

# Necessary and Sufficient Conditions for the Trumping Relation

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Entanglement catalysis allows one to convert certain entangled states into others by the temporary involvement of another entangled state (so-called catalyst), where after the conversion the catalyst is returned to the same state. For bipartite pure entangled states that can be transformed in this way with unit probability, the respective Schmidt coefficients are said to satisfy the trumping relation, a mathematical relation which is an extension of the majorization relation. This article provides all necessary and sufficient conditions for the trumping relation in terms of the Schmidt coefficients. The coefficients should satisfy strict inequalities for the entropy of entanglement and for power means excluding the special power 1.

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## I. INTRODUCTION

An important problem in quantum information theory is to understand the conditions for transforming a given entangled state into another desired state by using only local quantum operations assisted with classical communication (LOCC). Significant development has been achieved for the case of pure bipartite states. In the asymptotic limit, where an infinite number of copies of entangled pairs are to be transformed into each other, it is found that the conversion is possible with the probability of success approaching unity as long as the entropy of entanglement does not increase.[1]

However, away from the asymptotic limit, where a single copy of a given state is to be transformed into another given state, such a simple conversion criterion cannot be found and investigations have unearthed a deep connection of the problem to the mathematical theory of majorization.[2] For two sequences of  $n$  real numbers  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$ , we say that  $x$  is *majorized* by  $y$  (written  $x \prec y$ ) when

$$x_1^\uparrow + x_2^\uparrow + \dots + x_m^\uparrow \geq y_1^\uparrow + y_2^\uparrow + \dots + y_m^\uparrow \quad (1)$$

for  $m = 1, 2, \dots, n-1$  and the sequences have the same sum ( $\sum x_i = \sum y_i$ ). Here,  $x^\uparrow$  represents the sequence  $x$  when its elements are arranged in non-decreasing order ( $x_1^\uparrow \leq x_2^\uparrow \leq \dots \leq x_n^\uparrow$ ) and similarly for  $y^\uparrow$ . Nielsen has shown that for two given states having the Schmidt forms

$$|\psi\rangle = \sum_{i=1}^n \sqrt{x_i} |i_A \otimes i_B\rangle \quad , \quad (2)$$

$$|\phi\rangle = \sum_{i=1}^n \sqrt{y_i} |i'_A \otimes i'_B\rangle \quad , \quad (3)$$

where  $x$  and  $y$  are the respective Schmidt coefficients ( $\sum x_i = \sum y_i = 1$ ), the state  $|\psi\rangle$  can be converted into  $|\phi\rangle$  by LOCC with unit probability of success if and

only if  $x \prec y$ . [3] Subsequently, Vidal extended this result to probabilistic transformations where the probability of success is related to the violation of the majorization inequalities in (1). [4]

Soon afterwards, Jonathan and Plenio have demonstrated an interesting effect that is termed as catalysis or entanglement assisted local transformation. [5] There are some cases where  $|\psi\rangle$  cannot be converted into  $|\phi\rangle$  with certainty (conversion is possible only with a probability less than 1), but with the involvement of another entangled pair (a catalyst), the conversion is made possible. In other words, if  $|\chi\rangle = \sum_{\ell=1}^N \sqrt{c_\ell} |\ell_A \otimes \ell_B\rangle$  is the state of the catalyst, then  $|\psi\rangle \otimes |\chi\rangle$  can be converted into  $|\phi\rangle \otimes |\chi\rangle$  with complete success. In such a transformation, the entanglement of the catalyst is not consumed, although it takes part in the transformation. Catalysis is also useful in almost all conversion processes where it improves upon the conversion probability.

Expressing in terms of the Schmidt coefficients and considering only the cases where catalysis helps achieve unit probability of success, we basically have situations where  $x$  is not majorized by  $y$ , but there is a sequence  $c$  such that  $x \otimes c$  is majorized by  $y \otimes c$ . Following Nielsen, for two sequences of non-negative numbers  $x$  and  $y$  with  $n$ -elements, we will say that  $x$  is *trumped* by  $y$  (written  $x \prec_T y$ ) if there exists another sequence  $c$  such that  $x \otimes c \prec y \otimes c$ . It is easy to see that in such cases the catalyst sequence  $c$  can be chosen from strictly positive numbers.

A lot of research has been directed to understand the catalytic transformations [6] and to analyze the mathematical structure of the trumping relation. [7, 8, 9, 10] One of the open problems is to find a way to decide if two given sequences  $x$  and  $y$  satisfies the trumping relation. The purpose of this article is to give all necessary and sufficient conditions for this relation. This problem has been partially solved by Aubrun and Nechita [11, 12], who work with stochastic tools to describe the closure of a set constructed with the trumping relation. The methods used in this article, however, are quite similar to those of a recent study that provided an expression for the catalytic conversion probability. [13] But, as the trumping relation necessarily implies that the two sequences have

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the same sum, the mathematical details of the proofs given below are more complicated than the ones in that article.

It appears that the necessary and sufficient conditions for the trumping relation can be expressed in terms of the power means, which are defined as

$$A_\nu(x) = \left( \frac{\sum_{i=1}^n x_i^\nu}{n} \right)^{\frac{1}{\nu}}, \quad (4)$$

and the entropy of entanglement,

$$\sigma(x) = - \sum_{i=1}^n x_i \ln x_i. \quad (5)$$

The main theorem that we would like to prove is the following.

**Theorem 1.** For two  $n$ -element sequences of non-negative numbers  $x$  and  $y$  such that  $x$  has non-zero elements and the sequences are distinct (i.e.,  $x^\uparrow \neq y^\uparrow$ ), the relation  $x \prec_T y$  is equivalent to the following three inequalities

$$A_\nu(x) > A_\nu(y), \quad \forall \nu \in (-\infty, 1), \quad (6)$$

$$A_\nu(x) < A_\nu(y), \quad \forall \nu \in (1, +\infty), \quad (7)$$

$$\sigma(x) > \sigma(y), \quad (8)$$

where all inequalities are strict.

Note that by the continuity of the power mean function against  $\nu$ , the requirement that the sequences  $x$  and  $y$  have the same sum is included in the conditions (6) and (7). Moreover, the limits of these inequalities at  $\nu = -\infty$  and  $\nu = +\infty$  imply that the minimum and maximum elements of the sequences satisfy the respective inequalities  $x_1^\uparrow \geq y_1^\uparrow$  and  $x_n^\uparrow \leq y_n^\uparrow$ , but these do not have to be strict. Note also that all of these three conditions can be equivalently expressed as the strict positivity of the function

$$R_\nu = \frac{1}{\nu - 1} \ln \frac{A_\nu(y)}{A_\nu(x)} \quad (9)$$

for all finite values of  $\nu$ , where the value  $R_1$  corresponds to the difference of the entropies.

Notice that some of the results in Ref. 14 about the connection between the multiple-copy entanglement transformation, a related phenomenon discovered by Badyopadhyay *et al.*[15], and the trumping relation can be easily understood in view of the conditions (6-8). For any sequence  $x$  and any integer  $k > 1$ , we have  $A_\nu(x^{\otimes k}) = A_\nu(x)^k$  and  $\sigma(x^{\otimes k}) = k\sigma(x)$ . As a result, if  $k$  copies of a state with coefficients  $x$  can be transformed into  $k$  copies of another state with coefficients  $y$ , either with or without catalysis, then  $x$  must be trumped by  $y$ .

The article is organized as follows: In section II, a few relations related to majorization are given and a key lemma is proved. Then, in section III, the theorem is proved. Finally, section IV contains the conclusions.

## II. A FEW RELATIONS AND A KEY LEMMA

The following facts about the majorization and the trumping relation will be used occasionally.

- (1) If  $x \prec y$ , then for any convex function  $f$ , we have

$$\sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n f(y_i). \quad (10)$$

Moreover, if  $x^\uparrow \neq y^\uparrow$  and  $f$  is strictly convex, then the inequality above is strict.

- (2) For any sequence  $x$ , we define the *characteristic function*  $H_x(t) = \sum_{i=1}^n (t - x_i)^+$  where  $(\alpha)^+ = \max(\alpha, 0)$  denotes the positive-part function. For non-negative sequences  $x$  and  $y$  which have the same sum ( $\sum x_i = \sum y_i$ ), the relation  $x \prec y$  can be equivalently stated as

$$H_x(t) \leq H_y(t) \quad \forall t \geq 0. \quad (11)$$

- (3) For the cross-product of two sequences we have  $H_{x \otimes c} = \sum_\ell c_\ell H_x(t/c_\ell)$ .
- (4) Relation  $\prec$  and  $\prec_T$  are partial orders on all  $n$ -element sequences (up to equivalence under rearrangement). Moreover,  $x \prec y$  implies  $x \prec_T y$ .
- (5) Let  $z$  be any sequence of strictly positive numbers and let  $x \oplus z$  denote the sequence obtained by concatenating the elements of  $z$  to those of  $x$ . Then  $x \prec_T y$  iff  $x \oplus z \prec_T y \oplus z$ . Moreover,  $x$  and  $y$  satisfies the inequalities (6-8) iff  $x \oplus z$  and  $y \oplus z$  satisfies the same inequalities. As a result, nothing will be lost from generality if a proof of the Theorem 1 is given for sequences that have no common elements.

The proof of the sufficiency of the conditions (6-8) is based on the following key lemma.

**Lemma:** If a polynomial  $\gamma(s)$  has no positive roots and  $\gamma(0) > 0$  then

- (a) it can be expressed as  $\gamma(s) = b(s)/a(s)$  where  $a(s)$  and  $b(s)$  are polynomials with non-negative coefficients.
- (b) Moreover,  $a(s)$  can be chosen as a polynomial with integer coefficients.

*Proof:* The lemma can easily be generalized to polynomials which have a root at  $s = 0$ , but for our purposes the above form is sufficient. For part (a), we first provide the proof for a second degree polynomial with complex roots, e.g.,  $\gamma(s) = 1 - 2\xi s + \lambda s^2$  where  $\lambda > \xi^2$ . For  $\xi \leq 0$ , there is nothing to be proved as  $\gamma$  has already non-negative coefficients. For  $\xi > 0$ ,  $\gamma(s)$  can be written as the ratio  $b(s)/a(s)$  where

$$a(s) = \sum_{k=0}^{2N-1} (1 + \lambda s^2)^k (2\xi s)^{2N-1-k}, \quad (12)$$

$$b(s) = (1 + \lambda s^2)^{2N} - (2\xi s)^{2N}, \quad (13)$$

and  $N$  is a sufficiently large integer so that the inequality

$$\frac{1}{4} \left( \frac{(2N)!}{N!^2} \right)^{\frac{1}{N}} \geq \frac{\xi^2}{\lambda} \quad , \quad (14)$$

is satisfied. It is always possible to find such an  $N$  since the left-hand side has a limit 1 when  $N$  goes to infinity and the right-hand side is strictly less than 1. For such a choice of  $N$ , both  $a(s)$  and  $b(s)$  will have non-negative coefficients.

For a general polynomial  $\gamma$  which has no positive root, we first express it as a product of its irreducible factors as

$$\gamma(s) = A \prod_i (1 + \zeta_i s) \prod_i (1 - 2\xi_i s + (\xi_i^2 + \eta_i^2)s^2) \quad , \quad (15)$$

where  $A > 0$ ,  $-1/\zeta_i$  are the real roots of  $\gamma$  (therefore  $\zeta_i > 0$ ) and  $(\xi_i \pm i\eta_i)^{-1}$  are the complex roots of  $\gamma$ . As the quadratic factors can be expressed as a ratio and the rest is simply a polynomial with non-negative coefficients,  $\gamma$  can be expressed as a ratio of two polynomials with non-negative coefficients. Note that, as  $\gamma$  has no root at 0, both  $a(s)$  and  $b(s)$  can be chosen as polynomials having a non-zero constant term.

Before passing on to the proof of the statement (b), we first show that the polynomial  $b(s)$  can always be chosen such that all of its coefficients are strictly positive. For this, consider a degree  $m$  solution for  $b(s)$ , i.e.,  $b(s) = \sum_{k=0}^m b_k s^k$  where  $b_0 > 0$ ,  $b_m > 0$  and  $b_k \geq 0$  for all  $1 \leq k < m$ . Let  $e(s) = 1 + s + \dots + s^{m-1}$ . Then  $e(s)b(s)$  is a polynomial with degree  $2m-1$  and all of its  $2m$  coefficients are positive. Moreover, the polynomials  $e(s)b(s)$  and  $e(s)a(s)$  satisfy the conditions of part (a). Therefore,  $b(s)$  can be chosen to have non-zero coefficients.

For the proof of part (b), suppose that  $b(s)$  is a degree  $m$  polynomial with positive coefficients and let  $\beta = \min_{0 \leq k \leq m} b_k$  be the minimum of those. Let

$$\epsilon = \frac{\beta}{\sum_k |\gamma_k|} \quad , \quad (16)$$

where  $\gamma_n$  are the coefficients of the polynomial  $\gamma(s)$ . Define a new polynomial  $\bar{a}(s)$  such that it has the same degree as  $a(s)$  and its coefficients are chosen from rational numbers such that

$$|\bar{a}_k - a_k| \leq \epsilon \quad k = 0, 1, \dots, N \quad , \quad (17)$$

where  $\bar{a}_k$  and  $a_k$  are the coefficients of  $\bar{a}(s)$  and  $a(s)$  respectively. As the rational numbers are dense, this can always be done. If  $\bar{a}(s)\gamma(s) = \bar{b}(s)$ , then the coefficients  $\bar{b}(s)$  satisfy

$$\bar{b}_k - b_k = \sum_{\ell} (\bar{a}_{\ell} - a_{\ell}) \gamma_{k-\ell} \geq -\epsilon \sum_{\ell} |\gamma_{\ell}| \geq -\beta \quad (18)$$

Therefore,  $\bar{b}_k \geq b_k - \beta \geq 0$ , i.e.,  $\bar{b}(s)$  has non-negative coefficients as desired. Multiplying  $\bar{a}(s)$  by the common denominator of its coefficients gives a polynomial with integer coefficients.  $\square$

### III. PROOF OF THEOREM 1

Proof of the necessity of the conditions (6-8) for the trumping relation is trivial. Given that there is a catalyst  $c$  so that we have  $x \otimes c \prec y \otimes c$ , we use the strict inequality of (10) for the following strictly convex functions:  $f(t) = t^{\nu}$  for  $\nu > 1$  and  $\nu < 0$ ,  $f(t) = -t^{\nu}$  for  $0 < t < 1$ ,  $f(t) = -\ln t$  and  $f(t) = t \ln t$ . All inequalities (6-8) follow from these.

The proof of the sufficiency of the conditions (6-8) is lengthy and requires us to separate it into three special cases. The key proof is for case A, where only the sequences which can be expressed as integer powers of a common number is considered. Case B concentrates on non-zero sequences and uses the stability of the sufficiency conditions under small changes of sequences to reduce the problem to case A. Finally, case C deals with the situation where  $y$  has zero elements.

**Case A.**  $y$  has strictly positive elements such that  $y_i = K\omega^{\alpha_i}$  and  $x_i = K\omega^{\beta_i}$  for some integers  $\alpha_i$  and  $\beta_i$  and for some numbers  $K > 0$  and  $\omega > 1$ .

*Proof:* Without loss of generality, it is assumed that  $x$  and  $y$  have no common elements and they are arranged in non-decreasing order. The smallest of the exponents is  $\alpha_1$  which can be set equal to 0 by a redefinition of  $K$ . Finally, both  $x$  and  $y$  can be divided by  $K$  which is equivalent to setting  $K = 1$ . As a result, it is not required that the sequences are normalized (i.e., they do not add up to 1). Since  $\alpha_1 = 0$ , all other exponents satisfy  $\alpha_i \geq 0$  and  $\beta_i > 0$ .

Let the polynomial  $\Gamma(s)$  be defined as

$$\Gamma(s) = \sum_{i=1}^n (s^{\alpha_i} - s^{\beta_i}) = \sum_k \Gamma_k s^k \quad . \quad (19)$$

First, note that  $\Gamma(s)$  has simple roots at  $s = 1$  and  $s = \omega$ . This can be simply seen by evaluating its derivative at these points,

$$\Gamma'(1) = \sum_{i=1}^n (\alpha_i - \beta_i) < 0 \quad , \quad (20)$$

$$\Gamma'(\omega) = \frac{\sigma(x) - \sigma(y)}{\ln \omega} > 0 \quad , \quad (21)$$

where the former strict inequality follows from (6) at  $\nu = 0$  and the latter follows from (8). The fact that  $x$  and  $y$  are not normalized do not invalidate the latter inequality.

Therefore,  $\gamma(s) = \Gamma(s)/((1-s)(1-s/\omega))$  is a polynomial. It can be seen that  $\gamma(0) > 0$ . Moreover, we will show that  $\gamma(s)$  has no positive root. For this purpose let  $s = \omega^{\nu}$  where  $\nu$  is any real number ( $\nu = 0$  and  $\nu = 1$  can be excluded if desired). Then

$$\gamma(\omega^{\nu}) = \frac{1}{(1-\omega^{\nu})(1-\omega^{\nu-1})} \sum_{i=1}^n (y_i^{\nu} - x_i^{\nu}) \quad , \quad (22)$$

which can be seen to be strictly positive by virtue of (6) and (7) for all values of  $\nu$ . (For  $\nu = 0$  and  $\nu = 1$ , we have seen above that  $\gamma(s)$  has no root at 1 and  $\omega$ ).

By the lemma, there are two polynomials  $a(s)$  and  $b(s)$  with non-negative coefficients such that  $a(s)\gamma(s) = b(s)$  and  $a(s)$  has integral coefficients. The constant coefficients  $a(0)$  and  $b(0)$  will also be chosen to be non-zero. In terms of  $\Gamma$ , the relation can be expressed as

$$a(s)\Gamma(s) = (1-s)(1-s/\omega)b(s) \quad (23)$$

Let  $a(s)$  has degree  $N$ . The catalyst sequence  $c$  will be chosen from the numbers  $\omega^k$  which are repeated  $a_k$  times ( $k = 0, 1, \dots, N$ ). In that case, the characteristic function of  $c$  is

$$H_c(t) = \sum_{k=0}^N a_k(t - \omega^k)^+ \quad (24)$$

We would like to show that the function

$$\Delta(t) = H_{y \otimes c}(t) - H_{x \otimes c}(t) \quad (25)$$

$$= \sum_{i=1}^n y_i H_c(t/y_i) - x_i H_c(t/x_i) \quad (26)$$

$$= \sum_{\ell} \Gamma_{\ell} \omega^{\ell} H_c(t\omega^{-\ell}) \quad (27)$$

$$= \sum_{k,\ell} a_k \Gamma_{\ell} (t - \omega^{k+\ell})^+ \quad (28)$$

is non-negative for all  $t \geq 0$ . First note that

$$\Delta(t) = \sum_{m=0}^{M+1} (f_m - f_{m-1})(t - \omega^m)^+ \quad (29)$$

where  $f(s) = (1-s/\omega)b(s)$ ,  $f_m$  are coefficients of the polynomial  $f(s)$  and we have chosen  $f_{-1} = 0$  for simplicity. Here  $M$  is the degree of  $f$  ( $M+1$  is the degree of  $a(s)\Gamma(s)$ ). Since  $\Delta(t)$  is a piecewise linear function, for showing its positivity, it is sufficient to look at its value at the turning points and at the 0 and  $\infty$  limits. First note that  $\Delta(t) = 0$  for  $t \leq 1$  and  $\Delta(t)$  is constant for  $t \geq \omega^{M+1}$ . As a result, we only need to check the values of  $\Delta(t)$  at  $t = \omega, \omega^2, \dots, \omega^{M+1}$ . For any  $1 \leq k \leq M+1$ ,

$$\Delta(\omega^k) = \sum_{m=0}^{k-1} (f_m - f_{m-1})(\omega^k - \omega^m) \quad (30)$$

$$= (\omega - 1) \sum_{m=0}^{k-1} (f_m - f_{m-1}) \sum_{p=m}^{k-1} \omega^p \quad (31)$$

$$= (\omega - 1) \sum_{p=0}^{k-1} \omega^p \sum_{m=0}^p (f_m - f_{m-1}) \quad (32)$$

$$= (\omega - 1) \sum_{p=0}^{k-1} f_p \omega^p \quad (33)$$

Finally, as  $f(s) = (1-s/\omega)b(s)$ , the coefficients of these polynomials satisfy

$$f_p = b_p - \frac{b_{p-1}}{\omega} \quad (34)$$

where  $b_{-1} = 0$ , which leads to

$$\Delta(\omega^k) = (\omega - 1)b_{k-1}\omega^{k-1} \geq 0 \quad (35)$$

This completes the proof of  $\Delta(t) \geq 0$  for all  $t \geq 0$ . It also shows that  $x \otimes c \prec y \otimes c$ . Therefore,  $x \prec_T y$ .  $\square$

Before passing on to the next case, we first state another theorem that shows the stability of the inequalities (6-8) against small variations in sequences  $x$  and  $y$ . Since only sequences with non-zero elements will be considered in the next case, the distance between two sequences will be measured by the deviation of the ratio of the corresponding elements from 1. For two sequences  $x$  and  $\bar{x}$  which has no zero elements, the distance between them is defined as

$$D(x; \bar{x}) = \max_i \left| \ln \frac{x_i}{\bar{x}_i} \right| \quad (36)$$

The following theorem expresses the stability of the conditions (6-8).

**Theorem 2.** Let  $x$  and  $y$  be  $n$ -element sequences formed from positive numbers such that  $x_1^\dagger > y_1^\dagger$  and  $x_n^\dagger < y_n^\dagger$ . If  $x$  and  $y$  satisfy the inequalities (6), (7) and (8), then there is a positive number  $\epsilon$  such that whenever  $D(x; \bar{x}) \leq \epsilon$  and  $D(y; \bar{y}) \leq \epsilon$ , and  $\sum \bar{x}_i = \sum \bar{y}_i = \sum x_i$ , the sequences  $\bar{x}$  and  $\bar{y}$  satisfy the same strict inequalities.

The proof of Theorem 2 is postponed to Appendix A. This result will be used in the proof of the next case.

**Case B.**  $y$  has strictly positive elements.

The proof will be carried out by choosing two new sequences  $\bar{x}$  and  $\bar{y}$  which are sufficiently near to  $x$  and  $y$  such that Theorem 2 can be invoked, and it will be made sure that  $\bar{x}$  and  $\bar{y}$  satisfy the conditions considered in case A. Without loss of generality, it is assumed that  $x$  and  $y$  are normalized ( $\sum x_i = \sum y_i = 1$ ) and they are arranged in non-decreasing order. Let  $H = \sigma(x) - \sigma(y) > 0$  be the entropy difference of these sequences and let  $L = |\ln y_1^\dagger|$ . Note that the logarithm of all elements are bounded by  $L$ , i.e.,  $|\ln y_i| \leq L$  and  $|\ln x_i| \leq L$ . Let  $\epsilon_0$  be a positive number such that whenever  $D(x, \bar{x}) \leq \epsilon_0$  and  $D(y, \bar{y}) \leq \epsilon_0$ , the sequences  $\bar{x}$  and  $\bar{y}$  satisfy all the inequalities in (6-8). The positive number  $\epsilon$  is chosen such that

$$\epsilon < \min \left( \frac{\epsilon_0}{2}, \frac{1}{8n}, \frac{1}{n^2}, \frac{H}{96nL} \right) \quad (37)$$

First note that the definition above implies that  $\epsilon < L$ , an inequality that will be used below.

We will define  $\alpha_i$  and  $\beta_i$  to be some rational approximations to numbers  $\ln y_i$  and  $\ln x_i$  respectively. Let  $\phi_i$  and  $\theta_i$  represent the deviation of these rational approximations from the true values,

$$\alpha_i = \ln y_i + \phi_i \quad (38)$$

$$\beta_i = \ln x_i + \theta_i \quad (39)$$

As the rational numbers are dense, these deviations can be chosen essentially arbitrarily. But, for our purposes,

we are going to choose them as

$$\frac{\epsilon}{2n} \leq \phi_i \leq \frac{\epsilon}{n} \quad \text{for } 1 \leq i \leq n-1, \quad (40)$$

$$\left| \sum_{i=1}^n y_i \phi_i \right| \leq \epsilon^2. \quad (41)$$

In other words, the rational approximations  $\alpha_i$  for all elements excepting the last one are to be chosen such that the corresponding deviations  $\phi_i$  are positive and small, but they are also required to be sufficiently far away from zero. The last element is an exception. In that case  $\alpha_n$  has to be chosen as a rational number so that this time the sum in (41) is made very small. In that case,  $\phi_n$  does not need to be positive. Note that the conditions (40) and (41) provides  $n$  separate intervals to choose  $\alpha_i$  from. As rational numbers are dense, all of  $\alpha_i$  can be chosen as rational numbers. Similarly, we define  $\beta_i$  and the corresponding deviations  $\theta_i$  such that

$$-\frac{\epsilon}{n} \leq \theta_i \leq -\frac{\epsilon}{2n} \quad \text{for } 1 \leq i \leq n-1, \quad (42)$$

$$\left| \sum_{i=1}^n x_i \theta_i \right| \leq \epsilon^2, \quad (43)$$

where the deviations for the first  $n-1$  elements are chosen this time to be negative. Similar comments apply in here.

Below, however, we will need a uniform bound on all the deviations. For this purpose, note the following bound on  $\phi_n$

$$y_n |\phi_n| \leq \epsilon^2 + \sum_{i=1}^{n-1} y_i |\phi_i| \leq \epsilon^2 + (1 - y_n) \frac{\epsilon}{n} \quad (44)$$

$$|\phi_n| \leq \frac{\epsilon^2}{y_n} + \left( \frac{1}{y_n} - 1 \right) \frac{\epsilon}{n} \quad (45)$$

$$\leq n\epsilon^2 + (n-1) \frac{\epsilon}{n} \leq \epsilon \quad (46)$$

where we have used the fact that  $y_n \geq 1/n$  for the maximum element of  $y$ . Therefore, the following uniform bounds can be placed on all deviations

$$|\phi_i| \leq \epsilon, \quad |\theta_i| \leq \epsilon \quad \text{for } i = 1, 2, \dots, n, \quad (47)$$

where the bounds on  $\theta_i$  follow by a similar analysis. For most of the following, we will use these uniform bounds. The stricter bounds given in (40) and (42) will only be necessary at the very end. The following bounds on the rational approximations will be occasionally used:  $|\alpha_i| \leq |\ln y_i| + |\phi_i| \leq L + \epsilon \leq 2L$  and similarly  $|\beta_i| \leq 2L$ .

Consider the following function

$$F(\lambda) = \sum_{i=1}^n (e^{\lambda \alpha_i} - e^{\lambda \beta_i}) \quad (48)$$

Our first job is to establish that this function has a root near 1, i.e., there is a number  $\lambda_0$ , which is very close to

1 such that  $F(\lambda_0) = 0$ . Once this problem is solved, the two new sequences  $\bar{x}$  and  $\bar{y}$  can be defined as

$$\bar{x}_i = \frac{e^{\lambda_0 \beta_i}}{Z_0}, \quad (49)$$

$$\bar{y}_i = \frac{e^{\lambda_0 \alpha_i}}{Z_0}, \quad (50)$$

where  $Z_0 = \sum_{i=1}^n e^{\lambda_0 \alpha_i} = \sum_{i=1}^n e^{\lambda_0 \beta_i}$ . In that case, both  $\bar{x}$  and  $\bar{y}$  are normalized sequences. However, in order to reach to the final conclusion, we also need to place bounds on the deviation of both  $\lambda_0$  and  $Z_0$  from 1. Therefore, the following analysis of bounds is needed.

First, we must show that  $F(\lambda)$  has a root somewhere near 1. For this purpose, we look at the value of  $F(1)$ . By using the following inequalities satisfied by the exponential function,  $1 + t \leq e^t \leq 1 + t + t^2$  for all  $|t| \leq 1$ , the following bounds can be placed on the first term of  $F(1)$ ,

$$\sum_{i=1}^n e^{\alpha_i} = \sum_{i=1}^n y_i e^{\phi_i} \quad (51)$$

$$\geq \sum_{i=1}^n y_i (1 + \phi_i) \geq 1 - \epsilon^2, \quad (52)$$

$$\sum_{i=1}^n e^{\alpha_i} \leq \sum_{i=1}^n y_i (1 + \phi_i + \phi_i^2) \leq 1 + 2\epsilon^2. \quad (53)$$

Same bounds can also be placed for the second term as well, which lead to

$$|F(1)| \leq 3\epsilon^2, \quad (54)$$

a very small quantity, which indicates that a root is very close to 1.

However, to verify that there is root around 1 and to place a bound on the deviation of the root from 1, we must make sure that the derivative  $F'(\lambda)$  does not rapidly go to zero around  $\lambda = 1$ . For this purpose, a lower bound will be placed on the derivative for  $|\lambda - 1| \leq \epsilon/L$ . First, note that

$$\sum_{i=1}^n \alpha_i e^{\lambda \alpha_i} = -\sigma(y) + \sum_{i=1}^n y_i \phi_i \quad (55)$$

$$+ \sum_{i=1}^n y_i \alpha_i \left( e^{(\lambda-1) \ln y_i + \lambda \phi_i} - 1 \right) \quad (56)$$

and the argument of the exponential is small as

$$|(\lambda - 1) \ln y_i + \lambda \phi_i| \leq \frac{\epsilon}{L} L + \left( 1 + \frac{\epsilon}{L} \right) \epsilon \leq 3\epsilon. \quad (57)$$

Now, using  $|e^t - 1| \leq |t| + t^2 \leq 2|t|$  for all  $|t| \leq 1$ , we can find the following lower bound on the expression above

$$\sum_{i=1}^n \alpha_i e^{\lambda \alpha_i} \geq -\sigma(y) - \epsilon^2 - 2L \cdot 6\epsilon \quad (58)$$

$$\geq -\sigma(y) - 13L\epsilon \quad (59)$$

Similar analysis for the second term of  $F'(\lambda)$  gives

$$\sum_{i=1}^n \beta_i e^{\lambda \beta_i} \leq -\sigma(x) - 13L\epsilon \quad . \quad (60)$$

Both of these give the following lower bound on the derivative  $F'(\lambda)$  for  $|\lambda - 1| \leq \epsilon/L$ ,

$$F'(\lambda) \geq H - 26L\epsilon > \frac{1}{2}H \quad . \quad (61)$$

By using the lower bound given above it is possible to see that  $F(1 + \epsilon/L)$  is positive and  $F(1 - \epsilon/L)$  is negative. This guarantees the presence of the root in the specified interval. But, this interval is too large for our purposes, and we need to find a better bound on the place of the root. Using  $F(\lambda_0) = 0$ , we can get

$$-F(1) = \int_1^{\lambda_0} F'(\lambda) d\lambda \quad , \quad (62)$$

$$|F(1)| \geq |\lambda_0 - 1| \frac{H}{2} \quad , \quad (63)$$

$$|\lambda_0 - 1| \leq \frac{2|F(1)|}{H} \leq \frac{6\epsilon^2}{H} \quad . \quad (64)$$

In other words, the root is very close to the value 1.

One final bound, this time a bound on  $\ln Z_0$  will be needed. For this, we first note that

$$Z_0 = \sum_{i=1}^n e^{\alpha_i} e^{(\lambda_0 - 1)\alpha_i} \leq \left( \sum_{i=1}^n e^{\alpha_i} \right) e^{+2|\lambda_0 - 1|L} \quad , \quad (65)$$

and a similar analysis for the lower bound gives

$$|\ln Z_0| \leq \left| \ln \left( \sum_{i=1}^n e^{\alpha_i} \right) \right| + 2|\lambda_0 - 1|L \quad . \quad (66)$$

Finally, (52) and (53) gives

$$\left| \ln \left( \sum_{i=1}^n e^{\alpha_i} \right) \right| \leq 2\epsilon^2 \quad (67)$$

where we have used the fact that  $t - 1 \geq \ln t \geq (t - 1)/t$ . As a result, we get

$$|\ln Z_0| \leq \left( 2 + \frac{12L}{H} \right) \epsilon^2 \quad . \quad (68)$$

Now, it is possible to show that the sequences  $\bar{x}$  and  $\bar{y}$  satisfy all the required properties to complete the proof. First, we will show that  $x$  is majorized by  $\bar{x}$ . For this reason, we will look at the ratio  $x_i/\bar{x}_i$  for  $i = 1, 2, \dots, n-1$ , i.e., for all elements except the last one. Here, we will make use of the upper bounds given in (42) as

$$\ln \frac{x_i}{\bar{x}_i} = -\theta_i + (1 - \lambda_0)\beta_i + \ln Z_0 \quad (69)$$

$$\geq \frac{\epsilon}{2n} - \left( 2 + \frac{24L}{H} \right) \epsilon^2 \geq 0 \quad , \quad (70)$$

where the last inequality can be obtained simply by inspecting (37). In other words, we have  $x_i \geq \bar{x}_i$  for all  $i < n$ . The conclusion  $x \prec \bar{x}$  then follows. By the same method, it can be shown that  $\bar{y}$  is majorized by  $y$  as

$$\ln \frac{\bar{y}_i}{y_i} = \phi_i + (\lambda_0 - 1)\alpha_i - \ln Z_0 \quad (71)$$

$$\geq \frac{\epsilon}{2n} - \left( 2 + \frac{24L}{H} \right) \epsilon^2 \geq 0 \quad , \quad (72)$$

in other words  $\bar{y}_i \geq y_i$  for all  $i < n$  and therefore  $x \prec_T y$ . Finally, we have

$$D(x; \bar{x}) = \max_i |\theta_i - (1 - \lambda_0)\beta_i - \ln Z_0| \quad (73)$$

$$\leq \epsilon + \left( 2 + \frac{24L}{H} \right) \epsilon^2 < \epsilon_0 \quad , \quad (74)$$

and similarly  $D(y; \bar{y}) < \epsilon_0$ . Therefore, the inequalities (6-8) are also satisfied by  $\bar{x}$  and  $\bar{y}$ . It is easy to see that  $\bar{x}$  and  $\bar{y}$  satisfy the conditions of case A. The number  $\omega$  is given as  $\exp(\lambda_0/\mathcal{N})$  where  $\mathcal{N}$  is the common denominator of the rational numbers  $\alpha_i$  and  $\beta_i$ . As a result, the conclusion  $\bar{x} \prec_T \bar{y}$  follows. Combined with  $x \prec \bar{x}$  and  $\bar{y} \prec y$ , it leads to the desired result  $x \prec_T y$ .  $\square$

**Case C.**  $y$  has zero components.

Without loss of generality, it is supposed that  $x$  and  $y$  are normalized, they are arranged in non-decreasing order and have no common elements. Let  $y$  have  $m$  zeros, i.e.,  $y_1 = y_2 = \dots = y_m = 0$  and  $0 < y_{m+1} \leq \dots \leq y_n$ . Let  $z^\epsilon$  be a sequence defined as follows,

$$z_i^\epsilon = \epsilon \quad \text{for } i = 1, 2, \dots, m \quad , \quad (75)$$

$$z_i^\epsilon = (1 - m\epsilon)y_i \quad \text{for } i = m+1, \dots, n \quad . \quad (76)$$

where  $\epsilon$  is a non-negative parameter. We will only be interested in the values of  $\epsilon$  in the range  $\epsilon \leq (y_{m+1}^{-1} + m)^{-1}$ , where  $z^\epsilon$  is arranged in increasing order. It is easy to see that all such sequences are related to each other by the majorization relation, i.e., if  $\epsilon_A > \epsilon_B$  then  $z^{\epsilon_A} \prec z^{\epsilon_B}$ . As  $z^0 = y$ , we have  $z^\epsilon \prec y$  for all values of  $\epsilon$  in the range considered.

Our job is to show that if  $\epsilon$  is sufficiently small, then  $x$  and  $z^\epsilon$  satisfy the inequalities (6-8). This is a straightforward but laborious procedure which is detailed below. For this purpose, different intervals of  $\nu$  values will be considered separately and for each interval, the existence of a separate upper bound for  $\epsilon$  will be provided.

(a) For  $\nu \leq 0$ : The quantity  $\epsilon_1 = y_n(x_1/y_n)^{n/m}$  is a possible upper bound for this range. Let  $\epsilon < \epsilon_1$ . For the special case  $\nu = 0$ , we have

$$\frac{A_0(x)}{A_0(z^\epsilon)} = \left( \frac{\prod_{i=1}^n x_i}{\epsilon^m (1 - m\epsilon)^{n-m} \prod_{i=m+1}^n y_i} \right)^{\frac{1}{n}} \quad (77)$$

$$\geq \frac{x_1}{y_n} \left( \frac{y_n}{\epsilon} \right)^{\frac{m}{n}} > 1 \quad . \quad (78)$$

For all negative values of  $\nu$  we make use of Bernoulli's inequality, which states that  $\alpha^r - 1 \geq r(\alpha - 1)$  for any

$r \geq 1$  and any positive number  $\alpha$ , to reach

$$m(\epsilon^\nu - y_n^\nu) > n(x_1^\nu - y_n^\nu) \quad . \quad (79)$$

This then leads to

$$\sum_{i=1}^n (z_i^\epsilon)^\nu = m\epsilon^\nu + (1-m\epsilon)^\nu \sum_{i=m+1}^n y_i^\nu \quad (80)$$

$$> m\epsilon^\nu + (n-m)y_n^\nu > nx_1^\nu \geq \sum_{i=1}^n x_i^\nu \quad (81)$$

As a result, we conclude that  $A_\nu(x) > A_\nu(z^\epsilon)$  for all  $\nu \leq 0$  whenever  $\epsilon < \epsilon_1$ .

(b) For  $0 < \nu \leq 1/2$ : The function

$$J_\nu = \left( \frac{\sum_{i=1}^n x_i^\nu - \sum_{i=m+1}^n y_i^\nu}{m} \right)^{\frac{1}{\nu}} \quad (82)$$

is strictly positive in the interval  $(0, 1/2]$  and moreover it has a strictly positive limit at  $\nu = 0$ . Therefore,  $\epsilon_2 = \min_{\nu \in [0, 1/2]} J_\nu$  is a positive number. If  $\epsilon < \epsilon_2$ , we have

$$\sum_{i=1}^n x_i^\nu > m\epsilon^\nu + \sum_{i=m+1}^n y_i^\nu > \sum_{i=1}^n (z_i^\epsilon)^\nu \quad , \quad (83)$$

which leads to  $A_\nu(x) > A_\nu(z^\epsilon)$  in this interval.

(c) For  $2 \leq \nu$ : Let  $K$  be defined as

$$K = \max_{\nu \in [2, \infty]} \frac{A_\nu(x)}{A_\nu(y)} \quad , \quad (84)$$

which is a positive number such that  $K < 1$ . Note that, as  $x$  and  $y$  have no common elements, the ratio above at  $\nu = +\infty$  gives  $x_n^\dagger/y_n^\dagger$  which is smaller than 1. Let  $\epsilon_3 = (1-K)/m$ . Then, for any  $\epsilon < \epsilon_3$  and for all  $\nu \geq 2$  we have

$$\sum_{i=1}^n (z_i^\epsilon)^\nu > (1-m\epsilon)^\nu \sum_{i=m+1}^n y_i^\nu \quad (85)$$

$$> K^\nu \sum_{i=m+1}^n y_i^\nu > \sum_{i=1}^n x_i^\nu \quad . \quad (86)$$

This shows the desired inequality,  $A_\nu(z^\epsilon) > A_\nu(x)$ .

(d) For  $1/2 \leq \nu \leq 2$ : Let

$$R_\nu = \frac{1}{\nu-1} \ln \frac{A_\nu(y)}{A_\nu(x)} \quad . \quad (87)$$

The inequalities (6-8) imply that  $R_\nu$  is a strictly positive continuous function in the interval considered. Therefore, the minimum  $M = \min_{\nu \in [1/2, 2]} R_\nu$  is a positive number. Let

$$R_\nu(\epsilon) = \frac{1}{\nu-1} \ln \frac{A_\nu(z^\epsilon)}{A_\nu(x)} \quad (88)$$

Since all sequences  $z^\epsilon$  are related into each other by the majorization relation, for any  $\epsilon_A > \epsilon_B$  we have  $R_\nu(\epsilon_A) \leq$

$R_\nu(\epsilon_B)$  for all  $\nu$ . In other words, as  $\epsilon$  decreases, the function  $R_\nu(\epsilon)$  monotonically increases. Finally, we note that  $R_\nu(\epsilon)$  converges pointwise to  $R_\nu$  as  $\epsilon$  goes to zero. At this point, we invoke Dini's theorem, which states that a sequence of monotonically increasing, continuous and pointwise convergent functions on a compact space are uniformly convergent. Therefore, there is a positive number  $\epsilon_4$  such that whenever  $\epsilon < \epsilon_4$ , we have  $R_\nu(\epsilon) > M/2$ .

For such values of  $\epsilon$ , the inequalities (6) and (7) are satisfied for all  $\nu \in [1/2, 2]$ . Moreover, the inequality (8) is also satisfied, as  $R_1(\epsilon) = \sigma(x) - \sigma(z^\epsilon) > M/2 > 0$ .

As a result, if  $\epsilon < \min(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ , then the sequences  $x$  and  $z^\epsilon$  satisfies all the inequalities (6-8). The proof of case B enables us to conclude that  $x \prec_T z^\epsilon$ . Finally, by  $z^\epsilon \prec y$  we reach to the desired result  $x \prec_T y$ .

#### IV. DISCUSSION AND CONCLUSION

A set of necessary and sufficient conditions are given for the trumping relation. The conditions involve a continuous variable, but they are easy to verify for concrete examples.

Conditions (6-8) can be easily adopted to the sequences in the closure of  $T(y)$ , where

$$T(y) = \{x : x \prec_T y\} \quad , \quad (89)$$

is the set of sequences trumped by  $y$ . In that case, if  $x \in \overline{T(y)}$  then the conditions (6-8) must be satisfied but with strict inequalities replaced with non-strict ones.

If  $x \in \overline{T(y)}$ , but  $x$  is not trumped by  $y$ , it means that no catalyst can achieve the conversion of  $x$  into  $y$  with probability 1, but it is possible to find a sequence of catalysts (with growingly large Schmidt numbers) such that the conversion probability is made to approach 1. Interestingly, this property is also shared by states that are far from the boundary of  $T(y)$ . Consider the example,

$$x = \left( \frac{2}{9}, \frac{3}{9}, \frac{4}{9} \right) \quad , \quad (90)$$

$$y = \left( \frac{1}{5}, \frac{2}{5}, \frac{2}{5} \right) \quad . \quad (91)$$

As  $x_3^\dagger > y_3^\dagger$ ,  $x$  is not in the closure of  $T(y)$ . However, it can be verified that  $A_\nu(x) > A_\nu(y)$  for all  $\nu < 1$ . This then implies that, any given probability less than 1 can be achieved by a suitable catalyst in the conversion of  $x$  into  $y$ . [13] However, the elements of  $\overline{T(y)}$  satisfy an additional property, i.e., they can be catalytically converted *with unit probability* to another state only slightly different from  $y$ . It is puzzling to see that this property is not shared by the pair  $x, y$  given in the example above.

Once it is understood that catalysis is possible, the problem of finding a suitable catalyst can in principle be solved by going backwards along the proofs. Although possible solutions of the problem posed in the Lemma

in Section II can be found by the well-established procedures of linear programming, carrying out the whole procedure for realistic cases might be forbidding, as the degree of the polynomial  $\gamma(s)$  and of the sought for polynomial  $a(s)$  might be very large. However, the method used in the proof of the Lemma can be used to place an upper bound on the degree of  $a(s)$  (but not on the Schmidt number). This also suggests a conjecture that the complex roots,  $\nu$ , of the equation  $A_\nu(x) = A_\nu(y)$ , and their closeness to the real line could be used for estimating the minimum amount of resources the catalysts should have.

## APPENDIX A: PROOF OF THEOREM 2

The most troublesome part of the proof of Theorem 2 is the neighborhood of  $\nu = 1$ . This part can be handled with the following theorem.

**Theorem 3.** For any positive sequence  $x$  and any given  $\delta > 0$ , there is a positive number  $\epsilon$  such that, for any  $\bar{x}$  with  $\sum_i \bar{x}_i = \sum x_i$ , and  $D(x; \bar{x}) \leq \epsilon$  we have

$$e^{-\delta|\nu-1|} \leq \frac{A_\nu(\bar{x})}{A_\nu(x)} \leq e^{\delta|\nu-1|} \quad \forall \nu \in [1/2, 2] \quad . \quad (\text{A1})$$

*Proof of Theorem 3.* Without loss of generality, it is supposed that  $x = x^\uparrow$ . Let  $S_\nu(x) = \sum_i x_i^\nu$  be the  $\nu$ th power sum and  $K_\nu$  be defined as

$$K_\nu = \ln \frac{S_\nu(\bar{x})}{S_\nu(x)} \quad . \quad (\text{A2})$$

Note that  $K_1 = 0$ . We place the following bound on the absolute value of  $\nu$  derivative of  $K_\nu$ ,

$$\left| \frac{dK_\nu}{d\nu} \right| = \frac{1}{S_\nu(x)S_\nu(\bar{x})} \left| \sum_{ij} \bar{x}_i^\nu x_j^\nu \ln \frac{\bar{x}_i}{x_j} \right| \quad (\text{A3})$$

$$\leq \epsilon + \frac{1}{S_\nu(x)S_\nu(\bar{x})} \left| \sum_{ij} \bar{x}_i^\nu x_j^\nu \ln \frac{x_i}{x_j} \right| \quad (\text{A4})$$

$$\leq \epsilon + \frac{1}{S_\nu(x)S_\nu(\bar{x})} \left| \sum_{i>j} (\bar{x}_i^\nu x_j^\nu - \bar{x}_j^\nu x_i^\nu) \ln \frac{x_i}{x_j} \right|$$

Since, for  $i > j$  we have  $x_i \geq x_j$ , all of the logarithmic terms are non-negative in the expression above. As a result, for any positive  $\nu$ ,

$$\left| \frac{dK_\nu}{d\nu} \right| \leq \epsilon + \frac{e^{\nu\epsilon} - e^{-\nu\epsilon}}{S_\nu(x)S_\nu(\bar{x})} \sum_{i>j} x_i^\nu x_j^\nu \ln \frac{x_i}{x_j} \quad (\text{A5})$$

$$\leq \epsilon + \frac{e^{\nu\epsilon} - e^{-\nu\epsilon}}{2S_\nu(x)S_\nu(\bar{x})} \sum_{i,j} x_i^\nu x_j^\nu \left| \ln \frac{x_i}{x_j} \right| \quad (\text{A6})$$

$$\leq \epsilon + \frac{e^{\nu\epsilon} - e^{-\nu\epsilon}}{2} \frac{S_\nu(x)}{S_\nu(\bar{x})} \ln \frac{x_n}{x_1} \quad (\text{A7})$$

Finally, we can apply  $S_\nu(\bar{x}) \geq e^{-\nu\epsilon} S_\nu(x)$  to the last line which gives

$$\left| \frac{dK_\nu}{d\nu} \right| \leq \epsilon + \frac{e^{2\nu\epsilon} - 1}{2} \ln \frac{x_n}{x_1} \quad (\text{A8})$$

$$\leq \epsilon + \frac{e^{4\epsilon} - 1}{2} \ln \frac{x_n}{x_1} \quad (\text{A9})$$

where the last inequality is valid for all  $0 < \nu \leq 2$ . Note that the right-hand side of the last expression has zero limit as  $\epsilon \rightarrow 0$ . This enables us to choose the value of  $\epsilon$  so small that the right hand side is less than  $\delta/2$ . In other words,  $|dK_\nu/d\nu| \leq \delta/2$ .

Next, we express  $K_\nu$  as

$$K_\nu = \int_1^\nu \frac{dK_\nu}{d\nu} d\nu \quad . \quad (\text{A10})$$

The inequality above then implies that

$$|K_\nu| \leq \frac{1}{2} \delta |\nu - 1| \quad . \quad (\text{A11})$$

Finally, considering only the values of  $\nu$  in the interval  $[1/2, 2]$ , we have

$$\left| \ln \frac{A_\nu(\bar{x})}{A_\nu(x)} \right| = \frac{|K_\nu|}{\nu} \leq \frac{2\delta|\nu-1|}{\nu} \leq \delta|\nu-1| \quad , \quad (\text{A12})$$

which is the desired result.  $\square$

*Proof of Theorem 2:* Without loss of generality suppose that  $x$  and  $y$  are normalized, i.e.,  $\sum x_i = \sum y_i = 1$ . Since the minimum and maximum values of the these sequences are different by the assumptions of the theorem, the strict inequalities are valid at the infinities, i.e.,  $A_{-\infty}(x) > A_{-\infty}(y)$  and  $A_{\infty}(x) < A_{\infty}(y)$ . Let  $G_\nu$  be defined as

$$G_\nu = \ln \frac{A_\nu(y)}{A_\nu(x)} \quad . \quad (\text{A13})$$

By the inequalities (6-8), we have  $G_\nu < 0$  for all  $\nu < 1$  and  $G_\nu > 0$  for all  $\nu > 1$ . At infinities  $G_\nu$  approaches to non-zero limits. Moreover, the derivative of  $G_\nu$  at  $\nu = 1$  is

$$G'_1 = \sigma(x) - \sigma(y) > 0 \quad . \quad (\text{A14})$$

Therefore, both of the following quantities are strictly positive,

$$B = \min_{\nu \in [-\infty, 1/2] \cup [2, \infty]} |G_\nu| \quad , \quad (\text{A15})$$

$$M = \min_{\nu \in [1/2, 2]} \frac{G_\nu}{\nu - 1} \quad . \quad (\text{A16})$$

By Theorem 3, there are numbers  $\epsilon_1$  and  $\epsilon_2$  such that  $D(x; \bar{x}) \leq \epsilon_1$  implies that  $|\ln A_\nu(\bar{x})/A_\nu(x)| \leq M|\nu-1|/3$  and  $D(y; \bar{y}) \leq \epsilon_2$  implies that  $|\ln A_\nu(\bar{y})/A_\nu(y)| \leq M|\nu-1|/3$ . We choose  $\epsilon = \min(\epsilon_1, \epsilon_2, B/3)$ .



Let  $\bar{x}$  and  $\bar{y}$  be arbitrary sequences such that  $\sum \bar{x}_i = \sum \bar{y}_i = 1$ ,  $D(x; \bar{x}) \leq \epsilon$  and  $D(y; \bar{y}) \leq \epsilon$ . Let

$$\bar{G}_\nu = \ln \frac{A_\nu(\bar{y})}{A_\nu(\bar{x})} = G_\nu + \ln \frac{A_\nu(\bar{y})}{A_\nu(x)} + \ln \frac{A_\nu(x)}{A_\nu(\bar{x})} \quad . \quad (\text{A17})$$

Our purpose is to show that  $\bar{G}_\nu$  satisfies the desired properties, i.e., it is negative for  $\nu < 1$ , positive for  $\nu > 1$  and has a simple zero at  $\nu = 1$ . Note that  $D(x; \bar{x}) \leq \epsilon$  implies that  $|\ln A_\nu(x)/A_\nu(\bar{x})| \leq \epsilon$  for all  $\nu$ .

We consider the following ranges of  $\nu$  values separately,

- (a) For  $\nu \leq 1/2$ , we have  $\bar{G}_\nu \leq -B + 2\epsilon \leq -B/3 < 0$ .
- (b) For  $\nu \geq 2$ , we have  $\bar{G}_\nu \geq B - 2\epsilon \geq B/3 > 0$ .
- (c) For  $\nu \in [1/2, 2]$  we have

$$\frac{\bar{G}_\nu}{\nu - 1} \geq M - \frac{2M}{3} > 0 \quad . \quad (\text{A18})$$

As a result,  $\bar{G}_\nu$  satisfies the desired properties in this interval as well.

Finally, for the inequality (8), we note that for  $\nu \in [1/2, 1)$ , we have

$$\frac{\bar{G}_\nu}{\nu - 1} \geq \frac{M}{3} \quad . \quad (\text{A19})$$

Taking  $\nu \rightarrow 1$  limit gives the derivative of  $\bar{G}_\nu$  which is

$$\bar{G}'_1 = \sigma(\bar{x}) - \sigma(\bar{y}) \geq \frac{M}{3} \quad . \quad (\text{A20})$$

This completes the proof of Theorem 2.

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