5. ★ Let $a, b \in \mathbb{R}$ with $a > 0$. If S is nonempty and bounded above, prove that

$$\sup_{x \in S}(ax + b) = a \sup(S) + b.$$

I. Let $y := \sup(S) \Rightarrow y \geq s \ \forall s \in S \Rightarrow y \in ub(S)$

$x \leq y \quad \forall x \in S$

$x \leq y \ | \cdot a \neq 0$

$ax \leq ay \ | + b$

$ax + b \leq ay + b$

$ax+b \leq a \sup(S) + b \quad (1) \Rightarrow a \cdot \sup(S) + b$ is an upperbound for $\{ax+b \mid x \in S\}$

Let $u \in ub(ax+b)$ (any upperbound of the set $ax+b$)

$ax + b \leq u \ | - b$

$ax \leq u - b \ | : a \neq 0$

$x \leq \frac{u-b}{a}$ - bounded above

$\sup(S) \leq \frac{u-b}{a} \Rightarrow a \sup(S) + b \leq u \quad (2)$

Using (1) and (2) $\Rightarrow$ $a \sup(S) + b$ is the least upperbound for $\{ax+b \mid x \in S\} \Rightarrow$

$\Rightarrow \sup(ax+b) = a \sup(S) + b$

II. $$\sup_{x \in D}\big(f(x) + g(x)\big) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x).$$

based on this relation proved during seminar:

let us take $\left.\begin{array}{l} f(x) = ax \ , a \in \mathbb{R} \\ g(x) = b \ , b \in \mathbb{R} \end{array}\right\} \Rightarrow \sup(ax+b) \leq \sup(ax) + \sup(b)$

$\sup f(x) = a \cdot \sup(S)$ since a is a constant and $x \in S$

$\sup g(x) = b$ since b is a constant

$\Rightarrow \sup(ax+b) \leq a \sup(S) + b$

★8. Let $a, b \in \mathbb{R}$. Prove that there exist neighborhoods $U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t. $U \cap V = \emptyset$.

contra exemple

Assume that $\nexists\ U \in \mathcal{V}(a)$ and $V \in \mathcal{V}(b)$ s.t. $U \cap V = \emptyset$

$\exists\ \varepsilon > 0$ s.t. $(a-\varepsilon, a+\varepsilon) \subseteq U$

$\exists\ \varepsilon > 0$ s.t. $(b-\varepsilon, b+\varepsilon) \subseteq V$

Let us take $a = 1$, $b = 2$, $\varepsilon_1 = 0.5$

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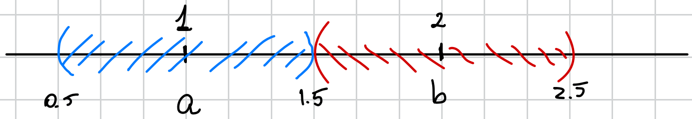

Even if $\left.\begin{array}{l} (\frac{1}{2}, \frac{3}{2}) \in \mathcal{V}(1) \\ (\frac{3}{2}, \frac{5}{2}) \in \mathcal{V}_2 \end{array}\right\} \Rightarrow (\frac{1}{2}, \frac{3}{2}) \cap (\frac{3}{2}, \frac{5}{2}) = \emptyset$

$\Rightarrow$ For any $\varepsilon > \frac{a+b}{2}$ $U \cap V \neq \emptyset$, but for any $\varepsilon \le \frac{a+b}{2} \Rightarrow U \cap V = \emptyset$

10. ★ Let $A = (0, 1) \cap \mathbb{Q}$. Show that $\inf A = 0$, $\sup A = 1$, $\text{int}A = \emptyset$ and $\text{cl}A = [0, 1]$.

$A = (0,1) \cap \mathbb{Q}$

$lb(A) = (-\infty, 0]$

$ub(A) = [1, +\infty)$

1) $x = \inf(A) = 0 \le a\ \forall a \in A$, $x \in lb(A)$

$x \ge u,\ \forall u \in lb(A)$

2) $y = \sup(A) = 1 \ge a\ \forall a \in A$, $y \in ub(A)$

$y \le u,\ \forall u \in ub(A)$

3) $\text{int } A = \emptyset$

$\text{int}(A) := \{x \in \mathbb{R} \mid \exists\ V \in \mathcal{V}(x) \text{ s.t. } V \subseteq A\}$

$A = (0,1) \cap \mathbb{Q} \Rightarrow \nexists\ V \in \mathcal{V}(x),\ x \in A,$ s.t. $V \subseteq A$

4) $cl(A) = [0,1]$

$cl(A) := \{x \in \mathbb{R} \mid \forall\ V \in \mathcal{V}(x),\ V \cap A \neq 0\}$

$A = (0,1) \cap \mathbb{Q} \Rightarrow [0,1] \cap (0,1) \cap \mathbb{Q} \neq \emptyset$