tangent plane of

a) the elliptic paraboloid $\frac{x^2}{5}+\frac{y^2}{3}=z$ and of

b) the hyperbolic paraboloid $x^2-\frac{y^2}{4}=z$

which are parallel to the plane $x-3y+2z-1=0$.

a) $u=\langle(1,-3,2)\rangle$

$$T_*P: \quad \frac{xx_0}{5}+\frac{yy_0}{3}-\frac{1}{2}z-\frac{1}{2}z_0=0$$

$$T_*P \parallel \pi \Rightarrow \frac{\frac{x_0}{5}}{1}=\frac{\frac{y_0}{3}}{3}=\frac{-\frac{1}{2}}{2}=-\frac{1}{4}$$

$$\frac{x_0}{5}=\frac{y_0}{9}=\frac{-1}{4}$$

$$\Rightarrow x_0 = \frac{-5}{4} \qquad y_0 = \frac{-9}{4}$$

$$\frac{\left(\frac{-5}{4}\right)^2}{5} + \frac{\left(\frac{-9}{4}\right)^2}{3} = z_0$$

$$\frac{\cancel{25}\,5}{16\cdot\cancel{5}} + \frac{\cancel{81}\,27}{16\cdot\cancel{3}} = z_0 \Rightarrow \frac{32}{16} = z_0 = 2$$

$$T_A P: \frac{-1}{4}x + \frac{-3}{4}y - \frac{1}{2}z - 1 = 0 \quad | \cdot 4$$

$$-x - 3y - 2z - 4 = 0$$

sau $\nabla F_1 = \langle\left(\frac{2x}{5}, \frac{2y}{5}, -1\right) = k\, n$?

b) $T_A P: x_0 x - \frac{y_0 y}{4} - \frac{1}{2}z - \frac{1}{2}z_0 = 0$

$$T_A P \parallel \Pi: \quad \frac{x_0}{1} = \frac{\frac{y_0}{4}}{3} = \frac{-\frac{1}{2}}{2}$$

$$x_0 = \frac{y_0}{12} = \frac{-1}{4}$$

$$y_0 = \frac{-12}{4} = -3$$

$$\frac{1}{16} - \frac{\cancel{5}\,9}{4} = z$$

$$1 - 36 = 16z$$

$$35 = 16z \Rightarrow z = \frac{35}{16}$$

$$T_A P: \frac{-1}{4}x + \frac{3}{4}y - \frac{1}{2}z - \frac{1}{2}\cdot\frac{35}{16} = 0 \quad | \cdot 4$$

$$-x + 3y - 2z - \frac{35}{8} = 0$$