Seminar 1

1. Which ones of the usual symbols of addition, subtraction, multiplication and division define an operation (composition law) on the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$?

2. Let $A = \{a_1, a_2, a_3\}$. Determine the number of:
(i) operations on A;
(ii) commutative operations on A;
(iii) operations on A with identity element.
Generalization for a set A with n elements $(n \in \mathbb{N}^*)$.

3. Decide which ones of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are groups together with the usual addition or multiplication.

4. Let "$*$" be the operation defined on $\mathbb{R}$ by $x * y = x + y + xy$. Prove that:
(i) $(\mathbb{R}, *)$ is a commutative monoid.
(ii) The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

5. Let "$*$" be the operation defined on $\mathbb{N}$ by $x * y = \text{g.c.d.}(x, y)$.
(i) Prove that $(\mathbb{N}, *)$ is a commutative monoid.
(ii) Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ $(n \in \mathbb{N}^*)$ is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid.
(iii) Fill in the table of the operation "$*$" on D_6.

6. Determine the finite stable subsets of $(\mathbb{Z}, \cdot)$.

7. Let $(G, \cdot)$ be a group. Show that:
(i) G is abelian $\iff \forall x, y \in G$, $(xy)^2 = x^2y^2$.
(ii) If $x^2 = 1$ for every $x \in G$, then G is abelian.

8. Let "$\cdot$" be an operation on a set A and let $X, Y \subseteq A$. Define an operation "$*$" on the power set $\mathcal{P}(A)$ by

$$X * Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

Prove that: $\mapsto$ { set / ass / el. neutru
(i) If $(A, \cdot)$ is a monoid, then $(\mathcal{P}(A), *)$ is a monoid.
(ii) If $(A, \cdot)$ is a group, then in general $(\mathcal{P}(A), *)$ is not a group.
↳ + inv. ↳ take identity el. ⇒ if has e as its inverse but $\emptyset$ has no inverse

1. Addition: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
Subtraction: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
Multiplication: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$
Division: $\mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$

2. $A = \{a_1, a_2, a_3\}$
(i) 3 elements on $3 \times 3 = 9$ spaces $\Rightarrow 3^9$ $\quad n^{n^2}$
(ii) commutative

	a	b	c
a		X	y
b	X		z
c	y	z	

3^6 $\quad n^n \cdot n^{\frac{n(n-1)}{2}}$

(iii)

	e	b	c
e	e	b	c
b	b		
c	c		

3^4 $\quad n^{(n-1)^2+1}$

3. Decide which ones of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are groups together with the usual addition or multiplication.

for $(G,+)$ - group $\Rightarrow$ $\left\{\begin{array}{l}\text{set}\\ \text{associative}\\ \text{neutral el.}\\ \text{invertable el.}\end{array}\right.$ $\Rightarrow (\mathbb{Z},+), (\mathbb{Q},+), (\mathbb{R},+), (\mathbb{C},+)$

$(\mathbb{Q}^*,\cdot)\ (\mathbb{R}^*,\cdot), (\mathbb{C}^*,\cdot)$

4. Let "$*$" be the operation defined on $\mathbb{R}$ by $x * y = x + y + xy$. Prove that:
(*i*) $(\mathbb{R}, *)$ is a commutative monoid.
(*ii*) The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

(i) $(\mathbb{R},*)$ commutative monoid $\Leftrightarrow$ $\left\{\begin{array}{l}\text{set}\\ \text{associative}\\ \text{neutral el}\\ \text{com.}\end{array}\right.$

$\forall\, x,y \in \mathbb{R} \Rightarrow x*y = x+y+xy = x(y+1)+y+1-1 = (x+1)(y+1)-1 \in \mathbb{R}$

$\forall\, x,y,z \in \mathbb{R} \Rightarrow (x*y)*z = x*(y*z)$

$[(x+1)(y+1)-1]*z = x*[(y+1)(z+1)-1]$

$(x+1)(y+1)(z+1)-1 = (x+1)(y+1)(z+1)-1 \rightarrow$ True

$\exists\, e \in \mathbb{R}$ s.t. $x*e = e*x = x$

$(x+1)(e+1)-1 = x$

$(x+1)(e+1) = x+1$

$e(x+1) = x+1 - x+1$

$\left.\begin{array}{l} e(x+1) = 0\\ \forall x \in \mathbb{R}\end{array}\right\} \Rightarrow e = 0$

$\forall\, x,y \in \mathbb{R} \Rightarrow \quad x*y = y*x$

$(x+1)(y+1) \ -1 = (y+1)(x+1)-1 \rightarrow$ True

$\Rightarrow (\mathbb{R},*)$ monoid

(ii) Let $A = [-1,\infty)$, $A \leq (\mathbb{R},*)$ iff $\forall\, x,y \in A \Rightarrow x*y \in A$

$x \geq -1 \ |+1$

$x+1 \geq 0$

$y+1 \geq 0$

$(x+1)(y+1) \geq 0 \ |-1$

$(x+1)(y+1)-1 \geq -1 \ \Rightarrow \ x*y \geq -1 \ \forall\, x,y \in A \Rightarrow$ stable subset

5. Let " $*$ " be the operation defined on $\mathbb{N}$ by $x * y = \text{g.c.d.}(x, y)$.
(i) Prove that $(\mathbb{N}, *)$ is a commutative monoid.
(ii) Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ $(n \in \mathbb{N}^*)$ is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid.
(iii) Fill in the table of the operation " $*$ " on D_6.

(i) $(N, *)$ commutative monoid $\left\{ \begin{array}{l} \text{set} \\ \text{assoc,} \\ \text{neutral} \\ \text{com} \end{array} \right.$

(ii) $D_n = \{x \in N \mid x/n\}$

$\forall x, y \in D_n \Rightarrow x \mid n$ and $y \mid n \Rightarrow \begin{array}{l} n = x \cdot d_1 \\ n = y \cdot d_2 \end{array}$

$x * y = \gcd(x, y) = \alpha \Rightarrow \begin{array}{l} x = \alpha \cdot x_1 \\ y = \alpha \cdot x_2 \end{array}$

$\Rightarrow \left. \begin{array}{l} n = \alpha x_1 d_1 \\ n = \alpha x_2 d_2 \end{array} \right\} \Rightarrow \alpha \mid n \Rightarrow \gcd(x, y) \mid n \Rightarrow x * y \in D_n$

(iii) $D_6 = \{1, 2, 3, 6\}$

	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

6. Determine the finite stable subsets of $(\mathbb{Z}, \cdot)$.

Let $H \subseteq \mathbb{Z}$

H - stable subset $\Leftrightarrow \exists x \in H \Rightarrow \forall n \in N^*$ $x^n \in H$

$\{0\}, \{1\}, \{0, 1\}, \{-1, 1\}, \{-1, 0, 1\}$

7. Let $(G, \cdot)$ be a group. Show that:
(i) G is abelian $\Longleftrightarrow \forall x, y \in G$, $(xy)^2 = x^2y^2$.
(ii) If $x^2 = 1$ for every $x \in G$, then G is abelian.

$\forall x, y \in G \Rightarrow x \cdot y = y \cdot x$

$(x \cdot y)^2 = xy\,xy = xx\,yy = x^2 y^2$

$\Leftrightarrow \forall x, y \in G$ (group) $\exists x^{-1}, y^{-1} \in H$

$\Rightarrow x^{-1} \mid (x \cdot y)^2 = x^2 \cdot y^2 \mid y^{-1}$

$x \cdot y = y \cdot x \Rightarrow$ abelian

$R: \forall x \in A \quad x r x$
$S: \forall x, y \in A \quad x s y \Rightarrow y s x$
$T: \forall x, y, z \in A \quad x t y$ and $y t z \Rightarrow x t z$
$h = (R, M, H)$ -relation : $H \subseteq R \times M$
h-function if $\forall x \in R : |h\langle x \rangle| = 1$
(injective)

Seminar 2

$r = (A, B, R) \quad A = B$

1. Let r, s, t, v be the homogeneous relations defined on the set $M = \{2, 3, 4, 5, 6\}$ by

$$x\, r\, y \iff x < y \quad \{(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),\dots\}$$
$$x\, s\, y \iff x|y \quad \{(2,4),(2,6),(3,6),(2,2),(3,3),\dots\}$$
$$x\, t\, y \iff g.c.d.(x, y) = 1 \quad \{(2,3),(2,5),(3,2),(3,4),(3,5)\dots\}$$
$$x\, v\, y \iff x \equiv y \pmod 3\,. \quad \{(3,6),(6,3),(2,5)\dots\}$$

Write the graphs R, S, T, V of the given relations.

2. Let A and B be sets with n and m elements respectively ($m, n \in \mathbb{N}^*$). Determine the number of:
(i) relations having the domain A (n) and the codomain B (m); form pairs $((a,b),(b,a))$ each $\Rightarrow 2^{mn}$
(ii) homogeneous relations on A. $2^{|A \times A|} = 2^{n^2}$

3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

3. $R = \{1, 2, 3\}$
$r = \{(1,1),(2,2),(3,3)\}$
$t = \{(1,2),(2,3),(1,3)\}$
$s = \{(1,2),(2,1)\}$

4. Which ones of the properties of reflexivity, transitivity and symmetry hold for the following homogeneous relations: the strict inequality relations on $\mathbb{R}$, the divisibility relation on $\mathbb{N}$ and on $\mathbb{Z}$, the perpendicularity relation of lines in space ($x \perp x$ false), the parallelism relation of lines in space ($x \parallel x$), the congruence of triangles in a plane (true), the similarity of triangles in a plane (true)?

4. $r = (\mathbb{R}, \neq)$
$\forall x \in \mathbb{R} \quad x \neq x$ false (r)
$(\mathbb{N}, |)$
$\forall x \in \mathbb{N} \quad x | x$ true
$\forall x, y \in \mathbb{N} \quad x|y \Rightarrow y|x$ false
$\forall x, y, z \in \mathbb{N} \quad y|x, z|y \Rightarrow z|x$ true

5. Let $M = \{1, 2, 3, 4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$, $R_2 = \Delta_M \cup \{(1, 2), (1, 3)\}$, $\pi_1 = \{\{1\}, \{2\}, \{3, 4\}\}$, $\pi_2 = \{\{1\}, \{1, 2\}, \{3, 4\}\}$.
(i) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.
(ii) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

6. Define on $\mathbb{C}$ the relations r and s by:

$$z_1\, r\, z_2 \iff |z_1| = |z_2|\,; \qquad z_1\, s\, z_2 \iff arg\, z_1 = arg\, z_2 \text{ or } z_1 = z_2 = 0\,.$$

Prove that r and s are equivalence relations on $\mathbb{C}$ and determine the quotient sets (partitions) $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometric interpretation).

7. Let $n \in \mathbb{N}$. Consider the relation ρ_n on $\mathbb{Z}$, called the *congruence modulo* n, defined by:

$$x\, \rho_n\, y \iff n|(x - y)\,.$$

Prove that ρ_n is an equivalence relation on $\mathbb{Z}$ and determine the quotient set (partition) $\mathbb{Z}/\rho_n$. Discuss the cases $n = 0$ and $n = 1$.

8. $M = \{1, 2, 3\}$
part:
$\{\{1\}, \{2\}, \{3\}\} = \{(1,1),(2,2),(3,3)\}$
$\{\{1,2\}, \{3\}\} = \Delta_M \cup \{(1,2),(2,1)\}$
$\{\{1,3\}, \{2\}\} = \Delta_M \cup \{(1,3),(3,1)\}$
$\{\{1\}, \{2,3\}\} = \Delta_M \cup \{(2,3),(3,2)\}$
$\{\{1,2,3\}\} =$

8. Determine all equivalence relations and all partitions on the set $M = \{1, 2, 3\}$.

9. Let $M = \{0, 1, 2, 3\}$ and let $h = (\mathbb{Z}, M, H)$ be a relation, where

$$H = \{(x, y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : x = 4z + y\}\,.$$

Is h a function?

10. Consider the following homogeneous relations on $\mathbb{N}$, defined by:

$$m\, r\, n \iff \exists a \in \mathbb{N} : m = 2^a n\,,$$

$$m\, s\, n \iff (m = n \text{ or } m = n^2 \text{ or } n = m^2)\,.$$

Are r and s equivalence relations?

5. Let $M = \{1,2,3,4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.

(*i*) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.

(*ii*) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

$R_1 = \Delta_M \cup \{(1,2)\ (2,1)\ (1,3)\ (3,1)\ (2,3)\ (3,2)\}$

$\pi_1 = \{\underbrace{\{1\}, \{2\}, \{3,4\}}_{\text{all } \cap \text{ are } \emptyset}\} \Rightarrow R_{\pi_1} = \{(1,1), (2,2), (3,3), (3,4), (4,4), (4,3)\}$

$M/r_1 = \{r_1\langle x\rangle \mid x \in M\}$ $r\langle 1\rangle = \{1,2,3\}$, $r\langle 4\rangle = \{4\}$ $\Big\} \Rightarrow M/r_1 = \{\{1,2,3\}, \{4\}\}$

$\Rightarrow r_1$ - equivalence

$R_2 = \underbrace{\Delta_M}_{\text{reflex}} \cup \{(1,2), (1,3)\}$

$(1,1), (1,2), (2,2)$ but not symmetric

$\pi_2 = \{\{1\}, \{1,2\}, \{3,4\}\}$

$\{1\} \cap \{1,2\} = \{1\} \neq \emptyset \Rightarrow$ not a partition

6. Define on $\mathbb{C}$ the relations r and s by:

$$z_1\, r\, z_2 \Longleftrightarrow |z_1| = |z_2|\,; \qquad z_1\, s\, z_2 \Longleftrightarrow arg\, z_1 = arg\, z_2 \text{ or } z_1 = z_2 = 0\,.$$

Prove that r and s are equivalence relations on $\mathbb{C}$ and determine the quotient sets (partitions) $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometric interpretation).

$r:\ z\, r\, z \Leftrightarrow |z| = |z|$ - true

$\mathbb{C}/r = \{r(z) \mid z \in \mathbb{C}\} = \{z\, r\, z \mid z \in \mathbb{C}\} = \{r(z) \mid |z| = |\bar{z}|\ z \in \mathbb{C}\} = \{0\} \cup \{C(0,|z|)\}$

7. Let $n \in \mathbb{N}$. Consider the relation ρ_n on $\mathbb{Z}$, called the *congruence modulo n*, defined by:

$$x\, \rho_n\, y \Longleftrightarrow n|(x-y)\,.$$

Prove that ρ_n is an equivalence relation on $\mathbb{Z}$ and determine the quotient set (partition) $\mathbb{Z}/\rho_n$. Discuss the cases $n = 0$ and $n = 1$.

$x\, \rho_n\, y \Leftrightarrow n \mid (x-y)$

$R: \forall x \in \mathbb{Z} \Rightarrow x\, \rho_n\, x \Leftrightarrow n \mid x - x$ true

$T: \forall x,y,z \in \mathbb{Z} \Rightarrow x\, \rho_n\, y$ and $y\, \rho_n\, z \Rightarrow x\, \rho_n\, z$

$n \mid (x-y)$ $n \mid (y-z)$ $\Rightarrow n \mid (x-y+y-z) = n \mid (x-z)$ true

$S: \forall x,y \in \mathbb{Z} \Rightarrow x\, \rho_n (y) \Rightarrow y\, \rho_n (x)$

$n \mid (x-y) \Rightarrow n \mid (y-x)$ true

$\mathbb{Z}/\rho_n = \{\hat{0}, \hat{1}, \dots, \widehat{n-1}\}$

$\mathbb{Z}/\rho_0 = \{\{x\} \mid x \in \mathbb{Z}\}$ $\quad 0 \mid x-y \Rightarrow x=y$

$\mathbb{Z}/\rho_1 = \{\mathbb{Z}\}$ $\quad 1 \mid x-y$

9. Let $M = \{0,1,2,3\}$ and let $h = (\mathbb{Z}, M, H)$ be a relation, where

$$H = \{(x,y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : x = 4z + y\}. \qquad x \equiv y \pmod 4$$

Is h a function?

for h to be a function $\Rightarrow$ $|h\langle x\rangle| = 1 \quad \forall x \in \mathbb{Z}$

$h\langle x\rangle = \{y \in M \mid (x,y) \in \mathbb{Z}\times M\} = \{y \in M \mid \exists z \in M : x = 4z+y\}$

$= \{y \in M \mid x \equiv y \pmod 4\}$ which is uniquely det.

$\Rightarrow h\langle x\rangle = 1 \ \forall x \in \mathbb{Z} \Rightarrow h$ is a function

10. Consider the following homogeneous relations on $\mathbb{N}$, defined by:

$$m\, r\, n \iff \exists a \in \mathbb{N} : m = 2^a n,$$

$$m\, s\, n \iff (m = n \text{ or } m = n^2 \text{ or } n = m^2).$$

Are r and s equivalence relations?

$r = (\mathbb{N}, \mathbb{N}, R) \qquad m\, r\, n \Leftrightarrow \exists a \in \mathbb{N} \quad m = 2^a n$

$S:$ $\left.\begin{matrix} m\, r\, n \Leftrightarrow m = 2^a n \\ n\, r\, m \Leftrightarrow n = 2^b m \end{matrix}\right\} \Rightarrow m = 2^a 2^b m \Rightarrow \left.\begin{matrix} 2^{a+b} = 1 \Rightarrow a+b=0 \\ a,b \in \mathbb{N} \end{matrix}\right\} \Rightarrow$

$\Rightarrow a=b=0 \Rightarrow m$ must be $= n$

$\Rightarrow$ not symmetric

$s = (\mathbb{N}, S) \Leftrightarrow m = n$ or $m = n^2$ or $n = m^2$

$r:$ $m\, s\, m \Leftrightarrow m = m \Rightarrow$ reflex

$t:$ $\left.\begin{matrix} m\, s\, n \\ n\, s\, q \end{matrix}\right\} \Rightarrow \left.\begin{matrix} m = n^2 \\ n^2 = q \end{matrix}\right\} \Rightarrow m = n^2 q \Rightarrow m = q$ (true)

$S:$ $\left.\begin{matrix} m\, s\, n & m = n \\ n\, s\, m & n = m \end{matrix}\right\} \Rightarrow m = n$ symetric

$(G, *)$ group $\Leftrightarrow$ $\begin{cases} \text{associative} \\ \text{identity el} \\ \text{all el. have asymetric} \end{cases}$

$(R,+,\cdot)$ ring $\Leftrightarrow$ $\begin{cases} (R,+) \text{ abelian group} \\ (R,\cdot) \text{ semigroup} \\ \text{distributivity law holds} \end{cases}$

$f:(G_1,\circ)\to(G_2,*)$ homo $\Rightarrow$ $f(x\circ y)=f(x)*f(y)$

isomorph. if f bijective

Seminar 3

1. Let M be a non-empty set and let $S_M = \{f : M \to M \mid f \text{ is bijective}\}$. Show that $(S_M, \circ)$ is a group, called the *symmetric group* of M.

2. Let M be a non-empty set and let $(R, +, \cdot)$ be a ring. Define on $R^M = \{f \mid f : M \to R\}$ two operations by: $\forall f, g \in R^M$,

distributivity: $\forall f,g,h \in R^M$: $(f\cdot(g+h))(x)=(f\cdot g)(x)+(f\cdot h)(x)$

$$f + g : M \to R, \quad (f + g)(x) = f(x) + g(x), \quad \forall x \in M,$$

$$f \cdot g : M \to R, \quad (f \cdot g)(x) = f(x) \cdot g(x), \quad \forall x \in M.$$

Show that $(R^M, +, \cdot)$ is a ring. If R is commutative or has identity, does R^M have the same property?

yes yes yes but $\theta(x)=0$ and $E(x)=1$

3. Prove that $H = \{z \in \mathbb{C} \mid |z| = 1\}$ is a subgroup of $(\mathbb{C}^*, \cdot)$, but not of $(\mathbb{C}, +)$.

4. Let $U_n = \{z \in \mathbb{C} \mid z^n = 1\}$ $(n \in \mathbb{N}^*)$ be the *set of n-th roots of unity*. Prove that U_n is a subgroup of $(\mathbb{C}^*, \cdot)$.

5. Let $n \in \mathbb{N}$, $n \geq 2$. Prove that:
(i) $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) \neq 0\}$ is a stable subset of the monoid $(M_n(\mathbb{C}), \cdot)$;
(ii) $(GL_n(\mathbb{C}), \cdot)$ is a group, called the *general linear group of rank* n;
(iii) $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) = 1\}$ is a subgroup of the group $(GL_n(\mathbb{C}), \cdot)$.

6. Show that the following sets are subrings of the corresponding rings:
(i) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ in $(\mathbb{C}, +, \cdot)$.
(ii) $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$ in $(M_2(\mathbb{R}), +, \cdot)$.

7. (i) Let $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = |z|$. Show that f is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

(ii) Let $g : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $g(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that g is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}), \cdot)$.

8. Let $n \in \mathbb{N}$, $n \geq 2$. Prove that the groups $(\mathbb{Z}_n, +)$ of residue classes modulo n and $(U_n, \cdot)$ of n-th roots of unity are isomorphic.

9. Let $n \in \mathbb{N}$, $n \geq 2$. Consider the ring $(\mathbb{Z}_n, +, \cdot)$ and let $\widehat{a} \in \mathbb{Z}_n^*$.
(i) Prove that $\widehat{a}$ is invertible $\Longleftrightarrow$ $(a, n) = 1$.
(ii) Deduce that $(\mathbb{Z}_n, +, \cdot)$ is a field $\Longleftrightarrow$ n is prime.

10. Let $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$. Show that $(\mathcal{M}, +, \cdot)$ is a field isomorphic to $(\mathbb{C}, +, \cdot)$.

1. Let M be a non-empty set and let $S_M = \{f : M \to M \mid f \text{ is bijective}\}$. Show that $(S_M, \circ)$ is a group, called the *symmetric group* of M.

assoc. $\forall f_1 f_2 f_3 \in S_M \Rightarrow ((f_1 \circ f_2) \circ f_3)(x) = f_1 \circ (f_2 \circ f_3)(x) \ \forall x \in M$

$f_1(f_2(f_3(x))) = f_1((f_2 \circ f_3)(x))$ true

neutral: $\exists e \in S_M$ st $\forall f \in S_M : (e \circ f)(x) = (f \circ e)(x) = f(x) \ \forall x \in M$

$e(x) = x \Rightarrow$ bijective true

symmetric: for a bij. function it's inverse is also bijective $\Rightarrow \forall f \in S_M \ \exists f^{-1} \in S_M$ s.t. $(f \circ f^{-1})(x) = e(x)$

$\rightarrow$ group.

3. Prove that $H = \{z \in \mathbb{C} \mid |z| = 1\}$ is a subgroup of $(\mathbb{C}^*, \cdot)$, but not of $(\mathbb{C}, +)$.

I $H \leq (\mathbb{C}^*, \cdot) \Leftrightarrow \begin{cases} H \neq \emptyset \\ \forall x, y \in H : x \cdot y^{-1} \in H \end{cases}$

Let $z_1 = 1 \Rightarrow |z| = 1 \in H \Rightarrow H \neq 0$

$z_1, z_2 \in H : z_1 \cdot z_2^{-1} \in H$

$z_2^{-1} = \frac{1}{z_2} \Rightarrow |z_2^{-1}| = \frac{1}{|z_2|} = \frac{1}{1} = 1 \in H$

$z_1 \cdot z_2^{-1} = z_1 \cdot \frac{1}{z_2} = \frac{z_1}{z_2}$

$|z_1 \cdot z_2^{-1}| = \frac{|z_1|}{|z_2|} = \frac{1}{1} = 1 \in H$

$\Rightarrow H \leq (\mathbb{C}^*, \cdot)$

II $(H, +) \not\leq (\mathbb{C}, +)$

$z_1 = 1$ and $z_2 = i \in H$

$z_1 + z_2 = 1 + i$

$|z_1 + z_2| = \sqrt{1+1} = \sqrt{2} \neq 1 \notin H$

$\Rightarrow (H, +) \not\leq (\mathbb{C}, +)$

4. Let $U_n = \{z \in \mathbb{C} \mid z^n = 1\}$ $(n \in \mathbb{N}^*)$ be the *set of n-th roots of unity.* Prove that U_n is a subgroup of $(\mathbb{C}^*, \cdot)$.

$U_n \neq \emptyset$

$\forall x, y \in U_n \Rightarrow x \cdot y^{-1} \in U_n$

$z = 1 \to z^n = 1^n = 1$ true

let $z_1, z_2 \in U_n$ st. $z_1^n = z_2^n = 1$ $\quad z_2^{-1} = \frac{1}{z_2} \in \mathbb{C}$

$z_2^{-n} = \frac{1}{z_2^n} = 1 \in U_n$

$z_1 \cdot z_2^{-1} = \frac{z_1^n}{z_2^n} = \frac{1}{1} = 1 \in U_n \Rightarrow$ subgroup

5. Let $n \in \mathbb{N}$, $n \geq 2$. Prove that:
(i) $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) \neq 0\}$ is a stable subset of the monoid $(M_n(\mathbb{C}), \cdot)$;
(ii) $(GL_n(\mathbb{C}), \cdot)$ is a group, called the *general linear group of rank n*;
(iii) $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) = 1\}$ is a subgroup of the group $(GL_n(\mathbb{C}), \cdot)$.

(i) $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) \neq 0\} \leq (M_n(\mathbb{C}), \cdot)$

$\forall A, B \in GL_n(\mathbb{C}) \Rightarrow \det A \neq 0$, $\det B \neq 0$ $\Rightarrow \det A \cdot \det B \neq 0 \Rightarrow \det(A \cdot B) \neq 0 \Rightarrow$
$\Rightarrow A \cdot B \in GL_n(\mathbb{C})$

(ii) $(GL_n(\mathbb{C}), \cdot)$ - group
$\hookrightarrow$ assoc.
neutral element : I_n for matrix mult. $\det I_n \neq 0$
inverse

$\forall A \in GL_n(\mathbb{C})\ \exists A^{-1}$ s.t. $A \cdot A^{-1} = I_n$

$\det(A \cdot A^{-1}) = \det I_n$ $\Rightarrow$ group

$\left.\begin{array}{l}\underbrace{\det A}_{\neq 0} \cdot \det A^{-1} = 1\end{array}\right\} \Rightarrow \det A^{-1} \neq 0 \Rightarrow A^{-1} \in GL_n(\mathbb{C})$

(iii) $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) = 1\} \leq GL_n(\mathbb{C})$
$\hookrightarrow$ stable subset
$(SL_n(\mathbb{C}), \cdot)$ group - easy to prove

$\forall A, B \in SL_n(\mathbb{C}) \overset{?}{\Rightarrow} A \cdot B \in SL_n(\mathbb{C})$ $\mapsto$ true

$\det A \cdot \det B = \det(A \cdot B) = 1$

6. Show that the following sets are subrings of the corresponding rings:
(i) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ in $(\mathbb{C}, +, \cdot)$.
(ii) $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$ in $(M_2(\mathbb{R}), +, \cdot)$.

(i) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \leq (\mathbb{C}, +, \cdot)$

subring $\Leftrightarrow$ $\begin{cases} |\mathbb{Z}[i]| \geq 2 \text{ (has at least 2 el.)} \\ \forall x, y \in \mathbb{Z}[i] : x - y \in \mathbb{Z}[i] \\ \forall x, y \in \mathbb{Z}[i] : x \cdot y \in \mathbb{Z}[i] \end{cases}$

$z_1 = 0 = 0 + 0 \cdot i \in \mathbb{Z}[i]$
$z_2 = 1 = 1 + 0 \cdot i \in \mathbb{Z}[i]$

$\forall x, y \in \mathbb{Z}[i]$ $\quad x - y \in \mathbb{Z}[i]$

$x = a_1 + b_1 i$ $\quad x - y = a_1 + b_1 i - a_2 - b_2 i$
$y = a_2 + b_2 i$ $\quad x - y = (a_1 - a_2) + (b_1 - b_2) i \in \mathbb{Z}[i]$

$x \cdot y = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i \in \mathbb{Z}[i]$ $\Rightarrow$ subring

(ii) $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$ in $(\mathcal{M}_2(\mathbb{R}), +, \cdot)$

$m_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}$

$m_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}$

$\forall A, B \in \mathcal{M} \Rightarrow A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$ $\Rightarrow A - B = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ 0 & c_1 - c_2 \end{pmatrix} \in \mathcal{M}$

$A \cdot B = \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 c_2 \\ 0 & c_1 c_2 \end{pmatrix} \in \mathcal{M}$

$\Rightarrow (\mathcal{M}, +, \cdot)$ subring of $(\mathcal{M}_2(\mathbb{R}), +, \cdot)$

7. (*i*) Let $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = |z|$. Show that f is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

(*ii*) Let $g : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $g(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that g is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}), \cdot)$.

(i) group homomorphism $\Leftrightarrow \forall z_1, z_2 \in \mathbb{C}^* \Rightarrow f(z_1 \cdot z_2) = f(z_1) \cdot f(z_2)$

$f(z_1, z_2) = |z_1 \cdot z_2| = |z_1| \cdot |z_2| = f(z_1) \cdot f(z_2)$ true

(ii) $g(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

$\forall z_1, z_2 \in \mathbb{C}^*$ s.t. $\left.\begin{matrix} z_1 = a_1 + b_1 i \\ z_2 = a_2 + b_2 i \end{matrix}\right\} \Rightarrow g(z_1 \cdot z_2) = g(z_1) \cdot g(z_2)$

$g(z_1 \cdot z_2) = g(a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1)) = \begin{pmatrix} a_1 a_2 - b_1 b_2 & a_1 b_2 + a_2 b_1 \\ -a_1 b_2 - a_2 b_1 & a_1 a_2 - b_1 b_2 \end{pmatrix}$

$g(z_1) \cdot g(z_2) = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_1 \\ -b_2 & a_2 \end{pmatrix} = \nearrow \Rightarrow$ group homo

8. Let $n \in \mathbb{N}$, $n \geq 2$. Prove that the groups $(\mathbb{Z}_n, +)$ of residue classes modulo n and $(U_n, \cdot)$ of n-th roots of unity are <u>isomorphic.</u>

$(\mathbb{Z}_n, +)$ isomorphic $\Leftrightarrow$ find a function b.w. which is a group isomorphism

$f : U_n \to \mathbb{Z}_n \qquad f(z^k) = k \quad \forall k \in \mathbb{Z}_n$

group homo: + bijective $\quad f(z^{k_1} \cdot z^{k_2}) = f(z^{k_1 + k_2}) = k_1 + k_2 = f(z^{k_1}) + f(z^{k_2})$

9. Let $n \in \mathbb{N}$, $n \geq 2$. Consider the ring $(\mathbb{Z}_n, +, \cdot)$ and let $\widehat{a} \in \mathbb{Z}_n^*$.
(i) Prove that $\widehat{a}$ is invertible $\iff (a, n) = 1$.
(ii) Deduce that $(\mathbb{Z}_n, +, \cdot)$ is a field $\iff n$ is prime.

(i) $\hat{a}$ inv. $\iff (a,n)=1$

$\exists\, \hat{b} \in \mathbb{Z}_n$ s.t. $\hat{a}\cdot\hat{b} = \hat{1} \Rightarrow \widehat{ab} = \hat{1} \Rightarrow n \mid ab-1 \iff \exists\, k \in \mathbb{Z}$ s.t. $ab-1=nk$

$\iff a\cdot b + n\cdot(-k) = 1 \Rightarrow (a,n)=1$

(ii) $(\mathbb{Z}_n,+,\cdot)$ - ring

$\hookrightarrow$ field $\iff$ + invertible $\longrightarrow \forall\, \hat{a} \in \mathbb{Z}_n$ inv. $\iff \hat{1}, \hat{2}, \dots \widehat{(n-1)}$ inv. (i) $\Rightarrow (a,n)=1 \Rightarrow n$ prime

+ neutral ✓

+ com ✓

10. Let $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$. Show that $(\mathcal{M}, +, \cdot)$ is a field isomorphic to $(\mathbb{C}, +, \cdot)$.

$f : \mathbb{C} \to M$ $\quad f(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

$(M,+)$ abelian group (matrix addition)

identity O_3

symetric $\begin{pmatrix} -a & -b \\ b & -a \end{pmatrix}$

$(M^*, \cdot)$ group

identity I_3

$\det A = a^2+b^2 \geq 0$, $A \neq O_2 \Rightarrow$ inv.

distr. holds

$\Rightarrow$ field

+ bijective homomorphism

$f((a+bi)+(c+di)) = f((a+c)+i(b+d)) = \begin{pmatrix} a+c & b+d \\ -b-d & a+c \end{pmatrix} = f(a+bi) + f(c+di)$

$f((a+bi)\cdot(c+di)) = f(ac-bd+(ad+bc)i) = \begin{pmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{pmatrix} = f(a+bi)\; f(c+di)$

$\Rightarrow$ homomorphism

$\exists!\, \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ s.t. $f(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} =$ bijective $\Rightarrow$ isomorphism

$(K, *, \circ)$ field
$(V, \perp)$ abelian group
V - Vector space with $\odot$ external op.

Seminar 4

1. Let K be a field. Show that $K[X]$ is a K-vector space, where the addition is the usual addition of polynomials and the scalar multiplication is defined as follows: $\forall\, k \in K$, $\forall f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$,

$$k \cdot f = (ka_0) + (ka_1)X + \cdots + (ka_n)X^n\,.$$

2. Let K be a field and $m, n \in \mathbb{N}$, $m, n \geq 2$. Show that $M_{m,n}(K)$ is a K-vector space, with the usual addition and scalar multiplication of matrices.

3. Let K be a field, $A \neq \emptyset$ and denote $K^A = \{f \mid f : A \to K\}$. Show that K^A is a K-vector space, where the addition and the scalar multiplication are defined as follows: $\forall f, g \in K^A$, $\forall k \in K$, $f + g \in K^A$, $kf \in K^A$,

$$(f + g)(x) = f(x) + g(x)\,, \quad (k \cdot f)(x) = k \cdot f(x)\,, \forall x \in A\,.$$

* Check everything:
• $(V, \perp)$ abelian group
• „$\top$" is a well defined op.
• axioms

4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define the operations: $x \perp y = xy$ and $k \top x = x^k$, $\forall k \in \mathbb{R}$ and $\forall x, y \in V$. Prove that V is a vector space over $\mathbb{R}$.

5. Let K be a field and let $V = K \times K$. Decide whether V is a K-vector space with respect to the following addition and scalar multiplication:
(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + 2y_2)$ and $k \cdot (x_1, y_1) = (kx_1, ky_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.
(ii) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k \cdot (x_1, y_1) = (kx_1, y_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

6. Let p be a prime number and let V be a vector space over the field $\mathbb{Z}_p$.
(i) Prove that $\underbrace{x + \cdots + x}_{p \text{ times}} = 0$, $\forall x \in V$.
(ii) Is there a scalar multiplication endowing $(\mathbb{Z}, +)$ with a structure of a vector space over $\mathbb{Z}_p$?

7. Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;
(iv) $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(v) $E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;
(vi) $F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$?

8. Which ones of the following sets are subspaces:
(i) $[-1, 1]$ of the real vector space $\mathbb{R}$;
(ii) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ of the real vector space $\mathbb{R}^2$;
(iii) $\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Q} \right\}$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;
(iv) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous}\}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

9. Which ones of the following sets are subspaces of the K-vector space $K[X]$:
(i) $K_n[X] = \{f \in K[X] \mid \text{degree}(f) \leq n\}$ $(n \in \mathbb{N})$;
(ii) $K'_n[X] = \{f \in K[X] \mid \text{degree}(f) = n\}$ $(n \in \mathbb{N})$.

10. Show that the set of all solutions of a homogeneous system of two equations and two unknowns with real coefficients is a subspace of the real vector space $\mathbb{R}^2$.

1. Let K be a field. Show that $K[X]$ is a K-vector space, where the addition is the usual addition of polynomials and the scalar multiplication is defined as follows: $\forall\, k \in K$, $\forall f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$,

$$k \cdot f = (ka_0) + (ka_1)X + \cdots + (ka_n)X^n .$$

(L1) $k\cdot(v_1+v_2) = kv_1 + kv_2$

$k\cdot(f_1+f_2) = k\left[a_{10}+a_{20} + (a_{11}+a_{21})X + \ldots + (a_{1n}+a_{2n})X^n\right] =$

$= ka_{10} + ka_{11}X + \ldots + ka_{1n}X^n + ka_{20} + \ldots + ka_{2n}X^n = kf_1 + kf_2$

(L2) $(k_1+k_2)v = k_1v + k_2v$

$(k_1+k_2)f = (k_1+k_2)a_0 + (k_1+k_2)a_1X + \ldots + (k_1+k_2)a_nX^n =$

$= k_1a_0 + k_1a_1X + \ldots + k_1a_nX^n + k_2a_0 + k_2a_1X + \ldots + k_2a_nX^n = k_1f + k_2f$

(L3) $(k_1\cdot k_2)\cdot v = k_1(k_2 v)$

$(k_1\cdot k_2)f = k_1k_2a_0 + k_1k_2a_1X + \ldots + k_1k_2a_nX^n = k_1(k_2f)$

(L4) $1\cdot f = 1\cdot(a_0 + a_1X + \ldots + a_nX^n) = f$

ed op. **5.** Let K be a field and let $V = K \times K$. Decide whether V is a K-vector space with respect to the following addition and scalar multiplication:

(*i*) $(x_1, y_1)+(x_2, y_2) = (x_1+x_2, y_1+2y_2)$ and $k\cdot(x_1, y_1) = (kx_1, ky_1)$, $\forall(x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

(*ii*) $(x_1, y_1)+(x_2, y_2) = (x_1+x_2, y_1+y_2)$ and $k\cdot(x_1, y_1) = (kx_1, y_1)$, $\forall(x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

(i) $(x_1, y_1) + (x_2, y_2) = (x_1+x_2, y_1+2y_2)$

$k\cdot(x_1,y_1) = (kx_1, ky_1)$

$(V,+)$ - abelian group $\Leftrightarrow$ $\left\{\begin{array}{l}\text{assoc.}\\ \text{identity el}\\ \text{inversable}\\ \text{comutative}\end{array}\right.$

* assoc. $\forall z,x,y \in V \Rightarrow (x+y)+z = x+(y+z)$ - true
* identity $\exists e \in V$ s.t. $x+e = e+x = x$ $\Rightarrow e = 0$ true
* inv. $\forall x \in V \Rightarrow \exists -x \in V$ s.t. $-x+x = e$ true $\Rightarrow$ abelian group
* com. $\forall x,y \in V \Rightarrow x+y = y+x$ true

check if "$\cdot$" well def: $\forall v \in V$, $k \in K$

$\Rightarrow v\cdot k \in V$

$\left.\begin{array}{l} k(x,y) = (kx, ky)\\ kx \in K\\ ky \in K\end{array}\right\} \Rightarrow (kx, ky) \in V \Rightarrow$ well def.

axioms:

(L1) $\forall v_1, v_2 \in V$, $\alpha \in K$

$\alpha(v_1+v_2) = \alpha v_1 + \alpha v_2 = \alpha(x_1,y_1) + \alpha(x_2,y_2)$

(L_2) $\forall v \in V$ $\alpha, \beta \in K$

$(\alpha+\beta)v = ((\alpha+\beta)x, y) \neq \alpha v + \beta v = ((\alpha+\beta)x, y+2y)$ $\Rightarrow$ not a vector space

7. Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;
(iv) $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(v) $E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;
(vi) $F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$?

$$S \leq \mathbb{R}^3 \Leftrightarrow \begin{cases} S \neq \emptyset \\ \forall k_1, k_2 \in K \ \forall x, y \in S \\ k_1 x + k_2 y \in S \end{cases}$$

* the cond. needs to be a homogeneous linear sistem

(i) a) $A \neq \emptyset$ $(0,1,2) \in A$

b) $\forall x, y \in A$, $x+y \in A$

$(0, x_1, x_2)$
$(0, y_1, y_2)$ $\Rightarrow x+y = (0, x_1+y_1, x_2+y_2) \in A$

c) $\forall k \in \mathbb{R}$, $x \in A$

$k \cdot x = k(0, y, z) = (0, ky, kz) \in A$ $\Rightarrow$ subspace of $\mathbb{R}^3$

(ii) a) $B \neq \emptyset$ $(0,1,1) \in B$

b) let $x = (0, a, b)$
$y = (c, d, 0)$ $\Rightarrow x+y = (c, a+d, b) \notin B$

$\Rightarrow$ not a subspace of $\mathbb{R}^3$

ex: $(0,1,2) + (2,1,0) = (2,2,2) \notin B$

(iii) a) $C \neq \emptyset$ $(0,1,1) \in C$

b) $x, y \in C \Rightarrow x+y = (x_1+x_2, y_1+y_2, z_1+z_2) \in C$

c) $k \in \mathbb{R}$, $x \in C \Rightarrow k \cdot x = (ka, kb, kc) \notin C$ $\Rightarrow$ not a subspace

(iv) b) $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$

but $x+y+z=0$ $\Rightarrow \in D$
$x_1+y_1+z_1+x_2+y_2+z_2=0$

c) $k(x,y,z) = (kx, ky, kz)$ $\in D$ $\Rightarrow$ subspace of $\mathbb{R}^3$
$k(x+y+z)=0$

(v) b)
$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$

but $x+y+z=1$ $\Rightarrow \notin E$ $\Rightarrow$ not a subspace
$x_1+y_1+z_1+x_2+y_2+z_2=2 \neq 1$

(vi) b) $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2)$ $\in F$
$x_1+x_2 = y_1+y_2 = z_1+z_2$

c) $k(x,y,z) = (kx, ky, kz)$ $\in F$ $\Rightarrow$ subspace of $\mathbb{R}^3$
$x=y=z$

8. Which ones of the following sets are subspaces:
(*i*) $[-1,1]$ of the real vector space $\mathbb{R}$;
(*ii*) $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 1\}$ of the real vector space $\mathbb{R}^2$;
(*iii*) $\left\{\begin{pmatrix} a & b\\ 0 & c\end{pmatrix} \Big| a,b,c\in\mathbb{Q}\right\}$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;
(*iv*) $\{f:\mathbb{R}\to\mathbb{R}\mid f \text{ continuous}\}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

(i) $[-1,1] \leq_K \mathbb{R}$

$x,y \in [-1,1] \Rightarrow x+y \in [-2,2] \Rightarrow$ not a subspace

(ii) $B = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \leq 1\}$

a) $B \neq \emptyset$

b) $\alpha,\beta \in B$

$\alpha=(x_1,y_1)$
$\beta=(x_2,y_2)$ $\Rightarrow \alpha+\beta=(x_1+x_2, y_1+y_2)$

$(x_1+x_2)^2+(y_1+y_2)^2 \overset{?}{\leq} 1 \Rightarrow$ not a subspace

take $(1,0)+(0,1)=(1,1)$
$1^2+1^2=2 \not\leq 1$

(iii) $C=\left\{\begin{pmatrix} a & b\\ 0 & c\end{pmatrix} \Big| a,b,c\in\mathbb{Q}\right\} \leq_{\mathbb{Q}} M_2(\mathbb{Q})$ or ${}_{\mathbb{R}}M_2(\mathbb{R})$

$$\alpha\begin{pmatrix} a & b\\ 0 & c\end{pmatrix}+\beta\begin{pmatrix} d & e\\ 0 & f\end{pmatrix}=\begin{pmatrix} \alpha a+\beta d & \alpha b+\beta e\\ 0 & \alpha c+\beta f\end{pmatrix}$$
$\in M_2(\mathbb{Q})$ for $\alpha,\beta\in\mathbb{Q} \Rightarrow$ subspace but
for $\alpha,\beta\in\mathbb{R}$ $\notin M_2(\mathbb{Q}) \Rightarrow$ not subspace

(iv) $D=\{f:\mathbb{R}\to\mathbb{R} \mid f \text{ cont}\}$

$\alpha f_1+\beta f_2 =$ cont. $\Rightarrow$ subspace

9. Which ones of the following sets are subspaces of the K-vector space $K[X]$:
(*i*) $K_n[X]=\{f\in K[X]\mid \text{degree}(f)\leq n\}$ $(n\in\mathbb{N})$;
(*ii*) $K'_n[X]=\{f\in K[X]\mid \text{degree}(f)=n\}$ $(n\in\mathbb{N})$.

(i) $k,l \leq n$

$f=a_0+a_1X+\ldots+a_kX^k$

$g=b_0+b_1X+\ldots+b_lX^l$

let $k<l$

$\alpha,\beta\in K \Rightarrow \alpha f+\beta g=(\alpha a_0+\beta b_0)+\ldots+(\alpha a_k+\beta b_k)X^k+\ldots+\beta b_lX^l \Rightarrow$ degree $(\alpha f+\beta g)=l\leq n$

$\Rightarrow$ subspace

(ii) same for $\alpha f+\beta g=(\alpha a_0+\beta b_0)+\ldots+(\alpha a_n+\beta b_n)X^n \Rightarrow$ degree $(\alpha f+\beta g)\leq n \Rightarrow$ not a subspace

10. Show that the set of all solutions of a homogeneous system of two equations and two unknowns with real coefficients is a subspace of the real vector space $\mathbb{R}^2$.

$S=\{(a,b) \mid (a,b) \text{ system of sol}\}$

$\alpha(a,b)+\beta(c,d)=(\alpha a+\beta c, \alpha b+\beta d)$?

$V = A \oplus B$ if $V = A+B$ and $A \cap B = \{0\}$
$\forall v \in V \ \exists! s \in S, t \in T$ s.t. $v = s+t$

$\text{Ker} f = \{x \in \mathbb{R} \mid f(x) = 0\}$
$\text{im} f = \{f(x) \mid x \in \mathbb{R}\}$

Seminar 5

1. Determine the following generated subspaces:
(i) $< 1, X, X^2 >$ in the real vector space $\mathbb{R}[X]$.
(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in the real vector space $M_2(\mathbb{R})$.

2. Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces with a minimal number of generators.

3. Consider the following vectors in the real vector space $\mathbb{R}^3$:

$$a = (-2, 1, 3), b = (3, -2, -1), c = (1, -1, 2), d = (-5, 3, 4), e = (-9, 5, 10).$$

Show that $\langle a, b\rangle = \langle c, d, e\rangle$.

4. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that S and T are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

5. Let S and T be the set of all even functions and of all odd functions in $\mathbb{R}^\mathbb{R}$ respectively. Prove that S and T are subspaces of the real vector space $\mathbb{R}^\mathbb{R}$ and $\mathbb{R}^\mathbb{R} = S \oplus T$.

6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x + y, x - y),$$
$$g(x, y) = (2x - y, 4x - 2y),$$
$$h(x, y, z) = (x - y, y - z, z - x).$$

Show that $f, g \in End_\mathbb{R}(\mathbb{R}^2)$ and $h \in End_\mathbb{R}(\mathbb{R}^3)$.

7. Which ones of the following functions are endomorphisms of the real vector space $\mathbb{R}^2$:
(i) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (ax + by, cx + dy)$, where $a, b, c, d \in \mathbb{R}$;
(ii) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (a + x, b + y)$, where $a, b \in \mathbb{R}$?

8. Let $a \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x \cos a - y \sin a, x \sin a + y \cos a) .$$

Prove that $f \in End_\mathbb{R}(\mathbb{R}^2)$.

9. Determine the kernel and the image of the endomorphisms from Exercise **6.**

10. Let V be a vector space over K and $f \in End_K(V)$. Show that the set

$$S = \{x \in V \mid f(x) = x\}$$

of fixed points of f is a subspace of V.

1. Determine the following generated subspaces:
(i) $< 1, X, X^2 >$ in the real vector space $\mathbb{R}[X]$.
(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in the real vector space $M_2(\mathbb{R})$.

(i) $\langle 1, x, x^2 \rangle = \{ a + bx + cx^2 \mid a, b, c \in \mathbb{R} \} = \mathbb{R}_2[x]$

(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle = \left\{ a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\} = \mathcal{M}_2(\mathbb{R})$

2. Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces with a <u>minimal number of generators.</u>

(i) $A = \{ (x, y, z) \in \mathbb{R}^3 \mid x = 0 \} = (0, a, b) = a \cdot (0,1,0) + b(0,0,1) = \langle (0,1,0), (0,0,1) \rangle$

(ii) $B = \{ (x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0 \}$

$x = -(y+z) \Rightarrow (-(y+z), y, z) = (-y, y, 0) + (-z, 0, z) = \langle (-1,1,0), (-1,0,1) \rangle$

(iii) $C = \{ (x, y, z) \in \mathbb{R}^3 \mid x = y = z \} = (x, x, x) = \langle (1,1,1) \rangle$

3. Consider the following vectors in the real vector space $\mathbb{R}^3$:

$$a = (-2, 1, 3), b = (3, -2, -1), c = (1, -1, 2), d = (-5, 3, 4), e = (-9, 5, 10).$$

Show that $\langle a, b \rangle = \langle c, d, e \rangle$.

$$\begin{cases} c = a + b \\ d = a - b \\ e = 3a - b \end{cases}$$

4. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that S and T are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

$S = \{ (x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0 \} = (-y - z, y, z) = y(-1, 1, 0) + z(-1, 0, 1) = \langle (-1,1,0), (-1,0,1) \rangle$

$T = \langle (1,1,1) \rangle$

$S, T \leq_K \mathbb{R} \begin{cases} S, T \neq \emptyset \\ \forall k_1, k_2 \ \forall v_1, v_2 \Rightarrow k_1 v_1 + k_2 v_2 \in S, T \end{cases}$

To prove $\mathbb{R}^3 = S \oplus T$: $S+T = \mathbb{R}^3$ and $S \cap T = \{0_3\}$

$\forall v \in \mathbb{R}^3$ $\exists!$ $s \in S, t \in T$ s.t. $v = s+t \Leftrightarrow (v_1, v_2, v_3) = a s_1 + b s_2 + c \cdot t =$

$= a(-1,1,0) + b(-1,0,1) + c(1,1,1) \Rightarrow \begin{cases} v_1 = -a-b+c \\ v_2 = a+c \\ v_3 = b+c \end{cases}$

$\Rightarrow c = (v_1+v_2+v_3) \cdot \frac{1}{3} \Rightarrow b = \frac{-1}{3} v_1 - \frac{1}{3} v_2 + \frac{2}{3} v_3$

$a = \frac{-1}{3} v_1 + \frac{2}{3} v_2 - \frac{1}{3} v_3$ $\Rightarrow$ unique

$\to f(-x) = f(x)$ $\mapsto f(-x) = -f(x)$

5. Let S and T be the set of all even functions and of all odd functions in $\mathbb{R}^{\mathbb{R}}$ respectively. Prove that S and T are subspaces of the real vector space $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{R}} = S \oplus T$.

$S, T \leq_K \mathbb{R}^{\mathbb{R}}$ $\qquad$ $\mathbb{R}^{\mathbb{R}} = S \oplus T$

$S \neq \emptyset$ $\qquad$ $T \neq \emptyset$

$f, g \in S \Rightarrow (af + bg)(-x) = (af)(-x) + (bg)(-x) = -af(x) - bg(x) = -(af+bg)(x) \in S \leq \mathbb{R}^{\mathbb{R}}$

same for T

finish mine

6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x + y, x - y),$$

$$\to g(x, y) = (2x - y, 4x - 2y),$$

$$\to h(x, y, z) = (x - y, y - z, z - x).$$

$g \in End_{\mathbb{R}}(\mathbb{R}^2) \Leftrightarrow$

$g = \{ f : V \to V \mid f \text{ is } K\text{-linear} \}$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

$f(x,y) = (x+y, x-y)$

f - linear map $\Leftrightarrow \begin{cases} f(v_1+v_2) = f(v_1) + f(v_2) \\ f(kv) = k f(v) \end{cases}$

$f((x_1,y_1)+(x_2,y_2)) = f(x_1+x_2, y_1+y_2) = (x_1+x_2+y_1+y_2, x_1+x_2-y_1-y_2) = (x_1+y_1, x_1-y_1) +$

$+ (x_2+y_2, x_2-y_2) = f(x_1,y_1) + f(x_2,y_2)$

$f(k(x,y)) = f(kx, ky) = (k(x+y), k(x-y)) = k(x+y, x-y) = k f(x,y)$

7. Which ones of the following functions are endomorphisms of the real vector space $\mathbb{R}^2$:
(*i*) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (ax + by, cx + dy)$, where $a, b, c, d \in \mathbb{R}$;
(*ii*) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (a + x, b + y)$, where $a, b \in \mathbb{R}$?

endomorphism $f: V \to V$, K-lin. map
$f(v_1+v_2) = f(v_1)+f(v_2)$
$f(kv) = k(f(v))$

(ii) $g: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad g(x,y) = (a+x, b+y)$

is K-linear map ?

$$g(v_1+v_2) \stackrel{?}{=} g(v_1)+g(v_2)$$

$$g(v_1+v_2) = g((x_1,y_1)+(x_2,y_2)) = (a+x_1+x_2,\ b+x_1+x_2) = (a+x_1, b+y_1) + (x_2,y_2) =$$
$$= g(x_1,y_1) + (x_2,y_2) \text{ – Not.}$$

8. Let $a \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x \cos a - y \sin a, x \sin a + y \cos a).$$

Prove that $f \in End_{\mathbb{R}}(\mathbb{R}^2)$.

$$f(x,y) = (x\cos a - y\sin a,\ x\sin a + y\cos a)$$

$$f(x_1+x_2, y_1+y_2) = ((x_1+x_2)\cos a - (y_1+y_2)\sin a,\ (x_1+x_2)\sin a + (y_1+y_2)\cos a)$$

yes

$$f(kv) = k f(v)$$

$$f(kx, ky) = [k(x\cos a - y\sin a),\ k(x\sin a + y\cos a)] = k f(x,y) \to \text{True}$$

9. Determine the kernel and the image of the endomorphisms from Exercise **6**.

$$g(x, y) = (2x - y, 4x - 2y),$$

$\ker(g) = \{(x,y) \mid (2x-y, 4x-2y) = (0,0)\} \Rightarrow$ $\quad 2x - y = 0$
$2x = y$
let $x = a \in \mathbb{R} \Rightarrow y = 2a \in \mathbb{R}$

$\Rightarrow \ker(g) = \{(a, 2a) \mid a \in \mathbb{R}\} = \langle (1,2) \rangle$

$\operatorname{Im}(g) = \{(x,y) \mid x,y \in \mathbb{R}\} = \{(2a-b,\ 4a-2b)\} = \{(2a,4a)+(-b,-2b)\} = \{2a(1,2) - b(1,2)\}$
$= \langle (1,2) \rangle$

10. Let V be a vector space over K and $f \in End_K(V)$. Show that the set

$$S = \{x \in V \mid f(x) = x\}$$

of fixed points of f is a subspace of V.

$$\begin{cases} S \neq \emptyset \\ \forall\, a,b \in K,\ v_1,v_2 \in V \Rightarrow av_1 + bv_2 \in S \end{cases}$$

$S \neq \emptyset$ $\quad f(0) = 0$

$End_K(V) \Rightarrow \quad f(v_1+v_2) = v_1 + v_2 = f(v_1) + f(v_2)$ true

$f(k v_1) = k v_1 = k\, f(v_1)$ $\quad \Rightarrow$ subspace

V K-vector space

$v_1, v_2, \dots, v_n \in V$ linearly independent if $\left.\begin{matrix}\forall \alpha_1, \dots, \alpha_n \in K \\ \alpha_1 v_1 + \dots + \alpha_n v_n = 0\end{matrix}\right\} \Rightarrow \alpha_1 = \dots = \alpha_n = 0$

# of lin. ind. vectors is the rank $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$\Rightarrow$ lin. indep. $\Leftrightarrow$ rank$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = n$

Seminar 6

1. Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:
(i) v_1, v_2, v_3 are linearly dependent and determine a dependence relationship.
(ii) v_1, v_2 are linearly independent.

2. Prove that the following vectors are linearly independent:
(i) $v_1 = (1, 0, 2)$, $v_2 = (-1, 2, 1)$, $v_3 = (3, 1, 1)$ in ${}_{\mathbb{R}}\mathbb{R}^3$.
(ii) $v_1 = (1, 2, 3, 4)$, $v_2 = (2, 3, 4, 1)$, $v_3 = (3, 4, 1, 2)$, $v_4 = (4, 1, 2, 3)$ in ${}_{\mathbb{R}}\mathbb{R}^4$.

3. Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly independent.

4. Let $v_1 = (1, -2, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (0, a, 1, 2)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^4$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly dependent.

5. Let $v_1 = (1, 1, 0)$, $v_2 = (-1, 0, 2)$, $v_3 = (1, 1, 1)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$.
(i) Show that the list (v_1, v_2, v_3) is a basis of the real vector space $\mathbb{R}^3$.
(ii) Express the vectors of the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$ as a linear combination of the vectors v_1, v_2 and v_3.
(iii) Determine the coordinates of $u = (1, -1, 2)$ in each of the two bases.

6. Let $n \in \mathbb{N}^*$. Show that the vectors

$$v_1 = (1, \dots, 1, 1), v_2 = (1, \dots, 1, 2), v_3 = (1, \dots, 1, 2, 3), \dots, v_n = (1, 2, \dots, n-1, n)$$

form a basis of the real vector space $\mathbb{R}^n$ and write the coordinates of a vector $(x_1, \dots, x_n)$ in this basis.

7. Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the lists (E_1, E_2, E_3, E_4) and (A_1, A_2, A_3, A_4) are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in each of the two bases.

8. Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X - a, (X - a)^2)$ $(a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial $f = a_0 + a_1 X + a_2 X^2 \in \mathbb{R}_2[X]$ in each basis.

9. Determine the number of bases of the vector space $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$.

10. Determine the number of elements of the general linear group $(GL_3(\mathbb{Z}_2), \cdot)$ of invertible 3×3-matrices over $\mathbb{Z}_2$.

1. Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:

(i) v_1, v_2, v_3 are linearly <u>dependent</u> and determine a dependence relationship.

(ii) v_1, v_2 are linearly independent.

(i) $$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0 \Rightarrow \begin{cases} a_1 + 2a_2 + a_3 = 0 \Rightarrow \\ -a_1 + a_2 + 5a_3 = 0 \Rightarrow a_1 = 3a_3 \\ a_2 + 2a_3 = 0 \Rightarrow a_2 = -2a_3 \end{cases}$$

$$\Rightarrow S = \{(3a, -2a, a) \mid a \in \mathbb{R}\} \Rightarrow \text{lin. dep.}$$

$$\text{if } a = 1 \Rightarrow (3, -2, 1) \in S$$

(ii) $$a(1,-1,0) + b(2,1,1) = (0,0,0)$$

$$\left.\begin{array}{l} a + 2b = 0 \\ -a + b = 0 \\ b = 0 \end{array}\right\} \Rightarrow a = 0 \Rightarrow \text{lin. ind.}$$

2. Prove that the following vectors are linearly independent:

(i) $v_1 = (1, 0, 2)$, $v_2 = (-1, 2, 1)$, $v_3 = (3, 1, 1)$ in $_{\mathbb{R}}\mathbb{R}^3$.

(ii) $v_1 = (1, 2, 3, 4)$, $v_2 = (2, 3, 4, 1)$, $v_3 = (3, 4, 1, 2)$, $v_4 = (4, 1, 2, 3)$ in $_{\mathbb{R}}\mathbb{R}^4$.

(i) $$\begin{vmatrix} 1 & 0 & 2 \\ -1 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = \cancel{2} \cancel{-2} + 0 - 12 - 1 + 0 = -13 \neq 0 \Rightarrow \text{indep.}$$

(ii) det.

3. Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in $_{\mathbb{R}}\mathbb{R}^3$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly independent.

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = (0,0,0)$$

$$\begin{cases} \alpha_1 + a\alpha_2 + \alpha_3 = 0 \\ a\alpha_1 + \alpha_2 \\ \alpha_2 + a\alpha_3 = 0 \end{cases} \Rightarrow A = \begin{pmatrix} 1 & a & 1 \\ a & 1 & 0 \\ 0 & 1 & a \end{pmatrix} = a(2 - a^2) \neq 0 \begin{cases} a \neq 0 \\ 2 - a^2 \neq 0 \end{cases}$$

$$\Rightarrow a \neq \pm\sqrt{2}$$

$V \to K$ vector space

$B = (v_1, \dots, v_n)$

B basis for V $\Leftrightarrow$ (i) $\langle v_1, \dots, v_n \rangle = V$ system of gen. $\Leftrightarrow \forall v \in V\ \exists!\ \alpha_1, \dots \alpha_n \in K$

(ii) $v_1, \dots, v_n$ lin. indep.

$u = \alpha_1 v_1 + \dots + \alpha_n v_n$

5. Let $v_1 = (1,1,0)$, $v_2 = (-1,0,2)$, $v_3 = (1,1,1)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$.
(i) Show that the list (v_1, v_2, v_3) is a basis of the real vector space $\mathbb{R}^3$.
(ii) Express the vectors of the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$ as a linear combination of the vectors v_1, v_2 and v_3.
(iii) Determine the coordinates of $u = (1,-1,2)$ in each of the two bases.

(i) basis $\Leftrightarrow$ v_1, v_2, v_3 – lin. indep.

$\langle v_1, v_2, v_3 \rangle$ – system

$$\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \text{lin. ind.}$$

let $u = (u_1, u_2, u_3) \in \mathbb{R}^3\ \exists!\ a_1, a_2\ a_3 \in \mathbb{R}$

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = u$$

$$\begin{cases} a_1 - a_2 + a_3 = u_1 & \Rightarrow a_2 = u_2 - u_1 \\ a_1 + a_3 = u_2 & \Rightarrow a_1 = u_2 - a_3 = u_2 - u_3 + 2u_2 - 2u_1 \\ 2a_2 + a_3 = u_3 & \Rightarrow a_3 = u_3 - 2a_2 = u_3 - 2u_2 + 2u_1 \end{cases}$$

$\Rightarrow$ unique α $\Rightarrow$ basis

(ii) $a_1 v_1 + a_2 v_2 + a_3 v_3 = e_1$

$$\begin{cases} a_1 - a_2 + a_3 = 1 & \Rightarrow -a_2 = 1 \Rightarrow a = -1 \\ a_1 + a_3 = 0 & \Rightarrow a_1 = -a_3 \Rightarrow a_1 = -2 \\ 2a_2 + a_3 = 0 & \Rightarrow 2a_2 = -a_3 \Rightarrow a_1 = 2a_2 \\ & \Rightarrow a_3 = 2 \end{cases}$$

...

iii) $u = (1,-1,2)$

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = (1,-1,2)$$

$$\begin{cases} a_1 - a_2 + a_3 = 1 \\ a_1 \quad + a_3 = -1 \\ 2a_2 + a_3 = 2 \end{cases} \Rightarrow \begin{matrix} a_1 = -7 \\ a_2 = -2 \\ a_3 = 6 \end{matrix}$$

7. Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the lists (E_1, E_2, E_3, E_4) and (A_1, A_2, A_3, A_4) are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in each of the two bases.

basis $\Leftrightarrow \begin{cases} \langle v_1, \dots v_n \rangle \text{ system of gen.} \\ v_1, \dots v_n \text{ lin. indep.} \end{cases}$

(E_1, E_2, E_3, E_4) - basis of $M_2(\mathbb{R})$

coord.: $a \cdot E_1 + b \cdot E_2 + c \cdot E_3 + d \cdot E_4 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow (a,b,c,d) = (2,1,1,0)$

Let $a,b,c,a \in \mathbb{R}$

$a \cdot A_1 + b \cdot A_2 + c \cdot A_3 + d \cdot A_4 = M \qquad M \in M_2(\mathbb{R})$

$$\begin{bmatrix} a+b+c+d & b+c+d \\ c+d & a+d \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

$$\begin{cases} a+b+c+d = x \\ b+c+d = y \\ c+d = z \\ a \quad + d = t \end{cases} \begin{matrix} \Rightarrow a = x-y \\ \Rightarrow b = y-z \\ \Rightarrow c = t-z+x-y \\ d = t-x+y \end{matrix}$$

unique sol. $\Rightarrow \beta$-basis

$$\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_B = \begin{pmatrix} x-y \\ y-z \\ t-z+x-y \\ t-x+y \end{pmatrix} = \begin{pmatrix} 2-1 \\ 1-1 \\ 0-1+2-1 \\ 0-2+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$

8. Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X - a, (X - a)^2)$ $(a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial $f = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]$ in each basis.

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Let $\alpha_1, \alpha_2 \alpha_3 \in \mathbb{R}$ $\quad u = \alpha_1 + \alpha_2 \cdot X + \alpha_3 X^2 \Rightarrow (1, X, X^2)$ - basis of $\mathbb{R}_2[X]$

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

$$u = \alpha_1 + \alpha_2 (X-a) + \alpha_3 (X-a)^2 =$$
$$= \alpha_1 + \alpha_2 X - \alpha_2 a + \alpha_3 X^2 - 2\alpha_3 a X + \alpha_3 a^2 =$$
$$= (\alpha_1 - \alpha_2 a + \alpha_3 a^2) + (\alpha_2 - 2\alpha_3 a) X + \alpha_3 X^2 =$$

$$\begin{cases} \alpha_1 - \alpha_2 a + \alpha_3 a^2 = a_0 \Rightarrow \alpha_1 = a_0 - a_2 a^2 + (a_1 + 2a_2 a) \cdot a \\ \alpha_2 - 2\alpha_3 a = a_1 \Rightarrow \alpha_2 = a_1 + 2\alpha_3 \cdot a = a_1 - 2 a_2 \cdot a \\ \alpha_3 = a_2 \end{cases}$$

unique α $\Rightarrow$ Basis

$$[f]_B = \begin{pmatrix} a_0 + a \cdot a_1 + a^2 a_2 \\ a_1 + 2a_2 a \\ a_2 \end{pmatrix}$$

the dimention is given by the number of vectors in the basis

$f: V \to V$ $\quad \dim(V) = \dim(\ker(f)) + \dim(\operatorname{Im}(f))$

$A \subseteq B$ $\quad \dim(A) \leq \dim(B)$

$\bar{S}$ complement of S, $S \oplus \bar{S} = V$

Seminar 7

1. Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}\,.$$

2. Let K be a field and $S = \{(x_1, \ldots, x_n) \in K^n \mid x_1 + \cdots + x_n = 0\}$.
(*i*) Prove that S is a subspace of the canonical vector space K^n over K.
(*ii*) Determine a basis and the dimension of S.

3. Determine a basis and the dimensions of the vector spaces $\mathbb{C}$ over $\mathbb{C}$ and $\mathbb{C}$ over $\mathbb{R}$. Prove that the set $\{1, i\}$ is linearly dependent in the vector space $\mathbb{C}$ over $\mathbb{C}$ and linearly independent in the vector space $\mathbb{C}$ over $\mathbb{R}$.

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (y, -x)$. Prove that f is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

5. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x, y, z) = (-y + 5z, x, y - 5z)$. Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

6. Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

7. Determine a complement for the following subspaces:
(*i*) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ in the real vector space $\mathbb{R}^3$;
(*ii*) $B = \{aX + bX^3 \mid a, b \in \mathbb{R}\}$ in the real vector space $\mathbb{R}_3[X]$.

8. Let V be a vector space over K and let S, T and U be subspaces of V such that $dim(S \cap U) = dim(T \cap U)$ and $dim(S + U) = dim(T + U)$. Prove that if $S \subseteq T$, then $S = T$.

9. Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}\,,$$

$$T =< (0, 1, 1), (1, 1, 0) >$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

10. Determine the dimensions of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $M_2(\mathbb{R})$, where

$$S = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle, \qquad T = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle.$$

1. Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

$A = \{(x,y,z) \in \mathbb{R}^3 \mid z=0\} = \{(x,y,0) \mid x,y \in \mathbb{R}\} = \{x(1,0,0) + y(0,1,1) \mid x,y \in \mathbb{R}\} =$
$= \langle (1,0,0),(0,1,0) \rangle$ – basis $\Leftrightarrow \alpha(1,0,0) + \beta(0,1,0) = (0,0,0)$ true $\Leftrightarrow \alpha = \beta = 0$
$\Rightarrow$ basis
$\dim(A) = 2$ add 1 more . ex: $(0,1,1)$

$B = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\} = \{x(1,0,-1) + y(0,1,-1) \mid x,y \in \mathbb{R}\} = \langle (1,0,-1),(0,1,-1) \rangle$
– basis $\Rightarrow \dim(B) = 2$

$C = \{(x,y,z) \in \mathbb{R}^3 \mid x=y=z\} = \langle (1,1,1) \rangle$ (has one vect $\Rightarrow$ ind.)
$\Rightarrow \dim(C) = 1$

2. Let K be a field and $S = \{(x_1, \ldots, x_n) \in K^n \mid x_1 + \cdots + x_n = 0\}$.
(i) Prove that S is a subspace of the canonical vector space K^n over K.
(ii) Determine a basis and the dimension of S.

$S = \{(x_1, \ldots, x_n) \in K^n \mid x_1 + \ldots + x_n = 0\}$

(i) $S \leq_K K^n \Leftrightarrow \begin{cases} S \neq \emptyset & ((0,\ldots,0) \in S \neq \emptyset) \\ \left.\begin{array}{l} \forall a,b \in K \\ \forall m,n \in S \end{array}\right\} \Rightarrow am + bn \in S \end{cases}$

$am + bn = a(m_1, \ldots, m_n) + b(n_1, \ldots, n_n) = (am_1 + bn_1, \ldots, am_n + bn_n) \in S$
$\Leftrightarrow am_1 + bn_1 + \ldots + am_n + bn_n = 0$ true $\Rightarrow S \leq_K K^n$

(ii) $x_n = -x_1 - \ldots - x_{n-1}$

$\Rightarrow S = \{(x_1, \ldots, x_{n-1}, -x_1 - \ldots - x_{n-1}) \mid x_1 + \ldots + x_n = 0\} = \langle (1, \ldots, -1), (0,1,\ldots,-1) \ldots$
$(0,0,\ldots,1,-1) \rangle$

$\Rightarrow \dim S = n-1$

3. Determine a basis and the dimensions of the vector spaces $\mathbb{C}$ over $\mathbb{C}$ and $\mathbb{C}$ over $\mathbb{R}$. Prove that the set $\{1, i\}$ is linearly dependent in the vector space $\mathbb{C}$ over $\mathbb{C}$ and linearly independent in the vector space $\mathbb{C}$ over $\mathbb{R}$.

$$\left.\begin{array}{l} (\mathbb{C},+) - \text{abelian group} \\ \text{"}\cdot\text{" external op} \\ (k_1+k_2)\cdot z = k_1\cdot z + k_2\cdot z \\ k(z_1+z_2) = kz_1 + kz_2 \\ (k_1\cdot k_2)\cdot z = k_1(k_2\cdot z) \\ 1\cdot z = z \end{array}\right\} \Rightarrow \text{v.s. over } \mathbb{C} \text{ and } \mathbb{R}$$

$\forall z \in \mathbb{C}$, $\exists a,b \in \mathbb{R}$ s.t $z = a\cdot 1 + b\cdot i \Rightarrow \mathbb{C} = \langle 1, i\rangle$ basis $\Rightarrow \dim \mathbb{C} = 2$

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (y, -x)$. Prove that f is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

$f \to$ R-linear map if $f(ax+by) = af(x)+bf(y)$ $\forall a,b \in \mathbb{R}$, $x,y \in \mathbb{R}^3$

$f(a(x_1,x_2,x_3)+b(y_1,y_2,y_3)) = f(ax_1+by_1, ax_2+by_2, ax_3+by_3) = (ax_2+by_2, -ax_1-by_1) =$

$= a(x_2,-x_1)+b(y_2,-y_1) = af(x)+bf(y)$ - true

$\ker f = \{(x,y,z) \in \mathbb{R}^3 \mid (y,-x) = (0,0)\} = \{(0,0,z) \mid z \in \mathbb{R}\} = \langle 0,0,1\rangle \Rightarrow \dim \operatorname{Ker} f = 1$

$\operatorname{Im} f = \{(y,-x) \in \mathbb{R}^2 \mid f(x,y,z) = (y,-x)\} = \{(y,0)+(0,-x)\} = \langle (1,0),(0,-1)\rangle$

$\Rightarrow \dim \operatorname{Im} f = 2$

5. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x, y, z) = (-y + 5z, x, y - 5z)$. Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

$\ker f = \{(x,y,z) \in \mathbb{R}^3 \mid (-y+5z, x, y-5z) = (0,0,0)\}$

$$\left.\begin{array}{l} -y+5z=0 \\ x=0 \\ y-5z=0 \Rightarrow y=5z \end{array}\right\} \Rightarrow \ker f = \langle (0,5,1)\rangle \Rightarrow \dim \ker f = 1$$

$\operatorname{Im} f = \{ (-y+5z, x, y-5z) \mid f(x,y,z) = y(-1,0,1) + x(0,1,0) + z(5,0,-5) =$

$\langle (-1,0,1), (0,1,0), (5,0,-5) \rangle$

but $(5,0,-5) = -5 \cdot (-1,0,1) \Rightarrow$

$\Rightarrow \operatorname{Im} f = \langle (-1,0,1), (0,1,0) \rangle$ basis

$\dim \operatorname{Im} f = 2$

7. Determine a complement for the following subspaces:
(*i*) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$ in the real vector space $\mathbb{R}^3$;
(*ii*) $B = \{aX + bX^3 \mid a, b \in \mathbb{R}\}$ in the real vector space $\mathbb{R}_3[X]$.

(i) $A = \{ (x,y,z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0 \}$ in $\mathbb{R}^3$

$x + 2y + 3z = 0 \Rightarrow x = -2y - 3z$

$A = \{ y(-2,1,0) + z(-3,0,1) \mid y, z \in \mathbb{R} \} =$

$= \langle (-2,1,0), (-3,0,1) \rangle$ - basis of A

$\left.\begin{array}{l} \dim A = 2 \\ \dim \mathbb{R}^3 = 3 \end{array}\right\} \Rightarrow$ we need to add 1 more vector

let $v = \mathbb{R}^3 \setminus \langle (-2,1,0), (-3,0,1) \rangle$

$v = (0,0,1)$

Suppose $v \in A \Rightarrow \exists \alpha$ s.t. $\alpha \cdot v = (-2,1,0)$ or $\alpha \cdot v = (-3,0,1)$

$\left.\begin{array}{l} (0,0,\alpha) = (-2,1,0) \Rightarrow -2 = 0 \text{ false} \\ (0,0,\alpha) = (-3,0,1) \Rightarrow -3 = 0 \text{ false} \end{array}\right\} \Rightarrow$ lin. ind. $\Rightarrow$

$\Rightarrow \langle (-2,1,0), (-3,0,1), (0,0,1) \rangle$ basis of A

(ii) $B = \{ aX + bX^3 \mid a, b \in \mathbb{R} \}$ in $\mathbb{R}_3[X]$ }

$B = \langle X, X^3 \rangle$

$\left.\begin{array}{l} \dim B = 2 \\ \dim \mathbb{R}_3[X] = 4 \end{array}\right\} \Rightarrow$ add 2

$\Rightarrow \langle 1, X, X^2, X^3 \rangle$ basis of B, $\overline{B} = \langle 1, X^2 \rangle$

8. Let V be a vector space over K and let S, T and U be subspaces of V such that $dim(S \cap U) = dim(T \cap U)$ and $dim(S + U) = dim(T + U)$. Prove that if $S \subseteq T$, then $S = T$.

$S, T, U \leq_K V$

2^{nd} dimension th.: $\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$

$S+T = \langle S \cup T \rangle$

$\dim S + \dim T = \dim(S \cap T) + \dim(S+T)$

$S \subseteq T$

$\dim S \leq \dim T$

$\dim U + \dim T = \dim(T \cap U) + \dim(T+U)$

$\Rightarrow \dim S = \dim T \qquad \Rightarrow S = T$

9. Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\},$$

$$T =< (0, 1, 1), (1, 1, 0) >$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

$S = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\} = \langle (0,1,0), (0,0,1) \rangle$

$T = \langle (0,1,1), (1,1,0) \rangle = \{(x,y,z) \in \mathbb{R}^3 \mid x - y + z = 0\}$

$S \cap T = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0 \text{ and } x - y + z = 0\}$

$= \{(0,y,z) \in \mathbb{R}^3 \mid y = z\} = \langle (0,1,1) \rangle \quad \dim(S \cap T) = 1$

$S+T \subseteq \mathbb{R}^3, \ \dim(S+T) \leq \dim \mathbb{R}^3 = 3$

$\dim(S) + \dim(T) = \dim(S+T) + \dim(S \cap T)$

$2 + 2 = \dim(S+T) + 1 \Rightarrow \dim(S+T) = 3 = \dim \mathbb{R}^3$

$\Rightarrow S+T = \mathbb{R}^3$

10. Determine the dimensions of the subspaces S, T, $S+T$ and $S\cap T$ of the real vector space $M_2(\mathbb{R})$, where

$$S=\left\langle \underbrace{\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix}}_{\text{ind.}\Rightarrow\ \dim(S)=2}\right\rangle,\qquad T=\left\langle \underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix}}_{\text{ind}\Rightarrow\ \dim(T)=2}\right\rangle.$$

$$S+T=\left\langle \begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix}\right\rangle$$

$$a\begin{pmatrix}1&1\\0&0\end{pmatrix}+b\begin{pmatrix}1&0\\1&1\end{pmatrix}+c\begin{pmatrix}0&1\\1&0\end{pmatrix}+d\begin{pmatrix}0&0\\1&1\end{pmatrix}=\begin{pmatrix}x&y\\z&t\end{pmatrix}$$

$$\begin{cases}a+b=x \Rightarrow b=x-y+z-t\\ a+c=y \Rightarrow a=y-z+t\\ b+c+d=z \Rightarrow c=z-t\\ b+d=t \Rightarrow d=t-x+y-z+t\\ \qquad\qquad\qquad\quad =-x+y-z+2t\end{cases}$$

$=0 \Rightarrow a=b=c=d=0 \Rightarrow$ lin. ind. $\Rightarrow \dim(S+T)=4$

$\dim(S)+\dim(T)=\dim(S+T)+\dim(S\cap T) \Rightarrow \dim(S\cap T)=4$

__Extra__ $f:\mathbb{R}^2\to\mathbb{R}^2$ lin. map $\left(\text{homomorphism}: \begin{cases} f(v_1+v_2)=f(v_1)+f(v_2)\\ f(kv)=kf(v)\end{cases}\right)$

$f(1,2)=(3,1)$

$f(4,3)=(2,5)$

Find (x,y) $\forall (x,y)\in\mathbb{R}^2$

$\left.\begin{array}{l}(1,2) \text{ and } (4,3) \text{ - lin. ind}\\ \dim\mathbb{R}^2=2\end{array}\right\}\Rightarrow \langle(1,2),(4,3)\rangle$ basis of $\mathbb{R}^2$

$\forall (x,y)\in\mathbb{R}^2 \quad \exists\, \alpha,\beta\in\mathbb{R}: \quad (x,y)=\alpha(1,2)+\beta(4,3)$

$f(x,y)=f(\alpha(1,2)+\beta(4,3))=\alpha f(1,2)+\beta f(4,3)=\alpha(3,1)+\beta(2,5)$

$$\cancel{\Rightarrow \begin{cases} x=3\alpha+2\beta \Rightarrow x=3y-15\beta+2\beta=3y-13\beta \Rightarrow \beta=\dfrac{3y-x}{13}\\ y=\alpha+5\beta \Rightarrow \alpha=y-5\beta=\dfrac{13y-15y+5x}{13}=\dfrac{5x-2y}{13}\end{cases}}$$

$$(x,y) = \alpha(1,2) + \beta(4,3)$$

$$\begin{cases} x = \alpha + 4\beta \Rightarrow \alpha = x - 4\beta \Rightarrow \alpha = \frac{x - 8x + 4y}{5} = \frac{-7x+4y}{5} \\ y = 2\alpha + 3\beta \Rightarrow y = 2x - 5\beta \Rightarrow \beta = \frac{2x-y}{5} \end{cases}$$

$$f(x,y) = \left(3 \cdot \frac{-7x+4y}{5} + \frac{2x-y}{5},\ 2 \cdot \frac{-7x+4y}{5} + 2x - y\right)$$