ENPM 667 - CONTROL OF ROBOTIC SYSTEMS
FINAL PROJECT

THE UNIVERSITY OF MARYLAND,
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ENPM 667 Final Project

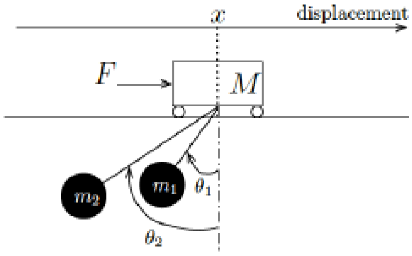
University of Maryland, College Park

Due Date: Dec 20 (at 3am) in ELMS

INSTRUCTIONS

- Please include all relevant calculations, simulation plots, simulation schemes, and algorithms as part of your solutions. Please submit your solution in a pdf file. Also, provide all your code for the grader to test.
- This project is worth **20%** of the course grade.
- The project will be submitted via ELMS.
- Two students are allowed (not required) to work as a group on this project. If you decide to work as a group please submit one technical report with both of your names clearly written at the top.

First Component (100 points): Consider a crane that moves along an one-dimensional track. It behaves as a frictionless cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m_1 and m_2, and the lengths of the cables are l_1 and l_2, respectively. The following figure depicts the crane and associated variables used throughout this project.



A) (25 points) Obtain the equations of motion for the system and the corresponding nonlinear state-space representation.

B) (25 points) Obtain the linearized system around the equilibrium point specified by $x = 0$ and $\theta_1 = \theta_2 = 0$. Write the state-space representation of the linearized system.

C) (25 points) Obtain conditions on M, m_1, m_2, l_1, l_2 for which the linearized system is controllable.

D) (25 points) Choose $M = 1000Kg$, $m_1 = m_2 = 100Kg$, $l_1 = 20m$ and $l_2 = 10m$. Check that the system is controllable and obtain an LQR controller. Simulate the resulting response to initial conditions when the controller is applied to the linearized system and also to the original nonlinear system. Adjust the parameters of the LQR cost until you obtain a suitable response. Use Lyapunov's indirect method to certify stability (locally or globally) of the closed-loop system.

Second Component (100 points): Consider the parameters selected in C) above.

E) Suppose that you can select the following output vectors: $x(t)$, $(\theta_1(t), \theta_2(t))$, $(x(t), \theta_2(t))$ or $(x(t), \theta_1(t), \theta_2(t))$. Determine for which output vectors the linearized system is observable.

F) Obtain your "best" Luenberger observer for each one of the output vectors for which the system is observable and simulate its response to initial conditions and unit step input. The simulation should be done for the observer applied to both the linearized system and the original nonlinear system.

G) Design an output feedback controller for your choice of the "smallest" output vector. Use the LQG method and apply the resulting output feedback controller to the original nonlinear system. Obtain your best design and illustrate its performance in simulation. How would you reconfigure your controller to asymptotically track a constant reference on x ? Will your design reject constant force disturbances applied on the cart ?

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1. First Component

1.1. Modelling the system

The given crane moves along a one-dimensional track, it can be seen as a frictionless cart with mass M, and external force F acting on it, which can be considered as the input of the system. There are two loads suspended from cables attached to the crane.

The loads have mass:

- m_1, m_2

The lengths of the cables:

- l_1, l_2

The vertical & horizontal displacement of the loads can be described by the horizontal displacement of the crane (x) and the angle between the loads and the vertical axis of the crane:

- θ_1, θ_2

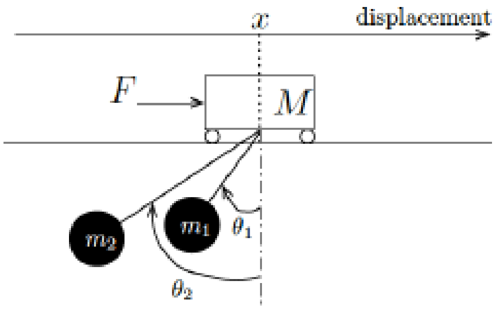

Figure 1: Inverted Pendulum

As a result, the displacement of the loads can be written up as follows:

$$X_{m1} = x - l_1 sin(\theta_1)$$
$$Y_{m1} = -l_1 cos(\theta_1)$$
$$X_{m2} = x - l_2 sin(\theta_2)$$
$$X_{m2} = -l_2 cos(\theta_2)$$

where, X_{m1} and Y_{m1} are the positions of load 1 in terms of x and y-direction
X_{m2} and Y_{m2} are the positions of load 2 in terms of x and y-direction

Taking their time derivative the horizontal and vertical velocity of the loads can be written up as follows:

$$\dot{X_{m1}} = \dot{x} - l_1 cos(\theta_1)\dot{\theta_1} \quad (1)$$
$$\dot{Y_{m1}} = l_1 sin(\theta_1)\dot{\theta_1} \quad (2)$$
$$\dot{X_{m2}} = \dot{x} - l_2 cos(\theta_2)\dot{\theta_2} \quad (3)$$
$$\dot{Y_{m2}} = l_2 sin(\theta_2)\dot{\theta_2} \quad (4)$$

Calculating their vectorial sum, the velocity of the loads from equations (1), (2), (3) and (4) is given by the following:

$$v_1^2 = \dot{X}_{m1}^2 + \dot{Y}_{m1}^2 = \left(\dot{x} - l_1 cos(\theta_1)\dot{\theta_1}\right)^2 + \left(l_1 sin(\theta_1)\dot{\theta_1}\right)^2$$
$$= \dot{x}^2 + l_1\dot{\theta_1}^2 - 2l_1 cos(\theta_1)\dot{x}\dot{\theta}_1 \text{ (5)}$$

$$v_2^2 = \dot{X}_{m2}^2 + \dot{Y}_{m2}^2 = \left(\dot{x} - l_2 cos(\theta_2)\dot{\theta_2}\right)^2 + \left(l_2 sin(\theta_2)\dot{\theta_2}\right)^2$$
$$= \dot{x}^2 + l_2\dot{\theta_2}^2 - 2l_2 cos(\theta_2)\dot{x}\dot{\theta}_2 \text{ (6)}$$

1.2. Equations of motion

The Euler-Lagrange method can be used to calculate the system equations of the motion.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = f \quad (7)$$

where, $L = K - P$

L, the Lagrange operator is calculated using the difference of the Potential and Kinetic Energy of the system, P and K respectively; f accounts for the forces acting on the system, t represents the time and q are the used generalized coordinates.

The system was modelled, and the displacements and the velocities were calculated in the previous section, based on

Generalized coordinates: x, θ_1, θ_2

The kinetic energy of the system can be calculated by the means of the sum of the kinetic energy associated with the crane and the kinetic energy of the loads, given as:

$$K = \frac{1}{2}\dot{x}^2(M) + \frac{1}{2}(m_1 v_1^2) + \frac{1}{2}(m_2 v_2^2) \text{ (8)}$$

Using x, v_1, v_2 from the modelled system and substituting velocity values from equations (5) and (6),

$$K = \frac{1}{2}\dot{x}(m_1 + m_2 + M) + \frac{1}{2}\left(m_1 l_1 \dot{\theta_1}^2\right) + \frac{1}{2}\left(m_2 l_2 \dot{\theta_2}^2\right) - m_1 l_1 cos(\theta_1)\dot{x}\dot{\theta}_1 - m_2 l_2 cos(\theta_2)\dot{x}\dot{\theta}_2 \text{ (9)}$$

The potential energy of the system can be calculated as the sum of the potential energy of the loads, since the crane does not move vertically and the reference was taken from the connection point.

$$P = -m_1 g l_1 cos(\theta_1) - m_2 g l_2 cos(\theta_2) \text{ (10)}$$

Using the components above the Lagrange operator of the system is the following:

$$L = K - P$$

Substituting equations (9) and (10) in the above equation,

$$L = \frac{1}{2}\dot{x}(m_1 + m_2 + M) + \frac{1}{2}\left(m_1 l_1 \dot{\theta_1}^2\right) + \frac{1}{2}\left(m_2 l_2 \dot{\theta_2}^2\right) - m_1 l_1 cos(\theta_1)\dot{x}\dot{\theta}_1 - m_2 l_2 cos(\theta_2)\dot{x}\dot{\theta}_2 + m_1 g l_1 cos(\theta_1) + m_2 g l_2 cos(\theta_2) \text{ (11)}$$

The Lagrangian equation can be obtained by differentiating L by $\dot{q} = \left[\dot{x}, \dot{\theta_1}, \dot{\theta_2}\right]$ and then t, and finally, the second part is obtained by differentiating by $q = [x, \theta_1, \theta_2]$:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} = F \text{ (12)} \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta_1}}\right] - \frac{\partial L}{\partial \theta_1} = 0 \text{ (13)} \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}_2}\right] - \frac{\partial L}{\partial \theta_2} = 0 \text{ (14)}$$

First, the partial differentiation of L with respect to $'\dot{x}'$,

$$\frac{\partial L}{\partial \dot{x}} = \dot{x}(m_1 + m_2 + M) - m_1 l_1 cos(\theta_1)\dot{\theta_1} - m_2 l_2 cos(\theta_2)\dot{\theta_2} \text{ (15)}$$

Differentiating the above equation with respect to time $'t'$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{x}}\right] = \ddot{x}(m_1 + m_2 + M) - m_1 l_1\left(\ddot{\theta_1} cos(\theta_1) - sin(\theta_1)\dot{\theta_1}^2\right) - m_1 l_1\left(\ddot{\theta_2} cos(\theta_2) - sin(\theta_2)\dot{\theta_2}^2\right) \text{ (16)}$$

The partial differentiation of L with respect to $'x'$,

$$\frac{\partial L}{\partial x} = 0 \text{ (17)}$$

The partial differentiation of L with respect to $'\dot{\theta}_1'$,

$$\frac{\partial L}{\partial \dot{\theta_1}} = m_1 l_1^2 \dot{\theta_1} - m_1 l_1 cos(\theta_1)\dot{x} \text{ (18)}$$

Differentiating the above equation with respect to time 't',

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 \left(cos(\theta_1)\ddot{x} - sin(\theta_1) x \dot{\theta}_1 \right) \text{ (19)}$$

The partial differentiation of L with respect to 'θ_1',

$$\frac{\partial L}{\partial \theta_1} = sin(\theta_1) m_1 l_1 x \dot{\theta}_1 - m_1 l_1 g sin(\theta_1) \text{ (20)}$$

Similarly, the partial differentiation of L with respect to '$\dot{\theta}_2$',

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 - m_2 l_2 cos(\theta_2)\dot{x} \text{ (21)}$$

Differentiating the above equation with respect to time 't',

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 \left(cos(\theta_2)\ddot{x} - sin(\theta_2) x \dot{\theta}_2 \right) \text{ (22)}$$

The partial differentiation of L with respect to 'θ_2',

$$\frac{\partial L}{\partial \theta_2} = sin(\theta_2) m_2 l_2 x \dot{\theta}_2 - m_2 l_2 g sin(\theta_2) \text{ (23)}$$

Now, substituting the derivatives from equations (15), (16) and (17) in equation (12):

$$\frac{d}{dt}\left[\dot{x}(M + m_1 + m_2) - m_1 \mathrm{l}_1 \dot{\theta}_1 \mathrm{Cos}\theta_1 - m_2 \mathrm{l}_2 \dot{\theta}_2 \mathrm{Cos}\theta_2 + 0 \right] - 0 = F$$
$$\ddot{x}(M + m_1 + m_2) - m_1 \mathrm{l}_1 \ddot{\theta}_1 \mathrm{Cos}\theta_1 + m_1 \mathrm{l}_1 \dot{\theta}_1^{\,2} \mathrm{Sin}\theta_1 - m_2 \mathrm{l}_2 \ddot{\theta}_2 \mathrm{Cos}\theta_2 + m_2 \mathrm{l}_2 \dot{\theta}_2^{\,2} \mathrm{Sin}\theta_2 = \mathrm{F} \text{ (24)}$$

Substituting the derivatives from equations (18), (19) and (20) in equation (13):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 \left(cos(\theta_1)\ddot{x} - sin(\theta_1) x \dot{\theta}_1 \right) - sin(\theta_1) m_1 l_1 x \dot{\theta}_1 + m_1 g l_1 sin(\theta_1)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 cos(\theta_1)\ddot{x} + m_1 g l_1 sin(\theta_1)$$
$$= m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 \ddot{x} + m_1 g l_1 \theta_1 = 0 \text{ (25)}$$

Substituting the derivatives from equations (21), (22) and (23) in equation (14):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)-\frac{\partial L}{\partial\theta_2}=m_2l_2^2\ddot{\theta}_2-m_2l_2\left(cos(\theta_2)\ddot{x}-sin(\theta_2)x\dot{\theta}_2\right)-sin(\theta_2)m_2l_2x\dot{\theta}_2+m_2gl_2sin(\theta_2)$$

$$=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)-\frac{\partial L}{\partial\theta_2}=m_2l_2^2\ddot{\theta}_2-m_2l_2cos(\theta_2)\ddot{x}+m_2gl_2sin(\theta_2)=\ 0$$

$$m_2l_2^2\ddot{\theta}_2-m_2l_2\ddot{x}+m_2gl_2\theta_2=0\ (26)$$

Now, the Lagrange equations of the system can be given as:

$$F=\ddot{x}(m_1+m_2+M)-m_1l_1\left(\ddot{\theta}_1cos(\theta_1)-sin(\theta_1)\theta_1^2\right)-m_2l_2\left(\ddot{\theta}_2cos(\theta_2)-sin(\theta_2)\theta_2^2\right)$$

$$m_1l_1^2\ddot{\theta}_1-m_1l_1cos(\theta_1)\ddot{x}+m_1gl_1sin(\theta_1)=0$$

$$m_2l_2^2\ddot{\theta}_2-m_2l_2cos(\theta_2)\ddot{x}+m_2gl_2sin(\theta_2)=0$$

The values of $\ddot{\theta}_1$ and $\ddot{\theta}_2$ can be obtained from equations (25) and (26) as follows,

$$\ddot{\theta}_1=\frac{cos(\theta_1)\ddot{x}-gsin(\theta_1)}{l_1}\ (27)$$

$$\ddot{\theta}_2=\frac{cos(\theta_2)\ddot{x}-gsin(\theta_2)}{l_2}\ (28)$$

The value of $\ddot{x}$ can be obtained from equation (24),

$$\ddot{x}=\frac{F+m_1l_1\left(\ddot{\theta}_1cos(\theta_1)-sin(\theta_1)\{\dot{\theta}_1^2)\right\}+m_2l_2\left(\ddot{\theta}_2cos(\theta_2)-sin(\theta_2)\{\dot{\theta}_2^2)\right\}}{m_1+m_2+M}$$

$$\ddot{x}=\frac{F+m_1\left(cos(\theta_1)^2\ddot{x}-gsin(\theta_1)cos(\theta_1)-l_1sin(\theta_1)\dot{\theta_1}^2\right)+m_2\left(cos(\theta_2)^2\ddot{x}-gsin(\theta_2)cos(\theta_2)-l_2sin(\theta_2)\dot{\theta_2}^2\right)}{m_1+m_2+M}$$

$$(m_1+m_2+M)\ddot{x}-m_1cos(\theta_1)^2\ddot{x}-m_2cos(\theta_2)^2\ddot{x}=F-m_1\left(gsin(\theta_1)cos(\theta_1)+l_1sin(\theta_1)\dot{\theta_1}^2\right)-m_2\left(gsin(\theta_2)cos(\theta_2)+l_2sin(\theta_2)\dot{\theta_2}^2\right)$$

$$\left(M+m_1(1-cos(\theta_1)^2)+m_2(1-cos(\theta_2)^2)\right)\ddot{x}=F-m_1\left(gsin(\theta_1)cos(\theta_1)+l_1sin(\theta_1)\dot{\theta_1}^2\right)\backslash -m_2\left(gsin(\theta_2)cos(\theta_2)+l_2sin(\theta_2)\dot{\theta_2}^2\right)$$

$$\ddot{x}=\frac{F-m_1\left(gsin(\theta_1)cos(\theta_1)+l_1sin(\theta_1)\dot{\theta_1}^2\right)-m_2\left(gsin(\theta_2)cos(\theta_2)+l_2sin(\theta_2)\ddot{\theta_2}^2\right)}{\left(M+m_1\left(sin(\theta_1)\right)^2+m_2\left(sin(\theta_2)\right)^2\right)}\ (29)$$

Now, substituting the above equation (29) in equations (27) and (28), we get:

$$\ddot{\theta}_1=\frac{\cos(\theta_1)}{l_1}\left[\frac{F-m_1\left(g\sin(\theta_1)\cos(\theta_1)+l_1\sin(\theta_1)\dot{\theta_1}^2\right)-m_2\left(g\sin(\theta_2)\cos(\theta_2)+l_2\sin(\theta_2)\ddot{\theta_2}^2\right)}{\left(M+m_1\left(\sin(\theta_1)\right)^2+m_2\left(\sin(\theta_2)\right)^2\right)}\right]-g\frac{sin(\theta_1)}{l_1}$$

$$\ddot{\theta}_2 = \frac{cos(\theta_2)}{l_2}\left[\frac{F - m_1\left(gsin(\theta_1)cos(\theta_1) + l_1 sin(\theta_1)\dot{\theta}_1^2\right) - m_2\left(gsin(\theta_2)cos(\theta_2) + l_2 sin(\theta_2)\ddot{\theta}_2^2\right)}{\left(M + m_1(sin(\theta_1))^2 + m_2(sin(\theta_2))^2\right)}\right] - g\frac{sin(\theta_2)}{l_2}$$

Now, the non-linear state representation of the system can be given in the form in terms of initial state and initial input. It can be written in the matrix form as:

$$\dot{x} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{F - m_1(g\text{Sin}\theta_1\text{Cos}\theta_1 + l_1\dot{\theta}_1^2\text{Sin}\theta_1) - m_2(g\text{Sin}\theta_2\text{Cos}\theta_2 + l_2\dot{\theta}_2^2\text{Sin}\theta_2)}{M + m_1\text{Sin}^2\theta_1 + m_2\text{Sin}^2\theta_2} \\ \dot{\theta}_1 \\ \frac{\text{Cos}\theta_1}{l_1}\left[\frac{F - m_1(g\text{Sin}\theta_1\text{Cos}\theta_1 + l_1\dot{\theta}_1^2\text{Sin}\theta_1) - m_2(g\text{Sin}\theta_2\text{Cos}\theta_2 + l_2\dot{\theta}_2^2\text{Sin}\theta_2)}{M + m_1\text{Sin}^2\theta_1 + m_2\text{Sin}^2\theta_2}\right] - \frac{g\text{Sin}\theta_1}{l_1} \\ \dot{\theta}_2 \\ \frac{\text{Cos}\theta_2}{l_2}\left[\frac{F - m_1(g\text{Sin}\theta_1\text{Cos}\theta_1 + l_1\dot{\theta}_1^2\text{Sin}\theta_1) - m_2(g\text{Sin}\theta_2\text{Cos}\theta_2 + l_2\dot{\theta}_2^2\text{Sin}\theta_2)}{M + m_1\text{Sin}^2\theta_1 + m_2\text{Sin}^2\theta_2}\right] - \frac{g\text{Sin}\theta_2}{l_2} \end{bmatrix} \quad (30)$$

1.3. *Linearizing the system around equilibrium point*

Linearization at x=0, θ_1=0 and θ_2=0 can be done by two methods, which are:

- Linearization by small angle approximation
- Jacobian Linearization

For the small angle method, the assumption of small angle values will be taken so the equations will be linearized. Therefore, the system can be simplified with the following substitutions,

$$cos(\theta_1) = 1, \qquad sin(\theta_1) = \theta_1, \qquad \dot{\theta}_1^2 = 0, \qquad \dot{\theta}_2^2 = 0$$

Based on these assumptions, we can equate equations (24), (25) and (26) as:

$$F = \ddot{x}(m_1 + m_2 + M) - m_1 l_1 \ddot{\theta}_1 - m_2 l_2 \ddot{\theta}_2 \quad (31)$$

$$m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 \ddot{x} + m_1 g l_1 \theta_1 = 0$$

$$\ddot{\theta}_1 = \frac{m_1 l_1 \ddot{x} - m_1 g l_1 \theta_1}{m_1 l_1^2}$$

$$\ddot{\theta}_1 = \frac{\ddot{x} - g\theta_1}{l_1} \quad (32)$$

$$m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 cos(\theta_2)\ddot{x} + m_2 g l_2 sin(\theta_2) = 0$$

$$\ddot{\theta}_2 = \frac{m_2 l_2 \ddot{x} - m_2 g l_2 \theta_2}{m_2 l_2^2}$$

$$\ddot{\theta}_2 = \frac{\ddot{x} - g\theta_2}{l_2} \quad (33)$$

Now, substituting equations (32) and (33) in (31),

$$F = \ddot{x}(m_1 + m_2 + M) - m_1 l_1 \left(\frac{\ddot{x} - g\theta_1}{l_1}\right) - m_2 l_2 \left(\frac{\ddot{x} - g\theta_2}{l_2}\right)$$

$$\mathrm{F} = m_1\ddot{x} + m_2\ddot{x} + M\ddot{x} - m_1\ddot{x} + m_1\, g\theta_1 - m_2\ddot{x} + m_2\, g\theta_2$$

$$\mathrm{F} = M\ddot{x} + m_1\, g\theta_1 + m_2\, g\theta_2$$

It can be written in terms of $\ddot{x}$ as,

$$\ddot{x} = \frac{\mathrm{F} - m_1\, g\theta_1 - m_2\, g\theta_2}{M} \quad (34)$$

Now, substituting equation (34) in (32) and (33),

$$\ddot{\theta}_1 = \frac{\frac{\mathrm{F} - m_1\, g\theta_1 - m_2\, g\theta_2}{M} - g\theta_1}{l_1}$$

$$\ddot{\theta}_1 = \frac{\mathrm{F} - m_1\, g\theta_1 - m_2\, g\theta_2 - Mg\theta_1}{Ml_1}$$

$$\ddot{\theta}_1 = \frac{-(m_1 + \mathrm{M})\, g\theta_1}{Ml_1} - \frac{m_2\, g\theta_2}{Ml_1} + \frac{F}{Ml_1} \quad (35)$$

$$\ddot{\theta}_2 = \frac{\frac{\mathrm{F} - m_1\, g\theta_1 - m_2\, g\theta_2}{M} - g\theta_2}{l_2}$$

$$\ddot{\theta}_2 = \frac{\mathrm{F} - m_1\, g\theta_1 - m_2\, g\theta_2 - Mg\theta_2}{Ml_2}$$

$$\ddot{\theta}_2 = \frac{-(m_1 + \mathrm{M})\, g\theta_2}{Ml_2} - \frac{m_2\, g\theta_2}{Ml_2} + \frac{F}{Ml_2} \quad (36)$$

Now, the linearized system can be written in matrix form from the values of equations (34), (35) and (36),

$$\begin{bmatrix} \dot{\mathrm{x}} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_1} & 0 & \frac{-gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} \mathrm{F} \quad (37)$$

Now, the Jacobian linearization method, at the equilibrium point $(x, \theta_1, \theta_2)_{(0,0,0)}$ is given as follows,

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} & \frac{\partial f_1}{\partial x_5} & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} & \frac{\partial f_2}{\partial x_5} & \frac{\partial f_2}{\partial x_6} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} & \frac{\partial f_3}{\partial x_5} & \frac{\partial f_3}{\partial x_6} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} & \frac{\partial f_4}{\partial x_5} & \frac{\partial f_4}{\partial x_6} \\ \frac{\partial f_5}{\partial x_1} & \frac{\partial f_5}{\partial x_2} & \frac{\partial f_5}{\partial x_3} & \frac{\partial f_5}{\partial x_4} & \frac{\partial f_5}{\partial x_5} & \frac{\partial f_5}{\partial x_6} \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \frac{\partial f_6}{\partial x_3} & \frac{\partial f_6}{\partial x_4} & \frac{\partial f_6}{\partial x_5} & \frac{\partial f_6}{\partial x_6} \end{bmatrix}_{(0,0,0)}$$

where,

$$\mathrm{x} = \begin{bmatrix} \mathrm{x} \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}, \ \mathrm{f} = \begin{bmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \\ \mathrm{f}_3 \\ \mathrm{f}_4 \\ \mathrm{f}_5 \\ \mathrm{f}_6 \end{bmatrix}$$

$\mathrm{f}_1 = \dot{\mathrm{x}}$

$\mathrm{f}_2 = \frac{F - m_1\left(g sin(\theta_1) cos(\theta_1) + l_1 sin(\theta_1) \dot{\theta}_1^{2}\right) - m_2\left(g sin(\theta_2) cos(\theta_2) + l_2 sin(\theta_2) \ddot{\theta}_2^{2}\right)}{\left(M + m_1(sin(\theta_1))^2 + m_2(sin(\theta_2))^2\right)}$ [From equation (29)]

$\mathrm{f}_3 = \dot{\theta}_1$

$\mathrm{f}_4 = \frac{cos(\theta_1)\mathrm{f}_2 - g sin(\theta_1)}{l_1}$ [From equation (27)]

$\mathrm{f}_5 = \dot{\theta}_2$

$\mathrm{f}_6 = \frac{cos(\theta_2)\mathrm{f}_2 - g sin(\theta_2)}{l_2}$ [From equation (28)]

It can be written in terms of our non-linear system using our MATLAB code linearizable.m, as shown below,

$$A = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial \dot{x}} & \frac{\partial \dot{x}}{\partial \theta_1} & \frac{\partial \dot{x}}{\partial \dot{\theta}_1} & \frac{\partial \dot{x}}{\partial \theta_2} & \frac{\partial \dot{x}}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{x}}{\partial x} & \frac{\partial \ddot{x}}{\partial \dot{x}} & \frac{\partial \ddot{x}}{\partial \theta_1} & \frac{\partial \ddot{x}}{\partial \dot{\theta}_1} & \frac{\partial \ddot{x}}{\partial \theta_2} & \frac{\partial \ddot{x}}{\partial \dot{\theta}_2} \\ \frac{\partial \dot{\theta}_1}{\partial x} & \frac{\partial \dot{\theta}_1}{\partial \dot{x}} & \frac{\partial \dot{\theta}_1}{\partial \theta_1} & \frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_1} & \frac{\partial \dot{\theta}_1}{\partial \theta_2} & \frac{\partial \dot{\theta}_1}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{\theta}_1}{\partial x} & \frac{\partial \ddot{\theta}_1}{\partial \dot{x}} & \frac{\partial \ddot{\theta}_1}{\partial \theta_1} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_1} & \frac{\partial \ddot{\theta}_1}{\partial \theta_2} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_2} \\ \frac{\partial \dot{\theta}_2}{\partial x} & \frac{\partial \dot{\theta}_2}{\partial \dot{x}} & \frac{\partial \dot{\theta}_2}{\partial \theta_1} & \frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_1} & \frac{\partial \dot{\theta}_2}{\partial \theta_2} & \frac{\partial \dot{\theta}_2}{\partial \dot{\theta}_2} \\ \frac{\partial \ddot{\theta}_1}{\partial x} & \frac{\partial \ddot{\theta}_1}{\partial \dot{x}} & \frac{\partial \ddot{\theta}_1}{\partial \theta_1} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_1} & \frac{\partial \ddot{\theta}_1}{\partial \theta_2} & \frac{\partial \ddot{\theta}_1}{\partial \dot{\theta}_2} \end{bmatrix}_{(0,0,0)}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_1} & 0 & \frac{-gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} F \quad (31)$$

Similarily, the Jacobian linearization for B is given as follows,

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial F} \\ \frac{\partial f_2}{\partial F} \\ \frac{\partial f_3}{\partial F} \\ \frac{\partial f_4}{\partial F} \\ \frac{\partial f_5}{\partial F} \\ \frac{\partial f_6}{\partial F} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{M+m_1\left(sin(\theta_1)\right)^2+m_2\left(sin(\theta_2)\right)^2} \\ 0 \\ \frac{1}{(M+m_1\left(sin(\theta_1)\right)^2+m_2\left(sin(\theta_2)\right)^2)l_1} \\ 0 \\ \frac{1}{(M+m_1\left(sin(\theta_1)\right)^2+m_2\left(sin(\theta_2)\right)^2)l_2} \end{bmatrix}$$

1.4 Controllability:

The A and B matrices obtained equation (31) above are independent of time and thus, the system is an LTI system. An LTI system is controllable if the controllability matrix obtained has full rank condition, i.e., its rank should be equal to n.

rank (C) = rank [B AB A^2B A^3B A^4B A^5B] = n

Controllability matrix is obtained using MATLAB as shown below,

$$\begin{pmatrix}
0 & \frac{1}{M} & 0 & -\frac{g\,m_1}{M^2\,l_1}-\frac{g\,m_2}{M^2\,l_2} & 0 & \frac{\frac{g\,m_1\,(M\,g+g\,m_1)}{M^2\,l_1}+\frac{g^2\,m_1\,m_2}{M^2\,l_2}}{M\,l_1}+\frac{\frac{g\,m_2\,(M\,g+g\,m_2)}{M^2\,l_2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1}}{M\,l_2} \\
\frac{1}{M} & 0 & -\frac{g\,m_1}{M^2\,l_1}-\frac{g\,m_2}{M^2\,l_2} & 0 & \frac{\frac{g\,m_1\,(M\,g+g\,m_1)}{M^2\,l_1}+\frac{g^2\,m_1\,m_2}{M^2\,l_2}}{M\,l_1}+\frac{\frac{g\,m_2\,(M\,g+g\,m_2)}{M^2\,l_2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1}}{M\,l_2} & 0 \\
0 & \frac{1}{M\,l_1} & 0 & -\frac{M\,g+g\,m_1}{M^2\,{l_1}^2}-\frac{g\,m_2}{M^2\,l_1\,l_2} & 0 & \frac{\frac{(M\,g+g\,m_1)^2}{M^2\,{l_1}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2}}{M\,l_1}+\frac{\frac{g\,m_2\,(M\,g+g\,m_1)}{M^2\,{l_1}^2}+\frac{g\,m_2\,(M\,g+g\,m_2)}{M^2\,l_1\,l_2}}{M\,l_2} \\
\frac{1}{M\,l_1} & 0 & -\frac{M\,g+g\,m_1}{M^2\,{l_1}^2}-\frac{g\,m_2}{M^2\,l_1\,l_2} & 0 & \frac{\frac{(M\,g+g\,m_1)^2}{M^2\,{l_1}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2}}{M\,l_1}+\frac{\frac{g\,m_2\,(M\,g+g\,m_1)}{M^2\,{l_1}^2}+\frac{g\,m_2\,(M\,g+g\,m_2)}{M^2\,l_1\,l_2}}{M\,l_2} & 0 \\
0 & \frac{1}{M\,l_2} & 0 & -\frac{M\,g+g\,m_2}{M^2\,{l_2}^2}-\frac{g\,m_1}{M^2\,l_1\,l_2} & 0 & \frac{\frac{(M\,g+g\,m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2}}{M\,l_2}+\frac{\frac{g\,m_1\,(M\,g+g\,m_2)}{M^2\,{l_2}^2}+\frac{g\,m_1\,(M\,g+g\,m_1)}{M^2\,l_1\,l_2}}{M\,l_1} \\
\frac{1}{M\,l_2} & 0 & -\frac{M\,g+g\,m_2}{M^2\,{l_2}^2}-\frac{g\,m_1}{M^2\,l_1\,l_2} & 0 & \frac{\frac{(M\,g+g\,m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2}}{M\,l_2}+\frac{\frac{g\,m_1\,(M\,g+g\,m_2)}{M^2\,{l_2}^2}+\frac{g\,m_1\,(M\,g+g\,m_1)}{M^2\,l_1\,l_2}}{M\,l_1} & 0
\end{pmatrix}$$

The determinant of the controllability matrix is found as follows,

$$\det(C) = \frac{-g^6(l_1^2-l_2^2)}{M^6 l_1^6 l_2^6} \quad (32)$$

From the given controllability matrix above, for it to have a full rank, its determinant should not be equal to be zero, therefore, from equation (32), the determinant of C matrix won't be equal to zero only when $l_1^2 - l_2^2$ is not equal to 0 ,i.e, $l_1^2 - l_2^2 \neq 0 \Rightarrow l_1 \neq l_2$.

As a result, the given system is controllable except when the lengths of the cables of the crane are equal.

1.5 LQR Controller:

First, we find the controllability of the system with the given values: M=1000kg, m1 = m2 = 100 kg, l1 = 20 m and l2 = 10 m. Now, the A and B matrices are obtained as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{49}{50} & 0 & -\frac{49}{50} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{539}{1000} & 0 & -\frac{49}{1000} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{49}{500} & 0 & -\frac{539}{500} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{1000} \\ 0 \\ \frac{1}{2000} \\ 0 \\ \frac{1}{10000} \end{bmatrix}$$

The controllable matrix(C) is obtained as:

$$\begin{pmatrix} 0 & \frac{1}{1000} & 0 & -\frac{338958922354413}{2305843009213693952} & 0 & \frac{1306573659368811}{9223372036854775808} \\ \frac{1}{1000} & 0 & -\frac{338958922354413}{2305843009213693952} & 0 & \frac{1306573659368811}{9223372036854775808} & 0 \\ 0 & \frac{1}{20000} & 0 & -\frac{2350115194990597}{73786976294838206464} & 0 & \frac{1674185909292917}{73786976294838206464} \\ \frac{1}{20000} & 0 & -\frac{2350115194990597}{73786976294838206464} & 0 & \frac{1674185909292917}{73786976294838206464} & 0 \\ 0 & \frac{1}{10000} & 0 & -\frac{8315792228428267}{73786976294838206464} & 0 & \frac{4597367655677375}{36893488147419103232} \\ \frac{1}{10000} & 0 & -\frac{8315792228428267}{73786976294838206464} & 0 & \frac{4597367655677375}{36893488147419103232} & 0 \end{pmatrix}$$

The rank of this matrix returns 6 and is hence, controllable. Now, the LQR controller system is called so due to the controller (L) being added to a linear system which gives us:

U= -KX (33)

Hence, the state space system can be written in terms of equation (33) as:

$$\dot{X} = AX + BU$$

$$\dot{X} = AX - BKX$$

$$\dot{X} = (A - BK)X \quad (34)$$

The values or Q and R are found by applying Bryson's rule in an iterative manner, which is given as,

$$J = \int_0^{\infty} (\vec{X}(t)^T Q \vec{X}(t) + (\vec{U}(t)^T R \vec{U}(t))\, dt$$

The Q and R is obtained as :

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 800 \end{bmatrix}$$

$$R = \frac{1}{10000}$$

The K value obtained after passing it through the LQR controller is obtained as:

$$\begin{pmatrix} 1000 & \frac{5121854189432495}{2199023255552} & \frac{7171671554565829}{1099511627776} & \frac{6019350694942567}{2199023255552} & \frac{4293372334677357}{549755813888} & -\frac{3297302936202093}{2199023255552} \end{pmatrix}$$

The eigen values of the new (A-BK) matrix are obtained after substituting the value of K in equation (34) and is obtained as follows:

$$\begin{pmatrix} -\frac{7456516615864955}{9007199254740992} + \frac{1713402849343347}{2251799813685248}\mathrm{i} \\ -\frac{7456516615864955}{9007199254740992} - \frac{1713402849343347}{2251799813685248}\mathrm{i} \\ -\frac{245238728675145}{1125899906842624} + \frac{3865318523206987}{4503599627370496}\mathrm{i} \\ -\frac{245238728675145}{1125899906842624} - \frac{3865318523206987}{4503599627370496}\mathrm{i} \\ -\frac{8097798436156567}{72057594037927936} + \frac{1546683431060665}{2251799813685248}\mathrm{i} \\ -\frac{8097798436156567}{72057594037927936} - \frac{1546683431060665}{2251799813685248}\mathrm{i} \end{pmatrix}$$

According to Lyapunov's Indirect method, since the Eigen values are negative, the closed loop system is at least locally stable.

The initial conditions can be given as follows,

$$X = \begin{bmatrix} 2 \\ 0 \\ 0.1 \\ 0 \\ -0.1 \\ 0 \end{bmatrix}$$

The output step response for the given initial conditions and the non-linear system output using LQR controller are given below.

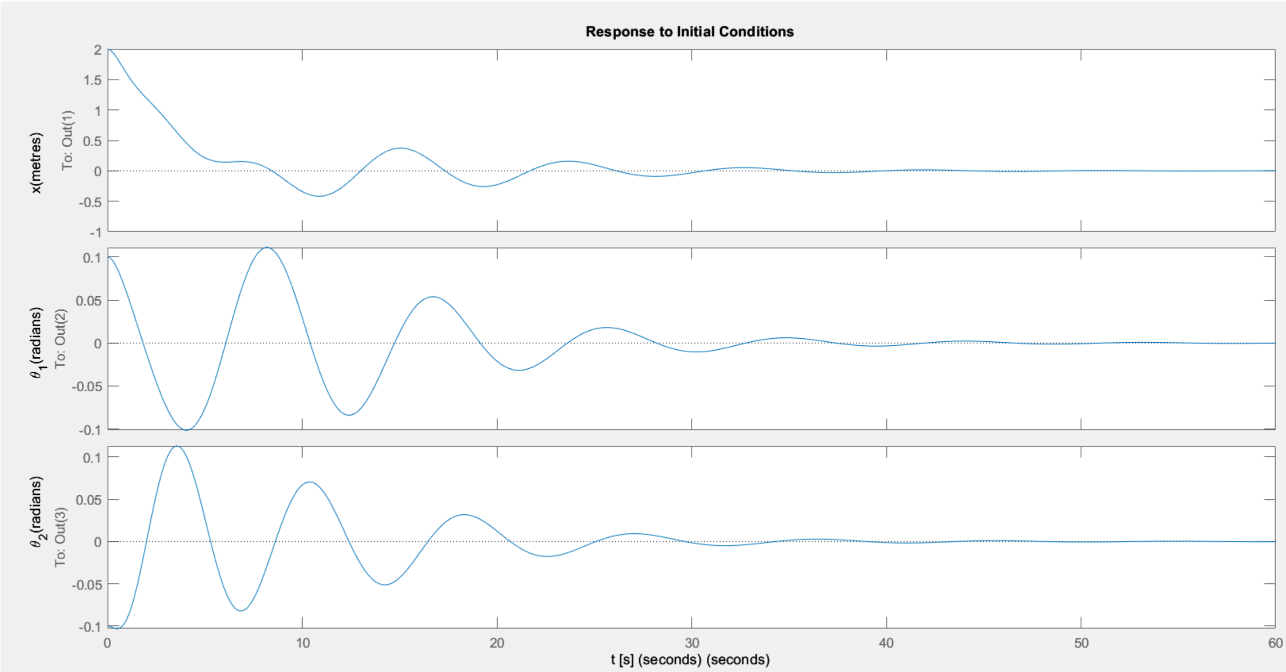

Figure 2: Response to Initial conditions

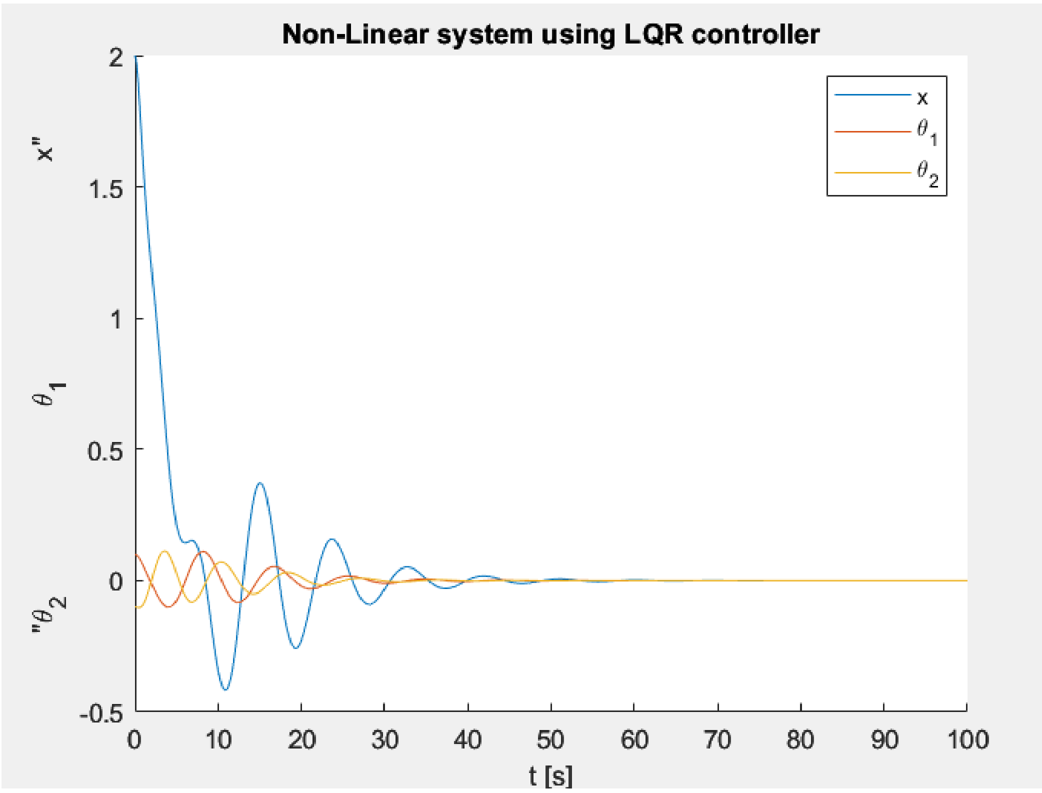

Figure 3: Output for Non-Linear system using LQR Controller

2 *Second Component*

1.1 Observable Vectors:

The given output vectors are x(t), $(\theta_1(t), \theta_2(t))$, $(x, \theta_2(t))$ and $(x, \theta_1(t), \theta_2(t))$. It can be written in matrix form as follows they can be calculated by the Jacobian linearization as well, the results are the same, refer to the code attached.

For x:

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $(\theta_1(t), \theta_2(t))$:

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For $(x, \theta_2(t))$:

$$C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

For $(x, \theta_1(t), \theta_2(t))$:

$$C_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The observability of an output vector can be obtained by finding rank of the O matrix,

$$O = \text{rank}[C_i \;\; C_i A \;\; C_i A^2 \;\; C_i A^3 \;\; C_i A^4 \;\; C_i A^5]$$

where, i =1,2,3,4

The O matrices are obtained as:

O_1 =

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1} + \frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2} + \frac{g^2\,m_1\,m_2}{M^2\,l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1} + \frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2} + \frac{g^2\,m_1\,m_2}{M^2\,l_1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

O_2 =

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{g\,(M+m_1)}{M\,l_1} & 0 & -\frac{g\,m_2}{M\,l_1} & 0 \\
0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{g\,(M+m_1)}{M\,l_1} & 0 & -\frac{g\,m_2}{M\,l_1} \\
0 & 0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{g^2\,(M+m_1)^2}{M^2\,{l_1}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_1)}{M^2\,{l_1}^2}+\frac{g^2\,m_2\,(M+m_2)}{M^2\,l_1\,l_2} & 0 \\
0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{g^2\,(M+m_1)^2}{M^2\,{l_1}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_1)}{M^2\,{l_1}^2}+\frac{g^2\,m_2\,(M+m_2)}{M^2\,l_1\,l_2} \\
0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2}
\end{pmatrix}
$$

$O_3 =$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} & 0 \\
0 & 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} \\
0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1}+\frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 \\
0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1}+\frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2}
\end{pmatrix}
$$

O_4 =

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} & 0 \\
0 & 0 & -\frac{g\,(M+m_1)}{M\,l_1} & 0 & -\frac{g\,m_2}{M\,l_1} & 0 \\
0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} & 0 \\
0 & 0 & 0 & -\frac{g\,m_1}{M} & 0 & -\frac{g\,m_2}{M} \\
0 & 0 & 0 & -\frac{g\,(M+m_1)}{M\,l_1} & 0 & -\frac{g\,m_2}{M\,l_1} \\
0 & 0 & 0 & -\frac{g\,m_1}{M\,l_2} & 0 & -\frac{g\,(M+m_2)}{M\,l_2} \\
0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1}+\frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1} & 0 \\
0 & 0 & \frac{g^2\,(M+m_1)^2}{M^2\,{l_1}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_1)}{M^2\,{l_1}^2}+\frac{g^2\,m_2\,(M+m_2)}{M^2\,l_1\,l_2} & 0 \\
0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 \\
0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1}+\frac{g^2\,m_1\,m_2}{M^2\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_2)}{M^2\,l_2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1} \\
0 & 0 & 0 & \frac{g^2\,(M+m_1)^2}{M^2\,{l_1}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,m_2\,(M+m_1)}{M^2\,{l_1}^2}+\frac{g^2\,m_2\,(M+m_2)}{M^2\,l_1\,l_2} \\
0 & 0 & 0 & \frac{g^2\,m_1\,(M+m_2)}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,(M+m_1)}{M^2\,l_1\,l_2} & 0 & \frac{g^2\,(M+m_2)^2}{M^2\,{l_2}^2}+\frac{g^2\,m_1\,m_2}{M^2\,l_1\,l_2}
\end{pmatrix}$$

Solving this in MATLAB, we get the full rank, i.e., 6, for O_1, O_3 and O_4 vectors which means they are observable. But we get rank=4 for O_2, which means its not observable at $\theta_1(t), \theta_2(t)$.

1.2 Luenberger Observer Design

The Luenberger Observer state space equation is as follows:

$$\dot{\hat{X}}(t) = A\hat{x}(t) + B_k U_k(t) + L(Y(t) - C\hat{x}(t)) \quad (35)$$

Where, $\hat{x}(t)$ is the state estimator, L is the observer gain matrix, $Y(t) - C\hat{x}(t)$ is the correction term and $\hat{x}(0)$ = 0.

The estimation error $\dot{X}_e(t)$ state space representation is given as,

$$\dot{X}_e(t) = \dot{X}(t) - \dot{\hat{X}}(t)$$

Putting equation (35) in the above equation,

$$\dot{X}_e(t) = AX_e(t) - L(Y(t) - C\hat{x}(t)) + B_d U_d(t) \quad (36)$$

Assuming D=0 and Y(t) = Cx(t), equation (36) can be written as:

$$\dot{X}_e(t) = (A-LC)X_e(t) + B_d U_d(t)$$

Where, $B_d U_d(t)$ is the disturbance in the system.

This disturbance is filtered out using a Kalman filter before proceeding with the transformations. After getting the filtered L gain matrix, the A matrix is updated as shown,

$$Ac = A-LC$$

Using this new A_c matrix we obtain the state space matrix for the estimation error $\dot{X}_e(t)$ in the system. Now, we find the linear and non-linear output response for each of the observable output vectors, which are (x_1,0,0), (x_1,0,θ_2) and (x_1, θ_1, θ_2). The K matrix for obtaining the linear system is obtained from the LQR controller. The state response for (x_1,0,0), (x_1,0,θ_2) and (x_1, θ_1, θ_2) respectively is:

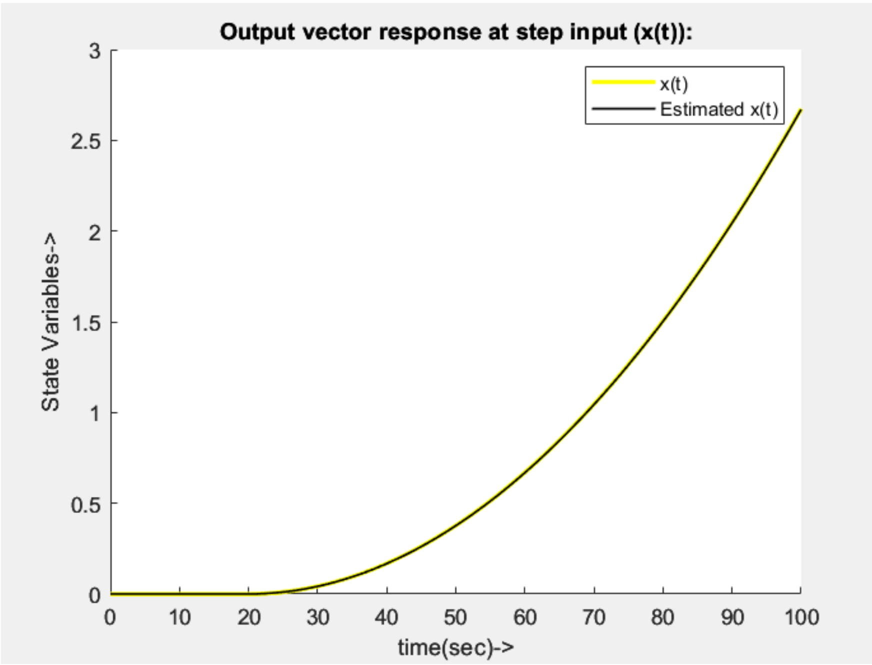

Figure 4: Step input response at x(t)

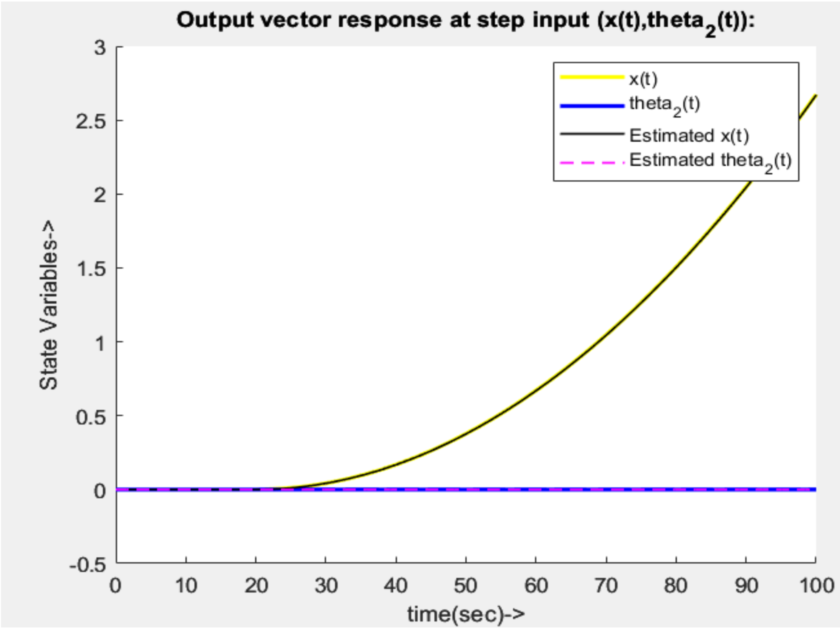

Figure 5: Step input response at (x(t),theta2(t))

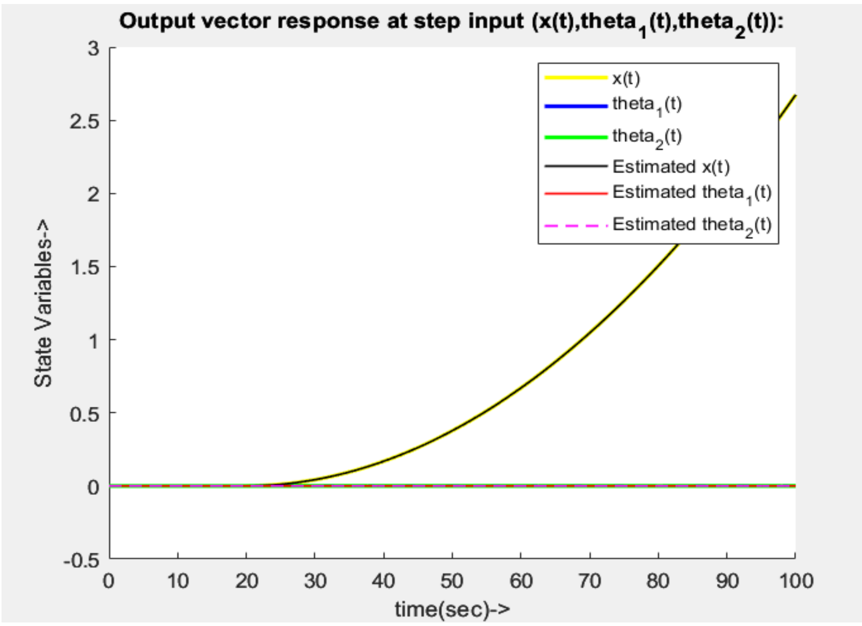

Figure 6: Step input response at (x(t),theta1(t),theta2(t))

The Luenberger Observer outputs for the linear system is as follows for (x_1,0,0), (x_1,0,θ_2) and (x_1, θ_1, θ_2) respectively is obtained as shown below. We can see that the output converges to zero due to the gain from the LQR controller.

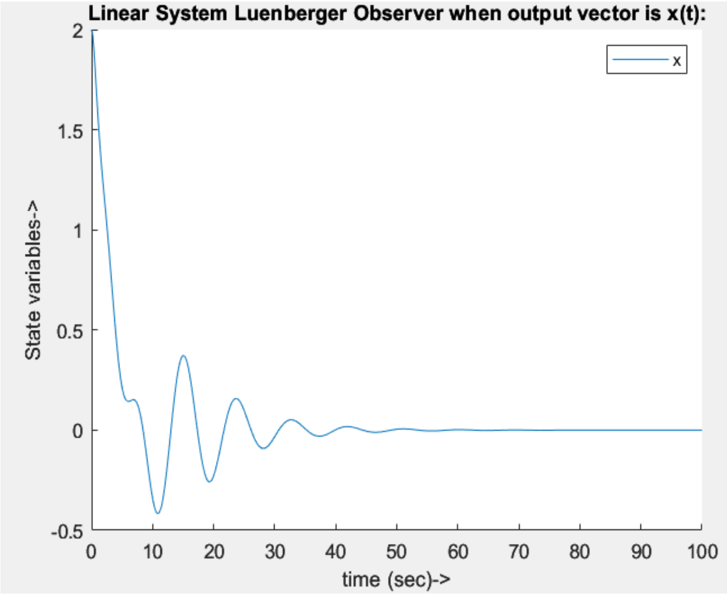

Figure 7: Linear system output at x(t)

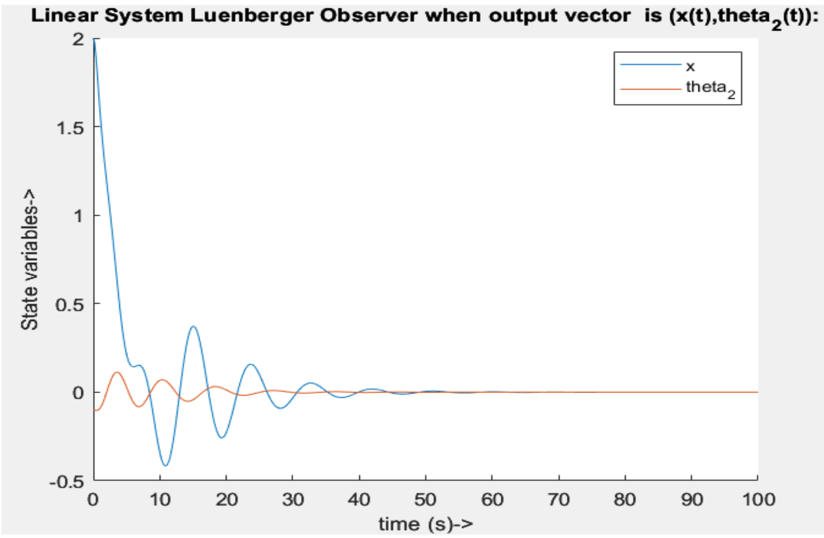

Figure 8: Linear system output at (x(t),theta2(t))

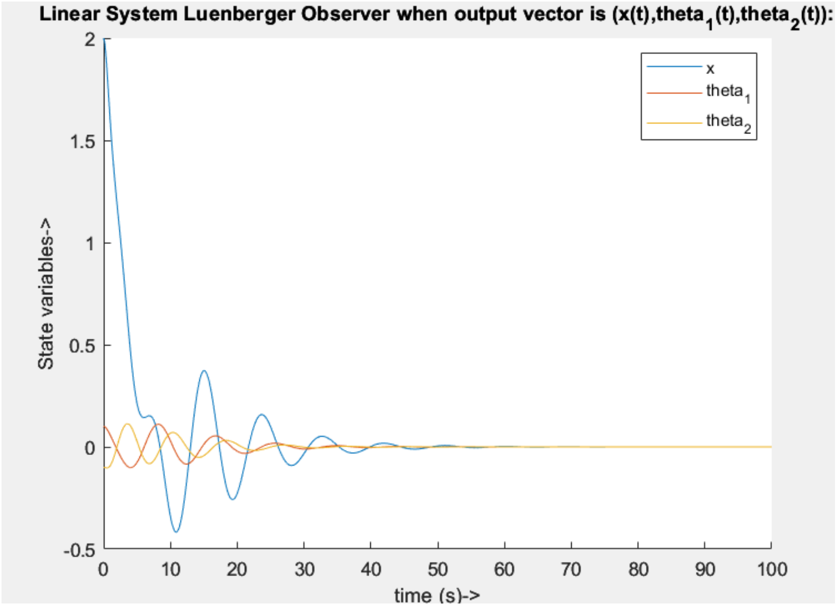

Figure 9: Linear system output at (x(t),theta1(t),theta2(t))

The Luenberger Observer outputs for the non-linear system is as follows for (x_1,0,0), (x_1,0,θ_2) and (x_1, θ_1, θ_2) respectively is obtained as shown below. We can see that the output increase exponentially due to an applied force of 5 N acting on the cart.

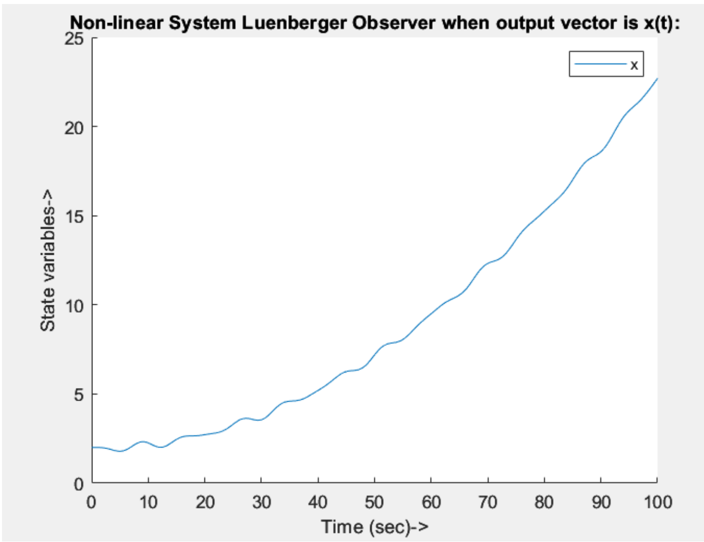

Figure 10:Non-linear system output at x(t)

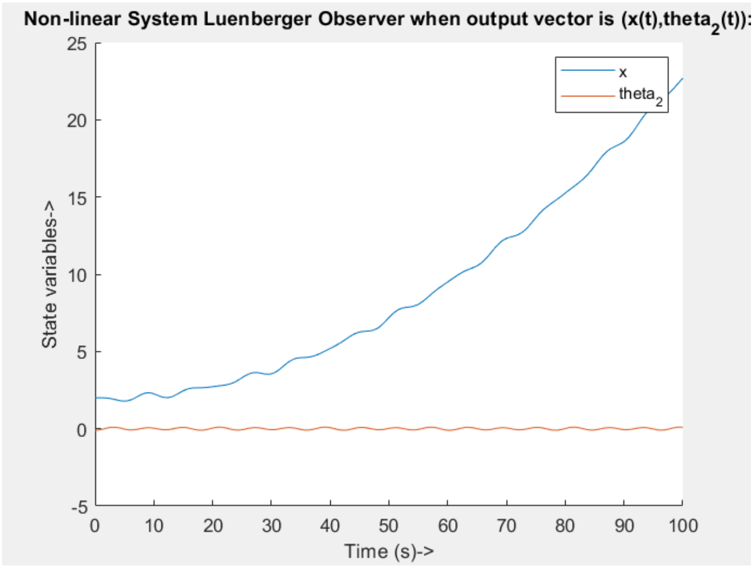

Figure 11:Non-linear system output at (x(t),theta2(t))

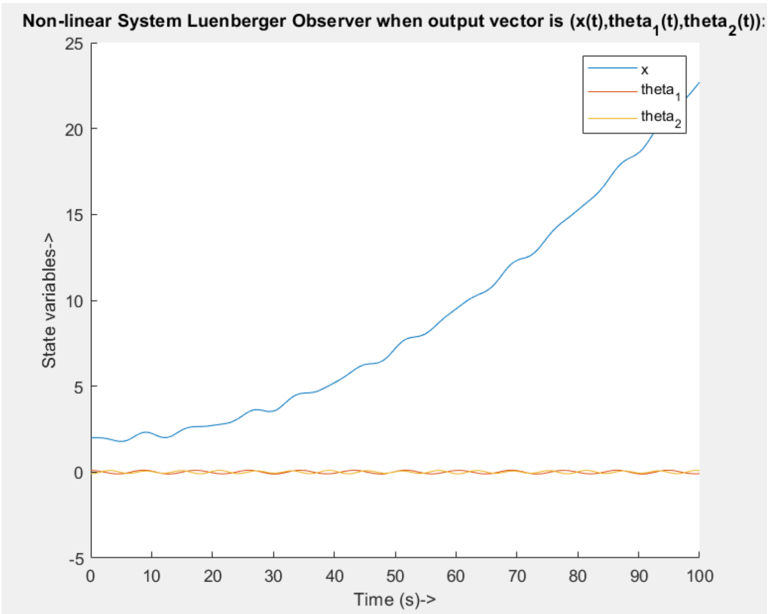

Figure 12:Non-linear system output at (x(t),theta1(t),theta2(t))

1.3 LQG Design

The LQG controller is a combination of a LQR controller with a Kalman filter. Its state space equation is given as:

$$\dot{x} = Ax - BK\,\hat{x} + B_D U_D$$

where, $\hat{x}$ is the state estimate and $B_D U_D$ is the disturbance in this system.

The LQG controller output for smallest output vector (x(t)) is shown below. Reconfiguration of the controller for asymptotically tracking constant reference on x, is can be done by iterating the parameters of or LQR controller according to the reference, therefore a feedback will be provided which can achieve the desired output (figure 13) .

The designed controller's behaviour for constant force disturbances on the cart is satisfactory, the instability, due to the increased disturbance in the Kalman filter, will be eliminated in finite time, therefore the position of the crane will be stabilized (figure 14).

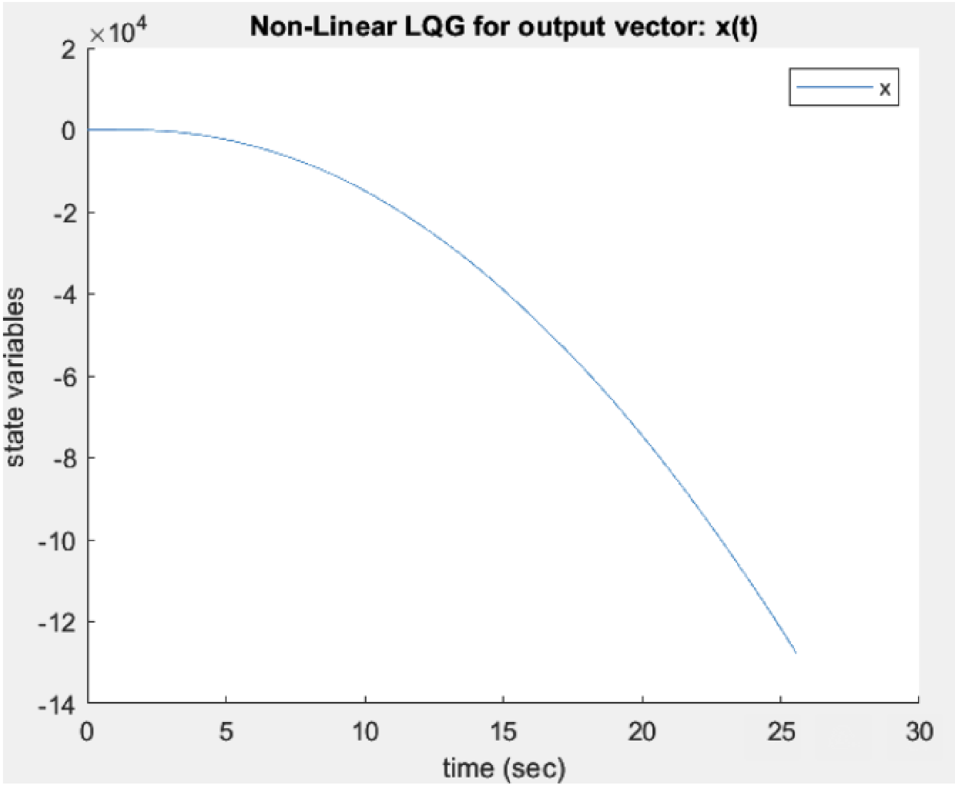

Figure 13:Non-linear system output at x(t)

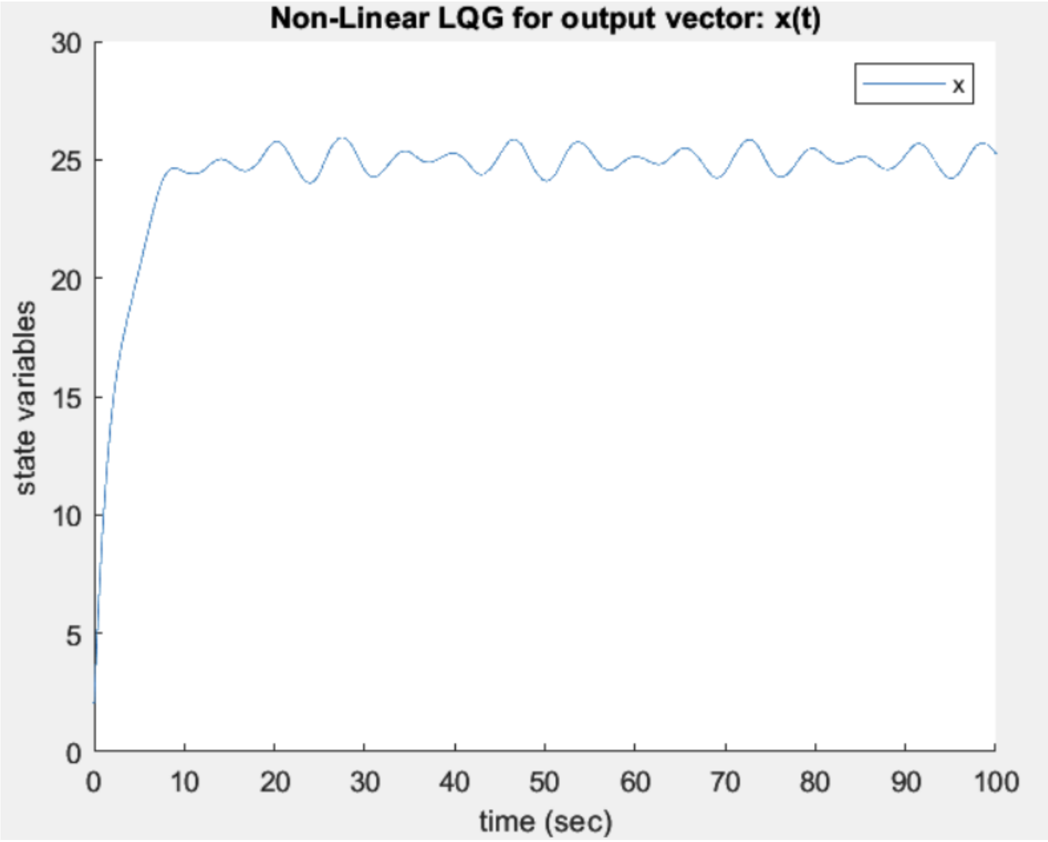

Figure 14:Non-linear system output at x(t) after trajectory tracking

3.References:

firstpart.m

```
clc
clear all
syms  g M m_1 m_2 l_2 l_1
%Initial Conditions

A=[0 1 0 0 0 0;
   0 0 (-g*m_1)/M 0 (-g*m_2)/M 0;
   0 0 0 1 0 0;
   0 0 (-g*(M+m_1))/(l_1*M) 0 (-g*m_2)/(M*l_1) 0;
   0 0 0 0 0 1;
   0 0 (-g*m_1)/(M*l_2) 0 (-g*(M+m_2))/(M*l_2) 0];
B=[0;
    1/M;
    0;
    1/(M*l_1);
    0;
    1/(M*l_2)];
D=0;
C_1=[1 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 0 0];

Q = eye(6,6);
%This has to be penalized
Q(1,1)=10;
Q(2,2)=1;
Q(3,3)=0;
Q(4,4)=0;
Q(5,5)=0;
Q(6,6)=0;
R= 0.0001;
X_0 = [2; 0; 0.1; 0; 0.1; 0];
R= 0.0001;
%Substituting values of m1 l1 l2 and M in A
A_1=double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
B_1 = double(subs(B,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));

[K1,S,P] = lqr(A_1, B_1, Q, R);

sys = ss(A_1-B_1*K1,B_1,C_1,D);

%Kalman Estimator Design
Bd = 0.01*eye(6); %disturbance
Vn = 0.001;      %Gaussian White Noise
[L,P,E] = lqe(A_1,Bd,C_1,Bd,Vn*eye(3)); %Considering vector output: x(t)
Ac1 = A_1-(L*C_1);
Xf = [25;0;0;0;0;0]
e_sys1 = ss(Ac1,[B_1 L],C_1,0);
ts = 0:0.01:100;
[t,state1] = ode45(@(t,state)nonLinear1_LQG(t,state,-K1*(state-Xf),L),ts,X_0);
figure();
hold on
plot(t,state1(:,1))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear LQG for output vector: x(t)')
legend('x')
```

```
hold off



%% LQG Non Linear
function ds1 = nonLinear1_LQG(~,x,f,l1)
    m1 = 100; m2 = 100; M=1000; L1 = 20; L2 = 10; g = 9.81;
    X_ = x(1);
    X_dot = x(2);
    theta_1 = x(3);
    theta_dot_1 = x(4);
    theta_2 = x(5);
    theta_dot2 = x(6);
    ds1 = zeros(6,1);
    y1 = [X_;0;0];
    c_1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
    iter = l1*(y1-c_1*x);
    ds1(1) = X_dot+iter(1);
    ds1(2) = (f - m1*L1*sin(theta_1)*theta_dot_1^2 -
(m2*L2*sin(theta_2)*theta_dot2^2 - m1*g*sin(theta_1)*cos(theta_1) -
m2*g*sin(theta_2)*cos(theta_2))/(M + m1 + m2 - m1*cos(theta_1)^2 -
m2*cos(theta_2)^2))+iter(2);
    ds1(3) = theta_dot_1+iter(3);
    ds1(4) = cos(theta_1)*ds1(2)/L1 - (g*sin(theta_1)/L1)+iter(4);
    ds1(5) = theta_dot2+iter(5);
    ds1(6) = cos(theta_2)*ds1(2)/L2 - (g*sin(theta_2)/L2)+iter(6);
end
```

LQG.m

```
syms  g M m_1 m_2 l_2 l_1
%Constants

A=[0 1 0 0 0 0;
   0 0 (-g*m_1)/M 0 (-g*m_2)/M 0;
   0 0 0 1 0 0;
   0 0 (-g*(M+m_1))/(l_1*M) 0 (-g*m_2)/(M*l_1) 0;
   0 0 0 0 0 1;
   0 0 (-g*m_1)/(M*l_2) 0 (-g*(M+m_2))/(M*l_2) 0];
B=[0;1/M;0;1/(M*l_1);0;1/(M*l_2)];
D=0;
C_1=[1 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 0 0];

Q = eye(6,6);
Q(1,1)=10;
Q(2,2)=1;
Q(3,3)=100;
Q(4,4)=1000;
Q(5,5)=1;
Q(6,6)=800;
R= 0.0001;
X_0 = [2; 0; 0.1; 0; 0.1; 0];

%Substituting
A_1=double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
```

```
B_1 = double(subs(B,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));

[K1,S,~] = lqr(A_1, B_1, Q, R);

sys = ss(A_1-B_1*K1,B_1,C_1,D);
X_final = [0;0;0;0;0;0];

%Kalman Estimator
Bd = 0.01*eye(6); %disturbance
Vn = 0.001;     %Gaussian White Noise
U = @(state) -K1*(X_0 - X_final);
[L,P,E] = lqe(A_1,Bd,C_1,Bd,Vn*eye(3));
Ac1 = A_1-(L*C_1);
e_sys1 = ss(Ac1,[B_1 L],C_1,0);
ts = 0:0.01:100;
[t,state1] = ode45(@(t,state)nonLinear1_LQG(t,state,U(state),L),ts,X_0);
figure();
hold on
plot(t,state1(:,1))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear LQG for output vector: x(t)')
legend('x')
hold off




%% LQG Non Linear
function ds1 = nonLinear1_LQG(t,x,f,l1)
    m1 = 100; m2 = 100; M=1000; L1 = 20; L2 = 10; g = 9.81;
    X_ = x(1);
    X_dot = x(2);
    theta_1 = x(3);
    theta_dot_1 = x(4);
    theta_2 = x(5);
    theta_dot2 = x(6);
    ds1 = zeros(6,1);
    y1 = [X_;0;0];
    c_1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
    iter = l1*(y1-c_1*x);
    ds1(1) = X_dot+iter(1);
    ds1(2) = (f - m1*L1*sin(theta_1)*theta_dot_1^2 -
(m2*L2*sin(theta_2)*theta_dot2^2 - m1*g*sin(theta_1)*cos(theta_1) -
m2*g*sin(theta_2)*cos(theta_2))/(M + m1 + m2 - m1*cos(theta_1)^2 -
m2*cos(theta_2)^2))+iter(2);
    ds1(3) = theta_dot_1+iter(3);
    ds1(4) = cos(theta_1)*ds1(2)/L1 - (g*sin(theta_1)/L1)+iter(4);
    ds1(5) = theta_dot2+iter(5);
    ds1(6) = cos(theta_2)*ds1(2)/L2 - (g*sin(theta_2)/L2)+iter(6);
end
```

LQG_ref_tracking.m

```
syms  g M m_1 m_2 l_2 l_1
%Constants

A=[0 1 0 0 0 0;
```

```
   0 0 (-g*m_1)/M 0 (-g*m_2)/M 0;
   0 0 0 1 0 0;
   0 0 (-g*(M+m_1))/(l_1*M) 0 (-g*m_2)/(M*l_1) 0;
   0 0 0 0 0 1;
   0 0 (-g*m_1)/(M*l_2) 0 (-g*(M+m_2))/(M*l_2) 0];

B=[0;1/M;0;1/(M*l_1);0;1/(M*l_2)];
D=0;
C_1=[1 0 0 0 0 0;
    0 0 0 0 0 0;
    0 0 0 0 0 0];

Q = eye(6,6);
%This has to be penalized
Q(1,1)=10;
Q(2,2)=1;
Q(3,3)=0;
Q(4,4)=0;
Q(5,5)=0;
Q(6,6)=0;
R= 0.0001;
X_0 = [2; 0; 0.1; 0; 0.1; 0];
R= 0.0001;

%Substituting
A_1=double(subs(A,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));
B_1 = double(subs(B,[M m_1 m_2 l_1 l_2 g], [1000 100 100 20 10 9.8]));

[K1,S,P] = lqr(A_1, B_1, Q, R);

sys = ss(A_1-B_1*K1,B_1,C_1,D);

%Kalman Estimator
Bd = 0.01*eye(6); %disturbance
Vn = 0.001;      %Gaussian White Noise
[L,P,E] = lqe(A_1,Bd,C_1,Bd,Vn*eye(3));
Ac1 = A_1-(L*C_1);
Xf = [25;0;0;0;0;0]
e_sys1 = ss(Ac1,[B_1 L],C_1,0);
ts = 0:0.01:100;
[t,state1] = ode45(@(t,state)nonLinear1_LQG(t,state,-K1*(state-Xf),L),ts,X_0);
figure();
hold on
plot(t,state1(:,1))
ylabel('state variables')
xlabel('time (sec)')
title('Non-Linear LQG for output vector: x(t)')
legend('x')
hold off




%% LQG Non Linear
function ds1 = nonLinear1_LQG(~,x,f,l1)
    m1 = 100; m2 = 100; M=1000; L1 = 20; L2 = 10; g = 9.81;
    X_ = x(1);
    X_dot = x(2);
    theta_1 = x(3);
```

```
    theta_dot_1 = x(4);
    theta_2 = x(5);
    theta_dot2 = x(6);
    ds1 = zeros(6,1);
    y1 = [X_;0;0];
    c_1 = [1 0 0 0 0 0; 0 0 0 0 0 0; 0 0 0 0 0 0];
    iter = l1*(y1-c_1*x);
    ds1(1) = X_dot+iter(1);
    ds1(2) = (f - m1*L1*sin(theta_1)*theta_dot_1^2 -
(m2*L2*sin(theta_2)*theta_dot2^2 - m1*g*sin(theta_1)*cos(theta_1) -
m2*g*sin(theta_2)*cos(theta_2))/(M + m1 + m2 - m1*cos(theta_1)^2 -
m2*cos(theta_2)^2))+iter(2);
    ds1(3) = theta_dot_1+iter(3);
    ds1(4) = cos(theta_1)*ds1(2)/L1 - (g*sin(theta_1)/L1)+iter(4);
    ds1(5) = theta_dot2+iter(5);
    ds1(6) = cos(theta_2)*ds1(2)/L2 - (g*sin(theta_2)/L2)+iter(6);
end
```