The Resource Theoretic Paradigm of Quantum Thermodynamics with Control

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The resource theory of quantum thermodynamics has been a very successful theory and has generated much follow up work in the community. It requires energy preserving unitary operations to be implemented over a system, bath, and catalyst as part of its paradigm. So far, such unitary operations have been considered a "free" resource of the theory. However, this is only an idealisation of a necessarily inexact process. Here, we include an additional auxiliary control system which can autonomously implement the unitary by turning "on/off" an interaction. However, the control system will inevitably be degraded by the back-action caused by the implementation of the unitary. We derive conditions on the quality of the control device so that the laws of thermodynamics do not change; and prove — by utilising a good quantum clock — that the laws of quantum mechanics allow the back-reaction to be small enough so that these conditions are satisfiable. Our inclusion of non-idealised control into the resource framework also rises interesting prospects, which were absent when considering idealised control. Namely: 1) the emergence of a 3rd law — without the need for the assumption of a light-cone. 2) the inability to apply the 2nd laws out of equilibrium.

Our results and framework unify the field of autonomous thermal machines with the thermodynamic quantum resource theoretic one, and lay the groundwork for all quantum processing devices to be unified with fully autonomous machines.

I. INTRODUCTION

Thermodynamics has been tremendously successful in describing the world around us. It has also been at the heart of developing new technologies, such as heat engines which powered the industrial revolution, jet and space rocket propulsion — just to name a few. In more recent times, scientists have been developing a theoretical understanding of thermodynamics for tiny systems for which often quantum effects cannot be ignored. These ongoing developments are influential in optimising current quantum technologies, or understanding important physical processes. Take for example, molecular machines or nano-machines such as molecular motors [1], which are important in biological processes [2], or distant technologies such as nanorobots [3], where quantum effects on the control mechanism, and the back-reaction they incur, are likely to be significant due to their small size.

The modern quantum thermodynamics literature tends to be on two types of precesses: those which are fully autonomous and those which assume implicit external control at no extra cost. An example of the former is the Brownian ratchet, popularised by Feynman [6], which simply sits between two thermal baths and extracts work in situ. There are many autonomous quantum thermal machines built on similar principles [4, 7–17]. However, there are a number of processes, such as quantum Carnot cycles, that require external control. This is true both in theory [18–22] and in practice [23].

A popular model for implicit externally controlled devices, on which a number of resource theories are based [5, 24–26], and from which one can derive the quantum version of the second law [5, 27, 28], is when the assumed control necessitates the application of energy preserving
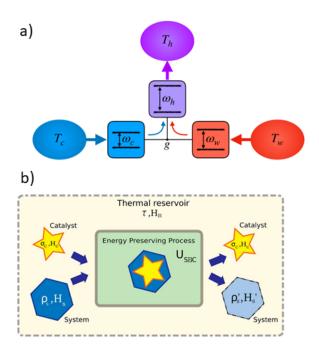

FIG. 1: **Fully autonomous absorption machines vs. resource theoretic cycle-based machines. a)** Depiction of a quantum absorption refrigerator from [4]. Quantum absorption machines do not need external control to operate. Given enough time, they settle into a functioning steady state. **b)** Schematic of a catalytic thermal operation from [5] — one discrete time step of a resource theory based thermal machine. Such devices require fine-tuned control over energy preserving unitaries U_{SRC}. For macroscopic systems, fine grained control is not needed and is thermodynamically free. In this manuscript, we set out to bridge the divide by studying how well resource theory based machines can be implemented via continuous time dynamical models.

unitaries. Unlike autonomous thermal machines which can be implemented via continuous dynamical models, an externally controlled machine requires a time-dependent Hamiltonian. Moreover, in order to implement the energy preserving unitary in the resource theory approach, an interaction Hamiltonian must be switched on and allowed to run for a specific amount of time before being turned off again. As such, thermal processes modelled on the resource theory approach, are not based on dynamical models, but on a discrete set of unitaries instead — much like the quantum circuit model of computation. Fig. 1 compares autonomous and non-autonomous machines.

Allowing for such fine tuned control at the quantum scale is highly contentious — especially in resource theories, where the cost of such control could in principle dwarf the other resources the theory is aiming to quantify. Imagine, for argument's sake, that the predicted efficiency of a heat engine undergoing an Otto cycle is dwarfed by the energy consumed or entropy generated by the external control needed for the machine to operate. Such machines may have been of little practical use and new laws of thermodynamics taking into account the thermodynamic costs of control would have been developed.

The resource-theoretic 2$^{\text{nd}}$ laws of quantum thermodynamics [5] are yet to be realised experimentally. In part, this can be attributed to the difficulty in obtaining the high level of control required. While some theoretical proposals have made the level of control more experimentally tractable [29, 30], external control was still assumed to be a free resource. Furthermore, the external control in resource theoretic thermodynamics is considered to be "perfect", yet such control is an idealisation in that it necessitates infinite energy[1] in order to circumvent physical back-reaction from the control device [32]. In fact, [32] uses the idealised momentum clock as the control device. It was first noted by Pauli that such clocks are unphysical [33].

To date, their is no rigorous work which validates the assumption that the costs of external control can be neglected in thermodynamic resource theories. Furthermore, their are two reasons why the answer is not a priori obvious.

On the one hand, consider traditionally classical disciplines of physics which have now been studied in the quantum regime. Often one finds that control at the quantum scale is vastly more complicated than it was in the classical theory. A case in point is computation. Modern day classical computing devices employ billions of transistors all functioning in tandem — often without the need for error correction. However, quantum computation is extremely fragile and error correction with significant overheads will undoubtedly be required due to imperfect control, among other things. Even beyond error correction, the implementation of gates has costs beyond Landauer erasure [34–36].

On the other hand, there is the issue of catalysis. In quantum thermodynamic resource theories, often a catalyst is employed to allow certain transformations which would otherwise be thermodynamically forbidden. The catalyst system, however, has to be returned to its initial state after the transition. It is well known that this process is highly unstable in that a small amount of error in its final state can yield vastly different transformation laws [5, 37].

The large popularity of thermodynamic resource theories (e.g. [25, 27, 28, 38–51] or reviews [52–54]) in addition to these observations, make the need to study the costs of control in the resource theoretic quantum thermodynamics even more pressing.

II. SETTING

A. t- Catalytic Thermal Operations

Resource theories have been applied to the study of quantum thermodynamics. In this setting, one considers a thermal operation (TO) from a state ρ_{A}^0 to ρ_{A}^1 as allowed iff there exists a unitary U_{AG} over system A and a Gibbs state τ_{G} i.e. $\rho_{\text{A}}^1 = \text{tr}_{\text{G}}[U_{\text{AG}} (\rho_{\text{A}}^0 \otimes \tau_{\text{G}}) U_{\text{AG}}^\dagger]$. This setup is entropy preserving since it is a unitary transformation. We further require the precess to be energy preserving, namely $[U_{\text{AG}}, \hat{H}_{\text{A}} + \hat{H}_{\text{G}}] = 0$, where $\hat{H}_{\text{A}}$ is the local Hamiltonian of the A system and $\hat{H}_{\text{G}}$ that of the thermal bath.[2] These operations can be extended to the strictly larger class of catalytic TOs (CTOs) by considering additional "free" objects called catalysts ρ_{Cat}^0. In this case the A system is bipartite with the requirement that the catalyst is returned to its initial state after the transformation; $\rho_{\text{S}}^1 \otimes \rho_{\text{Cat}}^0 = \text{tr}_{\text{G}}[U_{\text{SCatG}} (\rho_{\text{S}}^0 \otimes \rho_{\text{Cat}}^0 \otimes \tau_{\text{G}}) U_{\text{SCatG}}^\dagger]$, with a Hamiltonian $\hat{H}_{\text{A}}$ of the form $\hat{H}_{\text{S}} + \hat{H}_{\text{Cat}}$. The bath provides a source of entropy and heat. In the special case in which its Hamiltonian is completely degenerate, its Gibbs state τ_{G} becomes the maximally mixed state $\tau_{\text{G}} \propto \mathbb{1}_{\text{G}}$ and the bath can now only provide entropy. These are known as catalytic noisy operations (CNOs), or simply noisy operations (NOs) when there is no catalyst involved [55, 56]. It is known that CNOs allow for transitions that are not possible by NOs [57, 58].

In these frameworks, the operations (NOs, CNOs,

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[1] This is to say, Hamiltonians with no ground states or with unphysical boundary conditions [31].

[2] We often omit tensor products with the identity when adding operators on different spaces, e.g. $\hat{H}_{\text{A}} + \hat{H}_{\text{G}} \equiv \hat{H}_{\text{A}} \otimes \mathbb{1}_{\text{G}} + \mathbb{1}_{\text{A}} \otimes \hat{H}_{\text{G}}$.

TOs, CTOs) are considered to be *free* from the resource perspective, since they preserve entropy and energy over system A and the bath G — the two resources in thermodynamics. However, note that there is the assumption that the external control (i.e. the ability to apply energy preserving unitaries over the setup) is "perfect". In order to challenge this perspective, we will now introduce an auxiliary system to represent explicitly the system which implements the external control, while aiming to show to what extent it can be free, from the resource theory perspective.

If the control system is a thermodynamically free resource, its final state after the transition must be as useful as the state it would have been in had it not implemented the unitary, and instead evolved unitarily according to its free Hamiltonian. One way to realise this within the resource theoretic paradigm, is to choose a control device whose free evolution is periodic and let the time taken to apply the unitary be an integer multiple of its period. In this scenario the control device fits nicely within the resource theory framework, since when viewed at integer multiples of the period, the control device is a catalyst according to CTOs.

The downside with this approach is that the times corresponding to multiples of the period are a measure zero of all possible times. Consequently, not only would one need an idealised clock which can tell the time with zero uncertainty to discern these particular times, but one would like to be able to say whether the transition was thermodynamically allowed during proper intervals of time. Fortunately, there is a simple generalisation of CTOs,[3] which naturally resolves this issue. We introduce t-CTO which take into account that the transition is not instantaneous, but moreover occurs over a finite time interval. In the following definition, one should think of the catalyst system as playing the role of the external control device.

Definition 1 (t-CTO). *A transition from $\rho_S^0(t_1)$ to $\rho_S^1(t_2)$ with $t_1 \leq t_2$ is possible under t-CTO iff there exists a finite dimensional quantum state ρ_{Cat} with Hamiltonian $\hat{H}_{\mathrm{Cat}}$ such that*

$$\rho_S^0(0) \otimes \rho_{\mathrm{Cat}}^0(0) \xrightarrow[\mathrm{TO}]{} \bar{\sigma}_S(t) \otimes \rho_{\mathrm{Cat}}^0(t), \qquad (1)$$

where

$$\bar{\sigma}_S(t) = \begin{cases} \rho_S^0(t) & \text{if } t \leq t_1 \\ \rho_S^1(t) & \text{if } t \geq t_2, \end{cases} \qquad (2)$$

$\rho_D^n(t) := e^{-it\hat{H}_D} \rho_D^n e^{it\hat{H}_D}$, $D \in \{S, \mathrm{Cat}\}, n \in \{0, 1\}$, and t_1 is called "the time when the TO began" while t_2 "the time

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at which the TO was finalised". *If the bath is necessarily maximally mixed, $\tau_G \propto \mathbb{1}_G$, it will be denoted $\tilde{\tau}_G$ and we will call the transition a t-CNO.*

Unless stated otherwise, we will always use the notation $\rho_D^n(t)$, $n \in \{0, 1\}$, to denote the free evolution of a normalised quantum state ρ_D^n on some Hilbert space $\mathcal{H}_D$ according to its free Hamiltonian $\hat{H}_D$; namely $\rho_D^n(t) = e^{-it\hat{H}_D} \rho_D^n e^{it\hat{H}_D}$.

Definition 1 captures two notions. On the one hand that the individual subsystems are effectively non interacting before and after the transition has taken place. On the other hand, that during the time interval (t_1, t_2), in which the transition occurs, arbitrarily strong interactions could be realised. Note that there are two special cases for which t-CTOs reduce to CTOs at times t_1, t_2 — when the Hamiltonian of the catalyst is trivial, (i.e. if $\hat{H}_{\mathrm{Cat}} \propto \mathbb{1}_{\mathrm{Cat}}$), and when the catalyst is periodic with t_1, t_2 integer multiples of its period (i.e. if $\rho_{\mathrm{Cat}}^0(t_1) = \rho_{\mathrm{Cat}}^0(t_2) = \rho_{\mathrm{Cat}}^0(T_0)$).

From the resource theoretic perspective, the characterisation of t-CTOs is the same as CTOs as the following proposition shows.

Proposition 2 (t-CTO & CTO operational equivalence). *A t-CTO from $\rho_S^0(t_1)$ to $\rho_S^1(t_2)$ using a catalyst $\rho_{\mathrm{Cat}}^0(0)$, exists iff a CTO from ρ_S^0 to ρ_S^1 exists using catalyst $\rho_{\mathrm{Cat}}^0(0)$. In other words*

$$\rho_S^0(0) \otimes \rho_{\mathrm{Cat}}^0(0) \xrightarrow[\mathrm{TO}]{} \bar{\sigma}_S(t) \otimes \rho_{\mathrm{Cat}}^0(t), \qquad (3)$$

where $\bar{\sigma}_S(t)$ is defined in Eq. (2), if and only if

$$\rho_S^0 \otimes \rho_{\mathrm{Cat}}^0(0) \xrightarrow[\mathrm{TO}]{} \rho_S^1 \otimes \rho_{\mathrm{Cat}}^0(0). \qquad (4)$$

Proof. It is simple. The only difference between Eqs. (3) and (4) for $t \geq t_1$ is an energy preserving unitary transformation on the catalyst state on the r.h.s. However, all energy preserving unitary translations are TOs. Therefore one can always go from the r.h.s. of Eq. (3) to the r.h.s. of Eq. (4) via a TO. This proves the "if" part of the Proposition. Conversely, since the inverse of an energy preserving unitary is another energy preserving unitary, one can always go from the r.h.s. of Eq. (4) to the r.h.s. of Eq. (3) via a TO. ∎

While the generalisation to t-CTOs is admittedly quite trivial in nature, it is nevertheless important when considering the autonomous implementation of CTOs. So far, the t-CTOs have only allowed us to include the external control mechanism explicitly into the CTOs paradigm in such a way that they constitute a free resource. In the next section, we will see how this free resource unfortunately corresponds to unphysical time evolution governed by an idealised clock. It will however set the benchmark for what we should be aiming to achieve, if only approximately, with a more realistic control device.

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[3] Note that this generalisation also generalises NOs, CNOs and CTOs by allowing for the inclusion of a catalyst in the initial and final state of the transition and or specialising to the case of a maximally mixed Gibbs state.

B. Idealised Control, Clocks and Embezzling Catalysts

When a dynamical catalyst in a t-CTO is responsible for autonomously implementing the transition, it must have its own internal notion of time in order to implement the unitary between times t_1 and t_2. While in practice, the clock part may only form a small part of the full dynamical catalyst system, for convenience of expression, we refer to such dynamical catalysts as a clock and denote the state of the clock with the subscript Cl. Specifically, we would require the clock to induce dynamics on a system A which corresponds to a t-CTO on A. In other words, evolution of the form $\rho_{\text{AClG}}^F(t) = e^{-it\hat{H}_{\text{AClG}}}\left(\rho_A^0 \otimes \rho_{\text{Cl}}^0 \otimes \tau_G\right)e^{it\hat{H}_{\text{AClG}}}$ where for times $t \notin (t_1, t_2)$, one has that $\rho_{\text{AClG}}^F(t)$ satisfies [4]

$$\rho_{\text{ACl}}^F(t) = \rho_A^F(t) \otimes \rho_{\text{Cl}}^0(t), \quad \rho_A^F(t) = \begin{cases} \rho_A^0(t) \text{ if } t \leq t_1 \\ \rho_A^1(t) \text{ if } t \geq t_2 \end{cases} \tag{5}$$

Here $\rho_{\text{Cl}}^0(t)$ denotes the free evolution of the clock,

$$\rho_{\text{Cl}}^0(t) = e^{-it\hat{H}_{\text{Cl}}}\rho_{\text{Cl}}^0 e^{it\hat{H}_{\text{Cl}}}. \tag{6}$$

In the case in which the clock aims to implement autonomously a TO, we would have $\rho_A^n(t) = \rho_S^n(t); n = 0, 1$ while in the case of a CTO, $\rho_A^n(t) = \rho_S^n(t) \otimes \rho_{\text{Cat}}^0(t); n = 0, 1$. In this latter case, we see that we have two catalysts. The 1st one, ρ_{Cat}^0 simply allows for a transition on S which would otherwise be forbidden under TOs, while the second one, ρ_{Cl}^0 is the clock which implements autonomously the transition. Furthermore, note that while the r.h.s. of Eq. (5) is evolving according to the free Hamiltonian $\hat{H}_A + \hat{H}_{\text{Cl}}$, the Hamiltonian $\hat{H}_{\text{AClG}}$ can be in principle of any form such that Eq. (5) holds.

If the clock is not perfect, then we would require the equality in Eq (5) to hold only approximately. The following rules this out for a wide class of clock Hamiltonians even when Eq. (5) is relaxed to include correlations between system A and the clock.

Proposition 3 (Idealised Control No-Go). *Consider a time independent Hamiltonian $\hat{H}_{\text{AClG}}$ on $\mathcal{H}_{\text{AG}} \otimes \mathcal{H}_{\text{Cl}}$ where $\mathcal{H}_{\text{AG}}$ is finite dimensional, and $\mathcal{H}_{\text{Cl}}$ arbitrary; which w.l.o.g. we expand in the form $\hat{H}_{\text{AClG}} = \hat{H}_{\text{AG}} \otimes \mathbb{1}_{\text{Cl}} + \sum_{l,m=1}^{d_A d_G} |E_l\rangle\langle E_m|_{\text{AG}} \otimes \hat{H}_{\text{Cl}}^{(l,m)}$, where $\{|E_l\rangle_{\text{AG}}\}_{l=1}^{d_A d_G}$ are the energy eigenstates of $\hat{H}_{\text{AG}} = \hat{H}_A + \hat{H}_G$; the free Hamiltonian on $\mathcal{H}_A$ and the bath. Both of the following two assertions cannot simultaneously hold:*

1) *For all $k, l = 1, 2, \ldots, d_A d_G$; $k \neq l$, the power series expansion in t*

$$\text{tr}\left[e^{-it\hat{H}_{\text{Cl}}^{(k,k)}}\rho_{\text{Cl}}^0 e^{it\hat{H}_{\text{Cl}}^{(l,l)}}\right] \tag{7}$$

$$= \sum_{n,m=0}^{\infty} \text{tr}\left[\frac{(-i\hat{H}_{\text{Cl}}^{(k,k)})^n}{n!}\rho_{\text{Cl}}^0 \frac{(i\hat{H}_{\text{Cl}}^{(l,l)})^m}{m!}\right] t^{n+m} \tag{8}$$

has a radius of convergence $r > t_2$.

2) *For some $0 < t_1 < t_2 < t_3$ there exists a TO from $\rho_A^0(t)$ to*

$$\rho_A^F(t) = \begin{cases} \rho_A^0(t) & \text{for } t \in [0, t_1] \\ \text{tr}_G[U_{\text{AG}}(\rho_A^0(t) \otimes \tau_G)U_{\text{AG}}^\dagger] & \text{for } t \in [t_2, t_3], \end{cases} \tag{9}$$

(where U_{AG} has non-degenerate spectrum, and is an energy preserving unitary, namely $[U_{\text{AG}} \otimes \mathbb{1}_{\text{Cl}}, \hat{H}_{\text{AGClG}}] = 0$) which is implementable via unitary dynamics of the form

$$\rho_A^F(t) = \text{tr}_{\text{GClG}}\left[e^{-it\hat{H}_{\text{AClG}}}\left(\rho_A^0 \otimes \rho_{\text{Cl}}^0 \otimes \tau_G\right)e^{it\hat{H}_{\text{AClG}}}\right]. \tag{10}$$

The proof is by contradiction and found in Sec. A. The requirement of non degenerate spectrum in 2) for U_{AG} allows for exclusion of the trivial cases $U_{\text{AG}} \propto \mathbb{1}_{\text{AG}}$ for which 1) and 2) can simultaneously hold[5]. Furthermore, the no-go theorem also covers the more relaxed setting in which the clock (or any catalyst included in A) is allowed to become correlated with the system. The correlated scenario is also important and studied within the context of idealised control in [49, 59, 60].

The no-go proposition rules out physical implementation of idealised control for a number of cases. Prop. 3, case 1) holds when ρ_{Cl}^0 is an analytic vector [61]. The simplest examples of this, is when ρ_{Cl}^0 has bounded support on the spectral measures of the Hamiltonians $\{\hat{H}_{\text{Cl}}^{(k,k)}\}_{k=1}^{d_A d_{\text{Cl}}}$, such as in the finite dimensional clock case. It can also be seen that the contradicting statements 1) and 2) in Prop. 3 are not due to discontinuity, since the unitary U_{AG} facilitating the TO from ρ_A^0 to ρ_A^1 can be implemented via a smooth function of t, namely $U_{\text{AG}}(t) = \exp\left[-i\hat{H}_u \int_{t_1}^t \bar{\delta}(x)dx\right]$, with $\bar{\delta}(t)$ a normalised bump function with support on some interval $\subseteq [t_1, t_2]$ and $\hat{H}_u$ an appropriately chosen time independent Hamiltonian.

One can however find examples for Prop. 3 in which 2) is fulfilled while 1) is not. This corresponds to the

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[4]For any bipartite state ρ_{AB}, we use the notation of reduced states $\rho_A := \text{tr}_B(\rho_{\text{AB}})$, $\rho_B := \text{tr}_A(\rho_{\text{BA}})$.

[5]It is likely that the no-go theorem holds for all non-trivial U_{AG}, i.e. all cases for which there exists $t \in [0, t_1] \cup [t_2, t_3]$ such that $\rho_A^F(t) \neq \rho_A^0(t)$. However, the point of the no-go theorem is simply to show that the problem is non-trivial for most cases of interest.

case of the idealised momentum clock used for control in [32]. In this case $\hat{H}_{\mathrm{AGCl}}$ from Prop. 3 can be written in the form $\hat{H}_{\mathrm{AGCl}} = \hat{H}_{\mathrm{AG}} \otimes \mathbb{1}_{\mathrm{Cl}} + \sum_{n=1}^{d_A d_G} \Omega_n \, |E_n\rangle\langle E_n|_{\mathrm{AG}} \otimes g(\hat{x}_{\mathrm{Cl}}) + \mathbb{1}_{\mathrm{AG}} \otimes \hat{p}_{\mathrm{Cl}}$, with $\hat{x}_{\mathrm{Cl}}$, $\hat{p}_{\mathrm{Cl}}$ canonical position and momentum operators of a particle on a line. When g and the initial clock state have bounded support in position, 2) in Prop. 3 is satisfied, but 1) is not. Unfortunately such a clock state is so spread out in momentum, the power series expansion $\mathrm{Exp}[-\mathrm{i}t\hat{p}_{\mathrm{Cl}}] = \sum_{n=0}^{\infty}(-\mathrm{i}t\hat{p}_{\mathrm{Cl}})^n/n!$ diverges in norm when evaluated on it. Such clock states are also unphysical, since the Hamiltonian has no ground state, as 1st pointed out by Pauli [33]. We will also see how this idealised control allows us to violate the 3rd law or thermodynamics in Sec. III B — something which should not be possible with control coming from a physical system. We will thus refer to dynamics for which $\rho_{\mathrm{ACl}}^F(t)$ satisfies Eq. (5) as *idealised dynamics*.

At first sight, these observations may appear to be of little practical relevance, since indeed, one does not care about implementing the transition from ρ_{S}^0 to ρ_{S}^1 exactly, but only to a good approximation. Furthermore, for a sufficiently large clock, one might reasonably envisage being able to implement all transformations whose final states $\rho_{\mathrm{S}}^F(t)$ are in an epsilon ball of those reachable under t-CNO (and not a larger set) to arbitrary small epsilon as long as the final clock state becomes arbitrarily close in trace distance to the idealised case, namely if $\|\rho_{\mathrm{Cl}}^F(t) - \rho_{\mathrm{Cl}}^0(t)\|_1$ tends to zero as the dimension of the clock becomes large and approaches an idealised clock of infinite energy. Unfortunately, this intuitive reasoning may be false due to a phenomenon known as embezzlement. Indeed, when Eq. (5) is not satisfied the clock is disturbed by the act of implementing the unitary. As such, it is no longer a catalyst, but only an inexact one. Inexact catalysis has been studied in the literature with some counter intuitive findings. In [37] an inexact catalysis pair ρ_{Cat}^0, ρ_{Cat}^1 of dimension d_{Cat} were found such that for any d_{S} dimensional system, their trace distance vanished in the large d_{Cat} limit:

$$\|\rho_{\mathrm{Cat}}^0 - \rho_{\mathrm{Cat}}^1\|_1 = \frac{d_{\mathrm{S}}}{1 + (d_{\mathrm{S}} - 1)\log_{d_{\mathrm{S}}} d_{\mathrm{Cat}}}. \qquad (11)$$

Yet the noisy operation $\rho_{\mathrm{S}}^0 \otimes \rho_{\mathrm{Cat}}^0 \xrightarrow[\mathrm{NO}]{} \rho_{\mathrm{S}}^1 \otimes \rho_{\mathrm{Cat}}^1$ becomes valid for *all* states ρ_{S}^0, ρ_{S}^1 in the large d_{Cat} limit. In other words, they showed that the actual transition laws for the achievable state ρ_{S}^1 given an initial state ρ_{S}^0 cannot be approximated by those of CNOs — they are completely trivial, since all transformations are allowed. This paradoxical phenomena is known as work embezzlement[6] and stems from the concept of entanglement embezzlement [62].
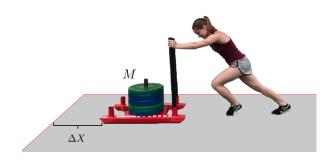
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[6] This is because in order for all states in the system Hilbert space to be reachable by an initial state under CNOs, the initial state needs to be supplemented with a work bit which is depleted in the process.

FIG. 2: **The counter intuitive phenomenon of embezzlement.** Consider a thought experiment in which an athlete who has to push a mass M a distance ΔX against a resistive force $F = Mg$ due to gravity pushing down on the weight. Suppose the distance the athlete has to push the weight is given by $\Delta X = f(M)$, where $f(M) \to 0$ as $M \to \infty$. The work done by the athlete pushing the weight is $W = \mu_0 F \Delta X = \mu_0 g M f(M)$, for some coefficient of resistance μ_0. One might be inclined to reason that the amount of work the athlete has to do in the limit of infinite mass M is zero, since the distance ΔX the weight has to be pushed is zero in this limit. However, a closer analysis would reveal that this is only correct if $f(M)$ decays sufficiently quickly — quicker than an inverse power. An analogous phenomenon is at play in our control setting. There, in the case of the idealised clock, Eq. (5) holds, yet this is unachievable since it requires infinite energy. However, all finite clocks, suffer a minimal back-reaction and even though this back-reaction can vanish in the large dimension/energy limit (c.f. Eq. 11), this is not sufficient to conclude that the set of implementable transformations are close to those implementable via the idealised clock. Moreover, the rate at which the error needs to vanish, and whether this is physically achievable; were (prior to this work) completely unknown.

By virtue of Prop 2, the above example shows that simply finding a clock satisfying $\|\rho_{\mathrm{Cl}}^F(t) - \rho_{\mathrm{Cl}}^0(t)\|_1 \to 0$ as $d_{\mathrm{Cl}} \to \infty$ is *not* sufficient to conclude that the set of allowed transformations generated by t-CNOs (and thus CTOs) corresponds to the set of transformations which can actually be implemented with physical control systems. A thought experiment illustrating such phenomena can be found at the classical level in Fig. 2.

III. RESULTS

We will start with the easier case of CNOs in Sec. III A before moving on to the more demanding setting of CTOs in Sec. III B.

A. Autonomous control for Catalytic Noisy Operations

In this section we will provide two theorems which together show that there exist clocks which are sufficiently accurate to allow the full realisation of t-CNOs to arbitrarily high precision. Our first result will give a sufficient

condition on the clock so as to be guaranteed that the achieved dynamics of the system are close to a transition permitted under t-CNOs. It can be viewed as a converse theorem to the result in [37] discussed at the end of Sec. II B.

In the following Theorem, let $V_{\mathrm{SCatClG}}(t) = \mathrm{e}^{-it\hat{H}_{\mathrm{SCatClG}}}$ be an arbitrary unitary implemented via a time *independent* Hamiltonian $\hat{H}_{\mathrm{SCatClG}}$, over $\rho_{\mathrm{S}}^0 \otimes \rho_{\mathrm{Cat}}^0 \otimes \rho_{\mathrm{Cl}}^0 \otimes \tilde{\tau}_{\mathrm{G}}$ and suppose that the final state at time $t \geq 0$,

$$\rho_{\mathrm{SCatClG}}^F(t) = V_{\mathrm{SCatClG}}(t) \left(\rho_{\mathrm{S}}^0 \otimes \rho_{\mathrm{Cat}}^0 \otimes \rho_{\mathrm{Cl}}^0 \otimes \tilde{\tau}_{\mathrm{G}} \right) V_{\mathrm{SCatClG}}^\dagger(t) \tag{12}$$

deviates from the idealised dynamics by an amount

$$\|\rho_{\mathrm{SCatCl}}^F(t) - \rho_{\mathrm{S}}^F(t) \otimes \rho_{\mathrm{Cat}}^0(t) \otimes \rho_{\mathrm{Cl}}^0(t)\|_1 \leq \epsilon_{emb}(t; d_{\mathrm{S}}, d_{\mathrm{Cat}} d_{\mathrm{Cl}}), \tag{13}$$

where recall $\rho_{\mathrm{Cl}}^0(t)$ is the free evolution of the clock according to its free, time independent, Hamiltonian $\hat{H}_{\mathrm{Cl}}$ (Eq. 6) and likewise for $\rho_{\mathrm{Cat}}^0(t)$ with arbitrary Hamiltonian $\hat{H}_{\mathrm{Cat}}$.

Theorem 1 (Sufficient conditions for t-CNOs). *For all states ρ_{S}^0 not of full rank, and for all catalysts ρ_{Cat}^0, clocks ρ_{Cl}^0 and maximally mixed states $\tilde{\tau}_{\mathrm{G}}$, there exists a state $\sigma_{\mathrm{S}}(t)$ which is ϵ_{res} close to $\rho_{\mathrm{S}}^F(t)$,*

$$\|\sigma_{\mathrm{S}}(t) - \rho_{\mathrm{S}}^F(t)\|_1 \leq \epsilon_{res}\left(d_{\mathrm{S}}, d_{\mathrm{Cat}} d_{\mathrm{Cl}}, \epsilon_{emb}(t; d_{\mathrm{S}}, d_{\mathrm{Cat}} d_{\mathrm{Cl}})\right), \tag{14}$$

such that for all times $t \geq 0$, a transition from

$$\rho_{\mathrm{S}}^0 \otimes \rho_{\mathrm{Cat}}^0 \otimes \rho_{\mathrm{Cl}}^0 \quad to \quad \sigma_{\mathrm{S}}(t) \otimes \rho_{\mathrm{Cat}}^0(t) \otimes \rho_{\mathrm{Cl}}^0(t) \tag{15}$$

is possible via a NO (i.e. ρ_{S}^0 to $\sigma_{\mathrm{S}}(t)$ via t-CNO). Specifically,

$$\epsilon_{res}(\epsilon_{emb}, d_{\mathrm{S}}, d_{\mathrm{Cat}} d_{\mathrm{Cl}}) \tag{16}$$

$$= 5 \left[\frac{d_{\mathrm{S}}^{5/3} + 4(\ln d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{Cl}}) \ln d_{\mathrm{S}}}{-\ln \epsilon_{emb}} + d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{Cl}} \epsilon_{emb}^{1/6} \right.$$

$$\left. + 5 \left((d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{Cl}})^2 \sqrt{\frac{\epsilon_{emb}}{d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{Cl}}}} \ln \sqrt{\frac{d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{Cl}}}{\epsilon_{emb}}} \right)^{\frac{2}{3}} \right].$$

Explicitly, one possible choice for $\sigma_{\mathrm{S}}(t)$ is

$$\sigma_{\mathrm{S}}(t) =$$
$$\begin{cases} \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}} & \text{if } \|\rho_{\mathrm{S}}^F(t) - \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}}\|_1 < \epsilon_{res} \\ (1 - \epsilon_{res})\rho_{\mathrm{S}}^F(t) + \epsilon_{res} \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}} & \text{if } \|\rho_{\mathrm{S}}^F(t) - \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}}\|_1 \geq \epsilon_{res} \end{cases}$$

See Sec. C for a proof. Note that this Theorem also holds more generally if one replaces $\hat{H}_{\mathrm{SCatClG}}$ with any time dependent Hamiltonian. However, the time independent Hamiltonian case is better physically motivated.

Before we move on, let us understand the physical meaning of the terms $\epsilon_{emb}, \epsilon_{res}$. By comparing the definition of ϵ_{emb} in Eq. (13) with that of Eq. (5), we see that it is the difference in trace distance between the dynamics achieved with the idealised clock, and the actual dynamics achieved by the clock. The quantity ϵ_{emb}
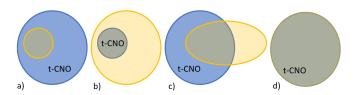

FIG. 3: **Possible scenarios resulting from the physical implementation of t-CNOs.** Given a state ρ_{S}^0, the above blue Venn diagrams represent the set of states ρ_{S}^1 which can be reached under t-CNOs. The orange Venn diagrams represent possible scenarios of reachable states when attempting to implement a t-CNO, while grey represents the intersection of the two sets. Due to the apparent impossibility of perfect control and that embezzlement can occur (see Eq. (11)), all options a) to d) are in principle open. Theorem 1 gives sufficient conditions on the control (clock) so that either a) or d) occur. Theorem 2 shows that transitions implemented via the Quasi-Ideal Clock can achieve d) under reasonable circumstances.

upper bounds how much one can embezzle from the resulting unavoidable inexact catalysis of the clock. Then ϵ_{res} (which is a function of ϵ_{emb}) characterises the resolution, i.e. how far from a t-CNO transition one can achieve due to embezzlement from the inexact catalysis. For example, consider a hypothetical clock for which ϵ_{emb} decays as an inverse power with d_{Cl}. Then, ϵ_{res} would diverge with increasing d_{Cl} and Theorem 1 would not tell us anything useful. On the other hand, if we had a more precise clock with, for example, ϵ_{emb} exponentially small in d_{Cl} then Theorem 1 would tell us that ϵ_{res} converges to zero as d_{Cl} increases.

Whether or not ϵ_{emb} and ϵ_{res} can both be simultaneously small, depends on the transition in question. Two examples at opposite extremes are as follows. Both ϵ_{emb} and ϵ_{res} are trivially arbitrarily small (zero in fact), and conditions in Theorem 1 are satisfied, when the t-CTO transition is the identity transition (i.e. ρ_{S}^0 to ρ_{S}^0). At the opposite extreme, both ϵ_{emb} and ϵ_{res} cannot be small or vanishing when one attempts a non-trivial t-CNO transition which occurs instantaneously, i.e. one for which $\rho_{\mathrm{S}}^F(t) = \rho_{\mathrm{S}}^0$ for $t \leq t_1$ and $\rho_{\mathrm{S}}^F(t) = \rho_{\mathrm{S}}^1$ for $t > t_1$.

Our next theorem shows how one can implement to arbitrary approximation all t-CTO transitions, over any fixed time interval, yet without allowing for a larger class — as the examples in Eq. 11 and Fig. 3 b) do. To achieve this, one must choose the time independent Hamiltonian $\hat{H}_{\mathrm{SCatClG}}$ and initial clock state ρ_{Cl}^0 appropriately. The theorem will use the *Quasi-Ideal Clock* [63] discussed in detail in Sec. D 1 a for the clock system on $\mathcal{H}_{\mathrm{Cl}}$. The Quasi-Ideal Clock has been proven to be optimal for some tasks [64, 65] and is believed to be for others [66, 67]. In the following, T_0 denotes the period of the Quasi-Ideal Clock, $\rho_{\mathrm{Cl}}^0(T_0) = \rho_{\mathrm{Cl}}^0(0)$.

Theorem 2 (Achieving t-CNOs). *Consider the Quasi-Ideal Clock [63] detailed in Sec. D 1 a with a time independent Hamiltonian of the form $\hat{H}_{\mathrm{SCatClG}} = \hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{Cat}} +$*

$\hat{H}_{\rm G} + \hat{I}_{\rm SCatClG} + \hat{H}_{\rm Cl}$, *giving rise to unitary dynamics*

$$\rho^F_{\rm SCatClG}(t) = \\ V_{\rm SCatClG}(t)\left(\rho^0_{\rm S} \otimes \rho^0_{\rm Cat} \otimes \rho^0_{\rm Cl} \otimes \tilde{\tau}_{\rm G}\right) V^\dagger_{\rm SCatClG}(t).$$

For every pair $\rho^0_{\rm S}$, $\rho^1_{\rm S}$ for which there exists a t-CNO from $\rho^0_{\rm S}$ to $\rho^1_{\rm S}$ using a catalyst $\rho^0_{\rm Cat}$, there exists an interaction term $\hat{I}_{\rm SCatClG}$ such that the following hold.

1) *$\sigma_{\rm S}(t)$ satisfies Eq. (15) and is of the form:*

$$\sigma_{\rm S}(t) = \\ \begin{cases} \rho^0_{\rm S}(t) \text{ for times } t \in [0, t_1] \text{ ``before'' the transition} \\ \rho^1_{\rm S}(t) \text{ for times } t \in [t_2, T_0] \text{ ``after'' the transition} \end{cases}$$

2) *ϵ_{emb} (satisfying Eq. (13)) is given by*

$$\epsilon_{emb} = \left(2 + 5(d_{\rm S} d_{\rm Cat})^{1/4}\right)\sqrt{\varepsilon_{\rm Cl}(d_{\rm Cl})}, \qquad (17)$$

for all $t \in [0, t_1] \cup [t_2, T_0]$, where $\varepsilon_{\rm Cl}(\cdot)$ is independent of $d_{\rm S}, d_{\rm Cat}, d_{\rm G}$ and is of order

$$\varepsilon(d_{\rm Cl}) = \mathcal{O}\left(poly(d_{\rm Cl})\exp\left[-c_0 d_{\rm Cl}^{1/4}\sqrt{\ln d_{\rm Cl}}\right]\right), \ (18)$$

as $d_{\rm Cl} \to \infty$, with $c_0 = c_0(t_1, t_2, T_0) > 0$ for all $0 < t_1 < t_2 < T_0$ and independent of $d_{\rm Cl}$.

See Sec. D for a proof.

As a direct consequence of Theorem 1, in the scenario described in Theorem 2, ϵ_{res} is of order of an inverse power in $d_{\rm Cl}$ as $d_{\rm Cl} \to \infty$ and thus both ϵ_{emb} and ϵ_{res} are simultaneously small. Therefore the Quasi-Ideal Clock allows *all* t-CNOs to be implemented without additional costs not captured by the resource theory.

The property that $\tilde{\tau}_{\rm G}$ is a maximally mixed state for CNOs is at the heart of two important aspects involved in proving Theorems 1 and 2. On the one hand, all CNOs (and hence all t-CNOs by virtue of Prop. 2), which are implemented via an arbitrary finite dimensional catalyst $\rho_{\rm Cat}$; can be done so with maximally mixed states $\tilde{\tau}_{\rm G}$ of *finite* dimension[7] [68]. The other relevant aspect is that they are the only states which are not "disturbed" by the action of a unitary, namely $U_{\rm G}\tilde{\tau}_{\rm G}U^\dagger_{\rm G} = \tilde{\tau}_{\rm G}$ for all unitaries $U_{\rm G}$. Together these mean that the clock only needed to control a system of finite size, and thus the back-reaction it experiences is limited and independent of the dimension $d_{\rm G}$.[8]

One would like to prove analogous theorems to Theorems 1 and 2 for t-CTOs. Unfortunately, their Gibbs

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[7]More precisely, the dimension $d_{\rm G}$ is uniformly upper bounded for all t-CNOs on a fixed $d_{\rm S}$ dimensional system Hilbert space $\mathcal{H}_{\rm S}$.
[8]Indeed, observe how the dimension $d_{\rm G}$ does not enter in either of the bounds in Theorems 1 or 2.

states satisfy neither of these two aforementioned properties. Indeed, there exists CTOs on finite dimensional systems $\mathcal{H}_{\rm S}$ which require infinite dimensional Gibbs states of infinite mean energy to implement them [68–70]. This observation, combined with the fact that Gibbs states are also generally disturbed by the CTO in the sense that $U_{\rm G}\tau_{\rm G}U^\dagger_{\rm G} \neq \tau_{\rm G}$ for some $U_{\rm G}$, suggests that a theorem like Theorem 2 for which ϵ_{res} from Theorem 1 vanishes, is not possible; since the back-reaction on any finite energy or dimensional clock would be infinite in some cases. Furthermore, there is a technical problem which prevents such theorems. The proof of Theorem 1 uses the known, necessary and sufficient transformation laws for noisy operations (the non increase of the so-called Rényi α-entropies). However, only necessary (but not sufficient) 2nd laws are known for CTOs (the most well-know of which are the non increase of the so-called Rényi α-divergences [5]).

B. Autonomous control for Catalytic Thermal Operations

In order to circumvent the dilemma explained at the end of the previous section, we now examine how well the energy preserving unitary of t-CTOs can be implemented when one restricts to attempting to implement t-CTOs which can be implemented with finite baths. We will also assume we know a lot more about the Hamiltonian which implements the t-CTO by specialising to a particular family of time independent Hamiltonians. Specifically we consider

$$\hat{H}_{\rm SCatClG} = \hat{H}_{\rm S} + \hat{H}_{\rm Cat} + \hat{H}_{\rm G} + \hat{H}^{\rm int}_{\rm SCatG} \otimes \hat{H}^{\rm int}_{\rm Cl} + \hat{H}_{\rm Cl}, \tag{19}$$

where

$$\left[\hat{H}_{\rm S} + \hat{H}_{\rm Cat} + \hat{H}_{\rm G}, \hat{H}^{\rm int}_{\rm SCatG}\right] = 0 \tag{20}$$

and the interaction term has eigenvalues bounded by pi: $\|\hat{H}^{\rm int}_{\rm SCatG}\|_\infty \leq \pi$. Given any finite $d_{\rm G}$ dimensional Gibbs state $\tau_{\rm G}$, and arbitrary catalyst system states, $\rho^0_{\rm Cat}, \rho^0_{\rm S}$, this defines a new joint system catalyst state

$$\sigma^1_{\rm SCat} := {\rm tr}_{\rm G}\left[e^{-i\hat{H}^{\rm int}_{\rm SCatG}}(\rho^0_{\rm S} \otimes \rho^0_{\rm Cat} \otimes \tau_{\rm G})e^{i\hat{H}^{\rm int}_{\rm SCatG}}\right], \tag{21}$$

which is ϵ_σ far from the initial catalyst state $\rho^0_{\rm Cat}$ and an arbitrary system state $\rho^1_{\rm S}$, namely

$$\epsilon_\sigma := \|\sigma^1_{\rm SCat} - \rho^1_{\rm S} \otimes \rho^0_{\rm Cat}\|_1. \tag{22}$$

The motivation for the definition of $\sigma^1_{\rm SCat}$ and ϵ_σ is that one would like to prepare the Hamiltonian in Eq. (19) with $\hat{H}^{\rm int}_{\rm SCatG}$ such that $\epsilon_\sigma = 0$, but due to engineering imperfections, this may not be possible, and thus ϵ_σ quantifies this imperfection. We can bound how large ϵ_σ can be in terms of small deviations in the Hamiltonian preparation. Specifically, let $\delta\hat{I}^{\rm int}_{\rm SCatG} := \hat{H}^{\rm int}_{\rm SCatG} - \hat{I}^{\rm int}_{\rm SCatG}$ characterise small imperfections between $\hat{H}^{\rm int}_{\rm SCatG}$ and $\hat{I}^{\rm int}_{\rm SCatG}$,

where the latter we define as any Hermitian operator satisfying the relation

$$\rho_S^1 \otimes \rho_{Cat}^0 = \text{tr}_G \left[e^{-i\hat{I}_{SCatG}^{int}} (\rho_S^0 \otimes \rho_{Cat}^0 \otimes \tau_G) e^{i\hat{I}_{SCatG}^{int}} \right]. \tag{23}$$

Proposition 37 states that ϵ_σ is upper bounded by

$$\epsilon_\sigma \leq 2\|\delta\hat{I}_{SCatG}^{int}\|_\infty + \|\delta\hat{I}_{SCatG}^{int}\|_\infty^2, \tag{24}$$

where $\|\delta\hat{I}_{SCatG}^{int}\|_\infty$ denotes the largest eigenvalue in magnitude of the imperfection $\delta\hat{I}_{SCatG}^{int}$. Note that there is also some freedom in the definition of $\hat{I}_{SCatG}^{int}$ in Eq. (23) since the final state of the bath is traced-out and hence irrelevant. One can minimise $\|\delta\hat{I}_{SCatG}^{int}\|_\infty$ over this degree of freedom, reducing the control requirements over the bath degrees of freedom and improving the bounds on ϵ_σ.

We will aim to use the interaction term $\hat{H}_{SCatG}^{int}$ from Eq. (19) to implement the unitary in Eq. (21) over a time interval $t \in [t_1, t_2]$. In other words, we define the *target* state to be

$$\rho_{SCatG}^{target}(t) = U_{SCatG}^{target}(t) \left(\rho_S^0(t) \otimes \rho_{Cat}^0(t) \otimes \tau_G \right) U_{SCatG}^{target\,\dagger}(t) \tag{25}$$

where $U_{SCatG}^{target}(t) = e^{-i\theta(t)\hat{H}_{SCatG}^{int}}$ with

$$\theta(t) = \begin{cases} 0 & \text{for } t \in [0, t_1] \\ 1 & \text{for } t \in [t_2, t_3]. \end{cases} \tag{26}$$

We now define a quantity $\Delta(t; x, y)$ which *only* depends on properties of the clock system:

$$\Delta(t; x, y) := \langle \rho_{Cl}^0 | \hat{\Gamma}_{Cl}^\dagger(x, t) \hat{\Gamma}_{Cl}(y, t) | \rho_{Cl}^0 \rangle, \tag{27}$$

$$\hat{\Gamma}_{Cl}(x, t) := e^{-it\hat{H}_{Cl} + ix(\theta(t)\mathbb{1}_{Cl} - t\hat{H}_{Cl}^{int})}, \quad x, t \in \mathbb{R}. \tag{28}$$

The following theorem states that if $\Delta(t; x, y)$ is small for all $x, y \in [-\pi, \pi]$, and the dimension of the bath d_G is not too large, then the clock can implement a unitary over SCatCl which is close to a t-CTO using the time independent Hamiltonian in Eq. (19). Furthermore, the clock itself is not disturbed much during the process.

Theorem 3 (Sufficient conditions for t-CTOs). *For all states ρ_S^0 and ρ_{Cat}^0, consider unitary dynamics $V_{SCatClG}(t) = e^{-it\hat{H}_{SCatClG}}$ implemented via Hamiltonian Eq. (19); $\rho_{SCatClG}^F(t) = V_{SCatClG}(t)(\rho_S^0 \otimes \rho_{Cat}^0 \otimes \rho_{Cl}^0 \otimes \tau_G)V_{SCatClG}^\dagger(t)$, with an initial pure clock state $\rho_{Cl}^0 = |\rho_{Cl}^0\rangle\langle\rho_{Cl}^0|$. Then the following hold:*

1) *The deviation from the idealised dynamics is bounded by*

$$\|\rho_{SCatCl}^F(t) - \rho_S^F(t) \otimes \rho_{Cat}^0(t) \otimes \rho_{Cl}^0(t)\|_1 \leq 2\epsilon_\sigma\,\theta(t) +$$
$$10\left(d_S\text{tr}\left[\rho_S^{0\,2}\right]d_{Cat}d_G\text{tr}\left[\tau_G^2\right]\max_{x,y\in[-\pi,\pi]} \left|1 - \Delta^2(t; x, y)\right|\right)^{1/4}. \tag{29}$$

2) *The final state $\rho_S^F(t)$ is*

$$\|\rho_S^F(t) - \rho_S^{target}(t)\|_1 \leq \epsilon_\sigma\,\theta(t) +$$
$$\sqrt{d_S\text{tr}\left[\rho_S^{0\,2}\right]d_{Cat}d_G\text{tr}\left[\tau_G^2\right]} \max_{x,y\in[-\pi,\pi]} \left|1 - \Delta^2(t; x, y)\right| \tag{30}$$

close to one which can be reached via t-CTO: For all $t \in [0, t_1] \cup [t_2, t_3]$ the transition

$$\rho_S^0 \otimes \rho_{Cat}^0 \otimes \rho_{Cl}^0 \quad to \quad \rho_S^{target}(t) \otimes \rho_{Cat}^0(t) \otimes \rho_{Cl}^0(t)$$

is possible via a TO i.e. ρ_S^0 to ρ_S^{target} via a t-CTO.

A proof can be found in Sec. F.

Since the definition of the target state in Eq. (25) allows for all t-CTOs implementable with a d_G dimensional bath,[9] Theorem 3 provides sufficient conditions (up to a small error) for *all* t-CTOs which are implementable via such baths. As long as the set of CTOs with finite bath size is a dense subset of the set of all CTOs, Theorem 3 provides sufficient conditions for implementing a dense subset of CTOs.

Intuitively, in order for $\Delta(t; x, y) \approx 1$ for all $x, y \in [-\pi, \pi]$, we see from Eq. (28) that we want the initial clock state $|\rho_{Cl}^0\rangle$ to be orthogonal to the interaction term $\hat{H}_{Cl}^{int}$ initially, and that the dynamics of the clock according to its free Hamiltonian $\hat{H}_{Cl}$ to "rotate" the initial clock state $|\rho_{Cl}^0\rangle$ to a state which is no longer orthogonal to $\hat{H}_{Cl}^{int}$ after a time t_1 when the interaction starts to happen. Similarly, the evolution induced by $\hat{H}_{Cl}$ should make the state $|\rho_{Cl}^0\rangle$ orthogonal to $\hat{H}_{Cl}^{int}$ after time t_2. Meanwhile, the interaction term $\hat{H}_{Cl}^{int}$ should have imprinted a phase of approximately e^{-ix} onto the state $|\rho_{Cl}^0\rangle$ during the time interval (t_1, t_2) to cancel out the phase factor $e^{ix\theta(t)}$ in Eq. (28). So we can think of the quantity $\Delta(t, x, y)$ as a formal mathematical expression which quantifies the intuitive physical picture of "turning on and off an interaction".

It turns out that the idealised momentum clock discussed in Sec. II B, satisfies $\Delta(t; x, y) = 1$ for all $x, y \in [-\pi, \pi]$ for an appropriate parameter choice in which case 1) in prop. 3 fails (see Sec. G in appendix) Thus the r.h.s. of Eqs. (29), (30) are exactly zero in this case.

On the other hand, since the Quasi-Ideal Clock can mimic the dynamics of the idealised momentum clock, up to an exponentially small correction term for finite $t_1 < t_2$, we also find that $\Delta(t; x, y)$ is of the same order as the r.h.s. of Eq. (18) in the case of the Quasi-Ideal Clock.

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[9]This is by construction, cf. Eq. (25) and definitions of CTO and t-CTO in Sec. II A.

The fact that the r.h.s. of Eqs. (29), (30) are exactly zero for all $t_1 < t_2$ in the case of the idealised momentum clock, highlights another point of failure for this clock: it allows for the violation of the 3rd law of thermodynamics. The 3rd law states that any system cannot be cooled to absolute zero (its ground state) in finite time. In [68, 69], it was shown that under CTOs, both the mean energy and dimension d_G of the bath need to diverge to infinity in order to cool a d_S dimensional system to the ground state. The inability to do this in finite time by any realistic control system on $\mathcal{H}_{Cl}$ manifests itself in that $\max_{x,y \in [-\pi, \pi]} |1 - \Delta^2(t; x, y)|$ cannot be exactly zero in this case, so that the r.h.s. of Eq. (30) becomes large due to the factor $d_G \text{tr}[\tau_G^2]$ diverging.[10] However, for the idealised momentum clock, the r.h.s. of Eqs. (29), (30) are exactly zero even in the limit $d_G \text{tr}[\tau_G^2] \to \infty$, thus allowing one to cool the system on $\mathcal{H}_S$ to absolute zero in any finite time interval $[t_1, t_2]$.

IV. DISCUSSION

Other than the fact that Theorem 1 provides necessary conditions for implementation of t-CNOs while Theorem 3 for implementation of t-CTOs, there are two main differences between them. The first is that Theorem 1 applied to any time independent Hamiltonian while Theorem 3 to Hamiltonians of a particular form. The other main difference, is that Theorem 1 provides bounds in terms of how close the catalyst and clock are in *trace distance* to their desired states, while Theorem 3 provides bounds in terms of how close $\Delta(t; x, y)$ is to unity. While the latter condition implies small trace distance between the clock and its free evolution, the converse is not necessarily true. Fortunately, while $\Delta(t; x, y) \approx 1$ is a stronger constraint, we have shown that it can be readily satisfied by the Quasi-Ideal clock. However, from a practical point of view, its fulfilment is likely harder to verify experimentally, since quantum measurements can be used to evaluate trace distances, while the ability to experimentally determine $\max_{x,y \in [-\pi, \pi]} \Delta(t; x, y)$ is less clear.

Observe how the bounds in Theorems 1 and 3 increase with d_{Cat}, the dimension of the catalyst. This aspect of the bound is also relevant in some important cases. Most exemplary is the setting of the important results of [49] which show that if one allows the catalysts to become correlated, then — up to an arbitrarily small error ϵ — there exists a catalyst and energy preserving unitary which achieves any TO between states block diagonal in

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[10]Note that the only case in which $d_G \text{tr}[\tau_G^2]$ does not diverge in the large d_G limit, is when the purity of the Gibbs state τ_G converges (in purity) to the maximally mixed state, since in that case $\text{tr}[\tau_G^2] = 1/d_G$. This is not the case for the baths needed to cool to absolute zero in which $\text{tr}[\tau_G^2]$ converges to a positive constant [68].

the energy basis if and only if the second law (non increase of von Neumann free energy) is satisfied. Here, the dimension of the catalyst diverges as ϵ converges to zero. The setting considered was that of idealised control, and thus the divergence of the catalyst did not affect the implementation of transitions. However, if one were to consider realistic control such as in our paradigm, the rate at which the catalyst diverges would be an important factor in determining how much back-reaction the clock would receive and consequently how large it would have to be to counteract this effect, and achieve small errors in the implementation of the control.

There are various results regarding the costs of implementing unitary operations [34, 35, 71–77]. These all have in common the assumption of implicit external control, while only restricting the set of allowed unitaries which is implemented by the external control. The allowed set of unitaries is motivated physically by demanding that they obey conservation laws (such as energy conservation), or by comparing unitaries which allow for coherent vs. incoherent operations. So while these works consider interesting paradigms, the questions they can address are of a very different nature to those posed and answered in this manuscript. In particular, the assumption of perfect control on the allowed set of unitaries means that effects such as back-reaction or degradation of the control device are neglected.

While other bounds do impose limitations arising from dynamics, these bounds are not of the right form to address the problem at hand in the manuscript. Perhaps one of the most well-known results in this direction is the so-called quantum speed limit which characterises the minimum time required for a quantum state to become orthogonal to itself or more generally, to within a certain trace-distance of itself. Indeed, such results have been applied to thermodynamics, metrology and the study of the rate at which information can be transmitted from a quantum system to an observer [78, 79]. In our context, the promise is of a different form, namely rather than the final state being a certain distance away from the initial state, we need it to be a state which is close to one permissible via the transformations laws of the resource theory (t-CNOs or t-CTOs). Similar difficulties arise when aiming to apply other results from the literature. Perhaps most markedly is [80]. Here necessary conditions in terms of bounds on the fidelity to which a unitary can be performed on a system, via a control device, is derived. Unfortunately, this result is unsuitable for our purposes for two reasons. Firstly, their bounds become trivial in the case that the unitary over the system to be implemented commutes with the Hamiltonian of the system (as is the case in this manuscript). Secondly, since catalysis is involved in our setting, bounds in trace distance for the precision of how well the unitary was implemented, are not meaningful, due to the embezzling problem discussed in Sec. II B. The latter problem is also why one cannot arrive at the conclusions of this

manuscript from [63] alone.

This work opens up interesting new questions for future research. In macroscopic thermodynamics, the 2nd law applies to transitions between states which are in thermodynamic equilibrium. Such a notion is not present in the CTOs, since the 2nd laws governing transition apply always, regardless of the nature of the state. One intriguing possibility which comes in to view with the results in this manuscript, is that we have only proven that the transition laws for t-CTOs hold for times $t \in [0, t_1] \cup [t_2, t_3]$ where the unitary implementing the transition occurs within the time interval (t_1, t_2). It would appear that CTOs are not satisfied for the state during the transition period (t_1, t_2). If this can be confirmed and proven to hold in general, then this would be the proof that the CTOs actually only hold in equilibrium, and the apparent absence of this property had been hidden in the unrealistic assumption of idealised control. A potential physical mechanism explaining this could be that at times around t_1 the clock sucks up entropy from the system it is controlling — allowing it to become more pure — after finally releasing entropy back around the t_2 time — so that the system can them become mixed enough to satisfy the 2nd laws.

Another aspect which the introduction of the paradigm of physical control into the paradigm of CTOs has given rise naturally to, is the variant of the 3rd law of thermodynamics stating that one cannot cool to absolute zero in finite time. It is noticeably absent from the CTO formalism. Future work could now investigate this property in more depth. Previous characterisations of the 3rd law [70] had to assume that the spatial area which the unitaries in the idealised control could act upon, satisfied a light-cone bound. While this is indeed a realistic assumption, it did not arise from the mathematics. Here it arises naturally even without the need for a light-cone bound assumption.

Introducing similar non idealised control for other resource theories [81, 82] could allow us to understand the requirements of these paradigms.

V. CONCLUSIONS

The resource theory approach to quantum thermodynamics has been immensely popular over the last few years. However, to date the conditions under which its underlying assumptions of idealised external control can be fulfilled, have not been justified. While it is generally appreciated that they cannot be achieved perfectly, to what extend and under what circumstances they can be approximately achieved, remained elusive. Our manuscript addresses this issue, providing sufficient

conditions which we prove are satisfiable. In doing so, our work has united two very popular yet starkly different paradigms: fully autonomous thermal machines and resource theoretic non-autonomous ones. Our approach and methods set the groundwork for future unifications of generic quantum processing machines — of which resource theoretic thermal machines can be seen as a particular example — with generic autonomous quantum processes.

Not only could these results be instrumental for future experimental realisations of the 2nd laws of quantum thermodynamics, but they can also open up new avenues of research into the 3rd law of thermodynamics and the role of non-equilibrium physics.

In particular, we have introduced a paradigm in which the cost of control in the resource theory approach of quantum thermodynamics using CNOs and CTOs can be characterised. This was achieved via the introduction of t-CNOs and t-CTOs in which control devices fit naturally into this thermodynamic setting as dynamic catalysts.

We have then derived sufficient conditions on how much the global dynamics including the control device can deviate from the idealised case, in order for the achieved state transition to be close to one permissible via CNOs. This is followed by examples of a control device which achieves this level of precision.

Finally, we introduced Hamiltonians which led us to a criteria for all CTOs with a finite dimensional bath. The bound captures the requirement of better quality control, as the bath size needed to implement the CTO gets larger. This has physical consequences for the 3rd law.

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Contents

Appendix A: Proof of Proposition 3

We will here prove Proposition 3. We will assume the assertions under both bullet points in the preposition, and culminate in a contradiction hence showing that assertions cannot simultaneously hold. To start with, we denote the unitary transformation implementing the TO from $\rho_{\text{AG}}^0(t)$ to $\rho_{\text{AG}}^1(t)$ by $U_{\text{AG}}(t) = e^{-i\delta(t)\hat{H}_u}$ where

$$\delta(t) = \begin{cases} 0 & \text{if } t \in [0, t_1] \\ 1 & \text{if } t \in [t_2, t_3]. \end{cases} \tag{A1}$$

By definition of TOs, $U_{\text{AG}}(t)$ is an energy preserving unitary which must commute with $\hat{H}_{\text{AG}} = \hat{H}_{\text{A}} \otimes \mathbb{1}_{\text{G}} + \mathbb{1}_{\text{A}} \otimes \hat{H}_{\text{G}} = \sum_{n=1}^{d_A d_G} E_n |E_n\rangle\langle E_n|_{\text{AG}}$ and can therefore be chosen to be of the form $\hat{H}_u = \sum_{n=0}^{d_A d_G} \Omega_n |E_n\rangle\langle E_n|_{\text{AG}}$ with $\Omega_n \in [-\pi, \pi)$. In order to avoid trivial unitaries, we have also assumed that the phases are non-degenerate, $\Omega_n \neq \Omega_p$ for $n \neq p$. It then follows from $[U_{\text{AG}} \otimes \mathbb{1}_{\text{Cl}}, \hat{H}_{\text{AGCl}}] = 0$ that

$$\hat{H}_{\text{Cl}}^{(k,l)} = 0, \tag{A2}$$

for $k \neq l$. Using the expansion of $\hat{H}_{\text{AGCl}}$ from the preposition, it then follows that

$$\hat{H}_{\text{AGCl}} = \hat{H}_{\text{AG}} \otimes \mathbb{1}_{\text{Cl}} + \sum_{n=1}^{d_A d_G} |E_n\rangle\langle E_n|_{\text{AG}} \otimes \hat{H}_{\text{Cl}}^{(n,n)}. \tag{A3}$$

Expanding the state ρ_{AG} in the energy basis, $\rho_{\text{AG}} = \sum_{l,m=1}^{d_A d_G} A_{l,m} |E_l\rangle\langle E_m|_{\text{AG}}$, we find from the definition of $\rho_{\text{AGCl}}^F(t)$

$$\langle E_l|\rho_{\text{AG}}^F(t)|E_m\rangle = A_{l,m}(t) \, \text{tr}\left[e^{-it\hat{H}_{\text{Cl}}^{(l,l)}} \rho_{\text{Cl}} \, e^{it\hat{H}_{\text{Cl}}^{(m,m)}} \right], \tag{A4}$$

where the time dependency of the coefficients $A_{l,m}(t)$ is defined via $\rho_{\text{AG}}(t) = e^{-it\hat{H}_{\text{AG}}} \rho_{\text{AG}} \, e^{it\hat{H}_{\text{AG}}} = \sum_{l,m=1}^{d_A d_G} A_{l,m}(t) |E_l\rangle\langle E_m|_{\text{AG}}$. On the other hand,

$$\langle E_l|U_{\text{AG}}(t)\rho_{\text{AG}}(t)U_{\text{AG}}^\dagger(t)|E_m\rangle = A_{l,m}(t) \, e^{-it(\Omega_m - \Omega_l)\delta(t)}. \tag{A5}$$

We will now proceed to show the contradicting statement. Let us assume we can equate Eqs. (9), (10) and furthermore assume that the power series in Eq. (8) is convergent in the neighbourhood of either t_1 or t_2. Since Eq. (A5) holds in the case of Eq. (9), and Eq. (A4) holds in the case of Eq. (10), we find by equating these equations for all $m \neq l$, $m, l = 1, 2, \ldots, d_A d_G$:

$$e^{-it(\Omega_m - \Omega_l)\delta(t)} = \text{tr}\left[e^{-it\hat{H}_{\text{Cl}}^{(l,l)}} \rho_{\text{Cl}}^0 \, e^{it\hat{H}_{\text{Cl}}^{(m,m)}} \right]. \tag{A6}$$

Hence if the power series expansion Eq. 8 holds,

$$e^{-it(\Omega_m - \Omega_l)\delta(t)} = \sum_{n,p=0}^{\infty} \text{tr}\left[\frac{\left(-i\hat{H}_{\text{Cl}}^{(l,l)}\right)^n}{n!} \rho_{\text{Cl}}^0 \frac{\left(i\hat{H}_{\text{Cl}}^{(m,m)}\right)^p}{p!} \right] t^{n+p}. \tag{A7}$$

However, for $t \in [0, t_1]$ we have that $\delta(t) = 0$, thus since $0 < t_1 < r$, with r the radius of convergence of the power series, for any $\tilde{t} \in (0, t_1)$, we find [11]

$$\left. \frac{d^q}{dt^q} e^{-it(\Omega_m - \Omega_l)\delta(t)} \right|_{t=\tilde{t}} = 0 \quad \text{for} \quad q \in \mathbb{N}^+. \tag{A8}$$

If we take derivatives of the r.h.s. of Eq. (A7), evaluate at $t = \tilde{t}$ and set to zero, we find

$$\text{tr}\left[\frac{\left(-i\hat{H}_{\text{Cl}}^{(l,l)}\right)^n}{n!} \rho_{\text{Cl}}^0 \frac{\left(i\hat{H}_{\text{Cl}}^{(m,m)}\right)^p}{p!} \right] = \delta_{0,n}\delta_{0,p}, \tag{A9}$$

where $\delta_{n,p}$ denotes the Kronecker-Delta function. Yet if we plug this solution into the r.h.s. of (A7), we find a contradiction for $t \in [t_2, r) \neq \emptyset$.

---

[11] Note that in order to make this conclusion, we interchange derivatives with the infinite sum, which is well known to hold for power series.

Appendix B: Preliminaries for proof of Theorem 1

1. Entropies, divergences: definitions and properties

In this section (and throughout this appendix unless stated otherwise), $\mathcal{P}_d$ will denote the set of normalised probability vectors in dimension d. Vectors $p, q \in \mathcal{P}_d$ will have entries denoted p_k, q_k respectively. We will also let $I_d \in \mathcal{P}_d$ be the uniform probability vector, namely $[I_d]_k = 1$, for $k = 1, \ldots, d$.

Definition 4 (Rényi α-entropies). *The Rényi α-entropies for $\alpha \in \mathbb{R}$ are defined to be*

$$S_\alpha(p) \;=\; \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \ln \sum_{k=1}^d p_k^\alpha, \tag{B1}$$

$$\mathrm{sgn}(\alpha) \;=\; \begin{cases} 1 & \text{for } \alpha \geq 0 \\ -1 & \text{for } \alpha < 0 \end{cases} \tag{B2}$$

where the singular point at $\alpha = 1$ is defined by demanding that the Rényi α-entropies are continuous in $\alpha \in \mathbb{R}$, and we use the conventions $\frac{a}{0} = \infty$, for $a > 0$ and $0 \ln 0 = 0$, $0^0 = 0$.

We will use the Rényi entropies evaluated on quantum states ρ in d dimensions. In which case $S_\alpha(\rho) := S_\alpha(p_\rho)$, where $p_\rho \in \mathcal{P}_d$ denotes the eigenvalues of ρ. This convention for extending the definition of functions evaluated on $\mathcal{P}_d$ to functions evaluated on quantum states of dimension d, will be used throughout.

The $\alpha = 1$ value is of particular interest, since it corresponds to the Shannon entropy, namely

$$S_1(p) := \lim_{\alpha \to 1} S_\alpha(p) = -\sum_{k=1}^d p_k \ln p_k. \tag{B3}$$

Note that the Rényi α-entropies were originally defined in [83] for $\alpha \geq 0$ only, but later extended to $\alpha \in \mathbb{R}$ for convenience in [5]. Note that $S_\alpha(p)$ can be infinite. The other functions defined in this section also have this property.

For $p \in \mathcal{P}_d$ define

$$f_\alpha(p) = \sum_{i=1}^d p_i^\alpha, \tag{B4}$$

for $\alpha \notin \{0, 1\}$ and

$$f_0(p) = -\sum_{i=1}^d \ln p_i, \quad f_1(p) = S_1(p) = -\sum_{i=1}^d p_i \ln p_i. \tag{B5}$$

If some of p_k is equal to zero, the value of f_α for $\alpha \leq 0$ is set to infinity. For $\alpha \in [0, 1]$ these functions are concave, and for $\alpha > 1$ convex.

Definition 5 (Tsallis Entropy). *Tsallis-Aczel-Daroczy entropy is as follows.*

$$T_\alpha(p) = \mathrm{sgn}(\alpha) \frac{1 - \sum_i p_i^\alpha}{\alpha - 1}, \tag{B6}$$

for $\alpha \neq 0, 1$ and $\alpha > 0$.

The Tsallis Entropy is convex, subadditive (but not additive), and for $\alpha = 1$, through a limit, it gives Shannon entropy.

Definition 6 (Hellinger Relative Entropy). *Hellinger divergence for $\alpha \in R$ is as follows*

$$\mathcal{H}_\alpha(p|q) = \frac{sgn(\alpha)}{\alpha - 1} \left(\sum_i p_i^\alpha q_i^{1-\alpha} - 1 \right), \tag{B7}$$

where the singular points are defined by continuity and, in addition to the conventions in Def. (4), we have $\frac{0}{0} = 0$.

We have

$$\lim_{\alpha \to 1} \mathcal{H}_\alpha(p|q) = D(p|q) \tag{B8}$$

where

$$D(p|q) = \sum_i p_i \ln \frac{p_i}{q_i} \tag{B9}$$

is *Kulback-Leibler entropy.* Moreover $\mathcal{H}_\alpha$ is monotonically increasing in α for $\alpha \in (0, \infty)$. In particular For $\alpha \geq 1$ we have

$$\mathcal{H}_\alpha(p|q) \geq D(p|q). \tag{B10}$$

We have also Pinsker inequality

$$D(p|q) \geq \frac{1}{2} \|p - q\|_1^2 \tag{B11}$$

We have

$$\mathcal{H}_\alpha(p|I_d/d) = d^{\alpha-1} \left(T_\alpha(I_d/d) - T_\alpha(p) \right), \tag{B12}$$

for $\alpha \neq 0, 1$ and $\alpha > 0$.

Lemma 7 (poor Subadditivity). *Let f be Schur convex, i.e. $x \succ y$ implies $f(x) \geq f(y)$. Then*

$$f(\rho_{\mathrm{AB}}) \geq f\left(\rho_{\mathrm{A}} \otimes \frac{\mathbb{1}_{\mathrm{B}}}{d_{\mathrm{B}}} \right), \tag{B13}$$

where $\mathbb{1}_{\mathrm{B}}$ is the identity operator on B.

Proof. Note that the state

$$\rho_{\mathrm{A}} \otimes \frac{\mathbb{1}_{\mathrm{B}}}{d_{\mathrm{B}}} \tag{B14}$$

can be obtained from ρ_{AB} by a mixture of unitaries (applying Haar random or discrete 2-design family) unitary on subsystem B). Thus by Uhlmann, the spectrum of original state ρ_{AB} majorizes the spectrum of the state (B14). Thus by Schur convexity of f we get (B13). ∎

2. Noisy operations, catalythic noisy operations, majorization and trumping

So called noisy operations [26] are a subclass of thermal operations introduced earlier in [24, 84]. As explained in Sec. II A of the main text, these are all operations that can be composed of: (i) adding the free resource with a maximally mixed state. (ii) applying an arbitrary unitary transformation. (iii) taking the partial trace.

It was shown that when the input and output state belong to a Hilbert space of the same dimension, the class of noisy operations is equivalent to mixture of unitaries. Therefore the condition that ρ can be transformed into σ is equivalent to majorization: ρ can be transformed into σ iff the spectrum p of ρ majorizes the spectrum q of σ. We say that $p \in \mathcal{P}_d$ majorizes $q \in \mathcal{P}_d$ if for all $l = 1, \ldots, d$

$$\sum_{i=1}^{l} p_i^\downarrow \geq \sum_{i=1}^{l} q_i^\downarrow, \tag{B15}$$

where $p^\downarrow$ is a vector obtained by arranging the components of p in decreasing order: $p^\downarrow = (p_1^\downarrow, \ldots, p_k^\downarrow)$ where $p_1^\downarrow \geq \ldots \geq p_k^\downarrow$. We now explain how catalytic noisy operations can be understood in terms of so called "trumping". As mentioned in Sec. II A of the main text, these are the noisy operations for which one is allowed to use an additional system as a catalyst — namely the additional system has to be returned to its initial state after the process. This idea, was first introduced to quantum information theory in the context of entanglement transformations [57].

Namely, we say that $p \in \mathcal{P}_d$ can be trumped into $q \in \mathcal{P}_d$ (or, that p catalytically majorizes q) if there exists some $k \in \mathbb{N}^+$ and $r \in \mathcal{P}_k$ such that

$$p \otimes r \succ q \otimes r. \tag{B16}$$

Klimesh [85] and Turgut [86] provided necessary and sufficient conditions for p to be trumped into q. Here we present conditions in the form provided by Klimesh.

Theorem 4 (Klimesh [85]). *Consider $p \in \mathcal{P}_d$ and $q \in \mathcal{P}_d$ which do not have any component simultaneously equal to zero (i.e. $p_i = 0 \implies q_i \neq 0$ and $q_i = 0 \implies p_i \neq 0 \; \forall i = 1, \ldots, d$), and let $p \neq q$. Then p can be trumped into q if and only if for all $\alpha \in (-\infty, \infty)$ we have*

$$f_\alpha(p) > f_\alpha(q) \tag{B17}$$

where the functions f_α are given by Eq. (B4) and (B5).

Appendix C: Proof of Theorem 1

1. Main proof section

In this section we prove Theorem 1. We will also need the results from subsections B 1 to C 5 to aid the proof.

The below Theorem, is the same as Theorem 1 but without writing the explicit time dependency, making the identification of ρ^0_{Cat} in the below Theorem with $\rho^0_{\mathrm{Cat}} \otimes \rho^0_{\mathrm{Cl}}$ in Theorem 1, and letting $D_{\mathrm{Cat}} = d_{\mathrm{Cat}} d_{\mathrm{Cl}}$. The motivation for this relabelling, is that for the purposes of this proof, there is not point in distinguishing between the clock catalyst (which controls the interaction) and the other catalyst, which allows for thermodynamics transitions, which would otherwise not be permitted under TOs. In other words, it is only later when we care about actual dynamics where the distinction between the two types of catalysts is important.

Theorem 5 (Sufficient conditions for implementing CNOs). *Consider arbitrary initial state ρ^0_{S} of not full rank and arbitrary catalysts ρ^0_{Cat}. Consider arbitrary unitary V_{SCatG} over $\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} \otimes \tilde{\tau}_{\mathrm{G}}$, and suppose that the final state, $\rho^F_{\mathrm{SCatG}} = V_{\mathrm{SCatG}}(\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} \otimes \tilde{\tau}_{\mathrm{G}})V^\dagger_{\mathrm{SCatG}}$ satisfies*

$$\|\rho^F_{\mathrm{SCat}} - \rho^F_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}}\|_1 \leq \epsilon_{emb} \tag{C1}$$

Then there exists a state σ_{S} which is close to ρ^F_{S}

$$\|\sigma_{\mathrm{S}} - \rho^F_{\mathrm{S}}\|_1 \leq \epsilon_{res} \tag{C2}$$

such that

$$\rho^0_{\mathrm{S}} \otimes \tilde{\rho}_{\mathrm{Cat}} \succ \sigma_{\mathrm{S}} \otimes \tilde{\rho}_{\mathrm{Cat}}, \tag{C3}$$

for some finite dimesioanl catalyst $\tilde{\rho}_{\mathrm{Cat}}$. Here $\epsilon_{res} = \epsilon_{res}(\epsilon_{emb}, d_{\mathrm{S}}, D_{\mathrm{Cat}})$ where $d_{\mathrm{S}}, D_{\mathrm{Cat}}$ are the dimensions of system ρ^0_{S} and catalyst ρ^0_{Cat} respectively. Specifically,

$$\epsilon_{res}(\epsilon_{emb}, d_{\mathrm{S}}, D_{\mathrm{Cat}}) = 5 \sqrt{\frac{d_{\mathrm{S}}^{5/3} + 4(\ln d_{\mathrm{S}} D_{\mathrm{Cat}}) \ln d_{\mathrm{S}}}{-\ln \epsilon_{emb}} + d_{\mathrm{S}} D_{\mathrm{Cat}} \epsilon_{emb}^{1/6} + 5 \left((d_{\mathrm{S}} D_{\mathrm{Cat}})^2 \sqrt{\frac{\epsilon_{emb}}{d_{\mathrm{S}} D_{\mathrm{Cat}}}} \ln \sqrt{\frac{d_{\mathrm{S}} D_{\mathrm{Cat}}}{\epsilon_{emb}}} \right)^{\frac{2}{3}}}. \tag{C4}$$

Explicitly one possible choice for σ_{S} is

$$\sigma_{\mathrm{S}} = \begin{cases} \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}} & \text{if } \|\rho^F_{\mathrm{S}} - \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}}\|_1 < \epsilon_{res} \\ (1 - \epsilon_{res})\rho^F_{\mathrm{S}} + \epsilon_{res} \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}} & \text{if } \|\rho^F_{\mathrm{S}} - \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}}\|_1 \geq \epsilon_{res} \end{cases} \tag{C5}$$

Remark 8. We use capital D_{Cat}, because for the paradigm we have formulated, the clock and the catalyst together constitute a total catalyst.

Proof. Since ρ^0_{S} is not of full rank, and the final state σ_{S} is by definition of full rank, we need only to consider Klimesh conditions from Theorem 4 for $\alpha > 0$. Consider first $\alpha > 0$, $\alpha \neq 1$. If for some unitary U we have

$$U\rho^0_S \otimes \rho^0_{\mathrm{Cat}} \otimes \tilde{\tau}_{\mathrm{G}} U^\dagger = \rho^F_{SCatG} \tag{C6}$$

then

$$f_\alpha(\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} \otimes \tilde{\tau}_{\mathrm{G}}) = f_\alpha(\rho^F_{\mathrm{SCatG}}), \tag{C7}$$

where f_α is defined in Sec. B 1. Due to convexity (concavity) of f_α and their multiplicativity, by lemma 7, putting $A = \mathrm{SCat}$ and $B = \mathrm{G}$ we obtain

$$\begin{aligned} f_\alpha(\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}}) &\geq f_\alpha(\rho^F_{\mathrm{SCat}}), && \text{for } \alpha > 1 \\ f_\alpha(\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}}) &\leq f_\alpha(\rho^F_{\mathrm{SCat}}), && \text{for } \alpha \in (0, 1). \end{aligned} \tag{C8}$$

This implies, by definition of Tsallis entropy T_α, Eq. (B6), that for all $\alpha > 0$, $\alpha \neq 1$ we have

$$T_\alpha(\rho_S^0 \otimes \rho_{\text{Cat}}^0) \leq T_\alpha(\rho_{\text{SCat}}^F), \tag{C9}$$

We now use $\|\rho_{\text{SCat}}^F - \rho_S^F \otimes \rho_{\text{Cat}}^0\|_1 \leq \epsilon_{emb}$ and the continuity lemma 18 to find for $\alpha > 0$

$$T_\alpha(\rho_{\text{SCat}}^F) \leq T_\alpha(\rho_S^F \otimes \rho_{\text{Cat}}^0) + \eta_\alpha, \tag{C10}$$

for all η_α satisfying

$$\eta_\alpha \geq 6D \left(\frac{\epsilon_{emb}}{D}\right)^\alpha \quad \text{for } \alpha \in (0, 1/2] \tag{C11}$$

$$\eta_\alpha \geq -32D\sqrt{\frac{\epsilon_{emb}}{D}} \ln \sqrt{\frac{\epsilon_{emb}}{D}} \quad \text{for } \epsilon_{emb} \leq \frac{1}{32D^2}, \quad \alpha \in (1/2, 2) \tag{C12}$$

$$\eta_\alpha \geq 6\sqrt{D\epsilon_{emb}} \quad \text{for } \alpha \in [2, \infty), \tag{C13}$$

where $D = d_S D_{\text{Cat}}$. Now we rewrite the above equation back in terms of functions f_α, which gives

$$f_\alpha(\rho_{\text{SCat}}^F) \geq f_\alpha(\rho_S^F \otimes \rho_{\text{Cat}}^0) - (\alpha - 1)\eta_\alpha \quad \text{for } \alpha > 1 \tag{C14}$$
$$f_\alpha(\rho_{\text{SCat}}^F) \leq f_\alpha(\rho_S^F \otimes \rho_{\text{Cat}}^0) - (\alpha - 1)\eta_\alpha \quad \text{for } \alpha \in (0, 1). \tag{C15}$$

Then by using Eq. (C8) followed by the multiplicity of the f_α's, we obtain from the above equations

$$f_\alpha(\rho_S^0) \geq f_\alpha(\rho_S^F) - \frac{(\alpha - 1)}{f_\alpha(\rho_{\text{Cat}}^0)}\eta_\alpha \quad \text{for } \alpha > 1 \tag{C16}$$

$$f_\alpha(\rho_S^0) \leq f_\alpha(\rho_S^F) - \frac{(\alpha - 1)}{f_\alpha(\rho_{\text{Cat}}^0)}\eta_\alpha \quad \text{for } \alpha \in (0, 1). \tag{C17}$$

Finally using $f_\alpha(p) \geq d^{1-\alpha}$ for $\alpha > 1$ and $f_\alpha(p) \geq 1$ for $\alpha \in (0, 1)$ (These inequalities follow from setting $r = 1, p = \alpha$ and $r = \alpha, p = 1$ respectively in Eq. (C296), in Lemma (25)) rewriting back in terms of T_α's we obtain

$$T_\alpha(\rho_S^0) \leq T_\alpha(\rho_S^F) + \eta_\alpha D_{\text{Cat}}^{\alpha-1} \quad \text{for } \alpha \geq 1 \tag{C18}$$
$$T_\alpha(\rho_S^0) \leq T_\alpha(\rho_S^F) + \eta_\alpha \quad \text{for } \alpha \in (0, 1). \tag{C19}$$

Here we have included the case $\alpha = 1$, which is obtained by taking the limit $\alpha \to 1$.[12] We can somewhat crudely unify this equation into

$$T_\alpha(\rho_S^0) \leq T_\alpha(\rho_S^F) + \eta_\alpha D_{\text{Cat}}^\alpha \quad \text{for } \alpha > 0 \tag{C20}$$

Furthermore, (C8) implies that for $\alpha > 1$

$$S_\alpha(\rho_S^0 \otimes \rho_{\text{Cat}}^0) \leq S_\alpha(\rho_{\text{SCat}}^F) \tag{C21}$$

and by taking limit $\alpha \to \infty$ we get

$$S_\infty(\rho_S^0 \otimes \rho_{\text{Cat}}^0) \leq S_\infty(\rho_{\text{SCat}}^F) \tag{C22}$$

which by Lemma 18 and additivity of S_∞ gives

$$S_\infty(\rho_S^0) \leq S_\infty(\rho_S^F) + \eta_\infty, \tag{C23}$$

where

$$\eta_\infty = D_{\text{Cat}}\epsilon_{emb}. \tag{C24}$$

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[12] This extension of the domain of α for which the inequality holds, follows trivially using proof by contradiction and noting that the functions in Eq. (C16) are continuous for $\alpha \in [1, \infty)$.

Let us now define as in Proposition 16

$$\sigma_{\mathrm{S}}^F(\epsilon) = \begin{cases} \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}} & \text{when } \|\rho_{\mathrm{S}}^F - \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}}\|_1 < \epsilon \\ (1-\epsilon)\rho_{\mathrm{S}}^F + \epsilon\mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}} & \text{when } \|\rho_{\mathrm{S}}^F - \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}}\|_1 \geq \epsilon. \end{cases} \tag{C25}$$

Now Eqs. (C20) and (C23) by using Proposition 16 lead to the following conclusion: for

$$\tilde{\epsilon}_T(\alpha) = \begin{cases} (16\eta_\alpha D_{\mathrm{Cat}}^\alpha d_{\mathrm{S}}^{\alpha-1})^{\frac{1}{3}} & \text{for } \alpha \geq 1 \\ (16\eta_\alpha D_{\mathrm{Cat}}^\alpha d_{\mathrm{S}}^{\alpha-1}\alpha^{-1})^{\frac{1}{3}} & \text{for } \alpha \in (0,1) \end{cases} \tag{C26}$$

$$\epsilon_\infty(\alpha) = 4\sqrt{\frac{\ln d_{\mathrm{S}}}{\alpha}} + \eta_\infty \quad \text{for } \alpha > 1 \tag{C27}$$

$$\epsilon_0(\alpha) = \left(1 - \frac{1}{d_{\mathrm{S}}}\right)^{\frac{1}{2\alpha}} \quad \text{for } \alpha \in (0,1), \tag{C28}$$

we have

$$T_\alpha(\rho_{\mathrm{S}}^0) \leq T_\alpha(\sigma_{\mathrm{S}}^F(\tilde{\epsilon}_T(\alpha))) - \min\left\{D_{\mathrm{Cat}}^\alpha \eta_\alpha,\ T_\alpha(\mathbb{1}/d_{\mathrm{S}}) - T_\alpha(\rho_{\mathrm{S}}^0)\right\} \quad \text{for } \alpha > 0 \tag{C29}$$

$$S_\alpha(\rho_{\mathrm{S}}^0) \leq S_\alpha(\sigma_{\mathrm{S}}^F(\epsilon_\infty(\alpha))) - \min\left\{D_{\mathrm{Cat}}^\alpha \eta_\alpha,\ \ln d_{\mathrm{S}} - S_1(\rho_{\mathrm{S}}^0)\right\} \quad \text{for } \alpha > 1 \tag{C30}$$

$$S_\alpha(\rho_{\mathrm{S}}^0) \leq S_\alpha(\sigma_{\mathrm{S}}^F(\epsilon_0(\alpha))) - \frac{1}{2}\ln\left(\frac{d_{\mathrm{S}}}{d_{\mathrm{S}}-1}\right) \quad \text{for } \alpha \in (0,1). \tag{C31}$$

From which we achieve

$$T_\alpha(\rho_{\mathrm{S}}^0) < T_\alpha(\sigma_{\mathrm{S}}^F(\tilde{\epsilon}_T(\alpha))) \quad \text{for } \alpha > 0 \tag{C32}$$

$$S_\alpha(\rho_{\mathrm{S}}^0) < S_\alpha(\sigma_{\mathrm{S}}^F(\epsilon_\infty(\alpha))) \quad \text{for } \alpha > 1 \tag{C33}$$

$$S_\alpha(\rho_{\mathrm{S}}^0) < S_\alpha(\sigma_{\mathrm{S}}^F(\epsilon_0(\alpha))) \quad \text{for } \alpha \in (0,1). \tag{C34}$$

Let us now insert explicitly the η's from Eqs. (C11)-(C13) and Eq. (C24) into Eqs. (C26)-(C28), for

$$\epsilon_{emb} \leq \frac{1}{32D^2}, \tag{C35}$$

we achieve the upper bounds

$$\tilde{\epsilon}_T(\alpha) \leq \begin{cases} \left(96D\,\frac{\epsilon_{emb}^\alpha}{\alpha}\right)^{\frac{1}{3}} =: \bar{\epsilon}_{Tmin}(\alpha) & \text{for } \alpha \in (0,1/2] \\ \left(-1024\,D^2\sqrt{\frac{\epsilon_{emb}}{D}}\ln\sqrt{\frac{\epsilon_{emb}}{D}}\right)^{\frac{1}{3}} =: \bar{\epsilon}_{Tmid} & \text{for all } \alpha \in (1/2,2] \\ \left(96\sqrt{D\epsilon_{emb}}D^\alpha\right)^{\frac{1}{3}} =: \bar{\epsilon}_{Tmax}(\alpha) & \text{for } \alpha \in (2,\infty) \end{cases} \tag{C36}$$

$$\epsilon_\infty(\alpha) \leq 4\sqrt{\frac{\ln d_{\mathrm{S}}}{\alpha}} + D\epsilon_{emb} =: \bar{\epsilon}_\infty(\alpha) \quad \text{for } \alpha \in [1,\infty) \tag{C37}$$

$$\epsilon_0(\alpha) \leq \left(\frac{d_{\mathrm{S}}-1}{d_{\mathrm{S}}}\right)^{\frac{1}{2\alpha}} =: \bar{\epsilon}_0(\alpha) \quad \text{for } \alpha \in (0,1) \tag{C38}$$

where $D = d_{\mathrm{S}}D_{\mathrm{Cat}}$. We now divide the set $(0,\infty)$ into five sub intervals (some of which may be empty). For $\alpha_{\min} \in (0,1)$, $\alpha_{\max} \in [2,\infty)$,

$$(0,\infty) = (0,\alpha_{\min}] \cup (\alpha_{\min},1/2] \cup (1/2,2] \cup [2,\alpha_{\max}] \cup (\alpha_{\max},\infty) \tag{C39}$$

For each of these intervals, we compute upper bounds on our epsilons. Specifically, from Eqs. (C36), (C37), (C38), we observe that:

$$\epsilon_0(\alpha) \leq \bar{\epsilon}_0(\alpha_{\min}) \quad \forall\,\alpha \in (0,\alpha_{\min}), \forall\,\alpha_{\min} \in (0,1) \tag{C40}$$

$$\tilde{\epsilon}_T(\alpha) \leq \begin{cases} \bar{\epsilon}_{Tmin}(\alpha_{\min}) & \forall\,\alpha \in (\alpha_{\min},1/2], \forall\,\alpha_{\min} \in (0,1/2] \\ \bar{\epsilon}_{Tmid} & \forall\,\alpha \in (1/2,2] \\ \bar{\epsilon}_{Tmax}(\alpha_{\max}) & \forall\,\alpha \in [2,\alpha_{\max}], \forall\,\alpha_{\max} \in [2,\infty) \end{cases} \tag{C41}$$

$$\epsilon_\infty(\alpha) \leq \bar{\epsilon}_\infty(\alpha_{\max}) \quad \forall\,\alpha \in (\alpha_{\max},\infty), \forall\,\alpha_{\max} \in [1,\infty). \tag{C42}$$

Now we define ϵ_{res} as any value satisfying

$$\epsilon_{res}(\alpha_{\min}, \alpha_{\max}) \geq \max\left\{\varepsilon_{\min}(\alpha_{\min}), \varepsilon_{\max}(\alpha_{\max}), \bar{\epsilon}_{Tmid}\right\} \tag{C43}$$

where

$$\varepsilon_{\min}(\alpha_{\min}) = \max\left\{\bar{\epsilon}_{Tmin}(\alpha_{\min}), \bar{\epsilon}_0(\alpha_{\min})\right\} \tag{C44}$$

$$\varepsilon_{\max}(\alpha_{\max}) = \max\left\{\bar{\epsilon}_{Tmax}(\alpha_{\max}), \bar{\epsilon}_\infty(\alpha_{\max})\right\}. \tag{C45}$$

Thus using Lemma 17, we have

$$S_\alpha(\rho_{\rm S}^0) < S_\alpha\left(\sigma_{\rm S}^F(\bar{\epsilon}_0(\alpha_{\min}))\right) \qquad < S_\alpha\left(\sigma_{\rm S}^F(\epsilon_{res}(\alpha_{\min}, \alpha_{\max}))\right) \quad \forall\, \alpha \in (0, \alpha_{\min}) \tag{C46}$$

$$T_\alpha(\rho_{\rm S}^0) < T_\alpha\left(\sigma_{\rm S}^F(\bar{\epsilon}_{Tmin}(\alpha_{\min}))\right) < T_\alpha\left(\sigma_{\rm S}^F(\epsilon_{res}(\alpha_{\min}, \alpha_{\max}))\right) \quad \forall\, \alpha \in (\alpha_{\min}, 1/2) \tag{C47}$$

$$T_\alpha(\rho_{\rm S}^0) < T_\alpha\left(\sigma_{\rm S}^F(\bar{\epsilon}_{Tmid})\right) \qquad < T_\alpha\left(\sigma_{\rm S}^F(\epsilon_{res}(\alpha_{\min}, \alpha_{\max}))\right) \quad \forall\, \alpha \in [1/2, 2] \tag{C48}$$

$$T_\alpha(\rho_{\rm S}^0) < T_\alpha\left(\sigma_{\rm S}^F(\bar{\epsilon}_{Tmax}(\alpha_{\max}))\right) < T_\alpha\left(\sigma_{\rm S}^F(\epsilon_{res}(\alpha_{\min}, \alpha_{\max}))\right) \quad \forall\, \alpha \in [2, \alpha_{\max}) \tag{C49}$$

$$S_\alpha(\rho_{\rm S}^0) < S_\alpha\left(\sigma_{\rm S}^F(\bar{\epsilon}_\infty(\alpha_{\max}))\right) \qquad < S_\alpha\left(\sigma_{\rm S}^F(\epsilon_{res}(\alpha_{\min}, \alpha_{\max}))\right) \quad \forall\, \alpha \in [\alpha_{\max}, \infty) \tag{C50}$$

holds for all $\alpha_{\min} \in (0,1)$, $\alpha_{\max} \in (2, \infty)$.[13]

Thus for any particular choice of $\alpha_{\min} \in (0,1)$ and $\alpha_{\max} \in (2, \infty)$, $\epsilon_{res}(\alpha_{\min}, \alpha_{\max})$ is such that the Klimesh conditions are satisfied, so that for any $\rho_{\rm S}^0$ there exists catalyst $\tilde{\rho}_{\rm Cat}$ such that

$$\rho_{\rm S}^0 \otimes \tilde{\rho}_{\rm Cat} \succ \sigma_{\rm S}^F(\epsilon_{res}) \otimes \tilde{\rho}_{\rm Cat} \tag{C51}$$

Our next aim is to find an explicit expression for $\epsilon_{res}(\alpha_{\min}, \alpha_{\max})$ with the aim of choosing the parameters $\alpha_{\min}, \alpha_{\max}$, so that $\epsilon_{res}(\alpha_{\min}, \alpha_{\max})$ is not too large. We will start with what appears to be the most significant term, $\varepsilon_{\max}(\alpha_{\max})$. Writing it explicitly, using Eqs. (C36), (C37) we have

$$\varepsilon_{\max}(\alpha_{\max}) = \max\left\{\left(96\sqrt{D\epsilon_{emb}}D^{\alpha_{\max}}\right)^{\frac{1}{3}}, 4\sqrt{\frac{\ln d_{\rm S}}{\alpha_{\max}} + D\epsilon_{emb}}\right\}. \tag{C52}$$

We now re-parametrizing $\alpha_{\max}$ in terms of a parameter $\beta_0 > 0$ via $\alpha_{\max} = -\beta_0(\ln \epsilon_{emb})/(2\ln D)$, to find

$$\max\left\{\left(96\sqrt{D\epsilon_{emb}}D^{\alpha_{\max}}\right)^{\frac{1}{3}}, 4\sqrt{\frac{\ln d_{\rm S}}{\alpha_{\max}} + D\epsilon_{emb}}\right\} = \max\left\{(96)^{1/3}D^{1/6}\epsilon_{emb}^{(1-\beta_0)/6}, 4\sqrt{\frac{2(\ln D)(\ln d_{\rm S})}{-\beta_0\ln\epsilon_{emb}} + D\epsilon_{emb}}\right\}. \tag{C53}$$

With this parametrisation, we see that we need $(1 - \beta_0) > 0$ if the 1st term in the square brackets is to tend to zero as ϵ_{emb} goes to zero. Taking this and the requirement $\alpha_{\max} \geq 2$ into account we have

$$\frac{4\ln D}{-\ln\epsilon_{emb}} \leq \beta_0 < 1. \tag{C54}$$

From Eqs. (C54) and (C53) we see that we need ϵ_{emb} to decay faster than any power of D. Specifically, it has to be of the form $\epsilon_{emb}(D) = {\rm e}^{-f(D)(\ln D)}$ where $\lim_{D\to+\infty} f(D) = +\infty$. Taking this into account, a reasonably good bound can be deduced by observing

$$(96)^{1/3}D^{1/6}\epsilon_{emb}^{(1-\beta_0)/6} \leq 5\sqrt{D}\sqrt{\epsilon_{emb}^{(1-\beta_0)/3}} < 5\sqrt{\frac{2(\ln D)(\ln d_{\rm S})}{-\beta_0\ln\epsilon_{emb}} + D\epsilon_{emb}^{(1-\beta_0)/3}} \tag{C55}$$

and

$$4\sqrt{\frac{2(\ln D)(\ln d_{\rm S})}{-\beta_0\ln\epsilon_{emb}} + D\epsilon_{emb}} < 5\sqrt{\frac{2(\ln D)(\ln d_{\rm S})}{-\beta_0\ln\epsilon_{emb}} + D\epsilon_{emb}^{(1-\beta_0)/3}}. \tag{C56}$$

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[13]Note that the range of α for which Eq. (C47) holds is the empty set if $\alpha_{\min}$ is large enough. Under such circumstances, this equation contains no information.

Given the constraints which β_0 must satisfy, its exact choice is of little relevance. We therefore set it to $\beta_0 = 1/2$. Thus we have for the appropriate $\alpha_{\max}$,

$$\varepsilon_{\max}(\alpha_{\max}) \leq 5\sqrt{\frac{2(\ln D)(\ln d_S)}{-\beta_0 \ln \epsilon_{emb}} + D\epsilon_{emb}^{(1-\beta_0)/3}} = 5\sqrt{\frac{4(\ln D)(\ln d_S)}{-\ln \epsilon_{emb}} + D\epsilon_{emb}^{1/6}}, \quad \text{if} \quad \frac{8\ln D}{-\ln \epsilon_{emb}} \leq 1. \tag{C57}$$

We will now deal with the term $\varepsilon_{\min}(\alpha_{\min})$ which plunging in is Eqs. (C36), (C38) reads.

$$\varepsilon_{\min}(\alpha_{\min})^3 = \max\left\{96D\,\frac{\epsilon_{emb}^{\alpha_{\min}}}{\alpha_{\min}}, \left[\left(\frac{d_S-1}{d_S}\right)^{3/2}\right]^{1/\alpha_{\min}}\right\} \leq \frac{1}{\alpha_{\min}}\max\left\{96D\,\epsilon_{emb}^{\alpha_{\min}}, \left[\left(\frac{d_S-1}{d_S}\right)^{3/2}\right]^{1/\alpha_{\min}}\right\}. \tag{C58}$$

We will now solve for $\alpha_{\min}$ the equation

$$96D\,\epsilon_{emb}^{\alpha_{\min}} = \left[\left(\frac{d_S-1}{d_S}\right)^{3/2}\right]^{1/\alpha_{\min}}, \tag{C59}$$

which can be written as

$$\alpha_{\min}^2 \ln(\epsilon_{emb}) + \alpha_{\min} \ln(96D) - \frac{3}{2}\ln\left(\frac{d_S-1}{d_S}\right). \tag{C60}$$

Noting that $\alpha_{\min} \in (0,1)$ we take only the non-negative root, namely

$$\alpha_{\min}^* = \frac{\ln(96D) + \sqrt{\ln^2(96D) + 6(\ln \epsilon_{emb})(\ln(1 - 1/d_S))}}{-2\ln(\epsilon_{emb})}. \tag{C61}$$

Now note that the l.h.s. of Eq. (C59) is monotonically increasing with $\alpha_{\min}$ while the r.h.s. is monotonically decreasing with $\alpha_{\min}$. Therefore, since $\alpha_{\min} \in (0,1)$, we conclude

$$\varepsilon_{\min}(\alpha_{\max})^3 \leq \frac{96D\epsilon_{emb}^\gamma}{\gamma}, \quad \gamma = \begin{cases} \alpha_{\min}^* & \text{if } \alpha_{\min}^* < 1 \\ 1 & \text{otherwise.} \end{cases} \tag{C62}$$

Finally we will derive conditions for when $\alpha_{\min}^* < 1$. To do so, we generalise the constraint in Eq. C57 to

$$\frac{\beta_1 \ln D}{-\ln \epsilon_{emb}} \leq 1, \quad 8 \leq \beta_1. \tag{C63}$$

Eq. (C61) can now be upper bounded by

$$\alpha_{\min}^* = \frac{\ln(96D)}{-2\ln \epsilon_{emb}} + \frac{1}{2}\sqrt{\left(\frac{\ln(96D)}{\ln \epsilon_{emb}}\right)^2 + 6\frac{\ln(1-1/d_S)}{\ln \epsilon_{emb}}} \leq \frac{\ln(96D)}{2\beta_1 \ln D} + \frac{1}{2}\sqrt{\left(\frac{\ln(96D)}{\beta_1 \ln D}\right)^2 - 6\frac{\ln(1-1/d_S)}{\beta_1 \ln D}} \tag{C64}$$

$$= \frac{\ln(96)}{2\beta_1 \ln(D_{Cat}d_S)} + \frac{1}{2\beta_1} + \frac{1}{2}\sqrt{\left(\frac{\ln(96)}{\beta_1 \ln(D_{Cat}d_S)} + \frac{1}{\beta_1}\right)^2 - 6\frac{\ln(1-1/d_S)}{\beta_1 \ln(D_{Cat}d_S)}} \tag{C65}$$

Thus recalling $D = D_{Cat}d_S$ and noting that Eq. (C65) is monotonically decreasing in D and d_S, we conclude that for all $D_{Cat} \geq D_{Cat}^*$ and $d_S \geq d_S^*$,

$$\alpha_{\min}^* \leq \frac{\ln(96)}{2\beta_1 \ln(D_{Cat}^* d_S^*)} + \frac{1}{2\beta_1} + \frac{1}{2}\sqrt{\left(\frac{\ln(96)}{\beta_1 \ln(D_{Cat}^* d_S^*)} + \frac{1}{\beta_1}\right)^2 - 6\frac{\ln(1-1/d_S^*)}{\beta_1 \ln(D_{Cat}^* d_S^*)}}. \tag{C66}$$

Therefore, for $\beta_1 = 10$, $D_{Cat}^* = 1$, $d_S^* = 2$; Eq. (C66) gives $\alpha_{\min}^* \leq 0.921\ldots$. For larger values of D_{Cat}^*, d_S^*, we can use $\beta_1 = 8$ and still achieve $\alpha_{\min}^* \leq 1$. We will thus assume

$$\frac{10\ln D}{-\ln \epsilon_{emb}} \leq 1 \tag{C67}$$

in the rest of this proof. We can now write Eq. (C62) in the form

$$\varepsilon_{\min}(\alpha_{\min})^3 \le \frac{96 D \epsilon_{emb}^{\alpha_{\min}^*}}{\alpha_{\min}^*} = \frac{96 D (-2) \ln(\epsilon_{emb})}{\ln(96D) + \sqrt{\ln^2(96D) + 6\left(\ln \epsilon_{emb}\right)\left(\ln(1 - 1/d_{\rm S})\right)}} \, e^{-\alpha_{\min}^* \ln \epsilon_{emb}} \tag{C68}$$

$$= \frac{96 D (-2) \ln(\epsilon_{emb})}{\ln(96D) + \sqrt{\ln^2(96D) + 6\left(\ln \epsilon_{emb}\right)\left(\ln(1 - 1/d_{\rm S})\right)}} \sqrt{e^{-\ln(96D) - \sqrt{\ln^2(96D) + 6(\ln \epsilon_{emb})(\ln(1 - 1/d_{\rm S}))}}}. \tag{C69}$$

We now observe that in the large D limit, if $\ln^2(96D) + 6\left(\ln \epsilon_{emb}\right)\left(\ln(1 - 1/d_{\rm S})\right) \approx \ln^2(96D)$ then the upper bound Eq. (C69) is approximately proportional to $(-\ln \epsilon_{emb})/\ln D$. This quantity is necessarily greater than 10 due to constraint Eq. (C67), and thus cannot be arbitrarily small. We will thus demand

$$\left(\ln \epsilon_{emb}\right)\left(\ln(1 - 1/d_{\rm S})\right) \ge \ln^2(D), \tag{C70}$$

[14] in order to have a non-trivial bound. Thus using Lemma 19, if follows from Eq. (C69),

$$\varepsilon_{\min}(\alpha_{\min})^3 \le \frac{96 D (-2) \ln(\epsilon_{emb})}{\sqrt{6\left(\ln \epsilon_{emb}\right)\left(\ln(1 - 1/d_{\rm S})\right)}} \sqrt{e^{-\ln(96D)} \, e^{-\sqrt{\ln^2(96D) + 6(\ln \epsilon_{emb})(\ln(1 - 1/d_{\rm S}))}}} \tag{C71}$$

$$\le \frac{96 D (-2) \ln(\epsilon_{emb})}{\sqrt{6\left(\ln \epsilon_{emb}\right)\left(\ln(1 - 1/d_{\rm S})\right)}} \sqrt{e^{-\ln(96D)} \frac{5^6 \, e^{-\sqrt{\ln^2(96D)}}}{\left((\ln \epsilon_{emb})(\ln(1 - 1/d_{\rm S}))\right)^4}} \tag{C72}$$

$$= 2 \frac{5^3}{\sqrt{6}} \frac{1}{(-\ln(1 - 1/d_{\rm S}))^{5/2} (-\ln \epsilon_{emb})^{3/2}} \tag{C73}$$

$$\tag{C74}$$

Furthmore, we can use the standard inequality $\ln(1 - x) \le -x$ for all $x < 1$ (which is sharp for small x.) with the identidication $x = 1/d_{\rm S}$, to achieve

$$\varepsilon_{\min}(\alpha_{\min})^3 \le 2 \frac{5^3}{\sqrt{6}} \frac{d_{\rm S}^{5/2}}{(-\ln \epsilon_{emb})^{3/2}} \tag{C75}$$

thus

$$\varepsilon_{\min}(\alpha_{\min}) \le 5 \frac{d_{\rm S}^{5/6}}{\sqrt{-\ln \epsilon_{emb}}}. \tag{C76}$$

Therefore, by comparing Eqs.(C57), (C76) we see that

$$\max\left\{\varepsilon_{\min}(\alpha_{\min}), \varepsilon_{\max}(\alpha_{\max})\right\} \le 5 \sqrt{\frac{f(D_{\rm Cat}, d_{\rm S})}{-\ln \epsilon_{emb}} + D \epsilon_{emb}^{1/6}}, \tag{C77}$$

where

$$f(D_{\rm Cat}, d_{\rm S}) := \max\left\{4\ln(D_{\rm Cat} d_{\rm S})\ln d_{\rm S}, \, d_{\rm S}^{5/3}\right\} = \begin{cases} 4\ln(D_{\rm Cat} d_{\rm S})\ln d_{\rm S} & \text{if } D_{\rm Cat} \ge \frac{1}{d_{\rm S}} \exp\left[\frac{d_{\rm S}^{5/3}}{4 \ln d_{\rm S}}\right] \\ d_{\rm S}^{5/3} & \text{otherwise} \end{cases} \tag{C78}$$

as long as constraints Eqs. (C67), (C70) are both satisfied. Taking into account the expression for $\bar{\epsilon}_{Tmid}$ in Eq. (C36),

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[14]This choice is so that we can use lemma 19, but one could make other choices if one made a different version of the bound.

have the bound,

$$\max\left\{\varepsilon_{\min}(\alpha_{\min}), \varepsilon_{\max}(\alpha_{\max}), \bar{\epsilon}_{Tmid}\right\} \leq 5\sqrt{\frac{f(D_{\mathrm{Cat}}, d_{\mathrm{S}})}{-\ln \epsilon_{emb}} + D\epsilon_{emb}^{1/6} + 5^{-2}\left(-1024\,D^2\sqrt{\frac{\epsilon_{emb}}{D}}\ln\sqrt{\frac{\epsilon_{emb}}{D}}\right)^{\frac{2}{3}}} \tag{C79}$$

$$< 5\sqrt{\frac{f(D_{\mathrm{Cat}}, d_{\mathrm{S}})}{-\ln \epsilon_{emb}} + D\epsilon_{emb}^{1/6} + 5\left(D^2\sqrt{\frac{\epsilon_{emb}}{D}}\ln\sqrt{\frac{D}{\epsilon_{emb}}}\right)^{\frac{2}{3}}} \tag{C80}$$

$$< 5\sqrt{\frac{d_{\mathrm{S}}^{5/3} + 4(\ln D)\ln d_{\mathrm{S}}}{-\ln \epsilon_{emb}} + D\epsilon_{emb}^{1/6} + 5\left(D^2\sqrt{\frac{\epsilon_{emb}}{D}}\ln\sqrt{\frac{D}{\epsilon_{emb}}}\right)^{\frac{2}{3}}}, \tag{C81}$$

if constraints Eqs. (C67), (C70) are satisfied. Thus by setting ϵ_{res} (defined via Eq. C43) to

$$\epsilon_{res} = 5\sqrt{\frac{d_{\mathrm{S}}^{5/3} + 4(\ln D)\ln d_{\mathrm{S}}}{-\ln \epsilon_{emb}} + D\epsilon_{emb}^{1/6} + 5\left(D^2\sqrt{\frac{\epsilon_{emb}}{D}}\ln\sqrt{\frac{D}{\epsilon_{emb}}}\right)^{\frac{2}{3}}}, \tag{C82}$$

we conclude the proof. ∎

2. Lemmas on norms and fidelity

This lemma says that if a state is close to a product, then it is also close to a product of its reductions.

Lemma 9. *We have for arbitrary states ρ_{AB}, η_{A}, η_{B} and pure state ψ_{B},*

$$\|\rho_{\mathrm{AB}} - \rho_{\mathrm{A}} \otimes \eta_{\mathrm{B}}\|_1 \leq 2\|\rho_{\mathrm{AB}} - \eta_{\mathrm{A}} \otimes \eta_{\mathrm{B}}\|_1 \tag{C83}$$

and

$$\|\rho_{\mathrm{AB}} - \rho_{\mathrm{A}} \otimes \rho_{\mathrm{B}}\|_1 \leq 3\|\rho_{\mathrm{AB}} - \eta_{\mathrm{A}} \otimes \eta_{\mathrm{B}}\|_1 \tag{C84}$$

Proof. We have

$$\|\rho_{\mathrm{AB}} - \rho_{\mathrm{A}} \otimes \eta_{\mathrm{B}}\|_1 \leq \|\rho_{\mathrm{AB}} - \eta_{\mathrm{A}} \otimes \eta_{\mathrm{B}}\|_1 + \|\eta_{\mathrm{A}} \otimes \eta_{\mathrm{B}} - \rho_{\mathrm{A}} \otimes \eta_{\mathrm{B}}\|_1 =$$
$$= \|\rho_{\mathrm{AB}} - \eta_{\mathrm{A}} \otimes \eta_{\mathrm{B}}\|_1 + \|\eta_{\mathrm{A}} - \rho_{\mathrm{A}}\|_1 \leq 2\|\rho_{\mathrm{AB}} - \eta_{\mathrm{A}} \otimes \eta_{\mathrm{B}}\|_1 \tag{C85}$$

The second inequality we prove in a similar way. ∎

Next lemma says that, if a reduced state is close to a pure state then the total state is close to a product (of its reduction tensored with the pure state)

Lemma 10. *We have*

$$\|\rho_{\mathrm{AB}} - \rho_{\mathrm{A}} \otimes \psi_{\mathrm{B}}\|_1 \leq 2\sqrt{\|\rho_{\mathrm{B}} - \psi_{\mathrm{B}}\|_1} \tag{C86}$$

Proof. Consider $F(\rho_{\mathrm{B}}, \psi_{\mathrm{B}})$. We have

$$F(\rho_{\mathrm{B}}, \psi_{\mathrm{B}}) = F(\phi_{\mathrm{ABC}}, \psi_{\mathrm{AC}} \otimes \psi_{\mathrm{B}}) \tag{C87}$$

where ϕ_{ABC} is a purification of ρ_{B} which we are free to choose the way we want, and ψ_{AC} is some pure state. By data processing we have

$$F(\phi_{\mathrm{ABC}}, \psi_{\mathrm{AC}} \otimes \psi_{\mathrm{B}}) \leq F(\rho_{\mathrm{AB}}, \sigma_{\mathrm{A}} \otimes \psi_{\mathrm{B}}) \tag{C88}$$

where σ_{A} is reduction of ψ_{AC}. Using it and twice Fuchs-Graaf we thus get:

$$\|\rho_{\mathrm{AB}} - \sigma_{\mathrm{A}} \otimes \psi_{\mathrm{B}}\|_1 \leq 2\sqrt{1 - F^2(\rho_{\mathrm{AB}}, \sigma_{\mathrm{A}} \otimes \psi_{\mathrm{B}})} \leq 2\sqrt{1 - F^2(\rho_{\mathrm{B}}, \psi_{\mathrm{B}})} \leq 2\sqrt{\|\rho_{\mathrm{B}} - \psi_{\mathrm{B}}\|_1} \tag{C89}$$

Now, the proposition says that if two states have closed corresponding reductions, and one of the reductions is close to a pure state, then the states are close to one another.

∎

Proposition 11. *Suppose that* $\|\rho_A - \sigma_A\|_1 \leq \epsilon_1$, $\|\rho_B - \sigma_B\|_1 \leq \epsilon_2$, $\|\sigma_B - \psi_B\|_1 \leq \epsilon_3$. *Then*

$$\|\rho_{AB} - \sigma_{AB}\|_1 \leq 2\sqrt{\epsilon_1} + 2\sqrt{\epsilon_1 + \epsilon_2} + \epsilon_3. \tag{C90}$$

Proof. By triangle inequality we have

$$\|\rho_B - \psi_B\|_1 \leq \epsilon_1 + \epsilon_2. \tag{C91}$$

By lemma 10 we have

$$\|\sigma_{AB} - \sigma_A \otimes \psi_B\|_1 \leq 2\sqrt{\|\sigma_B - \psi_B\|_1}$$
$$\|\rho_{AB} - \rho_A \otimes \psi_B\|_1 \leq 2\sqrt{\|\rho_B - \psi_B\|_1}$$
$$\tag{C92}$$

Sandwiching $\|\rho_{AB} - \sigma_{AB}\|_1$ with the above, we finish the proof. ∎

3. From approximate to strict inequalities

a. Main lemmas

Lemma 12 (From approximate to strict inequalities through smoothing)**.** *Let f be a concave, non negative function of $p \in \mathcal{P}_d$ such that $f(p) < f(I_d/d)$ for any $p \neq I_d/d$. Suppose that for some $\eta > 0$ we have*

$$f(p) \leq f(q) + \eta \tag{C93}$$

Then for ϵ satisfying $\epsilon \leq 1/2$ and

$$\epsilon \geq \min_{\delta > 0} \max\left\{\delta, \frac{2\eta}{f(I_d/d) - \max_{\|\rho - I_d/d\|_1 \geq \delta/2} f(\rho)}\right\} \tag{C94}$$

we have

$$f(p) \leq f(\tilde{q}(\epsilon)) - \min\{\eta, f(I_d/d) - f(p)\} \tag{C95}$$

where $\tilde{q}(\epsilon)$ is given by

$$\tilde{q}(\epsilon) = \begin{cases} I_d/d & \text{when } \|q - I_d/d\|_1 < \epsilon \\ (1-\epsilon)q + \epsilon I_d/d & \text{when } \|q - I_d/d\|_1 \geq \epsilon, \end{cases} \tag{C96}$$

Proof. If state q satisfies $\|q - I_d/d\|_1 < \epsilon$ then $\tilde{q}(\epsilon) = I_d/d$. Then,

$$f(p) = f(I_d/d) + f(p) - f(I_d/d) = f(\tilde{q}(\epsilon)) + (f(p) - f(I_d/d)). \tag{C97}$$

Thus trivially

$$f(p) \leq f(\tilde{q}(\epsilon)) - (f(I_d/d) - f(p)) \leq f(\tilde{q}(\epsilon)) - \min\{\eta, f(I_d/d) - f(p)\}. \tag{C98}$$

Now suppose that

$$\|q - I_d/d\|_1 \geq \epsilon. \tag{C99}$$

From concavity of f we have

$$f(\tilde{q}_\epsilon) \geq (1-\epsilon)f(q) + \epsilon f(I_d/d) \tag{C100}$$

hence

$$f(q) \leq \frac{f(q_\epsilon)}{1-\epsilon} - \frac{\epsilon}{1-\epsilon}f(I_d/d). \tag{C101}$$

Then from (C93)

$$f(p) \leq f(q) + \eta \leq \frac{f(\tilde{q}_\epsilon)}{1-\epsilon} - \frac{\epsilon}{1-\epsilon} f(I_d/d) + \eta = f(\tilde{q}(\epsilon)) + \frac{\epsilon}{1-\epsilon} \big(f(\tilde{q}(\epsilon)) - f(I_d/d)\big) + \eta \leq f(\tilde{q}(\epsilon)) + \epsilon \big(f(\tilde{q}(\epsilon)) - f(I_d/d)\big) + \eta.$$
(C102)

Thus it remains to show that the ϵ satisfying (C94) and $\epsilon \leq 1/2$ satisfies

$$\epsilon \big(f(I_d/d) - f(\tilde{q}(\epsilon))\big) \geq 2\eta.$$
(C103)

To this end, note that (C94) implies

$$\epsilon \geq \frac{2\eta}{f(I_d/d) - \max_{\|\rho - I_d/d\|_1 \geq \epsilon/2} f(\rho)}.$$
(C104)

Then, note that

$$\|\tilde{q}(\epsilon) - I_d/d\|_1 = \|(1-\epsilon)q + \epsilon I_d/d - I_d/d\|_1 = (1-\epsilon)\|q - I_d/d\|_1 \geq (1-\epsilon)\epsilon.$$
(C105)

Thus, for $\epsilon \leq 1/2$ we have $\|\tilde{q}_\epsilon - I_d/d\|_1 \geq \epsilon/2$, so that

$$f(\tilde{q}(\epsilon)) \leq \max_{\|\rho - I_d/d\|_1 \geq \epsilon/2} f(\rho),$$
(C106)

hence (C104) implies

$$\epsilon \geq \frac{2\eta}{f(I_d/d) - f(\tilde{q}(\epsilon))},$$
(C107)

which is equivalent to (C103). ∎

Lemma 13. *Let g be convex, non-negative function with the domain $D \in R$. Let also g be multiplicative, i.e. $g(xy) = g(x)g(y)$ and $g(1) = 1$. Let us denote $f_g(p) = \sum_{i=1}^d g(p_i)$. Then for any probability distribution $p \in \mathcal{P}_d$ satisfying $\|p - I_d/d\|_1 \geq \delta$ we have*

$$f_g(p) - f_g(I_d/d) \geq \frac{f_g(I_d/d)}{f_g(I_2/2)} \left(f_g(p_{(2)}^\delta) - f_g(I_2)\right),$$
(C108)

where

$$p_{(2)}^\delta = \left\{ \frac{1 + \delta/2}{2}, \frac{1 - \delta/2}{2} \right\}.$$
(C109)

If g is concave, and otherwise satisfies all the above assumptions we have

$$f_g(I_d/d) - f_g(p) \geq \frac{f_g(I_d/d)}{f_g(I_2/2)} \left(f_g(I_2/2) - f_g(p_{(2)}^\delta)\right).$$
(C110)

Remark 14. Lemma 13 applies to $g(x) = x^\alpha$ for $\alpha > 0$. The function f_g for such g we will denote by $\tilde{f}_\alpha$. We then obtain:

$$\tilde{f}_\alpha(p) - \tilde{f}_\alpha(I_d/d) \geq \frac{d^{1-\alpha}}{2^{1-\alpha}} \left(\tilde{f}_\alpha(p_{(2)}^\delta) - \tilde{f}_\alpha(I_2/2)\right)$$
(C111)

for $\alpha > 1$ (i.e. when x^α is convex), and

$$\tilde{f}_\alpha(I_d/d) - \tilde{f}_\alpha(p) \geq \frac{d^{1-\alpha}}{2^{1-\alpha}} \left(\tilde{f}_\alpha(I_2/2) - \tilde{f}_\alpha(p_{(2)}^\delta)\right)$$
(C112)

for $\alpha \in (0, 1)$ (i.e. when x^α is concave). From these, one gets:

$$T_\alpha(I_2/d) - T_\alpha(p) \geq \frac{d^{(1-\alpha)}}{2^{(1-\alpha)}} \left(T_\alpha(I_2/2) - T_\alpha(p_{(2)}^\delta)\right)$$
(C113)

for all $\alpha > 0$ (for $\alpha = 1$ it is obtained by continuity, and gives inequality for Shannon entropies).

Proof of Lemma 13. Let g be concave. Then $f_g(p)$ is concave as a function of probability distribution $p \in \mathcal{P}_d$. For any distribution p we consider its twirled version, that depends on just two parameters: the number k of p_i's greater than or equal to $1/d$ and $\delta = \|p - I_d/d\|_1$.

$$\tilde{p} = \left\{ \underbrace{\frac{1}{d} + \frac{\delta/2}{k}, \ldots, \frac{1}{d} + \frac{\delta/2}{k}}_{k}, \underbrace{\frac{1}{d} - \frac{\delta/2}{d-k}, \ldots, \frac{1}{d} + \frac{\delta/2}{d-k}}_{d-k} \right\}. \tag{C114}$$

The $\tilde{p}$ can be obtained from p by mixture of permutations (we consider two subset of p_i': those larger than $1/d$ and those smaller than or equal, and randomly permute elements within each of the subsets. Hence by concavity we have

$$f_g(p) \leq f_g(\tilde{p}) = d(r_1 g(x_1) + r_2 g(x_2)) \tag{C115}$$

where we denoted $r_1 = k/d, r_2 = (d-k)/d$ and $x_1 = 1/d + \delta/2d, x_2 = 1/d - \delta/(2(d-k))$. Note that $r_1 + r_2 = 1$, $r_1 x_1 + r_2 x_2 = 1/d$. One finds (see Fig. 4) that if $x_2 \leq \tilde{x}_2 \leq 1/d \leq \tilde{x}_1 \leq x_1$ then, due to concavity of the function f_g,
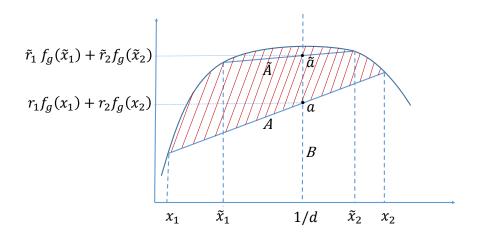
FIG. 4: Geometric proof of the inequality (C116). Due to concavity of f the interval A between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ together with the part of the graph of the function laying between these two points enclose a convex body (indicated in red). Therefore the interval $\tilde{A}$ between the points $(\tilde{x}_1, f(\tilde{x}_1))$ and $(\tilde{x}_2, f(\tilde{x}_2))$ must lie within the body. By assumption, the latter interval has nonempty intersection $\tilde{a}$ with the line $x = 1/d$. This intersection must be therefore above the intersection a of the latter line with the interval A, which means that inequality (C116) holds.

we have

$$r_1 f_g(x_1) + r_2 f_g(x_2) \leq \tilde{r}_1 f_g(\tilde{x}_1) + \tilde{r}_2 f_g(\tilde{x}_2) \tag{C116}$$

provided $\tilde{r}_1 + \tilde{r}_2 = 1$, $\tilde{r}_1 \tilde{x}_1 + \tilde{r}_2 \tilde{x}_2 = 1/d$. Since $1 \leq k \leq d$ we can choose

$$\tilde{x}_1 = \frac{1}{d} + \frac{\delta/2}{d}, \quad \tilde{x}_2 = \frac{1}{d} - \frac{\delta/2}{d} \tag{C117}$$

and $\tilde{r}_1 = \tilde{r}_2 = \frac{1}{2}$. We thus obtain

$$\begin{aligned} f_g(p) &\leq d \left[\frac{1}{2} g \left(\frac{1}{d} + \frac{\delta/2}{d} \right) + \frac{1}{2} g \left(\frac{1}{d} - \frac{\delta/2}{d} \right) \right] = \\ &= \frac{d}{2} \frac{g(2)}{g(d)} \left[g \left(\frac{1+\delta/2}{2} \right) + g \left(\frac{1-\delta/2}{2} \right) \right] = \\ &= \frac{f_g(I_d/d)}{f_g(I_2/2)} f_g(p_{(2)}^\delta) \end{aligned} \tag{C118}$$

where multiplicativity of g was used. From this we obtain (C112). Eq. (C111) is proven similarly. ∎

Lemma 15. *Let $p_{\min}(p) = \min_i p_i$ and $p_{\max}(p) = \max_i p_i$. Then for arbitrary p such that $\|p - I_d/d\|_1 = \delta$ we have*

$$p_{\min}(p) \leq \frac{p_{\min}(I_d/d)}{p_{\min}(I_2/2)} p_{\min}(p_{(2)}^\delta) = \frac{2}{d} p_{\min}(p_{(2)}^\delta) \tag{C119}$$

and

$$p_{\max}(p) \geq \frac{p_{\max}(I_d/d)}{p_{\max}(I_2/2)} p_{\max}(p_{(2)}^\delta) = \frac{2}{d} p_{\max}(p_{(2)}^\delta) \tag{C120}$$

where

$$p_{(2)}^\delta = \left\{ \frac{1 + \delta/2}{2}, \frac{1 - \delta/2}{2} \right\}. \tag{C121}$$

Proof. We use again the twirled version of p of Eq. (C114) By convexity of $p_{\max}$, we have

$$p_{\max}(p) \geq p_{\max}(\tilde{p}) \geq \frac{1}{d} + \frac{\delta/2}{d} = \frac{2}{d} p_{\max} \tag{C122}$$

Similarly one proves for $p_{\min}$. ■

Proposition 16. *Let $p, q \in \mathcal{P}_d$ and define*

$$\tilde{q}(\epsilon) = \begin{cases} I_d/d & \text{when } \|q - I_d/d\|_1 < \epsilon \\ (1 - \epsilon)q + \epsilon I_d/d & \text{when } \|q - I_d/d\|_1 \geq \epsilon \end{cases} \tag{C123}$$

Denote also

$$\epsilon_T(\alpha) = (16 \, \eta_\alpha \, d^{\alpha-1})^{\frac{1}{3}} \quad \text{for } \alpha \geq 1 \tag{C124}$$

$$\epsilon_T(\alpha) = (16 \, \eta_\alpha \, d^{\alpha-1} \alpha^{-1})^{\frac{1}{3}} \quad \text{for } \alpha \in (0, 1) \tag{C125}$$

$$\epsilon_\infty(\alpha) = 4\sqrt{\frac{\ln d}{\alpha} + \eta_\infty} \quad \text{for } \alpha > 1 \tag{C126}$$

$$\epsilon_0(\alpha) = \left(\frac{d-1}{d} \right)^{\frac{1}{2\alpha}} \quad \text{for } \alpha \in (0, 1). \tag{C127}$$

Now assuming, that all the above epsilons ($\epsilon_T, \epsilon_\infty, \epsilon_0$) are no greater than $1/2$ we have

(i) for $\alpha > 1$

$$S_\infty(p) \leq S_\infty(q) + \eta_\infty, \quad \text{implies} \quad S_\alpha(p) \leq S_\alpha(\tilde{q}(\epsilon_\infty(\alpha)) - \min\{\eta_\infty, \ln d - S_1(p)\} \tag{C128}$$

(ii) For $\alpha > 0$

$$T_\alpha(p) \leq T_\alpha(q) + \eta_\alpha, \quad \text{implies} \quad T_\alpha(p) \leq T_\alpha(\tilde{q}(\epsilon_T(\alpha)) - \min\{\eta_\alpha, T_\alpha(I_d/d) - T_\alpha(p)\} \tag{C129}$$

(iii) For $\alpha \in (0, 1)$, for p not full rank we have

$$S_\alpha(p) \leq S_\alpha(\tilde{q}(\epsilon_0(\alpha))) - \frac{1}{2} \ln \frac{d}{d-1} \tag{C130}$$

Proof. Proof of (i). Fix some ϵ, and consider $\tilde{q}(\epsilon)$ given by (C123). Consider first the case when $\|q - I_d/d\|_1 \leq \epsilon$. Let us show, that in this case, (i) holds for any ϵ. Indeed, we have $\tilde{q}(\epsilon) = I_d/d$, hence

$$S_\alpha(p) = S_\alpha(\tilde{q}(\epsilon)) - (\ln d - S_\alpha(p)). \tag{C131}$$

Since S_α is monotonically decreasing in α, we can replace on the right-hand-side $S_\alpha(p)$ with $S_1(p)$

$$S_\alpha(p) \leq S_\alpha(\tilde{q}(\epsilon)) - (S(I_d/d) - S_1(p)) \tag{C132}$$

which is what we want. Now we turn to less trivial case when $\|q - I_d/d\|_1 \geq \epsilon$. We write

$$S_\infty(\tilde{q}(\epsilon)) - S_\infty(q) = -\ln\left((1-\epsilon)q_{\max} + \epsilon/d\right) + \ln q_{\max} = -\ln\left((1-\epsilon) + \frac{\epsilon}{dq_{\max}}\right). \tag{C133}$$

By lemma 15 and $e^{-x} \geq 1 - x$ we bound it further as follows

$$S_\infty(\tilde{q}(\epsilon)) - S_\infty(q) \geq -\ln\left(1 - \epsilon\left(1 - \frac{1}{1+\epsilon/2}\right)\right) \geq \epsilon\frac{\epsilon/2}{1+\epsilon/2} \geq \frac{\epsilon^2}{4} \tag{C134}$$

We now use the fact that for $\alpha > 1$

$$S_\alpha(p) \leq \frac{\alpha}{1-\alpha}S_\infty(p). \tag{C135}$$

We get

$$S_\alpha(q_\epsilon) \geq S_\infty(q_\epsilon) \geq S_\infty(q) + \frac{\epsilon^2}{4} \geq S_\infty(p) + \frac{\epsilon^2}{4} - \eta_\infty \geq \frac{\alpha-1}{\alpha}S_\alpha(p) + \frac{\epsilon^2}{4} - \eta_\infty \geq S_\alpha(p) - \frac{\ln d}{\alpha} + \frac{\epsilon^2}{4} - \eta_\infty, \tag{C136}$$

where the second inequality holds by assumption of (i), while the last inequality comes from $S_\alpha(p) \leq \ln d$ for any $p \in \mathcal{P}_d$. Thus, provided

$$-\frac{\ln d}{\alpha} + \frac{\epsilon^2}{4} > 2\eta_\infty, \tag{C137}$$

we have $S_\alpha(q_\epsilon) > S_\alpha(p) + \eta_\infty$. If we now take $\epsilon = \epsilon_\infty(\alpha)$ we see that (C137) is satsified, hence we obtain that for $\|q - I_d/d\|_1 \geq \epsilon_\infty(\alpha)$

$$S_\alpha(p) \leq S_\alpha(\tilde{q}(\epsilon)) - \eta_\infty. \tag{C138}$$

Using this and (C132) we get that for arbitrary q we have

$$S_\alpha(p) \leq S_\alpha(\tilde{q}(\epsilon)) - \min\left\{\eta_\infty, \ln d - S_1(p)\right\}. \tag{C139}$$

This ends the proof of part (i).

Proof of (ii). Since T_α is concave function (in probability distributions) for all $\alpha > 0$ from lemma 12 we obtain that for ϵ given by

$$\epsilon = \min_\delta \max\left\{\delta, \frac{2\eta}{T_\alpha(I_d/d) - \max_p T_\alpha(p)}\right\}, \tag{C140}$$

where maximum is taken over all $p \in \mathcal{P}_d$ satisfying $\|p - I_d/d\|_1 \geq \delta/2$, we have

$$T_\alpha(p) \leq T_\alpha(\tilde{q}_\epsilon) - \min\left\{\eta_\alpha, T_\alpha(I_d/d) - T_\alpha(p)\right\}. \tag{C141}$$

By Eq. C113 (a consequence of lemma 13) we have for such p

$$T_\alpha(I_d/d) - \max_p T_\alpha(p) \geq \left(\frac{d}{2}\right)^{1-\alpha}\left(T_\alpha(I_2/2) - T_\alpha(p_{(2)}^\delta)\right) \tag{C142}$$

for all $\alpha > 0$. The right hand side can be expressed in terms of Hellinger relative entropy (Eq. B7) by virtue of Eq. (B12)

$$\left(\frac{d}{2}\right)^{1-\alpha}\left(T_\alpha(I_2/2) - T_\alpha(p_{(2)}^\delta)\right) = d^{1-\alpha}\mathcal{H}_\alpha(p_{(2)}^\delta|I_2/2). \tag{C143}$$

We thus obtained that for all $\alpha > 0$

$$\epsilon \leq \min_\delta \max\left\{\delta, \frac{2\eta}{d^{1-\alpha}\mathcal{H}_\alpha(p_{(2)}^\delta|I_2/2)}\right\}. \tag{C144}$$

Now from lemma 21 we have

$$\mathcal{H}_\alpha(p_{(2)}^\delta|I_2/2) \geq \frac{\alpha}{8}\delta^2 \tag{C145}$$

for $\alpha \in (0,1)$ and

$$\mathcal{H}_\alpha(p_{(2)}^\delta|I_2/2) \geq \frac{1}{8}\delta^2 \tag{C146}$$

for $\alpha > 1$. Thus for $\alpha > 1$

$$\epsilon \leq \min_\delta\left\{\delta, (16\eta d^{\alpha-1})/\delta^2\right\} = (16\eta d^{\alpha-1})^{1/3} \tag{C147}$$

and for $\alpha \in (0,1)$

$$\epsilon \leq \min_\delta\left\{\delta, (16\eta d^{\alpha-1}\alpha^{-1})/\delta^2\right\} = (16\eta d^{\alpha-1}\alpha^{-1})^{1/3}. \tag{C148}$$

The case $\alpha = 1$ we get by taking the limit $\alpha \to 1$.

Proof of (iii). For $\alpha > 1$ for all full rank distributions $p \in \mathcal{P}_d$ we have

$$S_\alpha(p) \geq dp_{\min}^\alpha \tag{C149}$$

where $p_{\min}$ is the minimal element of p. For distribution $\tilde{q}(\epsilon_0) = (1 - \epsilon_0)q + \epsilon_0 I_d/d$, since $p_{\min}(\tilde{q}(\epsilon_0)) \geq \epsilon_0/d$, irrespectively of what was q we obtain

$$S_\alpha(\tilde{q}(\epsilon_0)) \geq \frac{1}{1-\alpha}\ln\left(d\left(\frac{\epsilon_0}{d}\right)^\alpha\right) \tag{C150}$$

Now, since p was assumed to be not full rank, we have

$$S_\alpha(p) \leq \ln(d-1) \tag{C151}$$

Now, (C127) assures that

$$\frac{1}{1-\alpha}\ln\left(d\left(\frac{\epsilon_0}{d}\right)^\alpha\right) - \ln(d-1) \geq \frac{1}{2}\ln\frac{d}{d-1} \tag{C152}$$

for $\alpha \leq \alpha_0$. Using it, we then get

$$S_\alpha(p) = S_\alpha(\tilde{q}(\epsilon_0)) - (S_\alpha(\tilde{q}(\epsilon_0)) - S_\alpha(p)) \leq S_\alpha(\tilde{q}(\epsilon_0)) - \left(\frac{1}{1-\alpha}\ln\left(d\left(\frac{\epsilon_0}{d}\right)^\alpha\right) - \ln(d-1)\right) \leq S_\alpha(\tilde{q}(\epsilon_0)) - \frac{1}{2}\ln\frac{d}{d-1} \tag{C153}$$

i.e. $S_\alpha(p) \geq S_\alpha(\tilde{q}) + \frac{1}{2}\ln\frac{d}{d-1}$. ∎

Lemma 17. *For all probability distributions q, and $\epsilon \leq \epsilon'$, $\epsilon, \epsilon' \in (0,1)$ the Renyi and Tsalis entropies satisfy*

$$T_\alpha(\tilde{q}(\epsilon)) \leq T_\alpha(\tilde{q}(\epsilon')) \tag{C154}$$
$$S_\alpha(\tilde{q}(\epsilon)) \leq S_\alpha(\tilde{q}(\epsilon')) \tag{C155}$$

for all $\alpha \geq 0$, where $\tilde{q}(\epsilon)$ (introduced in Prop. 16), is given by

$$\tilde{q}(\epsilon) = \begin{cases} I_d/d & \text{if } \|q - I_d/d\|_1 < \epsilon \\ (1-\epsilon)q + \epsilon I_d/d & \text{if } \|q - I_d/d\|_1 \geq \epsilon \end{cases} \tag{C156}$$

Proof. We will start by proving Eq. (C155) first.

The proof of Eq. (C155) will be divided into two sub cases. We start with the easiest case.

Case 1: $\|q - I_d/d\|_1 < \epsilon$

It follows that $S_\alpha(\tilde{q}(\epsilon)) = S_\alpha(I_d/d)$ and also since $\|q - I_d/d\|_1 < \epsilon'$ we have $S_\alpha(\tilde{q}(\epsilon')) = S_\alpha(I_d/d)$ and thus Eq. (C155) holds for this case.

Case 2: $\|q - I_d/d\|_1 \geq \epsilon$.

It follows that $S_\alpha(\tilde{q}(\epsilon)) = S_\alpha((1-\epsilon)q + \epsilon I_d/d)$. We now have to further sub-divide into two possibilities.

Case 2.1: $\|q - I_d/d\|_1 < \epsilon' \implies S_\alpha(\tilde{q}(\epsilon')) = S_\alpha(I_d/d)$

Case 2.2: $\|q - I_d/d\|_1 \geq \epsilon' \implies S_\alpha(\tilde{q}(\epsilon')) = S_\alpha((1-\epsilon')q + \epsilon' I_d/d)$.

Therefore, for Case 2 we have to prove that

$$S_\alpha((1-\epsilon)q + \epsilon I_d/d) \leq S_\alpha((1-\epsilon')q + \epsilon' I_d/d) \tag{C157}$$
$$S_\alpha((1-\epsilon)q + \epsilon I_d/d) \leq S_\alpha(I_d/d) \tag{C158}$$

both hold under the quantifies stated in the Lemma. We first observe that Eq. (C157) implies Eq. (C158) by setting $\epsilon' = 1$. Thus we only need to prove Eq. (C157). For this, we first observe that

$$(1-\epsilon')q + \epsilon' I_d/d = \gamma((1-\epsilon)q + \epsilon I_d/d) + (1-\gamma)I_d/d, \tag{C159}$$

where $\gamma := (1-\epsilon')/(1-\epsilon) \in [0,1]$. Hence the vector $(1-\epsilon')q + \epsilon' I_d/d$ is a mixture of $(1-\epsilon)q + \epsilon I_d/d$ with the uniform distribution I_d/d. As such $(1-\epsilon)q + \epsilon I_d/d$ majorises $(1-\epsilon')q + \epsilon' I_d/d$, and since the Renyi entropy is Schur concave for all $\alpha \geq 0$, Eq. (C157) follows by Schur concavity.

We now need to prove Eq. (C154) to complete the proof of the lemma. From the definitions of the Renyi and Tsalis entropy, it follows

$$T_\alpha(S_\alpha) = \frac{\exp\left(\left(\frac{1-\alpha}{\alpha}\right)S_\alpha\right) - 1}{1-\alpha}, \tag{C160}$$

which is manifestly a non-decreasing function for all $\alpha > 0$, thus $S_\alpha(p) \leq S_\alpha(p')$ iff $T_\alpha(p) \leq T_\alpha(p')$, and Eq. (C154) follows from Eq. (C155). ∎

Lemma 18. *Let $p, p' \in \mathcal{P}_d$ Denote $\epsilon = \|p - p'\|_1$. Then*

- *For $\alpha \in (0, 1/2]$ we have*

$$|T_\alpha(p) - T_\alpha(p')| \leq 6d\left(\frac{\epsilon}{d}\right)^\alpha \tag{C161}$$

- *for $\alpha \in [1/2, 2]$ we have*

$$|T_\alpha(p) - T_\alpha(p')| \leq -32d\sqrt{\frac{\epsilon}{d}}\ln\sqrt{\frac{\epsilon}{d}} \quad \text{if } \epsilon \leq \frac{1}{32d^2} \tag{C162}$$

- *for $\alpha \in [2, \infty)$ we have*

$$|T_\alpha(p) - T_\alpha(p')| \leq 6\sqrt{d\epsilon} \tag{C163}$$

- *We have*

$$|S_\infty(p) - S_\infty(p')| \leq d\epsilon \tag{C164}$$

Proof. We will prove Eqs. (C161) to (C164) individual.

Proof of (C161). from (C193) we have for $\alpha \in (0, 1)$

$$|T_\alpha(p) - T_\alpha(p')| \leq 2\frac{\lceil\alpha\rceil}{|\alpha - 1|}d\left(\frac{\epsilon}{d}\right)^{\alpha/\lceil\alpha\rceil} \tag{C165}$$

We then estimate for $\alpha \in (0, 1/2]$

$$2\frac{\lceil\alpha\rceil}{|\alpha - 1|}d\left(\frac{\epsilon}{d}\right)^{\alpha/\lceil\alpha\rceil} \leq 4d\left(\frac{\epsilon}{d}\right)^\alpha \tag{C166}$$

Proof of (C162). From (C195) we have for $\alpha > 1$ and $\epsilon \leq \frac{1}{2e\lceil\alpha\rceil d^\alpha}$

$$|T_\alpha(p) - T_\alpha(p')| \leq 8\left[\epsilon \ln\left(\frac{d^{3/2}}{4}\right) - \epsilon \ln \epsilon\right] \tag{C167}$$

We then have

$$8\left[\epsilon \ln\left(\frac{d^{3/2}}{4}\right) - \epsilon \ln \epsilon\right] \leq -12d\frac{\epsilon}{d}\ln\frac{\epsilon}{d} \tag{C168}$$

so that for $\epsilon \leq \frac{1}{8d^2}$ and $1 \leq \alpha \leq 2$

$$|T_\alpha(p) - T_\alpha(p')| \leq -12d\frac{\epsilon}{d}\ln\frac{\epsilon}{d} \tag{C169}$$

We now combine it with (C194) which for $\alpha \in [1/2, 1]$ implies that for $\epsilon \leq d\left(\frac{1}{2ed}\right)^2 \leq \frac{1}{30d}$,

$$|T_\alpha(p) - T_\alpha(p')| \leq 4d\left[\left(\frac{3}{2\alpha} + 1\right)\left(\frac{\epsilon}{d}\right)^\alpha \ln d - \left(\frac{\epsilon}{d}\right)^\alpha \ln \epsilon\right] \tag{C170}$$

$$\leq -32d\sqrt{\frac{\epsilon}{d}}\ln\sqrt{\frac{\epsilon}{d}}. \tag{C171}$$

Taking worse of the two bounds we get that for $\alpha \in [1/2, 2]$ and $\epsilon \leq 1/(32d^2)$

$$|T_\alpha(p) - T_\alpha(p')| \leq -32d\sqrt{\frac{\epsilon}{d}}\ln\sqrt{\frac{\epsilon}{d}}. \tag{C172}$$

Proof of (C163). From (C193) for all $\alpha \in (0, 1) \cup (1, \infty)$ we have

$$|T_\alpha(p) - T_\alpha(p')| \leq 2\frac{\lceil\alpha\rceil}{|\alpha - 1|}d\left(\frac{\epsilon}{d}\right)^{\alpha/\lceil\alpha\rceil} \tag{C173}$$

Using $\alpha \geq 2$ and $\epsilon/d \leq 1$ (always true, since $d \geq 2$ and $\epsilon \leq 2$) we have

$$2\frac{\lceil\alpha\rceil}{|\alpha - 1|}d\left(\frac{\epsilon}{d}\right)^{\alpha/\lceil\alpha\rceil} \leq 6d\left(\frac{\epsilon}{d}\right)^{\alpha/(\alpha+1)} \leq 6d\sqrt{\frac{\epsilon}{d}} = 6\sqrt{\epsilon d} \tag{C174}$$

Proof of (C164).

$$|\ln p_{\max} - \ln p'_{\max}| = \left|\ln\left(\frac{|p_{\max} - p'_{\max}|}{\min\{p_{\max}, p'_{\max}\}} + 1\right)\right| \leq \frac{|p_{\max} - p'_{\max}|}{\min\{p_{\max}, p'_{\max}\}} \leq d\epsilon \tag{C175}$$

■

b. *Auxiliary lemmas*

Lemma 19. *Let $x \geq \ln^2(D)$, then for all $D \geq 2$,*

$$e^{-\sqrt{\ln^2(96D)+6x}} \leq 5^6\frac{e^{-\sqrt{\ln^2(96D)}}}{x^4}. \tag{C176}$$

Proof. We have that for all $D \geq 2$,

$$\frac{x^4}{5^6}e^{-\sqrt{\ln^2(96D)+6x}+\ln(96)+\ln(D)} \leq \frac{x^4}{5^6}e^{-\sqrt{\ln^2(96\cdot2)+6x}+\ln(96)+\sqrt{x}} \leq \frac{96}{5^6}x^4 e^{-\sqrt{27+6x}+\sqrt{x}} \tag{C177}$$

$$= \frac{96}{5^6}F(x). \tag{C178}$$

We now aim to find the maximum of $F(x) := x^4 \, e^{-\sqrt{27+6x}+\sqrt{x}}$ with domain $x \in [0,\infty)$. Since the extremal points are both zero, namely $F(0) = \lim_{x\to\infty} F(x) = 0$, the maximum will be one of the stationary points, we therefore want the solutions to

$$\frac{d}{dx}F(x) = \frac{e^{\sqrt{x}-\sqrt{6x+27}}x^3\left(-2\sqrt{3}x + \sqrt{x}\sqrt{2x+9} + 8\sqrt{2x+9}\right)}{2\sqrt{2x+9}} = 0. \tag{C179}$$

The only solution to $-2\sqrt{3}x + \sqrt{x}\sqrt{2x+9} + 8\sqrt{2x+9} = 0$ can be found analytically by hand (or using Mathematica's *Solve* routine) giving

$$x_0 :=$$

$$\frac{1}{2}\left(-\frac{\sqrt[3]{1719926784\sqrt{1149814}+17320375304957}}{60\ 5^{2/3}} - \frac{1}{300}\sqrt[3]{86601876524785 - 8599633920\sqrt{1149814}} + \frac{354075648}{625}\times\right.$$

$$\times\sqrt{\frac{3}{25\sqrt[3]{5\left(1719926784\sqrt{1149814}+17320375304957\right)} + 25\sqrt[3]{86601876524785 - 8599633920\sqrt{1149814}} + 2754793}}$$

$$\left.+\frac{2754793}{3750}\right)^{1/2} + \frac{941}{100} + \frac{1}{100\sqrt{\frac{3}{25\sqrt[3]{5\left(1719926784\sqrt{1149814}+17320375304957\right)}+25\sqrt[3]{86601876524785-8599633920\sqrt{1149814}}+2754793}}},$$

$$\tag{C180}$$

since x_0 is within the end points, namely $0 < x_0 < \infty$ and F evaluated at x_0 is larger than at the end points, i.e. $F(x_0) > F(0) = 0$, and $F(x_0) > \lim_{x\to\infty} F(x) = 0$; it must be a global maximum. Thus to conclude the proof, we use Eq. (C178) to find

$$\frac{x^4}{5^6}\, e^{-\sqrt{\ln^2(96D)+6x}+\ln(96)+\ln(D)} < \frac{96}{5^6}F(x_0) = 0.707818\ldots < 1, \tag{C181}$$

for all $D \geq 2$ and for all $x \geq \ln^2(D)$. ∎

Conjecture 20. $\mathcal{H}_\alpha(p|q)$ is convex in α for $\alpha < 0$ and $\alpha > 1$ and concave in α for $\alpha \in (0,1)$.

Remark. From the plot it follows at least for $\mathcal{H}_\alpha(p|I/2)$ for binary distributions, which is enough for us. We have not proven the conjecture, but we are able to prove the following

Lemma 21. *For arbitrary binary probability distribution $p \in \mathcal{P}_2$.*

- *For $\alpha \in (0,1)$*

$$\mathcal{H}_\alpha(p|I/2) \geq \alpha D(p|I/2) \geq \frac{\alpha}{8}\delta^2 \tag{C182}$$

- *For $\alpha > 1$*

$$\mathcal{H}_\alpha(p|I/2) \geq D(p|I/2). \tag{C183}$$

Proof. The inequality (C182) comes from the lemma 22 below and Pinsker inequality (B11). The inequality (C183) comes from (B10) and Pinsker inequality. ∎

Lemma 22. *For any probability distributions $p, q \in \mathcal{P}_d$ and for all $\alpha \in [0,1)$ and $\delta \in (0,1)$ we have*

$$H_\alpha(p|I/2) \geq \alpha D(p|I/2) \tag{C184}$$

Equivalently for all $\alpha \in (0,1)$ and $\delta < 1$ we have

$$\frac{(1+\delta)^\alpha + (1-\delta)^\alpha - 2}{\alpha - 1} \geq \alpha((1+\delta)\ln(1+\delta) + (1-\delta)\ln(1-\delta)) \tag{C185}$$

Proof. We will prove that

$$G(\alpha, \delta) := (1+\delta)^\alpha + (1-\delta)^\alpha - 2 - \alpha(\alpha-1)\big((1+\delta)\ln(1+\delta) + (1-\delta)\ln(1-\delta)\big) \geq 0. \tag{C186}$$

For $|x| < 1$ we have

$$(1+x)^\alpha = 1 + \alpha x + \sum_{n=2}^\infty \frac{a_n(\alpha)}{n!} x^n, \quad \ln(1+x) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} x^n \tag{C187}$$

where

$$a_n(\alpha) = \alpha(\alpha-1)\ldots(\alpha-(n-1)). \tag{C188}$$

One then finds that

$$(1+\delta)^\alpha + (1-\delta)^\alpha = 2 + 2\sum_{n=2,\,even}^\infty \frac{a_n(\alpha)}{n!}\delta^n, \quad (1+\delta)\ln(1+\delta) + (1-\delta)\ln(1-\delta) = 2\sum_{n=2,\,even}^\infty \frac{1}{n(n-1)}\delta^n \tag{C189}$$

Then

$$G(\alpha, \delta) = 2\sum_{n=2,\,even}^\infty \left(\frac{a_n(\alpha)}{n!} - \alpha(\alpha-1)\frac{1}{n(n-1)}\right)\delta^n. \tag{C190}$$

Now, it is enough to show that

$$(\alpha-2)\ldots(\alpha-(n-1)) \geq (n-2)!, \tag{C191}$$

for all $n \geq 2$ and even. Since there is even number of terms on the left hand side, all negative, the above inequality can be rewritten as

$$(2-\alpha)(3-\alpha)\ldots(n-1-\alpha) \geq (n-2)! \tag{C192}$$

Since $\alpha \in [0,1)$, lhs is no smaller than $(n-2)!$ and therefore the inequality holds.

∎

4. Tsalis continuity Theorem

Theorem 6 (Tsalis uniform continuity). *Let $p, p' \in \mathcal{P}_d$ have entries denoted p_k. For the following parameters, we have the following Tsalis entropy (Eq. B6) continuity bounds:*

0) For $\alpha \in (0,1) \cup (1, \infty]$,

$$|T_\alpha(p) - T_\alpha(p')| \leq \frac{2\lceil\alpha\rceil}{|\alpha-1|} d^{1-\alpha/\lceil\alpha\rceil}\left(\|p-p'\|_1\right)^{\alpha/\lceil\alpha\rceil} \leq 2\frac{(\alpha+1)}{|\alpha-1|}d\left(\frac{\|p-p'\|_1}{d}\right)^{\alpha/(1+\alpha)}. \tag{C193}$$

1) For $\alpha \in (0,1]$ and $\|p-p'\|_1 \leq d\left(\frac{1}{2e\,d}\right)^{1/\alpha}$,

$$|T_\alpha(p) - T_\alpha(p')| \leq 4\,d^{1-\alpha}\left[\left(\frac{3}{2\alpha}+1\right)\left(\|p-p'\|_1\right)^\alpha \ln d - \left(\|p-p'\|_1\right)^\alpha \ln\|p-p'\|_1\right]. \tag{C194}$$

2) For $\alpha \in [1,\infty)$ and $\|p-p'\|_1 \leq \frac{1}{2e\lceil\alpha\rceil d^\alpha}$,

$$|T_\alpha(p) - T_\alpha(p')| \leq 8\left[\|p-p'\|_1 \ln\left(\frac{d^{3/2}}{4}\right) - \|p-p'\|_1 \ln\left(\|p-p'\|_1\right)\right]. \tag{C195}$$

Remark 23. Eqs. C193, (C194), (C195), provide continuity bounds for the Tsalis entropy, which are uniform in $\alpha > 0$ bounded away from zero. For α near zero Eq. (C193) is best, while for α in the vicinity of 1, Eqs. (C194), (C195) are optimal. For large α one can either use Eq. C193 or Eq. (C195) depending on the circumstances. If the condition $\|p - p'\|_1 \leq \frac{1}{2e\lceil \alpha \rceil d^\alpha}$ is fulfilled (which becomes more stringent the larger α is), then it is likely to be preferable to use Eq. (C195) which only grows logarithmically with d. On the other hand, if $\|p - p'\|_1 \leq \frac{1}{2e\lceil \alpha \rceil d^\alpha}$ cannot be guaranteed to be fulfilled (such as in the limiting case $\alpha \to \infty$), then Eq. C193, with sub-linear scaling with d, is the only option.

Proof. We start by proving Eq. (C193). From the definition of the Tsalis entropy, Eq. (B6), it follows

$$|T_\alpha(p) - T_\alpha(p')| = \frac{1}{|1 - \alpha|} \left| \sum_i p_i^\alpha - p_i'^\alpha \right| \leq \frac{1}{|1 - \alpha|} \sum_i |p_i^\alpha - p_i'^\alpha|. \tag{C196}$$

We now apply Lemma 28 to find

$$|T_\alpha(p) - T_\alpha(p')| \leq \frac{2\lceil \alpha \rceil}{|1 - \alpha|} d^{(1 - \alpha/\lceil \alpha \rceil)} \left(\|p - p'\|_1 \right)^{\alpha/\lceil \alpha \rceil}, \tag{C197}$$

from which the bound follows by noting $\lceil \alpha \rceil \leq \alpha + 1$, and $\|p - p'\|/d \leq 1$. We will now prove Eqs. (C194), (C195). To start with,

$$|T_\alpha(p) - T_\alpha(p')| = \frac{1}{|1 - \alpha|} \left| \sum_i p_i^\alpha - p_i'^\alpha \right| \leq \frac{1}{|\alpha - 1|} |G_\alpha(p, p')|, \tag{C198}$$

where we have defined

$$G_\alpha(p, p') = \|p\|_\alpha^\alpha - \|p'\|_\alpha^\alpha. \tag{C199}$$

From the definition of $\| \cdot \|_p$ we see that for all $\alpha \in [0, \infty)$, $G_\alpha(p, p')$ is continuous. Furthermore, from Eq. (C269) we observe that $G_\alpha(p, p')$ is differentiable for $\alpha \in (0, \infty)$. As such, we can apply the mean value theorem to it as follows.
   Using the notation a, b, c, from the mean value theorem 8, we have

1) For $a = 1$, $b = \alpha$, $\alpha > 1$

$$G_\alpha(p, p') = G_1(p, p') + G_c'(p, p')(\alpha - 1) = G_c'(p, p')(\alpha - 1), \quad \text{for some } c \in (1, \alpha). \tag{C200}$$

2) For $b = 1$, $a = \alpha$, $0 \leq \alpha < 1$

$$G_\alpha(p, p') = G_1(p, p') + G_c'(p, p')(-1 + \alpha) = G_c'(p, p')(\alpha - 1), \quad \text{for some } c \in (\alpha, 1). \tag{C201}$$

Where in both cases we have used $\|p\|_1 = \|p'\|_1 = 1$. We thus conclude

$$g_\alpha(p, p') = g_c'(p, p')(\alpha - 1), \quad \text{for some } c \in \begin{cases} (\alpha, 1) & \text{if } 0 \leq \alpha < 1 \\ (1, \alpha) & \text{if } \alpha > 1. \end{cases} \tag{C202}$$

Plugging in to Eq. (C198), we thus have for all $\alpha > 0$,

$$|T_\alpha(p) - T_\alpha(p')| \leq \left| \frac{d}{d\alpha} \|p\|_\alpha^\alpha - \frac{d}{d\alpha} \|p'\|_\alpha^\alpha \right|_{\alpha=c} = c \left| \|p\|_\alpha^{\alpha-1} \frac{d}{d\alpha} \|p\|_\alpha - \|p'\|_\alpha^{\alpha-1} \frac{d}{d\alpha} \|p'\|_\alpha \right|_{\alpha=c}. \tag{C203}$$

Now plugging in Eq. C272,

$$|T_\alpha(p) - T_\alpha(p')| \leq \frac{1}{c} \left| \|p\|_\alpha^\alpha S_1 \left(q_\alpha(p) \right) - \|p'\|_\alpha^\alpha S_1 \left(q_\alpha(p') \right) \right|_{\alpha=c}, \tag{C204}$$

where

$$[q_\alpha(x)]_i := \frac{|x_i|^\alpha}{\|x\|_\alpha^\alpha}, \quad i = 1, 2, 3, \ldots, d. \tag{C205}$$

Thus we find,

$$|T_\alpha(p) - T_\alpha(p')| \leq \frac{1}{c}\|p\|_c^c \left| \left(\frac{\|p'\|_c^c}{\|p\|_c^c} - 1 \right) S_1\left(q_c(p')\right) + S_1\left(q_c(p')\right) - S_1\left(q_c(p)\right) \right| \tag{C206}$$

$$\leq \frac{1}{c}\|p\|_c^c \left(\left| \frac{\|p'\|_c^c}{\|p\|_c^c} - 1 \right| |S_1\left(q_c(p')\right)| + |S_1\left(q_c(p')\right) - S_1\left(q_c(p)\right)| \right) \tag{C207}$$

$$= \frac{1}{c} \left(|S_1\left(q_c(p')\right)| \left| \|p'\|_c^c - \|p\|_c^c \right| + \|p\|_c^c \left| S_1\left(q_c(p')\right) - S_1\left(q_c(p)\right) \right| \right) \tag{C208}$$

$$\leq \frac{1}{c} \left(\left(\max_{q \in \mathcal{P}_d} |S_1(q)| \right) \left| \|p'\|_c^c - \|p\|_c^c \right| + \|p\|_c^c \left| S_1\left(q_c(p')\right) - S_1\left(q_c(p)\right) \right| \right) \tag{C209}$$

$$= \frac{1}{c} \left(\ln d \left| \|p'\|_c^c - \|p\|_c^c \right| + \|p\|_c^c \left| S_1\left(q_c(p')\right) - S_1\left(q_c(p)\right) \right| \right). \tag{C210}$$

Applying the Fannes inequality (Lemma 26), we find

$$|T_\alpha(p) - T_\alpha(p')| \leq \frac{1}{c} \left(\left| \|p'\|_c^c - \|p\|_c^c \right| \ln d \ + \|p\|_c^c \left(\|q_c(p) - q_c(p')\|_1 \ln d - \|q_c(p) - q_c(p')\|_1 \ln \left(\|q_c(p) - q_c(p')\|_1 \right) \right) \right) \tag{C211}$$

$$\tag{C212}$$

We now pause a moment to bound $\|q_\alpha(p) - q_\alpha(p')\|_1$. Using the definition of $q_\alpha(p)$, we have

$$\|q_\alpha(p) - q_\alpha(p')\|_1 = \sum_{i=1}^{d} \left| \frac{p_i^\alpha}{\|p\|_\alpha^\alpha} + \frac{p_i'^\alpha}{\|p'\|_\alpha^\alpha} \right| = \sum_{i=1}^{d} \frac{1}{\|p\|_\alpha^\alpha} \left| p_i^\alpha - p_i'^\alpha + p_i'^\alpha \left(1 - \frac{\|p\|_\alpha^\alpha}{\|p'\|_\alpha^\alpha} \right) \right| \tag{C213}$$

$$\leq \sum_{i=1}^{d} \frac{1}{\|p\|_\alpha^\alpha} \left(|p_i^\alpha - p_i'^\alpha| + p_i'^\alpha \left| 1 - \frac{\|p\|_\alpha^\alpha}{\|p'\|_\alpha^\alpha} \right| \right) = \frac{1}{\|p\|_\alpha^\alpha} \left(\left| \|p\|_\alpha^\alpha - \|p'\|_\alpha^\alpha \right| + \sum_{i=1}^{d} |p_i^\alpha - p_i'^\alpha| \right) \tag{C214}$$

$$\leq \frac{2}{\|p\|_\alpha^\alpha} \left(\sum_{i=1}^{d} |p_i^\alpha - p_i'^\alpha| \right) = \frac{\Delta_\alpha(p,p')}{\|p\|_\alpha^\alpha}, \tag{C215}$$

where in the last line, we have used Lemma 28 and defined,

$$\Delta_\alpha(p,p') := 2 \sum_{i=1}^{d} \left| p_i^\alpha - p_i'^\alpha \right|. \tag{C216}$$

Plugging this into Eq. (C210), we find for $\|q_c(p) - q_c(p')\|_1 \leq 1/e \approx 0.37$,

$$|T_\alpha(p) - T_\alpha(p')| \leq \frac{1}{c} \left(\left| \|p'\|_c^c - \|p\|_c^c \right| \ln d \ + \|p\|_c^c \left(\frac{\Delta_c(p,p')}{\|p\|_c^c} \ln d - \frac{\Delta_c(p,p')}{\|p\|_c^c} \ln \left(\frac{\Delta_c(p,p')}{\|p\|_c^c} \right) \right) \right) \tag{C217}$$

$$\leq \frac{1}{c} \left(\frac{\Delta_c(p,p')}{2} \ln d \ + \Delta_c(p,p') \ln d - \Delta_c(p,p') \ln \left(\frac{\Delta_c(p,p')}{\|p\|_c^c} \right) \right), \tag{C218}$$

$$= \frac{1}{c} \left(\Delta_c(p,p') \ln \left(d^{3/2} \|p\|_c^c \right) - \Delta_c(p,p') \ln \Delta_c(p,p') \right). \tag{C219}$$

We now find bounds for $\Delta_c(p,p')$. To start with, from Lemma 28 we have

$$\Delta_c(p,p') \leq 4\lceil c \rceil \, d \left(\frac{\|p - p'\|_1}{d} \right)^{c/\lceil c \rceil}, \tag{C220}$$

where $\|p - p'\|_1/d \leq 1$ since $d \geq 2$ and $\|p - p'\|_1 \leq 2$ holds for all $p, p' \in \mathcal{P}_d$.
For $\alpha \in [0,1)$, $c \in (\alpha, 1)$ we find

$$\Delta_c(p,p') \leq 4d \left(\frac{\|p - p'\|_1}{d} \right)^c \leq 4d \left(\frac{\|p - p'\|_1}{d} \right)^\alpha \quad \forall\, c \in (\alpha, 1). \tag{C221}$$

For $\alpha > 1$, $c \in (1, \alpha)$ and we find

$$\Delta_c(p, p') \leq 4\lceil c \rceil d \left(\frac{\|p - p'\|_1}{d} \right) = 4\lceil c \rceil \|p - p'\|_1 \quad \forall c \in (1, \alpha). \tag{C222}$$

We will now upper bound $\|p\|_c^c$ using Eq. (C296) from Lemma 25.
For $\alpha \in (0, 1)$, $c \in (\alpha, 1)$:

1) Noting that $0 \leq p_i \leq 1$ for all i,

$$\|p\|_c^c = \sum_{i=1}^{d} p_i^c \leq \lim_{c \to 0^+} \sum_{i=1}^{d} p_i^c \leq d \quad \forall c \in (\alpha, 1) \tag{C223}$$

2) Setting $r = c$, $p = 1$, in Eq. (C296), we have

$$1 = \|p\|_1 \leq \|p\|_c \implies 1 \leq \|p\|_c^c \tag{C224}$$

For $\alpha \in (1, \infty)$, $c \in (1, \alpha)$:

1) Setting $r = 1$, $p = c$, in Eq. (C296), we have

$$\|p\|_c \leq \|p\|_r = 1 \quad \forall c \in (1, \alpha) \implies \ln \|p\|_c^c = c \ln \|p\|_c \leq 0. \tag{C225}$$

2) Seetting $r = c$, $p = \infty$, in Eq. (C296), and noting that $\max_i \{p_i\} \geq 1/d$ we have

$$\|p\|_\infty = \max_i \{p_i\} \leq \|p\|_c \quad \forall c \in (1, \alpha) \implies \left(\frac{1}{d} \right)^c \leq \|p\|_c^c \tag{C226}$$

Plugging Eqs. C221, C223 into Eq. (C219), we find for $\alpha \in (0, 1)$ and for all $c \in (\alpha, 1)$,

$$|T_\alpha(p) - T_\alpha(p')| \leq \frac{1}{c} \left[4d \left(\frac{\|p - p'\|_1}{d} \right)^\alpha \ln \left(d^{3/2} d \right) - 4d \left(\frac{\|p - p'\|_1}{d} \right)^\alpha \ln \left(4d \left(\frac{\|p - p'\|_1}{d} \right)^\alpha \right) \right] \tag{C227}$$

$$\leq 4 \frac{d}{\alpha} \left(\frac{\|p - p'\|_1}{d} \right)^\alpha \ln \left(\frac{d^{3/2}}{4} \right) - 4d \left(\frac{\|p - p'\|_1}{d} \right)^\alpha \ln \left(\frac{\|p - p'\|_1}{d} \right) \tag{C228}$$

$$\leq 4 \frac{d}{\alpha} \left(\frac{\|p - p'\|_1}{d} \right)^\alpha \ln \left(d^{3/2} \right) - 4d \left(\frac{\|p - p'\|_1}{d} \right)^\alpha \ln \left(\frac{\|p - p'\|_1}{d} \right) \tag{C229}$$

$$= 4 d^{1-\alpha} \left[\left(\frac{3}{2\alpha} + 1 \right) \left(\|p - p'\|_1 \right)^\alpha \ln d - \left(\|p - p'\|_1 \right)^\alpha \ln \|p - p'\|_1 \right]. \tag{C230}$$

Since the above equation is continuous around $\alpha = 1$, it must also hold for $\alpha \in (0, 1]$[15] which is Eq. (C194) of the Theorem. For $\alpha \in (0, 1)$ we can also find an appropriate bound on $\|p - p'\|_1$ so that the constraint $\|q_c(p) - q_c(p')\|_1 \leq 1/e$ is satisfied. Using Eqs, (C215), (C221), (C224), we find

$$\|q_c(p) - q_c(p')\|_1 \leq \frac{\Delta_c(p, p')}{\|p\|_c^c} \leq \frac{2d \left(\|p - p'\|_1 / d \right)^\alpha}{\|p\|_c^c} \leq 2d \left(\|p - p'\|_1 / d \right)^\alpha. \tag{C231}$$

Thus imposing the constraint $2d \left(\|p - p'\|_1 / d \right)^\alpha \leq 1/e$ we achieve the condition

$$\|p - p'\|_1 \leq d \left(\frac{1}{2e\,d} \right)^{1/\alpha}, \tag{C232}$$

which is the condition in the text above Eq. (C194) in the Theorem.

---

[15] One can easily prove this by contradiction. Imagine that the bound Eq. (C230) does not hold for $\alpha = 1$. Then there must exit a $\epsilon > 0$ such that it also does not hold for $\alpha = 1 - \epsilon$, but this would be a contradiction.

Similarly, plugging Eqs. C222, C225 into Eq. (C219), we find for $\alpha \in (1, \infty)$

$$|T_\alpha(p) - T_\alpha(p')| \leq \frac{1}{c}\left(\Delta_c(p, p')\ln d^{3/2} - \Delta_c(p, p')\ln \Delta_c(p, p')\right) \tag{C233}$$

$$\leq \frac{4\lceil c \rceil}{c}\left(\|p - p'\|_1 \ln d^{3/2} - \|p - p'\|_1 \ln\left(4\lceil c \rceil \|p - p'\|_1\right)\right) \tag{C234}$$

$$\leq \frac{4(c+1)}{c}\left(\|p - p'\|_1 \ln\left(\frac{d^{3/2}}{4}\right) - \|p - p'\|_1 \ln\left(\|p - p'\|_1\right) - \|p - p'\|_1 \ln\left(\lceil c \rceil\right)\right) \tag{C235}$$

$$\leq 8\left(\|p - p'\|_1 \ln\left(\frac{d^{3/2}}{4}\right) - \|p - p'\|_1 \ln\left(\|p - p'\|_1\right)\right), \quad \forall c \in (1, \alpha) \tag{C236}$$

which is Eq. (C195) in the theorem. Similarly to above, for $\alpha \in (1, \infty)$ we can also find an appropriate bound on $\|p - p'\|_1$ so that the constraint $\|q_c(p) - q_c(p')\|_1 \leq 1/e$ is satisfied. Using Eqs, (C215), (C226), we find

$$\|q_c(p) - q_c(p')\|_1 \leq \frac{\Delta_c(p, p')}{\|p\|_c^c} \leq \frac{2\lceil c \rceil \|p - p'\|_1}{\left(\frac{1}{d}\right)^c} = 2\lceil c \rceil d^c \|p - p'\|_1 \quad \forall c \in (1, \alpha). \tag{C237}$$

Thus imposing the constraint $2\lceil c \rceil d^c \|p - p'\|_1 \leq 1/e$ for all $c \in (1, \alpha)$. We observe that this is satisfied for all $c \in (1, \alpha)$ if

$$\|p - p'\|_1 \leq \frac{1}{2e\lceil \alpha \rceil d^\alpha} \tag{C238}$$

which is the condition in the text above Eq. (C195) in the Theorem.

∎

5. Rényi entropy continuity Theorem

Theorem 7 (Rényi uniform continuity). *Let $p, p' \in \mathcal{P}_d$ have entries denoted p_k. For the following parameters, we have the following Rényi entropy (Eq. B1) continuity bounds:*

0) For $\alpha \in [-\infty, -1]$, we have

$$\frac{\alpha}{1 - \alpha}\ln\left(\frac{\min_k\{p_k\}}{\min_l\{p_l p_l'\}}\|p - p'\|_1 + 1\right) \leq S_\alpha(p) - S_\alpha(p') \leq \frac{|\alpha|}{1 - \alpha}\ln\left(\frac{\min_k\{p_k'\}}{\min_l\{p_l p_l'\}}\|p - p'\|_1 + 1\right) \tag{C239}$$

and

$$\frac{\alpha}{1 - \alpha}\ln\left(\frac{\|p - p'\|_1}{\min_l\{p_l'\}} + 1\right) \leq S_\alpha(p) - S_\alpha(p') \leq \frac{|\alpha|}{1 - \alpha}\ln\left(\frac{\|p - p'\|_1}{\min_l\{p_l\}} + 1\right) \tag{C240}$$

1) For $\alpha \in (0, 1)$, we have

$$|S_\alpha(p) - S_\alpha(p')| \leq \frac{e^{(\alpha-1)S_\alpha(p)}}{1 - \alpha} d^{(1-\alpha)}\left(\|p - p'\|_1\right)^\alpha. \tag{C241}$$

2) Let $\epsilon_0 > 1$. Then for all $\alpha \in [\epsilon_0, \infty]$, we have

$$|S_\alpha(p) - S_\alpha(p')| \leq \frac{\epsilon_0}{\epsilon_0 - 1}\ln\left(1 + \|p - p'\|_1 d\right). \tag{C242}$$

3.1) Let $\alpha \in [1/2, 1)$, and $\|p - p'\|_1 \leq 1/[(4e)^2 d]$, then

$$|S_\alpha(p) - S_\alpha(p')| \leq 8d\left(6\ln d - \ln 2\right)\sqrt{\|p - p'\|_1} - 4d\sqrt{\|p - p'\|_1}\ln\sqrt{\|p - p'\|_1} \tag{C243}$$

3.2) Let $\alpha \in [1, 2]$, and $\|p - p'\|_1 \leq d/(8e)^2$, then

$$|S_\alpha(p) - S_\alpha(p')| \leq 2\sqrt{d}\left(\|p - p'\|_1 \ln d + \sqrt{\|p - p'\|_1}\, 4d\ln(d/64) - 8d\sqrt{\|p - p'\|_1}\ln\sqrt{\|p - p'\|_1}\right). \tag{C244}$$

Proof. This proof is divided into subsections, one for each α regime, 0), 1), 2), 3).

a. Proof of 0). $\alpha \in [-\infty, -1]$

For $\alpha < 0$, we have

$$S_\alpha(p) - S_\alpha(p') = \frac{-\alpha}{1-\alpha} \ln \|p\|_\alpha - \frac{-\alpha}{1-\alpha} \ln \|p'\|_\alpha = \frac{|\alpha|}{1-\alpha} \ln \left(\frac{\|p^{-1}\|_{|\alpha|}}{\|p'^{-1}\|_{|\alpha|}} \right), \tag{C245}$$

where $\|x^{-1}\|_\alpha := \sum_{i=1}^d (1/x_i)^\alpha$. Therefore,

$$S_\alpha(p) - S_\alpha(p') \geq 0 \tag{C246}$$

iff $\|p^{-1}\|_{|\alpha|}/\|p'^{-1}\|_{|\alpha|} \geq 0$.

We will now provide a proof for $\alpha \in [-\infty, -1]$ and finalise the proof for the remaining negative interval afterwards. Assume $\|p^{-1}\|_{|\alpha|}/\|p'^{-1}\|_{|\alpha|} \geq 0$,[16] we then have using the p-norm triangle inequality (Lemma 25),

$$S_\alpha(p) - S_\alpha(p') = \frac{|\alpha|}{1-\alpha} \ln \left(\frac{\|p^{-1} - p'^{-1} + p'^{-1}\|_{|\alpha|}}{\|p'^{-1}\|_{|\alpha|}} \right) \leq \frac{|\alpha|}{1-\alpha} \ln \left(\frac{\|p^{-1} - p'^{-1}\|_{|\alpha|}}{\|p'^{-1}\|_{|\alpha|}} + 1 \right). \tag{C247}$$

Also from Lemma 25, it follows

$$\|p^{-1} - p'^{-1}\|_{|\alpha|} \leq \|p^{-1} - p'^{-1}\|_1 = \sum_{i=1}^d \frac{|p_i - p'_i|}{p_i p'_i} \leq \frac{1}{\min_k p_k p'_k} \|p - p'\|_1 \tag{C248}$$

$$\|p'^{-1}\|_{|\alpha|} \geq \|p'^{-1}\|_\infty = \max_i \{1/p'_i\} = 1/\min_i \{p'_i\}. \tag{C249}$$

Therefore, by plugging in the above inequalities,

$$S_\alpha(p) - S_\alpha(p') \leq \frac{|\alpha|}{1-\alpha} \ln \left(\frac{\min_k \{p'_k\}}{\min_l \{p_l p'_l\}} \|p - p'\|_1 + 1 \right) \tag{C250}$$

which is the R.H.S. of Eq. (C239). For the proof of the L.H.S. of Eq. (C239), we note that this term is negative. Therefore, it is trivially true if $S_\alpha(p) - S_\alpha(p') \geq 0$. When $S_\alpha(p) - S_\alpha(p') < 0$, we can write $S_\alpha(p) - S_\alpha(p') = -(S_\alpha(p') - S_\alpha(p))$, where the term in brackets in positive. Thus from Eq. (C246), we can use upper bound Eq. (C250) with $p \mapsto p'$, $p' \mapsto p$ to achieve the L.H.S. of Eq. (C239). Eqs. (C240) follow from noting $\min_k \{p_k p'_k\} \geq \min_k \{p_k\} \min_k \{p'_k\}$.

b. Proof of 1). $\alpha \in (0, 1)$

From notes from old bound [delete old notes and just put that section here], we have

$$|S_\alpha(p) - S_\alpha(p')| \leq \frac{1}{1-\alpha} \frac{1}{\sum_{i=1}^d p_i^\alpha} \sum_{i=1}^d |p_i^\alpha - p_i'^\alpha| = \frac{1}{1-\alpha} \frac{1}{\|p_i\|_\alpha^\alpha} \sum_{i=1}^d |p_i^\alpha - p_i'^\alpha|. \tag{C251}$$

Eq. (C241) now follows directly by bounding $\sum_{i=1}^d |p_i^\alpha - p_i'^\alpha|$ using Lemma 28, and the relationship between Rényi entropies, and $\|p\|_\alpha$, namely $(1-\alpha)S_\alpha(p) = \ln \|p\|_\alpha^\alpha$.

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[16] Note that the upper bound Eq. (C246) is non-negative. Therefore, if assumption $\|p^{-1}\|_{|\alpha|}/\|p'^{-1}\|_{|\alpha|} \geq 0$ does not hold, the bound will be trivially true since $S_\alpha(p) - S_\alpha(p')$ will be negative.

We start with proving Eq. (C242).

$$|S_\alpha(p) - S_\alpha(p')| = \left| \frac{\alpha}{1-\alpha} \ln \|p'\|_\alpha - \frac{\alpha}{1-\alpha} \ln \|p\|_\alpha \right| = \left| \frac{\alpha}{1-\alpha} \right| \left| \ln \left(\frac{\|p'\|_\alpha}{\|p\|_\alpha} \right) \right| \tag{C252}$$

$$= \begin{cases} \left| \frac{\alpha}{1-\alpha} \right| \ln \left(\frac{\|p'\|_\alpha - \|p\|_\alpha}{\|p\|_\alpha} + 1 \right), & \text{for } \|p'\|_\alpha - \|p\|_\alpha \geq 0, \\[2mm] \left| \frac{\alpha}{1-\alpha} \right| \ln \left(\frac{\|p\|_\alpha - \|p'\|_\alpha}{\|p'\|_\alpha} + 1 \right), & \text{for } \|p\|_\alpha - \|p'\|_\alpha \geq 0, \end{cases} \tag{C253}$$

$$= \left| \frac{\alpha}{1-\alpha} \right| \ln \left(\frac{\left| \|p\|_\alpha - \|p'\|_\alpha \right|}{\|p''\|_\alpha} + 1 \right), \tag{C254}$$

where

$$\|p''\|_\alpha = \begin{cases} \|p\|_\alpha & \text{if } \|p'\|_\alpha - \|p\|_\alpha \geq 0 \\ \|p'\|_\alpha & \text{otherwise.} \end{cases} \tag{C255}$$

For $\alpha > 1$, from Lemma 28, we have $\left| \|p\|_\alpha - \|p'\|_\alpha \right| \leq \|p - p'\|_1$. Furthermore, using Eq. (C296), we have $\|p''\|_\alpha \geq \|p''\|_\infty$. However, we also have that $\|p''\|_\infty = \max_i\{p_i''\} \geq 1/d$. Thus Eq. (C242) follows since the logarithm is an increasing function.

We now move on to the proof of Eqs. (C244),(C243). To start with, we define the function F to be any upper bound to

$$\left| \frac{1}{\alpha - 1} \frac{\|p'\|_\alpha - \|p\|_\alpha}{\|p''\|_\alpha} \right| \leq F_\alpha(p, p'). \tag{C256}$$

Using Eq. (C252), we can now write

$$|S_\alpha(p) - S_\alpha(p')| = \left| \frac{\alpha}{1-\alpha} \right| \ln \left(\frac{\left| \|p\|_\alpha - \|p'\|_\alpha \right|}{\|p''\|_\alpha} + 1 \right) \leq \frac{\alpha}{|\alpha - 1|} \ln \left(|\alpha - 1| F_\alpha(p, p') + 1 \right) \tag{C257}$$

$$\leq \alpha F_\alpha(p, p'). \tag{C258}$$

where in the last line we have used the inequality $\ln(x+1) \leq x$ for $x \geq 0$.

We now set out to find an appropriate expression for $F_\alpha(p, p')$. For this we will use the mean value theorem, Theorem 8. We start by finding two separate expressions for the function[17]

$$g_\alpha(p, p') := \|p'\|_\alpha - \|p\|_\alpha. \tag{C259}$$

Using the notation a, b, c, from the mean value theorem (Thm. 8), we have

1) $a = 1$, $b = \alpha$, $\alpha \geq 1$

$$g_\alpha(p, p') = g_1(p, p') + g_c'(p, p')(\alpha - 1) = g_c'(p, p')(\alpha - 1), \quad \text{for some } c \in (1, \alpha). \tag{C260}$$

2) $b = 1$, $a = \alpha$, $\alpha \leq 1$

$$g_\alpha(p, p') = g_1(p, p') + g_c'(p, p')(-1 + \alpha) = g_c'(p, p')(\alpha - 1), \quad \text{for some } c \in (\alpha, 1). \tag{C261}$$

———————

[17] Here it will be assumed that $g_\alpha(p, p')$ is differentiable w.r.t. α on the interval $(1, \alpha)$ for $\alpha \geq 1$ and $(\alpha, 1)$ for $\alpha \leq 1$. Later we will calculate explicitly its derivative, thus verifying this assumption.

Where in both cases we have used $\|p\|_1 = \|p'\|_1 = 1$. We thus conclude

$$g_\alpha(p,p') = g'_\beta(p,p')(\alpha - 1), \quad \text{for some } \beta \in \begin{cases} (\alpha, 1) & \text{if } \alpha < 1 \\ (1, \alpha) & \text{if } \alpha \geq 1. \end{cases} \tag{C262}$$

We thus have

$$\left| \frac{1}{\alpha - 1} \frac{\|p'\|_\alpha - \|p\|_\alpha}{\|p''\|_\alpha} \right| = \frac{|g'_\beta(p,p')|}{\|p''\|_\alpha} = \frac{1}{\|p''\|_\alpha} \left| \frac{d}{d\beta} \left(\|p'\|_\beta - \|p\|_\beta \right) \right| := F_\beta(p,p'). \tag{C263}$$

We thus have taking into account eq. (C257),

$$|S_\alpha(p) - S_\alpha(p')| \leq \alpha \sup_{\beta \in \mathcal{I}} F_\beta(p,p'), \tag{C264}$$

where $\mathcal{I} = [1/2, 1]$ when $\alpha \leq 1$, and $\mathcal{I} = [1, 2]$ when $\alpha \geq 1$. We have so-far managed to remove the singularity at $\alpha = 1$ in our upper bound to the Rényi entropies. We will now set out to prove a relationship between this upper bound and the distance $\|p - p'\|_1$. We start by find the derivative. For convenience, note that for $x \in \mathbb{C}^n$,

$$\|x\|_\alpha = \exp\left(\frac{1}{\alpha} \ln \sum_{i=1}^\infty |x_i|^\alpha \right). \tag{C265}$$

Hence the derivative is

$$\frac{d}{d\alpha} \|x\|_\alpha = \|x\|_\alpha \frac{d}{d\alpha} \left(\frac{1}{\alpha} \ln \sum_{i=1}^\infty |x_i|^\alpha \right) = \left(\left(\frac{d}{d\alpha} \frac{1}{\alpha} \right) \ln \sum_{i=1}^\infty |x_i|^\alpha + \frac{1}{\alpha} \frac{d}{d\alpha} \ln \sum_{i=1}^\infty |x_i|^\alpha \right) \tag{C266}$$

$$= \|x\|_\alpha \left(\frac{-1}{\alpha^2} \ln \sum_{i=1}^\infty |x_i|^\alpha + \frac{1}{\alpha} \frac{1}{\sum_{i=1}^\infty |x_i|^\alpha} \frac{d}{d\alpha} \sum_{i=1}^\infty |x_i|^\alpha \right). \tag{C267}$$

However, $\frac{d}{d\alpha} \sum_{i=1}^\infty |x_i|^\alpha = \frac{d}{d\alpha} \sum_{i=1}^\infty e^{\alpha \ln |x_i|} = \sum_{i=1}^\infty |x_i|^\alpha \ln |x_i|$, therefore

$$\frac{d}{d\alpha} \|x\|_\alpha = \|x\|_\alpha \left(\frac{-1}{\alpha^2} \ln \sum_{i=1}^\infty |x_i|^\alpha + \frac{1}{\alpha} \frac{1}{\sum_{i=1}^\infty |x_i|^\alpha} \sum_{i=1}^\infty |x_i|^\alpha \ln |x_i| \right) \tag{C268}$$

$$= \frac{\|x\|_\alpha}{\alpha} \left(-\ln \|x\|_\alpha + \frac{1}{\|x\|_\alpha^\alpha} \sum_{i=1}^d |x_i|^\alpha \ln |x_i| \right). \tag{C269}$$

Now, by direct calculation we observe that the above line can be written in terms of the S_1, Shannon entropy for a probability distribution which depends on α, namely[18]

$$S_1\left(q_\alpha(x)\right) = -\alpha \left(-\ln \|x\|_\alpha + \frac{1}{\|x\|_\alpha^\alpha} \sum_{i=1}^d |x_i|^\alpha \ln |x_i| \right), \tag{C270}$$

where the components of the normalised probability vector $q_\alpha(x)$ are

$$[q_\alpha(x)]_i := \frac{|x_i|^\alpha}{\|x\|_\alpha^\alpha}, \quad i = 1, 2, 3, \ldots, d. \tag{C271}$$

Therefore,

$$\frac{d}{d\alpha} \|x\|_\alpha = -\frac{\|x\|_\alpha}{\alpha^2} S_1\left(q_\alpha(x)\right). \tag{C272}$$

_____

[18] Recall that $S_1(x) = -\sum_{i=1}^d |x_i| \ln |x_i|$.

From Eq. (C263),

$$\beta^2 \|p''\|_\alpha F_\beta(p, p') = \beta^2 \left| \frac{d}{d\beta} \left(\|p'\|_\beta - \|p\|_\beta \right) \right| = \left| \|p'\|_\beta S_1 \left(q_\alpha(p') \right) - \|p\|_\beta S_1 \left(q_\alpha(p) \right) \right|, \tag{C273}$$

Thus,

$$\beta^2 \|p''\|_\alpha F_\beta(p, p') = \|p\|_\beta \left| \left(\frac{\|p'\|_\beta}{\|p\|_\beta} - 1 \right) S_1 \left(q_\alpha(p') \right) + S_1 \left(q_\alpha(p') \right) - S_1 \left(q_\alpha(p) \right) \right| \tag{C274}$$

$$\leq \|p\|_\beta \left(\left| \frac{\|p'\|_\beta}{\|p\|_\beta} - 1 \right| \left| S_1 \left(q_\alpha(p') \right) \right| + \left| S_1 \left(q_\alpha(p') \right) - S_1 \left(q_\alpha(p) \right) \right| \right) \tag{C275}$$

$$= \left| S_1 \left(q_\alpha(p') \right) \right| \left| \|p'\|_\beta - \|p\|_\beta \right| + \|p\|_\beta \left| S_1 \left(q_\alpha(p') \right) - S_1 \left(q_\alpha(p) \right) \right| \tag{C276}$$

$$\leq \left(\max_{q \in \mathcal{P}_d} |S_1(q)| \right) \left| \|p'\|_\beta - \|p\|_\beta \right| + \|p\|_\beta \left| S_1 \left(q_\beta(p') \right) - S_1 \left(q_\beta(p) \right) \right|. \tag{C277}$$

Therefore, noting that the Shannon entropy is maximized for the uniform distribution and applying the Fannes inequality (Lemma 26), we achieve

$$\beta^2 \|p''\|_\alpha F_\beta(p, p') \leq \ln d \left| \|p'\|_\beta - \|p\|_\beta \right| + \|p\|_\beta \left(\|q_\beta(p) - q_\beta(p')\|_1 \ln d - \|q_\beta(p) - q_\beta(p')\|_1 \ln \left(\|q_\beta(p) - q_\beta(p')\|_1 \right) \right). \tag{C278}$$

We now pause a moment to bound $\|q_\beta(p) - q_\beta(p')\|_1$. Using the definition of $q_\alpha(p)$, we have

$$\|q_\beta(p) - q_\beta(p')\|_1 = \sum_{i=1}^d \left| \frac{p_i^\beta}{\|p\|_\beta^\beta} + \frac{p_i'^\beta}{\|p'\|_\beta^\beta} \right| = \sum_{i=1}^d \frac{1}{\|p\|_\beta^\beta} \left| p_i^\beta - p_i'^\beta + p_i'^\beta \left(1 - \frac{\|p\|_\beta^\beta}{\|p'\|_\beta^\beta} \right) \right| \tag{C279}$$

$$\leq \sum_{i=1}^d \frac{1}{\|p\|_\beta^\beta} \left(\left| p_i^\beta - p_i'^\beta \right| + p_i'^\beta \left| 1 - \frac{\|p\|_\beta^\beta}{\|p'\|_\beta^\beta} \right| \right) = \frac{1}{\|p\|_\beta^\beta} \left(\left| \|p\|_\beta^\beta - \|p'\|_\beta^\beta \right| + \sum_{i=1}^d \left| p_i^\beta - p_i'^\beta \right| \right) \tag{C280}$$

$$\leq \frac{2}{\|p\|_\beta^\beta} \left(\sum_{i=1}^d \left| p_i^\beta - p_i'^\beta \right| \right) = \frac{\Delta(p, p')}{\|p\|_\beta^\beta}, \tag{C281}$$

where in the last line, we have used Lemma 28 and defined,

$$\Delta(p, p') := 2 \sum_{i=1}^d \left| p_i^\beta - p_i'^\beta \right|. \tag{C282}$$

Therefore, for $\|q_\beta(p) - q_\beta(p')\|_1 \leq 1/e$,

$$\beta^2 \|p''\|_\alpha F_\beta(p, p') \leq \left| \|p'\|_\beta - \|p\|_\beta \right| \ln d + \|p\|_\beta^{1-\beta} \left(\Delta(p, p') \ln d - \Delta(p, p') \ln \Delta(p, p') + \beta \, \Delta(p, p') \ln \|p\|_\beta \right). \tag{C283}$$

We will now proceed to bound Eq. (C282) separately for $\beta \in [1/2, 1)$, and $\beta \in [1, 2]$. We start with the easier of the two.

For $\beta \in [1, 2]$:
Setting $r = 1$, $p = \beta$ in Eq. (C296), and recalling $\|p\|_1 = 1$, it follows

$$d^{1/\beta - 1} \leq \|p\|_\beta \leq 1. \tag{C284}$$

Similarly, from Lemma 28, and assuming $\|p - p'\|_1 \leq d$, we have

$$\Delta(p, p') \leq 8 \, d^{1-\beta/2} \left(\|p - p'\|_1 \right)^{\beta/2} \leq 8 \, d \left(\frac{\|p - p'\|_1}{d} \right)^{\beta/2} \leq 8 \sqrt{d \, \|p - p'\|_1} \tag{C285}$$

$$\left| \|p'\|_\beta - \|p\|_\beta \right| \leq \|p - p'\|_1. \tag{C286}$$

Furthermore more, from Eq. (C284) if follows

$$\|p\|_\beta^{1-\beta} \le \frac{1}{\left(d^{1/\beta-1}\right)^{\beta-1}} \le \sqrt{d}. \tag{C287}$$

We will now see which of the two constraints, $\|p-p'\|_1 \le d$, and $\|q_\beta(p)-q_\beta(p')\|_1 \le 1/e$ is more demanding.

$$\|q_\beta(p)-q_\beta(p')\|_1 \le \frac{8}{d}\sqrt{d\|p-p'\|_1} \le \frac{1}{e} \implies \|p-p'\|_1 \le \frac{d}{(8e)^2} \le d. \tag{C288}$$

therefore $\|p-p'\|_1 \le d$, and $\|q_\beta(p)-q_\beta(p')\|_1 \le 1/e$ are both satisfied if $\|p-p'\|_1 \le d/(8e)^2$. From these bounds, Eq. (C244) follows.

For $\beta \in [1/2, 1)$:

Setting $r = \beta$, $p = 1$ in Eq. (C296), and recalling $\|p\|_1 = 1$, it follows

$$1 \le \|p\|_\beta \le d^{1/\beta-1}. \tag{C289}$$

From Eq. (C289)

$$\|p\|_\beta^{1-\beta} \le \left(d^{1/\beta-1}\right)^{1-\beta} = d^{-2+\beta+1/\beta} \le \sqrt{d}. \tag{C290}$$

Similarly, from Lemma 28, and assuming $\|p-p'\|_1 \le d$, we have

$$\Delta(p,p') \le 4\,d^{1-\beta}\left(\|p-p'\|_1\right)^\beta \le 4d\left(\frac{\|p-p'\|_1}{d}\right)^\beta \le 4\sqrt{d\,\|p-p'\|_1}, \tag{C291}$$

$$\left|\|p'\|_\beta - \|p\|_\beta\right| \le 8\,d^{1/\beta-1/(2\beta)+1/2-\beta}\left(\|p-p'\|_1\right)^{1/2} \le 8\,d\sqrt{\|p-p'\|_1}. \tag{C292}$$

We will now see which of the two constraints, $\|p-p'\|_1 \le d$, and $\|q_\beta(p)-q_\beta(p')\|_1 \le 1/e$ is more demanding.

$$\|q_\beta(p)-q_\beta(p')\|_1 \le \frac{\Delta(p,p)}{\|p\|_\beta^\beta} \le 4\sqrt{d\|p-p'\|_1} \le \frac{1}{e} \implies \|p-p'\|_1 \le \frac{1}{(4e)^2 d} \le d. \tag{C293}$$

therefore $\|p-p'\|_1 \le d$, and $\|q_\beta(p)-q_\beta(p')\|_1 \le 1/e$ are both satisfied if $\|p-p'\|_1 \le 1/(4e)^2 d$.

Plugging this all into Eq. (C283) and simplifying the resultant expression, followed by plugging int Eq. (C264), we arrive at Eq. (C243).

∎

6. Miscellaneous Lemmas and Theorems used in the proofs to the entropy continuity Theorems 6 and 7.

Lemma 24. *Let $0 < \alpha < 1$. Then $\forall\ x,y \ge 0,\ \epsilon > 0$,*

$$|x^\alpha - y^\alpha| \le \epsilon^\alpha + \epsilon^{\alpha-1}|x-y| \tag{C294}$$

Proof. See Lemma 5 in [87]. ∎

Lemma 25 (*p-norm inequalities*)**.** *For $x \in \mathbb{C}^n$ and $p \in (0, \infty]$, define*

$$\|x\|_p := \left(\sum_{q=1}^n |x_q|^p\right)^{1/p}. \tag{C295}$$

For $p \ge 1$, this is a norm, known as the p-norm.
For $0 < r \le p$ we have the p-norm interchange inequalities

$$\|x\|_p \le \|x\|_r \le n^{\left(\frac{1}{r}-\frac{1}{p}\right)}\|x\|_p. \tag{C296}$$

Furthermore, for $p \in [1, \infty]$, we have the p-norm triangle inequality,

$$\|x+y\|_p \le \|x\|_p + \|y\|_p \tag{C297}$$

Proof. See [88]. ∎

Lemma 26 (Fannes inequality [89]). *For any $p, p' \in \mathcal{P}_d$, the following continuity bounds hold.*

$$|S_1(p) - S_1(p')| \leq \begin{cases} \|p - p'\|_1 \ln(d) - \|p - p'\|_1 \ln(\|p - p'\|_1), & \text{for all } \|p - p'\|_1 \\ \|p - p'\|_1 \ln(d) + 1/(\mathrm{e} \ln(2)), & \text{if } \|p - p'\|_1 \leq 1/(2\mathrm{e}), \end{cases} \tag{C298}$$

where S_1 is the Shannon entropy.

Proof. See [89]. ∎

Remark 27. Also see [90] for a nice tightening of the bound and [91] for bounds for the infinite dimensional case.

Theorem 8 (Mean Value Theorem). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$, and differentiable on the open interval (a,b), where $a < b$. The there exists some $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \tag{C299}$$

where $f'(c) := \frac{d}{dx} f(x) \Big|_{x=c}$.

Proof. See any introductory book to calculus. ∎

Lemma 28 (Sum difference upper bounds). *Let $p, p' \in \mathcal{P}_d$.*

1) For all $\alpha > 0$:

$$\left| \|p\|_\alpha^\alpha - \|p'\|_\alpha^\alpha \right| \leq \sum_{i=1}^d |p_i^\alpha - p_i'^\alpha| \leq 2\lceil \alpha \rceil \left(\|p - p'\|_{\alpha/\lceil \alpha \rceil} \right)^{\alpha/\lceil \alpha \rceil} \leq 2\lceil \alpha \rceil \, d^{(1 - \alpha/\lceil \alpha \rceil)} \left(\|p - p'\|_1 \right)^{\alpha/\lceil \alpha \rceil} \tag{C300}$$

2)

$$\left| \|p\|_\alpha - \|p'\|_\alpha \right| \leq \begin{cases} 4\lceil \alpha^{-1} \rceil \, d^{(\alpha^{-1} - \alpha^{-1}/\lceil \alpha^{-1} \rceil + 1/\lceil \alpha^{-1} \rceil - \alpha)} \left(\|p - p'\|_1 \right)^{1/\lceil \alpha^{-1} \rceil} & \text{for } 0 < \alpha < 1 \\ \|p - p'\|_1 & \text{for } \alpha \geq 1. \end{cases} \tag{C301}$$

Proof. It will be partitioned into two subsections, one for Eq. (C300), the other for Eq. (C301).

<div align="center">Proof of 1)</div>

The first line in Eq. (C300) follows directly from the triangle inequality. The remainder of this subsection will refer to the proof of (C300) for $\alpha > 0$. For the second line, start by defining $\alpha_1 := \alpha/\lceil \alpha \rceil \leq 1$ where $\lceil x \rceil := \min y \in \mathbb{Z}$ s.t. $y > x$. Now using the identity $(x^n - y^n) = (x - y) \sum_{k=0}^{n-1} x^k y^{n-1-k}$ for $n = 1, 2, 3, \ldots$, we have

$$\left| p_k^\alpha - p_k'^\alpha \right| = \left| \left(p_k^{\alpha_1} \right)^{\lceil \alpha \rceil} - \left(p_k'^{\alpha_1} \right)^{\lceil \alpha \rceil} \right| = \left| p_k^{\alpha_1} - p_k'^{\alpha_1} \right| \left| \sum_{n=0}^{\lceil \alpha \rceil - 1} \left(p_k^{\alpha_1} \right)^n \left(p_k'^{\alpha_1} \right)^{\lceil \alpha \rceil - 1 - n} \right| \leq \lceil \alpha \rceil \left| p_k^{\alpha_1} - p_k'^{\alpha_1} \right|. \tag{C302}$$

If $\alpha \in \mathbb{N}^+$ the proof of the second inequality in Eq. (C300) is complete. Otherwise $\alpha_1 < 1$ and we can employ Lemma 24 with $\epsilon = |p_k - p_k'|$ to achieve

$$\left| p_k^\alpha - p_k'^\alpha \right| \leq 2 \lceil \alpha \rceil \left| p_k - p_k' \right|^{\alpha_1} = 2\lceil \alpha \rceil \left| p_k - p_k' \right|^{\alpha/\lceil \alpha \rceil}, \tag{C303}$$

from which the proof of second inequality in Eq. (C300) follows. To achieve the third inequality, we employ Lemma 25 with $r = \alpha/\lceil \alpha \rceil$, $p = 1$.

<div align="center">Proof of 2)</div>

For $\alpha \geq 1$, this is easy. Using the p-norm triangle inequality, Eq. (C297), twice we have

$$-\|p - p'\|_\alpha \leq \|p\|_\alpha - \|p' - p + p\|_\alpha = \|p\|_\alpha - \|p'\|_\alpha = \|p - p' + p'\|_\alpha - \|p'\|_\alpha \leq \|p - p'\|_\alpha. \tag{C304}$$

Therefore, from the monotonicity of the p-norm, Eq. (C296), we find

$$\left| \|p\|_\alpha - \|p'\|_\alpha \right| \leq \|p - p'\|_\alpha \leq \|p - p'\|_1. \tag{C305}$$

For $\alpha \in (0, 1)$, we have to do a bit more work since the p-triangle inequality does not apply. Define $\beta_1 := \beta^{-1}/\lceil \beta^{-1} \rceil \leq 1$. We can write

$$\left| \|p\|_\alpha - \|p'\|_\alpha \right| = \left| \left[(\|p\|_\alpha^\alpha)^{\beta_1} \right]^{\lceil \alpha^{-1} \rceil} - \left[(\|p'\|_\alpha^\alpha)^{\beta_1} \right]^{\lceil \alpha^{-1} \rceil} \right| \tag{C306}$$

$$= \left| \left((\|p\|_\alpha^\alpha)^{\beta_1} - (\|p'\|_\alpha^\alpha)^{\beta_1} \right) \left| \sum_{n=0}^{\lceil \alpha^{-1} \rceil - 1} \left[(\|p\|_\alpha^\alpha)^{\beta_1} \right]^n \left[(\|p'\|_\alpha^\alpha)^{\beta_1} \right]^{\lceil \alpha^{-1} \rceil - 1 - n} \right| \right|, \tag{C307}$$

where in the last line we have applied the identity $(x^n - y^n) = (x - y) \sum_{k=0}^{n-1} x^k y^{n-1-k}$ for $n = 1, 2, 3, \ldots$. Applying Eq. (C296) for $r = \alpha$, $p = 1$, and noting $\|p\|_1 = 1$, we find

$$(\|p\|_\alpha^\alpha)^{\beta_1} \leq \left(d^{\alpha^{-1} - 1} \right)^{\alpha \beta_1} = d^{(\alpha^{-1} - 1)/\lceil \alpha^{-1} \rceil}. \tag{C308}$$

Therefore, plugging in this upper bound we find

$$\left| \|p\|_\alpha - \|p'\|_\alpha \right| \leq \left| (\|p\|_\alpha^\alpha)^{\beta_1} - (\|p'\|_\alpha^\alpha)^{\beta_1} \right| \left| \sum_{n=0}^{\lceil \alpha^{-1} \rceil - 1} \left[d^{(\alpha^{-1} - 1)/\lceil \alpha^{-1} \rceil} \right]^n \left[d^{(\alpha^{-1} - 1)/\lceil \alpha^{-1} \rceil} \right]^{\lceil \alpha^{-1} \rceil - 1 - n} \right| \tag{C309}$$

$$\leq \left| (\|p\|_\alpha^\alpha)^{\beta_1} - (\|p'\|_\alpha^\alpha)^{\beta_1} \right| \lceil \alpha^{-1} \rceil \left[d^{(\alpha^{-1} - 1)/\lceil \alpha^{-1} \rceil} \right]^{\lceil \alpha^{-1} \rceil - 1} \tag{C310}$$

$$\leq \left| (\|p\|_\alpha^\alpha)^{\beta_1} - (\|p'\|_\alpha^\alpha)^{\beta_1} \right| \lceil \alpha^{-1} \rceil d^{\alpha^{-1} - \alpha^{-1}/\lceil \alpha^{-1} \rceil + 1/\lceil \alpha^{-1} \rceil - 1}. \tag{C311}$$

Now for $\beta_1 < 1$ apply Lemma 24, with $\epsilon = \|p - p'\|_\alpha$, $x = \|p\|_\alpha^\alpha$, $y = \|p'\|_\alpha^\alpha$ to achieve

$$\left| \|p\|_\alpha - \|p'\|_\alpha \right| \leq \left| (\|p\|_\alpha^\alpha)^{\beta_1} - (\|p'\|_\alpha^\alpha)^{\beta_1} \right| \lceil \alpha^{-1} \rceil d^{\alpha^{-1} - \alpha^{-1}/\lceil \alpha^{-1} \rceil + 1/\lceil \alpha^{-1} \rceil - 1} \tag{C312}$$

$$\leq \left| \|p\|_\alpha^\alpha - \|p'\|_\alpha^\alpha \right|^{\beta_1} 2 \lceil \alpha^{-1} \rceil d^{\alpha^{-1} - \alpha^{-1}/\lceil \alpha^{-1} \rceil + 1/\lceil \alpha^{-1} \rceil - 1}. \tag{C313}$$

By inspection, we observe that the inequality also holds when $\beta_1 = 1$. We now plug in Eq. (C301), to find

$$\left| \|p\|_\alpha - \|p'\|_\alpha \right| \leq (\|p - p'\|_1)^{1/(\lceil \alpha \rceil \lceil \alpha^{-1} \rceil)} (2\lceil \alpha \rceil)^{\beta_1} 2 \lceil \alpha^{-1} \rceil d^{\alpha^{-1} - \alpha^{-1}/\lceil \alpha^{-1} \rceil + 1/\lceil \alpha^{-1} \rceil - 1}. \tag{C314}$$

Thus noting that $(2\lceil \alpha \rceil)^{\beta_1} \leq 2$, we achieve Eq. (C301) for $0 < \alpha < 1$.

∎

Appendix D: Proof of Theorem 2

In this section we detail the proof of Theorem 2 located in the main text.

Proof. We start by demonstrating part 1). Define

$$U_{\text{SCatG}}(t) = \begin{cases} \mathbb{1}_{\text{SCatG}} & \text{if } t \in [0, t_1] \\ U'_{\text{SCatG}} & \text{if } t \in [t_2, T_0] \end{cases} \tag{D1}$$

where U'_{SCatG} satisfies $\text{tr}_{\text{G}}\left[U'_{\text{SCatG}}\left(\rho_{\text{S}}^0 \otimes \rho_{\text{Cat}}^0 \otimes \tilde{\tau}_{\text{G}}\right)U'^{\dagger}_{\text{SCatG}}\right] = \rho_{\text{S}}^1 \otimes \rho_{\text{Cat}}^0$. Define

$$\sigma_{\text{SCatG}}(t) := U'_{\text{SCatG}}(t)\left(\rho_{\text{S}}^0 \otimes \rho_{\text{Cat}}^0 \otimes \tilde{\tau}_{\text{G}}\right)U'^{\dagger}_{\text{SCatG}}(t). \tag{D2}$$

It follows by the definition of t-CNO (Def. 1) and Prop. 2 that for every pair ρ_{S}^0, ρ_{S}^1 for which there exists a t-CNO from ρ_{S}^0 to ρ_{S}^1, there exists a unitary U_{SCatG} satisfying the above criteria. Since the catalyst ρ_{Cat}^0 is arbitrary, this is true iff Eq. (15) holds. Therefore $\sigma_{\text{S}}(t)$ in Eq. (D2) fulfils part 1) of the Theorem.

Recalling Def. 29 and Prop. 30, and using the identifications $A = \text{SCatG}$, $U_{\text{A}}^{\text{target}} = U'_{\text{SCatG}}$, for every unitary U_{SCatG} above, there exists an interaction term $\hat{I}_{\text{SCatClG}}$ such that using the Quasi Ideal Clock we have

$$\|\rho_{\text{SCatG}}^F(t) - \sigma_{\text{SCatG}}(t)\|_1 \leq \varepsilon_{\text{A}} = \sqrt{d_{\text{S}}d_{\text{Cat}}d_{\text{G}}\text{tr}[(\rho_{\text{S}}^0 \otimes \rho_{\text{Cat}}^0 \otimes \tilde{\tau}_{\text{G}})^2]}\,\varepsilon_{\text{Cl}}(d_{\text{Cl}}) \tag{D3}$$

$$\leq \sqrt{d_{\text{S}}d_{\text{Cat}}}\,\varepsilon_{\text{Cl}}(d_{\text{Cl}}), \tag{D4}$$

$$\|\rho_{\text{Cl}}^F(t) - \rho_{\text{Cl}}^0(t)\|_1 \leq \varepsilon_{\text{Cl}}(d_{\text{Cl}}), \tag{D5}$$

where in the last line of (D3) we have taken into account that $\tilde{\tau}_{\text{G}} = \mathbb{1}/d_{\text{G}}$. Recall that an expression for $\varepsilon(d_{\text{Cl}})$ is given by Eq. (D27). We now apply Prop. 11 with the identifications

$$\rho_{\text{SCatG}}^F(t) =: \rho_{\text{A}}, \quad \rho_{\text{Cl}}^F(t) =: \rho_{\text{B}}, \quad \rho_{\text{SCatGCl}}^F(t) =: \rho_{\text{AB}} \tag{D6}$$

$$\sigma_{\text{SCatG}}(t) =: \sigma_{\text{A}}, \quad \rho_{\text{Cl}}^0(t) =: \sigma_{\text{B}}, \quad \sigma_{\text{SCatG}}(t) \otimes \rho_{\text{Cl}}^0(t) =: \sigma_{\text{AB}} \tag{D7}$$

to achieve

$$\|\rho_{\text{SCatGCl}}^F(t) - \sigma_{\text{SCatG}}(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 \leq 2\sqrt{\varepsilon_{\text{A}}} + 2\sqrt{\varepsilon_{\text{A}} + \varepsilon_{\text{Cl}}}, \tag{D8}$$

for all $t \in [0, t_1] \cup [t_2, T_0]$, and where $\varepsilon_{\text{A}}, \varepsilon_{\text{Cl}}$ are given in Eqs. (D3), (D5). Note that here we have used the fact that $\rho_{\text{Cl}}^0(t)$ is a pure state, and thus $\epsilon_1 = 0$ in Prop. 11). Applying the data processing inequality, we find

$$\|\rho_{\text{SCatCl}}^F(t) - \sigma_{\text{SCat}}(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 \leq 2\sqrt{\varepsilon_{\text{A}}} + 2\sqrt{\varepsilon_{\text{A}} + \varepsilon_{\text{Cl}}}, \tag{D9}$$

for all $t \in [0, t_1] \cup [t_2, T_0]$. Using the triangle inequality, we have

$$\|\rho_{\text{SCatCl}}^F(t) - \rho_{\text{S}}^F(t) \otimes \rho_{\text{Cat}}^0 \otimes \rho_{\text{Cl}}^0(t)\|_1 \tag{D10}$$

$$\leq \|\rho_{\text{SCatCl}}^F(t) - \sigma_{\text{SCat}}(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 + \|\sigma_{\text{SCat}}(t) \otimes \rho_{\text{Cl}}^0(t) - \rho_{\text{S}}^F(t) \otimes \rho_{\text{Cat}}^0 \otimes \rho_{\text{Cl}}^0(t)\|_1 \tag{D11}$$

$$\leq 2\sqrt{\varepsilon_{\text{A}}(t)} + 2\sqrt{\varepsilon_{\text{A}}(t) + \varepsilon_{\text{Cl}}(t)} + \|\sigma_{\text{SCat}}(t) - \rho_{\text{S}}^F(t) \otimes \rho_{\text{Cat}}^0\|_1. \tag{D12}$$

Now we note that by definition, it follows that $\sigma_{\text{SCat}}(t) = \sigma_{\text{S}}(t) \otimes \rho_{\text{Cat}}^0$ for all $t \in [0, t_1] \cup [t_2, T_0]$. Plugging into Eq. (D12) we achieve

$$\|\rho_{\text{SCatCl}}^F(t) - \rho_{\text{S}}^F(t) \otimes \rho_{\text{Cat}}^0 \otimes \rho_{\text{Cl}}^0(t)\|_1 \tag{D13}$$

$$\leq 2\sqrt{\varepsilon_{\text{A}}} + 2\sqrt{\varepsilon_{\text{A}} + \varepsilon_{\text{Cl}}} + \|\sigma_{\text{S}}(t) - \rho_{\text{S}}^F(t)\|_1 \tag{D14}$$

$$\leq 2\sqrt{\varepsilon_{\text{A}}} + 2\sqrt{\varepsilon_{\text{A}} + \varepsilon_{\text{Cl}}} + \varepsilon_{\text{A}}, \tag{D15}$$

for all $t \in [0, t_1] \cup [t_2, T_0]$ and where in the last line, we have used Eq. (D3) after applying the data processing inequality to it. W.l.o.g. assume that $\varepsilon_{\text{A}} \leq 1$ (If this does not hold, then the following bound holds anyway since the trace distance of any state is upper bounded by 1), we achieve

$$\|\rho_{\text{SCatCl}}^F(t) - \rho_{\text{S}}^F(t) \otimes \rho_{\text{Cat}}^0 \otimes \rho_{\text{Cl}}^0(t)\|_1 \leq 2\sqrt{\varepsilon_{\text{A}}} + 2\sqrt{\varepsilon_{\text{A}} + \varepsilon_{\text{Cl}}} + \varepsilon_{\text{A}} \tag{D16}$$

$$\leq 5\sqrt{\varepsilon_{\text{A}}} + 2\sqrt{\varepsilon_{\text{Cl}}} \tag{D17}$$

$$= \left(2 + 5(d_{\text{S}}d_{\text{Cat}})^{1/4}\right)\sqrt{\varepsilon_{\text{Cl}}} \tag{D18}$$

$$= \epsilon_{emb}, \tag{D19}$$

for all $t \in [0, t_1] \cup [t_2, T_0]$. Now, recalling that ε_{Cl} is independent of $d_{\text{S}}, d_{\text{Cat}}, d_{\text{G}}$, and only a function of $d_{\text{Cl}}, t_1, t_2, T_0$, we conclude part 2) of the proof.

∎

Definition 29 (Autonomous Control device error). *Let $\rho_A^{target}(t)$ denote the idealised/targeted control of system* A, *namely*

$$\rho_A^{target}(t) = \begin{cases} \rho_A^0 & \text{for } t \in [0, t_1] \\ U_A^{target} \rho_A^0 U_A^{target\dagger} & \text{for } t \in [t_2, T_0], \end{cases} \tag{D20}$$

where we associate the time interval $[t_1, t_2]$ with the time in which the CPTP map is being implemented in the ideal case. Furthmore, let $\rho_{ACl}^F(t)$ denote the autonomous evolution of A *and the control system (the clock* Cl*),*

$$\rho_{ACl}^F(t) = e^{-it\hat{H}_{ACl}} (\rho_A \otimes \rho_{Cl}) e^{it\hat{H}_{ACl}}. \tag{D21}$$

Let $\varepsilon_A(t, d_{Cl}, d_A)$, and $\varepsilon_{Cl}(t, d_{Cl})$ be defined by the relations

$$\|\rho_A^F(t) - \rho_A^{target}(t)\|_1 \leq \varepsilon_A(t, d_{Cl}, d_A), \tag{D22}$$

.

$$\|\rho_{Cl}^F(t) - \rho_{Cl}^0(t)\|_1 \leq \varepsilon_{Cl}(t, d_{Cl}), \tag{D23}$$

where $\rho_{Cl}^0(t)$ is the free evolution of the clock,

$$\rho_{Cl}^0(t) := e^{-it\hat{H}_{Cl}} \rho_{Cl} e^{it\hat{H}_{Cl}}. \tag{D24}$$

Proposition 30. *There exists a clock state ρ_{Cl} and time independent Hamiltonian, called the Quasi-Ideal Clock [63] such that for all $t \in [0, t_1] \cup [t_2, T_0]$ and for all fixed $0 < t_1 < t_2 < T_0$, the error terms ε_A, ε_{Cl} are given by,*

$$\varepsilon_A(t, d_{Cl}, d_A) = \sqrt{d_A \text{tr}[\rho_A^2(0)]}\, \varepsilon(d_{Cl}) \tag{D25}$$

$$\varepsilon_{Cl}(t, d_{Cl}) = \varepsilon(d_{Cl}), \tag{D26}$$

where $\varepsilon(d_{Cl})$ is independent of the system A parameters, and is of order

$$\varepsilon(d_{Cl}) = \mathcal{O}\left(poly(d_{Cl}) \exp\left[-c_0 d_{Cl}^{1/4} \sqrt{\ln d_{Cl}}\right]\right), \quad \text{as } d_{Cl} \to \infty \tag{D27}$$

where the constant $c_0 > 0$ depends on t_1, t_2 and $poly(d_{Cl})$ is a polynomial in d_{Cl}.

Proof. This proposition is a direct consequence of Theorem 9 and the results in [63]. The quantities $\tilde{\epsilon}_V$, ε_c, ε_ν in Theorem 9 were calculated for a specific parametrization of the clock parameters in [63]. For the following results, see Section *Examples: 2) System error faster than power-law decay*, page 47 in [63]. For constant $\gamma_\psi \in (0, 1)$, it was found that there exists a clock with $k_0 = 0$ such that

$$\tilde{\epsilon}_V = \mathcal{O}\left(poly(d_{Cl}) e^{-2c_0 d_{Cl}^{1/4} \sqrt{\ln d_{Cl}}}\right), \tag{D28}$$

$$\varepsilon_\nu = \mathcal{O}\left(poly(d_{Cl}) e^{-2c_0 d_{Cl}^{1/4} \sqrt{\ln d_{Cl}}}\right), \tag{D29}$$

$$\frac{\|\rho_A(t) - \sigma_A(t)\|_1}{\sqrt{d_A \text{tr}[\rho_A^2(0)]}} = \mathcal{O}\left(t\, poly(d_{Cl}) e^{-2c_0 d_{Cl}^{1/4} \sqrt{\ln d_{Cl}}}\right), \tag{D30}$$

for all $t \in [0, t_1] \cup [t_2, T_0]$ and for all fixed $0 < t_1 < t_2 < T_0$. Here we have defined $2c_0 := \frac{\pi}{4} \alpha_0^2 \chi_2^2$, where α_0, χ are constants defined in [63]. Since $\gamma_\psi \in (0, 1)$ is constant it follows that ε_{LR} is exponentially small in d_{Cl}. Furthmore, from Eq. (D44) we have that ε_c is exponentially small in d_{Cl}. As such, we can calculate the order of the R.H.S. of Eq. (D52) to find

$$2\sqrt{4(\pi\tilde{\epsilon}_V)^2 + 16\varepsilon_{LR} + 10\varepsilon_{LR}^2 + 4(\varepsilon_c + \varepsilon_\nu + \varepsilon_c^2\varepsilon_\nu) + 6\varepsilon_c\varepsilon_\nu} = \mathcal{O}\left(\sqrt{poly(d_{Cl}) e^{-2c_0 d_{Cl}^{1/4} \sqrt{\ln d_{Cl}}}}\right) \tag{D31}$$

$$= \mathcal{O}\left(poly(d_{Cl}) e^{-c_0 d_{Cl}^{1/4} \sqrt{\ln d_{Cl}}}\right). \tag{D32}$$

Thus the proposition follows form Eqs. (D28) and (D31). ∎

This proposition is a generalisation of the results from [63]. Specifically, these results were proven for the special case in which $t = T_0$ in Eq. (D26). A more refined version of this proposition which could be of independent interest is detailed in the appendix, see Theorem 9.

1. Proposition 30: background, refined version and proof

a. Brief overview of the Quasi-Ideal Clock

In this section, we will recall some of the definitions in [63] specialised to a particular case. This section will set the terminology and definitions needed for Theorem 9. It is from this theorem, in conjunction with the results from [63], that Proposition 30 follows.

The time independent total Hamiltonian over system $\rho_A \otimes \rho_{Cl}$ is

$$\hat{H}_{ACl} = \hat{H}_A \otimes \mathbb{1}_{Cl} + \hat{H}_A^{int} \otimes \hat{V}_d + \mathbb{1}_A \otimes \hat{H}_{Cl}, \tag{D33}$$

where $\hat{H}_A$ is the system Hamiltonian which commutes with the target unitary U_A^{target}. The term $\hat{H}_A^{int}$ encodes the target unitary via the relation $U_A^{target} = e^{-i\hat{H}_A^{int}}$, and is detailed in [63]. The clock's free Hamiltonian, $\hat{H}_{Cl}$ is a truncated Harmonic Oscillator Hamiltonian. Namely, $\hat{H}_{Cl} = \sum_{n=0}^{d-1} \omega n |n\rangle\langle n|$. The free evolution of any initial clock state under this Hamiltonian has a period of $T_0 = 2\pi/\omega$, specifically, $e^{-iT_0\hat{H}_{Cl}}\rho_{Cl}e^{iT_0\hat{H}_{Cl}} = \rho_{Cl}$ for all ρ_{Cl}. The clock interaction term $\hat{V}_d$, takes the form,

$$\hat{V}_d = \frac{d}{T_0} \sum_{k=0}^{d-1} V_d(k) |\theta_k\rangle\langle\theta_k|, \tag{D34}$$

where the basis $\{|\theta_k\rangle\}_{k=0}^{d-1}$ is the Fourier transform of the energy eigenbasis $\{|n\rangle\}_{n=0}^{d-1}$. The function $V_d : \mathbb{R} \mapsto \mathbb{R} \cup \mathbb{H}^-$ (where $\mathbb{H}^- := \{a_0 + ib_0 : a_0 \in \mathbb{R}, b_0 < 0\}$ denotes the lower-half complex plane) is defined by

$$V_d(x) = \frac{2\pi}{d} V_0\left(\frac{2\pi}{d}x\right), \tag{D35}$$

where V_0 is an infinitely differentiable periodic function of period 2π.

Recall that for the quasi-ideal clock, the initial state is pure $\rho_{Cl} = |\Psi_{nor}(k_0)\rangle\langle\Psi_{nor}(k_0)|$, where

$$|\Psi_{nor}(k_0)\rangle = \sum_{k \in \mathcal{S}_d(k_0)} \psi(k_0; k) |\theta_k\rangle, \tag{D36}$$

$$\psi(k_0; x) = A e^{-\frac{\pi}{\sigma^2}(x-k_0)^2} e^{i2\pi n_0(x-k_0)/d}, \quad x \in \mathbb{R}. \tag{D37}$$

with $\sigma \in (0, d)$, $n_0 \in (0, d-1)$, $k_0 \in \mathbb{R}$, $A \in \mathbb{R}^+$, and $\mathcal{S}_d(k_0)$ is the set of d integers closest to k_0, defined as

$$\mathcal{S}_d(k_0) = \left\{ k : k \in \mathbb{Z} \text{ and } -\frac{d}{2} \leq k_0 - k < \frac{d}{2} \right\}. \tag{D38}$$

A is defined so that the state is normalised, namely

$$A = A(\sigma; k_0) = \frac{1}{\sqrt{\sum_{k \in \mathcal{S}_d(k_0)} e^{-\frac{2\pi}{\sigma^2}(k-k_0)^2}}} = \mathcal{O}\left(\left(\frac{2}{\sigma^2}\right)^{1/4}\right). \tag{D39}$$

In the proof of Theorem 9, we will need the two core theorems in [63] (Theorems: VIII.1, page 19 and IX.1, page 35). Namely

$$e^{-it\hat{H}_{Cl}} |\Psi_{nor}(k_0)\rangle = |\Psi_{nor}(k_0 + td/T_0)\rangle + |\varepsilon_c\rangle = \sum_{k \in \mathcal{S}_d(k_0+td/T_0)} \psi(k_0 + td/T_0; k) |\theta_k\rangle + |\varepsilon_c\rangle, \tag{D40}$$

$$\Gamma_n(t) |\Psi_{nor}(k_0)\rangle = |\bar{\Psi}_{nor}(k_0 + td/T_0, td/T_0)\rangle + |\varepsilon_\nu\rangle = \sum_{k \in \mathcal{S}_d(k_0+td/T_0)} e^{-i\Theta(td/T_0;k)} \psi(k_0 + td/T_0; k) |\theta_k\rangle + |\varepsilon_\nu\rangle, \tag{D41}$$

where

$$\Gamma_n(t) = e^{-it(\Omega_n \hat{V}_d + \hat{H}_{Cl})}, \quad \Omega_n \in \mathbb{R} \tag{D42}$$

$$\Theta(\Delta; x) = \Omega_n \int_{x-\Delta}^{x} dy V_d(y), \tag{D43}$$

and for $\sigma = \sqrt{d}$ (the more general case is covered in [63]),

$$\| \, |\varepsilon_c\rangle \, \|_2 = \varepsilon_c(t, d_{\mathrm{Cl}}) = \mathcal{O}\left(t \, poly(d) \, e^{-\frac{\pi}{4}d}\right) \text{ as } d \to \infty, \tag{D44}$$

$$\| \, |\varepsilon_\nu\rangle \, \|_2 = \varepsilon_\nu(t, d_{\mathrm{Cl}}) = \mathcal{O}\left(t \, poly(d) \, e^{-\frac{\pi}{4}\frac{d}{\zeta}}\right) \text{ as } d \to \infty, \tag{D45}$$

where $\zeta \geq 1$ is a measure of the size of the derivatives of $V_0(x)$,

$$\zeta = \left(1 + \frac{0.792\,\pi}{\ln(\pi d)}b\right)^2, \quad \text{for any}$$

$$b \geq \sup_{k \in \mathbb{N}^+} \left(2 \max_{x \in [0, 2\pi]} \left|\Omega_n V_0^{(k-1)}(x)\right|\right)^{1/k}, \tag{D46}$$

where $V_0^{(k)}(x)$ is the kth derivative with respect to x of $V_0(x)$.

Here we will restrict V_0 to a special case. We start by choosing $V_0 : \mathbb{R} \mapsto \mathbb{R}_{\geq 0}$ the normalisation of V_0 such that the integral of V_d over one period is one,

$$\int_x^{x+d} dx V_d(x) = 1, \tag{D47}$$

and let V_0 have a unique global maximum in the interval $x \in [0, 2\pi]$ at $x = x_0$. Furthmore, we define $\tilde{\epsilon}_V$ and x_{vl}, x_{vr} as in Lemma X.0.1. page 43 in [63]. Namely, let $\tilde{\epsilon}_V$ be such that

$$1 - \tilde{\epsilon}_V = \int_{x_{vl}}^{x_{vr}} dx \, V_0(x + x_0) \tag{D48}$$

for some $-\pi \leq x_{vl} < 0 < x_{vr} \leq \pi$, where, for simplicity, we assume $x_{vl} = -x_{vr}$. As in Corollary X.0.2. page 45 in [63], we will set

$$x_0 = \frac{t_2 + t_1}{2} \frac{2\pi}{T_0}, \tag{D49}$$

and let $0 < \gamma_\psi \leq 1$, $0 < x_{vr} \leq \pi$ be any combination satisfying

$$x_{vr} + \pi \gamma_\psi = \frac{t_2 - t_1}{2} \frac{2\pi}{T_0}. \tag{D50}$$

For later purposes, we also introduce an error term ε_{LR}, as

$$\varepsilon_{LR} = A^2 \frac{e^{-\frac{2\pi}{\sigma^2}(\gamma_\psi d/2 - \bar{k}(t))^2}}{1 - e^{-4\pi|\gamma_\psi d/2 - \bar{k}(t)|/\sigma^2}} \tag{D51}$$

where $\bar{k}(t) := \lfloor -d/2 + k_0 + td/T_0 + 1 \rfloor + d/2 - k_0 - td/T_0 \in [0, 1]$.

b. Clock disturbance due to implementation of the unitary

Theorem 9. *Consider the specialised potential function V_0 and quasi-ideal clock described in Subsection D 1 a. Set $k_0 = 0$. For all $t \in [kT_0, kT_0 + t_1] \cup [kT_0 + t_2, (k+1)T_0]$, $k \in \mathbb{N}^0$ and for all fixed $0 < t_1 < t_2 < T_0$, the error term, $\varepsilon_{\mathrm{Cl}}$ is given by,*

$$\|\rho_{\mathrm{Cl}}^0(t) - \rho_{\mathrm{Cl}}(t)\|_1 \leq \varepsilon_{\mathrm{Cl}}(t, d_{\mathrm{Cl}}) = 2\sqrt{4\left(\pi\tilde{\epsilon}_V\right)^2 + 16\varepsilon_{LR} + 10\varepsilon_{LR}^2 + 4(\varepsilon_c + \varepsilon_\nu + \varepsilon_c^2\varepsilon_\nu) + 6\varepsilon_c\varepsilon_\nu}, \tag{D52}$$

where $\tilde{\epsilon}_V$ is given by Eq. (D48), ε_{LR} by Eq. (D50), $\varepsilon_c = \varepsilon_c(t, d_{\mathrm{Cl}})$ is given by Eq. (D44), and $\varepsilon_\nu = \varepsilon_\nu(t, d_{\mathrm{Cl}})$ by Eq. (D45).

Proof. The main ingredients will be the core theorems in [63]. To start with, we will need expressions for the evolution of the quantum states involved. In the case of the free clock, this is straightforward. As stated in Eq. (D24), it is simply given by $\rho_{\mathrm{Cl}}^0(t) := e^{-it\hat{H}_{\mathrm{Cl}}} \rho_{\mathrm{Cl}} e^{it\hat{H}_{\mathrm{Cl}}}$.

The expression for the reduced state $\rho_{\mathrm{Cl}}(t) = \mathrm{tr}_A[\rho_{\mathrm{ACl}}(t)] = \mathrm{tr}_A[\mathrm{e}^{-it\hat{H}_{\mathrm{ACl}}}(\rho_A \otimes \rho_{\mathrm{Cl}})\,\mathrm{e}^{it\hat{H}_{\mathrm{ACl}}}]$ is a bit more involved. It was shown in [63] (Eq. 374, page 48), that $\rho_{\mathrm{Cl}}(t)$ is given by[19]

$$\rho_{\mathrm{Cl}}(t) = \sum_{n=1}^{d_A} \rho_{n,n}(0)\,|\bar{\Phi}_n(t)\rangle\langle\bar{\Phi}_n(t)|_{\mathrm{Cl}}\,, \tag{D53}$$

where $\{\rho_{n,n}(0)\}_n$ are the eigenvalues of the initial A system state ρ_A, and thus also constitute a set of normalised probabilities. $|\bar{\Phi}_n(t)\rangle_{\mathrm{Cl}}$ is defined by,

$$|\bar{\Phi}_n(t)\rangle_{\mathrm{Cl}} = \hat{\Gamma}_n(t)\,|\Psi_{\mathrm{nor}}(k_0)\rangle_{\mathrm{Cl}}\,, \quad \hat{\Gamma}_n(t) = \mathrm{e}^{-it(\Omega_n \hat{V}_d + \hat{H}_{\mathrm{Cl}})}, \tag{D54}$$

where $\{\Omega_n \in [-\pi,\pi]\}_{n=1}^{d_A}$ are a set of phases which determine the target unitary one wishes to apply on the clock, and $\hat{V}_d$ is a Hermitian operator on the clock Hilbert Space. Their construction is detailed in [63]. Given this re-cap of expressions for the dynamical evolutions of the clock, we can proceed with the proof. We start by lower bounding the Quantum Fidelity F.[20] Taking into account that $\rho_{\mathrm{Cl}}^0(t)$ is pure and using the short hand $|\Psi_{\mathrm{Cl}}(t)\rangle = \mathrm{e}^{-it\hat{H}_{\mathrm{Cl}}}\,|\Psi_{\mathrm{nor}}(k_0)\rangle$,

$$F\left(\rho_{\mathrm{Cl}}^0(t),\rho_{\mathrm{Cl}}(t)\right) = \mathrm{tr}\left[\sqrt{\sqrt{\rho_{\mathrm{Cl}}^0(t)}\rho_{\mathrm{Cl}}(t)\sqrt{\rho_{\mathrm{Cl}}^0(t)}}\right] = \mathrm{tr}\left[\sqrt{|\Psi_{\mathrm{Cl}}(t)\rangle\langle\Psi_{\mathrm{Cl}}(t)|\,\rho_{\mathrm{Cl}}(t)\,|\Psi_{\mathrm{Cl}}(t)\rangle\langle\Psi_{\mathrm{Cl}}(t)|}\right] \tag{D55}$$

$$= \mathrm{tr}\left[\sqrt{\langle\Psi_{\mathrm{Cl}}(t)|\rho_{\mathrm{Cl}}(t)|\Psi_{\mathrm{Cl}}(t)\rangle}\,|\Psi_{\mathrm{Cl}}(t)\rangle\langle\Psi_{\mathrm{Cl}}(t)|\right] = \sqrt{\langle\Psi_{\mathrm{Cl}}(t)|\rho_{\mathrm{Cl}}(t)|\Psi_{\mathrm{Cl}}(t)\rangle} \tag{D56}$$

$$= \sum_{n=1}^{d_{\mathrm{Cl}}} \rho_{n,n}(0)\left|\langle\Psi_{\mathrm{nor}}(k_0)|\mathrm{e}^{it\hat{H}_{\mathrm{Cl}}}\hat{\Gamma}_n(t)|\Psi_{\mathrm{nor}}(k_0)\rangle\right|^2. \tag{D57}$$

Thus plugging Eqs. (D40), (D41) into Eq. (D57) we find,

$$F\left(\rho_{\mathrm{Cl}}^0(t),\rho_{\mathrm{Cl}}(t)\right) = \sum_{n=1}^{d_{\mathrm{Cl}}} \rho_{n,n}(0)\Big|\underbrace{\langle\Psi_{\mathrm{nor}}(k_0+td/T_0)|\bar{\Psi}_{\mathrm{nor}}(k_0+td/T_0)\rangle}_{=:\Delta_n} + \underbrace{\langle\varepsilon_c|\bar{\Psi}_{\mathrm{nor}}(k_0+td/T_0)\rangle}_{=:\bar{\varepsilon}_c} \tag{D58}$$

$$+ \underbrace{\langle\Psi_{\mathrm{nor}}(k_0+td/T_0)|\varepsilon_\nu\rangle}_{=:\bar{\varepsilon}_\nu} + \underbrace{\langle\varepsilon_c|\varepsilon_\nu\rangle}_{=:\bar{\varepsilon}_{c\nu}}\Big|^2 \tag{D59}$$

$$= \sum_{n=1}^{d_{\mathrm{Cl}}} \rho_{n,n}(0)\left(\Delta_n + \bar{\varepsilon}_c + \bar{\varepsilon}_\nu + \bar{\varepsilon}_{c\nu}\right)\left(\Delta_n^* + \bar{\varepsilon}_c^* + \bar{\varepsilon}_\nu^* + \bar{\varepsilon}_{c\nu}^*\right) \tag{D60}$$

$$= \sum_{n=1}^{d_{\mathrm{Cl}}} \rho_{n,n}(0)\Big(|\Delta_n|^2 + \Delta_n\left(\bar{\varepsilon}_c^* + \bar{\varepsilon}_\nu^* + \bar{\varepsilon}_{c\nu}^*\right) + |\bar{\varepsilon}_c|^2 + \bar{\varepsilon}_c\left(\Delta_n^* + \bar{\varepsilon}_\nu^* + \bar{\varepsilon}_{c\nu}^*\right) + |\bar{\varepsilon}_\nu|^2 + \bar{\varepsilon}_\nu\left(\Delta_n^* + \bar{\varepsilon}_c^* + \bar{\varepsilon}_{c\nu}^*\right)\Big) \tag{D61}$$

$$\geq \sum_{n=1}^{d_{\mathrm{Cl}}} \rho_{n,n}(0)\Big(|\Delta_n|^2 - |\Delta_n|\left(|\bar{\varepsilon}_c| + |\bar{\varepsilon}_\nu| + |\bar{\varepsilon}_{c\nu}|\right) - |\bar{\varepsilon}_c|\left(|\Delta_n| + |\bar{\varepsilon}_\nu| + |\bar{\varepsilon}_{c\nu}|\right) - |\bar{\varepsilon}_\nu|\left(|\Delta_n| + |\bar{\varepsilon}_c| + |\bar{\varepsilon}_{c\nu}|\right)\Big). \tag{D62}$$

Noting that Δ_n is the overlap between two normalised states, we have $|\Delta_n| \leq 1$. Furthermore, noting the normalisation of $|\varepsilon_c\rangle$, $|\varepsilon_\nu\rangle$ in Eqs. (D44), (D45), and that the $\rho_{n,n}(0)$'s sum to one,

$$F\left(\rho_{\mathrm{Cl}}^0(t),\rho_{\mathrm{Cl}}(t)\right) \geq \left(\min_{n=1,2,3,\ldots,d_{\mathrm{Cl}}}|\Delta_n|^2\right) - 2(\varepsilon_c + \varepsilon_\nu + \varepsilon_c^2\varepsilon_\nu) - 3\varepsilon_c\varepsilon_\nu. \tag{D63}$$

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[19] In the following Eq. we have changed the notation to match that used here. Specifically $d_s \mapsto d_A$, and $c \mapsto \mathrm{Cl}$
[20] Sometimes the quantum Fidelity is defined as the square of this quantity.

We will now seek a lower bound for $|\Delta_n|$. Plugging in Eqs. (D40),(D41) into the definition of Δ_n, we find[21]

$$\Delta_n = \langle \Psi_{\text{nor}}(k_0 + td/T_0)|\bar{\Psi}_{\text{nor}}(k_0 + td/T_0)\rangle = \sum_{k \in \mathcal{S}_d(k_0+td/T_0)} e^{-i\Omega_n \int_{k-td/T_0}^{k} dy V_d(y)} \left|\psi_{\text{nor}}(k_0 + td/T_0; k)\right|^2 \qquad (D64)$$

$$= \sum_{k=\min\{\mathcal{S}_d(k_0+td/T_0)\}-td/T_0}^{k=\max\{\mathcal{S}_d(k_0+td/T_0)\}-td/T_0} e^{-i\Omega_n \int_{k}^{k+td/T_0} dy V_d(y)} \left|\psi_{\text{nor}}(k_0; k)\right|^2, \qquad (D65)$$

where in the last line we have noted that $\psi_{\text{nor}}(k_0 + td/T_0; k) = \psi_{\text{nor}}(k_0; k-td/T_0)$ and performed a change of variable. Note that if $V_d(y)$ were constant over the range of integration, it would be a constant phase factor which could be taken outside of the sum as a common factor. This happens, for example, when t is an integer multiple of the clock period T_0, due to the periodic nature of V_d. In such circumstances, Δ_n is precisely $\exp(-i\Omega_n \int_0^{T_0} dy V_d(y))$, since the summation is precisely one due to normalisation. In such circumstances, our work would be almost done. However, the phase is approximately constant, for times "before" and "after" the unitary is applies, i.e. for times $t \in [0, t_1]$ and $t \in [t_2, T_0]$. The remainder fo the proof will be focused on bounding the errors involved in such approximations.

Solving the max and min, we find

$$\Delta_n = \sum_{k=\lfloor -d/2+k_0+1+td/T_0 \rfloor -td/T_0}^{\lfloor d/2+k_0+td/T_0 \rfloor -td/T_0} e^{-i\Omega_n \int_{k}^{k+td/T_0} dy V_d(y)} \left|\psi_{\text{nor}}(k_0; k)\right|^2 \qquad (D66)$$

$$= \sum_{k=k_0-d/2+\bar{k}(t)}^{k_0+d/2+\bar{k}(t)-1} e^{-i\Omega_n \int_{k}^{k+td/T_0} dy V_d(y)} \left|\psi_{\text{nor}}(k_0; k)\right|^2, \qquad (D67)$$

where $\bar{k}(t) := \lfloor -d/2 + k_0 + td/T_0 + 1 \rfloor + d/2 - k_0 - td/T_0 \in [0, 1]$. This follows from noting that $\bar{k}(t)$ is a solution to both equations $\lfloor -d/2+k_0+1+td/T_0 \rfloor - td/T_0 = k_0 - d/2 + \bar{k}(t)$ and $\lfloor d/2+k_0+td/T_0 \rfloor - td/T_0 = k_0 + d/2 + \bar{k}(t) - 1$, since we can use the identity $1 = \lfloor -x+y+1 \rfloor - \lfloor x+y \rfloor + 2x$, for $2x \in \mathbb{Z}$, $y \in \mathbb{R}$ and set $x = d/2$, $y = k_0 + td/T_0$. For simplicity, we will now assume that $k_0 = 0$. We will now break the sum up into three contributions where the first two will correspond to the "Gaussian tails" to the "left" (Δ_L) and "right" (Δ_R) of $k = k_0 = 0$, and a "central term" (Δ_C) corresponding to the region $k \approx k_0 = 0$. Namely,

$$\Delta_n = \Delta_L + \Delta_C + \Delta_R. \qquad (D68)$$

We introduce[22] $\gamma_\psi \in (0, 1)$ and start with abounding Δ_L:

$$\Delta_L := \sum_{k=-d/2+\bar{k}(t)}^{\bar{k}(t)-\gamma_\psi d/2} e^{-i\Omega_n \int_{k}^{k+td/T_0} dy V_d(y)} \left|\psi_{\text{nor}}(k_0; k)\right|^2. \qquad (D69)$$

Therefore,

$$|\Delta_L| \leq \sum_{k=-d/2+\bar{k}(t)}^{\bar{k}(t)-\gamma_\psi d/2} \left|\psi_{\text{nor}}(k_0; k)\right|^2 = \sum_{y=\gamma_\psi d/2-\bar{k}(t)}^{d/2-\bar{k}(t)} A^2 e^{-\frac{2\pi}{\sigma^2}(-y)^2} \qquad (D70)$$

$$\leq \sum_{y=\gamma_\psi d/2-\bar{k}(t)}^{\infty} A^2 e^{-\frac{2\pi}{\sigma^2}(-y)^2} \leq A^2 \frac{e^{-\frac{2\pi}{\sigma^2}(\bar{k}(t)-\gamma_\psi d/2)^2}}{1 - e^{-4\pi|\bar{k}(t)-\gamma_\psi d/2|/\sigma^2}} = \varepsilon_{LR}, \qquad (D71)$$

where in the last line we have used Lemma J.0.1, page 59 from [63] to bound the infinite summation. We have also used the definition of ε_{LR} in Eq. (D50). Similarly, for Δ_R,

$$\Delta_R := \sum_{k=\bar{k}(t)+\gamma_\psi d/2}^{\bar{k}(t)+d/2} e^{-i\Omega_n \int_{k}^{k+td/T_0} dy V_d(y)} \left|\psi_{\text{nor}}(k_0; k)\right|^2. \qquad (D72)$$

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[21] We will use the convention $\sum_{y=a}^{b} f(y) = f(a) + f(a+1) + \ldots + f(b)$, where $a, b \in \mathbb{R}$, $b - a \in \mathbb{Z}$ and the sum is defined to be zero if $b - a \leq 0$.

[22] In order for the following summations to make sense according to our convention, further constraints will have to be paced on γ_ψ. We will derive these in Eq. (D76).

Therefore,

$$|\Delta_R| \leq \sum_{k=\bar{k}(t)+\gamma_\psi d/2}^{\bar{k}(t)+d/2} |\psi_{\text{nor}}(k_0;k)|^2 \leq \sum_{k=\bar{k}(t)+\gamma_\psi d/2}^{\infty} |\psi_{\text{nor}}(k_0;k)|^2 \tag{D73}$$

$$\leq \sum_{k=\bar{k}(t)-\gamma_\psi d/2}^{\infty} |\psi_{\text{nor}}(k_0;k)|^2 \leq A^2 \frac{e^{-\frac{2\pi}{\sigma^2}(\bar{k}(t)-\gamma_\psi d/2)^2}}{1-e^{-4\pi|\bar{k}(t)-\gamma_\psi d/2|/\sigma^2}} = \varepsilon_{LR} \tag{D74}$$

For Δ_C, we have

$$\Delta_C := \sum_{k=\bar{k}(t)-\gamma_\psi d/2+1}^{\bar{k}(t)+\gamma_\psi d/2-1} e^{-i\Omega_n \int_k^{k+td/T_0} dy V_d(y)} |\psi_{\text{nor}}(k_0;k)|^2 . \tag{D75}$$

We can now lower bound $|\Delta_n|^2$. In order for the sums Δ_L, Δ_C, Δ_R to be well defined, we need the constraints, $-d\gamma_\psi/2 + \bar{k}(t) - (-d/2+\bar{k}(t)) = n_1$, $d\gamma_\psi/2 + \bar{k}(t) - 1 - (-d\gamma_\psi/2 + \bar{k}(t)+1) = n_2$, $d\gamma_\psi/2 + \bar{k}(t) - (d/2+\bar{k}(t)) = n_3$, for $n_1, n_2, n_3 \in \mathbb{Z}$. A solution for γ_ψ is

$$\gamma_\psi = \gamma_\psi(m) = \frac{m-2}{d} \in (0,1), \quad \text{where } m = \begin{cases} 4,6,8,\ldots,d+2 \text{ if } d=2,4,6,\ldots \\ 3,5,7,\ldots,d+2 \text{ if } d=3,5,7,\ldots \end{cases} \tag{D76}$$

Therefore, $\Delta_C = \Delta_n - \Delta_L - \Delta_R$, and thus

$$|\Delta_C| \leq |\Delta_n| + |\Delta_L| + |\Delta_R|, \tag{D77}$$

$$|\Delta_C|^2 \leq |\Delta_n|^2 + |\Delta_L|^2 + |\Delta_R|^2 + 2|\Delta_L\Delta_R| + 2|\Delta_n|(|\Delta_L| + |\Delta_R|) \tag{D78}$$

$$\leq |\Delta_n|^2 + |\Delta_L|^2 + |\Delta_R|^2 + 2|\Delta_L\Delta_R| + 2(|\Delta_L| + |\Delta_R|). \tag{D79}$$

Hence using Eq. (D63) and Eqs. (D70) ,(D73),

$$F\left(\rho_{\text{Cl}}^0(t), \rho_{\text{Cl}}(t)\right) \geq \left(\min_{n=1,2,3,\ldots,d_{\text{Cl}}} |\Delta_C(n)|^2\right) - 5\varepsilon_{LR}^2 - 4\varepsilon_{LR} - 2(\varepsilon_c + \varepsilon_\nu + \varepsilon_c^2\varepsilon_\nu) - 3\varepsilon_c\varepsilon_\nu, \tag{D80}$$

where recall that $\Delta_C = \Delta_C(n)$, and is given by Eq. (D75). Out next task will be to lower bound $|\Delta_C(n)|$ away from 1. For this, we will calculate the absolute value of $|\Delta_C(n)|$ explicitly adding a phase factor $\beta_n(t) \in \mathbb{R}$ in the process which will help us to achieve the desired bound.

$$|\Delta_C(n)|^2 = |\Delta_C(n) e^{i\beta_n(t)}|^2 \tag{D81}$$

$$= \left(\sum_{k=\bar{k}(t)-\gamma_\psi d/2+1}^{\bar{k}(t)+\gamma_\psi d/2-1} \cos(\beta_n(t) - \Theta_k(n,t)) |\psi_{\text{nor}}(k_0;k)|^2\right)^2 + \left(\sum_{k=\bar{k}(t)-\gamma_\psi d/2+1}^{\bar{k}(t)+\gamma_\psi d/2-1} \sin(\beta_n(t) - \Theta_k(n,t)) |\psi_{\text{nor}}(k_0;k)|^2\right)^2 \tag{D82}$$

$$\geq \left(\sum_{k=\bar{k}(t)-\gamma_\psi d/2+1}^{\bar{k}(t)+\gamma_\psi d/2-1} \cos(\beta_n(t) - \Theta_k(n,t)) |\psi_{\text{nor}}(k_0;k)|^2\right)^2 , \tag{D83}$$

where

$$\Theta_k(n,t) := \Omega_n \int_k^{k+td/T_0} dy V_d(y). \tag{D84}$$

Before proceeding to bound Eq. (D81), we will bound the common factor term which is approximately one. We find

$$\sum_{k=\bar{k}(t)-\gamma_\psi d/2+1}^{\bar{k}(t)+\gamma_\psi d/2-1} |\psi_{\text{nor}}(k_0;k)|^2 = \sum_{k\in\mathcal{S}_d(k_0+td/T_0)} |\psi_{\text{nor}}(k_0;k-td/T_0)|^2 - \sum_{k=\bar{k}(t)-d/2}^{\bar{k}(t)-\gamma_\psi d/2} |\psi_{\text{nor}}(k_0;k)|^2 - \sum_{k=\bar{k}(t)+\gamma_\psi d/2}^{\bar{k}(t)+d/2} |\psi_{\text{nor}}(k_0;k)|^2 \tag{D85}$$

$$= 1 - \sum_{k=\bar{k}(t)-d/2}^{\bar{k}(t)-\gamma_\psi d/2} |\psi_{\text{nor}}(k_0;k)|^2 - \sum_{k=\bar{k}(t)+\gamma_\psi d/2}^{\bar{k}(t)+d/2} |\psi_{\text{nor}}(k_0;k)|^2 \tag{D86}$$

$$\geq 1 - |\Delta_L| - |\Delta_R| \geq 1 - 2\varepsilon_{LR}. \tag{D87}$$

We now introduce condition 1) :

$$\cos\left(\beta_n(t) - \Theta_k(n,t)\right) \geq 0 \quad \forall\, k \in \mathcal{I}_{\gamma_\psi}, \tag{D88}$$

where $\mathcal{I}_{\gamma_\psi} := \{\bar{k}(t) - \gamma_\psi d/2 + 1, \bar{k}(t) - \gamma_\psi d/2 + 2, \ldots, \bar{k}(t) + \gamma_\psi d/2 - 1\}$. If condition Eq. (D88) is satisfied,

$$|\Delta_C(n)|^2 \geq \left(\min_{l \in \mathcal{I}_{\gamma_\psi}}\left\{\cos\left(\beta_n(t) - \Theta_l(n,t)\right)\right\}\right)^2 \left(\sum_{k=\bar{k}(t)-\gamma_\psi d/2+1}^{\bar{k}(t)+\gamma_\psi d/2-1} |\psi_{\mathrm{nor}}(k_0;k)|^2\right)^2 \tag{D89}$$

$$\geq \left(\min_{l \in \mathcal{I}_{\gamma_\psi}}\left\{\cos\left(\beta_n(t) - \Theta_l(n,t)\right)\right\}\right)^2 (1 - 2\varepsilon_{LR})^2 \tag{D90}$$

$$= \left(\min_{l \in \mathcal{I}_{\gamma_\psi}}\left\{\cos\left(\beta_n(t) - \Theta_l(n,t)\right)\right\}\right)^2 \left(1 - 4\varepsilon_{LR} + 4\varepsilon_{LR}^2\right) \tag{D91}$$

$$\geq \left(\min_{l \in \mathcal{I}_{\gamma_\psi}}\left\{\cos\left(\beta_n(t) - \Theta_l(n,t)\right)\right\}\right)^2 - 4\varepsilon_{LR}. \tag{D92}$$

Now set

$$\beta_n(t) = \begin{cases} k\Omega_n & \text{if } t \in [kT_0, kT_0 + t_1], \\ (k+1)\Omega_n & \text{if } t \in [kT_0 + t_2, (k+1)T_0], \end{cases} \tag{D93}$$

for $k \in \mathbb{N}^0$. Using the identity $\cos(x) \geq 1 - x^2$ for all $x \in \mathbb{R}$,

$$\min_{l \in \mathcal{I}_{\gamma_\psi}}\left\{\cos\left(\beta_n(t) - \Theta_l(n,t)\right)\right\} \geq 1 - \max_{l \in \mathcal{I}_{\gamma_\psi}}\left\{(\beta_n(t) - \Theta_l(n,t))^2\right\} = 1 - \max_{l \in \mathcal{I}_{\gamma_\psi}}\left\{\left(\beta_n(t) - \Omega_n \int_l^{l+td/T_0} dy V_d(y)\right)^2\right\}. \tag{D94}$$

Recalling that V_d has period d, is non-negative, and has an integral of one when integrated over one period, we can simplify the integral over V_d. Specifically, for $t \in [kT_0, (k+1)T_0]$, $k \in \mathbb{N}^0$, define $\tau \in [0, T_0]$ by $t = kT_0 + \tau$. It follows,

$$\int_l^{l+td/T_0} dy V_d(y) = \int_{l-kd}^{l+td/T_0-kd} dx V_d(kd + x) = \int_{l-kd}^{l+\tau d/T_0} dx V_d(x) = k + \int_l^{l+\tau d/T_0} dx V_d(x). \tag{D95}$$

Therefore, using Eq. (D93), we have for $k \in \mathbb{N}^0$,

$$\beta_n(t) - \Omega_n \int_l^{l+td/T_0} dy V_d(y) = \begin{cases} -\Omega_n \int_l^{l+\tau d/T_0} dy V_d(y) \leq 0, & \text{for } t \in [kT_0, kT_0 + t_1],\ \tau \in [0, t_1] \\ \Omega_n\left(1 - \int_l^{l+\tau d/T_0} dy V_d(y)\right) \geq 0, & \text{for } t \in [kT_0 + t_2, (k+1)T_0],\ \tau \in [t_2, T_0]. \end{cases} \tag{D96}$$

Therefore, plugging into Eq, (D94), we find

$$\min_{l \in \mathcal{I}_{\gamma_\psi}}\left\{\cos\left(\beta_n(t) - \Theta_l(n,t)\right)\right\} \tag{D97}$$

$$\geq \begin{cases} 1 - \Omega_n^2\left(\max_{l \in \mathcal{I}_{\gamma_\psi}}\left\{\int_l^{l+\tau d/T_0} dy V_d(y)\right\}\right)^2, & \text{for } t \in [kT_0, kT_0 + t_1],\ \tau \in [0, t_1] \\ 1 - \Omega_n^2\left(\max_{l \in \mathcal{I}_{\gamma_\psi}}\left\{1 - \int_l^{l+\tau d/T_0} dy V_d(y)\right\}\right)^2, & \text{for } t \in [kT_0 + t_2, (k+1)T_0],\ \tau \in [t_2, T_0]. \end{cases} \tag{D98}$$

Our next aim will be to find bounds on τ for which the solutions to the maximisations in Eq. (D98) is small. First note,

$$\int_y^{y+\tau d/T_0} dx V_d(x) = \frac{2\pi}{d} \int_y^{y+\tau d/T_0} dx V_0\left(\frac{2\pi}{d}x\right) = \int_{2\pi y/d - x_0}^{2\pi y/d + \tau 2\pi/T_0 - x_0} dx V_0\left(x + x_0\right). \tag{D99}$$
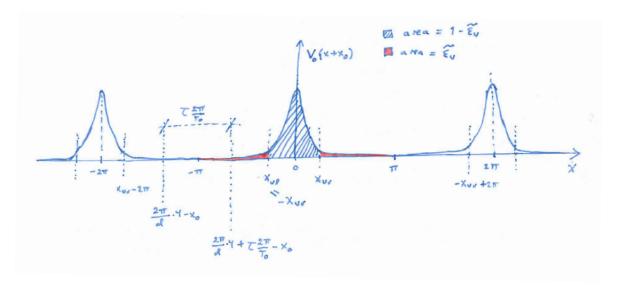
FIG. 5: Depiction of the periodic potential function $V_0(x + x_0)$ with the parameters x_{vl}, x_{vr} and $\tilde{\epsilon}_V$ introduced in Eq. (D48). Observe that the area under the potential $V_0(x + x_0)$ (i.e. it's integral) between $x = x_{vr} - 2\pi$ and $x = x_{vl}$ is $\tilde{\epsilon}_V$ due to symmetry. The quantities $\frac{2\pi}{d}y - x_0$ and $\frac{2\pi}{d}y + \tau\frac{2\pi}{T_0} - x_0$ are the lower and upper integration limits in Eq. (D99).

We can now use Eq. (D99) to find conditions for the time "before the unitary is applied", namely conditions for which

$$\int_y^{y + \tau d/T_0} dx V_d(x) \leq 1 - (1 - \tilde{\epsilon}_V) = \tilde{\epsilon}_V. \tag{D100}$$

From Fig. 5 and Eq. (D99) we observe that Eq. (D100) is satisfied for all $y \in \mathcal{I}_{\gamma_\psi}$ if

$$x_{vr} - 2\pi \leq \frac{2\pi}{d}y - x_0 \tag{D101}$$

$$\frac{2\pi}{d}y + \tau\frac{2\pi}{T_0} - x_0 \leq x_{vl} = -x_{vr}, \tag{D102}$$

for all $y \in \mathcal{I}_{\gamma_\psi}$. Or equivalently, if

$$\frac{2\pi}{d}\min_{y \in \mathcal{I}_{\gamma_\psi}}\{y\} \geq x_{vr} - 2\pi + x_0 \tag{D103}$$

$$\frac{2\pi}{d}\max_{y \in \mathcal{I}_{\gamma_\psi}}\{y\} \leq x_0 - \frac{2\pi}{T_0}\tau - x_{vr}, \tag{D104}$$

from which it follows,

$$x_0 \leq 2\pi - x_{vr} - \pi\gamma_\psi + \frac{2\pi}{d}(\bar{k}(t) + 1) \tag{D105}$$

$$0 \leq \tau\frac{2\pi}{T_0} \leq x_0 - x_{vr} - \pi\gamma_\psi - \frac{2\pi}{d}(\bar{k}(t) - 1). \tag{D106}$$

Thus recalling that $\bar{k}(t) \in [0, 1]$, sufficient conditions on x_0 and τ for Eq. (D100) to be satisfied are

$$x_0 \leq 2\pi - x_{vr} - \pi\gamma_\psi \tag{D107}$$

$$0 \leq \tau\frac{2\pi}{T_0} \leq x_0 - x_{vr} - \pi\gamma_\psi. \tag{D108}$$

Recalling definitions Eqs. (D49) , (D50), we see that Eq. (D107) and Eq. (D108) are equivalent to

$$\begin{aligned} t_2 &\leq T_0, \\ 0 \leq \tau &\leq t_1, \end{aligned} \tag{D109}$$

respectively. Therefore taking into account Eq. (D100), we can simplify Eq. (D97) to find

$$\min_{l \in \mathcal{I}_{\gamma_\psi}} \left\{ \cos \left(\beta_n(t) - \Theta_l(n, t) \right) \right\} \tag{D110}$$

$$\geq \begin{cases} 1 - (\Omega_n \tilde{\epsilon}_V)^2, & \text{for } t \in [kT_0, kT_0 + t_1], \ \tau \in [0, t_1] \\ 1 - \Omega_n^2 \left(\max_{l \in \mathcal{I}_{\gamma_\psi}} \left\{ 1 - \int_l^{l + \tau d/T_0} dy V_d(y) \right\} \right)^2, & \text{for } t \in [kT_0 + t_2, (k+1)T_0], \ \tau \in [t_2, T_0]. \end{cases} \tag{D111}$$

Similarly, we can work out conditions for the time "after a unitary has been applied". We find

$$\int_y^{y + \tau d/T_0} dx V_d(x) \geq 1 - \tilde{\epsilon}_V, \tag{D112}$$

for all $y \in \mathcal{I}_{\gamma_\psi}$ if

$$\frac{2\pi}{d} y - x_0 \leq -x_{vr} \tag{D113}$$

$$x_{vr} \leq \frac{2\pi}{d} y + \tau \frac{2\pi}{T_0} - x_0 \tag{D114}$$

for all $y \in \mathcal{I}_{\gamma_\psi}$. Or equivalently, if

$$x_{vr} + \max_{y \in \mathcal{I}_{\gamma_\psi}} \{y\} \frac{2\pi}{d} \leq x_0 \tag{D115}$$

$$x_{vr} + \frac{2\pi}{d} \max_{y \in \mathcal{I}_{\gamma_\psi}} \{-y\} + x_0 \leq \tau \frac{2\pi}{T_0}, \tag{D116}$$

from which it follows

$$x_{vr} + \frac{2\pi}{d} (\bar{k}(t) - 1) + \pi \gamma_\psi \leq x_0 \tag{D117}$$

$$x_{vr} - \frac{2\pi}{d} (\bar{k}(t) + 1) + \pi \gamma_\psi + x_0 \leq \tau \frac{2\pi}{T_0} \tag{D118}$$

Thus recalling that $\bar{k}(t) \in [0, 1]$, sufficient conditions on x_0 and τ for Eq. (D112) to be satisfied are

$$x_{vr} + \pi \gamma_\psi \leq x_0, \tag{D119}$$

$$x_{vr} + \pi \gamma_\psi + x_0 \leq \tau \frac{2\pi}{T_0}. \tag{D120}$$

Recalling definitions Eqs. (D49) , (D50), we see that Eq. (D119) and Eq. (D120) are equivalent to

$$\begin{aligned} 1 &\leq 1, \\ t_2 &\leq \tau, \end{aligned} \tag{D121}$$

respectively. Therefore, from Eqs. (D110), (D112), we have for all $k \in \mathbb{N}^0$ and for all $n = 1, 2, 3, \ldots, d_A$,

$$\min_{l \in \mathcal{I}_{\gamma_\psi}} \left\{ \cos \left(\beta_n(t) - \Theta_l(n, t) \right) \right\} \tag{D122}$$

$$\geq \begin{cases} 1 - (\Omega_n \tilde{\epsilon}_V)^2, & \text{for } t \in [kT_0, kT_0 + t_1], \\ 1 - (\Omega_n \tilde{\epsilon}_V)^2, & \text{for } t \in [kT_0 + t_2, (k+1)T_0], \end{cases} \tag{D123}$$

$$\geq 1 - (\pi \tilde{\epsilon}_V)^2, \quad \text{for } t \in [kT_0, kT_0 + t_1] \cup [kT_0 + t_2, (k+1)T_0]. \tag{D124}$$

Thus plugging into Eqs. (D89), (D80), we find for $\tilde{\epsilon}_V \leq 1/\pi$ and for all $t \in [kT_0, kT_0 + t_1] \cup [kT_0 + t_2, (k+1)T_0]$,

$$F \left(\rho_{\text{Cl}}^0(t), \rho_{\text{Cl}}(t) \right) \geq \left(1 - (\pi \tilde{\epsilon}_V)^2 \right)^2 - 4\varepsilon_{LR} - 5\varepsilon_{LR}^2 - 4\varepsilon_{LR} - 2(\varepsilon_c + \varepsilon_\nu + \varepsilon_c^2 \varepsilon_\nu) - 3\varepsilon_c \varepsilon_\nu \tag{D125}$$

$$\geq 1 - 2 \left(\pi \tilde{\epsilon}_V \right)^2 - 8\varepsilon_{LR} - 5\varepsilon_{LR}^2 - 2(\varepsilon_c + \varepsilon_\nu + \varepsilon_c^2 \varepsilon_\nu) - 3\varepsilon_c \varepsilon_\nu. \tag{D126}$$

Therefore, using the relationship between quantum fidelity and trace distance for two states ρ, σ, $\|\sigma - \rho\|_1/2 \leq \sqrt{1 - F^2(\sigma, \rho)}$ [92], and if line (D126) is non-negative and $\tilde{\epsilon}_V \leq 1/\pi$, we find

$$\|\rho_{\text{Cl}}^0(t) - \rho_{\text{Cl}}(t)\|_1 \leq 2\sqrt{(1 - F^2(\rho_{\text{Cl}}^0(t), \rho_{\text{Cl}}(t)))} \leq 2\sqrt{4(\pi\tilde{\varepsilon}_V)^2 + 16\varepsilon_{LR} + 10\varepsilon_{LR}^2 + 4(\varepsilon_c + \varepsilon_\nu + \varepsilon_c^2\varepsilon_\nu) + 6\varepsilon_c\varepsilon_\nu},$$
(D127)

for all $t \in [kT_0, kT_0 + t_1] \cup [kT_0 + t_2, (k+1)T_0]$, $k \in \mathbb{N}^0$. Finally, to remove the restrictions (i.e to remove the assumptions that either/or $\tilde{\epsilon}_V \leq 1/\pi$ and line (D126) being non-negative,) we note that if either of these restrictions do not hold, then the bound Eq. (D127) satisfies

$$2\sqrt{4(\pi\tilde{\varepsilon}_V)^2 + 16\varepsilon_{LR} + 10\varepsilon_{LR}^2 + 4(\varepsilon_c + \varepsilon_\nu + \varepsilon_c^2\varepsilon_\nu) + 6\varepsilon_c\varepsilon_\nu} \geq 2\sqrt{2}.$$
(D128)

However, $\|\rho - \sigma\|_1 \leq 2 \leq 2\sqrt{2}$ for all states ρ, σ. Therefore we conclude that Eq. (D127) holds in general.

∎

Appendix E: Continuity for perturbation: proof of Proposition 37

In this section we evaluate the norm of the difference between $e^{i(H+V)}$ and e^{iH} for Hermitian H and V, along the lines of [93]. For completeness we shall prove the result basing our proof solely on the version of mean value theorem for vector valued functions of [94], in the form taken from [95]:

Proposition 31. *Let f be defined on interval $[a, b] \subset \mathbb{R}$ with values in a d-dimensional linear space. Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exist numbers $\{c_k\}_{k=1}^d$ with $c_k \in (a, b)$ and $\{\lambda_k\}_{k=1}^d$ satisfying $\sum_k \lambda_k = 1$ such that such that*

$$f(b) - f(a) = (b - a)\sum_{k=1}^d \lambda_k f'(c_k).$$
(E1)

Using this we shall prove a version of Taylor's reminder theorem:

Proposition 32. *Let F a function defined on interval $[0, 1]$ with values in a d-dimensional linear space. Let F be $n + 1$ times differentiable on interval $(0, 1)$ and continuous on $[0, 1]$. We then have*

$$F(1) - \sum_{k=0}^n \frac{F^{(k)}(0)}{k!} = \sum_{l=1}^d \lambda_l \frac{(1 - t_l)^n}{n!} F^{(n+1)}(t_l),$$
(E2)

for some λ_l's satisfying $\sum_{l=1}^d \lambda_l = 1$ and some $t_l \in (0, 1)$, where $F^{(k)}(x) := \frac{d^k}{dx^k} F(x)$.

Proof. Following [93] we consider function G defined as

$$G(t) = F(t) + \sum_{k=1}^n \frac{(1 - t)^k}{k!} F^{(k)}(t).$$
(E3)

We see that

$$G(1) = F(1), \quad G(0) = \sum_{k=0}^n \frac{F^{(k)}(0)}{k!}$$
(E4)

and G satisfies assumptions of Prop. 31. Applying this Proposition to G for $a = 0$, $b = 1$, we obtain the desired result.

∎

We can apply the above proposition to the case $n = 1$ and get

$$F(1) - F(0) = F'(0) + \sum_{l=1}^d \lambda_l \frac{(1 - t_l)^2}{2} F''(t_l)$$
(E5)

which implies (by convexity of norm, and triangle inequality):

$$\|F(1) - F(0)\|_\infty \leq \|F'(0)\|_\infty + \frac{1}{2}\max_{t \in (0,1)} \|F''(t)\|_\infty.$$
(E6)

Consider now $F(t) = e^{i(H+tV)}$. We obtain the following.

Lemma 33. *Let $F(t) = e^{i(H+tV)}$ for Hermitian matrices H, V. We then have*

$$\|F'(t)\|_\infty \leq \|V\|_\infty, \quad \|F''(t)\|_\infty \leq \|V\|_\infty^2 \tag{E7}$$

for $t \in (0,1)$.

Proof. We use the following general formula

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{X(t)} = \int_0^1 e^{\alpha X(t)}\frac{\mathrm{d}X(t)}{\mathrm{d}t}e^{(1-\alpha)X(t)}\mathrm{d}\alpha. \tag{E8}$$

For $X = i(H + Vt)$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{i(H+Vt)} = \int_0^1 e^{\alpha i(H+Vt)}Ve^{(1-\alpha)i(H+Vt)}\mathrm{d}\alpha = \int_0^1 U_1 V U_2 \mathrm{d}\alpha \tag{E9}$$

with U_1, U_2 unitaries. Using convexity and multiplicativity of operator norm, and $\|U\|_\infty = 1$ for unitaries we get

$$F'(t) \leq \int_0^1 \|V\|_\infty \mathrm{d}\alpha = \|V\|_\infty. \tag{E10}$$

Similarly we have

$$
\begin{aligned}
F''(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}e^{i(H+Vt)} &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha i(H+Vt)}\right)iVe^{(1-\alpha)i(H+Vt)} + e^{\alpha i(H+Vt)}iV\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{(1-\alpha)i(H+Vt)}\right)\mathrm{d}\alpha = \\
&= \int_0^1 \Bigg\{ \int_0^1 e^{\beta\alpha i(H+Vt)}\alpha iV\, e^{(1-\beta)\alpha i(H+Vt)}\, iV\, e^{(1-\alpha)i(H+Vt)}\mathrm{d}\beta \\
&\quad + \int_0^1 e^{\alpha i(H+Vt)}\, iV\, e^{\beta\alpha i(H+Vt)}\alpha iV\, e^{(1-\beta)\alpha i(H+Vt)}\mathrm{d}\beta \Bigg\}\mathrm{d}\alpha \\
&= \int_0^1 \mathrm{d}\alpha\left\{ \int_0^1 i^2\alpha V_1 V V_2 V V_3 \mathrm{d}\beta + \int_0^1 i^2\alpha W_1 V W_2 V W_3 \mathrm{d}\beta \right\}
\end{aligned} \tag{E11}
$$

with V_i and W_j being unitary. As before, passing to norms, using convexity of norm, multiplicativity of norm and triangle inequality, we arrive at

$$\|F''(t)\|_\infty \leq 2\|V\|_\infty^2\int_0^1 \alpha\,\mathrm{d}\alpha = \|V\|_\infty^2. \tag{E12}$$

∎

Remark 34. Similarly one can prove that $\|F^{(k)}(t)\|_\infty \leq \|V\|_\infty^k$, $k \in \mathbb{N}$.

Now, combining Lemma 33 with formula (E6) we obtain

Proposition 35. *We have*

$$\|e^{i(H+V)} - e^{iH}\|_\infty \leq \|V\|_\infty + \frac{1}{2}\|V\|_\infty^2. \tag{E13}$$

Proof. We obtain the above equation by noting that for $F(t) = e^{i(H+tV)}$ we have $F(1) = e^{i(H+V)}$, $F(0) = e^{iH}$, and inserting these into (E6) and using (E7). ∎

To obtain bounds on states, we need the following well known fact (a special case of Hölder type inequalities [96]).

Lemma 36. *For arbitrary operators A, B in finite dimensional Hilbert space we have*

$$\|AB\|_1 \leq \|A\|_1\|B\|_\infty, \tag{E14}$$

where $\|\cdot\|_\infty$ denotes the infinity norm and $\|\cdot\|_1$ the one norm.

For the following proposition, we need to recall Eqs. (23), (21), (22) from the main text. We reproduce them here for convenience:

$$\rho_S^1 \otimes \rho_{Cat}^0 = \mathrm{tr}_G \left[e^{-i\hat{I}_{SCatG}^{int}} (\rho_S^0 \otimes \rho_{Cat}^0 \otimes \tau_G) e^{i\hat{I}_{SCatG}^{int}} \right], \tag{E15}$$

$$\sigma_{SCat}^1 := \mathrm{tr}_G \left[e^{-i\left(\hat{I}_{SCatG}^{int} + \delta\hat{I}_{SCatG}^{int}\right)} (\rho_S^0 \otimes \rho_{Cat}^0 \otimes \tau_G) e^{i\left(\hat{I}_{SCatG}^{int} + \delta\hat{I}_{SCatG}^{int}\right)} \right], \tag{E16}$$

$$\epsilon_\sigma := \|\sigma_{SCat}^1 - \rho_S^1 \otimes \rho_{Cat}^0\|_1. \tag{E17}$$

Proposition 37. *For all states $\rho_S^0, \sigma_S^1, \rho_{Cat}^0$ and Gibbs states τ_G, and for all Hermitian operators $\hat{I}_{SCatG}^{int}$, $\delta\hat{I}_{SCatG}^{int}$ satisfying Eq. (E15), the following bound on ϵ_σ, defined in Eq. (E17), holds:*

$$\epsilon_\sigma \le 2\|\delta\hat{I}_{SCatG}^{int}\|_\infty + \|\delta\hat{I}_{SCatG}^{int}\|_\infty^2. \tag{E18}$$

Proof. Let $U = e^{i(H+V)}$ and $U_0 = e^{iH}$. Then for any state ρ we have

$$\|U\rho U^\dagger - U_0\rho U_0^\dagger\|_1 = \|U\rho U^\dagger - U\rho U_0^\dagger + U\rho U_0^\dagger - U_0\rho U_0^\dagger\|_1 \le \|U\rho U^\dagger - U\rho U_0^\dagger\|_1 + \|U\rho U_0^\dagger - U_0\rho U_0^\dagger\|_1 \le$$

$$\le \|\rho(U^\dagger - U_0^\dagger)\|_1 + \|(U - U_0)\rho\|_1 \le \|\rho\|_1 \|U^\dagger - U_0^\dagger\|_\infty + \|\rho\|_1 \|U - U_0\|_\infty = 2\|U - U_0\|_\infty. \tag{E19}$$

We have here used triangle inequality for first inequality, invariance of trace norm under unitaries for the second one, and Eq. (E14) for the third one. Next, using Prop. 35 we obtain the needed relation

$$\|U\rho U^\dagger - U_0\rho U_0^\dagger\|_1 \le 2\|V\|_\infty + \|V\|_\infty^2. \tag{E20}$$

Now in the above equation, set $\rho = \rho_S^0 \otimes \rho_{Cat}^0 \otimes \tau_G$ and $U_0 = e^{-i\theta(t)\hat{I}_{SCatG}^{int}}$, $U = e^{-i\left(\hat{I}_{SCatG}^{int} + \delta\hat{I}_{SCatG}^{int}\right)}$. The bound Eq. (E18) now follows by applying the data processing inequality. ∎

Appendix F: Proof of Theorem 3

In this section, we prove Theorem 3 in the main text.

Proof of Theorem 3. The proof will be divided into two parts rebelled A and B.

a. *Part A of proof of Theorem 3*

We start with a comment on notation. We will denote $U_{SCatG}^{target}(t) = e^{-i\theta(t)\hat{H}_{SCatG}^{int}}$ from the main text by $U_{SCatG}^{target\,(\epsilon_\sigma)}(t) = e^{-i\theta(t)\hat{H}_{SCatG}^{int}}$ here to remind ourselves that $\hat{H}_{SCatG}^{int}$ induces a small error ϵ_σ onto the final catalyst and system state (see Eqs. (21), (22)). We will also denote $U_{SCatG}^{target\,(0)}(t) := e^{-i\theta(t)\hat{I}_{SCatG}^{int}}$, since $\hat{I}_{SCatG}^{int}$ corresponds to the case of no error, i.e. $\epsilon_\sigma = 0$, (see Eq. (23)).

Part A will consist in proving that the following holds. Let $\varepsilon_{SCatG}(\epsilon_\sigma; t) > 0$ and $\varepsilon_{Cl}(t) > 0$ satisfy

$$\|\rho_{SCatG}^F(t) - \rho_{SCatG}^{target\,(\epsilon_\sigma)}(t)\|_1 \le \varepsilon_{SCatG}(\epsilon_\sigma; t), \tag{F1}$$

$$\|\rho_{Cl}^F(t) - \rho_{Cl}^0(t)\|_1 \le \varepsilon_{Cl}(t). \tag{F2}$$

Recall

$$\theta(t) = \begin{cases} 0 & \text{if } t \in [0, t_1] \\ 1 & \text{if } t \in [t_2, T_0]. \end{cases} \tag{F3}$$

It follows that:

1) The deviation from the idealised dynamics is bounded by

$$\|\rho_{SCatCl}^F(t) - \rho_S^F(t) \otimes \rho_{Cat}^0(t) \otimes \rho_{Cl}^0(t)\|_1 \le 6\sqrt{\varepsilon_{SCatG}(\epsilon_\sigma; t)} + 2\sqrt{\varepsilon_{Cl}(t)} + 2\epsilon_\sigma\,\theta(t). \tag{F4}$$

2) The final state $\rho_S^F(t)$ is

$$\|\rho_S^F(t) - \rho_S^{\text{target}\,(0)}(t)\|_1 \leq \varepsilon_{\text{SCatG}}(\epsilon_\sigma; t) + \epsilon_\sigma\,\theta(t) \tag{F5}$$

close to one which can be reached via t-CTO: For all $t \in [0, t_1] \cup [t_2, T_0]$ the transition

$$\rho_S^0 \otimes \rho_{\text{Cat}}^0 \otimes \rho_{\text{Cl}}^0 \quad \text{to} \quad \rho_S^{\text{target}\,(0)}(t) \otimes \rho_{\text{Cat}}^0(t) \otimes \rho_{\text{Cl}}^0(t) \tag{F6}$$

is possible via a TO i.e. ρ_S^0 to $\rho_S^{\text{target}\,(0)}$ via a t-CTO.

To prove the Eqs. (F5),(F6), we start by extending the definition of $\rho_{\text{SCatG}}^{\text{target}\,(\epsilon_\sigma)}(t)$ above Eq. (26) to include the clock system:

$$\rho_{\text{SCatGCl}}^{\text{target}\,(\epsilon_\sigma)}(t) = \rho_{\text{SCatG}}^{\text{target}\,(\epsilon_\sigma)}(t) \otimes \rho_{\text{Cl}}^0(t), \tag{F7}$$

where $\rho_{\text{Cl}}^0(t)$ is the free evolution of the clock defined in Eq. 6. Due to Eqs. (21),(26), if follows that the reduced state after tracing out the Gibbs state on G is

$$\rho_{\text{SCatCl}}^{\text{target}\,(0)}(t) = \rho_S^{\text{target}\,(0)}(t) \otimes \rho_{\text{Cat}}^0(t) \otimes \rho_{\text{Cl}}^0(t), \tag{F8}$$

for $t \in [0, t_1] \cup [t_2, t_3]$. Thus taking into account property Eq. (20) it follows by definition of CTOs and t-CTOs that a transition from ρ_S^0 to $\rho_S^{\text{target}\,(0)}(t)$ is possible via a t-CTO. Finally, applying the data processing inequality to Eq. (F1), we achieve $\rho_S^F(t)$ is $\|\rho_S^F(t) - \rho_S^{\text{target}\,(\epsilon_\sigma)}(t)\|_1 \leq \varepsilon_{\text{SCatG}}(\epsilon_\sigma; t)$ while applying it to Eq. (22) we find $\|\rho_S^{\text{target}\,(0)}(t) - \rho_S^{\text{target}\,(\epsilon_\sigma)}(t)\|_1 \leq \epsilon_\sigma\,\theta(t)$. Eq. (F5) in 2) above follows from the triangle inequality:

$$\|\rho_S^F(t) - \rho_S^{\text{target}\,(0)}(t)\|_1 \leq \|\rho_S^F(t) - \rho_S^{\text{target}\,(\epsilon_\sigma)}(t)\|_1 + \|\rho_S^{\text{target}\,(\epsilon_\sigma)}(t) - \rho_S^{\text{target}\,(0)}(t)\|_1 \leq \varepsilon_{\text{SCatG}}(\epsilon_\sigma; t) + \epsilon_\sigma\,\theta(t). \tag{F9}$$

We now prove 1) of the theorem. We begin by using the triangle inequality followed by the identity $\rho_{\text{SCat}}^{\text{target}}(t) = \rho_S^{\text{target}}(t) \otimes \rho_{\text{Cat}}^0(t)$ which follows from Eq. (F8).

$$\|\rho_{\text{SCatCl}}^F(t) - \rho_S^F(t) \otimes \rho_{\text{Cat}}^0(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 \tag{F10}$$

$$= \|\rho_{\text{SCatCl}}^F(t) - \rho_{\text{SCat}}^{\text{target}\,(\epsilon_\sigma)}(t) \otimes \rho_{\text{Cl}}^0(t) + \rho_{\text{SCat}}^{\text{target}\,(\epsilon_\sigma)}(t) \otimes \rho_{\text{Cl}}^0(t) - \rho_S^F(t) \otimes \rho_{\text{Cat}}^0(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 \tag{F11}$$

$$\leq \|\rho_{\text{SCatCl}}^F(t) - \rho_{\text{SCat}}^{\text{target}\,(\epsilon_\sigma)}(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 + \|\rho_{\text{SCat}}^{\text{target}\,(\epsilon_\sigma)}(t) \otimes \rho_{\text{Cl}}^0(t) - \rho_{\text{SCat}}^{\text{target}\,(0)}(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 \tag{F12}$$

$$+ \|\rho_{\text{SCat}}^{\text{target}\,(0)}(t) \otimes \rho_{\text{Cl}}^0(t) - \rho_S^F(t) \otimes \rho_{\text{Cat}}^0(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 \tag{F13}$$

$$= \|\rho_{\text{SCatCl}}^F(t) - \rho_{\text{SCat}}^{\text{target}\,(\epsilon_\sigma)}(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 + \|\rho_{\text{SCat}}^{\text{target}\,(\epsilon_\sigma)}(t) - \rho_{\text{SCat}}^{\text{target}\,(0)}(t)\|_1 \tag{F14}$$

$$+ \|\rho_S^{\text{target}\,(0)} \otimes \rho_{\text{Cat}}^0(t) - \rho_S^F(t) \otimes \rho_{\text{Cat}}^0(t)\|_1$$

$$\leq \|\rho_{\text{SCatCl}}^F(t) - \rho_{\text{SCat}}^{\text{target}\,(\epsilon_\sigma)}(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 + \epsilon_\sigma\,\theta(t) + \|\rho_S^{\text{target}\,(0)} - \rho_S^F(t)\|_1 \tag{F15}$$

$$\leq \|\rho_{\text{SCatCl}}^F(t) - \rho_{\text{SCat}}^{\text{target}\,(\epsilon_\sigma)}(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 + 2\epsilon_\sigma\,\theta(t) + \varepsilon_{\text{SCatG}}(\epsilon_\sigma; t) \tag{F16}$$

where we have applied the data processing inequality to Eq. (F1) and used the resultant equation in the last line. Now we make the following identifications, noting that $\rho_{\text{Cl}}^0(t)$ all $t \in \mathbb{R}$ is pure by assumption of the theorem.

$$\rho_{\text{SCatG}}^F(t) =: \rho_A, \quad \rho_{\text{Cl}}^F(t) =: \rho_B, \quad \rho_{\text{SCatGCl}}^F(t) =: \rho_{AB} \tag{F17}$$

$$\rho_{\text{SCatG}}^{\text{target}\,(\epsilon_\sigma)}(t) =: \sigma_A, \quad \rho_{\text{Cl}}^0(t) =: \sigma_B, \quad \rho_{\text{SCatG}}^{\text{target}\,(\epsilon_\sigma)}(t) \otimes \rho_{\text{Cl}}^0(t) =: \sigma_{AB} \tag{F18}$$

and apply Prop. 11 to arrive at

$$\|\rho_{\text{SCatGCl}}^F(t) - \rho_{\text{SCatG}}^{\text{target}\,(\epsilon_\sigma)}(t) \otimes \rho_{\text{Cl}}^0(t)\| \leq 2\sqrt{\varepsilon_{\text{SCatG}}(\epsilon_\sigma; t)} + 2\sqrt{\varepsilon_{\text{SCatG}}(\epsilon_\sigma; t) + \varepsilon_{\text{Cl}}(t)}. \tag{F19}$$

Applying the data processing inequality to the above equation, followed by substituting into Eq. (F16), gives

$$\|\rho_{\text{SCatCl}}^F(t) - \rho_S^F(t) \otimes \rho_{\text{Cat}}^0(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 \leq 2\sqrt{\varepsilon_{\text{SCatG}}(\epsilon_\sigma; t)} + 2\sqrt{\varepsilon_{\text{SCatG}}(\epsilon_\sigma; t) + \varepsilon_{\text{Cl}}(t)} + \varepsilon_{\text{SCatG}}(\epsilon_\sigma; t) + 2\epsilon_\sigma\,\theta(t) \tag{F20}$$

$$\leq 6\sqrt{\varepsilon_{\text{SCatG}}(\epsilon_\sigma; t)} + 2\sqrt{\varepsilon_{\text{Cl}}(t)} + 2\epsilon_\sigma\,\theta(t), \tag{F21}$$

where the last inequality follows by noting that the trace distance between any two quantum states is always upper bounded by 2.

b. Part B of proof of Theorem 3

We now set out to prove the second part, which consists in deriving expressions for $\varepsilon_{\mathrm{SCatG}}(\epsilon_\sigma; t)$ and $\varepsilon_{\mathrm{Cl}}(t)$ such that the contents of sections 1) and 2) above are consistent with the claims in 1) and 2) of the Theorem.

To start with, since $\hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{Cat}} + \hat{H}_{\mathrm{G}}$ and $\hat{H}_{\mathrm{SCatG}}^{\mathrm{int}}$ commute, they share a common eigenbasis which we denote $\{|E_j\rangle\}_j$. We can write the interaction term in terms of this basis as follows, $\hat{H}_{\mathrm{SCatG}}^{\mathrm{int}} = \sum_{j=1}^{d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{G}}} \Omega_j |E_j\rangle\langle E_j|$ with eigenvalues Ω_j in the range $\Omega_j \in [-\pi, \pi]$ since $\|\hat{H}_{\mathrm{SCatG}}^{\mathrm{int}}\|_\infty \leq \pi$. We can also expanding the state $\rho_{\mathrm{S}}^0 \otimes \rho_{\mathrm{Cat}}^0 \otimes \tau_{\mathrm{G}}$ in the energy eigenbasis $\{|E_j\rangle\}_j$. Doing so allows one to simplify the expression for $\rho_{\mathrm{SCatGCl}}^F(t)$. We find

$$\rho_{\mathrm{SCatGCl}}^F(t) = \mathrm{e}^{-it\left(\hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{Cat}} + \hat{H}_{\mathrm{G}} + \hat{H}_{\mathrm{SCatG}}^{\mathrm{int}} \otimes \hat{H}_{\mathrm{Cl}}^{\mathrm{int}} + \hat{H}_{\mathrm{Cl}}\right)} \rho_{\mathrm{S}}^0 \otimes \rho_{\mathrm{Cat}}^0 \otimes \tau_{\mathrm{G}} \otimes |\rho_{\mathrm{Cl}}^0\rangle\langle\rho_{\mathrm{Cl}}^0| \; \mathrm{e}^{it\left(\hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{Cat}} + \hat{H}_{\mathrm{G}} + \hat{H}_{\mathrm{SCatG}}^{\mathrm{int}} \otimes \hat{H}_{\mathrm{Cl}}^{\mathrm{int}} + \hat{H}_{\mathrm{Cl}}\right)}$$
(F22)

$$= \sum_{j,j'=1}^{d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{G}}} \mathrm{e}^{-it\left(\hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{Cat}} + \hat{H}_{\mathrm{G}} + \Omega_j \hat{H}_{\mathrm{Cl}}^{\mathrm{int}} + \hat{H}_{\mathrm{Cl}}\right)} \rho_{\mathrm{SCatG},j,j'}^0 |E_j\rangle\langle E_{j'}| \otimes |\rho_{\mathrm{Cl}}^0\rangle\langle\rho_{\mathrm{Cl}}^0| \; \mathrm{e}^{it\left(\hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{Cat}} + \hat{H}_{\mathrm{G}} + \Omega_{j'} \hat{H}_{\mathrm{Cl}}^{\mathrm{int}} + \hat{H}_{\mathrm{Cl}}\right)}$$
(F23)

$$= \sum_{j,j'=1}^{d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{G}}} \rho_{\mathrm{SCatG},j,j'}^0(t) |E_j\rangle\langle E_{j'}| \otimes |\rho_{\mathrm{Cl},j}^0(t)\rangle\langle\rho_{\mathrm{Cl},j'}^0(t)| \,,$$
(F24)

where

$$\sum_{j,j'=1}^{d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{G}}} \rho_{\mathrm{SCatG},j,j'}^0(t) |E_j\rangle\langle E_{j'}| = \rho_{\mathrm{S}}(t) \otimes \rho_{\mathrm{Cat}}^0(t) \otimes \tau_{\mathrm{G}},$$
(F25)

$$|\rho_{\mathrm{Cl},j}^0(t)\rangle = \mathrm{e}^{-it\left(\Omega_j \hat{H}_{\mathrm{Cl}}^{\mathrm{int}} + \hat{H}_{\mathrm{Cl}}\right)} |\rho_{\mathrm{Cl}}^0\rangle \,.$$
(F26)

We thus have by taking partial traces

$$\rho_{\mathrm{SCatG}}^F(t) = \sum_{j,j'=1}^{d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{G}}} \rho_{\mathrm{SCatG},j,j'}^0(t) |E_j\rangle\langle E_{j'}| \langle\rho_{\mathrm{Cl},j'}^0(t)|\rho_{\mathrm{Cl},j}^0(t)\rangle$$
(F27)

$$\rho_{\mathrm{Cl}}^F(t) = \sum_{j=1}^{d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{G}}} \rho_{\mathrm{SCatG},j,j}^0 |\rho_{\mathrm{Cl},j}^0(t)\rangle\langle\rho_{\mathrm{Cl},j}^0(t)| \,,$$
(F28)

Similarly

$$\rho_{\mathrm{SCatG}}^{\mathrm{target}\,(\epsilon_\sigma)}(t) = U_{\mathrm{SCatG}}^{\mathrm{target}\,(\epsilon_\sigma)}(t) \left(\rho_{\mathrm{S}}^0(t) \otimes \rho_{\mathrm{Cat}}^0(t) \otimes \tau_{\mathrm{G}}\right) U_{\mathrm{SCatG}}^{\mathrm{target}\,(\epsilon_\sigma)\,\dagger}(t)$$
(F29)

$$= \mathrm{e}^{-it\theta(t)\hat{H}_{\mathrm{SCatG}}^{\mathrm{int}}} \left(\sum_{j,j'=1}^{d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{G}}} \rho_{\mathrm{SCatG},j,j'}^0(t) |E_j\rangle\langle E_{j'}|\right) \mathrm{e}^{it\theta(t)\hat{H}_{\mathrm{SCatG}}^{\mathrm{int}}}$$
(F30)

$$= \sum_{j,j'=1}^{d_{\mathrm{S}} d_{\mathrm{Cat}} d_{\mathrm{G}}} \rho_{\mathrm{SCatG},j,j'}^0(t) \mathrm{e}^{-it(\Omega_j - \Omega_{j'})\theta(t)} |E_j\rangle\langle E_{j'}| \,.$$
(F31)

Noting that the Frobenious norm $\|\cdot\|_F$ upper bounds the trace distance by the inequality $\|\cdot\|_F \geq \|\cdot\|_1/\sqrt{d}$ for a d

dimensional space, we find

$$\|\rho_{\text{SCatG}}^F(t) - \rho_{\text{SCatG}}^{\text{target}(\epsilon_\sigma)}(t)\|_1 \le \sqrt{d_S d_{\text{Cat}} d_G} \|\rho_{\text{SCatG}}^F(t) - \rho_{\text{SCatG}}^{\text{target}(\epsilon_\sigma)}(t)\|_F \tag{F32}$$

$$= \sqrt{d_S d_{\text{Cat}} d_G} \sqrt{\sum_{j,j'=1}^{d_S d_{\text{Cat}} d_G} \left|\rho_{\text{SCatG},j,j'}^0(t)\right|^2 \left|e^{-it(\Omega_j - \Omega_{j'})\theta(t)} - \langle\rho_{\text{Cl},j'}^0(t)|\rho_{\text{Cl},j}^0(t)\rangle\right|^2} \tag{F33}$$

$$\le \sqrt{d_S d_{\text{Cat}} d_G} \sqrt{\sum_{j,j'=1}^{d_S d_{\text{Cat}} d_G} \left|\rho_{\text{SCatG},j,j'}^0(t)\right|^2 \left(\max_{m,n}\left|e^{-it(\Omega_m - \Omega_n)\theta(t)} - \langle\rho_{\text{Cl},n}^0(t)|\rho_{\text{Cl},m}^0(t)\rangle\right|^2\right)} \tag{F34}$$

$$\le \sqrt{d_S d_{\text{Cat}} d_G} \sqrt{\text{tr}\left[\rho_S^0(t)^2 \otimes \rho_{\text{Cat}}^0(t)^2 \otimes \tau_G^2\right] \left(\max_{m,n}\left|e^{-it(\Omega_m - \Omega_n)\theta(t)} - \langle\rho_{\text{Cl},n}^0(t)|\rho_{\text{Cl},m}^0(t)\rangle\right|^2\right)} \tag{F35}$$

$$= \sqrt{d_S \text{tr}\left[\rho_S^{0\,2}\right] d_{\text{Cat}} \text{tr}\left[\rho_{\text{Cat}}^{0\,2}\right] d_G \text{tr}\left[\tau_G^2\right]} \max_{m,n}\left|e^{-it(\Omega_m - \Omega_n)\theta(t)} - \langle\rho_{\text{Cl},n}^0(t)|\rho_{\text{Cl},m}^0(t)\rangle\right| \tag{F36}$$

$$\le \sqrt{d_S \text{tr}\left[\rho_S^{0\,2}\right] d_{\text{Cat}} \text{tr}\left[\rho_{\text{Cat}}^{0\,2}\right] d_G \text{tr}\left[\tau_G^2\right]} \max_{x,y\in[-\pi,\pi]}\left|1 - \langle\rho_{\text{Cl}}^0|\hat{\Gamma}^\dagger(x,t)\hat{\Gamma}(y,t)|\rho_{\text{Cl}}^0\rangle\right|. \tag{F37}$$

Noting that the fidelity F between a pure state $|\rho_{\text{Cl}}^0(t)\rangle = e^{-it\hat{H}_{\text{Cl}}}|\rho_{\text{Cl}}^0\rangle$ and a state $\rho_{\text{Cl}}^F(t)$ is given by $F = \langle\rho_{\text{Cl}}^0(t)|\rho_{\text{Cl}}^F(t)|\rho_{\text{Cl}}^0(t)\rangle$, using Eq. (F28) and the usual bound for the trace distance in terms of the fidelity, we find

$$\|\rho_{\text{Cl}}^F(t) - \rho_{\text{Cl}}^0(t)\|_1 \le 2\sqrt{1 - F\left(\rho_{\text{Cl}}^F(t), |\rho_{\text{Cl}}^0(t)\rangle\right)} = 2\sqrt{1 - \langle\rho_{\text{Cl}}^0(t)|\rho_{\text{Cl}}^F(t)|\rho_{\text{Cl}}^0(t)\rangle} \tag{F38}$$

$$= 2\sqrt{1 - \sum_{j=1}^{d_S d_{\text{Cat}} d_G} \rho_{\text{SCatG},j,j}^0 \langle\rho_{\text{Cl}}^0(t)|\rho_{\text{Cl},j}^0(t)\rangle \langle\rho_{\text{Cl},j}^0(t)|\rho_{\text{Cl}}^0(t)\rangle} \tag{F39}$$

$$\le 2\sqrt{1 - \min_j\left|\langle\rho_{\text{Cl}}^0(t)|\rho_{\text{Cl},j}^0(t)\rangle\right|^2} \le 2\sqrt{1 - \min_{x\in[-\pi,\pi]}\left|\langle\rho_{\text{Cl}}^0(t)|e^{-ix\theta(t)}\Gamma(x,t)|\rho_{\text{Cl}}^0\rangle\right|^2} \tag{F40}$$

$$= 2\sqrt{1 - \min_{x\in[-\pi,\pi]}|\langle\rho_{\text{Cl}}^0|\Gamma^\dagger(0,t)\Gamma(x,t)|\rho_{\text{Cl}}^0\rangle|^2} \tag{F41}$$

$$\le 2\max_{x,y\in[-\pi,\pi]}\sqrt{1 - |\langle\rho_{\text{Cl}}^0|\Gamma^\dagger(y,t)\Gamma(x,t)|\rho_{\text{Cl}}^0\rangle|^2} \tag{F42}$$

Hence from Eqs. (F4), (F5), we conclude

$$\|\rho_{\text{SCatCl}}^F(t) - \rho_S^0(t) \otimes \rho_{\text{Cat}}^0(t) \otimes \rho_{\text{Cl}}^0(t)\|_1 \le 6\left(d_S \text{tr}\left[\rho_S^{0\,2}\right] d_{\text{Cat}} \text{tr}\left[\rho_{\text{Cat}}^{0\,2}\right] d_G \text{tr}\left[\tau_G^2\right]\right)^{1/4} \max_{x,y\in[-\pi,\pi]}\left(|1 - \Delta(t;x,y)|\right)^{1/2} \tag{F43}$$

$$+ 4\max_{\bar{x},\bar{y}\in[-\pi,\pi]}\left(1 - |\Delta(t;\bar{x},\bar{y})|^2\right)^{1/4} + 2\epsilon_\sigma\,\theta(t), \tag{F44}$$

and

$$\|\rho_S^F(t) - \rho_S^{\text{target}(0)}(t)\|_1 \le \epsilon_\sigma\,\theta(t) + \sqrt{d_S \text{tr}\left[\rho_S^{0\,2}\right] d_{\text{Cat}} \text{tr}\left[\rho_{\text{Cat}}^{0\,2}\right] d_G \text{tr}\left[\tau_G^2\right]} \max_{x,y\in[-\pi,\pi]}|1 - \Delta(t;x,y)|. \tag{F45}$$

where we have defined

$$\Delta(t;x,y) := \langle\rho_{\text{Cl}}^0|\Gamma^\dagger(x,t)\Gamma(y,t)|\rho_{\text{Cl}}^0\rangle. \tag{F46}$$

Finally, to finish the proof we need find some simplifying upper bounds to the r.h.s. of Eqs. (F44) and (F45) to conclude the bounds states in the paper. For this, just need the identities

$$1 - |c|^2 \le |1 - c^2|, \qquad\qquad |1 - c| \le |1 - c^2|, \tag{F47}$$

for all $c \in \mathbb{C}$ satisfying $|c| \leq 1$ and $|1 - c| \leq 1$.

To see this, write c in terms of real and imaginary parts: $c = a + ib$. The constraints $|c| \leq 1$ and $|1-c| \leq 1$ imply that a satisfies $0 \leq a \leq 1$. We find $(1-|c|^2)^2 = (1-a^2)^2 - 2(1-a^2)b^2 + b^4$ while $|1-c^2|^2 = (1-a^2)^2 + 2(1-a^2)b^2 + b^4 + 4a^2b^2$, thus concluding the 1st inequality in (F47). For the second inequality in Eq. F47, we start by observing that $(1-a^2)^2 \leq (1-a)^2$ and $b^2 \leq (2+2a^2)b^2 + b^4 = 2(1-a^2)b^2 + 4a^2b^2 + b^4$, and thus $(1-a^2)^2 + b^2 \leq (1-a)^2 + 2(1-a^2)b^2 + 4a^2b^2 + b^4$. On the other hand, we find $|1-c|^2 = (1-a)^2 + b^2$ and $|1-c^2|^2 = (1-a^2)^2 + 2(1-a^2)b^2 + 4a^2b^2 + b^4$, thus proving the 2nd inequality in Eq. (F47).

Now observe that we can make the identification $c = \Delta(t; x, y)$ since $|\Delta(t; x, y)| \leq 1$ follows from unitarity while in Eqs. (F44) and Eq. (F45) the upper bounds always hold true when the trace distance is upper bounded by one for all input states, so while $|1 - c| = |1 - \Delta(t; x, y)| \leq 1$ may not be always true, we can assume it w.l.o.g. when dealing with the bounds in Eqs. (F44), (F45). We thus have the bounds

$$\|\rho_{\mathrm{SCatCl}}^{F}(t) - \rho_{\mathrm{S}}^{F}(t) \otimes \rho_{\mathrm{Cat}}^{0}(t) \otimes \rho_{\mathrm{Cl}}^{0}(t)\|_{1} \tag{F48}$$

$$\leq 2\epsilon_{\sigma}\,\theta(t) + 6\left(d_{\mathrm{S}}\mathrm{tr}\left[\rho_{\mathrm{S}}^{0\,2}\right] d_{\mathrm{Cat}}\mathrm{tr}\left[\rho_{\mathrm{Cat}}^{0\,2}\right] d_{\mathrm{G}}\mathrm{tr}\left[\tau_{\mathrm{G}}^{2}\right]\right)^{1/4} \max_{x,y\in[-\pi,\pi]}\left(\left|1 - \Delta^{2}(t;x,y)\right|\right)^{1/2} \tag{F49}$$

$$+\,4 \max_{\bar{x},\bar{y}\in[-\pi,\pi]}\left(\left|1 - \Delta^{2}(t;\bar{x},\bar{y})\right|\right)^{1/4} \tag{F50}$$

$$\leq 2\epsilon_{\sigma}\,\theta(t) + \left[6\left(d_{\mathrm{S}}\mathrm{tr}\left[\rho_{\mathrm{S}}^{0\,2}\right] d_{\mathrm{Cat}}\mathrm{tr}\left[\rho_{\mathrm{Cat}}^{0\,2}\right] d_{\mathrm{G}}\mathrm{tr}\left[\tau_{\mathrm{G}}^{2}\right]\right)^{1/4} + 4\right] \max_{x,y\in[-\pi,\pi]}\left(\left|1 - \Delta^{2}(t;x,y)\right|\right)^{1/4} \tag{F51}$$

$$\leq 2\epsilon_{\sigma}\,\theta(t) + 10\left(d_{\mathrm{S}}\mathrm{tr}\left[\rho_{\mathrm{S}}^{0\,2}\right] d_{\mathrm{Cat}}\mathrm{tr}\left[\rho_{\mathrm{Cat}}^{0\,2}\right] d_{\mathrm{G}}\mathrm{tr}\left[\tau_{\mathrm{G}}^{2}\right] \max_{x,y\in[-\pi,\pi]}\left|1 - \Delta^{2}(t;x,y)\right|\right)^{1/4}, \tag{F52}$$

where in the last inequality we have used that $1 \leq d\,\mathrm{tr}[\rho^2]$ for all d dimesional states ρ. Similarly,

$$\|\rho_{\mathrm{S}}^{F}(t) - \rho_{\mathrm{S}}^{\mathrm{target}(0)}(t)\|_{1} \leq \epsilon_{\sigma}\,\theta(t) + \sqrt{d_{\mathrm{S}}\mathrm{tr}\left[\rho_{\mathrm{S}}^{0\,2}\right] d_{\mathrm{Cat}}\mathrm{tr}\left[\rho_{\mathrm{Cat}}^{0\,2}\right] d_{\mathrm{G}}\mathrm{tr}\left[\tau_{\mathrm{G}}^{2}\right] \max_{x,y\in[-\pi,\pi]}\left|1 - \Delta^{2}(t;x,y)\right|}. \tag{F53}$$

■

Appendix G: Calculating $\Delta(t; x, y)$ for the Idealised Momentum Clock

In the case of the idealised momentum clock, we have $\hat{H}_{\mathrm{Cl}} = \hat{p}$, $\hat{H}_{\mathrm{Cl}}^{\mathrm{int}} = g(\hat{x})$, where $\hat{x}$ and $\hat{p}$ are the position and momentum operators of a particle in one dimension satisfying the Weyl form of the canonical commutation relations, $[\hat{x}, \hat{p}] = \mathrm{i}$, while g is an integrable function from the reals to the reals, normalised such that $\int_{\mathbb{R}} g(x)dx = 1$.[23]

Therefore, we find for the idealised momentum clock, for $z, y \in \mathbb{R}$

$$\Delta(t; z, y) = \langle\rho_{\mathrm{Cl}}^{0}|\,\hat{\Gamma}_{\mathrm{Cl}}^{\dagger}(z, t)\hat{\Gamma}_{\mathrm{Cl}}(y, t)\,|\rho_{\mathrm{Cl}}^{0}\rangle = \mathrm{e}^{-\mathrm{i}(z-y)\theta(t)}\,\langle\rho_{\mathrm{Cl}}^{0}|\,\mathrm{e}^{\mathrm{i}t\hat{p}+\mathrm{i}ztg(\hat{x})}\mathrm{e}^{-\mathrm{i}t\hat{p}-\mathrm{i}ytg(\hat{x})}\,|\rho_{\mathrm{Cl}}^{0}\rangle \tag{G1}$$

$$= \mathrm{e}^{-\mathrm{i}(z-y)\theta(t)}\int_{\mathbb{R}} dx\,\langle\rho_{\mathrm{Cl}}^{0}|\,\mathrm{e}^{\mathrm{i}t\hat{p}+\mathrm{i}ztg(\hat{x})}\,|x\rangle\langle x|\,\mathrm{e}^{-\mathrm{i}t\hat{p}-\mathrm{i}ytg(\hat{x})}\,|\rho_{\mathrm{Cl}}^{0}\rangle\,. \tag{G2}$$

We can now use the relation $\hat{p} = -\mathrm{i}\frac{\partial}{\partial x}$ and solve the 1st order 2 variable differential equation resulting from the Schrödinger eq. for the Hamiltonian $\hat{p} + yg(\hat{x})$ and initial wave-function $\langle x|\rho_{\mathrm{Cl}}^{0}\rangle$. Plugging the solution into the above, we arrive at

$$\Delta(t; z, y) = \mathrm{e}^{-\mathrm{i}(z-y)\theta(t)}\int_{\mathbb{R}} dx\,\left|\langle x|\rho_{\mathrm{Cl}}^{0}\rangle\right|^{2}\mathrm{e}^{-\mathrm{i}(y-z)\int_{x}^{x+t} g(x')dx'}\,. \tag{G3}$$

We now choose the support of the initial wave-function $\langle x|\rho_{\mathrm{Cl}}^{0}\rangle$ to be $x \in [x_{\psi l}, x_{\psi r}]$ and the support of $g(x)$ to be $x \in [x_{gl}, x_{gr}]$. Noting that

$$\int_{x}^{x+t} g(x')dx' = \begin{cases} 0 & \text{if } x + t \leq x_{gl} \\ 1 & \text{if } x \leq x_{gl} \text{ and } x + t \geq x_{gr}, \end{cases} \tag{G4}$$

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[23] One can also come to the same conclusions for the idealised momentum clock on a circle, rather than a line. In this case, $[\hat{x}, \hat{p}]$ still satisfy the Heisenberg form of the canonical commutation relations, but not the Weyl form. See [31] for details.

and taking into account the support interval of $\langle x|\rho_{\text{Cl}}^0\rangle$, we conclude

$$\int_{\mathbb{R}} dx \, \left|\langle x|\rho_{\text{Cl}}^0\rangle\right|^2 \mathrm{e}^{-\mathrm{i}(y-z)\int_x^{x+t} g(x')dx'} = \begin{cases} 1 & \text{if } t \leq x_{gl} - x_{\psi r} \\ \mathrm{e}^{\mathrm{i}(z-y)} & \text{if } t \geq x_{gr} - x_{\psi l} \end{cases}. \tag{G5}$$

Therefore, choosing $t_1 = x_{gr} - x_{\psi l}$ and $t_2 = x_{gr} - x_{\psi l}$, from Eq. (G3) we arrive at

$$\Delta(t; x, y) = 1 \quad \forall x, y \in [-\pi, \pi], \tag{G6}$$

as claimed in Sec. III B of the main text. Furthermore, note that the derivation holds for all $t_1 < t_2$ by appropriately choosing the parameters $x_{gr}, x_{\psi l}, x_{gr}, x_{\psi l}$.