Seminar 9

Compute by applying elementary operations the ranks of the matrices:

~~**1.**~~ $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. ~~**3.**~~ $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Compute by applying elementary operations the inverses of the matrices:

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. ~~**5.**~~ $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

6. Let K be a field, let $B = (e_1, e_2, e_3, e_4)$ be a basis and let $X = (v_1, v_2, v_3)$ be a list in the canonical K-vector space K^4, where

$$v_1 = 3e_1 + 2e_2 - 5e_3 + 4e_4 \,,$$
$$v_2 = 3e_1 - e_2 + 3e_3 - 3e_4 \,,$$
$$v_3 = 3e_1 + 5e_2 - 13e_3 + 11e_4 \,.$$

$$\begin{pmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \end{pmatrix}$$

Write the matrix of the list X in the basis B, determine an echelon form for it and deduce that X is linearly dependent.

For the following exercises, for a list X of vectors in a canonical vector space $\mathbb{R}^n$, use that $dim < X >$ is equal to the rank of an echelon form C of the matrix consisting of the components of the vectors of X, and a basis of $< X >$ is given by the non-zero rows of C.

7. In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1, 0, 4)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 5, -36)$ and $v_4 = (2, 10, -72)$. Determine $dim < X >$ and a basis of $< X >$.

8. In the real vector space $\mathbb{R}^4$ consider the list $X = (v_1, v_2, v_3)$, where $v_1 = (1, 0, 4, 3)$, $v_2 = (0, 2, 3, 1)$ and $v_3 = (0, 4, 6, 2)$. Determine $dim < X >$ and a basis of $< X >$.

9. Determine the dimension of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S = < (1, 0, 4), (2, 1, 0), (1, 1, -4) > ,$$

$$T = < (-3, -2, 4), (5, 2, 4), (-2, 0, -8) > \;.$$

~~**10.**~~ Determine the dimension of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^4$ and a basis for the first three of them, where

$$S = < (1, 2, -1, -2), (3, 1, 1, 1), (-1, 0, 1, -1) > ,$$

$$T = < (2, 5, -6, -5), (-1, 2, -7, -3) > \;.$$

$A, B \in \mathcal{M}_{m,n}(K)$

If $A \sim B$, then $\operatorname{rank} A = \operatorname{rank} B$

So if E is the echalon form of A, then:

$\operatorname{rank} A = \operatorname{rank} E = \#$ of non-zero rows in E

Compute by applying elementary operations the ranks of the matrices:

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

1 $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_3 \leftrightarrow L_1}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 2\cdot L_1}{\overset{L_2 \leftarrow L_2 - 2\cdot L_1}{\sim}}$

$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \underset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \underset{L_4 \leftarrow L_4 - L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \operatorname{rank} A = 3$

3. $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{L_2 \leftrightarrow L_1}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{L_3 \leftarrow L_3 - 2L_1}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 0 & \alpha & -2 & 1 \end{pmatrix}$

$\underset{L_2 \leftarrow L_2 - \beta L_1}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\alpha\beta & 3(1-\beta) & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix} \underset{L_3 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3(1-\beta) & 4-3\beta \end{pmatrix}$

$$\xrightarrow[\sim]{L_3 \leftarrow L_3 + \beta L_2} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1 & 3-5\beta & 4-2\beta \end{pmatrix} \xrightarrow[\sim]{L_2 \to L_1 - L_2}$$

$$\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 1 & 0 & 5 & 2 \\ 0 & 1 & 3-5\beta & 4-2\beta \end{pmatrix} \sim$$

$$\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \xrightarrow[\sim]{L_2 \leftrightarrow L_1} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \xrightarrow[L_3 = L_3 - 2L_1]{L_2 \leftarrow L_2 - \beta L_1}$$

$$\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix} \xrightarrow[\sim]{L_2 \leftrightarrow L_3} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3(1-\beta) & 4-3\beta \end{pmatrix}$$

if $\alpha = 0$:

$$\begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 3-3\beta & 4-3\beta \end{pmatrix} \xrightarrow[\sim]{L_2 \leftrightarrow L_3} \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3(1-\beta) & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

$\Rightarrow$ rank $A = 3$

if $\alpha \neq 0$:

$$\xrightarrow[\sim]{L_2 = \frac{1}{\alpha} L_2} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & \frac{-2}{\alpha} & \frac{1}{\alpha} \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix} \xrightarrow[\sim]{L_3 \leftarrow L_3 - (1-\beta\alpha) L_2} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & \frac{-2}{\alpha} & \frac{1}{\alpha} \\ 0 & 0 & x & y \end{pmatrix}$$

$$x = 3 - 3\beta + \frac{2(1-\beta\alpha)}{\alpha} = 3 - 3\beta + \frac{2}{\alpha} - 2\beta = 3 - 5\beta + \frac{2}{\alpha}$$

$$y = 4 - 3\beta - \frac{1-\beta\alpha}{\alpha} = 4 - 3\beta - \frac{1}{\alpha} + \beta = 4 - 2\beta - \frac{1}{\alpha}$$

if x and $y \Rightarrow$ rank $A = 2$, else rank $A = 3$

$$\text{rank } A = \begin{cases} 2, & \text{if } x = y = 0 \\ 3 & \text{otherwise} \end{cases}$$

$$\begin{cases} 3 - 5\beta + \frac{2}{\alpha} = 0 \\ 4 - 2\beta - \frac{1}{\alpha} = 0 \quad \cdot 2 \end{cases} \quad (+)$$

$\Rightarrow \alpha = \frac{9}{14}$, $\beta = \frac{~~55~~}{45}$

$$\Rightarrow \text{rank } A = \begin{cases} 2, & \alpha = \frac{9}{14}, \ \beta = \frac{11}{9} \\ 3, & \text{otherwise} \end{cases}$$

inverting a matrix $A \in \mathcal{M}_n(K)$

$$(A \mid I_n) \overset{\text{Gauss-Jordan}}{\sim} \dots \sim (I_n \mid A^{-1})$$

you don't need to check beforehand if the matrix is invertable

5. $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix} = A \qquad A^{-1} = ?$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} L_2 \leftarrow L_2 - 2L_1 \\ \sim \\ L_3 \leftarrow L_3 - 3L_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \underset{\sim}{L_2 \leftarrow \frac{L_2}{-5}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & \frac{-1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \underset{\sim}{L_3 \leftarrow L_3 + 12 L_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & \frac{-1}{5} & 0 \\ 0 & 0 & -7+\frac{36}{5} & -3+\frac{24}{5} & \frac{-12}{5} & 1 \end{array}\right)$$ # poza final

Use Gaussian elimination to extract a basis out of a system of generators. Place the generators as rows in a matrix, bring it to the echelon form. The rows will form a basis.

10. Determine the dimension of the subspaces S, T, $S+T$ and $S \cap T$ of the real vector space $\mathbb{R}^4$ and a basis for the first three of them, where

$$S = < (1, 2, -1, -2), (3, 1, 1, 1), (-1, 0, 1, -1) >,$$

$$T = < (2, 5, -6, -5), (-1, 2, -7, -3) > .$$

$$M_S = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 3 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix} \underset{L_2 \leftarrow L_2 - 3L_1}{\overset{L_3 \leftarrow L_3 + L_1}{\sim}} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 2 & 0 & -3 \end{pmatrix} \underset{\sim}{L_3 \leftarrow L_3 + \frac{2}{5} L_2} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 0 & \frac{8}{5} & \frac{-1}{5} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & 0 & -3 \\ 0 & -5 & 4 & 7 \end{pmatrix} \underset{\sim}{L_3 \leftarrow \frac{+5}{2} L_2} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 4 & -\frac{1}{2} \end{pmatrix} \Rightarrow \dim S = 3$$

and $((1,2,-1,-2),(0,2,0,-3),(0,0,4,\frac{-1}{2}))$

$$M_T=\begin{pmatrix}2&5&-6&-5\\-1&2&-7&-3\end{pmatrix}\underset{\sim}{L_2\leftrightarrow L_1}\begin{pmatrix}-1&2&-7&-3\\2&5&-6&-5\end{pmatrix}\underset{\sim}{L_2\leftarrow L_2+2L_1}\begin{pmatrix}-1&2&-7&-3\\0&9&-20&-11\end{pmatrix}$$

$\Rightarrow \dim T=2 \Rightarrow ((-1,2,-7,-3),(0,9,-20,-11))$ - basis

$$\Rightarrow S+T=\begin{pmatrix}1&2&-1&-2\\0&2&0&-3\\0&0&4&\frac{-1}{2}\\-1&2&-7&-3\\0&9&-20&-11\end{pmatrix}\underset{\sim}{L_4+L_1}\begin{pmatrix}1&2&-1&-2\\0&2&0&-3\\0&0&4&\frac{-1}{2}\\0&4&-8&-5\\0&9&-20&-11\end{pmatrix}$$

$$\underset{\sim}{\substack{L_4-2L_2\\ L_5-\frac{9}{2}L_2}}\begin{pmatrix}1&2&-1&-2\\0&2&0&-3\\0&0&4&\frac{-1}{2}\\0&0&-8&1\\0&0&-20&\frac{5}{2}\end{pmatrix}\underset{\sim}{\substack{L_4\leftarrow L_4+2L_3\\ L_5\leftarrow L_5+5L_3}}\begin{pmatrix}1&2&-1&-2\\0&2&0&-3\\0&0&4&\frac{-1}{2}\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$\Rightarrow \text{rank}(S+T)=3 \Rightarrow ((1,2,-1,-2),(0,2,0,-3),(0,0,4,\frac{-1}{2}))$ basis

$\dim(S\cap T)=\dim S+\dim T-\dim(S+T)=3+2-3=2$