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(c) ★ $x^2 - y^2$ subject to $x^2 + y^2 = 1$.

$f(x,y) = x^2 - y^2 \qquad g(x,y) = x^2 + y^2 - 1$

$L(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y) = x^2 - y^2 + \lambda(x^2 + y^2 - 1)$

$\frac{\partial L}{\partial x} = 2x + 2\lambda x = 0 \Rightarrow \lambda = -1$ or $x = 0$

$\frac{\partial L}{\partial y} = -2y + 2\lambda y = 0 \Rightarrow \lambda = 1$ or $y = 0$

$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0 \Rightarrow x^2 + y^2 = 1$

i) $\lambda = -1 \Rightarrow y = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$ $\quad (x,y,\lambda) = [(1,0,-1), (-1,0,-1)]$

i.i) $f(x,y) = x^2 - y^2 = 1 - 0 = 1$

i.ii) $f(x,y) = x^2 - y^2 = (-1)^2 - 0 = 1$ $\longrightarrow$ max

ii) $\lambda = 1 \Rightarrow x = 0 \Rightarrow y^2 - 1 = 0 \Rightarrow y = \pm 1$ $\quad (x,y,\lambda) = [(0,1,1), (0,-1,1)]$

$f(x,y) = x^2 - y^2 = 0 - 1 = -1 \longrightarrow$ min

(f) ★ $x^3 + y^3 + z^3$ subject to $x^2 + y^2 + z^2 = 1$.

$f(x,y,z) = x^3 + y^3 + z^3 \qquad g(x,y,z) = x^2 + y^2 + z^2 - 1$

$L(x,y,z,\lambda) = f(x,y,z) + \lambda \cdot g(x,y,z) = x^3 + y^3 + z^3 + \lambda(x^2 + y^2 + z^2 - 1)$

$\frac{\partial L}{\partial x} = 3x^2 + 2\lambda x = x(3x + 2\lambda) = 0 \Rightarrow x = 0$ or $3x + 2\lambda = 0,\ 3x = -2\lambda,\ x = \frac{-2}{3}\lambda$

$\frac{\partial L}{\partial y} = 3y^2 + 2\lambda y = y(3y + 2\lambda) = 0 \Rightarrow y = 0$ or $3y + 2\lambda = 0,\ y = \frac{-2}{3}\lambda$

$\frac{\partial L}{\partial z} = 3z^2 + 2\lambda z = z(3z + 2\lambda) = 0 \Rightarrow z = 0$ or $3z + 2\lambda = 0,\ z = \frac{-2}{3}\lambda$

$\frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1 = 0 \Rightarrow x^2 + y^2 + z = 1$

1) $x = 0 \Rightarrow y^2 + z^2 = 1$

1.1) $y = 0 \Rightarrow z = \frac{-2}{3}\lambda$

$\frac{4}{9}\lambda^2 = 1 \Rightarrow \lambda^2 = \frac{9}{4} \Rightarrow \lambda = \pm\frac{3}{2} \Rightarrow (x,y,z) = [(0,0,-1), (0,0,1)]$

$f(x,y,z) = 0^3 + 0^3 + 1^3 = 1 \qquad f(x,y,z) = -1$

1.2) $z = 0 \Rightarrow y = \frac{-2}{3}\lambda$

$\frac{4}{9}\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{3}{2} \Rightarrow (x,y,z) = [(0,-1,0), (0,1,0)]$

$f(x,y,z) = \begin{cases} -1 \longrightarrow \text{min} \\ 1 \longrightarrow \text{max} \end{cases}$

1.3) $y = z = \frac{-2}{3}\lambda$

$\frac{8}{9}\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{3}{2\sqrt{2}} \Rightarrow y = z = \mp\frac{\sqrt{2}}{2} \Rightarrow (x,y,z) = [(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (0, \frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})]$

$$f(x,y,z) = 0^3 + 2\cdot\left(\frac{\sqrt{2}}{2}\right)^3 = \frac{\cancel{2}\sqrt{2}}{\cancel{2^3}}\cdot\cancel{2} = \frac{\sqrt{2}}{2}$$

$$f(x,y,z) = 0^3 + 2\cdot\left(\frac{-\sqrt{2}}{2}\right)^3 = \frac{-2\sqrt{2}}{\cancel{2^3}}\cdot\cancel{2} = \frac{-\sqrt{2}}{2}$$

2) $x = \frac{-2}{3}\lambda \Rightarrow y^2 + z^2 = 1 - \frac{4}{9}\lambda^2$

2.1) $y = 0 \Rightarrow z = \frac{-2}{3}\lambda$

$$\frac{8}{9}\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{3}{2\sqrt{2}} \Rightarrow x = z = \mp\frac{\sqrt{2}}{2} \Rightarrow (x,y,z) = \left[\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right), \left(\frac{-\sqrt{2}}{2}, 0, \frac{-\sqrt{2}}{2}\right)\right]$$

$$f(x,y,z) < \begin{matrix}\frac{\sqrt{2}}{2}\\ \frac{-\sqrt{2}}{2}\end{matrix}$$

2.2) $z = 0 \Rightarrow y = \frac{-2}{3}\lambda$

$$\frac{8}{9}\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{3}{2\sqrt{2}} \Rightarrow x = y = \mp\frac{\sqrt{2}}{2} \Rightarrow (x,y,z) = \left[\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), \left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, 0\right)\right]$$

$$f(x,y,z) = < \begin{matrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}\end{matrix}$$

2.3) $y = z = \frac{-2}{3}\lambda$

$$\frac{12}{9}\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{3}{2\sqrt{3}} \Rightarrow x = y = z = \mp\frac{\sqrt{3}}{3} \Rightarrow (x,y,z) = \left[\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \left(\frac{-\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right)\right]$$

$$f(x,y,z) = 3\cdot\left(\frac{\sqrt{3}}{3}\right)^3 = \frac{\cancel{3}\cdot\cancel{3}\sqrt{3}}{\cancel{3^3}} = \frac{\sqrt{3}}{3}$$

$$f(x,y,z) = 3\cdot\left(\frac{-\sqrt{3}}{3}\right)^3 = \frac{3\cdot(-3\sqrt{3})}{3^3} = \frac{-\sqrt{3}}{3}$$