Seminar 7

1. Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}\,.$$

2. Let K be a field and $S = \{(x_1, \dots, x_n) \in K^n \mid x_1 + \cdots + x_n = 0\}$.
(i) Prove that S is a subspace of the canonical vector space K^n over K.
(ii) Determine a basis and the dimension of S.

3. Determine a basis and the dimensions of the vector spaces $\mathbb{C}$ over $\mathbb{C}$ and $\mathbb{C}$ over $\mathbb{R}$. Prove that the set $\{1, i\}$ is linearly dependent in the vector space $\mathbb{C}$ over $\mathbb{C}$ and linearly independent in the vector space $\mathbb{C}$ over $\mathbb{R}$.

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (y, -x)$. Prove that f is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

~~5.~~ Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x, y, z) = (-y + 5z, x, y - 5z)$. Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

~~6.~~ Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

~~7.~~ Determine a complement for the following subspaces:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ in the real vector space $\mathbb{R}^3$;
(ii) $B = \{aX + bX^3 \mid a, b \in \mathbb{R}\}$ in the real vector space $\mathbb{R}_3[X]$.

8. Let V be a vector space over K and let S, T and U be subspaces of V such that $dim(S \cap U) = dim(T \cap U)$ and $dim(S + U) = dim(T + U)$. Prove that if $S \subseteq T$, then $S = T$.

9. Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}\,,$$

$$T = < (0, 1, 1), (1, 1, 0) >$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

~~10.~~ Determine the dimensions of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $M_2(\mathbb{R})$, where

$$S = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle, \qquad T = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle.$$

7* keeps popping in the midterm

$f: \mathbb{R}^2 \to \mathbb{R}^2$ linear map

$f(1,2) = (3,1)$

$f(4,3) = (2,5)$

Find $f(x,y)$ $\forall (x,y) \in \mathbb{R}^2$

Sol: $\left.\begin{array}{l} (1,2) \text{ and } (4,3) \text{ are linearly independent} \\ \dim \mathbb{R}^2 = 2 \end{array}\right\} \Rightarrow ((1,2),(4,3))$ basis of $\mathbb{R}^2$

$\forall (x,y) \in \mathbb{R}^2 \ \exists\, \alpha, \beta \in \mathbb{R}: (x,y) = \alpha \cdot (1,2) + \beta(4,3)$

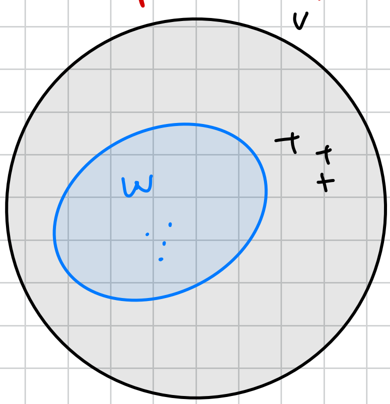
$$f(x,y) = f(\alpha(1,2) + \beta(4,3)) \underset{\uparrow}{=} \alpha f(1,2) + \beta f(4,3) = \alpha(3,1) + \beta(2,5) =$$

linear map → morphism between 2 vector spaces

$$\begin{cases} x = \alpha + 4\beta \\ y = 2\alpha + 3\beta \end{cases} \Leftrightarrow \begin{cases} \alpha = x - 4\beta \\ y = 2x - 8\beta + 3\beta \end{cases} \Leftrightarrow \begin{cases} \alpha = x - 4\beta \\ \beta = \dfrac{2x-y}{5} \end{cases} \Leftrightarrow \begin{cases} \alpha = \dfrac{-3x+4y}{5} \\ \beta = \dfrac{2x-y}{5} \end{cases}$$

$$f(x,y) = \left(3 \cdot \frac{-3x+4y}{5} + 2 \cdot \frac{2x-y}{5}, \frac{-3x+4y}{5} + 2x - y\right) = \left(-x + 2y, \frac{7x}{5} - \frac{y}{5}\right)$$

linear map = morphism between 2 vector spaces



7\. 5, 6 $f \in \underline{End_{\mathbb{R}}(\mathbb{R}^3)} \Rightarrow$ Ker f and Im f are subspaces

$f(x,y,z) = (-y+5z,\ x,\ y-5z)$

Determine a basis and the dimention for Ker f and Im f. Complete the basis to bases of the ambient space

Let $v = (x,y,z) \in \mathbb{R}^3$

Ker $f = \{ v \in \mathbb{R}^3 \mid f(v) = (0,0,0) \}$

$$\begin{cases} -y+5z = 0 \\ x = 0 \\ y-5z = 0 \end{cases} \Leftrightarrow \begin{cases} y = 5z \\ x = 0 \\ y = 5z \end{cases} \Leftrightarrow \begin{cases} y = 5z \\ x = 0 \end{cases} \Leftrightarrow \text{Ker} f = \{(0, 5z, z)\}$$

$v_1 = \langle (0,5,1) \rangle$ – linearly independent $\Rightarrow ((0,5,1))$ – basis for Ker f

$$\left.\begin{array}{l} \dim \text{Ker} f = 1 \\ \dim \mathbb{R}^3 = 3 \end{array}\right\} \Rightarrow \text{ we need 2 more vectors for it to form a basis}$$

We want to find v_2 & $v_3 \in \mathbb{R}^3$ s.t. v_1, v_2, v_3 linearly independent

First we choose $v_2 \in \mathbb{R}^3 \setminus \langle v_1 \rangle$

Let $v_2 = (1,2,3)$

Now we choose $v_3 \in \mathbb{R}^3 \setminus \langle v_1, v_2 \rangle$

$$\left.\begin{array}{l} \langle v_1, v_2 \rangle = a v_1 + b v_2 = \{ a\cdot(0,5,1) + b(1,2,3) \mid a,b \in \mathbb{R} \} = \{ (b, 5a+2b, a+3b) \mid a,b \in \mathbb{R} \} \\ (0,0,1) \notin \langle v_1, v_2 \rangle \end{array}\right\} \Rightarrow$$

$\Rightarrow v_3 = (0,0,1)$

$$\left.\begin{array}{l} \Rightarrow v_1, v_2, v_3 \text{ is linearly independent} \\ \dim \mathbb{R}^3 = 3 \end{array}\right\} \Rightarrow \langle v_1, v_2, v_3 \rangle \text{ basis for } \mathbb{R}^3$$

$\Rightarrow$ the complement of Ker $f = \langle v_2, v_3 \rangle$

Im f

$\text{Im} f = \{ (-y+5z,\ x,\ y-5z) \mid x,y,z \in \mathbb{R} \}$

$= \{(-y, 0, y) + (0, x, 0) + (5z, 0, -5z) \mid x, y, z \in \mathbb{R}\}$

$= \{y(-1, 0, 1) + x(0, 1, 0) + z(5, 0, -5) \mid x, y, z \in \mathbb{R}\}$

$= \langle (-1, 0, 1), (0, 1, 0), (5, 0, -5) \rangle$

Let $A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & -5 \end{pmatrix}$

rank $A = \begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & -5 \end{vmatrix} = 0$

$M_1 = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow ((-1, 0, 1), (0, 1, 0))$ are linearly independent $\Rightarrow$ they form the basis of the im$f$

$\left.\begin{array}{l} \dim((-1,0,1),(0,1,0)) = 2 \\ \dim \mathbb{R}^3 = 3 \end{array}\right\} \Rightarrow$ we need to add another vector $v_3 \in \underbrace{\mathbb{R}^3 \setminus \langle v_1, v_2 \rangle}_{\mathbb{R}^3 \setminus \mathrm{Im} f}$

Let $v_3 = (1, 0, 0) \notin \mathrm{Im} f \Rightarrow v_1, v_2, v_3$ basis of $\mathbb{R}^3$

7.7. **7.** Determine a complement for the following subspaces:
(*i*) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ in the real vector space $\mathbb{R}^3$;
(*ii*) $B = \{aX + bX^3 \mid a, b \in \mathbb{R}\}$ in the real vector space $\underline{\mathbb{R}_3[X]}$. $\rightarrow$ all polynomials deg $\mathbb{R}_3[X] \leq 3$

(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$

$\left.\begin{array}{l} x + 2y + 3z = 0 \\ x = -2y - 3z \end{array}\right\} \Rightarrow (x, y, z) = (-2y - 3z, y, z)$

Let $v = (x, y, z)$

$A = \{(-2y - 3z, y, z) \mid y, z \in \mathbb{R}\} =$

$= \{(-2y, y, 0) + (-3z, 0, z) \mid y, z \in \mathbb{R}\} =$

$= \{y(-2, 1, 0) + z(-3, 0, 1) \mid y, z \in \mathbb{R}\}$

$= \langle (-2, 1, 0), (-3, 0, 1) \rangle \Rightarrow ((-2, 1, 0), (-3, 0, 1))$ - basis of A

$\left.\begin{array}{l} \dim A = 2 \\ \dim \mathbb{R}^3 = 3 \end{array}\right\} \Rightarrow$ we need to add one more vector $v_3 \in \mathbb{R}^3 \setminus A$

$v_3 = (0,0,1) \in \mathbb{R}^3 \setminus A$

If $v_3 \in A \Rightarrow \begin{cases} -2y - 3z = 0 \\ y = 1 \\ x = 0 \end{cases} \Rightarrow -2 = 0$ – false $\Rightarrow v_3 \notin A$

$\Rightarrow (v_1, v_2, v_3)$ is a basis of A

$\Rightarrow$ The complement of A in $\mathbb{R}^3$ is $\langle v_3 \rangle$

(ii) $B = \{ aX + bX^3 \mid a, b \in \mathbb{R} \}$

$B = \langle X, X^3 \rangle$

If $ax + bx^3 = 0 \Rightarrow a = b = 0 \Rightarrow X, X^3$ – linearly independent $\Rightarrow (X, X^3)$ is a basis of B

* Know dimentions

$\dim \underline{\mathbb{R}_3[X] = 4} \Rightarrow$ we need 2 more vectors

$v_3 \in \mathbb{R}_3[x] \setminus \langle x, x^3 \rangle$

$v_3 = 1 \notin \langle \underset{v_1}{x}, \underset{v_2}{x^3} \rangle \Rightarrow v_1, v_2, v_3$ are linearly independent

$v_4 \in \mathbb{R}_3[x] \setminus \langle v_1, v_2, v_3 \rangle$

$\langle v_1, v_2, v_3 \rangle = \{ a \cdot 1 + b \cdot x + c \cdot x^3 \mid a, b, c \in \mathbb{R} \}$

Let $v_4 = x^2 \notin \langle v_1, v_2, v_3 \rangle \Rightarrow v_1, v_2, v_3, v_4$ are linearly ind.

$\Rightarrow \dim \mathbb{R}_3[x] = 4$

$\Rightarrow \langle v_1, v_2, v_3, v_4 \rangle$ basis of $\mathbb{R}_3[x]$

$\Rightarrow$ The complement of B is $\overline{B} = \langle v_3, v_4 \rangle = \langle 1, x^2 \rangle = \{ a + bx^2 \mid a, b \in \mathbb{R} \}$

$\forall$ V K – vect. space.

$\forall$ $W \leq_K V$ $\exists$ $W' \leq_K V$

$V = W \oplus W'$

$\Updownarrow$

- $V = W + W' \Leftrightarrow \forall v \in V$ $\exists w \in W, w' \in W'$ $v = w + w'$
- $W \cap W' = 0$

$\Leftrightarrow$ $\forall v \in V$ $\exists!$ $w \in W, w' \in W'$ $v = w + w'$

1^{st} dimention theorem

$f : V \to V'$

$\dim V = \underbrace{\dim(\operatorname{Ker} f)}_{\text{nullity}} + \underbrace{\dim \operatorname{Im} f}_{\text{rank}}$

2^{nd} dimention theorem

$\forall\, V_K$ - v.s. , $S, T \subseteq_K V$

$\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$

$S+T = \langle S \cup T \rangle$

7.10. Find the dimention of $S, T, S+T, S\cap T$

$S = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$

$T = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$

Sol

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ linearly indep. $\Rightarrow$ they form a basis $\Rightarrow \dim S = 2$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ linearly indep $\Rightarrow$ —— " —— $\Rightarrow \dim T = 2$

$S+T = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$

We check for linear indep. using the def.

Let $a, b, c, d \in \mathbb{R}$

$$a \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \cdot \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} a + b = 0 \Rightarrow b = 0 \\ a + c = 0 \Rightarrow a = 0 \\ b + c + d = 0 \Rightarrow c = 0 \\ b + d = 0 \Rightarrow d = 0 \end{cases}$$

$\Rightarrow a = b = c = d = 0 \Rightarrow$ the 4 matrices are lin. ind

$\Rightarrow \dim(S+T) = 4$

$\dim(S \cap T) = \dim S + \dim T - \dim(S+T) = 2 + 2 - 4 = 0 \Rightarrow S \cap T = 0$

$\Rightarrow \mathcal{M}_2(\mathbb{R}) = S \oplus T$