Øving 2 - Patrik Kjærran

① a) $F(s) = \frac{1}{s^2(s^2+1)}$

Metode 1 - Delbrøkoppspalting:

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = \underline{\underline{t - \sin t}}$$

Metode 2 - Convolution

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} \cdot \frac{1}{s^2+1} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t * \sin t = \underline{\underline{t - \sin t}}$$

b) $$F(s) = \frac{s}{s^2+2s+1} = \frac{s}{(s+1)^2} = \frac{(s+1)-1}{(s+1)^2}$$

$$= \frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{(s+1)^2}\right\} = \underline{\underline{e^{-t}(1-t)}}$$

c) $$F(s) = \frac{2s}{(s^2+1)^2} = \frac{2s}{s^2+1} \cdot \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} \cdot \frac{1}{s^2+1}\right\} = 2\cos t * \sin t = 2\int_0^t \cos(\tau)\cdot\sin(t-\tau)\,d\tau = \underline{\underline{t\sin t}}$$

d) $$F(s) = (s-3)^{-5} = \frac{1}{(s-3)^5}$$

$$\frac{1}{(s-3)^5} = \frac{4!}{4!} \cdot \frac{1}{(s-3)^5} = \frac{1}{4!} \cdot \frac{4!}{(s-3)^5}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{4!} \cdot \frac{4!}{(s-3)^5}\right\} = \frac{e^{3t}}{4!} \cdot t^4 = \underline{\underline{\frac{t^4 e^{3t}}{24}}}$$

② a) $f(t) = (U(t) - U_\pi(t))\cos t$

$= U(t)\cos t - U_\pi(t)\cos t$

$\Rightarrow F(s) = \mathcal{L}\{U(t)\cos t\} - \mathcal{L}\{U_\pi(t)\cos t\}$

$= \mathcal{L}\{\cos t\} - e^{-\pi s}\cdot\mathcal{L}\{\cos(t+\pi)\}$ | $\cos(t+\pi) = -\cos t$

$= \mathcal{L}\{\cos t\} + e^{-\pi s}\cdot\mathcal{L}\{\cos t\}$

$= \mathcal{L}\{\cos t\}\cdot(1+e^{-\pi s}) = \underline{\underline{\dfrac{s(1+e^{-\pi s})}{s^2+1}}}$

b) $f(t) = U_a(t)\cdot t^2$ for $a > 0$.

$\Rightarrow F(s) = e^{-as}\cdot\mathcal{L}\{(t+a)^2\}$

$= e^{-as}\cdot\mathcal{L}\{g(t)\}$

$g(t) = (t+a)^2$

$g'(t) = 2(t+a)$

$g''(t) = 2$

$\mathcal{L}\{f'\} = s\,\mathcal{L}\{f\} - f(0)$

$\Leftrightarrow \mathcal{L}\{f\} = \frac{1}{s}\left(\mathcal{L}\{f'\} + f(0)\right)$

$= \frac{1}{s}\left(\frac{1}{s}\left(\mathcal{L}\{f''\} + f'(0)\right) + f(0)\right)$

$= \frac{1}{s^2}\mathcal{L}\{f''\} + \frac{1}{s^2}f'(0) + \frac{1}{s}f(0)$

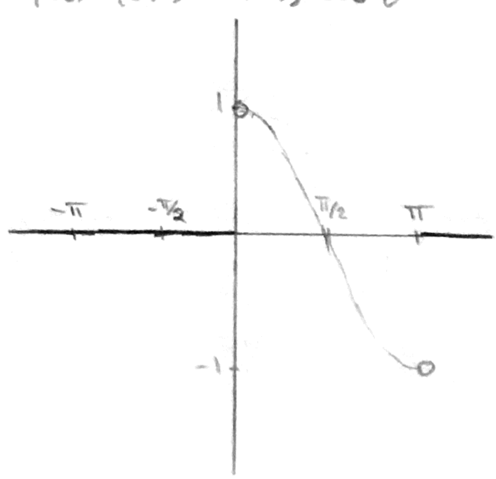
(*) $= \frac{1}{s^2}\left(\mathcal{L}\{f''\} + f'(0) + sf(0)\right)$

Innsatt (*) gir:

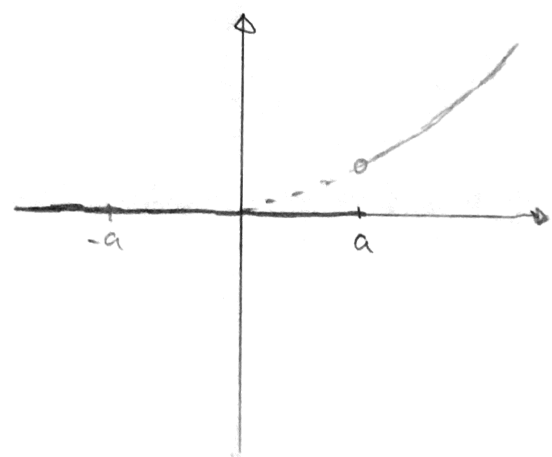
$F(s) = \dfrac{e^{-as}}{s^2}\left(\mathcal{L}\{2\} + f'(0) + sf(0)\right)$

$= \underline{\underline{\dfrac{e^{-as}}{s^2}\left(\dfrac{2}{s} + 2a + sa^2\right)}}$

a) $f(t) = (U(t) - U_\pi(t))\cdot\cos t$



b) $f(t) = U_a(t)\cdot t^2$



c) $f(t) = U(t) + 2\sum_{i=1}^{\infty} (-1)^i \cdot U(t-ia)$ for $a>0$

Vi har:

$\mathcal{L}\{U_c(t)\} = e^{-cs} \cdot \mathcal{L}\{1\} = \frac{e^{-cs}}{s}$ | Benytter $\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$ og gjør en leddvis transformasjon.

$$\Rightarrow F(s) = \mathcal{L}\{U(t)\} + 2\sum_{i=1}^{\infty} (-1)^i \cdot \mathcal{L}\{U_{ia}(t)\}$$

$$= \frac{1}{s} + 2\sum_{i=1}^{\infty} (-1)^i \cdot \frac{e^{-ias}}{s}$$

| ser at dette er en konvergent geometrisk rekke med $a_0 = \frac{-e^{-as}}{s}$ og $r = -e^{-as}$

(1): $S = a_0 + a_0 r + a_0 r^2 \ldots \quad | \cdot r$

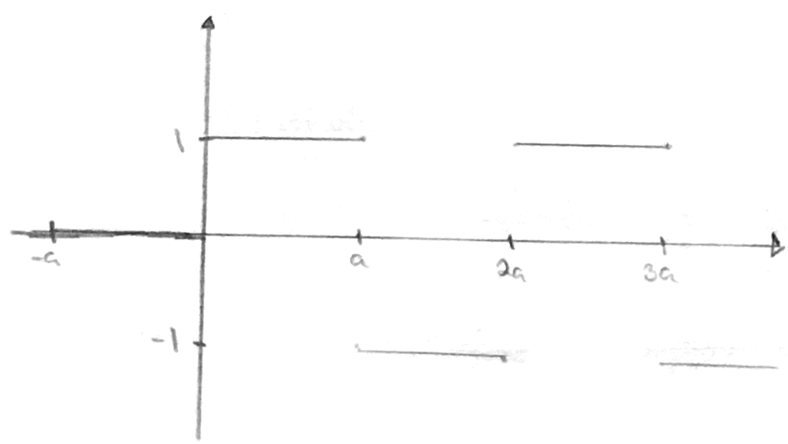
(2): $Sr = a_0 r + a_0 r^2 + a_0 r^3 \ldots$

(1)-(2): $\left.\begin{aligned} S - Sr &= a_0 \\ S(1-r) &= a_0 \end{aligned}\right\} S = \frac{a_0}{1-r}$

$$\Rightarrow F(s) = \frac{1}{s} + 2 \cdot \frac{-e^{-as}/s}{1-(-e^{-as})} = \frac{1}{s} \cdot \frac{1+e^{-as}}{1+e^{-as}} - \frac{2e^{-as}}{s(1+e^{-as})}$$

$$= \frac{1+e^{-as} - 2e^{-as}}{s(1+e^{-as})} = \underline{\underline{\frac{1-e^{-as}}{s(1+e^{-as})}}}$$

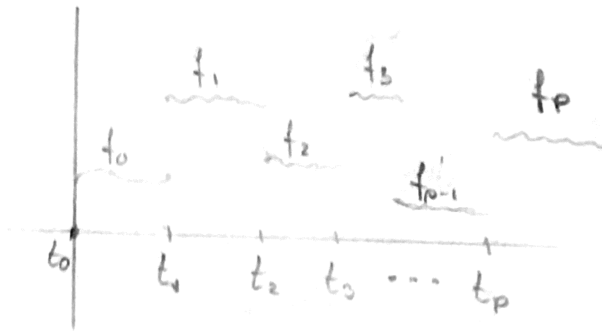
Plot av $f(t) = U(t) + 2\sum_{i=1}^{\infty} (-1)^i \cdot U(t-ia)$



③ $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} \cdot f'(t)\, dt$

$$f_n = \begin{cases} f(t) & \text{for } t_n < t < t_{n+1} \\ f(t_n^+) & \text{for } t = t_n \\ f(t_{n+1}^-) & \text{for } t = t_{n+1} \\ 0 & \text{ellers} \end{cases}$$

Innfører $p+1$ delfunksjoner $f_0, f_1 \dots f_p$:



Lar $t_0^+ = 0$ og $t_{p+1}^- = \infty$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} \cdot (f_0' + f_1' + f_2' + \dots + f_p')\, dt$$

$$= \sum_{i=0}^{p} \mathcal{L}\{f_i'\}$$

$$= \sum_{i=0}^{p} \int_0^\infty e^{-st} \cdot f_i'\, dt$$

$$= \sum_{i=0}^{p} \int_{t_i^+}^{t_{i+1}^-} e^{-st} \cdot f'\, dt \quad \Big| \text{ Delvis integrasjon}$$

$$= \sum_{i=0}^{p} \left(\left[e^{-st} \cdot f \right]_{t_i^+}^{t_{i+1}^-} + s \int_{t_i^+}^{t_{i+1}^-} e^{-st} \cdot f\, dt \right)$$

Innfører $g(t) = e^{-st} f(t)$:

$$= s\mathcal{L}\{f(t)\} + \left(\left[g(t)\right]_{t_0^+}^{t_1^-} + \left[g(t)\right]_{t_1^+}^{t_2^-} + \dots + \left[g(t)\right]_{t_p^+}^{t_{p+1}^-} \right)$$

$$= s\mathcal{L}\{f(t)\} + \left(g(t_1^-) - g(t_0^+) + g(t_2^-) - g(t_1^+) \dots + g(t_{p+1}^-) - g(t_p^+) \right)$$

$$= s\mathcal{L}\{f(t)\} - g(t_0^+) + \sum_{i=1}^{p} \left[g(t_i^-) - g(t_i^+) \right] + g(t_{p+1}^-)$$

$$= s\mathcal{L}\{f(t)\} - f(0) + \sum_{i=1}^{p} \left[e^{-st_i^-} \cdot f(t_i^-) - e^{-st_i^+} \cdot f(t_i^+) \right] + 0$$

$$= s\mathcal{L}\{f(t)\} - f(0) + \sum_{i=1}^{p} e^{-st_i} \cdot \left[f(t_i^-) - f(t_i^+) \right]$$

$$= s\mathcal{L}\{f(t)\} - f(0) - \sum_{i=1}^{p} e^{-st_i} \cdot \left[f(t_i^+) - f(t_i^-) \right]$$

4

a) Skal vise $\mathcal{L}^{-1}\{F(kt)\} = \frac{1}{k}\cdot f(\frac{t}{k})$: | sub. $t = \hat{s}$

$$F(k\hat{s}) = \int_0^\infty e^{-(k\hat{s})t}\cdot f(t)\,dt$$

$$= \int_{t=0}^{t=\infty} e^{-\hat{s}(kt)}\cdot f(t)\,dt$$ | sub. $u = kt \iff t = u/k$ og $du = k\cdot dt \iff dt = \frac{du}{k}$

$$= \int_{u=k\cdot 0}^{u=k\cdot\infty} e^{-\hat{s}u}\cdot f(\tfrac{u}{k})\cdot\frac{du}{k}$$

$$= \int_0^\infty e^{-\hat{s}u}\cdot\left(\frac{1}{k}\cdot f(\tfrac{u}{k})\right)du = \mathcal{L}\left\{\frac{1}{k}f(\tfrac{t}{k})\right\}(\hat{s})$$

$$F(kt) = \mathcal{L}\left\{\frac{1}{k}f(\tfrac{t}{k})\right\} \iff \underline{\underline{\mathcal{L}^{-1}\{F(kt)\} = \frac{1}{k}f(\tfrac{t}{k})}}$$

b) Skal vise $\mathcal{L}^{-1}\left\{\frac{F}{h_2}\right\}(t) = \int_0^t\int_0^u f(v)\,dv\,du$:

(1) $\mathcal{L}\{f*g\} = F(s)\cdot G(s)$

(2) $f*g = \mathcal{L}^{-1}\{F(s)\cdot G(s)\}$

(3) $\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}$

velger først å se på $\mathcal{L}^{-1}\left\{\frac{F}{h_1}\right\}(t) \overset{(h_1(s)=s)}{=} \mathcal{L}^{-1}\left\{\frac{F}{s}\right\}(t)$

Innfører $G(s) = \frac{1}{s}$ og $g(t) = 1$:

$$\mathcal{L}^{-1}\left\{\frac{F}{h_1}\right\}(t) = \mathcal{L}^{-1}\{F(s)\cdot G(s)\}(t)$$

$$= \mathcal{L}^{-1}\{F(s)\}(t) * \mathcal{L}^{-1}\{G(s)\}(t)$$ | (3)

$$= f(t) * g(t) = (f*g)(t)$$

$$= \int_0^t f(v)\cdot \cancel{g(t-v)}\,dv = \int_0^t f(v)\,dv$$

Analysens fundamentalteorem

$$\int_0^t f(v)\,dv = \hat{F}(t)$$

hvor $\hat{F}(t)$ beskriver den antideriverte til f, og er følgelig en funksjon av t.

$$\left(\hat{F}(t) = \int_0^t f(v)\,dv\right)$$

ser nå på $\mathcal{L}^{-1}\left\{\frac{F}{h_2}\right\}(t)$:

$$\mathcal{L}^{-1}\left\{\frac{F}{h_2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{F}{h_1}\cdot G(s)\right\}(t)$$

$$= \mathcal{L}^{-1}\left\{\frac{F}{h_1}\right\}(t) * \mathcal{L}^{-1}\{G(s)\}(t)$$

$$= \hat{F}(t) * g(t) = (\hat{F}*g)(t)$$

$$= \int_0^t \hat{F}(u)\cdot\cancel{g(t-u)}\cdot du$$

$$= \int_0^t\left(\int_0^u f(v)\,dv\right)du = \underline{\underline{\int_0^t\int_0^u f(v)\,dv\,du}}$$