TIØ4285 Production and Network Economics
Assignment 5 – Suggested solution

Exercise 1. Minimum cost flow with quadratic costs

a. The optimization problem

Set Nodes N, Supply nodes $P \subset N, P = \{1,2\}$, Demand nodes $D \subset N, D = \{6,7\}$ Transshipment nodes $T \subset N, T = \{3,4,5\}$. Variables Supply amounts s_i, flows x_{ij}. There are several ways to represent mass balances. Here, assume $s_i = 0, for\ i \notin P$, $d_i = 0\ for\ i \notin D$

Minimize the sum of total production costs and total transport costs:

$$\min \sum_{i \in P} \{c_i s_i + d_i (s_i)^2\} + \sum_{i \in N \setminus D} \sum_{j \in N \setminus P} f_{ij} x_{ij}^2$$

s.t.

$$s_i + \sum_{j \in N \setminus D} x_{ji} = d_i + \sum_{j \in N \setminus P} x_{ij}\ ,\ i \in N$$

$$s_i \leq cap_i$$

$$s_i, x_{ij} \geq 0$$

b. XPRESS & GAMS implementations.

See separate files

c. Production levels

The production by node 1 is 5.44 and by node 2 is 4.56.

It is not enough to just say: because the constant per unit production cost c_i at node 1 is lower than at node 2, and values for d_i are the same. You have to consider the total (marginal) supply costs for suppliers to demand nodes, including both production and transport costs.

Given the demand level of 6 at node 7, we know that both suppliers supply to node 7. This is only possible if the marginal supply costs are equal (Otherwise using the cheaper option more and the expensive option less would be cheaper.)

Additionally, all arcs carry a positive flow. This means that for supplier 2, the direct flow to node 7, and the routes via node 4 and via node 5, all have the same marginal cost. (Otherwise using the cheaper route more and the expensive route less would be cheaper.)

Supplier 1 also supplies to node 7 via node 4. The marginal transport costs from node 4 to node 7 are the same for both suppliers. Because the cost functions for flows on arc 1-4 and on arc 2-4 are the same, if the flows are the same the marginal costs are the same. But because the constant per unit production cost at node 1 is lower than at node 2, node 1 can supply more, and have a larger flow to node 4, since the higher marginal transport costs are compensated by lower marginal supply cost.

d. All positive flow values:

1.3 2.449, 1.4 1.764
1.6 1.225, 2.4 0.983
2.5 1.952, 2.7 1.627
3.6 2.449, 4.6 0.326
4.7 2.422, 5.7 1.952

A consequence of quadratic costs is, since marginal costs depend on the flow level, that flows are spread out over the arcs, and that there is in fact a unique solution (the problem is strictly convex).

e. The complementarity problem

Minimization objective: see above

Reorder constraints

$$s_i + \sum_{j \in N \setminus D} x_{ji} - d_i - \sum_{j \in N \setminus P} x_{ij} = 0, (\phi_i\ f.i.s.)$$
$$cap_i - s_i \geq 0\ (\lambda_i \geq 0)$$

Derivation of KKT-conditions

$$0 \leq s_i \perp c_i + d_i s_i - \phi_i + \lambda_i \geq 0$$
$$0 \leq x_{ij} \perp 2\ f_{ij} x_{ij} + \phi_i - \phi_j \geq 0$$
$$\phi_i\ f.i.s., s_i + \sum_{j \in N \setminus D} x_{ji} - d_i - \sum_{j \in N \setminus P} x_{ij} = 0$$
$$0 \leq \lambda_i \perp cap_i - s_i \geq 0$$

f. GAMS implementation

See separate file

g. Explain the dual to mass balance node 4

Pick either node 1 or node 2 and propagate known values through the equations:

$s_i > 0 \Rightarrow c_i + d_i s_i + \lambda_i = \phi_i$ and $x_{ij} > 0 \Rightarrow 2\ f_{ij} x_{ij} + \phi_i = \phi_j$.

The values for phi are available in the listing (*.lst) file.
$s_1 = 5.438 \Rightarrow 1 + 025 * 5.438 + 0 = \phi_1$ =2.3595 and $x_{14} = 1.764 \Rightarrow 2 * 0.5 * 1.764 + 2.3595 =$ ϕ_4 =4.1235

The value of the dual for the mass balance at a node indicates the marginal costs to supply at that node (accounting for bottlenecks in capacity if applicable.)

Exercise 2. Facility location

a. Optimization problem

Service nodes P, Demand nodes D

Minimize the sum of total establishment cots and transport costs.

This is literally a problem presented in the lecture:

$$\min \sum_{i\in P} \sum_{j\in D} c_{ij} x_{ij} + \sum_{i\in P} f_i y_i$$

s.t.

$$\sum_{j=1}^{n} x_{ij} \leq y_i s_i \, , i \in P$$

$$\sum_{i=1}^{n} x_{ij} = d_j \, , j \in D$$

$$y_i \; binary$$

$$x_{ij} \geq 0$$

b. XPRESS implementation

See separate file.

c. Locations with: facilities?

Location	Base	Double	Eightfold	Tenfold
1	200	200	200	
2	250	250	250	
3	300	300	300	300
4	300			350
5				
6		415	415	
7				
8				515
9	115			
10	600	650	650	650
Total cost	32630	36125	56225	62555

d. The larger the fixed costs, the fewer facilities will be established in a least-cost solution, because it becomes relatively cheaper to transport goods over longer distances.

Exercise 3. Set Coverage

This is literally a problem presented in the lecture, with

Fixed costs $f_i = 1$ and coverage parameters $c_{ij} = \begin{cases} 1 & dist_{ij} \leq cutoff \\ 0 & otherwise \end{cases}$

$$\min \sum_{i \in P} y_i$$

s.t.

$$\sum_{i \in P} c_{ij} y_i \geq 1, \; j \in D$$

$$y_i \; binary$$

See separate file for implemenations

a. **The tables below present all solutions – this is not however required to answer correctly.**

Cut-off	Num facilities	at locations	Comment
25	1	5	unique solution
24	2	(1 or 2 or 3 or 4) & (6 or 7 or 8 or 9 or 10)	Not unique
23	2	(1 or 2 or 3) & (7 or 8 or 9 or 10)	Not unique
22	2	(1 or 2) & (8 or 9 or 10)	Not unique
21	2	1 & (9 or 10)	Not unique

b. **For maximum coverage distances larger than 25 a single facility suffices.**

Note: for maximum coverage distances larger than 29 a single facility at any location suffices.

30	1	Any	Not interesting
29	1	Any location but 10	Not unique
28	1	Any but 1, 9 or 10	Not unique
27	1	3,4,5,6 or 7	Not unique
26	1	4, 5, or 6	Not unique

c. **For maximum coverage distances below 21 the problem is infeasible, since not service location is within 21 of demand location 11.**

<=20	infeasible	Location 11 cannot be covered by any potential facility for distances below 21