Øving 3 - Patrik Kjærran

① $y - y * t = t$

Laplace-transformerer og benytter $\mathcal{L}\{f * g\} = \mathcal{L}^{-1}\{F(s)\} \cdot \mathcal{L}^{-1}\{G(s)\}$

$$Y(s) - Y(s) \cdot \frac{1}{s^2} = \frac{1}{s^2}$$

$$Y(s)\left(1 - \frac{1}{s^2}\right) = \frac{1}{s^2} \quad \Big| \cdot s^2$$

$$Y(s)(s^2 - 1) = 1$$

$$Y(s) = \frac{1}{s^2-1}$$

$$\Rightarrow \underline{\underline{y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t}}$$

② $x' = 2x - y$, $y' = 3x - 2y$ for $x(0) = 0$, $y(0) = 1$ | Laplace-transformerer

$$\mathcal{L}\{x'\} = 2X(s) - Y(s)$$
$$\mathcal{L}\{y'\} = 3X(s) - 2Y(s)$$

| Benytter $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$.

$$sX(s) - x(0) = 2X(s) - Y(s)$$
$$sY(s) - y(0) = 3X(s) - 2Y(s)$$

| Løser for $X(s)$ og $Y(s)$

(1) $X(s) = \dfrac{-Y(s)}{s-2}$

(2) $Y(s) = \dfrac{3X(s) + 1}{s+2}$

| Innsatt (2) i (1) gir

$$X(s) = \frac{-(3X(s) + 1)}{(s+2)(s-2)}$$

| Løser for $X(s)$.

$$\underline{X(s) = \frac{-1}{s^2-1}}$$

$$Y(s) = \frac{3\left(\frac{-1}{s^2-1}\right) + 1}{s+2}$$

$$= \frac{-3}{(s^2-1)(s+2)} + \frac{1}{s+2} \cdot \frac{(s^2-1)}{(s^2-1)}$$

$$= \frac{s^2-4}{(s^2-1)(s+2)} = \frac{\cancel{(s+2)}(s-2)}{(s^2-1)\cancel{(s+2)}} = \frac{s-2}{s^2-1}$$

$$\underline{= \frac{s}{s^2-1} - \frac{2}{s^2-1}}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{-1}{s^2-1}\right\} = \underline{\underline{-\sinh t}}$$

og

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2-1} - \frac{2}{s^2-1}\right\} = \underline{\underline{\cosh t - 2\sinh t}}$$

③ a) Skal vise $\mathcal{L}\{t^n\}(s) = \frac{\Gamma(n+1)}{s^{n+1}}$, gitt $\Gamma(x) = \int_0^\infty t^{x-1} \cdot e^{-t}\,dt$ $(x>0)$

$$\mathcal{L}\{t^n\}(s) = \int_0^\infty e^{-st} \cdot t^n\,dt \quad \Big|\ \text{sub. } t = \frac{\hat{t}}{s} \Rightarrow dt = \frac{d\hat{t}}{s}$$

$$= \int_0^\infty e^{-s\left(\frac{\hat{t}}{s}\right)} \cdot \left(\frac{\hat{t}}{s}\right)^n \frac{d\hat{t}}{s}$$

$$= \int_0^\infty e^{-\hat{t}} \cdot \frac{\hat{t}^n}{s^n} \cdot \frac{d\hat{t}}{s}$$

$$= \frac{\int_0^\infty e^{-\hat{t}}\,\hat{t}^{(n+1)-1} \cdot d\hat{t}}{s^{n+1}} = \underline{\underline{\frac{\Gamma(n+1)}{s^{n+1}}}}$$

Fra 6.1.4 og 6.1.5 ser vi at $\Gamma(x)$ fungerer som en kontinuerlig variant av fakultet-operatoren, på formen $\Gamma(n+1) = n!$

b) 1) $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$ | Delvis integresjon $u = t^x$, $u' = xt^{x-1}$, $v = -e^{-t}$, $v' = e^{-t}$

$$= \left[-t^x e^{-t}\right]_0^\infty + \int_0^\infty e^{-t} \cdot xt^{x-1}\,dt$$

$$= (0-0) + x \cdot \int_0^\infty e^{-t} t^{x-1}\,dt$$

$$\underline{\underline{\Gamma(x+1) = x \cdot \Gamma(x)}} \quad (*)$$

2) $$\Gamma\left(\frac{2k+1}{2}\right) \overset{(*)}{=} \left(\frac{2k-1}{2}\right)\left(\frac{2k-3}{2}\right) \cdots \left(\frac{5}{2}\right) \cdot \left(\frac{3}{2}\right) \cdot \left(\frac{1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)$$

$$= \prod_{i=0}^{k-1} \left(\frac{2i+1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)$$

$$= \prod_{i=0}^{k-1} \left(\frac{2i+1}{2}\right) \cdot \sqrt{\pi} = \underline{\underline{\frac{(2k-1)!!}{2^k} \cdot \sqrt{\pi}}}$$

c) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1} \cdot e^{-t}\,dt = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\,dt$ | sub. $u = t^{\frac{1}{2}} \Rightarrow du = \frac{1}{2} t^{-\frac{1}{2}}\,dt$
$\Rightarrow (t = u^2)$ $dt = \frac{2du}{t^{\frac{1}{2}}} = \frac{2du}{u}$

$$= \int_0^\infty \cancel{u} \cdot e^{-(u^2)} \cdot \frac{2du}{\cancel{u}}$$

$$\underline{\underline{= 2\int_0^\infty e^{-u^2}\,du}}$$

4 a) $f(x) = x$ for $x \in (-\pi, \pi)$

$$= \sum_{n=-\infty}^{\infty} c_n \cdot e^{i\frac{n\pi x}{L}} \quad \text{for} \quad c_n = \frac{1}{2}(a_n - ib_n)$$

| $a_n = 0$ for alle n siden $f(x) = x$ er odde.

$$= \sum_{n=-\infty}^{\infty} \frac{-i}{2} \cdot b_n \cdot e^{i\frac{n\cancel{\pi} x}{\cancel{\pi}}}$$

$= \frac{-i}{2} \cdot b_n$

$$= \sum_{n=-\infty}^{\infty} \frac{-i}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\frac{n\cancel{\pi}x}{\cancel{\pi}}\,dx\right)\cdot e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} \frac{-i}{\cancel{2}}\left(\frac{\cancel{2}}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx\right)e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} \frac{-i}{\pi}\left(\int_{0}^{\pi} f(x)\sin nx\,dx\right)e^{inx}$$

| Delvis integrasjon:
$u = x$ $\quad u' = 1$
$v = \frac{-\cos nx}{n}$ $\quad v' = \sin nx$

$$= \sum_{n=-\infty}^{\infty} \frac{-i}{\pi}\left(\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)e^{inx}$$

| Benytter at $\cos n\pi = (-1)^n$

$$= \sum_{n=-\infty}^{\infty} \frac{-i}{\cancel{\pi}}\left(\frac{-\cancel{\pi}}{n}\cdot(-1)^n + \cancel{\frac{1}{n}\left[\frac{1}{n}\sin nx\right]_0^{\pi}}\right)e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} \frac{i(-1)^n}{n}\cdot e^{inx}$$

Siden summen ikke er definert for $n=0$, ekskluderer vi $n=0$ fra summen og evaluerer denne eksplisitt.

$$= c_0 + \sum_{n\neq 0} \frac{i(-1)^n}{n}\cdot e^{inx}$$

| $c_0 = \frac{-i}{2} b_0 = \frac{-i}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 0\,dx\right) = 0$

$$\Rightarrow f(x) = \sum_{n\neq 0} \frac{i(-1)^n}{n} e^{inx}$$

4 b) Definerer $g(x) = 2\pi \cdot x$ og $h(x) = x^2$ for $x \in (-\pi, \pi)$.

Dette gir $x(2\pi - x) = g(x) - h(x)$.

$g(x) = 2\pi \cdot x$ | Benytter resultatet i a

$\underline{= 2\pi \cdot f(x)}$

$$h(x) = x^2$$

$$= \sum_{n=-\infty}^{\infty} c_n \cdot e^{i\frac{n\pi x}{L}} \quad \text{for} \quad c_n = \frac{1}{2}(a_n - ib_n) = \frac{a_n}{2}$$

| $b_n = 0$ for alle n, siden $g(x) = x^2$ er jevn.

$$= \sum_{n=-\infty}^{\infty} \frac{a_n}{2} \cdot e^{i\frac{n\pi x}{\cancel{\pi}}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos\frac{n\pi x}{\cancel{\pi}}\,dx\right)e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx\right)e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx \cdot e^{inx}$$

| Delvis integrasjon:

$u_1 = x^2 \qquad u_1' = 2x$

$v_1 = \frac{\sin nx}{n} \qquad v_1' = \cos nx$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{\pi}\left(\cancel{\left[x^2\frac{\sin nx}{n}\right]_0^{\pi}} - \int_0^{\pi}\frac{2x\sin nx}{n}\,dx\right)e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} \frac{-2}{\pi n}\cdot\int_0^{\pi} x\sin nx\,dx \cdot e^{inx}$$

| Delvis integrasjon:

$u_2 = x \qquad u_2' = 1$

$v_2 = \frac{-\cos nx}{n} \qquad v_2' = \sin nx$

$$= \sum_{n=-\infty}^{\infty} \frac{-2}{\pi n}\left(\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)e^{inx}$$

$$= \sum_{n=-\infty}^{\infty} \frac{-2}{\cancel{\pi} n}\left(\frac{-\cancel{\pi}\cos n\pi}{n} + \cancel{\frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}}\right)e^{inx}$$

| Benytter at $\cos n\pi = (-1)^n$

$$= \sum_{n=-\infty}^{\infty} \frac{2\cdot(-1)^n}{n^2}\cdot e^{inx}$$

Siden summen ikke er definert for $n=0$, ekskluderer vi $n=0$ fra summen og evaluerer denne eksplisitt.

$$= c_0 + \sum_{n\neq 0}\frac{2\cdot(-1)^n}{n^2}\cdot e^{inx}$$

| $c_0 = \frac{a_0}{2} = \frac{1}{2}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cdot\cos 0\,dx$

$= \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$

$$\Rightarrow \underline{h(x) = \frac{\pi^2}{3} + \sum_{n\neq 0}\frac{2\cdot(-1)^n}{n^2}\cdot e^{inx}}$$

(forts. neste side)

Vi kan nå uttrykke $x(2\pi - x)$ gitt ved:

$$
\begin{aligned}
x(2\pi - x) &= g(x) - h(x) \\
&= 2\pi \cdot f(x) - h(x) \\
&= 2\pi \cdot \sum_{n\neq 0} \frac{i(-1)^n}{n} e^{inx} - \left(\frac{\pi^2}{3} + \sum_{n\neq 0} \frac{2(-1)^n}{n^2} e^{inx}\right) \\
&= -\frac{\pi^2}{3} + \sum_{n\neq 0} \frac{2\pi i(-1)^n}{n} e^{inx} - \sum_{n\neq 0} \frac{2(-1)^n}{n^2} e^{inx} \\
&= -\frac{\pi^2}{3} + \sum_{n\neq 0} \frac{2\pi(-1)^n}{n} e^{inx} + \sum_{n\neq 0} \frac{2(-1)^{n+1}}{n^2} e^{inx} \\
&= \underline{\underline{-\frac{\pi^2}{3} + \sum_{n\neq 0} \left(\frac{2\pi(-1)^n}{n} + \frac{2(-1)^{n+1}}{n^2}\right) e^{inx}}}
\end{aligned}
$$