Lecture 2: Inventory Models and Risk Pooling

TIØ4285 Production and Network Economics

Spring 2020

Norwegian University of Science and Technology

Outline

- Supply Chain Design under Uncertainty
- Classic inventory models
  - Economic Order Quantity (aka Economic Lot Size)
  - The (s,S) Inventory Policy
- Risk Pooling
- The Newsboy Problem

What is Stochastic Programming?

- Mathematical programming is about decision making
- Stochastic programming is about decision making under uncertainty
  - Mathematical programming with uncertain parameters
- Fundamental assumption: we know a (joint) probability distribution
  - We may not need to know the whole joint distribution (the focus is normally on some random variables)
  - A subjective specification of the joint distribution can also provide useful information / insight to the problem

Why Stochastic Programming?

- To find all the *explicit and implicit options* worth paying for
  - A spare bus in case of breakdowns
  - Extra ground time for a plane to absorb small delays
  - A financial instrument to reduce variations in income from trade in several currencies
  - A different schedule which is simply more robust in light of delays

- In order to do this, we must

  Model stages …

  … and thereby the information structure

  Study random phenomena …

  … and decide how to represent them

  Update our old modeling skills …

  … not the least how we treat constraints

Recourse Decisions

- Consider a problem with two stages. The following sequence of events occurs:
  1. We make a decisions now (first-stage decision)
  2. Nature makes a random decision (high/low, wet/dry, etc…)
  3. We make a second stage decision that attempts to correct some of the problems caused by (2)
  4. (in a multi-stage problem, nature makes another decision, we make another corrective action, and so on…)
- The second stage decision are called recourse decisions

- The goal of a two-stage model is to identify a first-stage decision that is well positioned against all possible realizations of the random variables

Remember last week?

- Consider the following optimization problem:

$$\max 3x + 2y + z$$

subject to

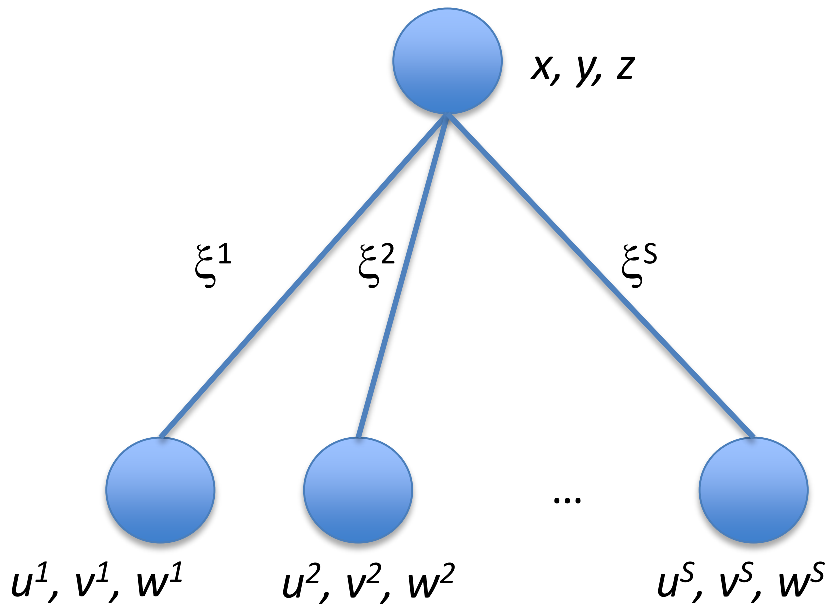
$$
\begin{aligned}
x &\leq \xi \\
y &\leq 1 - \xi \\
x + y + z &\leq 1 \\
x, y, z &\geq 0
\end{aligned}
$$

- ξ is an unknown parameter (demand) and *x, y, z* have to be chosen before ξ is known (production)

What happens in the real world?

- We first choose a production level (*x*,*y*,*z*)
- We then observe demand ξ
- We then sell our products according to demand (*u*, *v*, *w*)

Two-stage structure



Stochastic Programming formulation

- General two-stage SP model

$$\min c^{\mathsf{T}} x + Q(x)$$

subject to

$$Ax = b,$$
$$x \geq 0,$$

First stage (valid for all scenarios)

where

$$Q(x) = \sum_{s} p^s Q(x, \xi^s)$$

Second stage (valid for each scenario, *S* different problems)

and

$$Q(x, \xi) = \min\{q(\xi)^{\mathsf{T}} y | W(\xi) y = h(\xi) - T(\xi) x, y \geq 0\}$$

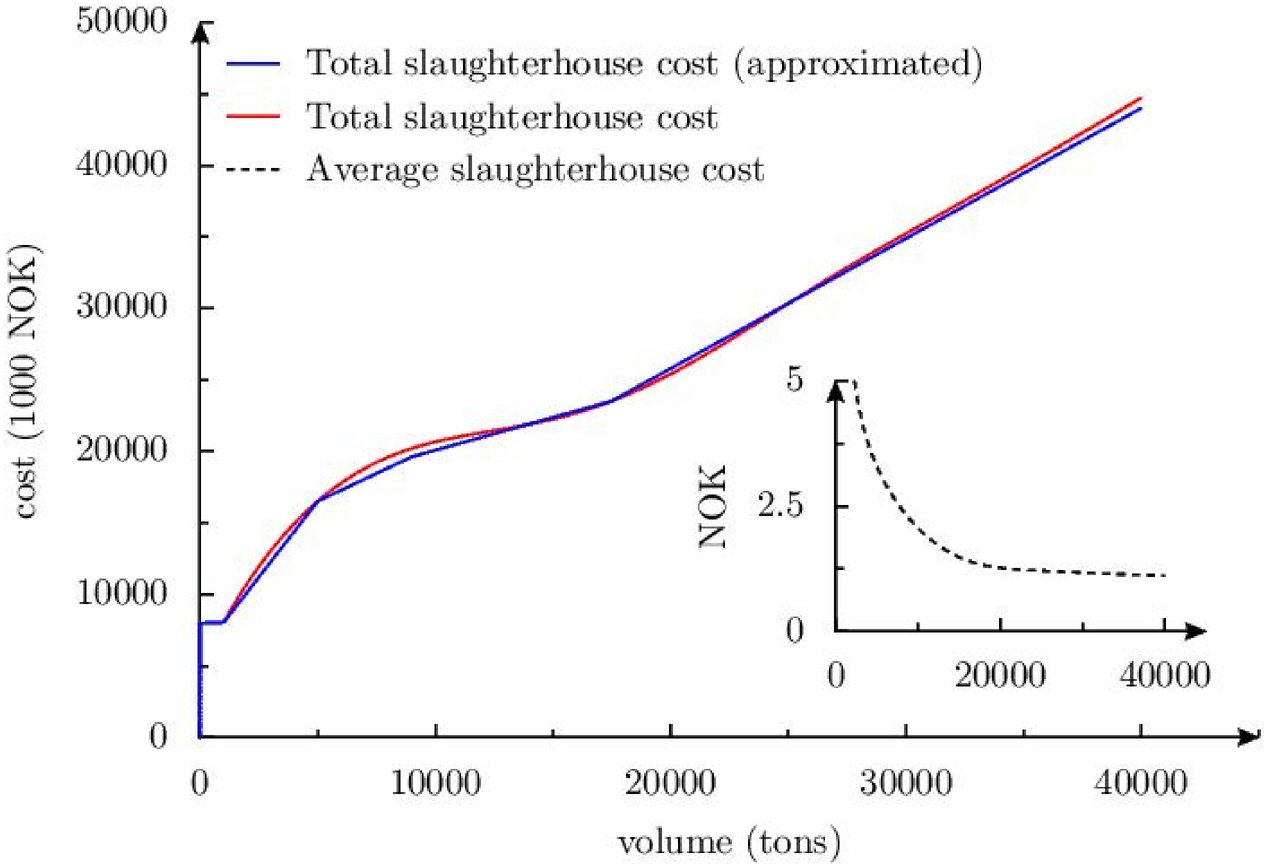
The slaughterhouses again…

- Let’s assume that future demand for slaughtering services is uncertain

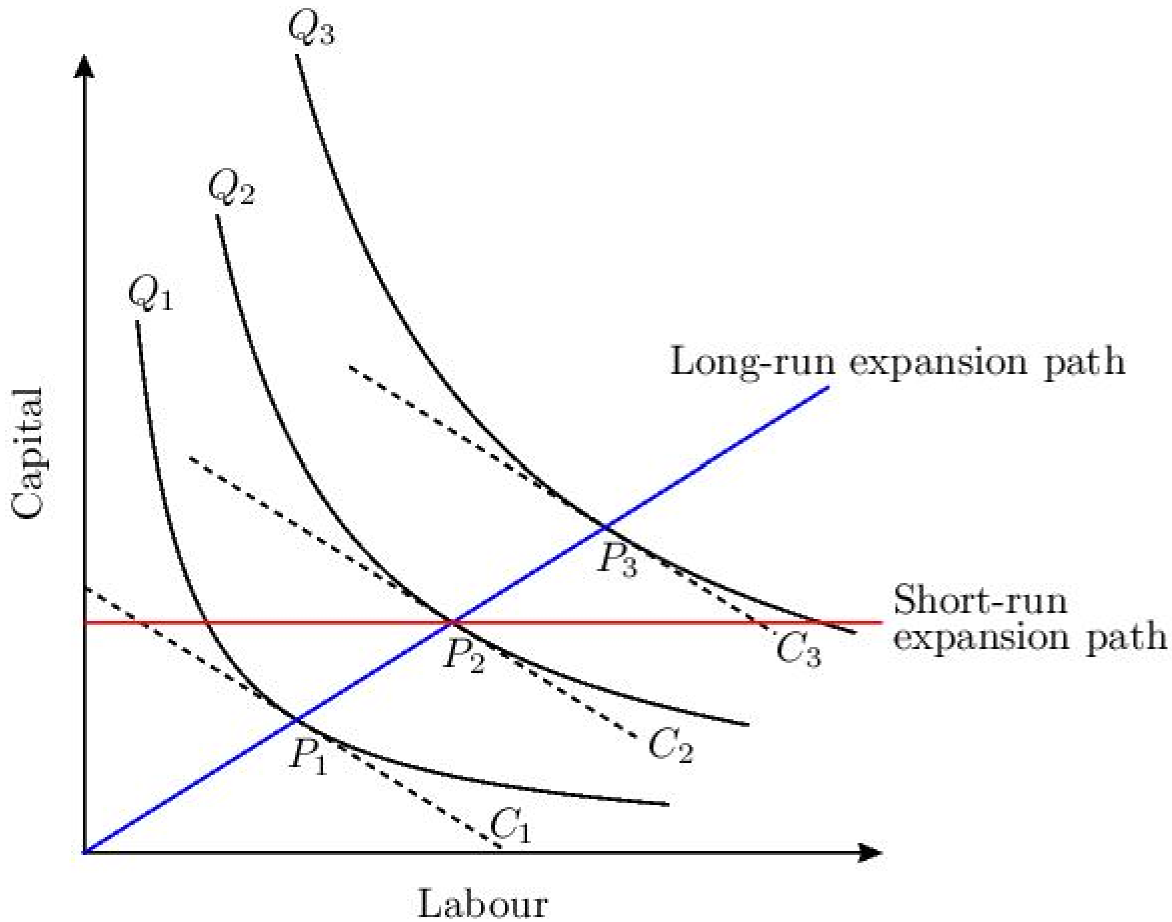
- How does the decision-information structure look like?
  - First, we need to build slaughterhouses
  - Then, we observe demand
  - Last, we satisfy demand

- This is a two stage structure, but what does this new structure mean for our model?
  - What happens if realized demand deviates from the one we planned for?

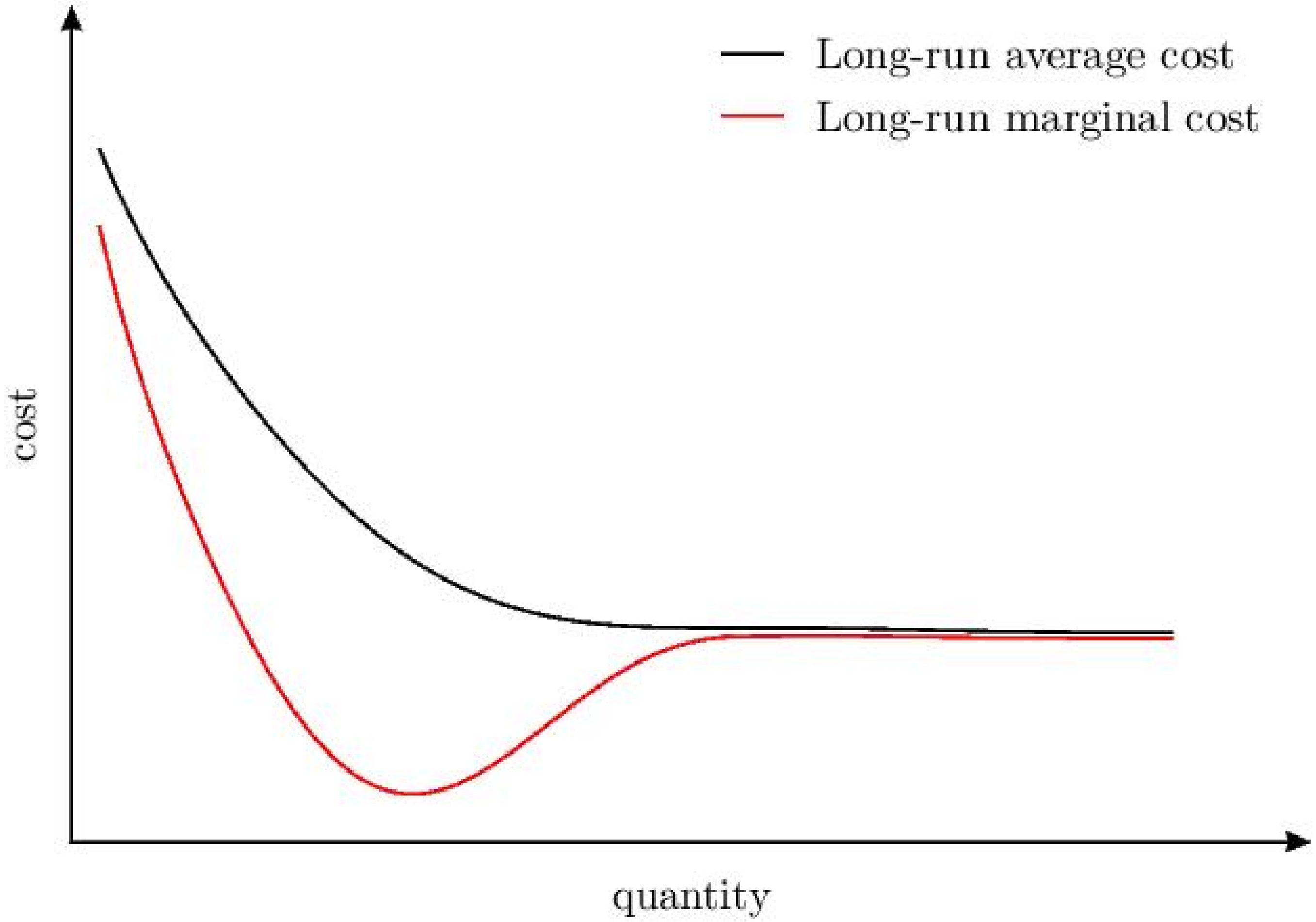
Total Slaughterhouse Costs
(without Transportation)



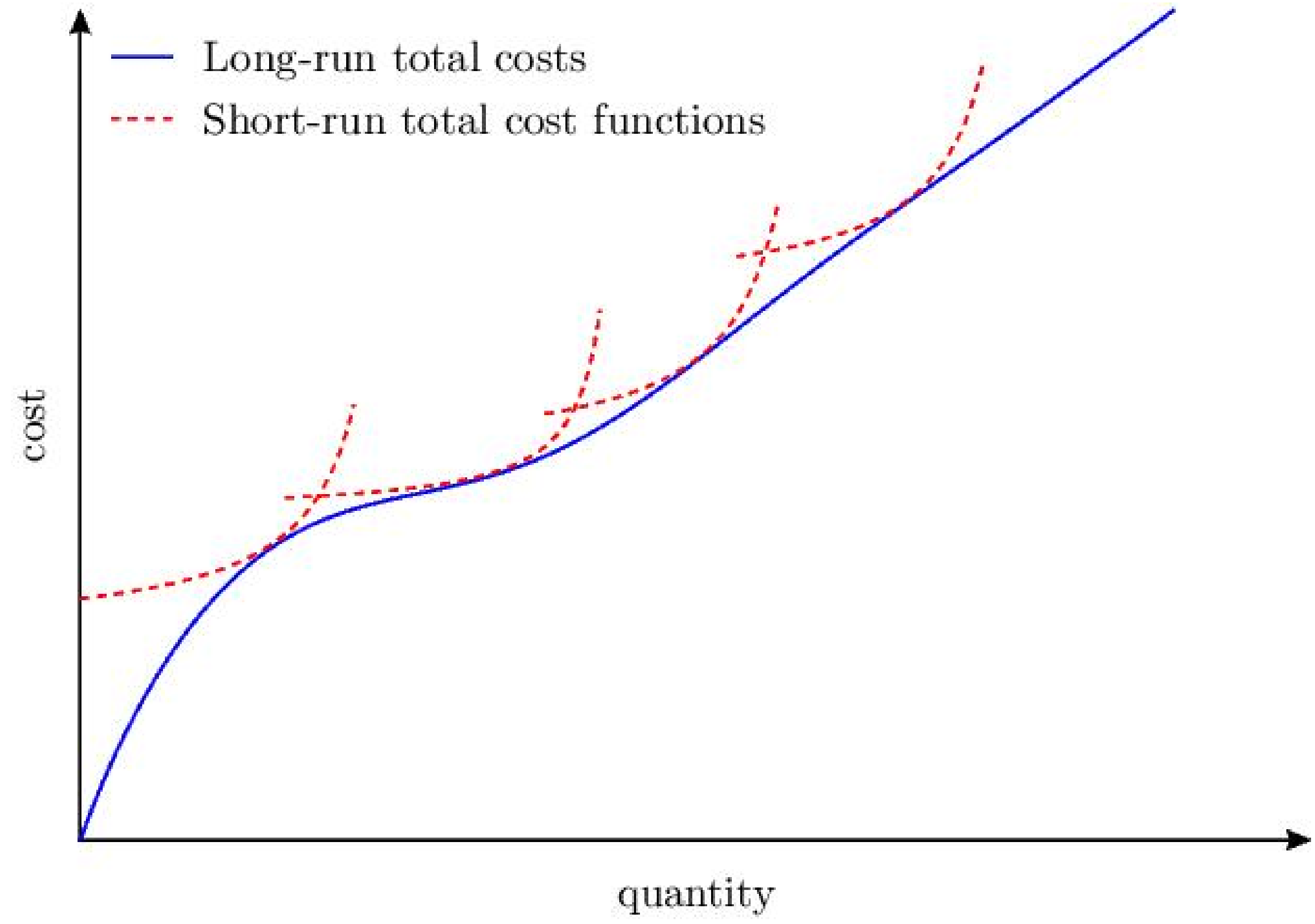
Long-Run and Short-Run Capacity Expansion



Economies of Scale



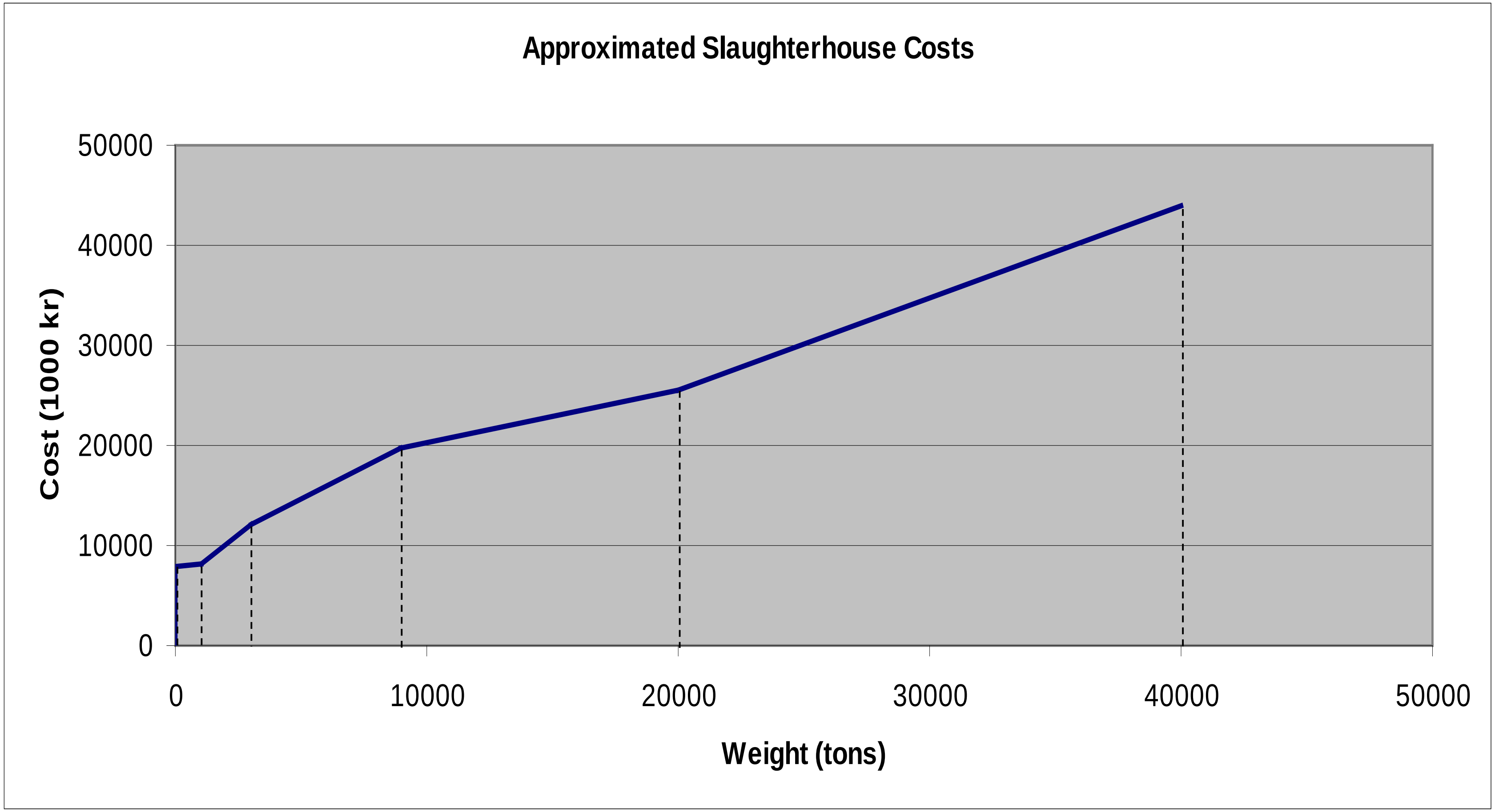
Long-Run vs. Short-Run Costs



Uncertain Demand

- 420 locations with stochastic demand
- Data generated to test the algorithm based on real-world animal population
- Demand is assumed to be normal distributed
- 2 datasets with 100 scenarios each
  - 1 set with low standard deviation
  - 1 set with high standard deviation

First stage: Building a facility…
or choose a capacity at location *j*



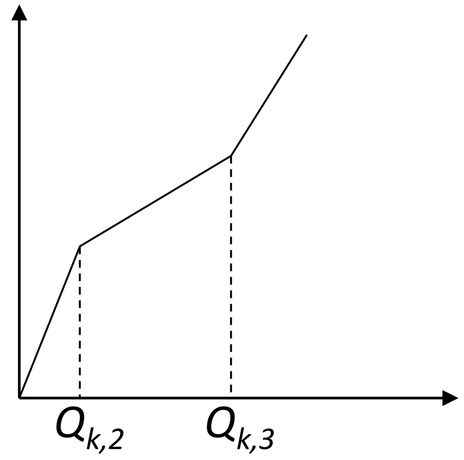
First Stage Problem

$$\min E\left[Q\left(y, \tilde{D}\right)\right]$$

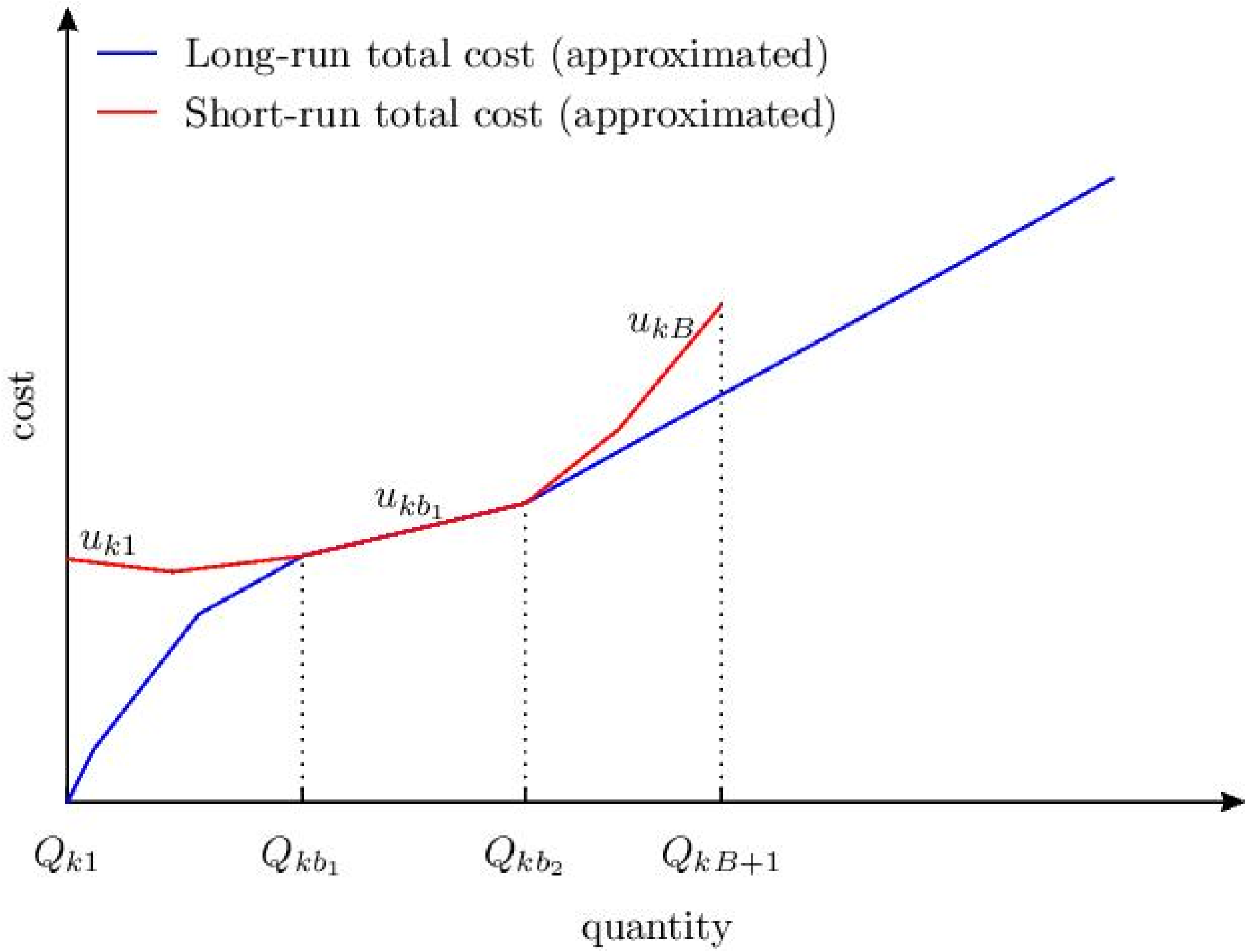
subject to

$$\sum_{k \in \mathcal{K}} y_{jk} = 1, \qquad SOS1, j \in \mathcal{J}$$

$$y_{jk} \in \{0, 1\}, \qquad j \in \mathcal{J}, k \in \mathcal{K}$$



Second stage: Satisfy demand…
or allocating animals given a capacity



Second Stage Problem (for a given scenario)

$$Q(y,D) = \min \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} T_{ij} x_{ij} + \sum_{j\in\mathcal{J}} \sum_{k\in\mathcal{K}} \sum_{b\in\mathcal{B}} C_{kb} \mu_{jkb}$$

subject to

$$\sum_{j\in\mathcal{J}} x_{ij} = D_i \qquad i \in \mathcal{I}$$

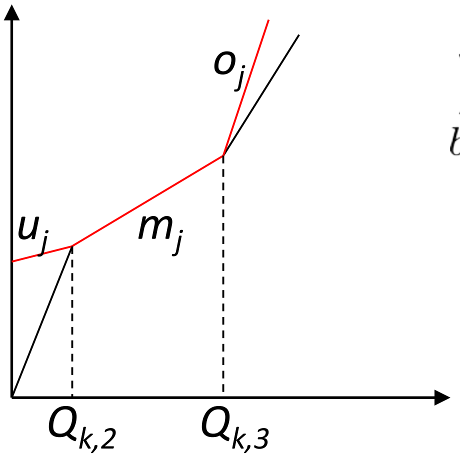
$$\sum_{i\in\mathcal{I}} x_{ij} = \sum_{k\in\mathcal{K}} \sum_{b\in\mathcal{B}} Q_{kb} \mu_{jkb} \qquad j \in \mathcal{J}$$

$$x_{ij} \le L_{ij} D_i \qquad i \in \mathcal{I}, j \in \mathcal{J}$$

$$\sum_{b\in\mathcal{B}} \mu_{jkb} = y_{jk} \qquad SOS2, j \in \mathcal{J}, k \in \mathcal{K}$$

$$x_{ij} \ge 0 \qquad i \in \mathcal{J}, j \in \mathcal{J}$$

$$\mu_{jkb} \ge 0 \qquad j \in \mathcal{J}, k \in \mathcal{K}, b \in \mathcal{B}$$



Results (using Lagrangean Relaxation)

- Results for $K = 5$ and initial $\lambda_i^s = 650$

η	IWI	S	σ	# of slaughterhouses in best feasible solution	Gap	Runtime [hh:mm]
2	150	1	---	10	5.3%	0:25
1	50	10	low	11	7.9%	3:51
1	50	10	high	10	8.4%	3:55
1	50	50	low	11	7.5%	18:14
1	50	50	high	10	7.6%	19:18
1	50	100	low	11	8.3%	39:53
1	50	100	high	10	6.8%	39:59

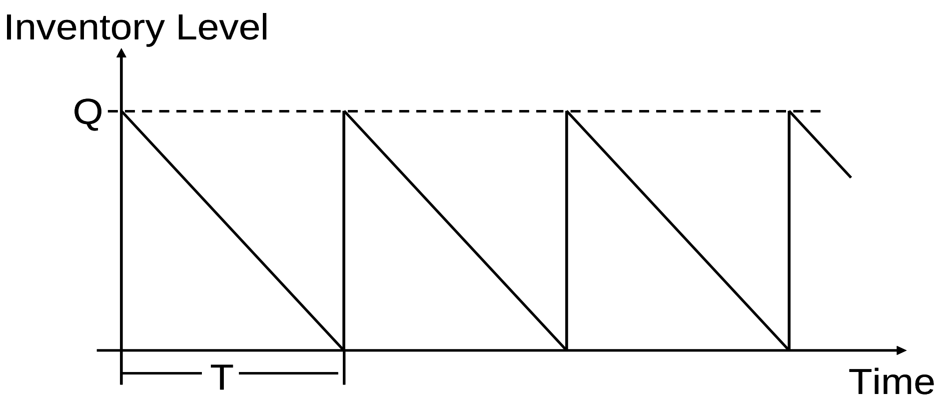
Classic inventory theory

The Economic Order Quantity

Assumptions:

- Constant demand rate of D items per day
- Fixed order quantity Q
- Fixed cost K every time an order is placed
- Inventory carrying cost h (holding cost) per unit and day
- Lead time between placing an order an receipt of goods is zero
- Initial inventory is zero
- Planning horizon is infinite

Determining the EOQ



- Total inventory cost per cycle *T*

$$K + \frac{hTQ}{2}$$

- Average cost per unit of time

$$\frac{KD}{Q} + \frac{hQ}{2}$$

- Cost minimizing quantity

$$Q^* = \sqrt{\frac{2KD}{h}}$$

Q* is known as the Economic Order Quantity (EOQ)

The (s,S) inventory policy, introduction

Basic description of the (s,S) inventory policy:

> Whenever our inventory level drops below a certain level, say *s,* we order (or produce) in order to increase the inventory level to *S*. Such a policy is referred to as an (*s,S*) or *min max* policy.
>
> *s* is called the reorder point and *S* the order-up-to-level

The (s,S) inventory policy, random demand

Additional Assumptions:

- Daily demand is random and normally distributed
- If an order is placed, it arrives after the appropriate lead time
- All orders that can’t be satisfied from stock are lost
- A service level is defined as the probability that the retailer is not stocking out during lead time

We also have:

- Fixed cost K for placing an order
- Inventory holding cost h is charged per item per unit time

The (s,S) policy with random demand

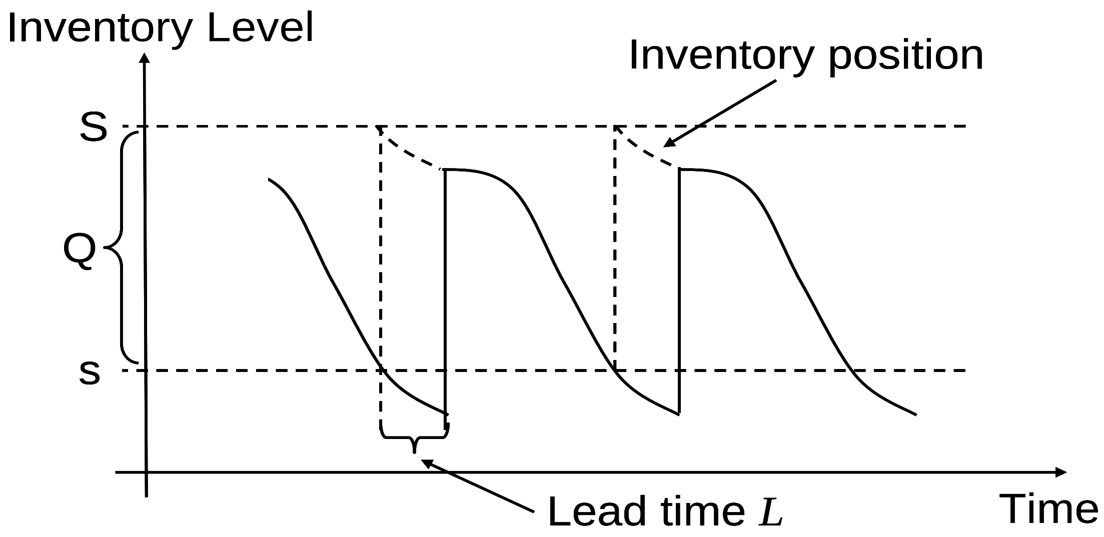
Additional information needed to calculate the inventory policy

AVG	=	Average daily demand
STD	=	Standard deviation of daily demand
L	=	Replenishment lead time
$L \times STD^2$	=	Cumulative variance over time

α = Service level (implying that the probability of stocking out is 1- α)

z = Safety factor associated with the service level (can be taken from tables, e. g. Table 2-2, p. 43 in SKS)

Reorder point s and Order-up-to-level S



- Average demand during lead time

$$L \cdot AVG$$

- Safety stock

$$z \cdot STD \cdot \sqrt{L}$$

- Reorder point *s* $L \cdot AVG + z \cdot STD \cdot \sqrt{L}$

- *z* is taken from a statistical table to ensure that the probability of a stock-out during lead time is 1-α

$$\Pr\{\text{demand during lead time} \geq L \cdot AVG + z \cdot \ STD \cdot \sqrt{L}\} = 1 - \alpha$$

Reorder point *s* and Order-up-to-level *S*

- Recall from EOQ

$$Q^* = \sqrt{\frac{2K \cdot AVG}{h}}$$

- Order-up-to-level S

$$S = Q^* + s$$

- Expected level of inventory

before receipt	after receipt	on average
$z \cdot STD \cdot \sqrt{L}$	$Q^* + z \cdot STD \cdot \sqrt{L}$	$\frac{Q^*}{2} + z \cdot STD \cdot \sqrt{L}$

Example 2-8 I

Define the inventory policy for a TV distributor given the following assumptions:

- Whenever an order is placed the distributor incurs a fixed cost of \$4500 (independent of order size)
- The cost of the TV to the distributor is \$250 and annual inventory holding cost is about 18% of the product cost
- Replenishment lead time is 2 weeks
- Desired service level is 97%
- Demand data for the last 12 months:

Month	Sept.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.
Sales	200	152	100	221	287	176	151	198	246	309	98	156

Example 2-8 II

- Average monthly demand is 191.17, standard deviation of monthly demand is 66.53

- Lead time is two weeks, transform monthly demand data into weekly demand data:

$$\text{Average weekly demand} = \frac{\text{Average monthly demand}}{4.3}$$
$$\text{Standard deviation of weekly demand} = \frac{\text{Monthly standard deviation}}{\sqrt{4.3}}$$

- Table 2-2 (SKS, p. 43) for determining the safety factor z for a 97% service level as 1.88

Example 2-8 III

Parameter	Average weekly demand	Standard deviation of weekly demand
Value	44.58	32.08

Calculate

- Average demand during lead time

$$L \cdot AVG = 2 \cdot 44.58 = 89.16$$

- Safety stock

$$z \cdot STD \cdot \sqrt{L} = 1.88 \cdot 32.08 \cdot \sqrt{2} = 85.29$$

- Reorder point

$$L \cdot AVG + z \cdot STD \cdot \sqrt{L} = 89.16 + 85.29 = 174.45 \approx 175$$

Example 2-8 IV

- Weekly inventory holding cost

$$\frac{0.18 \cdot 250}{52} = 0.87$$

- Order quantity Q

$$Q^* = \sqrt{\frac{2K \cdot AVG}{h}} = \sqrt{\frac{2 \cdot 4500 \cdot 44.58}{0.87}} = 679.1 \approx 679$$

- Order-up-to-level

$$S = Q^* + s = 679 + 175 = 854$$

- Average inventory level

$$\frac{Q^*}{2} + z \cdot STD \cdot \sqrt{L} = \frac{679}{2} + 85.29 = 424.79 \approx 425$$

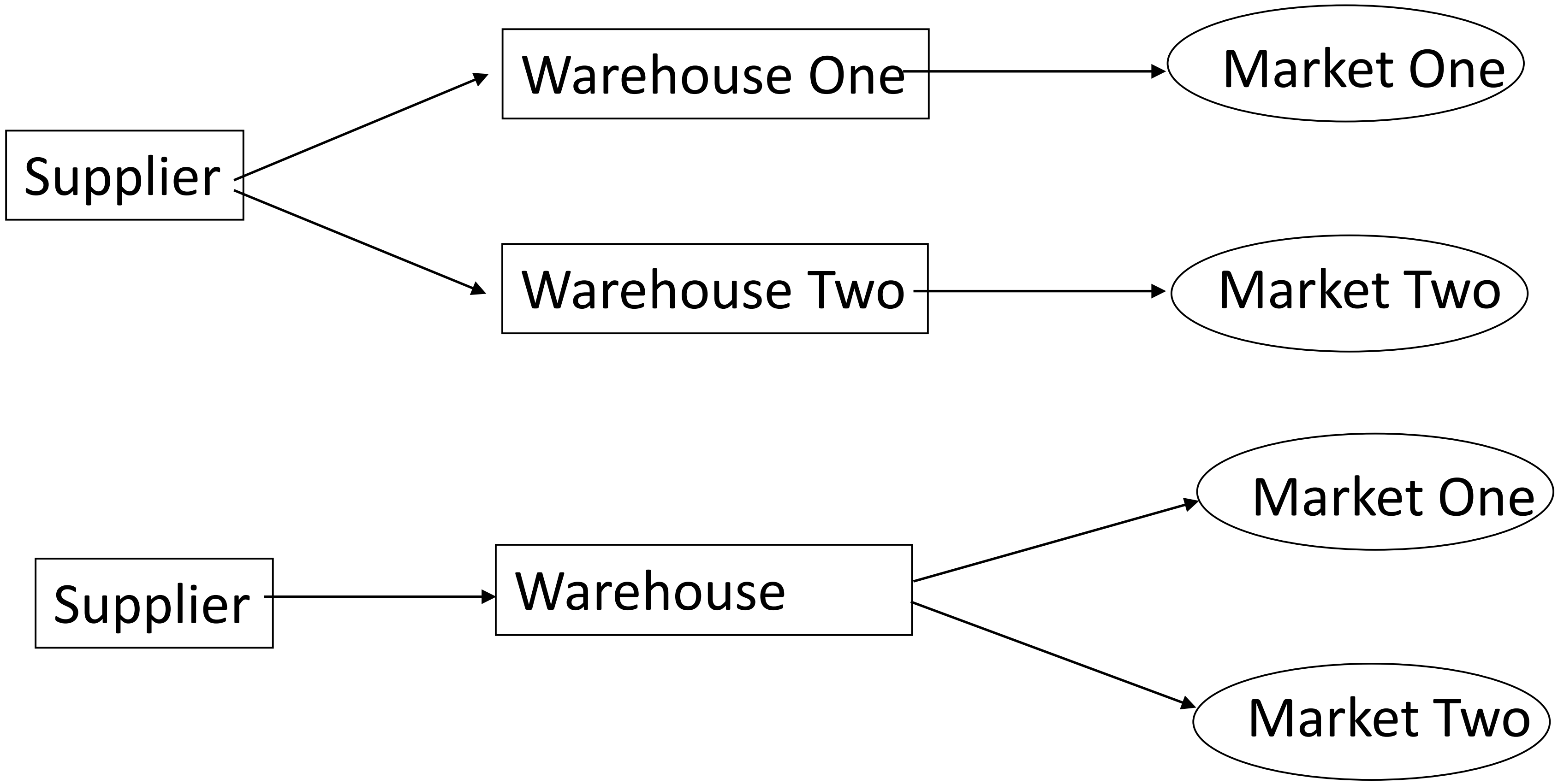
Risk Pooling

Central versus local facilities in supply chains

- Overhead: Economics of scale suggests few sites
- Lead time: more warehouses normally leads to shorter lead times
- Transportation costs: usually increases with fewer warehouses
- Service: Depends on how service is defined. Shipping time increases with fewer warehouses, while the probability that the products are in stock increases even with lower total inventory levels.
- Safety stock: risk pooling in centralized systems

Risk Pooling

- Consider these two systems:



Risk Pooling

- For the same service level, which system will require more inventory? Why?

- For the same total inventory level, which system will have better service? Why?

- What are the factors that affect these answers?

Risk Pooling Example

- Compare the two systems:
  - two products
  - maintain 97% service level
  - $60 order cost
  - $.27 weekly holding cost
  - $1.05 transportation cost per unit in decentralized system, $1.10 in centralized system
  - 1 week lead time

Risk Pooling Example

Week	1	2	3	4	5	6	7	8
Prod A, Market 1	33	45	37	38	55	30	18	58
Prod A, Market 2	46	35	41	40	26	48	18	55
Prod B, Market 1	0	2	3	0	0	1	3	0
Product B, Market 2	2	4	0	0	3	1	0	0

Risk Pooling Example

Warehouse	Product	AVG	STD	CV	s	S	Avg. Inven.	% Dec.
Market 1	A	39.3	13.2	.34	65	197	91	
Market 2	A	38.6	12.0	.31	62	193	88	
Market 1	B	1.125	1.36	1.21	4	29	14	
Market 2	B	1.25	1.58	1.26	5	29	15	
Cent.	A	77.9	20.7	.27	118	304	132	26%
Cent	B	2.375	1.9	.81	6	39	20	33%

$$\text{CV} = \text{Coefficient of variation} = \frac{\text{Standard deviation}}{\text{Average demand}}$$

Important Observations

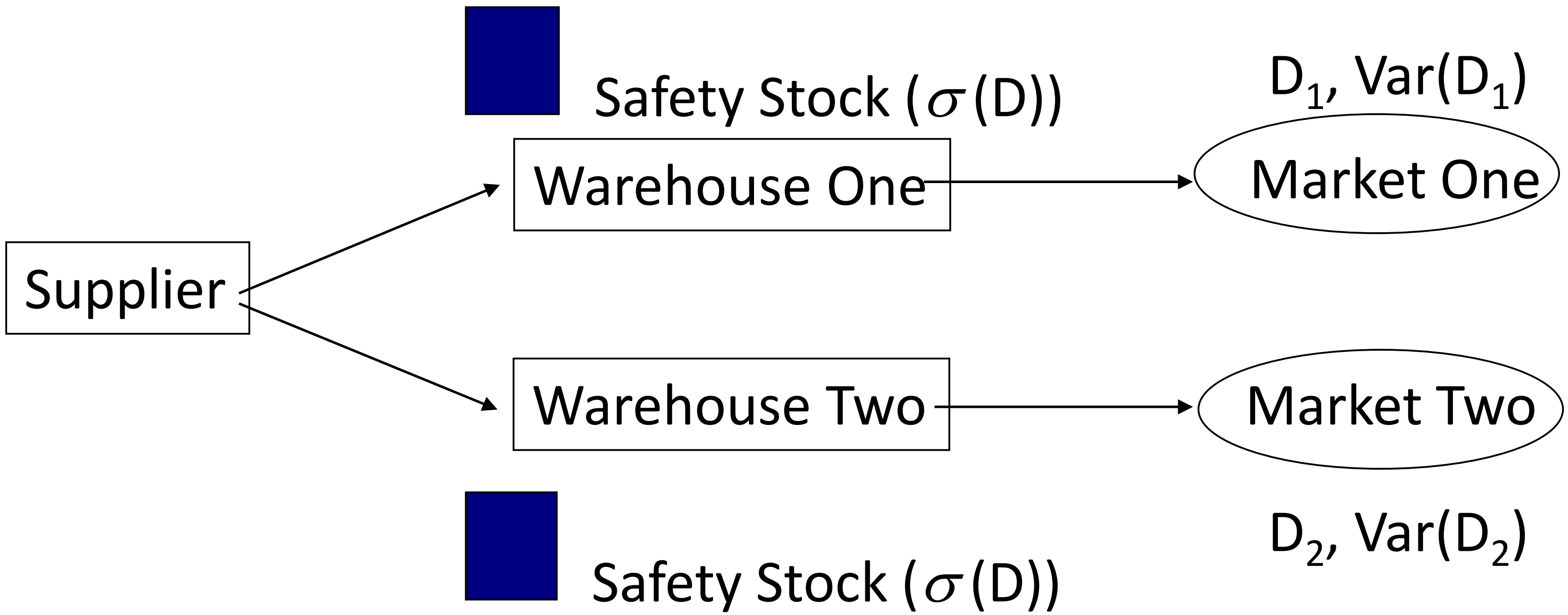
- Centralizing inventory control reduces both safety stock and average inventory level for the same service level.

- This works best for
  - High coefficient of variation, which reduces required safety stock.
  - Negatively correlated demand. Why?

- Risk Pooling only focuses on inventory levels, there might be a trade-off between inventory levels and transportation cost

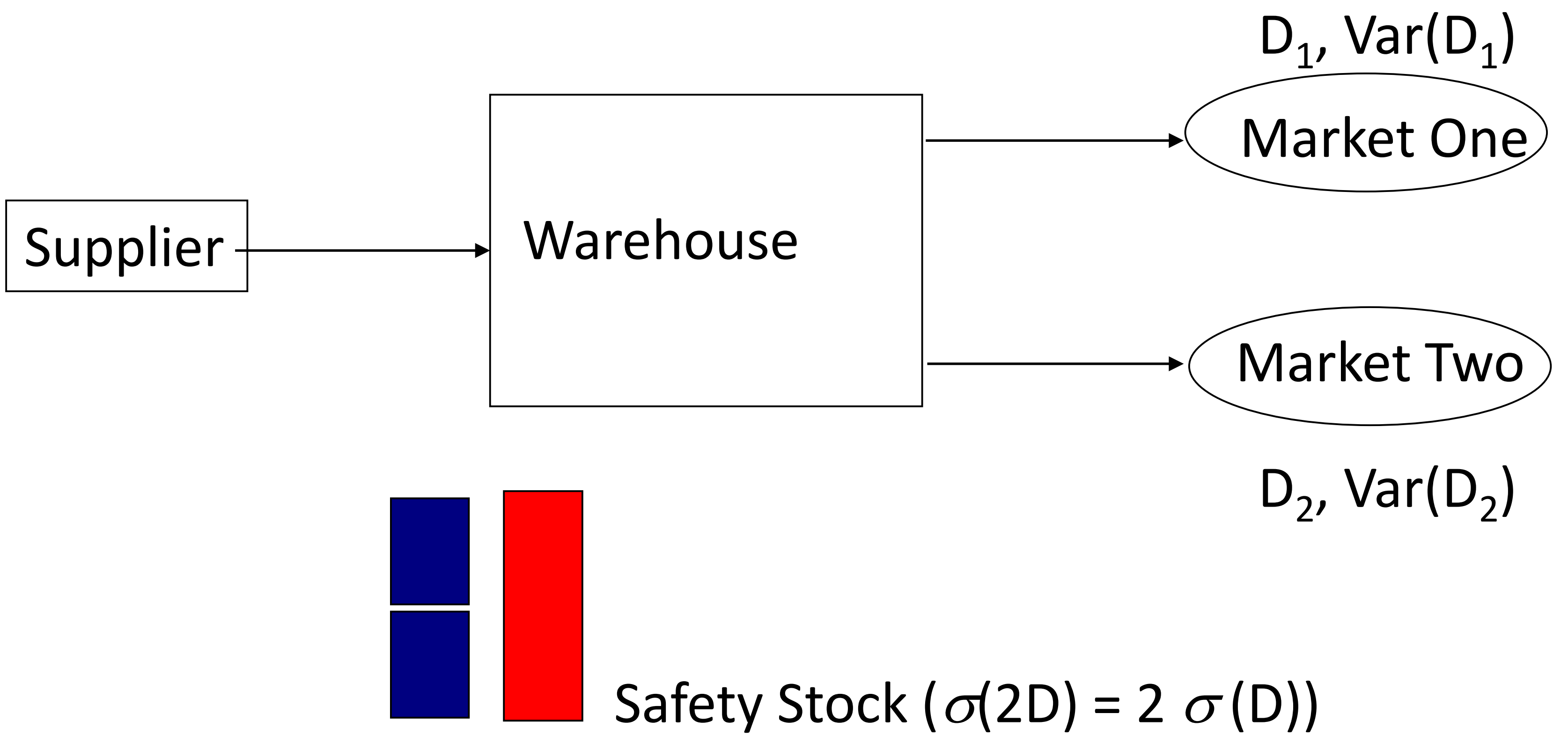
The Decentralized System

$\mathrm{Var}(D_1) = \mathrm{Var}(D_2) = \mathrm{Var}(D)$



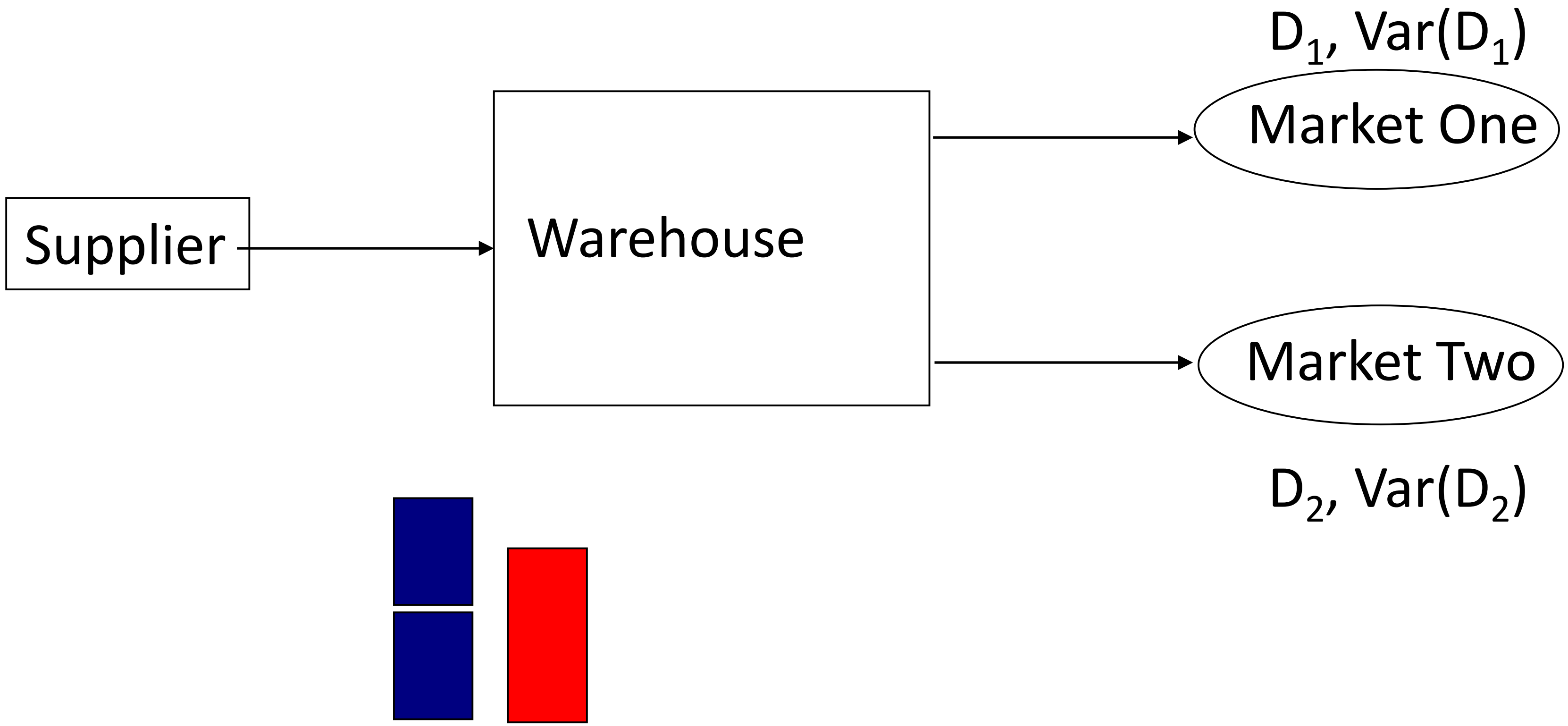
Perfectly Correlated Demand, $\rho_{12} = 1$ Centralized

$Var(D_1) = Var(D_2) = Var(D)$



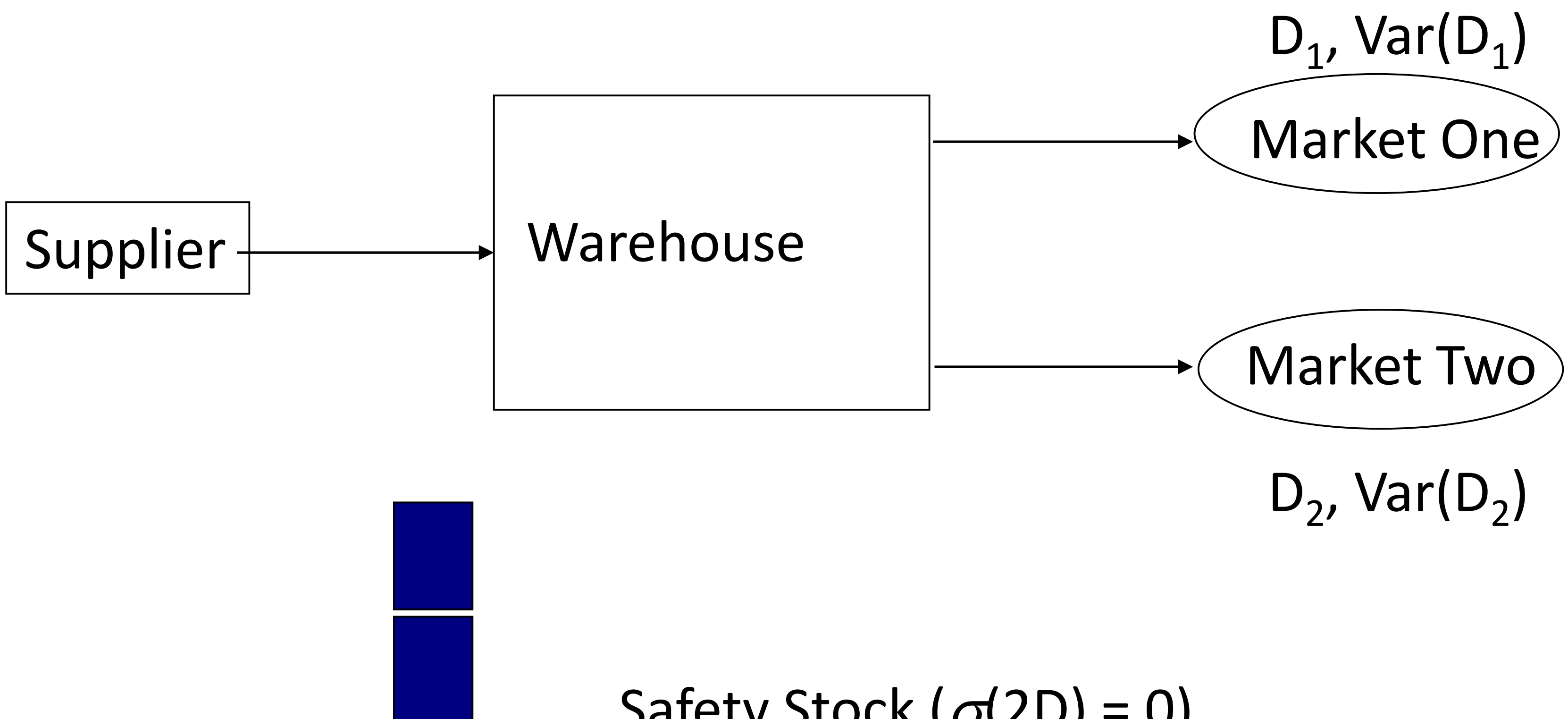
Uncorrelated Demand, $\rho_{12} = 0$ Centralized

$\mathrm{Var}(D_1) = \mathrm{Var}(D_2) = \mathrm{Var}(D)$



Negatively Correlated Demand, ρ_{12}= -1 Centralized

$Var(D_1) = Var(D_2) = Var(D)$



Safety Stock ($\sigma(2D) = 0$)

The Newsboy Problem

The Newsboy Problem

Short problem description:

You are selling newspapers. Each morning you go and buy the amount of newspapers you think you are going to sell in the course of the day. Unfortunately, you don't know exactly how many newspapers you are going to sell, your demand is uncertain.
In addition, you have no chance of buying additional newspapers if you realize you face a higher demand than what you anticipated. So all demand you cannot satisfy from stock is lost.
In the other case – you have bought too many newspapers – you will not be able to sell them later, because no one will be interested in yesterday's news. (But you can sell the papers with a loss as recycling material.)

How many newspapers do you buy every morning?

The Newsboy Problem, Formulation by Rudi and Pyke

We use the following notation here:+

W	-	cost of buying a newspaper
R	-	selling price of the newspaper ($R > W$)
S	-	return price (salvage value) of a newspaper ($S < W$)
Q	-	number of newspapers bought in the morning
D	-	stochastic demand

In addition, we will use the following definition:

$$y^+ = \begin{cases} y & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The Newsboy Problem, Formulation by Rudi and Pyke

- Define expected profit as function of Q

$$\Pi_r(Q) = E\left[R\min(D,Q) + S(Q-D)^+ - WQ\right]$$

- With

$$\min(D,Q) = D - (D-Q)^+$$

and

$$Q = D - (D-Q)^+ + (Q-D)^+$$

we reformulate $\Pi_r(Q)$

$$\Pi_r(Q) = (R-W)ED - \underbrace{E\left[(R-W)(D-Q)^+ + (W-S)(Q-D)^+\right]}_{G_r(Q)}$$

and call $G_r(Q)$ the cost of uncertainty

The Newsboy Problem, Formulation by Rudi and Pyke

- The first term

  $$(R-W)(D-Q)^+$$

  represents expected number of units short times the opportunity cost per unit short. Thus, define the unit underage cost C_u as: $C_u = (R-W)$

- The second term

  $$(W-S)(Q-D)^+$$

  represents expected number of left over units times the opportunity cost per left over unit. Thus, define the unit overage cost C_o as: $C_o = (W-S)$

- Rewrite $G_r(Q)$ as $G_r(Q) = E\left[C_u(D-Q)^+ + C_o(Q-D)^+\right]$

The Newsboy Problem, Formulation by Rudi and Pyke

In order to determine the order quantity that results in the highest expected profit, derive $G_r(Q)$ and find Q such that the derivative equals zero.

$$G_r(Q) = E\left[C_u(D-Q)^+ + C_o(Q-D)^+\right]$$

- Derivative of the first term of $G_r(Q)$:

$$-C_u \Pr(D > Q)$$

- Derivative of the second term of $G_r(Q)$:

$$C_o \Pr(Q > D)$$

- The derivative of $G_r(Q)$:

$$G'_r(Q) = -C_u \Pr(D > Q) + C_o \Pr(Q > D)$$

The Newsboy Problem, Formulation by Rudi and Pyke

- With $\Pr(D > Q) = 1 - \Pr(D < Q)$, we can rewrite $G'_r(Q)$ as

$$G'_r(Q) = -C_u[1 - \Pr(D < Q)] + C_o \Pr(D < Q)$$

- Equating this derivative to zero, we get the following Newsboy optimality condition

$$\Pr(D < Q) = \frac{C_u}{C_u + C_o}$$

- This allows to find the optimal order quantity $Q*$

Example: Homework Problem #3.1

NoNo is a producer of high-end YoYos (they only make one product) at **unit cost $10**. The YoYos only sell during the summer and a new model is launched very year. The total production lead time is 6 months.

The YoYo is currently sold by 100 different identical retailers. They pay a **wholesale price** of **$15** and sell at **unit revenue $30**. **Unit salvage value is $8**. Each retailer faces a **normal distribution** of demand with **mean of 100** and **standard deviation of 40**.

a) How many YoYo’s should each retailer order?
b) What is the expected profit for each retailer?
c) What is the manufacturer profit?

How many to order?

- Underage cost $C_u = R - W = 30 - 15 = 15$
- Overage cost $C_o = W - S = 15 - 8 = 7$

- Optimality criterion:

$$\Pr(D < Q) = \frac{C_u}{C_u + C_o} = 0.6818$$

- Invert normal distribution to find order quantity

$$Q^* = 119$$

Expected profit for each retailer

- Retailer profit given as

$$\Pi_r(Q) = (R-W)ED - \underbrace{E\left[(R-W)(D-Q)^+ + (W-S)(Q-D)^+\right]}_{G_r(Q)}$$

- Cost of uncertainty $G_r(Q)$ requires calculating expected shortage and expected salvage
  - Complex for normal distribution (use software!)
  - Fairly easy for uniform distribution
  - Here: $G_r(Q)$=314

- Retailer profits 1500-314=1186

Expected profit NoNo

- Profit margin per YoYo: $W - M = 15 - 10 = 5$

- Each retailer order 119 YoYos, 100 retailers

- Manufacturer profits
$$\Pi_m = (W - M) \cdot 119 \cdot 100 = 59500$$