Entropy Measures and Catalysis of Bipartite Quantum State Transformations

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Abstract — **Given probability vectors x and y, necessary and sufficient conditions are given for the existence of a probability vector z such that $x \otimes z$ is majorized by $y \otimes z$. This is of interest because the "catalytic majorization" relation is known to be useful in classifying which transformations of jointly held pure quantum states are possible using local operations and classical communication when an additional jointly held state may be specified to facilitate the transformation without being consumed.**

I. Introduction

Let $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ be d-dimensional vectors with real components. Let $x^\downarrow$ denote the vector obtained by arranging the components of x in decreasing order: $x^\downarrow = (x_1^\downarrow, \ldots, x_d^\downarrow)$ where $x_1^\downarrow \geq \cdots \geq x_d^\downarrow$. A vector x is said to be *majorized* by y, written $x \prec y$, if $\sum_{i=1}^d x_i = \sum_{i=1}^d y_i$ and for $1 \leq \ell < d$, $\sum_{i=1}^\ell x_i^\downarrow \leq \sum_{i=1}^\ell y_i^\downarrow$. The majorization relation has been well studied [1].

Recently, the following question has attracted interest: given two d-dimensional probability vectors x and y, does there exist a probability vector z, of any (finite) dimension, such that the tensor products of x and y with z satisfy $x \otimes z \prec y \otimes z$? The resulting relation between x and y may be referred to as catalytic majorization; we write $x \prec_T y$ when such a z exists. Clearly $x \prec y$ implies $x \prec_T y$; less obviously, there exist x and y for which $x \not\prec y$ but $x \prec_T y$ (see e.g. [2]). See [3, 4] for additional properties of this relation.

Majorization and catalytic majorization have been shown to arise in the study of transformations of entangled bipartite pure quantum states. Nielsen [5] has shown the following:

Theorem 1 *Suppose Alice and Bob are in joint possession of an entangled pure quantum state $|\psi_1\rangle$ that they wish to transform into another bipartite entangled pure state $|\psi_2\rangle$. Let $|\psi_1\rangle = \sum_{i=1}^d \sqrt{\alpha_i}|i_A\rangle|i_B\rangle$ and $|\psi_2\rangle = \sum_{i=1}^d \sqrt{\beta_i}|i'_A\rangle|i'_B\rangle$ be Schmidt decompositions of $|\psi_1\rangle$ and $|\psi_2\rangle$. Then $|\psi_1\rangle$ can be converted to $|\psi_2\rangle$ (with success guaranteed) using only local operations and classical communication if and only if the vector $\alpha = (\alpha_1, \ldots, \alpha_d)$ is majorized by $\beta = (\beta_1, \ldots, \beta_d)$.*

Jonathan and Plenio [2] have extended this result by showing that even if it is not possible to convert $|\psi_1\rangle$ to $|\psi_2\rangle$ directly, it may be possible to convert $|\psi_1\rangle|\phi\rangle$ to $|\psi_2\rangle|\phi\rangle$, where $|\phi\rangle$ is an additional bipartite state shared by Alice and Bob. If x, y, and z are the vectors of (squared) Schmidt coefficients of $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\phi\rangle$ respectively, then the Schmidt coefficients of $|\psi_1\rangle|\phi\rangle$ are the components of $x \otimes z$ and the Schmidt coefficients of $|\psi_2\rangle|\phi\rangle$ are the components of $y \otimes z$; thus Nielsen's Theorem implies that $|\psi_1\rangle|\phi\rangle$ can be converted to $|\psi_2\rangle|\phi\rangle$ when $x \otimes z \prec y \otimes z$. In this case the state $|\phi\rangle$ is referred to as a catalyst and $|\phi\rangle$ is said to catalyze the transformation from $|\psi_1\rangle$ to $|\psi_2\rangle$. Note that a state exists that can catalyze the transformation from $|\psi_1\rangle$ to $|\psi_2\rangle$ if and only if $x \prec_T y$.

II. Main Result

To state our main result we specify a family of functions, indexed by a real number r. For a d-dimensional probability vector x, let

$$f_r(x) = \begin{cases} \ln \sum_{i=1}^d x_i^r & (r > 1); \\ \sum_{i=1}^d x_i \ln x_i & (r = 1); \\ -\ln \sum_{i=1}^d x_i^r & (0 < r < 1); \\ -\sum_{i=1}^d \ln x_i & (r = 0); \\ \ln \sum_{i=1}^d x_i^r & (r < 0). \end{cases}$$

For $r > 0$, $r \neq 1$, the functions f_r are essentially (up to a constant factor) the negatives of the Rényi entropies. The function f_1 is the negative of the Shannon entropy.

Our main result is the following:

Theorem 2 *Let $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ be d-dimensional probability vectors. Suppose that x and y do not both contain components equal to 0 and that $x \neq y$. Then $x \prec_T y$ if and only if $f_r(x) < f_r(y)$ for all $r \in \mathbf{R}$.*

A function $f : \mathbf{R}^d \to \mathbf{R}$ is *Schur-convex* if $f(x) \leq f(y)$ whenever $x \prec y$ [1]. Nielsen [4] has introduced the notion of an *additive Schur-convex* function: A function f from probability vectors (of any dimension) to $\mathbf{R}$ is additive Schur-convex if f is Schur-convex and $f(x \otimes y) = f(x) + f(y)$. If $x \prec_T y$ then we must have $f(x) \leq f(y)$ for any additive Schur-convex function f. Nielsen [4] has conjectured that a family of additive Schur-convex functions exists that can be used to determine whether or not $x \prec_T y$. Our main result shows that this is indeed the case, as our functions f_r are additive Schur-convex.

Although the characterization of catalytic majorization provided by Theorem 2 involves an infinite number of inequalities, it may not be possible to simplify it significantly. We conjecture that all of the inequalities are needed, or more precisely that for any $r_0 \in \mathbf{R}$, there exists an x and y such that $f_r(x) < f_r(y)$ holds for all r except $r = r_0$.

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