Introduction
Block *Equilibrium Modeling* – lecture 1

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Ruud Egging-Bratseth

Prof. in Managerial Economics
- Master in Business Econometrics – Free Univ. Amsterdam
- ORTEC Consultants, Netherlands
- Energy Research Centre of the Netherlands
- PhD in Civil and Environmental Engineering, Univ. of Maryland, USA
- In Norway since Jan 2011 (Jeg snakker norsk)
- Affiliations: SINTEF Industry; German Institute for Economic Research: DIW Berlin
- Coordinator dual master's *Sustainable Energy Systems and Management*
- Vice-chair Gemini Centre *Economic Modeling and Analysis*

Research areas
- European and global energy markets
- Decision making under uncertainty
- Value chain optimization (biofuels, CO2, LNG)
- Transition to a low-carbon energy system and society
- Managing large shares of fluctuating renewable energy supply
- Smart neighborhoods and smart grids

Main teaching
- Industrial Economics Analysis
- Experts in Team "Renewable Energy Management"
- Energy Markets and Policy
- Production and Network Economics

Outline: three lectures

- Wk 6 - Lecture 1 Equilibrium modeling
  - Introduction, motivation and preliminaries
  - Multi-variable optimization, Lagrangian multipliers and Karush Kuhn Tucker conditions
  - Social welfare, perfect competition, monopoly, Bertrand & Cournot equilibrium problems
- Wk 7 - Lecture 2 Network modeling
  - Transportation problems
  - Facility location problems
- Wk 8 - Lecture 3 Markets with transport networks
  - Equilibrium problems with embedded transport networks
    - commodity markets relying on underlying network to transport goods from producers to consumers
  - Combines concepts of the previous two lectures

INTRODUCTION, MOTIVATION & PRELIMINARIES

Feedback from last years (2018)

- Many students need a refresher on the microeconomic foundations (students on exchange may not have learned about oligopolistic markets)
- Not clear for all why market power is relevant and needs attention in analysis
- Some students need time to get accustomed to my teaching style
- Applications derived from energy markets perceived to favor energy & environment specialization
- Derivations and illustrations on blackboard too quick
- Partial handouts make it hard to take notes

Feedback from last years (2019)

- my teaching style – see previous
- Micro foundations all but gone for most students – see previous
- Post all slides in advance
- GAMS: too quickly - confusing
- Mention “Game theory”
- Introduce notation in each example
- One person: clearly separate math and examples

Changes to the lectures

- 2019
  - More motivation "market power"
  - More attention microeconomic foundations and market structures such as oligopoly variants
  - Keep examples general
  - Write slower on blackboard
  - Post extensive handouts ahead of class
- 2020
  - Two → Three lectures
  - Shorter motivation / more Micro foundations
  - Separate intro of multi-agent problems and network aspects
  - Revised presentation of KKT-condition derivation to be less technical
  - Smaller, simpler stylized example problems
  - Implementations in both GAMS and XPRESS
  - (More) focus practical skills & implementations in assignments

Learning Objectives
Block Equilibrium Modeling

- Understand why & when equilibrium models are useful
- Develop & solve small-scale equilibrium problems by hand
- Implement & solve small-scale equilibrium problems using GAMS and XPRESS
- Have an intuition & able to interpret equations & results
- This block contributes to most objectives, knowledge & skills listed for TIØ4285, focusing on value chains & networks:
  - Explain difference operations research models and economic models and how they supplement each other.
  - Modeling and economic analysis in industrial value chains
  - Evaluate effects of price structures and incentives.
  - (Evaluate how challenges like uncertainty affect decisions)

WHAT IS MARKET POWER?
AND WHY DOES IT MATTER?

Market power

- "The ability – of either a seller or a buyer – to affect the price of a good"
  - Pindyck and Rubinfeld

Organization for Economic Cooperation and Development (OECD)

- The ability of a firm (or group of firms) to raise and maintain price above the level that would prevail under competition is referred to as market or monopoly power.

- The exercise of market power leads to reduced output and loss of economic welfare.

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Google

European Commission - Press release

Antitrust: Commission fines Google €4.34 billion for illegal practices regarding Android mobile devices to strengthen dominance of Google's search engine

Brussels, 18 July 2018

The European Commission has fined Google €4.34 billion for breaching EU antitrust rules. Since 2011, Google has imposed illegal restrictions on Android device manufacturers and mobile network operators to cement its dominant position in general internet search.

Prevented market access of competitors
Android as a vehicle to cement the dominance of its search engine.
practices have denied rivals chance to innovate and compete on the merits.
denied European consumers the benefits of effective competition in the important mobile sphere

In particular, Google:

- has required manufacturers to pre-install the Google Search app and browser app (Chrome), as a condition for licensing Google's app store (the Play Store);
- made payments to certain large manufacturers and mobile network operators on condition that they exclusively pre-installed the Google Search app on their devices; and
- has prevented manufacturers wishing to pre-install Google apps from selling even a single smart mobile device running on alternative versions of Android that were not approved by Google (so-

http://europa.eu/rapid/press-release_IP-18-4581_en.htm

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Mastercard

Mastercard ***artificially raised the costs of card payments,*** *harming consumers and retailers in the EU.*

European Commission - Press release

Antitrust: Commission fines Mastercard €570 million for obstructing merchants' access to cross-border card payment services

Brussels, 22 January 2019

The European Commission has fined the card scheme Mastercard €570 566 000 for limiting the possibility for merchants to benefit from better conditions offered by banks established elsewhere in the Single Market, in breach of EU antitrust rules.

Commissioner Margrethe **Vestager**, in charge of competition policy, said: *"European consumers use payment cards every day, when they buy food or clothes or make purchases online. By preventing merchants from shopping around for better conditions offered by banks in other Member States, Mastercard's rules artificially raised the costs of card payments, harming consumers and retailers in the EU."*

Mastercard is the second largest card scheme in the European Economic Area (EEA) in terms of consumer card issuing and value of

http://europa.eu/rapid/press-release_IP-19-582_en.htm

MAN, Volvo/Renault, Daimler, Iveco, DAF

European Commission - Press release

Antitrust: Commission fines truck producers € 2.93 billion for participating in a cartel

Brussels, 19 July 2016

The European Commission has found that MAN, Volvo/Renault, Daimler, Iveco, and DAF broke EU antitrust rules. These truck makers colluded for 14 years on truck pricing and on passing on the costs of compliance with stricter emission rules. The

truck manufacturers in the cartel produce more than 90% of medium and heavy trucks sold in Europe. These trucks account for around 75% of inland transport of goods in Europe and play a vital role in the European economy.

- **coordinating prices at "gross list" level** for medium and heavy trucks in the European Economic Area (EEA=EØS)
- **timing for the introduction of emission technologies** for medium and heavy trucks to comply with the increasingly strict European emissions standards (Euro III- VI)
- **passing on to customers of the costs for the emissions technologies** required to comply with Euro III-VI

http://europa.eu/rapid/press-release_IP-16-2582_en.htm

What are consequences of market power exertion?

- Consumers pay prices higher than marginal costs – *sketch this*
  - suppliers could provide additional units and still have (at least) their marginal costs covered.
- Unfair: surplus transfer from consumers to suppliers
- Inefficient: combined surplus producers and consumers is not maximized (*Dead Weight Loss*)

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MARKETS

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Markets

- Place or institution in which buyers and sellers of a good or asset meet (Oxford dictionary)
- Equilibrium: supply equals demand at a market price
- Characteristics: number and type buyers, number and type sellers, characteristics of goods and production processes, information exchange, …..

Market structures

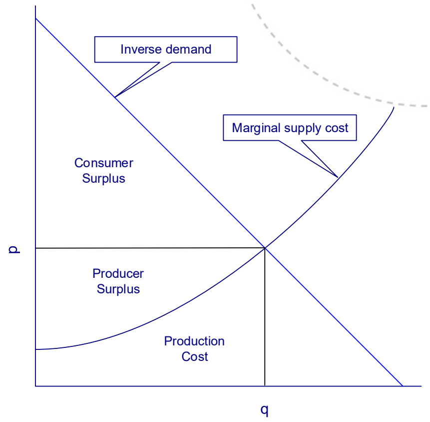
Perfect Competition
- ✓ *Homogenous goods*
- ✓ *No entry barriers*
- ✓ *No external effects*
- ✓ *Transparency*
- ✓ *All agents price takers*

Main market structures	Price taker(s)	Homogeneous products	Entry barriers
Perfect competition	Y	Y	N
Monopoly	N	…	…
Cournot oligopoly	N	…	…
Bertrand oligopoly	…	…	…
Dominant firm with fringe	…	…	…
Stackelberg oligopoly	…	…	…
Et cetera	…	…	…

Social welfare (maximization)

Ruud: draw & illustrate

- Demand
- Supply
- What is maximized?
- Consumer surplus
- Supplier profits
- MC=MR
- Efficiency
- Dead weight loss



Perfectly competitive markets

- Individual firms are price takers
- At the market level the price is defined via the inverse demand curve

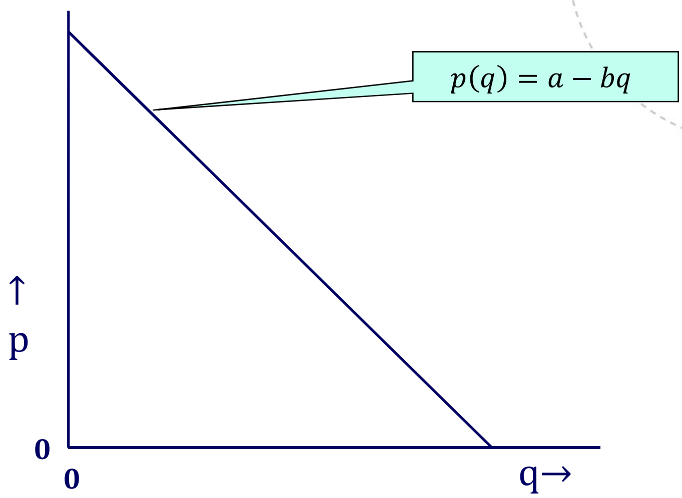
Perfectly competitive markets: profit maximization

- In markets with many suppliers, each takes the market price as given: $p = \hat{p}$
- Revenues from sales: $R(q) = \hat{p} \cdot q$
- Production cost: $c(q) = c \cdot q + d \cdot q^2$
- Profit $z(q) = \hat{p} \cdot q - c(q)$
- Maximizing profit…
  - find maximum of a quadratic function…
  - Solve …
  - Also show MC=MR

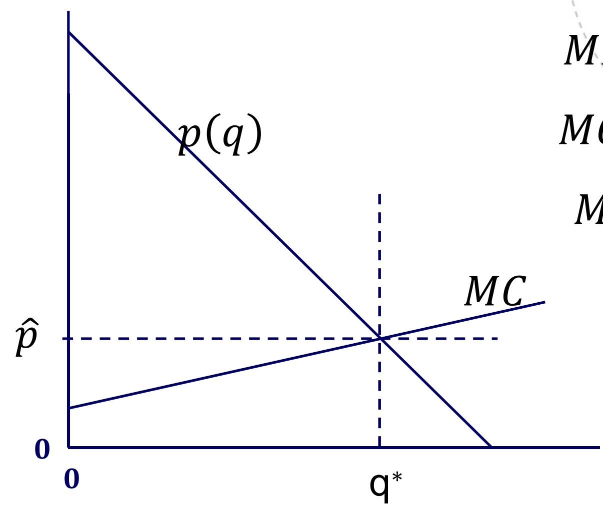
But …

- Generally people will buy less of a product, when price increases.
- What does that mean for the price-demand relation $p(q)$?
- Inverse demand curve

(Inverse) demand curve



Competitive market outcome:



$$MR = p(q)$$

$$MC = c'(q)$$

$$MC = MR$$

Here: c(q) and MC at the sector level

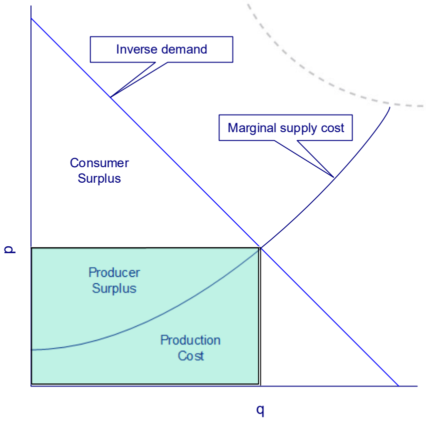
Competitive = efficient = socially optimal

- The outcome of a perfectly competitive market maximizes social welfare (sum of all surpluses) and is (therefore!) efficient.
- The individual profit maximization of agents in a perfectly-competitive environment (*agents are price takers*) will result in a socially optimal allocation.

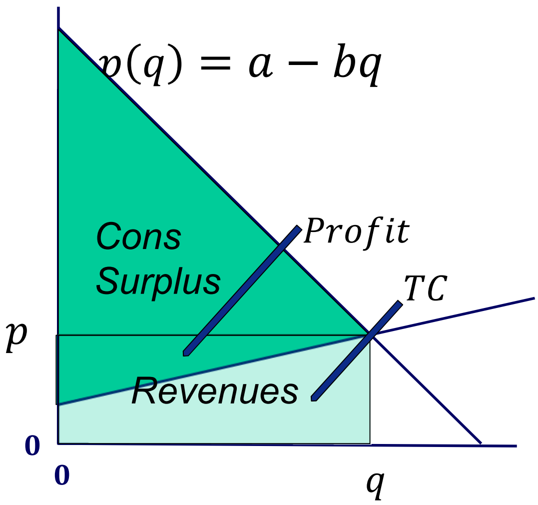
- That social welfare maximization and perfectly competition result in the same market equilibria will become more intuitive when we solve both problem types both as optimization and as equilibrium problems

Social welfare: SW=CS+PS

PS = profit =
revenues - costs



Competitive market outcome:



$$SW{=}TS{=}CS{+}PS$$
$$={CS}+{REV}-{TC}$$
$$CS = \frac{1}{2}(a - bq - p)q$$
$$= \frac{1}{2}bq^2$$
$$REV = pq$$
$$TC = (c + dq)q$$

Maximize total surplus =CS + PS = CS + profit

Competitive market - GAMS

$$
\begin{aligned}
SW &= CS + REV - TC \\
&= \frac{1}{2} bq^2 + pq - (c + dq)q \\
&= \frac{1}{2} bq^2 + (a - bq)q - (c + dq)q
\end{aligned}
$$

Use

$$a = 20, b = 1, c = 2, d = 0.5$$

$$\max \mathrm{SW}(q)\ s.t.\, q \geq 0$$

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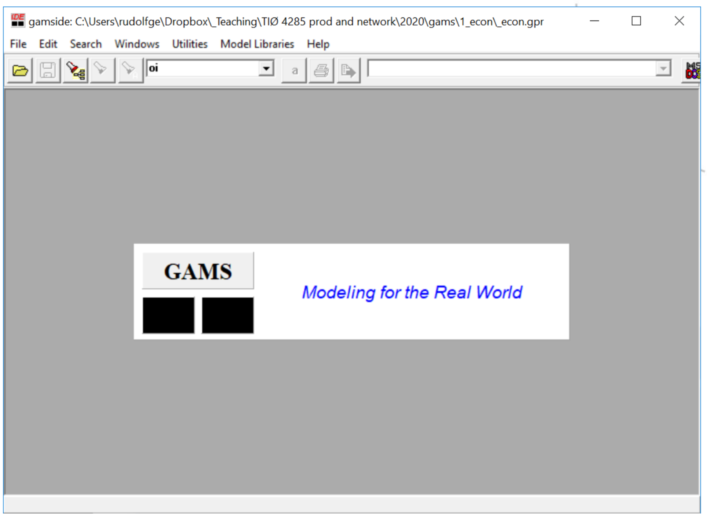

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$a = 20$
$b = 1$
$c = 2$
$d = 0.5$

```
parameter
   a  demand intercept          /20/
   b  demand slope              / 1/
   c  constant cost per unit    / 2/
   d  increasing cost per unit  / 0.5/
;

positive variable
   Q quantity supplied
;

free variable
   Z social welfare SW=CS+REV-TC
;

equations
   def_z definition of social welfare
;

def_z.. Z =E= 0.5*b*Q*Q + (a-b*Q)*Q - (c+d*Q)*Q;
```

$$Z = \frac{1}{2}bq^2 + (a - bq)q - (c + dq)q$$

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MULTI-VARIABLE (CONSTRAINED) OPTIMIZATION

A stepping stone towards Karush Kuhn Tucker conditions

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Optimization

- Unconstrained optimization
- Finding maximum (minimum) of function
- First Order Condition (FOC)
  - Stationary point: first order derivative zero
- Second Order Condition (SOC)
  - Second order derivative negative (positive)

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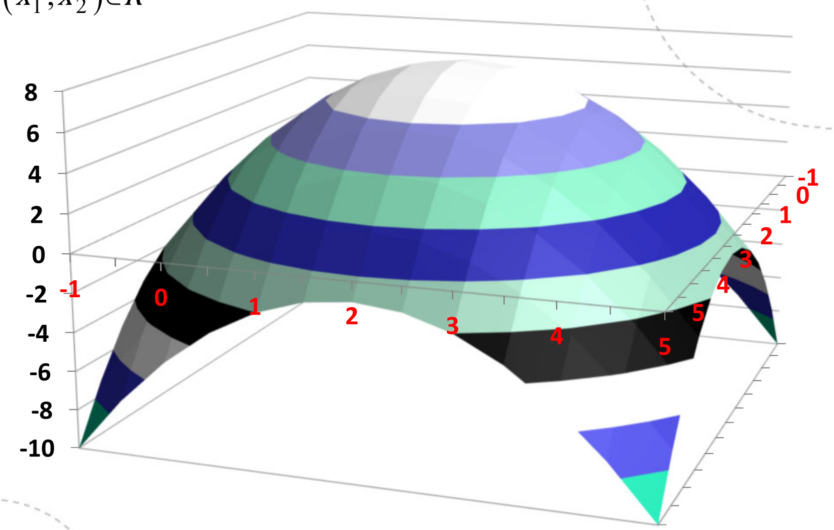
$$\max_{(x_1,x_2)\in R^2} f(x) = \left(4x_1 - x_1^{\ 2} + 4x_2 - x_2^{\ 2}\right)$$



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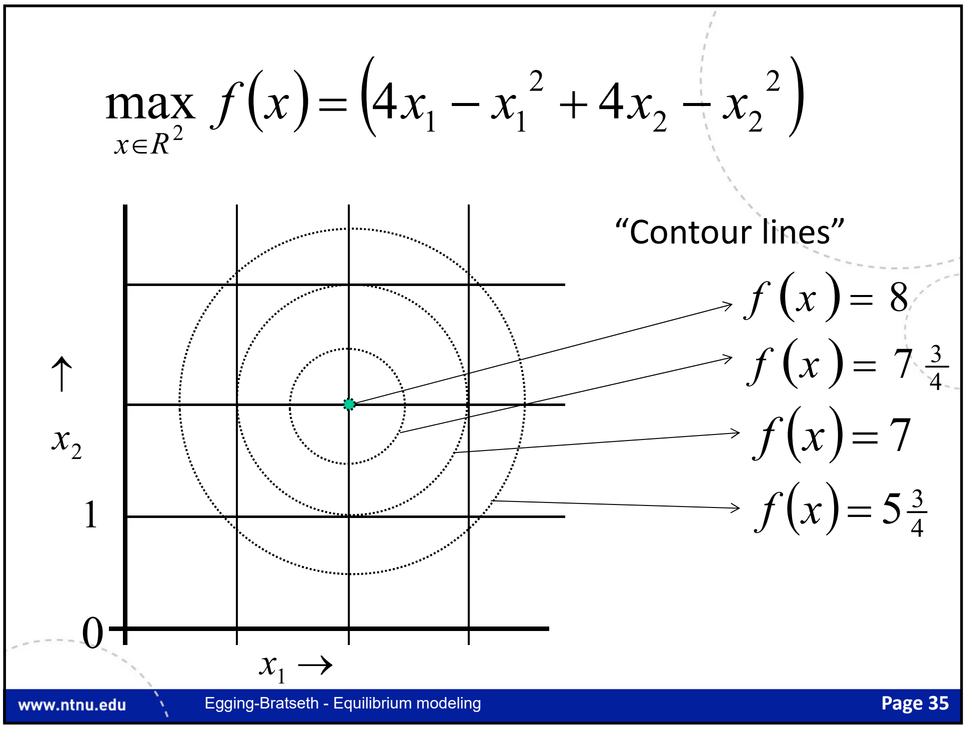
$\max_{x \in R^2} f(x) = \left(4x_1 - x_1^{\ 2} + 4x_2 - x_2^{\ 2}\right)$
"Contour lines"
$f(x) = 8$
$f(x) = 7\frac{3}{4}$
$f(x) = 7$
$f(x) = 5\frac{3}{4}$
$\uparrow$ x_2
1
0
$x_1 \rightarrow$
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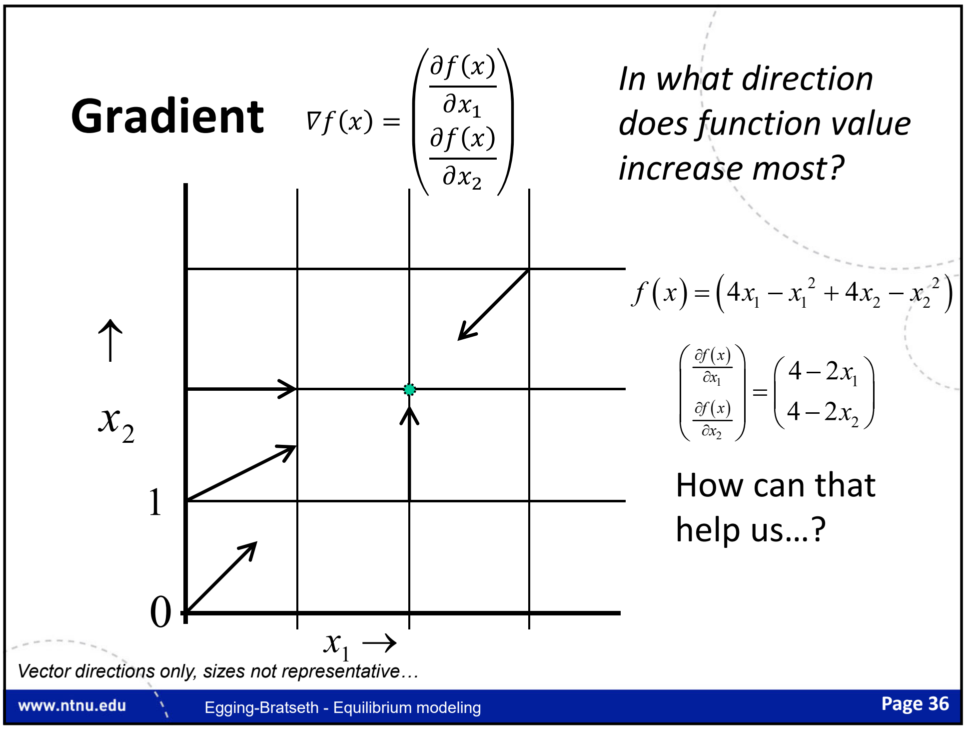
Gradient
$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \end{pmatrix}$
In what direction does function value increase most?
$f(x) = \left(4x_1 - x_1^{\ 2} + 4x_2 - x_2^{\ 2}\right)$
$\begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 4 - 2x_1 \\ 4 - 2x_2 \end{pmatrix}$
How can that help us…?
$\uparrow$ x_2
1
0
$x_1 \rightarrow$
Vector directions only, sizes not representative…
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Unconstrained optimization

$$\max\ f(x) = \left(4x_1 - x_1^{\ 2} + 4x_2 - x_2^{\ 2}\right)$$

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

There is no direction in which function value increases

So we must be at the highest point (at least locally)

$$\frac{\partial f(x)}{\partial x_1} = 4 - 2x_1 = 0 \Rightarrow x_1 = 2$$

$$\frac{\partial f(x)}{\partial x_2} = 4 - 2x_2 = 0 \Rightarrow x_2 = 2$$

Second order condition (S.O.C.)

$$J = \nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 4 - 2x_1 \\ 4 - 2x_2 \end{pmatrix}$$

First Order Condition (F.O.C.) Jacobian=0

$$H = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_1^{\ 2}} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^{\ 2}} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

S.O.C. Hessian negative (semi-) definite (n.s.d.)?

- Can verify definiteness using principal minors
- Here: -2 <0 and -2*-2=4>0, hence negative definite; therefore concave (quadratic: even strictly concave); therefore: solution via F.O.C. global and unique
- Regularity conditions beyond the scope of this class.
- *F.O.C. guarantee solutions for concave maximization & convex minimization over polyhedral feasible regions*

CONSTRAINED OPTIMIZATION

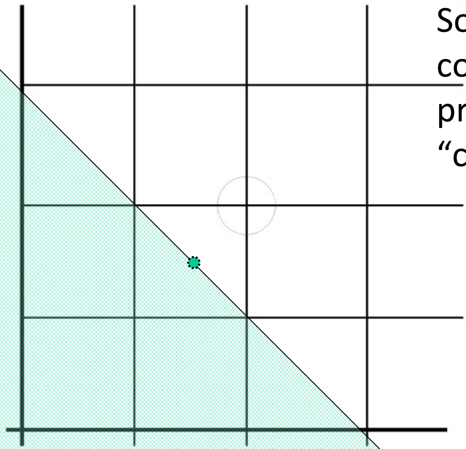
Optimization continued

- Add a constraint:

$$\begin{aligned} &\max \quad && f(x) \\ &s.t. \quad && x_1 + x_2 \le 3 \end{aligned}$$

Constrained Optimization cont'd

$$x_1 + x_2 \leq 3$$



Solution for constrained problem using "contour lines":

Constrained Optimization cont'd

- How do we solve this analytically?
- Lagrangian Multiplier Method: put constraint in objective, with penalty $\lambda \geq 0$.
- Reorder constraint *(skip "=" & slack variables)*:

$$x_1 + x_2 \leq 3 \Leftrightarrow g\left(x_1, x_2\right) = 3 - x_1 - x_2 \geq 0$$

$$\max L\left(x, \lambda\right) = f\left(x\right) + \lambda(3 - x_1 - x_2)$$

- intuition

  → too large x-values, second term negative

  → if original constraint binding in a solution it will be enforced

Lagrangian Multiplier Method

$$\max_{x \in R^2,\, \lambda \geq 0} L(x_1, x_2, \lambda) = \left(4x_1 - x_1^2 + 4x_2 - x_2^2\right) + \lambda\left(3 - x_1 - x_2\right)$$

$$\nabla L(x) \ldots$$

$$\frac{\partial L(x)}{\partial x_1} = 4 - 2x_1 - \lambda$$

$$\frac{\partial L(x)}{\partial x_2} = 4 - 2x_2 - \lambda$$

$$\frac{\partial L(x)}{\partial \lambda} = 3 - x_1 - x_2$$

Lagrangian **Multiplier Method** cont'd

$$\frac{\partial L(x)}{\partial x_1} = 4 - 2x_1 - \lambda$$

$$\frac{\partial L(x)}{\partial x_2} = 4 - 2x_2 - \lambda$$

$$\frac{\partial L(x)}{\partial \lambda} = 3 - x_1 - x_2$$

$$\nabla L(x) = \bar{0} \Rightarrow$$

$$x_1 = x_2 = 2 - \tfrac{1}{2}\lambda$$

$$x_1 + x_2 = 3 = 4 - \lambda$$

$$\lambda = 1, x_1 = x_2 = \tfrac{3}{2}$$

Optimization continued: two constraints

$$max\ f(x_1, x_2) = 4x_1 - (x_1)^2 + 4x_2 - (x_2)^2$$
$$s.t. \quad x_1 + x_2 \leq 3$$
$$x_1 \leq 2$$

$$g_1(x_1, x_2) = 3 - x_1 - x_2$$
$$g_2(x_1) = 2 - x_1$$

$$L(x_1, x_2, \lambda_1, \lambda_2) = f(x_1, x_2) + \lambda_1 g_1(x_1, x_2) + \lambda_2 g_2(x_1)$$

Solve this:

$$x_1 = x_2 = \frac{3}{2}, \lambda_1 = 1, \lambda_2 = \cdots$$

Lagr multiplier of not-binding constraint value…?
Basis for understanding KKT

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KARUSH-KUHN-TUCKER (KKT) CONDITIONS

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Karush-Kuhn-Tucker (KKT) conditions

- First-order necessary conditions for a solution in nonlinear programming to be optimal, provided that some regularity conditions are satisfied
- Allowing inequality constraints, the KKT approach generalizes the method of Lagrange multipliers (which allows only equality constraints)

Wikipedia 2019.01.23

Karush Kuhn Tucker conditions

- Finding stationary points for optimization problem:

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & g(x) \leq 0 \\ & h(x) = 0 \end{aligned}$$

- Introduce (dual) vectors u, v

KKT continued

$$\min f(x)$$
$$s.t. \quad g(x) \le 0$$
$$h(x) = 0$$

- KKT points:

$$(x', u', v'):$$

$$\nabla f(x') + \sum_{i=1}^{n} u_i' \nabla g_i(x') + \sum_{j=1}^{m} v_j' \nabla h_j(x') = 0$$

KKT continued

$$\min f(x)$$
$$s.t. \quad g(x) \le 0$$
$$h(x) = 0$$

- KKT points:

$$(x', u', v'):$$

$$g_i(x') \le 0, \; u_i' \ge 0, \; u_i' g_i(x') = 0, \quad \forall i = 1,..,n$$
$$h_j(x') = 0, \; v_j' \; free\ in\ sign \qquad \forall j = 1,..,m$$

KKT continued

$$\min \ f(x)$$
$$s.t. \quad g(x) \le 0$$
$$h(x) = 0$$

- KKT points: (x', u', v'):

$$h_j(x') = 0, v_j \ \ free\ in\ sign \qquad \forall j = 1,..,m$$

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KKT continued

- KKT points: (x', u', v'):

$$\nabla f(x') + \sum_{i=1}^{n} u_i' \nabla g_i(x') + \sum_{j=1}^{m} v_j' \nabla h_j(x') = 0$$
$$g_i(x') \le 0, \ u_i' \ge 0, \ u_i' g_i(x') = 0, \quad \forall i = 1,.., n$$
$$h_j(x') = 0, \ v_j' \ \ free\ \ in\ \ sign \qquad \forall j = 1,.., m$$

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KKT continued

- For (*strictly*) convex minimization problems on polyhedral feasible regions KKT points are (*unique*) global solutions

KKT continued

$$\nabla f(x') + \sum_{i=1}^{n} u_i' \nabla g_i(x') + \sum_{j=1}^{m} v_j' \nabla h_j(x') = 0$$
$$g_i(x') \le 0,\ u_i' \ge 0,\ u_i' g_i(x') = 0,\ \forall i = 1,.., n$$
$$h_j(x') = 0,\ v_j'\ \textit{free in sign} \qquad \forall j = 1,.., m$$

- Stationarity:

$$\nabla f(x) + \sum u_i \nabla g_i(x) + \sum v_j \nabla h_j(x) = 0$$

$$\Leftrightarrow$$

$$\nabla f(x) = -\sum u_i \nabla g_i(x) - \sum v_j \nabla h_j(x)$$

Intuition: function gradient and a weighted aggregate of binding restrictions push in opposite directions (cancel each other out)

Optimization – remember previous problem…

$$\begin{array}{ll} \max & \left(4x_1 - x_1^{\ 2} + 4x_2 - x_2^{\ 2}\right) \\ s.t. & x_1 + x_2 \le 3 \end{array}$$

$$\begin{array}{ll} \min & \left(x_1^{\ 2} - 4x_1 + x_2^{\ 2} - 4x_2\right) \\ s.t. & x_1 + x_2 - 3 \le 0 \end{array}$$

Standard form

Constrained optimization using KKT

$$\begin{array}{ll} \min & \left(x_1^{\ 2} - 4x_1 + x_2^{\ 2} - 4x_2\right) \\ s.t. & x_1 + x_2 - 3 \le 0 \end{array}$$

$$\nabla f(x') + \sum_{i=1}^{n} u_i^{'} \nabla g_i(x') + \sum_{j=1}^{m} v_j^{'} \nabla h_j(x') = 0$$

$$g_i(x') \le 0,\ u_i^{'} \ge 0,\ u_i^{'} g_i(x') = 0, \quad \forall i = 1,.., n$$

$$h_j(x') = 0,\ v_j^{'} \ \textit{free in sign} \qquad \forall j = 1,.., m$$

General

$$(i) \quad \nabla f(x') + \sum_{i=1}^{n} u_i^{'} \nabla g_i(x') = 0$$

$$(ii) \quad 0 \le u_i^{'} \perp g_i(x') \le 0$$

$$(i') \quad \nabla f(x') = -\sum_{i=1}^{n} u_i^{'} \nabla g_i(x')$$

Here

$$\begin{pmatrix} 2x_1 - 4 \\ 2x_2 - 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

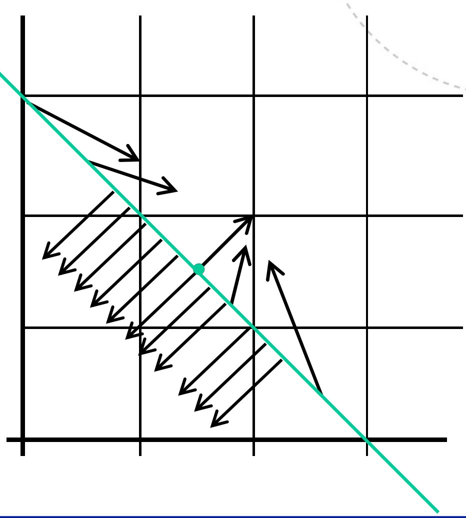
$$0 \le \lambda \perp x_1 + x_2 - 3 \le 0$$

$$\begin{pmatrix} 2x_1 - 4 \\ 2x_2 - 4 \end{pmatrix} = -\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Constrained optimization using KKT

$$\nabla f(x) = -\lambda \nabla g(x)$$

g(x) binding:
Where on the line $g(x) = 0$ are $\nabla f(x)$ and $\lambda \nabla g(x)$ in opposite directions?



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g(x) not binding in the solution?

$$\begin{pmatrix} 2x_1 - 4 \\ 2x_2 - 4 \end{pmatrix} = -\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Not binding: $\lambda = 0 \Rightarrow$

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EQUILIBRIUM PROBLEM = COMPLEMENTARITY PROBLEM

Complementarity - concept

- Consider dual prices of constraints or the complementarity slackness conditions from linear programming
- Pairs of variables and equations
- Variable can only be positive if constraint is binding
- If slack in constraint, variable has zero value

$$0 \leq q \perp \frac{\partial z}{\partial q} \geq 0$$

$$0 \leq \lambda \perp CAP - q \geq 0$$

Equilibrium models

- Implemented as "complementarity problems" using
- (slightly adjusted) KKT conditions for agent problems and
- market clearing conditions: connecting the agent problems into common markets

Complementarity: notiation

$$x.y = 0 \Leftrightarrow x \perp y$$

$$u_i^{'} g_i\left(x'\right) = 0 \Leftrightarrow g_i\left(x'\right) \perp u_i^{'}$$

Linear complementarity problem

- Find a non-negative vector
- For the set of equations
- Such that in each variable-equation pair the variable or the equation equals zero (complementarity).

$$\left.\begin{aligned} u_i' &\geq 0 \\ g_i(x') &\geq 0 \\ u_i' g_i(x') &= 0 \end{aligned}\right\}$$

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SINGLE AGENT EQUILIBRIUM PROBLEMS

Some examples of linear complementarity problems.

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From complementarity problem to optimization problem – simplified steps

- We find the complementarity problem via the Lagrangian Multiplier Method (however, we skip several steps)

Procedure:

- Write objective as minimization
- Reorder restrictions and assign dual variables
- Derive KKT for each variable
  - First Order Conditions based on Lagrangian (objective and “penalized” restrictions)
  - “<=“ : duals get a ‘+’ in stationarity conditions
  - “>=“ : duals get a ‘-’ in stationarity conditions
  - Include the restrictions
  - Non-negative variables imply non-negative stationarity conditions
  - Needed later. Equality restrictions: Sinks-Sources=0, “=“: dual gets ‘+’

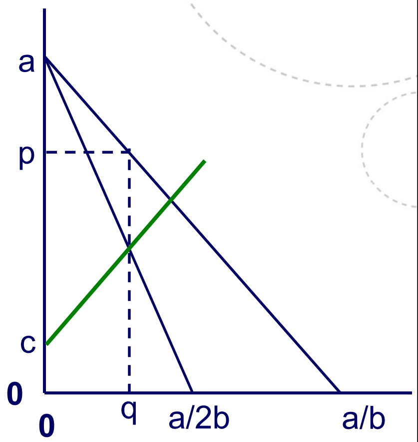
MONOPOLY

Example monopoly supplier

- Inverse demand curve $$p(q) = a - bq$$
- Supply costs $$c(q) = cq + dq^2$$
- Capacity constraint $$q \le cap$$
- Nonnegative quantities $$q \ge 0$$
- Solve first using a graph:

Monopoly supplier

- Inv dem: p=a-bq
- Marg revenues: a-**2b**q
- Marg prod cost: c+**2**dq
- MR=MC: a-2bq=c+2dq
  - q= (a-c)/[2(b+d)]
- Check capacity: q<=cap…



Monopoly

- Maximize profits under a capacity constraint

$$\max_{q\geq 0} (a-bq)q-(cq+dq^2)$$
$$s.t. \quad q \leq cap$$

- Write objective as minimization

$$\min \ (b+d)q^2-(a-c)q$$
$$s.t. \quad cap-q\geq 0 \qquad (\lambda \geq 0)$$

- Reorder restrictions and assign dual variables
- Derive KKT for each variable
  - F.O.C.s based on Lagrangian
  - <= duals get a '+'
  - >= : duals get a '-'
  - Include the restrictions

$$0 \leq q \perp -(a-c)+2(b+d)q+\lambda \geq 0$$
$$0 \leq \lambda \perp cap-q \geq 0$$

Do on the board

$$q>0 \Rightarrow (a-c)-2(b+d)q-\lambda=0$$
$$a-2bq=c+2dq+\lambda$$
$$MR=MC \ (!!)$$

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Solving the LCP

$$0 \leq q \perp -(a-c)+2(b+d)q+\lambda \geq 0 \quad (i)$$
$$0 \leq \lambda \perp cap-q \geq 0 \quad (ii)$$

- Use $a=20, b=1, c=2, d=0.5$
- Explicit enumeration. Try
  - q=0 in i. => λ=a-c>0 (but ii. not binding) NO
  - q>0 & λ=0 in i. (ii not binding) => q=6. (not feasible) NO
  - q>0 and λ>0 (ii binding): q=5. 18-15-λ=0. λ=3. KKT-point
- strictly convex: KKT point is unique solution

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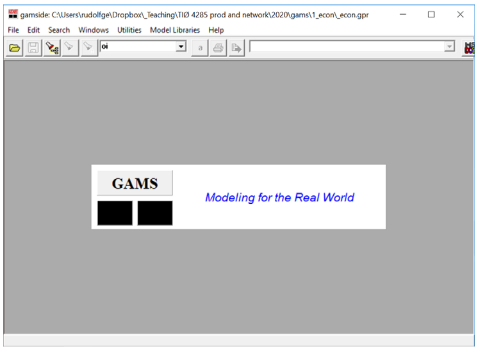
Implementing equilibrium problems

G A M S General Algebraic Modeling System

Implement & solve LCP & Optimization



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PERFECT COMPETITION

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Perfect competition

- Perfect competition = maximize social welfare

$$max\ SW = \frac{1}{2}bq^2 + (a - bq)q - (c + dq)q$$

- Set up the complementarity problem
  - Write as minimization
  - Reorder restrictions and assign dual variables
  - Derive KKT for each variable
    - F.O.C. objective & restrictions
    - <=, = : duals get a '+' in stationarity conditions
    - >= : duals get a '-' in stationarity conditions
    - Include the restrictions
  - Show in GAMS

COURNOT OLIGOPOLY

Cournot Oligopoly

Supplier i: max $z_i = \left(a - b\sum_j q_j\right) q_i - \left(c_i q_i + d_i q_i^2\right)$

Micro-economics / Industrial Economic Analysis: solve using *optimal response curves*

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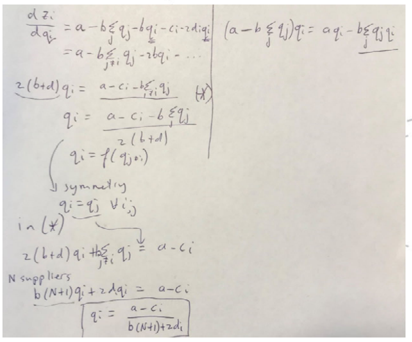
Cournot Oligopoly: optimal response curves

$$\frac{\partial z_i}{\partial q_i} = \left(a - b\sum_j q_j - bq_i\right) - \left(c_i + 2d_i q_i\right)$$

$$\frac{\partial z_i}{\partial q_i} = 0 \Rightarrow \left(2bq_i + 2d_i q_i\right) = a - c_i - b\sum_{j \neq i} q_j$$

$$\Rightarrow q_i = \frac{a - c_i - b\sum_{j \neq i} q_j}{2\left(b + d_i\right)}$$



Easily solved when only two suppliers or symmetric suppliers. What if many suppliers, production or network capacity constraints, multiple periods, …?

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Cournot Oligopoly: complementarity problem

- Set up complementarity problem:
  - Write as minimization
  - Derive KKT

$$\forall i: \quad \max z_i = \left(a - b\sum_j q_j\right) q_i - \left(c_i q_i + d_i q_i^2\right)$$

$$\forall i: \quad \min z_i = \left(c_i q_i + d_i q_i^2\right) - \left(a - b\sum_j q_j\right) q_i$$

$$\forall i: \quad 0 \le q_i \perp \left(c_i + 2d_i q_i\right) - \left(a - b\sum_j q_j - bq_i\right) \ge 0$$

GAMS

Equilibrium modeling in network economics

- Industries where a network is needed to transport goods and the agent behavior not necessarily lead to the system-wide "optimal" solution: imperfect competition

Complementarity models:

- capture aspects of markets, such as market power a là Cournot, that optimization models can't
- can be modeled using Karush-Kuhn-Tucker conditions

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