6.4 Exercises

6.1. Find the equation of the circle:

a) of diameter $[A,B]$, with $A(1,2)$ and $B(-3,-1)$,

b) with center $I(2,-3)$ and radius $R=7$,

c) with center $I(-1,2)$ and passing through $A(2,6)$,

d) centered at the origin and tangent to $\ell : 3x-4y+20=0$,

e) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $\ell : 3x-y-2=0$,

f) passing through $A(1,1)$, $B(1,-1)$ and $C(2,0)$,

g) tangent to both $\ell_1 : 2x+y-5=0$ and $\ell_2 : 2x+y+15=0$ if one tangency point is $M(3,-1)$.

6.2. For a circle $\mathcal{C}$ of radius R:

a) Use the parametrization $x \mapsto (x, \pm\sqrt{R^2-x^2})$ to deduce a parametrization of tangent lines to $\mathcal{C}$.

b) Use the parametrization $\theta \mapsto (R\cos(\theta), R\sin(\theta))$ to deduce a parametrization of tangent lines to $\mathcal{C}$.

c) Compare these to the equation of the tangent line $xx_0+yy_0=R^2$ where $(x_0,y_0)\in\mathcal{C}$.

6.3. Determine the foci (focal points) of the Ellipse $9x^2+25y^2-225=0$

6.4. Determine the intersection of the line $\ell : x+2y-7=0$ and the ellipse $\mathcal{E} : x^2+3y^2-25=0$.

6.5. Determine the position of the line $\ell : 2x+y-10=0$ relative to the ellipse $\mathcal{E} : \frac{x^2}{9}+\frac{y^2}{4}-1=0$.

6.6. Determine an equation of a line which is orthogonal to $\ell : 2x-2y-13=0$ and tangent to the ellipse $\mathcal{E} : x^2+4y^2-20=0$.

6.7. A *diameter* of an ellipse is the line segment determined by the intersection points of the ellipse with a line passing through the center of the ellipse. Show that the tangent lines to an ellipse at the endpoints of a diameter are parallel.

6.8. Consider the family of ellipses $\mathcal{E}_a : \frac{x^2}{a^2}+\frac{y^2}{16}=1$. For what value $a\in\mathbb{R}$ is $\mathcal{E}_a$ tangent to the line $\ell : x-y+5=0$?

6.9. Consider the family of lines $\ell_c : \sqrt{5}x-y+c=0$. For what values $c\in\mathbb{R}$ is ℓ_c tangent to the ellipse $\mathcal{E} : x^2+\frac{y^2}{4}=1$?

6.10. Determine the common tangents to the ellipses

$$\frac{x^2}{45}+\frac{y^2}{9}=1 \quad \text{and} \quad \frac{x^2}{9}+\frac{y^2}{18}=1.$$

Prob Cap 6: 1 $\longrightarrow$ 10 cu excepția pb 2

$(x-x_0)^2+(y-y_0)^2=r^2$ $\leftarrow$ ecuația cercului

coord. centrului

6.1 a) $A(1,2)$ $B(-3,-1)$

$O\left(\frac{1-3}{2}, \frac{2-1}{2}\right) \rightarrow O\left(-1, \frac{1}{2}\right)$

$AB = \sqrt{16+9} = \sqrt{25} = 5 \Rightarrow R = \frac{AB}{2} = \frac{5}{2}$

$(x+1)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{25}{4}$

b) $i(2,-3)$; $R=7$

$(x-2)^2+(y+3)^2=49$

c) $i(-1,2)$ $A(2,6)$

$i(-1,2)$

$R = iA = \sqrt{9+16} = \sqrt{25} = 5$

$(x+1)^2+(y-2)^2 = \left(\frac{5}{2}\right)^2$

d) $\ell: 3x-4y+20=0$

$d(O,\ell) = \frac{|0-0+20|}{\sqrt{9+16}} = \frac{20}{5} = 4$

$\left.\begin{array}{l} d(O,\ell)=R=4 \\ O(0,0) \end{array}\right\} \Rightarrow x^2+y^2=16$

e) $A(3,1)$ $B(-1,3)$ $O\in\ell$: $3x-y-2=0$

$O(x_0,y_0)$

$$\begin{cases} 3x_0-y_0-2=0 \\ (3-x_0)^2+(1-y_0)^2=R^2 \\ (-1-x_0)^2+(3-y_0)^2=R^2 \end{cases}$$

$$\begin{cases} \cancel{9}-6x_0+\cancel{x_0^2}+\cancel{1}-2y_0+\cancel{y_0^2}=R^2 \\ \cancel{1}+2x_0+\cancel{x_0^2}+\cancel{9}-6y_0+\cancel{y_0^2}=R^2 \end{cases} \quad (-)$$

$4y_0=8x_0$

$x_0=\frac{1}{2}y_0$

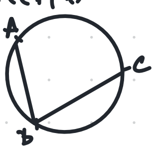
$\frac{3}{2}y_0-y_0-2=0 \;|\cdot 2$

$3y_0-2y_0-4=0$ $\qquad \Rightarrow OA=\sqrt{1+9}=\sqrt{10}$

$y_0=4 \Rightarrow x_0=2$

$\Rightarrow (x-2)^2+(y-4)^2=10$

f) $A(1,1)$ $B(1,-1)$ $C(2,0)$



$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & -1 & 1 \\ 4 & 2 & 0 & 1 \end{vmatrix} \sim \begin{vmatrix} x^2+y^2 & x & y & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 4 & 2 & 0 & 1 \end{vmatrix} = 2 \cdot \begin{vmatrix} x^2+y^2 & x & 1 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} =$$

$$= 2(x^2+y^2+\cancel{4}+4x-\cancel{4}-2x^2-2y^2-2x) =$$

$$= 2(-x^2-y^2+2x) = 0$$

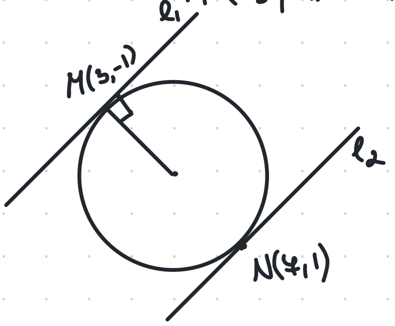
$$x^2+y^2-2x=0$$

$$(x-1)^2+y^2=1$$

g) tg to $\ell_1: 2x+y-5=0$

$\ell_2: 2x+y+15=0$

ℓ_1, $M(3,-1) \in \ell_1$



$$d(M,\ell_2) = D = \frac{|2\cdot3-1+15|}{\sqrt{4+1}} = \frac{\overset{5\sqrt5}{\cancel{20}}}{\sqrt5} = 4\sqrt5$$

$m_{\ell_1} = -2$

fie $d \perp \ell_1 \Rightarrow m_d = \frac{1}{2}$

$y+1 = \frac{1}{2}(x-3)$

$2y+2 = x-3$

$x-2y-5=0$

$x-2y-5=0 \cap 2x+y+15=0$

$$\begin{cases} x-2y-5=0 \quad |\cdot 2 \\ 2x+y+15=0 \end{cases} \quad (-)$$

$-5y=-5 \Rightarrow y=1$

$x = 5+2 = 7$ $\Big\} \Rightarrow N(7,1)$

(2) $O\left(\frac{7+3}{2}, \frac{-1+1}{2}\right) = O(5,0)$

$(x-5)^2+y^2=80$

Ariana: $4y+10+y+15=0$

$y=-5 \quad x=-5$

63 Determine the foci (focal points) of the Ellipse $9x^2+25y^2-225=0$

focal points $F_1(-c,0)$, $F_2(0,c)$, where $c=\sqrt{a^2-b^2}$

$\mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$9x^2 + 25y^2 - 225 = 0 \quad | : 225$$

$$\frac{9}{225}x^2 + \frac{25}{225}y^2 - 1 = 0$$

$$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1 \Rightarrow a = 5 \quad b = 3$$

$$\Rightarrow c = \sqrt{25 - 9} = \sqrt{16} = 4 \Rightarrow \begin{array}{l} F_1(-4,0) \\ F_2(0,4) \end{array}$$

6.6. Determine an equation of a line which is orthogonal to $\ell : 2x - 2y - 13 = 0$ and tangent to the ellipse $\mathcal{E} : x^2 + 4y^2 - 20 = 0$.

$m_\ell = \frac{2}{2} = 1 \Rightarrow m_t = -1$ since $\ell \perp t$

$\mathcal{E}: x^2 + 4y^2 - 20 = 0$

$x^2 + 4y^2 = 20 \quad | : 20$

$\mathcal{E}: \frac{x^2}{20} + \frac{y^2}{5} = 1$

If t tang. to $\mathcal{E} \Rightarrow b^2 = m^2 - a^2k^2$

$m = \pm\sqrt{b^2 + a^2k^2}$ where $y = kx \pm \sqrt{b^2 + a^2k^2}$ - eq of tan

$m = \pm\sqrt{20 + 5} = \pm 5$

$t: y = -x \pm 5$

6.5 $\left.\begin{array}{l} \ell: 2x + y - 10 = 0 \\ \mathcal{E}: \frac{x^2}{9} + \frac{y^2}{4} = 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \ell: y = -2x + 10 \\ \mathcal{E}: \frac{x^2}{9} + \frac{y^2}{4} = 1 \end{array}\right. \Rightarrow \frac{x^2}{9} + \frac{(-2x+10)^2}{4} = 1$

$$\Rightarrow 4x^2 + 9(-2x + 10)^2 = 36$$

$$4x^2 + 9(4x^2 - 40x + 100) = 36$$

$$4x^2 + 36x^2 - 360x + 900 = 36$$

$$40x^2 - 360x + 864 = 0 \quad | : 4$$

$$16x^2 - 90x + 216 = 0$$

$$8x^2 - 45x + 108 = 0$$

$\Delta = 2025 - \ldots < 0 \Rightarrow \ell \not\cap \mathcal{E}$

$\ell \cap \mathcal{E} = \emptyset$