Ch 6.

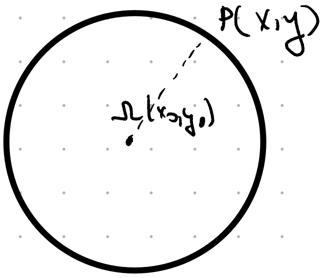
1, 3, 4, 5, 6, 7, 8, 9, 10

Quadratic curves (ellipses)

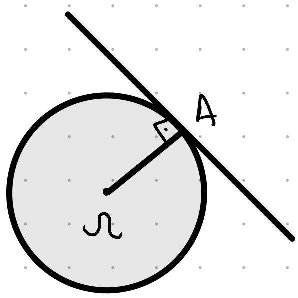
$Q: a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$

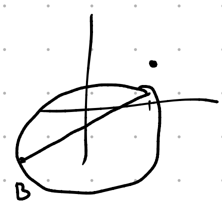
circle = locus of points whose distance to a fixed point $\Omega(x_0, y_0)$ (called the centre) is $R > 0$ (called the radius)

$\mathcal{C}(\Omega, R) : (x - x_0)^2 + (y - y_0)^2 = R^2$ (implicit equation)



$\mathcal{C}(\Omega, R) : \begin{cases} x = x_0 + R\cos t \\ y = y_0 + R\sin t \end{cases}$ (parametric equation) $t \in [0, 2\pi)$





6.1. Find the eq of the circle:

a) of diameter $[AB]$, $A(1,2)$, $B(-3,-1)$

e) circle passing through $A(3,1)$, $B(-1,3)$, having the centre on the line $\mathcal{C}: 3x - y - 2 = 0$

$|AB| = \sqrt{16 + 9} = \sqrt{25} = 5 \Rightarrow R = \frac{5}{2}$

$M = \left(\frac{1-3}{2}, \frac{2-1}{2}\right) = \left(-1, \frac{1}{2}\right)$

$\Rightarrow (x+1)^2 + \left(y - \frac{1}{2}\right) = \frac{25}{4}$

e) $\vec{AB}(-4,2)$

M - mid [AB] $\Rightarrow$ $M(1,2)$

$\Rightarrow$ $-4(x-1)+2(y-2)=0$

$-4x+\cancel{4}+2y-\cancel{4}=0$

$-4x+2y=0 \Rightarrow y=2x$

$\Rightarrow \begin{cases} 3x-y+2=0 \\ y=2x \end{cases} \Rightarrow \cancel{3x-2x}+2=0$

$x=+2$

$\Rightarrow y=+4$

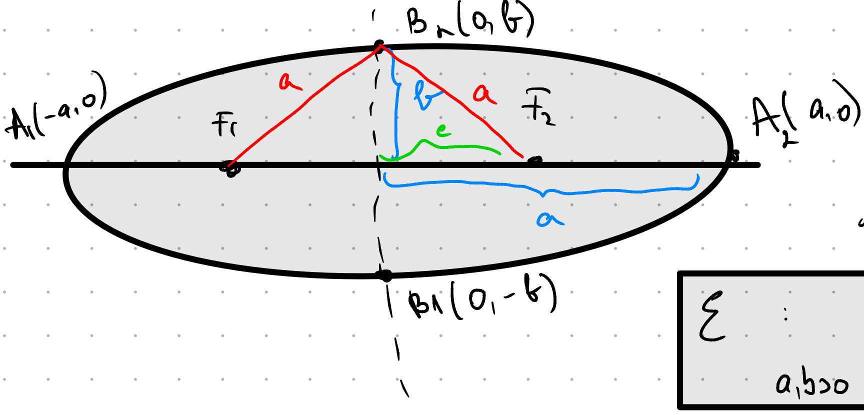
$\Rightarrow \Omega(+2,+4)$

$R=\sqrt{10}$

$\rightarrow \mathcal{C}(c,\sqrt{10}) : (x-2)^2+(y-4)^2=10$

ellipse = locus of points in the plane whose sum of distances to two fixed points f_1, f_2 (which are called the focal points or foci) is a constant $2a$.

If we fix a reference system $\mathcal{R}=(O,\vec{i},\vec{j})$ where O is the midpoint of $[F_1,F_2]$ and $\vec{i}\in D(F_1,F_2)$



then the eq of the ellipse is:

$$\boxed{\mathcal{E} : \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad a,b>0}$$

$F_1(-c,0)$

$F_2(c,0)$

$c=\sqrt{a^2-b^2}$

$\forall M\in\mathcal{E}: MF_1+MF_2=2a$

$e_i=\frac{c}{a}$ = eccentricity (how far is the ellipse from a circle)

$2\times c$ = the focal distance

6.3. Det. the foci of the ellipse $9x^2+25y^2=225$

$\Rightarrow \frac{x^2}{5^2}+\frac{y^2}{3^2}=1 \Rightarrow a=5, \ b=3$

$c=\sqrt{5^2-3^2}=4 \Rightarrow F_1(-4,0), F_2(4,0)$

6.4. Det. the intersection of the line $\ell : x + 2y - 7 = 0$

and the ellipse $\mathcal{E}: x^2 + 3y^2 - 25 = 0$

$\sqrt{112} =$

$56 | 2$

$28 | 2$

$14 | 2$

$7 \cdot 2^4 = 2^2\sqrt{7}$

$\Rightarrow x = 7 - 2y$

$\Rightarrow \cancel{49} - 28y + 4y^2 + 3y^2 - \cancel{25} = 0$

$24 - 28y + 7y^2 = 0$

$\Delta = 28^2 - 28 \cdot 24 = 28(28 - 24) = 28 \cdot 4 = 112$

$y_1 = \dfrac{28 - \sqrt{112}}{14} = \dfrac{28 - 4\sqrt{7}}{14} = \dfrac{14 - 2\sqrt{7}}{7}$

$y_2 = \dfrac{28 + \sqrt{112}}{14} = \dfrac{28 + 4\sqrt{7}}{14} = \dfrac{14 + 2\sqrt{7}}{7}$

$x_1 = \dfrac{\cancel{49} - \cancel{28} + 2\sqrt{7}}{7} = \dfrac{21 + 2\sqrt{7}}{7}$

$x_2 = \dfrac{21 - 2\sqrt{7}}{7}$

$\ell : y = kx + m$

$\mathcal{E} : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\ell \cap \mathcal{E} : \begin{cases} y = kx + m \\ \dfrac{x^2}{a^2} + \dfrac{(kx+m)^2}{b^2} = 1 \end{cases}$

$x^2\left(\dfrac{1}{a^2} + \dfrac{k^2}{b^2}\right) + \dfrac{2km}{b^2} \cdot x + \dfrac{m^2}{b^2} - 1 = 0$

$\Delta = \dfrac{4k^2m^2}{b^4} - 4\,\dfrac{b^2 + a^2k^2}{a^2b^2} \cdot \dfrac{m^2 - b^2}{b^2} =$

$= \dfrac{4}{a^2b^4}\left(a^2k^2m^2 - \dfrac{(b^2 + a^2k^2)\cdot(m^2 - b^2)}{a^2}\right)$

$= \dfrac{4}{a^2b^4}\left(\cancel{a^2k^2m^2} - b^2m^2 + b^4 - \cancel{a^2k^2m^2} + a^2k^2b^2\right)$

$= \dfrac{4}{a^2b^2}\left(-m^2 + b^2 + a^2k^2\right)$

$a^2k^2 + b^2 - m^2$	intersection
< 0	$\emptyset$
$= 0$	1 point (tangent line)
> 0	2 points (secant line)

If $\underline{a^2k^2 + b^2 - m^2 = 0}$

$\Rightarrow$ $m = \pm\sqrt{b^2 + a^2k^2}$ So the tangents are

$$y = kx \pm \sqrt{b^2 + a^2k^2}$$

The tangent to the ellipse $\mathcal{E}$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
in the point (x_0, y_0) is:

$$T(x_0, y_0), \mathcal{E} : \frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1$$

$$\Leftrightarrow \underbrace{\frac{x_0}{a^2}(x - x_0)}_{\frac{\partial f}{\partial x}(x_0, y_0)} + \underbrace{\frac{y_0}{b^2}(y - y_0)}_{\frac{\partial f}{\partial y}(x_0, y_0)} = 0$$

where $f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

EXAM.

6.6. Det the eq of a line which is orthogonal to the line $l: 2x - 2y - 13 = 0$ and tangent to the ellipse to the ellipse $x^2 + 4y^2 - 20 = 0 \;|:4$

$$\frac{x^2}{4} + \frac{y^2}{1} = 5 \;|:5$$

$$\frac{x^2}{20} + \frac{y^2}{5} = 1 \Rightarrow \left.\begin{array}{l} a = 2\sqrt{5} \Rightarrow A_2(2\sqrt{5}, 0) \\ b = \sqrt{5} \Rightarrow B_2(0, \sqrt{5}) \end{array}\right\} \Rightarrow$$ line that goes through the points.

$$ax + by + c = 0$$

$$a(2\sqrt{5}) + c = 0 \Rightarrow 2a\sqrt{5} + c = 0$$

$$b\sqrt{5} + c = 0$$

$$\Rightarrow c = -b\sqrt{5}$$

$$\Rightarrow 2a - b = 0$$

$$\Rightarrow b = 2a.$$

$$\Rightarrow ax + 2ay + c = 0 \quad D(l) = (a, 2a)$$

and this is orthogonal to

$$2x - 2y - 13.$$

$$\Rightarrow B(l) = (2, -2)$$

$$\Rightarrow$$

$$l_2 \cap E \Rightarrow \begin{cases} x = x_A + 2\lambda \\ y = y_A - 2\lambda \\ x^2 + 4y^2 - 20 = 0 \end{cases}$$

$$x_A^2 + 4x_A\lambda + 4\lambda^2 + 4y_A^2 - 16y_A\lambda + 16\lambda^2 - 20 = 0$$

$$x_A^2 + 4y_A^2 + \lambda(4x_A - 16y_A) + 20\lambda - 20 = 0$$

$$(4x_A - 16y_A)^2 - 4(x_A^2 + 4y_A^2 - 20) \cdot 20 = 0.$$

$$y^2 - 2yc + c^2 + 4y - 20 = 0$$

$$5y^2 - 2cy - 20 + c^2 = 0$$

$$D = 0 \Rightarrow c^2 = \frac{400}{16} \Rightarrow c = \pm 5$$

l_2 : $x + y + 5 = 0$

or : $x + y - 5 = 0$