Seminar 5

~~**1.**~~ Determine the following generated subspaces:
(i) $< 1, X, X^2 >$ in the real vector space $\mathbb{R}[X]$.
(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in the real vector space $M_2(\mathbb{R})$.

~~**2.**~~ Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces with a minimal number of generators.

3. Consider the following vectors in the real vector space $\mathbb{R}^3$:

$$a = (-2, 1, 3), b = (3, -2, -1), c = (1, -1, 2), d = (-5, 3, 4), e = (-9, 5, 10).$$

Show that $\langle a, b \rangle = \langle c, d, e \rangle$.

4. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that S and T are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

5. Let S and T be the set of all even functions and of all odd functions in $\mathbb{R}^{\mathbb{R}}$ respectively. Prove that S and T are subspaces of the real vector space $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{R}} = S \oplus T$.

6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x + y, x - y),$$
$$g(x, y) = (2x - y, 4x - 2y),$$
$$h(x, y, z) = (x - y, y - z, z - x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

7. Which ones of the following functions are endomorphisms of the real vector space $\mathbb{R}^2$:
(i) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (ax + by, cx + dy)$, where $a, b, c, d \in \mathbb{R}$;
(ii) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (a + x, b + y)$, where $a, b \in \mathbb{R}$?

8. Let $a \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x \cos a - y \sin a, x \sin a + y \cos a)\,.$$

Prove that $f \in End_{\mathbb{R}}(\mathbb{R}^2)$.

9. Determine the kernel and the image of the endomorphisms from Exercise **6.**

10. Let V be a vector space over K and $f \in End_K(V)$. Show that the set

$$S = \{x \in V \mid f(x) = x\}$$

of fixed points of f is a subspace of V.

★ important to get the concept of dimention

V k - vector space

$X \subseteq V$

$\langle X \rangle = \bigcap_{\substack{S \leq_k V \\ S \supseteq X}} S$

↳ the subspace of V generated by the set X

→ angle brackets = chevrons

$\langle X \rangle = \{ \sum_{i=1}^{n} \alpha_i x_i \mid n \in \mathbb{N}, x_i \in X, \alpha_i \in K \}$ ⟶ almost always finite

If X is finite $X = \{x_1, x_2, \dots, x_n\}$

$\Rightarrow \langle X \rangle = \{ \sum_{i=1}^{n} \alpha_i x_i \mid \alpha_i \in K, x_i \in X \}$

ex 1) Find the following generated subspaces :

(i) $\langle 1, x, x^2 \rangle \leq_{\mathbb{R}} \mathbb{R}[x]$

(ii) $\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$

(iii) $\langle (1,2,0), (0,-1,1) \rangle \leq_{\mathbb{R}} \mathbb{R}^3$

(i) $\langle 1, x, x^2 \rangle = \{ k_1 \cdot 1 + k_2 \cdot x + k_3 \cdot x^2 \mid k_1, k_2, k_3 \in \mathbb{R} \}$

$= \{ f \in \mathbb{R}[x] \mid \deg f \leq 2 \} = \mathbb{R}_2[x]$

(ii) $\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle = \{ k_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mid k_1, k_2, k_3 \in \mathbb{R} \}$

$= \{ \begin{pmatrix} k_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & k_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k_3 \end{pmatrix} \mid k_1, k_2, k_3 \in \mathbb{R} \}$

$= \{ \begin{pmatrix} k_1 & k_2 \\ 0 & k_3 \end{pmatrix} \mid k_1, k_2, k_3 \in \mathbb{R} \} = T_2(\mathbb{R})$ (uppertriangular matrix)

(iii) $\langle (1,2,0), (0,-1,1) \rangle = \{ k_1 (1,2,0) + k_2 (0,-1,1) \mid k_1, k_2 \in \mathbb{R} \}$

$= \{ (k_1, 2k_1, 0) + (0, -k_2, k_2) \mid k_1, k_2 \in \mathbb{R} \}$

$= \{ (k_1, 2k_1 - k_2, k_2) \mid k_1, k_2 \in \mathbb{R} \}$

$= \{ (x,y,z) \mid k_1 = x, 2k_1 - k_2 = y, k_2 = z \}$

$= \{ (x,y,z) \mid 2x - z = y \}$

ex 2) Consider the following subspaces:

(i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x=0\}$

(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$

(iii) $C = \{(x,y,z) \in \mathbb{R}^3 \mid x=y=z\}$

(vi) $D = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; \begin{cases} x+t=0 \\ y=0 \end{cases} \right\}$

(v) $E = \{a_0 + a_1 x + a_2 x^2 \in \mathbb{R}[x] \mid a_1 + 2a_2 = 0\}$

Write A, B, C, D, E as generated subspaces with a minimal .... of generators

(i) $\{(x,y,z) \in \mathbb{R}^3 \mid x=0\}$

let $a, b \in \mathbb{R} \Rightarrow a\cdot(0,1,0) + b(0,0,1) = (0,a,0) + (0,0,b) = (0,a,b) \in A$

$A = \langle (0,1,0), (0,0,1) \rangle \quad (0,0,1) \notin \langle 0,1,0 \rangle$ because if it were the case then we'd have $\alpha \in \mathbb{R}$ s.t.

$\alpha \cdot (0,1,0) = (0,0,1)$

$\Rightarrow \alpha = 0 \Rightarrow (0,0,1) = (0,0,0)$ false

$\Rightarrow \langle (0,1,0), (0,0,1) \rangle$ minimum set of gen.

(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$

$x = -y-z$

$(-y-z, y, z) \in B$

let $a, b \in \mathbb{R} \Rightarrow a(-1,1,0) + b(-1,0,1) =$

$= (-a, a, 0) + (-b, 0, b) =$

$= (-a-b, a, b) \in B$

(iii) $C = \{(x,y,z) \in \mathbb{R}^3 \mid x=y=z\} = \{(x,x,x) \mid x \in \mathbb{R}\} = \{x\cdot(1,1,1) \mid x \in \mathbb{R}\}$

(iv) $D = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; \begin{cases} x+t=0 \\ y=0 \end{cases} \right\} = \left\{ \begin{pmatrix} x & 0 \\ z & -x \end{pmatrix} \in M_2(\mathbb{R}) \right\} = \left\{ x\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;\middle|\; x,t \in \mathbb{R} \right\}$

$= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle$

(v) $E = \{a_0 + a_1 x + a_2 x^2 \in \mathbb{R}[x] \mid a_1 + 2a_2 = 0\} = \{a_0 - 2a_2 x + a_2 x^2 \in \mathbb{R}[x]\} = \{a_0 1 + a_2(x^2 - 2x) \in \mathbb{R}[x]\}$

$a_1 = -2a_2$

$= \langle 1, x^2 - 2x \rangle$

V_1, V_2 k - vector spaces

$f: V_1 \to V_2$ is a k-homomorphism of vector spaces (or a k-linear map)

if:
- $\forall v_1, v_2$: $f(v_1+v_2) = f(v_1) + f(v_2)$
- $\forall \alpha \in K, \forall v \in V$ $f(\alpha v) = \alpha \cdot f(v)$

Short version: $\forall \alpha_1, \alpha_2 \in K$ $\forall v_1, v_2 \in V$: $f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 \cdot f(v_1) + \alpha f(v_2)$

$\underset{\text{kernel}}{\mathrm{Ker}}\, f := \{ v \in V_2 \mid f(v) = 0_{V_2} \} \leq_{\mathbb{R}} V_1$

$\mathrm{Im}\, f = \{ f(v) \mid v \in V_1 \} \leq_{\mathbb{R}} V_2$

6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x,y) = (x+y, x-y),$$
$$g(x,y) = (2x-y, 4x-2y),$$
$$h(x,y,z) = (x-y, y-z, z-x).$$

Show that $f, g \in \underset{\text{endo morphism}}{End_{\mathbb{R}}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

$f(x,y) = (x+y, x-y)$

$\forall \alpha_1, \alpha_1 \in K$ $\forall v_1, v_2 \in V$: $f(\alpha_1 v_1 + \alpha_2 v_2) \overset{?}{=} \alpha_1 f(v_1) + \alpha_2 f(v_2)$

$f(\alpha_1 v_1 + \alpha_2 v_1) = f(\alpha_1(x_1,y_1) + \alpha_2(x_2,y_2)) = f((\alpha_1 x_1 + \alpha_2 x_2) + (\alpha_2 y_2 + \alpha_1 y_1)) =$

$= (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_1 y_1 + \alpha_2 y_2, \alpha_1 x_1 + \alpha_1 y_1 - \alpha_2 x_2 - \alpha_2 y_2) =$

$= (\alpha_1(x_1+y_1) + \alpha_2(x_2+y_2), \alpha_1(x_1+y_1) - \alpha_2(x_2+y_2))$

$= \alpha_1 f(v_1) + \alpha_2 f(v_2)$

$\Rightarrow$ $\mathrm{Ker}\, f = \{ v \in \mathbb{R}^2 \mid f(v) = (0,0) \}$

$f(v) = (0,0) \Leftrightarrow (x+y, x-y) = (0,0)$

$\Rightarrow \begin{cases} x+y = 0 \\ x-y = 0 \end{cases} \Rightarrow \begin{cases} x = -y \\ x = y \end{cases} \Rightarrow x = y = 0 \Rightarrow \mathrm{Ker}\, f = (0,0) \Rightarrow f$ is injective

$\mathrm{Im} f = \{ f(v) \mid v \in \mathbb{R}^2 \} = \{ (x+y, x-y) \mid x,y \in \mathbb{R}^2 \} = \{ x(1,1) + y(1,-1) \mid x,y \in \mathbb{R} \} = \langle (1,1), (1,-1) \rangle$

ex 2) $\mathrm{Ker}\, g = \{ v \in \mathbb{R}^2 \mid g(v) = 0 \}$

$g(v) = (0,0) \Leftrightarrow (2x-y, 4x-2y) = (0,0)$

$2x - y = 0 \Rightarrow y = 2x$

$\Rightarrow \quad \operatorname{Ker} g = \{(a, 2a) \mid a \in \mathbb{R}\} = \langle (1,2) \rangle$