Today $\underline{3.17}$, 3.30, 3.33, $\underline{3.34}$, $\underline{3.36}$, $\underline{3.40}$, 3.42, 4.11a, $\underline{4.13}$, 4.16, $\boxed{4.18}$ this for midterm

3.34. Solve exercise 2.16 using normal vectors.

2.16 → show that the paramise intersection of the planes

$\pi_1 : 3x + y + z - 5 = 0$

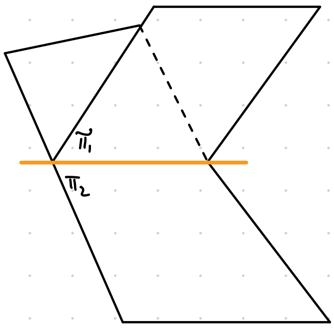
$\pi_2 : 2x + y + 3z + 2 = 0$

$\pi_3 : 5x + 2y + 4z + 1 = 0$ are parallel lines

* one eq. in 3D is a plane

2 eq. in 3D is a line

$$\ell : \begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$$



* directional vector can also be determined by the cross product of the normal vectors

$$\langle \vec{n}_{\pi_1} \times \vec{n}_{\pi_2} \rangle = D(\ell)$$

$\ell_1 = \pi_2 \cap \pi_3$

$\ell_2 = \pi_3 \cap \pi_1$

$\ell_3 = \pi_1 \cap \pi_2$

$$\ell_1 = \pi_2 \cap \pi_3 = \begin{cases} 2x + y + 3z + 2 = 0 \quad | \cdot (-2) \\ 5x + 2y + 4z + 1 = 0 \end{cases}$$

$\ell_1 = \pi_2 \cap \pi_3$

$$n_{\pi_2} \times n_{\pi_3} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 5 & 2 & 4 \end{vmatrix} = 4i + 4k + 15j - 5k - 6i - 8j = -2i + 7j - k = (-2, 7, -1)$$

$$\ell_2 = \pi_3 \times \pi_1 = \begin{vmatrix} i & j & k \\ 5 & 2 & 4 \\ 3 & 1 & 1 \end{vmatrix} = 2i + 5k + 12j - 6k - 4i - 5j = -2i + 7j - k = (-2, 7, -1)$$

$$n_{\pi_1} \times n_{\pi_2} = \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 3i + 3k + 2j - 2k - i - 9j = 2i - 7j + k = (2, -7, 1)$$

$$D(\ell_1) = D(\ell_2) = D(\ell_3) = \langle (2, -7, 1) \rangle \Rightarrow \ell_1 \parallel \ell_2 \parallel \ell_3$$

3.36 Determine the angles between the planes $\pi_1 : x - \sqrt{2} y + z - 1 = 0$

$\pi_2 : x + \sqrt{2} y - z + 3 = 0$

$$\cancel{n_{\pi_1} \times n_{\pi_2} = \begin{vmatrix} i & j & k \\ 1 & -\sqrt{2} & 1 \\ 1 & \sqrt{2} & -2 \end{vmatrix} = 2\sqrt{2} i + \sqrt{2} k + j + \sqrt{2} k - \sqrt{2} i + 2j = \sqrt{2} i + 3j + 2\sqrt{2} k}$$

$\| n_{\pi_1} \| = \sqrt{1 + 2 + 1} = 2$ $\qquad \| n_{\pi_2} \| = \sqrt{1 + 2 + 1} = 2$

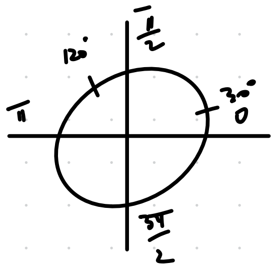
* for angles use dot product

$n_{\pi_1} \cdot n_{\pi_2} = \|n_{\pi_1}\| \cdot \|n_{\pi_2}\| \cdot \cos\alpha$

$1-2-1 = 4 \cdot \cos\alpha$

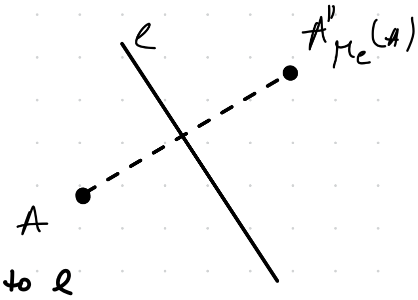
$-2 = 4 \cdot \cos\alpha$

$-\frac{1}{2} = \cos\alpha \Rightarrow \alpha = 120^\circ = \frac{2\pi}{3}$



3.40 $A(1,3,5)$ line ℓ.

$\ell: \underbrace{2x+y+z-1=0}_{\pi_1} \cap \underbrace{3x+y+2z-3=0}_{\pi_2}$



$\ell = \begin{cases} 2x+y+z-1=0 \\ 3x+y+2z-3=0 \end{cases}$

Ortogonal projection and reflection of A with respect to ℓ

$n_{\pi_1} \times n_{\pi_2} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 2i+2k+3j-3k-i-4j = i-j-k$

* the line is the midpoint between the projection & the reflection

π plane containing A and s.t. $\ell \perp \pi$, then $\{A'\} = \pi \cap \ell$

$\pi = \begin{cases} x-y-z+D=0 \\ A \in \pi \end{cases}$

$A(1,3,5)$: $1-3-5+D=0$
$D=7$

$\Rightarrow \pi: x-y-z+7=0$

$\ell \cap \pi = \begin{cases} 2x+y+z-1=0 \\ 3x+y+2z-3=0 \\ x-y-z+7=0 \end{cases} \Rightarrow \begin{cases} 3x+6=0 \Rightarrow x = \frac{-6}{3} = -2 \\ 3x+y+2z-3=0 \end{cases}$

$y+2z-3-6=0$

$y+2z=9 \Rightarrow y=9-2z$

$-2-y-z-7=0$

$-y-z=9$

$-y=9+z$

$-9+2z=9+z$

$z=18 \Rightarrow y=9-2\cdot 18 = 9-36=-27$

~~$\Rightarrow A'(-2,-27,18)$~~

$\left|\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 3 & 1 & 2 & 3 \\ 1 & -1 & -1 & -7 \end{array}\right|$

$\det A = 3$

$\Delta y = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 3 & 2 \\ 1 & -7 & -1 \end{vmatrix} = -6-21+2-\cancel{3}+28+\cancel{3} =$

$= -4+7 = 3 \Rightarrow y=1$

$$\Delta z = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & 3 \\ 1 & -1 & -7 \end{vmatrix} = -14 - \cancel{3} + \cancel{3} - 1 + 6 + 21 = 7 + 5 = 12$$

$$z = \frac{12}{3} = 4 \qquad \Rightarrow A'(-2, 1, 4)$$

or $D(\ell) = \langle (1,-1,-1) \rangle$

$$1 \cdot (x-1) + (-1)(y-3) + (-1)(z-5) = 0$$

$$x - y - z + 7 = 0$$

reflection:

$$-2 = \frac{1+x_r}{2} \Rightarrow x_r = -5$$

$$1 = \frac{3+y_r}{2} \Rightarrow y_r = -1$$

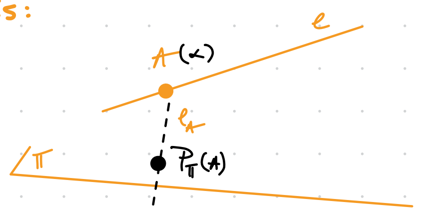
$$4 = \frac{5+z_r}{2} \Rightarrow z_r = 3$$

3.42 Determine the orthogonal projection of the line $\ell: x - y - 1 = 0 \cap x + y - z + 1 = 0$ on the plane $\pi: x + 2y - z = 0$

* take 2 points on the line, project them and write the line

or

use this:



normal vect. of the plane & intersect it with A

$$\ell: \begin{cases} 2x - y - 1 = 0 \Rightarrow y = 2x - 1 \\ x + y - z + 1 = 0 \Rightarrow 3x - z = 0 \Rightarrow \ell: 3x - z = 0 \end{cases}$$

$\Rightarrow x = \alpha$

$z = 3\alpha$

$2\alpha - y - 1 = 0$

$y = 2\alpha - 1$

$$\Rightarrow \ell: \begin{cases} x = \alpha \\ y = 2\alpha - 1 \\ z = 3\alpha \end{cases}$$

$n_\pi = (1, 2, -1)$

~~let $A(1, 1, 3) \in \ell$~~

~~$\ell_A: x = 1 + \alpha$, $y = 1 + 2\alpha$, $z = 3 - \alpha$~~

$$\ell_A = \begin{cases} x = \alpha + 1 \cdot \lambda \\ y = 2\alpha - 1 + 2 \cdot \lambda \\ z = 3\alpha - 1 \cdot \lambda \end{cases}$$

fixed

$$P_\pi(A) = \begin{cases} x = \alpha + \lambda \\ y = 2\alpha - 1 + 2\lambda \\ z = 3\alpha - \lambda \\ x + 2y - z = 0 \end{cases}$$

$$\alpha + \lambda + 4\alpha - 2 + 4\lambda - 3\alpha + \lambda = 0$$

$$2\alpha + 6\lambda - 2 = 0 \quad |:2$$

$$\lambda = \frac{1-\alpha}{3}$$

$$\Rightarrow P_\pi(A) = \begin{cases} x = \alpha + \frac{1-\alpha}{3} \\ y = 2\alpha - 1 + 2 \cdot \frac{1-\alpha}{3} \\ z = 3\alpha - \frac{1-\alpha}{3} \end{cases} = \begin{cases} x = \frac{2\alpha - 1}{3} \\ y = \frac{6\alpha - 3 + 2 - 2\alpha}{3} = \frac{4\alpha - 1}{3} \\ z = \frac{10\alpha - 1}{3} \end{cases}$$

$$D(P_\pi(\ell)) = \langle (\tfrac{2}{3}, \tfrac{4}{3}, \tfrac{10}{3}) \rangle$$

4.11 a Prove the Grassmann identity

$\vec{v_1}, \vec{v_2}, \vec{v_3}$

$$\vec{v_1} \times (\vec{v_2} \times \vec{v_3}) = \begin{vmatrix} \vec{v_2} & \vec{v_3} \\ \vec{v_1} \cdot \vec{v_2} & \vec{v_1} \cdot \vec{v_3} \end{vmatrix} = -(\vec{v_1} \cdot \vec{v_2})\, \vec{v_3} + (\vec{v_1} \cdot \vec{v_3})\, \vec{v_2}$$

<u>Proof</u> $\vec{v_i}\,(x_i, y_i, z_i)$

$i = \overline{1,3}$

$$\vec{v_2} \times \vec{v_3} = \begin{vmatrix} i & j & k \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = i \cdot \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} + \begin{vmatrix} z_2 & x_2 \\ z_3 & x_3 \end{vmatrix} \cdot j + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \cdot k$$

$$\vec{v_1} \times (v_2 \times v_3) = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} & \begin{vmatrix} z_2 & x_2 \\ z_3 & x_3 \end{vmatrix} & \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \end{vmatrix}$$

$$x_{\vec{v_1} \times (\vec{v_2} \times v_3)} = \begin{vmatrix} y_1 & z_1 \\ \begin{vmatrix} z_2 & x_2 \\ z_3 & x_3 \end{vmatrix} & \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \end{vmatrix} = y_1(x_2 y_3 - x_3 y_2) - z_1(z_2 x_3 - z_3 x_2) = x_2(y_1 y_3 + z_1 z_3) - x_3(y_1 y_2 + z_1 z_2)$$

$$= x_2(x_1 x_3 + y_1 y_3 + z_1 z_3) - x_3(x_1 x_2 + y_1 y_2 + z_1 z_2) =$$

$$-(\vec{v_1} \cdot \vec{v_3})\, x_2 - (\vec{v_1} \cdot \vec{v_2})\, x_3$$

* do the same for y & z