Øving 1 - TMA4135

① a) $f(t) = \sinh(At) = \dfrac{e^{At} - e^{-At}}{2}$

$$\Rightarrow F(s) = \int_0^\infty e^{-st} \cdot \frac{e^{At} - e^{-At}}{2} dt = \lim_{n\to\infty} \frac{1}{2} \int_0^n \left(e^{(A-s)t} - e^{-(A+s)t}\right) dt$$

$$= \lim_{n\to\infty} \frac{1}{2}\left[\frac{1}{(A-s)} e^{(A-s)t} + \frac{1}{(A+s)} e^{-(A+s)t}\right]_0^n \quad \Bigg| \text{ gitt } \begin{cases} A-s<0, \\ -(A+s)<0 \end{cases} \Rightarrow \begin{matrix} A<s \\ -A<s \end{matrix} \Rightarrow \boxed{|A|<s}$$

$$= \frac{1}{2}\left[\left(0 - \frac{1}{A-s}\right) + \left(0 - \frac{1}{A+s}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{A-s} - \frac{1}{A+s}\right] = \frac{1}{2}\left[\frac{-(A+s)}{A^2-s^2} - \frac{(A-s)}{A^2-s^2}\right] = \frac{1}{2}\left[\frac{-A \cancel{-s} - A \cancel{+s}}{A^2-s^2}\right] = \underline{\underline{\frac{A}{s^2-A^2}}}$$

b) $f(t) = \cosh(At) = \dfrac{e^{At} + e^{-At}}{2}$

$$\Rightarrow F(s) = \int_0^\infty e^{-st} \cdot \frac{e^{At} + e^{-At}}{2} dt = \frac{1}{2} \cdot \lim_{n\to\infty} \int_0^n \left(e^{(A-s)t} + e^{-(A+s)t}\right) dt$$

$$= \frac{1}{2} \lim_{n\to\infty} \left[\frac{1}{(A-s)} e^{(A-s)t} - \frac{1}{(A+s)} e^{-(A+s)t}\right]_0^n \quad \Bigg| \text{ gitt } \left.\begin{matrix} A-s<0 \\ -(A+s)<0 \end{matrix}\right\} \Rightarrow \boxed{|A|<s}$$

$$= \frac{1}{2}\left[\left(0 - \frac{1}{A-s}\right) - \left(0 - \frac{1}{A+s}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{A+s} - \frac{1}{A-s}\right] = \frac{1}{2}\left[\frac{(A-s) - (A+s)}{A^2-s^2}\right] = \frac{1}{\cancel{2}}\left[\frac{-\cancel{2}s}{A^2-s^2}\right] = \frac{-s}{A^2-s^2} = \underline{\underline{\frac{s}{s^2-A^2}}}$$

c) $f(t) = \begin{cases} 0 & \text{for } 0<t<\pi \\ 1 & \text{otherwise} \end{cases} = 1 - U_0(t) + U_\pi(t)$

$$= \mathcal{L}\{1 - U_0(t) + U_\pi(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{U_0(t)\} + \mathcal{L}\{U_\pi(t)\}$$

$$= \frac{1}{s} - \int_0^\infty e^{-st} \cdot \cancel{U_0(t)}\, dt + \int_0^\infty e^{-st} \cdot U_\pi(t)\, dt$$

$$= \frac{1}{s} - \int_0^\infty e^{-st} dt + \int_\pi^\infty e^{-st} dt$$

$$= \frac{1}{s} - \lim_{a\to\infty} \int_0^a e^{-st} dt + \lim_{b\to\infty} \int_\pi^b e^{-st} dt$$

$$= \frac{1}{s} - \lim_{a\to\infty} \left[\frac{1}{-s} e^{-st}\right]_0^a + \lim_{b\to\infty} \left[\frac{1}{-s} e^{-st}\right]_\pi^b$$

$$= \frac{1}{s} - \frac{1}{s} + \frac{1}{s} \cdot e^{-s\pi} = \underline{\underline{\frac{e^{-s\pi}}{s}}}$$

① d) $f(t) = \begin{cases} 0 & \text{for } 0 < t < \pi \\ \cos(t) & \text{otherwise} \end{cases} = \cos(t) \cdot U_\pi(t) \cdot$

$$\Rightarrow \cancel{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot \cos(t) \cdot \cancel{U_0(t)} \cdot (1 - U_\pi(t))\,dt = \int_0^\infty e^{-st} \cdot \cos(t)\,dt - \int_0^\infty e^{-st} \cos(t) \cdot U_\pi(t)\,dt}$$

$$\cancel{= \int_0^\infty e^{-st} \cos(t)\,dt - \int_\pi^\infty e^{-st} \cos(t)\,dt = \int_0^\pi e^{-st} \cos(t)\,dt} \quad \left| \cancel{\begin{matrix} u_1 = \cos(t) & v_1 = \frac{1}{-s} e^{-st} \\ u_1' = -\sin(t) & v_1' = e^{-st} \end{matrix}} \right.$$

$$\cancel{= \left[\frac{1}{-s} e^{-st} \cos(t)\right]_0^\pi - \frac{1}{s} \int_0^\pi e^{-st} \sin(t)\,dt} \quad \left| \cancel{\begin{matrix} u_2 = \sin(t) & v_2 = \frac{1}{-s} e^{-st} \\ u_2' = \cos(t) & v_2' = e^{-st} \end{matrix}} \right.$$

$$\cancel{= \left[\frac{1}{-s} e^{-st} \cos(t)\right]_0^\pi - \frac{1}{s}\left(\left[\frac{1}{-s} e^{-st} \cdot \sin(t)\right]_0^\pi + \frac{1}{s} \int_0^\pi e^{-st} \cos(t)\,dt\right)}$$

$$\mathcal{L}\{\cos(t) \cdot u_\pi(t)\} = \int_\pi^\infty e^{-st} \cdot \cos(t)\,dt \quad \left| \begin{matrix} \text{sub. } c = t - \pi \Rightarrow dc = dt \\ \Rightarrow t = c + \pi \end{matrix} \right.$$

$$= \lim_{a \to \infty} \int_0^\infty e^{-s(c+\pi)} \cos(c + \pi)\,dc$$

$$= e^{-s\pi} \cdot \lim_{a \to \infty} \int_0^\infty e^{-sc} \cdot \cos(c + \pi)\,dc = e^{-s\pi} \cdot \mathcal{L}\{\cos(c + \pi)\} \quad \left| \begin{matrix} \text{variabelskifte} \\ c \Rightarrow t \end{matrix} \right.$$

$$= e^{-s\pi} \cdot \lim_{a \to \infty} \int_0^\infty e^{-st} \cdot (\cos t \cdot \cos \pi - \cancel{\sin t \cdot \sin \pi})\,dt$$

$$= e^{-s\pi} \lim_{a \to \infty} \int_0^\infty e^{-st} \cos t \cdot (-1)\,dt = -e^{-s\pi} \cdot \mathcal{L}\{\cos(t)\}$$

$$= -e^{-s\pi} \cdot \frac{s}{s^2 + 1}$$

1 e) $f(t) = t^2 e^t \Rightarrow \mathcal{L}\{t^2 e^t\} = \int_0^\infty e^{-st} \cdot t^2 e^t \, dt = \int_0^\infty e^{-st+t} t^2 \, dt = \int_0^\infty e^{-(s-1)t} \cdot t^2 \, dt$

ser at $\mathcal{L}\{t^2 e^t\} = \mathcal{L}\{t^2\}$ for $s = s-1$:

Vi har:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \iff \mathcal{L}\{f(t)\} = \left(\mathcal{L}\{f'(t)\} + f(0)\right) \cdot \frac{1}{s}$$

$$\Rightarrow \mathcal{L}\{t^2\} = \frac{1}{s}\left(\mathcal{L}\{2t\} + 0\right) = \frac{1}{s} \cdot \mathcal{L}\{2t\}$$

$$= \frac{1}{s} \cdot \frac{1}{s}\left(\mathcal{L}\{2\} + 0\right) = \frac{1}{s^2} \cdot \mathcal{L}\{2\} = \underline{\frac{2}{s^3}}$$

Innsatt $s = s-1$ gir:

$$\mathcal{L}\{t^2 e^t\} = \underline{\underline{\frac{2}{(s-1)^3}}}$$

f) $f(t) = e^t \cos t \Rightarrow \mathcal{L}\{e^t \cos t\} = \int_0^\infty e^{-(s-1)t} \cos t \, dt$

ser at $\mathcal{L}\{e^t \cos t\} = \mathcal{L}\{\cos t\}$ for $s = s-1$:

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

Innsatt $s = s-1$ gir:

$$\mathcal{L}\{e^t \cos t\} = \frac{(s-1)}{(s-1)^2+1} = \underline{\underline{\frac{s-1}{s^2-2s+2}}}$$

g) $f(t) = e^t \sin t \Rightarrow \mathcal{L}\{e^t \sin t\} = \int_0^\infty e^{-(s-1)t} \sin t \, dt$

ser at $\mathcal{L}\{e^t \sin t\} = \mathcal{L}\{\sin t\}$ for $s = s-1$:

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

Innsatt $s = s-1$ gir:

$$\mathcal{L}\{e^t \sin t\} = \frac{1}{(s-1)^2+1} = \underline{\underline{\frac{1}{s^2-2s+2}}}$$

① a) $y'' - 2y' + 2y = 6e^{-t} \quad y(0) = 0, \; y'(0) = 1$

$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

$\mathcal{L}\{f''\} = s(s\mathcal{L}\{f\} - f(0)) - f'(0)$

$= s^2\mathcal{L}\{f\} - sf(0) - f'(0)$

$\Rightarrow s^2Y - sy(0) - y'(0) - 2(sY - y(0)) + 2Y = \mathcal{L}\{6e^{-t}\}$

$s^2Y - 2sY + 2Y = \frac{6}{s+1} + \cancel{sy(0)} + y'(0) + \cancel{2y(0)}$

$$\left.\begin{aligned} Y(s^2-2s+2) &= \frac{6}{s+1} + 1 \\ &= \frac{6}{s+1} + \frac{s+1}{s+1} \\ &= \frac{s+7}{s+1} \end{aligned}\right\} \quad Y(s) = \frac{s+7}{(s^2-2s+2)(s+1)}$$

$\cancel{y(t) = \mathcal{L}^{-1}\{Y(s)\} =}$

$$\frac{s+7}{(s^2-2s+2)(s+1)} = \frac{As+B}{s^2-2s+2} + \frac{C}{s+1} \Rightarrow s+7 = (As+B)(s+1) + C(s^2-2s+2)$$

$$\left.\begin{aligned} s^0 \text{ gir: } & 7 = B + 2C \\ s^1 \text{ gir: } & 1 = A + B - 2C \\ s^2 \text{ gir: } & 0 = A + C \end{aligned}\right\} \Leftrightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} \text{ for } \vec{x} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 2 & 7 \\ 1 & 1 & -2 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 1 & 2 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 6/5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6/5 \\ 0 & 1 & 0 & 23/5 \\ 0 & 0 & 1 & 6/5 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} -6/5 \\ 23/5 \\ 6/5 \end{bmatrix} \Rightarrow Y(s) = \frac{-\frac{6}{5}s + \frac{23}{5}}{(s^2-2s+1)+1} + \frac{\frac{6}{5}}{s+1}$$

$$= \frac{1}{5} \cdot \frac{-6(s+1-1)+23}{(s-1)^2+1^2} + \frac{6}{5} \cdot \frac{1}{s+1} = \frac{1}{5} \cdot \frac{-6(s-1)+23-6}{(s-1)^2+1^2} + \frac{6}{5} \cdot \frac{1}{s+1}$$

$$= -\frac{6}{5} \cdot \frac{s-1}{(s-1)^2+1^2} + \frac{17}{5} \cdot \frac{1}{(s-1)^2+1^2} + \frac{6}{5} \cdot \frac{1}{s+1}$$

| Innsatt resultat fra ① f,g får vi:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \underline{\underline{-\frac{6}{5}e^{t} \cdot \cos t + \frac{17}{5}e^{t} \sin t + \frac{6}{5}e^{-t}}}$$

② b) $y'' + y = f(t)$, $f(t) = \begin{cases} 0 & \text{if } 0 < t < \pi \\ 1 & \text{otherwise} \end{cases}$, $y(0) = 0$, $y'(0) = 0$

$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$

Skriver $f(t) = u_\pi(t)$ og transformerer begge sider

$s^2 Y(s) - \cancel{sy(0)} - \cancel{y'(0)} + Y(s) = \mathcal{L}\{u_\pi(t)\}$

$Y(s)(s^2+1) = e^{-\pi s} \cdot \mathcal{L}\{1\}$

$= \dfrac{e^{-\pi s}}{s}$

$\Leftrightarrow Y(s) = \dfrac{e^{-\pi s}}{s(s^2+1)} = e^{-\pi s}\left(\dfrac{1}{s(s^2+1)}\right)$

$\dfrac{1}{s(s^2+1)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1} \quad \cdot s(s^2+1)$

$1 = A(s^2+1) + (Bs+C)s$

s^0 gir: $1 = A + C$

s^1 gir: $0 = C$

s^2 gir: $0 = A + B$

$A = 1$, $B = -1$, $C = 0$

$Y(s) = e^{-\pi s}\left(\dfrac{1}{s} - \dfrac{s}{s^2+1}\right)$

$$\underline{\underline{y(t) = \mathcal{L}^{-1}\{Y(s)\} = u_\pi(t) \cdot (1 - \cos t)}}$$

③ $f(t)$ is periodic by period T if $f(t+T)=f(t)$.

for $T>0$, show that

$$\mathcal{L}\{f\}(s)=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

$$\mathcal{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)\,dt=\int_0^T e^{-st}f(t)\,dt+\int_T^\infty e^{-st}f(t)\,dt$$ sub $t=\hat{t}+T$

$$=\int_0^T e^{-st}f(t)\,dt+\int_0^\infty e^{-s(\hat{t}+T)}f(\hat{t}+T)\,d(\hat{t}+T)$$

$f(\hat{t}+T)=f(\hat{t})$

$d(\hat{t}+T)=d\hat{t}+dT=d\hat{t}$

α T konst.

$$=\int_0^T e^{-st}f(t)\,dt+e^{-sT}\int_0^\infty e^{-s\hat{t}}f(\hat{t})\,d\hat{t}$$

$\mathcal{L}\{f(t)\}=\mathcal{L}\{f(\hat{t})\}$ og:

$$\mathcal{L}\{f(t)\}=\int_0^T e^{-st}f(t)\,dt+e^{-sT}\cdot\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(t)\}\cdot(1-e^{-sT})=\int_0^T e^{-st}f(t)\,dt$$

$$\Rightarrow \mathcal{L}\{f(t)\}=\frac{1}{1-e^{-sT}}\cdot\int_0^T e^{-st}f(t)\,dt$$