Understanding LoRA: Low-Rank Adaptation for Fine-Tuning

1. What is LoRA?

LoRA (Low-Rank Adaptation) is a technique to fine-tune large models efficiently. Instead of updating a full weight matrix $W \in \mathbb{R}^{k\times d}$, we keep W **frozen** and learn a **low-rank update**

$$\Delta W = BA,$$

where

- $A \in \mathbb{R}^{r\times d}$ is a small trainable matrix,
- $B \in \mathbb{R}^{k\times r}$ is another small trainable matrix,
- $r \ll \min(k, d)$, where k is the input dimension of the layer and d is the output dimension.

The effective weight becomes

$$W_{\text{eff}} \;=\; W + \Delta W \;=\; W + BA,$$

so only A and B receive gradient updates.

2. Why Use LoRA?

- Full fine-tuning updates *every* weight tensor—expensive in memory and compute.
- LoRA reduces the number of trainable parameters by two or more orders of magnitude.
- In practice it achieves competitive downstream accuracy at a fraction of the cost.

3. Numerical Example

Let

- input dim $d = 4$, output dim $k = 4$, rank $r = 2$.

The initial matrices are:

$$W = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & -2 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ -0.1 & -0.2 & 0.0 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.5 & -0.5 \\ -1 & 1 \end{bmatrix}$$

The initial update matrix ΔW and the effective weight matrix W_{eff} are calculated as:

$$\Delta W = BA = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ -0.1 & -0.2 & 0.0 & 0.1 \\ 0.1 & 0.2 & 0.15 & 0.15 \\ -0.2 & -0.4 & -0.3 & -0.3 \end{bmatrix}, \qquad W_{\text{eff}} = W + \Delta W = \begin{bmatrix} 1.1 & 0.2 & -0.7 & 2.4 \\ -0.1 & 0.8 & 1.0 & -0.9 \\ 2.1 & -1.8 & 0.15 & 0.15 \\ 0.8 & 0.6 & 0.7 & 0.7 \end{bmatrix}.$$

3.1. A Single Gradient Update Step

Now, let's simulate one step of training. We need an input vector $\mathbf{x}$, a target output $\mathbf{y}_{\text{true}}$, a loss function L, and a learning rate α.

- Input: $\mathbf{x} = [1, 0, 0, 0]$
- Target: $\mathbf{y}_{\text{true}} = [1, 1, 1, 1]$
- Loss Function: $L = \frac{1}{2}\sum(y_i - y_{\text{true},i})^2$ (Sum of Squared Errors)
- Learning Rate: $\alpha = 0.1$

1. Forward Pass. First, we compute the predicted output $\mathbf{y} = \mathbf{x}W_{\text{eff}}$. With $\mathbf{x} = [1, 0, 0, 0]$, this simply selects the first row of W_{eff}:

$$\mathbf{y} = [1.1, 0.2, -0.7, 2.4]$$

2. Compute Loss and Gradients. We calculate the loss and then backpropagate to find the gradients of the loss with respect to A and B.

$$\text{Loss } L \approx 2.75$$

Crucially, since W is **frozen**, we do not compute its gradient. The gradients are computed only for the trainable parameters:

$$\nabla_A L = \frac{\partial L}{\partial A}, \qquad \nabla_B L = \frac{\partial L}{\partial B}, \qquad \nabla_W L = \mathbf{0}$$

Via backpropagation, we find the numerical gradients (values are approximate):

$$\nabla_A L \approx \begin{bmatrix} 0.1 & 0.2 & -1.7 & 1.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \nabla_B L \approx \begin{bmatrix} 0.01 & -0.01 \\ 0.02 & -0.02 \\ -0.17 & 0.17 \\ 0.14 & -0.14 \end{bmatrix}$$

3. Update Parameters. We update A and B using gradient descent. W is not updated.

$$A_{\text{new}} = A - \alpha\nabla_A L, \qquad B_{\text{new}} = B - \alpha\nabla_B L, \qquad W_{\text{new}} = W$$

With $\alpha = 0.1$, the new matrices are:

$$B_{\text{new}}A_{\text{new}} \approx \begin{bmatrix} 0.999 & 0.001 \\ -0.002 & 1.002 \\ 0.52 & -0.52 \\ -1.01 & 1.01 \end{bmatrix} \times \begin{bmatrix} 0.09 & 0.18 & 0.47 & 0.26 \\ -0.10 & -0.20 & 0.00 & 0.10 \end{bmatrix} = \begin{bmatrix} 0.09 & 0.18 & 0.47 & 0.26 \\ -0.10 & -0.20 & 0.00 & 0.10 \\ 0.0988 & 0.1976 & 0.2444 & 0.0832 \\ -0.1919 & -0.3838 & -0.4747 & -0.1616 \end{bmatrix}$$

The original weight matrix W remains **completely unchanged**.

4. New Effective Weight. Finally, we can see how the effective weight matrix has changed due to the updates to A and B only.

$$
\begin{aligned}
W_{\text{eff, new}} &= W + B_{\text{new}} A_{\text{new}} \\
&\approx \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & -2 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0.09 & 0.18 & 0.47 & 0.26 \\ -0.10 & -0.20 & 0.00 & 0.10 \\ 0.0988 & 0.1976 & 0.2444 & 0.0832 \\ -0.1919 & -0.3838 & -0.4747 & -0.1616 \end{bmatrix} \\
&\approx \begin{bmatrix} 1.09 & 0.18 & -0.53 & 2.26 \\ -0.10 & 0.80 & 1.00 & -0.90 \\ 2.10 & -1.80 & 0.24 & 0.08 \\ 0.81 & 0.62 & 0.52 & 0.84 \end{bmatrix}
\end{aligned}
$$

After just one update, only the small matrices A and B have been modified, changing the overall behavior of the layer while keeping the massive original weight matrix W frozen.

4. How Training Works

1. Forward: compute with $W_{\text{eff}} = W + BA$.
2. Compute loss.
3. Back-propagate gradients *only* into A and B.
4. Update A and B; keep W frozen.

5. Does LoRA Add New Layers?

No. A LoRA patch lives *inside* an existing `nn.Linear` layer.

- The layer's computation(during forward pass) is rewritten as

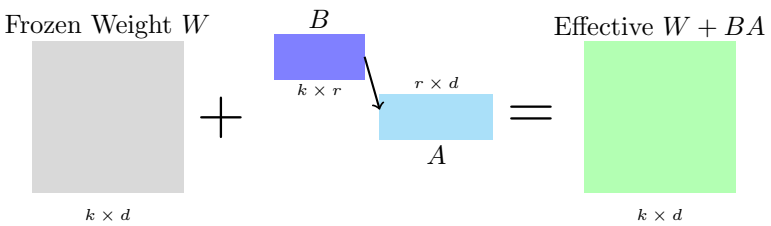
  $$y = (W + BA)\, x + b,$$

  but the call-graph still contains just the original layer.
- A and B are lightweight *matrices*, not brand-new modules.
- At inference you can *merge* them once into W and discard the patch: $W \leftarrow W + BA$.

$$\text{same layer} \quad \boxed{W_{\text{eff}} = W + BA} \quad \text{with} \quad \underbrace{W}_{\text{frozen}} + \underbrace{BA}_{\text{trainable, low-rank}}$$

Thus LoRA changes the *parameters* a layer holds, not the network topology.

6. LoRA Diagram



7. LoRA Parameter Savings Example

Assume:

- **LLaMA-7B** contains ≈ 7 billion parameters in total.
- We apply LoRA to its linear layers, which contain the vast majority of its parameters. The total number of weights in these layers is $\approx$ **6.48 billion**.
- LoRA rank is fixed at $r = 8$.

For a linear layer of shape $(k \times d)$ LoRA adds

$$A \in \mathbb{R}^{r \times d}, \quad B \in \mathbb{R}^{k \times r} \implies \text{extra params per layer} = r\,(k + d).$$

With a typical layer size of $k = d = 4096$:

$$\text{per layer} = 8\,(4096 + 4096) = 8 \times 8192 = 65,536.$$

Assuming we patch 100 of the model's main linear layers (a common practice):

$$100 \times 65,536 = 6.55 \text{ million parameters.}$$

Perspective.

$$\frac{6.55\text{ M}}{6.48\text{ B}} \approx 0.1\% \quad \text{(of targeted layers)}, \qquad \frac{6.55\text{ M}}{7\text{ B}} \approx 0.094\% \quad \text{(of the entire model)}.$$

LoRA therefore tunes about **one-tenth of one percent** of the weights it touches and below one-tenth of one percent of the full network—a truly dramatic savings in both memory and compute.

8. What Does *Rank* Mean in LoRA?

Think of the rank r as a **dial** that controls how much freedom the LoRA patch has.

Linear-algebra view. The **rank** of a matrix is the count of independent rows or columns, i.e. how many directions it can cover in space.

LoRA view. We replace a full (large) update matrix $\Delta W \in \mathbb{R}^{k \times d}$ with a product of two skinny ones:

$$\Delta W = BA, \qquad B \in \mathbb{R}^{k \times r},\ A \in \mathbb{R}^{r \times d}.$$

The middle dimension r is the **bottleneck size**—our dial.

- **Small** $r \to$ very few new parameters, runs fast, but can only make coarse adjustments.
- **Large** $r \to$ more parameters and compute, but can learn subtler changes.

In practice people pick

$$r \in \{4,\ 8,\ 16,\ 32\},$$

choosing a higher value for bigger models or harder tasks.

9. Summary

- LoRA injects low-rank *matrices*, not new layers.
- Only the small patches (A, B) are trained; W stays frozen.
- Widely used on `nn.Linear` layers in transformers to enable rapid, memory-efficient adaptation.
- Used for fine-tuning.