NTNU
Department of Industrial Economics and Technology Management
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TIØ4285 Production and Network Economics
Assignment 3 – Proposed Solution

Out: Thursday 30 January
In: Thursday 6 February, 6pm
Supervision: Monday 3 February, 16:15pm A31

Note that late exercises will not be approved.

Exercise

Chelski FC wants to build a new football stadium and intends to sell season tickets to help finance the build. The city subsidizes the stadium, but limits the number of season tickets to only 30,000.

Assume that the marginal cost of selling season tickets is 0 and that you only consider the club's income. You have just been hired as the Chelski FC's new CFO and your first job is to determine the right price and sales method for the season tickets. You have been in touch with 10 commercial wholesalers, who want to buy (and distribute) all of the 30,000 season tickets for Chelski FC's new stadium. The club assumes that the valuation of the tickets is uniformly distributed between 0 and 100 million and that the reservation price of all 10 commercial wholesalers is drawn from this distribution.

a) You have been asked to calculate the club's expected income from auctioning off the tickets. Justify your choice of auction design and provide all necessary assumptions for your calculations.

The choice of auction design will not affect the club's expected income, given that the assumptions for the *Revenue Equivalence Theorem* hold. We therefore assume that the following assumptions hold:

- The buyers' reservation prices are drawn from an interval $[L, U]$ with distribution $F_i(v_i)$. This distribution is the same for every buyer and known to all of them.
- Buyers are symmetric. This implies that the same reservation price leads to the same bidding strategy.
- Buyers are risk-neutral, i. e. they maximize their expected profits (utility).
- A buyer's reservation price only depends on this buyer's information and utility function, i. e. it is independent of other buyers' private information.

As auction design does not impact the expected income under these assumptions, it is sufficient to calculate the expected income from a first-price auction. We introduce the following notation:

v	Bidder's reservation price
$f(v)$	Probability density function for reservation prices
$F(v)$	Cumulative distribution function of reservation prices
L	Lower bound on reservation price

$b(v)$ Optimal bidding strategy (bid given a reservation price)
R Seller's income
N Number of bidders participating in the auction

From the text we know, $v \sim U[0, 100]$. This implies the following: $F(v) = \frac{v}{100}$ and $f(v) = \frac{1}{100}$. The expected revenue of the seller using a first price auction is given as (see slide 33, Lecture 7):

$$E(R) = N \int_0^{100} b(v) f(v) F^{N-1}(v) dv.$$

For reservation prices drawn from a uniform distribution, the optimal bid ($b(v)$) for each bidder is given as:

$$b(v) = v - \frac{v - L}{N}.$$

The lower bound L is 0 in this case and the seller's expected income is

$$E(R) = \frac{N-1}{N+1} \cdot 100.$$

With $N = 10$ bidders, Chelski FC has an expected income of $\frac{9}{11} \cdot 100 = 81.8$ million from the auction.

b) The club considers advertising the auction to attract additional bidders. How much should the club be willing to spend in order to attract n new bidders to the auction?

The increase in expected income after attracting n new bidders is simply given as

$$E(R_{N+n}) - E(R_N) = \left(\frac{9+n}{11+n} - \frac{9}{11} \right) \cdot 100.$$

This is also the maximum amount Chelski FC should spend in order to attract n new bidders.

c) Assume that it costs 500,000 to attract each new bidder. How many bidders should the club attract beyond the 10 already participating wholesalers? What is the club's new expected income?

It will profitable for the club to attract new bidders as long the marginal revenue of the last attracted bidder is greater than (or equal to) 500,000. This translates to

$$\frac{dE(R)}{dN} = \frac{200}{(N+1)^2} = 0.5.$$

This results in an optimal number of 19 bidders. Expected income for the club is then $E(R) = \frac{18}{20} \cdot 100 - 9 \cdot 0.5 = 85.5$. The auction's expected income increases by 8.2 million, but the club's increases its income by only 3.7 million.

The city is not happy for your suggestion and insists that the Chelski FC sells the tickets itself, without using commercial wholesalers. You decide to try a completely new approach to solve the problem. The demand for season tickets is uncertain and price dependent. Market research reveals that demand can be described by 3 scenarios: S1, S2, and S3 corresponding to low, medium and high demand. Each scenario provides demand for 3 different prices. You know that one of the scenarios is the correct one, but – unfortunately – not which one. The result from market research is given in Table 1. The probability for a scenario to be correct is given as $p = 0.3$, $p = 0.5$, and $p = 0.2$, respectively.

Table 1: Demand scenarios

	scenario		
price	S1 (p=0.3)	S2 (p=0.5)	S3 (p=0.2)
2500	20,000	35,000	45,000
3000	15,000	30,000	35,000
5000	5,000	15,000	30,000

d) Which ticket price should you choose given that you do not have any other information than the one in Table 1?

Without any additional information, the club should choose the price that results in the highest expected income:

$$\begin{aligned}
\text{Price 2500}: \quad & E(R) = 2,500 \cdot (0.3 \cdot 20' + 0.5 \cdot 35' + 0.2 \cdot 45') \\
& = 81,250' \\
\text{Price 3000}: \quad & E(R) = 3,000 \cdot (0.3 \cdot 15' + 0.5 \cdot 30' + 0.2 \cdot 35') \\
& = 79,500' \\
\text{Price 5000}: \quad & E(R) = 5,000 \cdot (0.3 \cdot 5' + 0.5 \cdot 15' + 0.2 \cdot 30') \\
& = 75,000'
\end{aligned}$$

The tickets should be sold for 2,500.

You now decide to sells the tickets by means of a modified Dutch auction (which is often used to buy back stocks from shareholders).

e) Which property of the modified Dutch auction makes it particularly attractive in this situation?

The modified Dutch auction reveals information about the buyers' demand. We will therefore be able to identify the correct scenario and can choose the price that will maximize income.

f) Determine optimal ticket sales and expected income from a modified Dutch auction given the demand scenarios in Table 1.

We know that we can identify the scenario if we carry out a modified Dutch auction. The optimal prices are

- Scenario 1: price 2500, income 50,000,000
- Scenario 2: price 3000, income 90,000,000
- Scenario 3: price 5000, income 150,000,000

The expected income from a modified Dutch auction is then
$0.3 \cdot 50,000' + 0.5 \cdot 90,000' + 0.2 \cdot 150,000' = 90,000'$.