NTNU
Department of Industrial Economics and Technology Management
Spring 2020

TIØ4285 Production and Network Economics
Assignment 1 – Proposed Solution

Out: Thursday 16 January
In: Thursday 23 January 6pm
Supervision: Monday 20 January, 4:15pm A31

Note that late answers will not be approved.

Exercise 1

Three brothers – Jim, Jack, and Johnnie – have each inherited a whisky distillery. They are experts on distilling whisky, but know nothing about the whisky market. They therefore signed an agreement with long-time family friend Glen who is running a blending and bottling facility.

Glen is using all of the whisky produced by Jim, Jack, and Johnnie, blends and bottles it and sells it to the customers. For his signature product, he uses 30% of Jim's whisky, 25% of Jack's whisky, and 45% of Johnnie's.

The three brothers and Glen want to expand their business and sell their product to Wales. For this expansion, they need to set up a new distribution center. Help them decide which distribution center to open. Cost and demand information is given in spreadsheet `exercise01.xlsx`.

a) Formulate the supply chain design problem as a mathematical programming problem. Remember to explain your formulation.

   Let us introduce the following notation: We use the following notation:

   Sets
   - $\mathcal{J}$ Set of distilleries.
   - $\mathcal{B}$ Set of bottling facilities.
   - $\mathcal{D}$ Set of possible locations for distribution centers.
   - $\mathcal{C}$ Set of customer locations.

   Indices
   - k, n, m location indices.

   Parameters
   - D_m Demand of customer m.
   - F_m Fixed costs of opening a distribution center at location m.
   - S_m Share of whisky produced at m used in the final blend.
   - T_{mn} Transportation cost between location m and location n.

   Decision variables
   - x_{mn} Amount of whisky transported from location m to location n.
   - y_m 1 if DC opens at location m, 0 otherwise.

   With that notation we can set up the following model:

$$\min \sum_{m \in \mathcal{D}} F_m y_m + \sum_{m \in \mathcal{J}} \sum_{n \in \mathcal{B}} T_{mn} x_{mn} + \sum_{m \in \mathcal{B}} \sum_{n \in \mathcal{D}} T_{mn} x_{mn} + \sum_{m \in \mathcal{D}} \sum_{n \in \mathcal{C}} T_{mn} x_{mn} \tag{1}$$

subject to

$$\sum_{k\in\mathcal{J}} x_{km} - \sum_{n\in\mathcal{D}} x_{mn} = 0 \qquad m\in\mathcal{B}, \tag{2}$$

$$\sum_{k\in\mathcal{B}} x_{km} - \sum_{n\in\mathcal{C}} x_{mn} = 0 \qquad m\in\mathcal{D}, \tag{3}$$

$$\sum_{k\in\mathcal{D}} x_{km} = D_m \qquad m\in\mathcal{C}, \tag{4}$$

$$x_{km} - S_k \sum_{n\in\mathcal{J}} x_{nm} = 0 \qquad k\in\mathcal{J}, m\in\mathcal{B}, \tag{5}$$

$$\sum_{k\in\mathcal{B}} x_k m \leq y_m \sum_{n\in\mathcal{C}} D_n \qquad m\in\mathcal{D}, \tag{6}$$

$$x_{mn} \geq 0, \qquad m\in\mathcal{J}\cup\mathcal{B}\cup\mathcal{D}, n\in\mathcal{B}\cup\mathcal{D}\cup\mathcal{C}, \tag{7}$$

$$y_m \in \{0,1\} \qquad m\in\mathcal{D}. \tag{8}$$

The objective function (1) is the sum of the fixed facility costs for opening a distribution center and the transportation costs between the different levels of the supply chain. Constraints (2) and (3) are mass balance constraints for the bottling facility and the distribution centers, repesctively. Equation (4) ensures that all customer demand is satisfied. Constraints (5) describe the blending of whisky. The amount of a particluar type whisky used at the blending facility has to be equal to a particular share of the total amount of whisky used there. Constraint (6) links the flow of whisky into a distribution center to the decision whether or not it is opened. We use the sum of all demands as *'Big M'*, a number large enough not to limit the flow of whisky. The final constraints (7) and (8) are the non-negativity and binary requirements for the decision variables.

b) Implement and solve the problem using commercial software.

Please see file `exercise01_solution.xlsx` for the implemented model and the solution.

c) Discuss the advantages and disadvantages of a centralized distribution system vs. a decentralized system. When would you prefer a centralized solution over a decentralized one and vice versa?

Topics that can be discussed here include (but are not limited to):

- Risk pooling
- Safety stocks
- Correlated vs. uncorrelated demands
- Customer closeness
- Warehousing costs (both facility and product)
- ...

The above points can also be used to illustrate when a centralized solution is preferable (e. g. in a situation with negatively correlated or uncorrelated demands) or a decentralized solution is the chosen one (e. g. cheap facility costs plus a requirement to be close to the customers).

Exercise 2

a) Formulate the Newsboy Problem as a two-stage stochastic programming problem.

Let us introduce the following notation: We use the following notation:

Sets

$\mathcal{S}$	Set of demand scenarios.

Indices

s	Scenario index.

Parameters

D^s	Customer demand in scenario s.
R	Revenue of a newspaper.
W	Wholesale price of a newspaper.
S	Salvage value of a newspaper.
p^s	Probability of scenario s.

Decision variables

x	Number of newspapers to buy every day.
y^s	Number of newspapers sold in scenario s (at price R).
z^s	Number of newspapers salvaged in scenario s (at salvage value S).

With that notation we can set up the following model:

$$\max -Wx + \sum_{s \in \mathcal{S}} p^s \cdot (Ry^s + Sz^s) \tag{9}$$

subject to

$$y^s \leq x \qquad s \in \mathcal{S}, \tag{10}$$

$$y^s \leq D^s \qquad s \in \mathcal{S}, \tag{11}$$

$$y^s + z^s = x \qquad s \in \mathcal{S}, \tag{12}$$

$$x \geq 0 \tag{13}$$

$$y^s, z^s \geq 0 \qquad s \in \mathcal{S}. \tag{14}$$

The objective function (9) maximizes the Newboy's expected profits. The first term represents the wholesale cost of buying newspapers. This is also the first stage cost. The second term, i. e. the sum, are the expected revenues from selling and salvaging the newspapers in each scenario.

Constraints (10) make sure that we do not sell more newspapers in a given scenario than we have available. Similarly, constraints (11) limit the number of sold newspapers, by preventing selling more than we have demand for. Equations (12) ensure that all newspapers are either sold or salvaged. Finally, constraints (13) and (14) are non-negativity constraints for the decision variables.

b) Implement the stochastic programming Newsboy formulation and solve it using the values provided in spreadsheet `exercise02.xlsx`. Assume that all scenarios are equally likely. How many newspapers should the newsboy buy and what are her expected profits?

Please see file `exercise02_solution.xlsx` for the implemented model and the solution. Optimal order quantity is $Q^* = 127$ and the newsboy's expected profits are given as $\Pi_r = 5280.6$.

c) Solve the Newsboy Problem analytically. Use the same values for R, W and S as in b), but assume that demand is uniformly distributed on the interval [50,150].

With $R = 100$, $W = 40$, and $S = 20$ we get the following underage cost $C_u = R - W = 60$ and overage cost $C_o = W - S = 20$. The Newsboy optimality criterion is thus given as

$$\Pr(D \leq Q) = \frac{C_u}{C_u + C_o} = \frac{60}{80} = 0.75.$$

Note that $\Pr(D \leq Q) = F(Q)$, where $F(\cdot)$ is the cumulative distribution function of D. The optimal order quantity can therefore be calculated as as $Q^* = F^{-1}\left(\frac{C_u}{C_u+C_o}\right)$.

The probability density for a uniform distribution is given as $f(x) = \frac{1}{b-a}$ for $x \in [a,b]$ and 0 otherwise. The cumulative distribution function (cdf) $F(x)$ is 0 for $x < a$, 1 for $x > b$ and $\frac{x-a}{b-a}$ for $x \in [a,b]$. We find the optimal order quantity by inverting the cdf for the given optimality criterion (here: 0.75):

$$\begin{aligned}
\frac{x-a}{b-a} &= 0.75 \\
\frac{x-50}{150-50} &= 0.75 \\
x &= 125
\end{aligned}$$

The Newsboy's expected profits are calculated using the formula given by Rudi & Pyke:

$$\Pi_r(Q^*) = (R-W)ED - E\left[(R-W)(D-Q^*)^+ + (W-S)(Q^*-D)^+\right]. \tag{15}$$

In order to calculate the cost of uncertainty, we need to calculate the expected number of salvaged newspapers and the expected shortfall, i.e. the expected number of newspapers we could have sold exceeding the bought ones. For the order quantity $Q^* = 125$, the expected number of salvaged newspapers is uniformly distributed on the interval $[0, 75]$, resulting in a conditional expectation (given that $Q > D$) of 37.5. Note that this will happen with probability $\Pr(D < Q) = 0.75$.

Similarly, the expected shortfall is uniformly distributed on the interval $[0, 25]$. The conditional expectation is 12.5, happening with probability 0.25. With this, we can calculate the Newsboy's expected profits:

$$\begin{aligned}
\Pi_r(Q^*) &= (R-W)ED - E\left[(R-W)(D-Q^*)^+ + (W-S)(Q^*-D)^+\right] \\
&= (100-40)\cdot 100 - (100-40)\cdot 12.5\cdot 0.25 - (40-20)\cdot 37.5\cdot 0.75 \\
&= 6000 - 187.5 - 562.5 \\
&= 5250.
\end{aligned}$$

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TIØ4285 Production and Network Economics
Assignment 2 – Proposed Solution

Out: Thursday 23 January
In: Thursday 30 January 6pm
Supervision: Monday 27 January, 4:15pm A31

Note that late answers will not be approved.

Exercise 1

The manager of the local football club is putting together the budget for next season. Attendance accounts for the largest portion of the revenues and the manager believes that attendance is directly related to the number of the team's wins. For the past 8 seasons, the following attendance figures are given:

Wins	Attendance
14	3,630
16	4,010
16	4,120
18	5,300
16	4,400
17	4,560
15	3,900
17	4,750

a) Given the players on the team, the manager strongly believes that the team will win at least 17 matches next season. Use a linear regression model to forecast next season's level of attendance.

With a linear regression, we describe a linear relationship, $y = a + b \cdot x$, between the number of wins as explanatory variable, x and attendance as dependent variable, y. Using the equations from Hillier & Lieberman (2015) for the method of least squares, we can estimate the parameters a and b:

$$b = \frac{\sum_{t=1}^{8}(x_t - \bar{x})(y_t - \bar{y})}{\sum_{t=1}^{8}(x_t - \bar{x})^2} = 406.09,$$
$$a = \bar{y} - b \cdot \bar{x} = -2214.5,$$

with $\bar{x} = \sum_{t=1}^{8} \frac{x_t}{8}$ and $\bar{y} = \sum_{t=1}^{8} \frac{y_t}{8}$.

The resulting least-squares estimate of attendance is thus $y = 406.09 \cdot x - 2214.5$. For next season, the manager forecasts an attendance level of 4689.

b) Analyze and discuss the quality of the forecast. Would you trust the forecast? (Do not discuss whether or not the manager is right in expecting at least 17 wins.)

We first plot the data for a visual examination of the data series, see Figure 1.

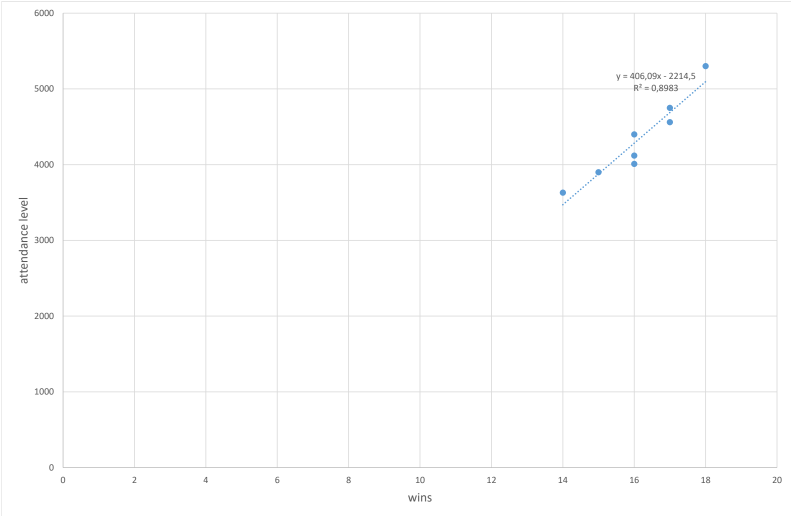

Figure 1: Attendance vs. number of wins, incl. regression line

In this case, the figure doesn't tell much. But keep in mind that the data here is not a time series, these are just realizations of wins and attendance levels. Trends and seasonality will not show up here. Still it might be possible to identify outliers (which in this case we don't have). What we do see is that the data points are somewhat nicely clustered, giving us a good chance that a linear regression produces a decent fit. The $R^2 = 0.8983$ confirms that.

Whether or not an $R^2 = 0.8983$ is good enough to use the regression for forecasting is somewhat relative. It is a good fit to the data, and a forecasted attendance level of 4689 for next season is definitely plausible. However, the linear regression has a negative intercept with the y-axis. In practice, negative attendance is not possible. As a consequence, the linear regression model will perform poorly when forecasting attendance levels in seasons with a very low number of wins. If the local football club never has fewer than, say, 7 or 8 wins, the model might still be (very) good. If low win seasons are likely, the linear regression might not be the best forecasting model for the club any longer.

(Note that checking the quality of a linear regression with mean squared error does not help with respect to finding a better linear regression. Remember that the parameters of the linear regression are usually estimated such that the sum of squared deviations is minimized. Therefore, use R^2 instead.)

Exercise 2

Good Bread AS is a large, successful bakery, selling bread to supermarkets all over the country. The company is organized in an independent sales division, distribution division and baking division. Over the past months, Good Bread AS has found itself more often in a situation with high inventory costs while at the same time being unable to deliver the ordered products in time. The figure below shows the customer demand for bread and the production orders issued by the baking division.

a) Explain to the CEO af Good Bread AS what might be causing the problems regarding inventory costs and service level.

   Good Bread's problems are could caused by the Bullwhip Effect. The variability of end-customer demand is amplified up the supply chain. This effect is influenced by the following factors (each point has to be discussed briefly):

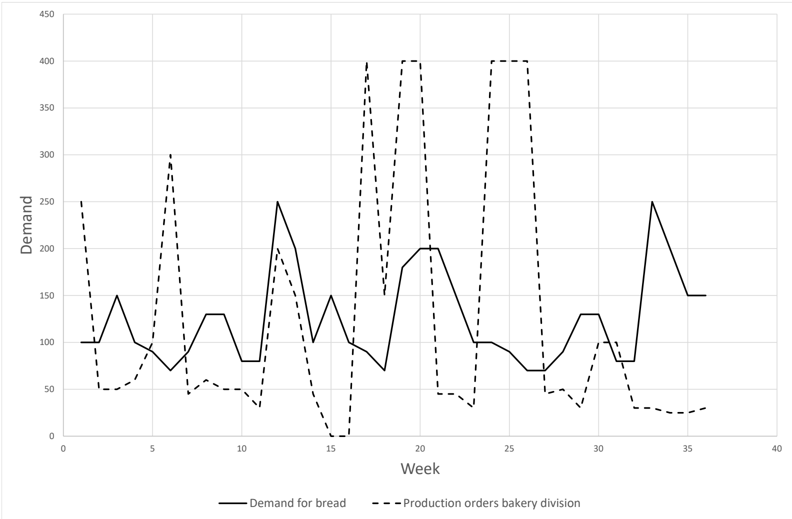

Figure 2: Customer demand and production orders by the baking division

- Demand forecasting
- Lead time
- Batch ordering
- Price fluctuations
- Inflated orders

After explaining the problem to the CEO, she wants to know more about the observed phenomenon and provides you with additional information: The demand for bread is independently and identically distributed in each week. The inventory is controlled by a policy using a safety buffer based on the variance of demand. Orders can be issued in each week.

b) There exist theoretic models to quantify the ratio between the variance of bread demand and the variance of orders issued by the bakery division. Please use one of these models to determine this ratio. Make sure you name all assumptions. Suggest and discuss measures to reduce this ratio.

We make the following assumptions:

- Decentralized demand information
- Moving-average forecasts with p observations
- Order-up-to-point determined according to SKS:

$$L \times AVG + z \times STD \times \sqrt{L}$$

- With
  - $Var\left(Q^k\right)$ – variance of orders placed by kth stage of supply chain
  - $Var(D)$ – variance of customer demand
  - L_i – lead time between stages i and $i+1$

  we get the following relationship (see SKS)

$$\frac{Var\left(Q^k\right)}{Var(D)} \geq \prod_{i=1}^{k}\left[1+\frac{2L_i}{p}+\frac{2L_i^2}{p^2}\right]$$

The baking division is the third stage in the supply chain (the sales division being the first). The ratio between the variance of orders issued by the baking division and the variance of customer demand is therefore given as

$$\frac{Var\left(Q^3\right)}{Var(D)} \geq \prod_{i=1}^{3}\left[1+\frac{2L_i}{p}+\frac{2L_i^2}{p^2}\right]$$

If we use centralized information instead of decentralized information, we can reduce the increase in the ratio to

$$\frac{Var\left(Q^2\right)}{Var(D)} \geq 1+\frac{2\sum_{i=1}^{3}L_i}{p}+\frac{2\left(\sum_{i=1}^{3}L_i\right)^2}{p^2}$$

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Spring 2020

TIØ4285 Production and Network Economics
Assignment 3 – Proposed Solution

Out: Thursday 30 January
In: Thursday 6 February, 6pm
Supervision: Monday 3 February, 16:15pm A31

Note that late exercises will not be approved.

Exercise

Chelski FC wants to build a new football stadium and intends to sell season tickets to help finance the build. The city subsidizes the stadium, but limits the number of season tickets to only 30,000.

Assume that the marginal cost of selling season tickets is 0 and that you only consider the club's income. You have just been hired as the Chelski FC's new CFO and your first job is to determine the right price and sales method for the season tickets. You have been in touch with 10 commercial wholesalers, who want to buy (and distribute) all of the 30,000 season tickets for Chelski FC's new stadium. The club assumes that the valuation of the tickets is uniformly distributed between 0 and 100 million and that the reservation price of all 10 commercial wholesalers is drawn from this distribution.

a) You have been asked to calculate the club's expected income from auctioning off the tickets. Justify your choice of auction design and provide all necessary assumptions for your calculations.

   The choice of auction design will not affect the club's expected income, given that the assumptions for the *Revenue Equivalence Theorem* hold. We therefore assume that the following assumptions hold:

   - The buyers' reservation prices are drawn from an interval $[L, U]$ with distribution $F_i(v_i)$. This distribution is the same for every buyer and known to all of them.
   - Buyers are symmetric. This implies that the same reservation price leads to the same bidding strategy.
   - Buyers are risk-neutral, i. e. they maximize their expected profits (utility).
   - A buyer's reservation price only depends on this buyer's information and utility function, i. e. it is independent of other buyers' private information.

   As auction design does not impact the expected income under these assumptions, it is sufficient to calculate the expected income from a first-price auction. We introduce the following notation:

v	Bidder's reservation price
$f(v)$	Probability density function for reservation prices
$F(v)$	Cumulative distribution function of reservation prices
L	Lower bound on reservation price

$b(v)$	Optimal bidding strategy (bid given a reservation price)
R	Seller's income
N	Number of bidders participating in the auction

From the text we know, $v \sim U[0, 100]$. This implies the following: $F(v) = \frac{v}{100}$ and $f(v) = \frac{1}{100}$. The expected revenue of the seller using a first price auction is given as (see slide 33, Lecture 7):

$$E(R) = N \int_0^{100} b(v) f(v) F^{N-1}(v) dv.$$

For reservation prices drawn from a uniform distribution, the optimal bid ($b(v)$) for each bidder is given as:

$$b(v) = v - \frac{v - L}{N}.$$

The lower bound L is 0 in this case and the seller's expected income is

$$E(R) = \frac{N-1}{N+1} \cdot 100.$$

With $N = 10$ bidders, Chelski FC has an expected income of $\frac{9}{11} \cdot 100 = 81.8$ million from the auction.

b) The club considers advertising the auction to attract additional bidders. How much should the club be willing to spend in order to attract n new bidders to the auction?

The increase in expected income after attracting n new bidders is simply given as

$$E(R_{N+n}) - E(R_N) = \left(\frac{9+n}{11+n} - \frac{9}{11} \right) \cdot 100.$$

This is also the maximum amount Chelski FC should spend in order to attract n new bidders.

c) Assume that it costs 500,000 to attract each new bidder. How many bidders should the club attract beyond the 10 already participating wholesalers? What is the club's new expected income?

It will profitable for the club to attract new bidders as long the marginal revenue of the last attracted bidder is greater than (or equal to) 500,000. This translates to

$$\frac{dE(R)}{dN} = \frac{200}{(N+1)^2} = 0.5.$$

This results in an optimal number of 19 bidders. Expected income for the club is then $E(R) = \frac{18}{20} \cdot 100 - 9 \cdot 0.5 = 85.5$. The auction's expected income increases by 8.2 million, but the club's increases its income by only 3.7 million.

The city is not happy for your suggestion and insists that the Chelski FC sells the tickets itself, without using commercial wholesalers. You decide to try a completely new approach to solve the problem. The demand for season tickets is uncertain and price dependent. Market research reveals that demand can be described by 3 scenarios: S1, S2, and S3 corresponding to low, medium and high demand. Each scenario provides demand for 3 different prices. You know that one of the scenarios is the correct one, but – unfortunately – not which one. The result from market research is given in Table 1. The probability for a scenario to be correct is given as $p = 0.3$, $p = 0.5$, and $p = 0.2$, respectively.

Table 1: Demand scenarios

	scenario		
price	S1 (p=0.3)	S2 (p=0.5)	S3 (p=0.2)
2500	20,000	35,000	45,000
3000	15,000	30,000	35,000
5000	5,000	15,000	30,000

d) Which ticket price should you choose given that you do not have any other information than the one in Table 1?

Without any additional information, the club should choose the price that results in the highest expected income:

$$\begin{aligned}
\text{Price } 2500: \quad & E(R) = 2,500 \cdot (0.3 \cdot 20' + 0.5 \cdot 35' + 0.2 \cdot 45') \\
& = 81,250' \\
\text{Price } 3000: \quad & E(R) = 3,000 \cdot (0.3 \cdot 15' + 0.5 \cdot 30' + 0.2 \cdot 35') \\
& = 79,500' \\
\text{Price } 5000: \quad & E(R) = 5,000 \cdot (0.3 \cdot 5' + 0.5 \cdot 15' + 0.2 \cdot 30') \\
& = 75,000'
\end{aligned}$$

The tickets should be sold for 2,500.

You now decide to sells the tickets by means of a modified Dutch auction (which is often used to buy back stocks from shareholders).

e) Which property of the modified Dutch auction makes it particularly attractive in this situation?

The modified Dutch auction reveals information about the buyers' demand. We will therefore be able to identify the correct scenario and can choose the price that will maximize income.

f) Determine optimal ticket sales and expected income from a modified Dutch auction given the demand scenarios in Table 1.

We know that we can identify the scenario if we carry out a modified Dutch auction. The optimal prices are

- Scenario 1: price 2500, income 50,000,000
- Scenario 2: price 3000, income 90,000,000
- Scenario 3: price 5000, income 150,000,000

The expected income from a modified Dutch auction is then
$0.3 \cdot 50,000' + 0.5 \cdot 90,000' + 0.2 \cdot 150,000' = 90,000'$.

TIØ4285 Production and Network Economics

Assignment 4 – Suggested solution

Exercise 1. Cournot oligopoly with capacity constraints

a. The optimization problems:

$$\forall\, i\text{: } \min\, z_i = \left(c_i q_i + d_i q_i^2\right) - \left(a - b\sum_j q_j\right) q_i$$
$$s.t. \quad cap_i - q_i \geq 0$$

Note: $\left(a - b\sum_j q_j\right) q_i = \left(a q_i - b q_i \sum_j q_j\right) = a q_i - b q_i \sum_{j \neq i} q_j - b q_i^2$

b. The complementarity problem:

$$\forall i\text{:} \qquad 0 \leq q_i \perp \left(c_i + 2 d_i q_i\right) - \left(a - b\sum_{j \neq i} q_j - 2 b q_i\right) \geq 0$$
$$0 \leq \lambda_i \perp cap_i - q_i \geq 0$$

Which is equivalent to:

$$\forall i\text{:} \qquad 0 \leq q_i \perp \left(c_i + 2 d_i q_i\right) - \left(a - b\sum_j q_j - b q_i\right) \geq 0$$
$$0 \leq \lambda_i \perp cap_i - q_i \geq 0$$

c. The equilibrium price and quantities for $N = 3, c_i = 2, d = \frac{1}{2}, a = 20, b = 1, cap_i = 5$ can be calculated via the optimal response function (see lecture slides), but also taking into account the capacity restrictions: $q_i = \min\left(\dfrac{a - c_i - b\sum_{j \neq i} q_j}{2\left(b + d_i\right)}, cap_i\right)$

Assume first that the capacities are not binding:

$$q_i = \frac{a - c_i - b\sum_{j \neq i} q_j}{2\left(b + d_i\right)} = \frac{20 - 2 - 1\sum_{j \neq i} q_j}{2\left(1 + 0.5\right)} = \frac{18 - \sum_{j \neq i} q_j}{3} = 6 - \frac{\sum_{j \neq i} q_j}{3}$$

Use symmetry: $q_i = q_j \Rightarrow q_j = \dfrac{18 - \left(N - 1\right) q_j}{3} \Rightarrow \left(N + 2\right) q_j = 18 \Rightarrow q_j = \frac{18}{5} = 3.6$

Since this is lower than the respective capacity limits of 5, this is the optimal solution.

d. See separate GAMS code

```
*Capacity-constrainted oligopoly on a single node. Partial solution assignment 4
*Ruud Egging-Bratseth, 2020. NTNU TIØ4285, Production and Network Economics.

set i suppliers /1*4/;
alias(i,j);

parameter a      intercept of the inv demand curve                     /20/
          b      negative slope of inverse demand curve                / 1/
          c(i)   constant cost term in productoin cost   c*q + d*q^2 /1*3  2/
          d(i)   increasing cost term in productoin cost c*q + d*q^2 /1*4  0.5/
          cap(i) capacity of supplier i                              /1*3  5/
;
c('4')= 99;

positive variable Q(i)    quantity supplied by supplier i
                  lam(i)  capacity dual price i
;
equations stat_q, eq_cap
;

stat_q(i) .. c(i) + 2*d(i)*Q(i) + lam(i) - (a - b*(sum(j,Q(j)) + Q(i))) =G= 0
;
eq_cap(i) ..  cap(i) - Q(i) =G= 0;
;
model olig / stat_q.Q
             eq_cap.lam
/
;
solve olig using mcp
;
parameter rep(*,*,*)  output report;
```

```
parameter rep(*,*,*)  output report;

rep('p','mkt',  '1')= a-b*sum(j,Q.l(j));
rep('q', i,     '1')= Q.l(i);
rep('lam',i,    '1')= lam.l(i);

c(i)$(ord(i)<4)= 1;
solve olig using mcp;
rep('p','mkt',  '2')= a-b*sum(j,Q.l(j));
rep('q', i,     '2')= Q.l(i);
rep('lam',i,    '2')= lam.l(i);

cap(i)$(ord(i)<4)=3;
solve olig using mcp;
rep('p','mkt',  '3')= a-b*sum(j,Q.l(j));
rep('q', i,     '3')= Q.l(i);
rep('lam',i,    '3')= lam.l(i);

cap('2')=4;
cap('3')=5;
solve olig using mcp;
rep('p','mkt',  '4')= a-b*sum(j,Q.l(j));
rep('q', i,     '4')= Q.l(i);
rep('lam',i,    '4')= lam.l(i);

c('4')=1;
cap(i)=3;
solve olig using mcp;
rep('p','mkt',  '5')= a-b*sum(j,Q.l(j));
rep('q', i,     '5')= Q.l(i);
rep('lam',i,    '5')= lam.l(i);

c(i)= 2;
solve olig using mcp;
rep('p','mkt',  '6')= a-b*sum(j,Q.l(j));
rep('q', i,     '6')= Q.l(i);
rep('lam',i,    '6')= lam.l(i);

option rep:2:2:1
display rep;
```

```
                 1          2          3          4          5          6
p   .mkt      9.20       8.60      11.00       9.00       8.00       8.00
q   .1        3.60       3.80       3.00       3.00       3.00       3.00
q   .2        3.60       3.80       3.00       4.00       3.00       3.00
q   .3        3.60       3.80       3.00       4.00       3.00       3.00
q   .4                                                    3.00       3.00
lam.1                               4.00       2.00       1.00
lam.2                               4.00                  1.00
lam.3                               4.00                  1.00
lam.4                                                     1.00
```

	1	2	3	4	5	6
parameter	**value**	**value**	**value**	**value**	**value**	**value**
N	3	3	3	3	4	4
a	20	20	20	20	20	20
b	1	1	1	1	1	1
c_i	2	1	1	1	1	2
cap_1	5	5	3	3	3	3
cap_2	5	5	3	4	3	3
cap_3	5	5	3	5	3	3
cap_4					3	3
variable						
p	9.2	8.6	11	9	8	8
q_1	3.6	3.8	3	3	3	3
q_2	3.6	3.8	3	4	3	3
q_3	3.6	3.8	3	4	3	3
q_4					3	3
λ_1	0	0	4	2	1	0
λ_2	0	0	4	0	1	0
λ_3	0	0	4	0	1	0
λ_4					1	0

Discuss briefly for each supplier how MR=MC in each outcome.

Column 1

- MR_i=a-(N+1)bq_I=20-4x3.6=5.6
- MC_i=c+2dq_I=2+q_i=2+3.6=5.6

Column 2

- MR_i=a-(N+1)bq_I=20-4x3.8=4.8
- MC_i=c+2dq_I=1+q_i=1+3.8=4.8

Column 3

- MR_i=20-4*3=8
- MC=1+3=4
- The difference between MR and MC = lambda = 4!

Column 4

- MR(1)=20-(2x3+4+4)=6
- MC(1)=1+3=4
- Lam(1)=2
- MR(2,3)= 20-(2x4+3+4)=5
- MC(2,3)=1+4=5

Column 5

- MR(i)=20-(3x3+2x3)=5
- MC=1+3=4
- Lam=1

Column 6

- MR(i)=20-(3x3+2x3)=5
- MC(i)2+3=5
- Lam=0

Is the capacity limit restrictive in the last column?

No. The shadow price on the constraint is zero. This means that increasing the capacities will not change the solution.

Exercise 2. Social welfare maximization

SW_assignment.mos

```
!SW maximization on a single node. Solution assignment 4.2
!Ruud Egging-Bratseth, 2020. Lecture notes NTNU TIØ4285, Production and Network Economics.

model "SW_opt"
 uses "mmxprs","mmquad"

parameters
   a=  20     ! intercept of the inverse demand curve
   b=   1     ! negative slope of inverse demand curve
   c=   2     ! constant cost term in productoin cost   c*q + d*q^2
   d=   0.5   ! increasing cost term in productoin cost c*q + d*q^2
end-parameters

declarations
    Q: mpvar;
    Z: qexp;
end-declarations

Q>=0

!def_sw :=
Z:= 0.5*b*Q*Q + (a - b*Q)*Q  - (c + d*Q)*Q

maximize(Z);

writeln("Supply         ", getsol(Q))
writeln("Social welfare: ", getobjval)
writeln("Market price   ", a - b*getsol(Q))

end-model
```

Supply 9
Social welfare: 81
Market price 11

MR(i)=20-9=11

MC(i)=2+2x0.5x9=11

TIØ4285 Production and Network Economics

Assignment 5 – Suggested solution

Exercise 1. Minimum cost flow with quadratic costs

a. The optimization problem

Set Nodes N, Supply nodes $P \subset N, P = \{1,2\}$, Demand nodes $D \subset N, D = \{6,7\}$ Transshipment nodes $T \subset N, T = \{3,4,5\}$. Variables Supply amounts s_i, flows x_{ij}. There are several ways to represent mass balances. Here, assume $s_i = 0, for\ i \notin P, d_i = 0\ for\ i \notin D$

Minimize the sum of total production costs and total transport costs:

$$\min \sum_{i \in P} \{c_i s_i + d_i (s_i)^2\} + \sum_{i \in N \setminus D} \sum_{j \in N \setminus P} f_{ij} x_{ij}^2$$

s.t.

$$s_i + \sum_{j \in N \setminus D} x_{ji} = d_i + \sum_{j \in N \setminus P} x_{ij}\ ,\ i \in N$$

$$s_i \leq cap_i$$

$$s_i, x_{ij} \geq 0$$

b. XPRESS & GAMS implementations.

See separate files

c. Production levels

The production by node 1 is 5.44 and by node 2 is 4.56.

It is not enough to just say: because the constant per unit production cost c_i at node 1 is lower than at node 2, and values for d_i are the same. You have to consider the total (marginal) supply costs for suppliers to demand nodes, including both production and transport costs.

Given the demand level of 6 at node 7, we know that both suppliers supply to node 7. This is only possible if the marginal supply costs are equal (Otherwise using the cheaper option more and the expensive option less would be cheaper.)

Additionally, all arcs carry a positive flow. This means that for supplier 2, the direct flow to node 7, and the routes via node 4 and via node 5, all have the same marginal cost. (Otherwise using the cheaper route more and the expensive route less would be cheaper.)

Supplier 1 also supplies to node 7 via node 4. The marginal transport costs from node 4 to node 7 are the same for both suppliers. Because the cost functions for flows on arc 1-4 and on arc 2-4 are the same, if the flows are the same the marginal costs are the same. But because the constant per unit production cost at node 1 is lower than at node 2, node 1 can supply more, and have a larger flow to node 4, since the higher marginal transport costs are compensated by lower marginal supply cost.

d. All positive flow values:

1.3 2.449, 1.4 1.764
1.6 1.225, 2.4 0.983
2.5 1.952, 2.7 1.627
3.6 2.449, 4.6 0.326
4.7 2.422, 5.7 1.952

A consequence of quadratic costs is, since marginal costs depend on the flow level, that flows are spread out over the arcs, and that there is in fact a unique solution (the problem is strictly convex).

e. The complementarity problem

Minimization objective: see above

Reorder constraints

$$s_i + \sum_{j \in N \setminus D} x_{ji} - d_i - \sum_{j \in N \setminus P} x_{ij} = 0, (\phi_i \; f.i.s.)$$
$$cap_i - s_i \geq 0 \; (\lambda_i \geq 0)$$

Derivation of KKT-conditions

$$0 \leq s_i \perp c_i + d_i s_i - \phi_i + \lambda_i \geq 0$$
$$0 \leq x_{ij} \perp 2 \, f_{ij} x_{ij} + \phi_i - \phi_j \geq 0$$
$$\phi_i \; f.i.s., s_i + \sum_{j \in N \setminus D} x_{ji} - d_i - \sum_{j \in N \setminus P} x_{ij} = 0$$
$$0 \leq \lambda_i \perp cap_i - s_i \geq 0$$

f. GAMS implementation

See separate file

g. Explain the dual to mass balance node 4

Pick either node 1 or node 2 and propagate known values through the equations:

$s_i > 0 \Rightarrow c_i + d_i s_i + \lambda_i = \phi_i$ and $x_{ij} > 0 \Rightarrow 2 \, f_{ij} x_{ij} + \phi_i = \phi_j$.

The values for phi are available in the listing (*.lst) file.
$s_1 = 5.438 \Rightarrow 1 + 025 * 5.438 + 0 = \phi_1$ =2.3595 and $x_{14} = 1.764 \Rightarrow 2 * 0.5 * 1.764 + 2.3595 =$ ϕ_4 =4.1235

The value of the dual for the mass balance at a node indicates the marginal costs to supply at that node (accounting for bottlenecks in capacity if applicable.)

Exercise 2. Facility location

a. Optimization problem

Service nodes P, Demand nodes D

Minimize the sum of total establishment cots and transport costs.

This is literally a problem presented in the lecture:

$$\min \sum_{i\in P}\sum_{j\in D} c_{ij}x_{ij} + \sum_{i\in P} f_i y_i$$

s.t.

$$\sum_{j=1}^{n} x_{ij} \le y_i s_i\,, i \in P$$

$$\sum_{i=1}^{n} x_{ij} = d_j\,, j \in D$$

$$y_i\ binary$$

$$x_{ij} \ge 0$$

b. XPRESS implementation

See separate file.

c. Locations with: facilities?

Location	Base	Double	Eightfold	Tenfold
1	200	200	200	
2	250	250	250	
3	300	300	300	300
4	300			350
5				
6		415	415	
7				
8				515
9	115			
10	600	650	650	650
Total cost	32630	36125	56225	62555

d. The larger the fixed costs, the fewer facilities will be established in a least-cost solution, because it becomes relatively cheaper to transport goods over longer distances.

Exercise 3. Set Coverage

This is literally a problem presented in the lecture, with

Fixed costs $f_i = 1$ and coverage parameters $c_{ij} = \begin{cases} 1 & dist_{ij} \leq cutoff \\ 0 & otherwise \end{cases}$

$$\min \sum_{i \in P} y_i$$

s.t.

$$\sum_{i \epsilon P} c_{ij} y_i \geq 1, \; j \in D$$

$$y_i \; binary$$

See separate file for implemenations

a. **The tables below present all solutions – this is not however required to answer correctly.**

Cut-off	Num facilities	at locations	Comment
25	1	5	unique solution
24	2	(1 or 2 or 3 or 4) & (6 or 7 or 8 or 9 or 10)	Not unique
23	2	(1 or 2 or 3) & (7 or 8 or 9 or 10)	Not unique
22	2	(1 or 2) & (8 or 9 or 10)	Not unique
21	2	1 & (9 or 10)	Not unique

b. **For maximum coverage distances larger than 25 a single facility suffices.**

Note: for maximum coverage distances larger than 29 a single facility at any location suffices.

30	1	Any	Not interesting
29	1	Any location but 10	Not unique
28	1	Any but 1, 9 or 10	Not unique
27	1	3,4,5,6 or 7	Not unique
26	1	4, 5, or 6	Not unique

c. **For maximum coverage distances below 21 the problem is infeasible, since not service location is within 21 of demand location 11.**

<=20	infeasible	Location 11 cannot be covered by any potential facility for distances below 21

NTNU Department of Industrial Economics and Technology Management
Spring 2020

TIØ4285 Production and Network Economics
Assignment 6 – Suggested solution

Exercise 1. Hybrid market

a. Firm D: considers the inverse demand curve $p(Q)$ - immediately in standard form:

$$\begin{aligned} min_{q_D} Z_D &= (c_D(q_D) - p(Q)q_D) \\ &= (c_D q_D + d_D q_D^2 - (a - b(q_D + q_F))q_D) \\ &= \left(c_D q_D + d_D q_D^2 - (a q_D - b(q_D + q_F)q_D)\right) \\ &= \left((c_D - a + b q_F)q_D + (b + d_D)q_D^2\right) \end{aligned}$$

Fringe F: considers market price $\hat{p}$ as given

$$\begin{aligned} min_{q_F} Z_F &= (c_F(q_F) - \hat{p} q_F) \\ &= (c_F q_F + d_F q_F^2 - \pi q_F) \\ &= \left((c_F - \pi)q_F + d_F q_F^2\right) \end{aligned}$$

b. MCP

- Firm D

$$0 \leq q_D \perp (c_D - a + b q_F) + 2(b + d_D)q_D \geq 0$$

- Fringe F:

$$0 \leq q_F \perp (c_F - \pi) + 2 d_F q_F$$

$$= \left(c_F - \left(a - b(q_D + q_F)\right)\right) + 2 d_F q_F$$

$$= (c_F - a + b q_D) + (b + 2 d_F)q_F \geq 0$$

c. To facilitate a compact and intuition supporting implementation, introduce conjectural variation parameter CV_i: $CV_F = 1, CV_D = 0$ and reorder conditions to an MC=MR perspective:

$(c_D - a + b q_F) + 2(b + d_D)q_D \geq 0$	$(c_F - a + b q_D) + (b + 2 d_F)q_F \geq 0$
$c_D + 2 d_D q_D \geq \mathrm{a} - b q_F - 2\mathrm{b} q_D$	$c_F + (b + 2 d_F)q_F \geq a - b q_D - b q_f$
$c_D + 2 d_D q_D \geq \mathrm{a} - b(q_F + q_D) - \mathrm{CV_D b} q_D$	$c_F + 2 d_F q_F \geq a - b(q_F + q_D) - \mathrm{CV_F} b q_D$

Both expressions can be written as: $c_i + 2 d_i q_i \geq a - b \sum_j q_j - \mathrm{CV_i} b q_i$. This is implemented in file *asn_3_2_hybrid.gms*.

d. You can use the model implementation to find the answer.
For illustrative purposes, here the calculated the results are presented:
Assume $q_F > 0$. Then: $(c_F - a + bq_D) + (b + 2d_F)q_F \ = 0 \Rightarrow q_F \ = \frac{(a-c_F-bq_D)}{(b+2d_F)}$ (*note: this is the optimal response curve of the fringe, which we need in assignment 3.*)
Substitute this into the other KKT, for the dominant firm, and assume its supply is positive:

$$q_D > 0 \Rightarrow (c_D - a + bq_F) + 2(b + d_D)q_D \ = 0$$
$$= \left(c_D - a + b\frac{(a - c_F - bq_D)}{(b + 2d_F)}\right) + 2(b + d_D)q_D$$
$$= \left(c_D - a + \frac{b(a - c_F)}{(b + 2d_F)}\right) + \left(2(b + d_D) - \frac{(b^2)}{(b + 2d_F)}\right)q_D$$
$$\Rightarrow q_D = \frac{\left(a - c_D - \frac{b(a - c_F)}{(b + 2d_F)}\right)}{\left(2(b + d_D) - \frac{(b^2)}{(b + 2d_F)}\right)}$$

Substitute in parameter values $a = 100, b = 2, c_D = 10, d_D = \frac{1}{2}, c_F = 20, d_F = 1.$

$$\boldsymbol{q_D} = \frac{100-10-\frac{2(100-20)}{(2+2)}}{2\left(2+\frac{1}{2}\right)-\frac{(2^2)}{(2+2)}} = \frac{50}{4} = \mathbf{12\tfrac{1}{2}},\ \boldsymbol{q_F} \ = \frac{\left(100-20-2\frac{50}{4}\right)}{(2+2)} = \frac{55}{4} = \mathbf{13\tfrac{3}{4}}, \boldsymbol{p} = 100 - 2\left(\frac{50}{4} + \frac{55}{4}\right) = \mathbf{47\tfrac{1}{2}}$$

Exercise 2. Dominant firm with a competitive fringe

Derive the optimal response function for the fringe (or take it from the solution from assignment 1 above) and substitute this into the optimization problem of the dominant firm.

$$q_F = \frac{(a - c_F - bq_D)}{(b + 2d_F)}$$

a. Firm D – immediately in standard form:

$$min_{q_D}\, Z_D = \left(c_D(q_D) - p(Q)q_D\right) =$$

$$= \left(c_D q_D + d_D q_D^2 - \left(a - b(q_D + q_F)\right)q_D\right)$$

$$= \left(c_D + d_D q_D - \left(a - b\left(q_D + \frac{(a - c_F - bq_D)}{(b + 2d_F)}\right)\right)\right)q_D =$$

$$= \left(c_D - a + \frac{b(a - c_F)}{(b + 2d_F)}\right)q_D + \left(d_D + b - \frac{b^2}{(b + 2d_F)}\right){q_D}^2$$

b. See file asn_3_2_dom_firm

c. MCP

- Firm D

$$0 \le q_D \perp \left(c_D - a + \frac{b(a - c_F)}{(b + 2d_F)}\right) + 2\left(d_D + b - \frac{b^2}{(b + 2d_F)}\right)q_D \ \ge 0$$

$$q_D > 0 \Rightarrow q_D \ = \frac{a - c_D - \dfrac{b(a - c_F)}{(b + 2d_F)}}{2\left(d_D + b - \dfrac{b^2}{(b + 2d_F)}\right)}$$

Fill out the parameter values $a = 100, b = 2, c_D = 10, d_D = \frac{1}{2}, c_F = 20, d_F = 1$:

$\boldsymbol{q_D} = \frac{50}{3} = \mathbf{16\frac{2}{3}}, \mathbf{q_F} = \frac{35}{3} = \mathbf{11\frac{2}{3}}, \boldsymbol{p} = \frac{130}{3} = \mathbf{43\frac{1}{3}}$

d. The dominant firm supplies more than in the previous assignment, because it knows that the fringe will respond by supplying less. Even though market price is lower, firm D makes higher profit.

Dominant firm's profit: $p(Q)q_D - c_D(q_D) = \left(\pi - 10 - \frac{1}{2}q_D\right)q_D =$

$$\left(\frac{130}{3} - 10 - \frac{1}{2}\frac{50}{3}\right)\frac{50}{3} = \left(\frac{260 - 60 - 50}{6}\right)\frac{50}{3} = \frac{150.50}{18} = 416\frac{2}{3}$$

Profit of market power exerting firm exercise 1: $p(Q)q_D - c_D(q_D) = \left(\pi - 10 - \frac{1}{2}q_D\right)q_D =$

$$\left(\frac{95}{2} - 10 - \frac{1}{2}\frac{25}{2}\right)\frac{25}{2} = \left(\frac{190 - 40 - 25}{4}\right)\frac{25}{2} == \frac{125.25}{8} = 390\frac{5}{8}$$

Exercise 3. Duopoly on a small network with transport losses

a. Optimization problem suppliers. We write in standard form, and assign dual variables. Account for losses by adjusting nodal inflows in the mass balance. Account for the losses on the inflows. f.i.s. means free in sign $\left(x\ f.i.s\right) \equiv \left(x \in \mathbb{R}\right)$

$$
\begin{aligned}
&\min_{q^P_{in}, q^S_{in}, f^P_{imn}} && \left\{ c_{in} q^P_{in} + \sum_{(n,m)} \left(c^A_{nm} + \tau^A_{nm}\right) f^P_{inm} - \sum_n \left(a_n - b_n \sum_j q^S_{jn} \right) q^S_{in} \right\}, && i = 1,2 \\
&s.t. && q^S_{in} + \sum_m \left(1 - l^A_{mn}\right) f^P_{imn} - q^P_{in} - \sum_m f^P_{inm} = 0 \qquad \left(\varphi^P_{in}\ f.i.s.\right), && n \in 1,2,3 \\
& && q^P_{in}, q^S_{in}, f^P_{inm} \geq 0
\end{aligned}
$$

b. Optimization problem TSO

$$
\begin{aligned}
&\min_{f^A_{nm}} && -\sum_{(n,m)} \left(\tau^A_{nm} f^A_{nm}\right) \\
&s.t. && f^A_{nm} - cap^A_{nm} \leq 0 \qquad \left(\lambda^A_{nm} \geq 0\right), \qquad \left(n,m\right) \in \left\{(1,2),(2,3)\right\} \\
& && f^A_{nm} \geq 0
\end{aligned}
$$

c. Market clearing condition for transportation services

$$
\sum_i f^P_{inm} - f^A_{nm} = 0 \qquad \left(\tau^A_{nm}\ f.i.s.\right), \qquad \left(n,m\right) \in \left\{(1,2),(2,3)\right\}
$$

d. Because supplier 2 cannot supply at node 1, supplier 1 is a single supplier with market power on node 1, hence a monopolist on node 1.

e. The complementarity conditions defining the equilibrium problem:

Suppliers

	$0 \leq q^P_{in} \perp c^P_{in} - \phi^P_{in} \geq 0$	(1)
	$0 \leq q^S_{in} \perp -a_n + b_n\left(\sum_j q^S_{jn} + q^S_{in}\right) + \phi^P_{in} \geq 0$	(2)
	$0 \leq f^P_{inm} \perp c^A_{nm} + \tau^A_{nm} + \phi^P_{in} - (1 - l^A_{nm})\phi^P_{im} \geq 0$	(3)
	$0 \leq \phi^P_{in} \perp q^S_{in} + \sum_m (1 - l^A_{nm}) f^P_{inm} - q^P_{inm} - \sum_m f^P_{inm} \geq 0$	(5)

TSO

	$0 \leq f^A_{nm} \perp -\tau^A_{nm} + \lambda^A_{nm} \geq 0$	(6)
	$0 \leq \lambda^A_{nm} \perp cap^A_{nm} - f^A_{nm} \geq 0$	(7)

MCC transport services

	τ^A_{nm} f.i.s., $\sum_i f^P_{inm} - f^A_{nm} = 0$	(8)

f. See *asn_3_3_duop_netw.gms.*

node	prod	flow+	sales	flow-	price
1	7.283		4	3.283	6
2	5.934	2.955	4.889	4	5.111
3		3.6	3.6		6.4

NTNU
Department of Industrial Economics and Technology Management
Spring 2020

TIØ4285 Production and Network Economics
Assignment 7 – Proposed Solution

Out: Thursday 27 February
In: Thursday 5 March, 6pm
Supervision: Monday 2 March, 4:15pm A31

Note that late exercises will not be approved.

Exercise 1

The company Bare Blomster AS has 2 divisions, the market garden that produces tulips and the retail division that sells these to the end customers. The market garden has the following cost function per batch of tulips q:

$$TC_G = 50 \cdot q + 2 \cdot q^2.$$

The retail division observes the following demand for batches of tulips in the market:

$$p = 150 - 0.5 \cdot q.$$

Its total cost function is given as

$$TC_B = 10 \cdot q.$$

a) Find the company's net marginal revenue curve as seen from the market garden. How should this curve be interpreted?

The net marginal revenue curve is given as $NMR = MR - MC_B = 150 - q - 10 = 140 - q$

Net Marginal Revenues represent Bare Blomster's marginal income of producing one additional unit as seen by the market garden (i. e. after the shop's marginal costs are deducted). Alternatively, it can be interpreted as the shop's demand function for "buying" tulips from the market garden.

b) Maximize the company's profits. What are the optimal profits? Which transfer price is used between the different divisions and what is the price for the end customer?

We get the following:

$$TC_T = TC_G + TC_B = 60 \cdot q + 2 \cdot q^2 \qquad \Rightarrow \qquad MC_T = 60 + 4 \cdot q$$
$$MR = 150 - q$$

Profit maximization for $MR = MC_T$: $150 - q = 60 + 4 \cdot q \quad \Rightarrow q^* = 18$
End-customer price: $P = 150 - 0.5 \cdot q = 141$.

Transfer price equal marginal cost of production: $MC_G(q^*) = 122$
Profits market garden: $\Pi_G = 122 \cdot 18 - 50 \cdot 18 - 2 \cdot 182 = 648$
Profits shop: $\Pi_B = 141 \cdot 18 - (10 + 122) \cdot 18 = 162$
Total profit: $\Pi_T = 810$

Exercise 2

Saftbolaget has a monopoly in the Nordic juice market. It consists of two departments: a production department and a retailing department. The manufacturing cost is given as

$$C_P = 250 + 8 \cdot q + \frac{1}{20} \cdot q^2,$$

whereas the retailing cost can be described by the following function

$$C_R = 225 + 9 \cdot q.$$

Demand in the Nordic juice market is given as

$$p(q) = 62 - \frac{1}{10} \cdot q.$$

a) How much juice should the company sell in the Nordic market to maximize profits? What are the optimal profits?

Marginal revenues are given as $MR = 62 - \frac{1}{5}q$. Saftbolagets marginal costs are simply the sum of the marginal cost of production and retailing: $MC = MC_P + MC_R = 8 + \frac{1}{10}q + 9 = 17 + \frac{1}{10}q$.

Using the optimality criterion ($MR = MC$), we get

$$\begin{aligned} 62 - \frac{1}{5}q &= 17 + \frac{1}{10}q \\ \frac{3}{10}q &= 45 \\ q &= 150 \end{aligned}$$

Selling $q^* = 150$ units of juice results in overall profits equal to $\Pi = p(q^*) \cdot q * -C_R(q^*) - C_P(q^*) = 2\,900$.

b) What is the transfer price for juice used between the production department and the retailing department?

The market price that end customers have to pay is $p(q^*) = 47$. To achieve optimal profits, we need to exchange juice at marginal cost of production, i. e. $TP = MC_P(q = 150) = 23$. Any other transfer price will distort the amount of juice ordered by the retailing division.

To increase profits, Saftbolaget considers the possibility of the production department selling juice to retailers in Southern markets. The demand function of these external retailers is given as

$$p_e(q) = 33 - \frac{1}{30} \cdot q_e.$$

Note that the demand of the external retailers does not influence the demand in the Nordic market. Saftbolaget does not incur retailing costs for juice sold to external retailers.

c) How much juice (if any) should be sold in the different markets? What are the optimal profits and which transfer prices should Saftbolaget use?

Marginal revenues from internal sales are given by the Net Marginal Revenue for producing juice, $NMR_P = MR - MC_R = 53 - \frac{1}{5}q_i$. Marginal revenues from external sales are derived from the external demand function $p_e(q_e)$, $MR_e = 33 - \frac{1}{15}q_e$. We determine the total net marginal revenue by horizontally summing NMR_P and MR_e. Note that we do not sell juice to the external market for outputs $q = q_i + q_e < 100$:

$$\left.\begin{aligned} q_i &= 265 - 5NMR_P \\ q_e &= 495 - 15MR_e \end{aligned}\right\} \quad \Rightarrow NMR_{Total} = 38 - \frac{1}{20}q$$

Setting these marginal revenues equal to marginal cost of production, we determine the optimal amount of produced juice:

$$
\begin{aligned}
NMR_{Total} &= MC_P \\
38 - \frac{1}{20}q &= 8 + \frac{1}{10}q \\
\frac{3}{20}q &= 30 \qquad \Rightarrow \qquad q^* = 200
\end{aligned}
$$

Sales to the external market

$$
\begin{aligned}
MR_e = MC_P(q = 200) \qquad \Rightarrow \qquad 33 - \frac{1}{15}q_e &= 28 \\
\frac{1}{15}q_e &= 5 \qquad \Rightarrow \qquad q_e = 75 \\
& \qquad\qquad\qquad\quad\; p_e = 30.5
\end{aligned}
$$

Sales to the internal market

$$
\begin{aligned}
NMR_P = MC_P(q = 200) \qquad \Rightarrow \qquad 53 - \frac{1}{5}q_i &= 28 \\
\frac{1}{5}q_i &= 25 \qquad \Rightarrow \qquad q_i = 125 \\
& \qquad\qquad\qquad\quad\; TP = 28 \\
& \qquad\qquad\qquad\quad\;\; p = 49.5
\end{aligned}
$$

resulting in the following overall profit

$$
\begin{aligned}
\Pi(q^*) &= 30.5 \cdot 75 + 49.5 \cdot 125 - 75 - 6 \cdot 125 - 150 - 3 \cdot 125 - 250 - 8 \cdot 200 - \frac{1}{20} \cdot 200^2 \\
&= 3\,275.
\end{aligned}
$$

Another way of determining the optimal production and sales quantities is to start by setting up the equation for total company profits:

$$\Pi(q) = p(q_i) \cdot q_i + p_e(q_e) \cdot q_e - C_R(q_i) - C_P(q_i + q_e).$$

Then you have to derive the profit function with respect to both q_i and q_e, set both equal to 0 and solve the resulting set of 2 equation with 2 unknowns.

NTNU
Department of Industrial Economics and Technology Management
Spring 2019

TIØ4285 Production and Network Economics
Assignment 8 – Solution

Out: Thursday 5 March
In: Thursday 12 March, 6pm
Supervision: Monday 9 March, 4:15pm A31

Note that late exercises will not be approved.

Exercise

PetrolAir offers fast and very luxurious flight connections for the top managers of Arabian oil companies. They plan to set up a new route between Dubai and Riyadh using small jets (capacity per jet is one passenger). In particular, they have to decide how many planes to use before they know how many tickets they will sell eventually.

The planes will be rented from Planes-R-Us for 10,000 Petrodollars (P$) per plane. PetrolAir charges 30,000 P$ for the exclusive journey from Dubai to Riyadh. As all airlines, they will sell tickets to everyone who wants one and deal with overbooking later. Daily ticket sales are uncertain and have a discrete probability distribution. The distribution has the following realizations that are equally likely:

$$6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25$$

In case they sell more tickets than they have planes, PetrolAir incurs a cost of 15,000 P$ in order to transfer the managers to other flights. To prevent planes from flying empty, PetrolAir sells free seats at Last-Minute prices of 5,000 P$. This Last-Minute market may be viewed as having perfect competition.

Use the following notation:

r – Revenues from selling tickets to the customers
p – Costs of overbooking and transferring passengers
w – Rental price per plane
s – Revenue per Last-Minute ticket in the perfect competition market
D – Demand / Number of tickets sold
C – Capacity / Number of planes rented from Planes-R-Us

The optimality condition for maximizing Petrol Air's profit is given by $\Pr(D < C) = \frac{p-w}{p-s}$.

You are put in charge of planning the new connection.

a) Please express the formula for PetrolAir's expected daily profit as a function of D and C.

$$\Pi = E\left[r \cdot D + s \cdot (C - D)^+ - p \cdot (D - C)^+ - w \cdot C\right]$$

b) How many planes should you rent from Planes-R-Us? What is the expected profit per day?

$$\Pr(D < C) = \frac{p-w}{p-s} = \frac{15-10}{15-5} = \frac{5}{10} = 0.5 \quad \Rightarrow \quad C = 15$$
$$\Pi(C = 15) = 285'.$$

Note that $C = 16$ will result in the same expected profits.

c) Your next task is to convince the board of directors that you did a good job determining the capacity requirements. Show that the optimality criterion is indeed given by $\Pr(D < C) = \frac{p-w}{p-s}$.

Using $C = D - (D-C)^+ + (C-D)^+$, we can reformulate the formula for expected profit:

$$\begin{aligned}
\Pi &= E\big[r \cdot D + s \cdot (C-D)^+ - p \cdot (D-C)^+ - w \cdot C\big] \\
&= E\big[r \cdot D + s \cdot (C-D)^+ - p \cdot (D-C)^+ - \\
&\quad w \cdot \big(D - (D-C)^+ + (C-D)^+\big)\big] \\
&= E\big[r \cdot D + s \cdot (C-D)^+ - p \cdot (D-C)^+ - w \cdot D + \\
&\quad w \cdot (D-C)^+ - w \cdot (C-D)^+\big] \\
&= (r-w)ED + E\big[s \cdot (C-D)^+ - p \cdot (D-C)^+ - w \cdot (D-C)^+ - \\
&\quad (C-D)^+\big] \\
&= (r-w)ED + E\big[(s-w) \cdot (C-D)^+ - (p-W) \cdot (D-C)^+\big] \\
\frac{d\Pi}{dC} &= (s-w)\Pr(D < C) + (p-w) \cdot \Pr(C < D) \\
&= (s-w)\Pr(D < C) + (p-w) \cdot (1 - \Pr(D < C)) \\
&= (s-w)\Pr(D < C) + (p-w) - (p-w) \cdot \Pr(D < C) \\
&= (p-w) + (s-w-p+w) \cdot \Pr(D < C) \\
&= (p-w) - (p-s) \cdot \Pr(D < C)
\end{aligned}$$

$$\frac{d\Pi}{dC} = 0 \quad \Rightarrow \quad \Pr(D < C) = \frac{p-w}{p-s}.$$

The board is impressed by your work. However, they believe the company should order 20 planes to capture a market share big enough to prevent competitors from entering into the market. You, of course, tell them that this would only be optimal if there is a shift in the price in the last-minute market.

d) What price would you need in the Last-Minute market to make an order of 20 planes optimal?

With $C = 20$, we have $\Pr(D < C) = 0.75 \quad \Rightarrow \quad = 8.333'$.

The board of directors considers such a price shift for Last-Minute tickets as unlikely. They suggest instead that PetrolAir and Planes-R-Us should coordinate their decisions and ask you to perform the analysis. Planes-R-Us supports the idea and shares their variable cost function with you. The variable operating costs of the planes are 7,500 P$. You don't have to consider their fixed costs.

e) Please suggest an optimal Profit-Sharing contract that both companies are willing to accept. Determine the capacity that PetrolAir should book in order to maximize the supply chain profits.

Optimal supply chain profits given for $\Pr(D < C) = \frac{p-c}{p-s}$, with c being the variable operating costs of the plane. Optimal order quantity and profits are $C = 20, \Pi = 330'$ ($C = 21$ results in the same profits).

Expected profits for Planes-R-Us:

$$\begin{aligned} \pi(C = 15) &= 37.5' \\ \pi(C = 16) &= 40' \end{aligned}$$

Expected supply chain profits:

$$\begin{aligned} \Pi(C = 15) &= 322.5' \\ \Pi(C = 16) &= 325' \end{aligned}$$

The Profit-Sharing contract uses marginal costs as wholesale price and distributes profits by PetrolAir paying a percentage of her profits to Planes-R-Us. In order to accept the contract, both companies must make at least as much profit as in the situation without the contract.

PetrolAir has to make at least $285'$, restricting the share paid to Plane-R-Us to

$$\left(1 - \frac{285}{330}\right) \cdot 100\% = 13.64\%$$

Planes-R-Us needs at least a profit of $37.5'$, thus requiring a share of at least

$$\frac{37.5}{330} \cdot 100\% = 11.36\%$$

Using a reservation profit of $40'$ (based on $C = 16$), is also correct.

f) Explain how Planes-R-Us' fixed costs influence the optimal number of planes.

Fixed costs do not affect the optimal solution.