Catalytic Entanglement

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Quantum entanglement of pure states is usually quantified via the entanglement entropy, the von Neumann entropy of the reduced state [1–3]. Entanglement entropy is closely related to entanglement distillation [1], a process for converting quantum states into singlets, which can then be used for various quantum technological tasks. The relation between entanglement entropy and entanglement distillation has been known only for the asymptotic setting, and the meaning of entanglement entropy in the single-copy regime has so far remained open. Here we close this gap by considering entanglement catalysis. We prove that entanglement entropy completely characterizes state transformations in the presence of entangled catalysts. Our results suggest that catalysis is useful for a broad range of quantum information protocols, giving asymptotic results an operational meaning also in the single-copy setup.

Originated in chemistry, catalysis allows to increase the rate of a chemical reaction. This is achieved by using a catalyst, a substance which is not consumed in the process, and can thus be used repeatedly without additional costs. Similarly, a quantum catalyst is a quantum system which is not changed by the process under consideration, giving access to transformations which are not achievable without it. As has been realized in the early days of quantum information science, catalysis can significantly improve our abilities to transform entangled quantum systems, when it comes to transformations via local operations and classical communication (LOCC) [4–6].

One of the first examples [5] demonstrating the power of catalysis in quantum theory involves pure entangled states shared by two parties, Alice and Bob. Two states, denoted by $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$, are chosen such that no LOCC procedure can convert $|\psi\rangle^{AB}$ into $|\phi\rangle^{AB}$. Such states can be found using conditions for LOCC transformations presented in [4]. Even if a direct conversion from $|\psi\rangle^{AB}$ to $|\phi\rangle^{AB}$ is not possible, in some cases conversion can still be achieved by using a catalyst. This is an additional quantum system in an entangled state $|\mu\rangle^{A'B'}$, enabling the transformation $|\psi\rangle^{AB} \otimes |\mu\rangle^{A'B'} \rightarrow |\phi\rangle^{AB} \otimes |\mu\rangle^{A'B'}$. Since the state of the catalyst remains unchanged in the process, it can be reused for another transformation in the future. A complete characterization of pure quantum states which can be transformed into each other via LOCC with a catalyst has so far remained open. Partial results addressing this question have been presented over the last decades [7–12].

Catalysis is also useful in quantum thermodynamics, lifting the well known second law in the classical domain to many second laws in the quantum regime [13]. Catalytic properties of quantum coherence [14, 15], purity [16], and theories having certain symmetries [17] have also been considered. There has also been significant interest in correlated catalysts [18–21]. Allowing a catalyst to build up correlations with the system has shown to enhance the transformation power of the corresponding procedure [22–24].

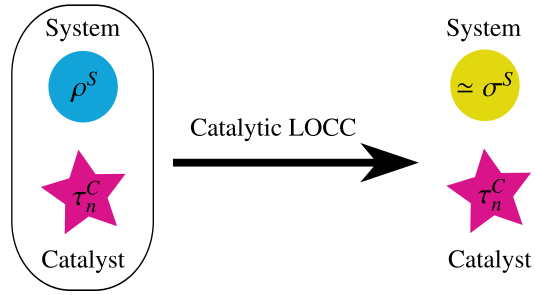

FIG. 1. Catalytic LOCC transformation from ρ^S to σ^S with a catalyst τ_n^C. The state of the catalyst does not change in the procedure, and the system becomes decoupled from the catalyst for $n \rightarrow \infty$. If ρ^S and σ^S are bipartite pure states, the transition is fully characterized by entanglement entropy of the states.

In this Letter we consider *catalytic LOCC transformations*. For a bipartite system $S = AB$ a catalytic LOCC transformation is defined as

$$\rho^S \rightarrow \lim_{n\rightarrow\infty} \mathrm{Tr}_C \left[\Lambda_n \left(\rho^S \otimes \tau_n^C\right)\right]. \tag{1}$$

Here, $C = A'B'$ is a bipartite system of the catalyst, $\left\{\tau_n^C\right\}$ is a sequence of catalyst states, and $\{\Lambda_n\}$ is a sequence of LOCC protocols. We require that the catalyst is unchanged for each n, and becomes decoupled from the system in the limit $n \rightarrow \infty$, see Fig. 1 and the methods section for more details.

We will now show that for pure states catalytic LOCC transformations are closely related to asymptotic LOCC transformations. In the asymptotic setting, the parties can operate on a large number of copies of the initial state simultaneously. The figure of merit for the process is the transformation rate, giving the maximal number of copies of $|\phi\rangle^{AB}$ achievable per copy of the initial state $|\psi\rangle^{AB}$. It has been shown in [1] that the optimal rate is given by $H\left(\psi^A\right)/H\left(\phi^A\right)$, where ψ^A and ϕ^A are the reduced states of $|\psi\rangle^{AB}$ and $|\phi\rangle^{AB}$, and $H(\rho) = -\mathrm{Tr}\left[\rho \log_2 \rho\right]$ is the von Neumann entropy. For pure states the von Neumann entropy of the reduced state is also a quantifier of entanglement, known as entanglement entropy [1–3]: $E\left(|\psi\rangle^{AB}\right) = H\left(\psi^A\right)$. If the initial and the target state have the same entanglement entropy, then conversion $|\psi\rangle^{AB} \rightarrow |\phi\rangle^{AB}$ is possible with unit rate.

We are now ready to present the first result of this Letter. Alice and Bob can convert $|\psi\rangle^{AB}$ into $|\phi\rangle^{AB}$ via catalytic LOCC

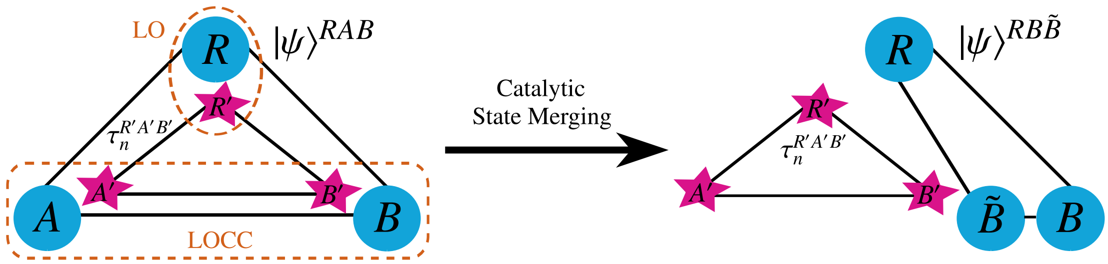

FIG. 2. Catalytic quantum state merging. Alice, Bob, and Referee share a single copy of $|\psi\rangle^{RAB}$. Alice aims to send her part of the state to Bob by using catalytic LOCC, and the Referee can apply local unitary transformations. The process is completely characterized by the quantum conditional entropy $H(A|B)$, see the main text for more details.

if and only if

$$H\left(\psi^A\right) \geq H\left(\phi^A\right). \tag{2}$$

This result means that for pure states catalysts enhance the transformation power of LOCC, making the transformations as powerful as in the asymptotic limit. The result in Eq. (2) is a consequence of the following theorem, concerning transformations from a general state ρ^S into a pure state $|\phi\rangle^S$ of a general multipartite system S via multipartite LOCC.

Theorem 1. *If ρ^S can be transformed into $|\phi\rangle^S$ via asymptotic LOCC with unit rate, then there exists a catalytic LOCC protocol transforming ρ^S into $|\phi\rangle^S$.*

We refer to the methods section for the proof which is inspired by techniques introduced very recently in quantum thermodynamics [24]. In the bipartite setting, this theorem directly implies that $|\psi\rangle^{AB}$ can be transformed into $|\phi\rangle^{AB}$ via catalytic LOCC if Eq. (2) is fulfilled. As we show in the methods section by using properties of entanglement quantifiers [25, 26], a transformation is not possible if Eq. (2) is violated.

Remarkably, our theorem holds not only for bipartite state transformations, but also for multipartite LOCC protocols. Here the goal is to convert a multipartite state ρ^S into a pure state $|\phi\rangle^S$ via multipartite LOCC. As a consequence, it allows us to translate a broad range of asymptotic results in entanglement theory to a corresponding result on the single-copy level. We show it explicitly for a variation of quantum state merging, which we term *catalytic quantum state merging*. Before we present this task, we review the standard quantum state merging procedure in the following.

In quantum state merging [27, 28], Alice, Bob, and Referee share asymptotically many copies of a quantum state $|\psi\rangle^{RAB}$. The goal of the process is to send Alice's part of the state to Bob, while preserving correlations with the Referee. Alice and Bob can perform LOCC protocols and share additional singlets. As was shown in [27, 28], the performance of this process is characterized by the quantum conditional entropy

$$H(A|B) = H\left(\psi^{AB}\right) - H\left(\psi^B\right). \tag{3}$$

For $H(A|B) > 0$ quantum state merging can be performed if Alice and Bob share additional singlets at rate $H(A|B)$, and merging is not possible if less singlets are available. For $H(A|B) \leq 0$ Alice and Bob can perform quantum state merging with LOCC, while additionally gaining singlets at rate $-H(A|B)$. Remarkably, quantum state merging gives an operational meaning to the quantum conditional entropy, regardless whether $H(A|B)$ is positive or negative.

We are now ready to define catalytic quantum state merging, giving the quantum conditional entropy an operational meaning also in the single-copy regime. Here, Alice, Bob, and Referee share one copy of the state $|\psi\rangle^{RAB}$, and can use additional catalysts in arbitrary states $\tau_n^{R'A'B'}$. While in standard quantum state merging the Referee is fully inactive, in catalytic quantum state merging we allow the Referee to perform local unitaries. However, communication between the Referee and the other parties is not required, see also Fig. 2. The goal is to merge the single copy of $|\psi\rangle^{RAB}$ on Bob's side without changing the state of the catalyst for all n, and with decoupling of the catalyst in the limit $n \to \infty$. We find that for $H(A|B) > 0$ catalytic quantum state merging can be performed if Alice and Bob additionally share a pure entangled state with entanglement entropy $H(A|B)$. This procedure is optimal: merging is not possible if a pure state with a smaller entanglement entropy is provided. If $H(A|B) \leq 0$, then catalytic state merging can be performed without extra entanglement. In the end of the process, Alice and Bob can gain an additional pure state with entanglement entropy $-H(A|B)$. Also this procedure is optimal: it is not possible to achieve merging and gain a pure state with entanglement entropy exceeding $-H(A|B)$.

As a final example we discuss assisted entanglement distillation [29, 30], where three parties, Alice, Bob, and Charlie, share a pure state $|\psi\rangle^{ABC}$. By performing LOCC involving all parties, their aim is to extract singlets between Alice and Bob. In the asymptotic setup, the optimal singlet rate is given by $\min\left\{H\left(\psi^A\right), H\left(\psi^B\right)\right\}$ [30]. Correspondingly, *catalytic assisted entanglement distillation* involves one copy of $|\psi\rangle^{ABC}$. By applying catalytic LOCC, the parties aim to establish a state $|\phi\rangle^{AB}$ shared by Alice and Bob, having entanglement entropy as large as possible. We find that $\min\left\{H\left(\psi^A\right), H\left(\psi^B\right)\right\}$ corresponds to the maximal entanglement entropy achievable from $|\psi\rangle^{ABC}$ in this procedure.

In summary, we have shown that catalysis offers a significant enhancement for entangled state transformations, leading to efficiencies previously known only for asymptotic setups. We have demonstrated this explicitly for bipartite pure state transitions, quantum state merging, and assisted entanglement distillation. It is reasonable to assume that similar results will hold for other quantum information protocols, including also the recently developed procedures for entangled state transformations in multipartite setups [31]. Our results suggest a full equivalence between asymptotic and catalytic entanglement theory. A rigorous proof of this equivalence is left open for future research.

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METHODS

Catalytic LOCC

A catalytic LOCC on the system in a state ρ^S is acting as in Eq. (1), where $\left\{\tau_n^C\right\}$ is a sequence of catalyst states and $\{\Lambda_n\}$ is a sequence of LOCC protocols. We require that for all n the catalyst remains unchanged in the process:

$$\mathrm{Tr}_S\left[\Lambda_n\left(\rho^S \otimes \tau_n^C\right)\right] = \tau_n^C. \tag{4}$$

In general, we do not bound the dimension of the catalyst, and we further require that the system decouples from the catalyst for large n. In particular, for a catalytic transformation from ρ^S to σ^S we require:

$$\lim_{n\to\infty} \left\|\mu_n^{SC} - \sigma^S \otimes \tau_n^C\right\|_1 = 0, \tag{5}$$

where $||M||_1 = \mathrm{Tr}\sqrt{M^\dagger M}$ is the trace norm and $\mu_n^{SC} = \Lambda_n\left(\rho^S \otimes \tau_n^C\right)$ is the total final state. We are now ready to prove Theorem 1.

Proof of Theorem 1

We consider a system S consisting of m-parties and restricted to LOCC. Let all the parties share a state ρ. Moreover, we assume that for any $\varepsilon > 0$ there exists an integer n and an LOCC transformation Λ such that

$$\Lambda\left(\rho^{\otimes n}\right) = \Gamma,\ \text{ s.t, } D\left(\Gamma, |\phi\rangle\langle\phi|^{\otimes n}\right) < \varepsilon, \tag{6}$$

where $D(\rho,\sigma) = \frac{1}{2}||\rho-\sigma||_1$ is the trace distance. The above relation implies that $|\phi\rangle$ is asymptotically achievable from ρ with unit rate. We will now show that in this case there also exists a catalytic LOCC procedure transforming ρ into $|\phi\rangle$. The following proof is inspired by techniques introduced very recently within quantum thermodynamics [24].

Consider a catalyst in the state

$$\tau = \frac{1}{n}\sum_{k=1}^{n} \rho^{\otimes k-1} \otimes \Gamma_{n-k} \otimes |k\rangle\langle k|. \tag{7}$$

The Hilbert space of the catalyst is in $S^{\otimes n-1} \otimes K$, where n is the integer introduced in Eq. (6) and K represents an auxiliary system of dimension n. For brevity, we denote the initial system S as S_1, and $n-1$ copies of the same system which belong to the catalyst are denoted by $S_2, \ldots, S_n$. Thus, the state of the catalyst acts on $S_2 \otimes \cdots \otimes S_n \otimes K$. Moreover, Γ is a quantum state on $S_1 \otimes S_2 \otimes \cdots \otimes S_n$, see also Eq. (6), and Γ_i is the reduced state of Γ on $S_1 \otimes S_2 \otimes \cdots \otimes S_i$. We further define $\Gamma_0 = 1$. The auxiliary system K is maintained by Alice, serving as a register with a Hilbert space of dimension n with basis $\{|k\rangle, k \in [1,n]\}$.

Consider now the following LOCC protocol acting on the system and the catalyst:

(i) Alice performs a rank-1 projective measurement on the auxiliary system K in the basis $|k\rangle$. She then communicates the outcome of the measurement to all the other parties. If Alice obtains the outcome n, all parties perform the LOCC protocol Λ given in Eq. (6) on $S_1 \otimes S_2 \otimes \cdots \otimes S_n$. For any other outcome of Alice's measurement the parties do nothing.

(ii) Alice applies a unitary on the auxiliary system that converts $|n\rangle \to |1\rangle$ and $|i\rangle \to |i+1\rangle$.

(iii) Finally, all the parties apply a SWAP unitary on their parts of (S_i, S_{i+1}) and (S_1, S_n), which shifts $S_i \to S_{i+1}$ and $S_n \to S_1$.

The initial state of the system and the catalyst is given by

$$\rho \otimes \tau = \frac{1}{n}\sum_{k=1}^{n} \rho^{\otimes k} \otimes \Gamma_{n-k} \otimes |k\rangle\langle k|. \tag{8}$$

After applying step (i), the initial state transforms to

$$\mu^i = \frac{1}{n}\sum_{k=1}^{n-1} \rho^{\otimes k} \otimes \Gamma_{n-k} \otimes |k\rangle\langle k| + \frac{1}{n}\Gamma \otimes |n\rangle\langle n|. \tag{9}$$

In step (ii), μ^i transforms to μ^{ii}, where

$$\mu^{ii} = \frac{1}{n}\sum_{k=1}^{n} \rho^{\otimes k-1} \otimes \Gamma_{n+1-k} \otimes |k\rangle\langle k|. \tag{10}$$

Note that tracing out S_n from μ^{ii} gives τ, which is the initial state of the catalyst, see Eq. (7). Therefore, using step (iii), we transform μ^{ii} to the final state μ having the property $\mathrm{Tr}_S[\mu] = \tau$. This proves that the state of the catalyst does not change in this procedure.

Now, we are left to show that $\mathrm{Tr}_C[\mu]$ is ε close to $|\phi\rangle$. $\mathrm{Tr}_C[\mu]$ can be expressed as

$$\mathrm{Tr}_C[\mu] = \frac{1}{n}\sum_{k=1}^{n} \gamma_k, \tag{11}$$

where

$$\gamma_k = \mathrm{Tr}_{1,2,\ldots,k-1,k+1,\ldots,n}[\Gamma] \tag{12}$$

is the reduced state of Γ on S_k. Using triangle inequality and monotonicity of the trace distance under partial trace we obtain

$$D\left(\frac{1}{n}\sum_{k=1}^{n}\gamma_k, |\phi\rangle\langle\phi|\right) \le \frac{1}{n}\sum_{k=1}^{n} D\left(\gamma_k, |\phi\rangle\langle\phi|\right) \tag{13}$$
$$\le \frac{1}{n}\sum_{k=1}^{n} D\left(\Gamma, |\phi\rangle\langle\phi|^{\otimes n}\right) < \varepsilon,$$

where in the last inequality we used Eq. (6).

The results just presented prove that if ρ can be converted into $|\phi\rangle$ with unit rate via asymptotic LOCC, then for any $\varepsilon > 0$ there exists a catalyst state τ and a LOCC protocol Λ such that

$$\sigma^{SC} = \Lambda\left(\rho^S \otimes \tau^C\right), \tag{14a}$$
$$\sigma^C = \tau^C, \quad D\left(\sigma^S, |\phi\rangle\langle\phi|^S\right) < \varepsilon. \tag{14b}$$

We will now show that the system and the catalyst decouple in this procedure, and moreover

$$D\left(\sigma^{SC}, |\phi\rangle\langle\phi|^S \otimes \tau^C\right) < \varepsilon + 3\sqrt{\varepsilon}. \tag{15}$$

In the first step, note that Eqs. (14) imply the inequality

$$F\left(\sigma^S, |\phi\rangle\langle\phi|^S\right) > \sqrt{1-\varepsilon}, \tag{16}$$

with fidelity $F(\rho,\sigma) = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$. The state σ^S has a purification

$$|\mu\rangle^{ST} = \sum_i \lambda_i |i\rangle^S |i\rangle^T \tag{17}$$

with the Schmidt coefficient λ_i sorted in decreasing order. Due to Eq. (16) we have

$$\lambda_0 > \sqrt{1-\varepsilon}. \tag{18}$$

Let now $|\nu\rangle^{SCD}$ be a purification of σ^{SC}, and observe that it can be written as

$$|\nu\rangle^{SCD} = \sum_i \lambda_i |i\rangle^S |\alpha_i\rangle^{CD}, \tag{19}$$

where λ_i are the same Schmidt coefficients as in Eq. (17) and $\{|\alpha_i\rangle\}$ is an orthonormal basis on CD. Noting that

$$F\left(|\nu\rangle\langle\nu|^{SCD}, |0\rangle\langle 0|^S \otimes |\alpha_0\rangle\langle\alpha_0|^{CD}\right) = \lambda_0, \tag{20}$$

and using the fact that the fidelity does not decrease under partial trace we obtain

$$F\left(\sigma^{SC}, |0\rangle\langle 0|^S \otimes \mathrm{Tr}_D\left[|\alpha_0\rangle\langle\alpha_0|^{CD}\right]\right) > \sqrt{1-\varepsilon}. \tag{21}$$

Using the inequality $D(\rho,\sigma) \le \sqrt{1-F(\rho,\sigma)^2}$ we arrive at

$$D\left(\sigma^{SC}, |0\rangle\langle 0|^S \otimes \mathrm{Tr}_D\left[|\alpha_0\rangle\langle\alpha_0|^{CD}\right]\right) < \sqrt{\varepsilon}. \tag{22}$$

Since ε can be chosen arbitrary small, this result shows that σ^{SC} can be made arbitrary close to a product state.

Noting that the trace norm does not increase under partial trace and using Eqs. (14) and (22) we obtain

$$D\left(\tau^C, \mathrm{Tr}_D\left[|\alpha_0\rangle\langle\alpha_0|^{CD}\right]\right) < \sqrt{\varepsilon}, \tag{23}$$

where τ^C is the state of the catalyst. We now use the triangle inequality, arriving at

$$D\left(\sigma^{SC}, |0\rangle\langle 0|^S \otimes \tau^C\right) \le D\left(\sigma^{SC}, |0\rangle\langle 0|^S \otimes \mathrm{Tr}_D\left[|\alpha_0\rangle\langle\alpha_0|^{CD}\right]\right)$$
$$+ D\left(|0\rangle\langle 0|^S \otimes \mathrm{Tr}_D\left[|\alpha_0\rangle\langle\alpha_0|^{CD}\right], |0\rangle\langle 0|^S \otimes \tau^C\right) < 2\sqrt{\varepsilon}. \tag{24}$$

Using again Eq. (22) we find

$$D\left(\sigma^S, |0\rangle\langle 0|^S\right) < \sqrt{\varepsilon}, \tag{25}$$

which together with Eqs. (14) and triangle inequality implies that

$$D\left(|\phi\rangle\langle\phi|^S, |0\rangle\langle 0|^S\right) < \varepsilon + \sqrt{\varepsilon}. \tag{26}$$

Using once again the triangle inequality we obtain Eq. (15):

$$D\left(\sigma^{SC}, |\phi\rangle\langle\phi|^S \otimes \tau^C\right) \le D\left(\sigma^{SC}, |0\rangle\langle 0|^S \otimes \tau^C\right)$$
$$+ D\left(|0\rangle\langle 0|^S \otimes \tau^C, |\phi\rangle\langle\phi|^S \otimes \tau^C\right) < \varepsilon + 3\sqrt{\varepsilon}. \tag{27}$$

This proves that the system and the catalyst decouple in the procedure, and that Eq. (5) is fulfilled. This completes the proof of Theorem 1.

Catalytic LOCC and squashed entanglement

Here we show that in the case of bipartite LOCC operations squashed entanglement, introduced in [25], is monotonic under catalytic LOCC transformations when the final state is pure. For bipartite quantum states ρ^{AB} the squashed entanglement is defined as [25]

$$E_{sq}\left(\rho^{AB}\right) = \inf\left\{\frac{1}{2}I(A;B|E) : \rho^{ABE} \text{ extension of } \rho^{AB}\right\}, \tag{28}$$

where the infimum is taken over all quantum states ρ^{ABE} with $\rho^{AB} = \mathrm{Tr}_E\left(\rho^{ABE}\right)$ and $I(A;B|E) = H\left(\rho^{AE}\right) + H\left(\rho^{BE}\right) - H\left(\rho^{ABE}\right) - H\left(\rho^E\right)$ is the quantum conditional mutual information of ρ^{ABE}.

We use the following properties of the squashed entanglement [25]:

(a) E_{sq} is an entanglement monotone, i.e. it does not increase under LOCC.

(b) E_{sq} is superadditive in general and additive on tensor products:

$$E_{sq}\left(\rho^{AA'BB'}\right) \ge E_{sq}\left(\rho^{AB}\right) + E_{sq}\left(\rho^{A'B'}\right) \tag{29}$$

and equality holds true if $\rho^{AA'BB'} = \rho^{AB} \otimes \rho^{A'B'}$.

(c) For a pure state $|\psi\rangle^{AB}$ squashed entanglement is equal to the entanglement entropy, i.e., the entropy of the reduced state:

$$E_{sq}\left(|\psi\rangle^{AB}\right) = H\left(\psi^{A}\right). \tag{30}$$

(d) Squashed entanglement is continuous in the vicinity of any pure state.

We are now ready to prove the following theorem.

Theorem 2. *If a bipartite state ρ^{AB} can be transformed into the pure state $|\phi\rangle^{AB}$ via catalytic LOCC, then*

$$E_{sq}\left(\rho^{AB}\right) \geq E_{sq}\left(|\phi\rangle^{AB}\right). \tag{31}$$

Proof. Assume that for any $\varepsilon > 0$ there exist a catalyst state $\tau^{A'B'}$ and an LOCC protocol Λ such that the final state $\sigma^{AA'BB'} = \Lambda(\rho^{AB} \otimes \tau^{A'B'})$ has the properties

$$D\left(\mathrm{Tr}_{A'B'}\left[\sigma^{AA'BB'}\right], |\phi\rangle\langle\phi|^{AB}\right) < \varepsilon,$$
$$\mathrm{Tr}_{AB}\left[\sigma^{AA'BB'}\right] = \tau^{A'B'}.$$

Using the properties (a) and (b) of the squashed entanglement, we find

$$E_{sq}\left(\sigma^{AA'BB'}\right) \leq E_{sq}\left(\rho^{AB}\right) + E_{sq}\left(\tau^{A'B'}\right) \tag{32}$$

and also

$$E_{sq}\left(\sigma^{AA'BB'}\right) \geq E_{sq}\left(\mathrm{Tr}_{A'B'}\left[\sigma^{AA'BB'}\right]\right) + E_{sq}\left(\tau^{A'B'}\right). \tag{33}$$

From Eqs. (32) and (33) it follows

$$E_{sq}\left(\rho^{AB}\right) \geq E_{sq}\left(\mathrm{Tr}_{A'B'}\left[\sigma^{AA'BB'}\right]\right). \tag{34}$$

If $\mathrm{Tr}_{A'B'}\left[\sigma^{AA'BB'}\right]$ can be made arbitrarily close to $|\phi\rangle\langle\phi|^{AB}$ in trace distance, then using the property (d) of the squashed entanglement we get $E_{sq}\left(\rho^{AB}\right) \geq E_{sq}\left(|\phi\rangle^{AB}\right)$, and the proof is complete. □

Combining Theorems 1 and 2, we conclude that a pure state $|\psi\rangle^{AB}$ can be transformed into another pure state $|\phi\rangle^{AB}$ via catalytic LOCC if and only if $H\left(\psi^{A}\right) \geq H\left(\phi^{A}\right)$, as claimed in the main text. This gives an operational interpretation for the von Neumann entropy in the single copy scenario.

Catalytic LOCC protocols in tripartite setups

We will now consider a tripartite setup, developing tools which will serve as a basis for catalytic quantum state merging. Similar to the state merging setup [27, 28], we consider three parties (Alice, Bob and Referee) sharing a tripartite state $\rho = \rho^{RAB}$. Assume now that by applying asymptotic LOCC between Alice and Bob, it is possible to asymptotically convert ρ into pure states $|\phi\rangle = |\phi\rangle^{RAB}$. The following theorem establishes a connection between this setup and catalytic LOCC.

Theorem 3. *If ρ can be converted into $|\phi\rangle$ via asymptotic LOCC between Alice and Bob with unit rate, then ρ can be converted into $|\phi\rangle$ by applying catalytic LOCC between Alice and Bob and a unitary on Referee's side.*

Proof. The proof follows similar reasoning as the proof of Theorem 1. If ρ can be converted into $|\phi\rangle$ via asymptotic LOCC between Alice and Bob with unit rate, then for any $\varepsilon > 0$ there exists an integer n and an LOCC protocol Λ between Alice and Bob such that

$$\Lambda\left[\rho^{\otimes n}\right] = \Gamma \quad \text{and} \quad D\left(\Gamma, |\phi\rangle\langle\phi|^{\otimes n}\right) < \varepsilon. \tag{35}$$

We now consider a catalyst in the state

$$\tau = \frac{1}{n}\sum_{k=1}^{n} \rho^{\otimes k-1} \otimes \Gamma_{n-k} \otimes |k\rangle\langle k|. \tag{36}$$

Also in this case the Hilbert space of the catalyst is in $S^{\otimes n-1} \otimes K$, where $S = RAB$ now corresponds to the tripartite system of Alice, Bob and Referee. Again, we denote the n copies of the same systems as $S_1, \ldots, S_n$, where S_1 corresponds to the system S, and the state of the catalyst is acting on $S_2 \otimes \cdots \otimes S_n \otimes K$. The operator Γ acts on $S_1 \otimes S_2 \otimes \cdots \otimes S_n$ and Γ_i ($i \in \{1, \ldots, n\}$) is the reduced state of Γ on $S_1 \otimes S_2 \otimes \cdots \otimes S_i$. Moreover, K is a register on Alice's side.

We now follow a procedure very similar to the one in the proof of Theorem 1.

(i) Alice performs a rank-1 projective measurement on the register K in the basis $|k\rangle$. She then communicates the outcome of the measurement to Bob. If Alice obtains the outcome n, Alice and Bob perform the LOCC protocol Λ given in Eq. (35). For any other outcome of Alice's measurement the parties do nothing.

(ii) Alice applies a unitary on the auxiliary system, that convert $|n\rangle \to |1\rangle$ and $|i\rangle \to |i+1\rangle$.

(iii) Alice, Bob and Referee apply a SWAP unitary on their parts of $(S_i,\ S_{i+1})$ and $(S_1,\ S_n)$ that shift $S_i \to S_{i+1}$ and $S_n \to S_1$. Note that communication with the Referee is not necessary, the Referee applies the SWAP operations independently of the procedure performed by Alice and Bob.

By the same reasoning as in the proof of Theorem 1, we see that the first subsystem S_1 of the final state is ε close to $|\phi\rangle$ while the catalyst remains unchanged. Moreover, the subsystem S_1 decouples from the catalyst in the limit $n \to \infty$, which is proven exactly in the same way as in Theorem 1, see also Eq. (15). □

Catalytic quantum state merging

In quantum state merging (QSM) [27, 28], we assume that Alice, Bob and Referee share asymptotically many copies of a pure quantum state $|\psi\rangle^{RAB}$. By applying LOCC operations, Alice and Bob aim to transfer the state of Alice to Bob while preserving correlations with Referee, i.e., the final state

$|\psi\rangle^{RBB'}$ is the same as $|\psi\rangle^{RAB}$ up to relabelling of A and B'. In the following, we describe three possible scenarios.

a) In the asymptotic limit where many copies of the state $|\psi\rangle^{RAB}$ are available, QSM is possible if the conditional entropy is zero [27, 28], i.e.,

$$H_\psi(A|B) = H\left(\psi^{AB}\right) - H\left(\psi^B\right) = 0. \tag{37}$$

This means, if $H_\psi(A|B) = 0$, there exists an LOCC protocol taking $|\psi\rangle^{RAB}$ arbitrarily close to $|\psi\rangle^{RBB'}$

$$\left(|\psi\rangle^{RAB}\right)^{\otimes n} \xrightarrow[LOCC]{\varepsilon} \left(|\psi\rangle^{RBB'}\right)^{\otimes n}, \tag{38}$$

where ε above the arrow represents that the final state is ε close in trace distance to $\left(|\psi\rangle^{RBB'}\right)^{\otimes n}$.

b) If $H_\psi(A|B) > 0$, then we define $|\psi'\rangle^{RA\tilde{A}B\tilde{B}} = |\psi\rangle^{RAB} \otimes |\phi_1\rangle^{\tilde{A}\tilde{B}}$, where $|\phi_1\rangle^{\tilde{A}\tilde{B}}$ is a shared entangled state between Alice and Bob with entanglement entropy $E\left(|\phi_1\rangle^{\tilde{A}\tilde{B}}\right) = H_\psi(A|B)$. This implies, $H_{\psi'}\left(A\tilde{A}|B\tilde{B}\right) = 0$. Therefore, from Eq. (38) we see that Alice and Bob can successfully merge the state $|\psi'\rangle^{RA\tilde{A}B\tilde{B}}$:

$$\left(|\psi'\rangle^{RA\tilde{A}B\tilde{B}}\right)^{\otimes n} \xrightarrow[LOCC]{\varepsilon} \left(|\psi\rangle^{RBB'}\right)^{\otimes n}. \tag{39}$$

c) If $H_\psi(A|B) < 0$, the following transformation is achievable via LOCC between Alice and Bob [27, 28]:

$$\left(|\psi\rangle^{RAB}\right)^{\otimes n} \xrightarrow[LOCC]{\varepsilon} \left(|\psi\rangle^{RBB'}\right)^{\otimes n} \otimes \left(|\phi^+\rangle^{\tilde{A}\tilde{B}}\right)^{\otimes -H_\psi(A|B)n}, \tag{40}$$

with the Bell state $|\phi^+\rangle = (|0\rangle|0\rangle + |1\rangle|1\rangle)/\sqrt{2}$. Additionally, we know that [1]

$$\left(|\phi^+\rangle^{\tilde{A}\tilde{B}}\right)^{\otimes -H_\psi(A|B)n} \xrightarrow[LOCC]{\varepsilon} \left(|\phi_2\rangle^{\tilde{A}\tilde{B}}\right)^{\otimes n}, \tag{41}$$

where $|\phi_2\rangle^{\tilde{A}\tilde{B}}$ is a bipartite pure state with entanglement entropy $E\left(|\phi_2\rangle^{\tilde{A}\tilde{B}}\right) = -H_\psi(A|B)$. Therefore, from Eqs. (40) and (41) we see that the following transformation is achievable via LOCC between Alice and Bob:

$$\left(|\psi\rangle^{RAB}\right)^{\otimes n} \xrightarrow[LOCC]{\varepsilon} \left(|\psi\rangle^{RBB'} \otimes |\phi_2\rangle^{\tilde{A}\tilde{B}}\right)^{\otimes n}. \tag{42}$$

Equipped with these tools we are now ready to discuss catalytic quantum state merging. Recall that in catalytic QSM, we allow LOCC operations between Alice and Bob, and the Referee can also perform local unitaries. However, no communication between the Referee and the other parties is allowed. This is exactly the setup considered in Theorem 3, and we will make use of this result in the following.

An immediate consequence of Theorem 3 and Eq. (42) is that for $H_\psi(A|B) < 0$ catalytic state merging from Alice to Bob is possible. Additionally, Alice and Bob can obtain an entangled state $|\phi_2\rangle^{\tilde{A}\tilde{B}}$ with entanglement entropy $E(|\phi_2\rangle^{\tilde{A}\tilde{B}}) = -H_\psi(A|B)$.

In the following, we prove by contradiction that this is the optimal value one can achieve. Let us assume that there exists a catalytic LOCC procedure such that

$$E\left(|\phi_2\rangle^{\tilde{A}\tilde{B}}\right) > H\left(\psi^B\right) - H\left(\psi^{AB}\right). \tag{43}$$

Consider now the squashed entanglement [25, 26] between Bob and the rest of the system in the initial state $|\psi\rangle^{RAB}$:

$$E_{sq}^{B|AR}\left(|\psi\rangle^{RAB}\right) = H\left(\psi^B\right). \tag{44}$$

On the other hand, the squashed entanglement between Bob and the rest of the system in the target state $|\psi\rangle^{RBB'} \otimes |\phi_2\rangle^{\tilde{A}\tilde{B}}$ is given by

$$E_{sq}^{BB'\tilde{B}|\tilde{A}R}\left(|\psi\rangle^{RBB'} \otimes |\phi_2\rangle^{\tilde{A}\tilde{B}}\right) = H\left(\psi^{AB}\right) + E\left(|\phi_2\rangle^{\tilde{A}\tilde{B}}\right). \tag{45}$$

From Eqs. (43), (44), and (45), we obtain

$$E_{sq}^{BB'\tilde{B}|\tilde{A}R}\left(|\psi\rangle^{RBB'} \otimes |\phi_2\rangle^{\tilde{A}\tilde{B}}\right) > H\left(\psi^B\right) = E_{sq}^{B|AR}(|\psi\rangle^{RAB}). \tag{46}$$

Recalling that squashed entanglement is continuous in the vicinity of pure states [25], this means that the squashed entanglement has increased in the process. This is a contradiction to Theorem 2, showing that for pure states squashed entanglement cannot increase under catalytic LOCC.

In the remaining case $H_\psi(A|B) \geq 0$, from Theorem 3 and Eq. (39) it directly follows catalytic QSM is possible when Alice and Bob are provided with an additional state $|\phi_1\rangle^{\tilde{A}\tilde{B}}$ with entanglement entropy $E\left(|\phi_1\rangle^{\tilde{A}\tilde{B}}\right) = H_\psi(A|B)$. We now show that this is the minimal entanglement entropy needed to perform catalytic QSM. Again we use the properties of the squashed entanglement to prove this. The squashed entanglement between Bob and the other parties in the initial state $|\psi\rangle^{RAB} \otimes |\phi_1\rangle^{\tilde{A}\tilde{B}}$ is given by

$$E_{sq}^{B\tilde{B}|A\tilde{A}R}\left(|\psi\rangle^{RAB} \otimes |\phi_1\rangle^{\tilde{A}\tilde{B}}\right) = H\left(\psi^B\right) + E\left(|\phi_1\rangle^{\tilde{A}\tilde{B}}\right). \tag{47}$$

For the target state $|\psi\rangle^{RBB'}$ we obtain

$$E_{sq}^{BB'|R}\left(|\psi\rangle^{RBB'}\right) = H\left(\psi^{AB}\right). \tag{48}$$

Using again the fact that squashed entanglement is continuous in the vicinity of pure states and cannot increase under catalytic LOCC, we have

$$E_{sq}^{B\tilde{B}|A\tilde{A}R}\left(|\psi\rangle^{RAB} \otimes |\phi_1\rangle^{\tilde{A}\tilde{B}}\right) \geq E_{sq}^{BB'|R}\left(|\psi\rangle^{RBB'}\right). \tag{49}$$

Hence, from Eqs. (47), (48) and (49), we get an achievable lower bound on the entanglement entropy of $|\phi_1\rangle^{\tilde{A}\tilde{B}}$:

$$E\left(|\phi_1\rangle^{\tilde{A}\tilde{B}}\right) \geq H\left(\psi^{AB}\right) - H\left(\psi^B\right). \tag{50}$$

Catalytic assisted entanglement distillation

Consider now three parties, Alice, Bob, and Charlie, sharing a pure state $|\psi\rangle^{ABC}$. By performing catalytic LOCC between all the parties, they aim to convert $|\psi\rangle^{ABC}$ into a state

$|\phi\rangle^{AB}$ which has maximal possible entanglement entropy. This task is analogous to assisted entanglement distillation, which has been previously studied in the asymptotic setting [30].

We will now show that the optimal procedure is for Charlie to merge his state either with Alice or with Bob. For this, consider the corresponding conditional mutual information

$$H_\psi(C|A) = H\left(\psi^{AC}\right) - H\left(\psi^{A}\right) = H\left(\psi^{B}\right) - H\left(\psi^{A}\right), \quad (51)$$

$$H_\psi(C|B) = H\left(\psi^{BC}\right) - H\left(\psi^{B}\right) = H\left(\psi^{A}\right) - H\left(\psi^{B}\right). \quad (52)$$

We immediately see that either $H_\psi(C|A)$ and $H_\psi(C|B)$ are both zero, or at least one of them is negative.

If $H_\psi(C|A) < 0$, then Charlie merges his system with Alice by using catalytic QSM. As a result, Alice and Bob will end up with a state having entanglement entropy $H\left(\psi^{B}\right)$. Note that $H_\psi(C|A) < 0$ is equivalent to $H\left(\psi^{B}\right) < H\left(\psi^{A}\right)$. On the other hand, if $H_\psi(C|B) \leq 0$, then Charlie merges his system with Bob, leaving Alice and Bob with a state having entanglement entropy $H\left(\psi^{A}\right)$. Since $H_\psi(C|B) \leq 0$ is equivalent to $H\left(\psi^{A}\right) \leq H\left(\psi^{B}\right)$, this proves that via catalytic LOCC it is possible to convert $|\psi\rangle^{RAB}$ into a quantum state $|\phi\rangle^{AB}$ having entanglement entropy

$$E\left(|\phi\rangle^{AB}\right) = \min\left\{H\left(\psi^{A}\right), H\left(\psi^{B}\right)\right\}. \quad (53)$$

The converse can be proven by using the properties of the squashed entanglement [25], in particular that for pure states it corresponds to the entanglement entropy and does not increase under catalytic LOCC, see Theorem 2. Since any tripartite LOCC protocol is also bipartite with respect to any bipartition, it must be that

$$E_{sq}^{A|BR}\left(|\psi\rangle^{RAB}\right) \geq E_{sq}^{A|B}\left(|\phi\rangle^{AB}\right), \quad (54)$$

$$E_{sq}^{B|AR}\left(|\psi\rangle^{RAB}\right) \geq E_{sq}^{A|B}\left(|\phi\rangle^{AB}\right). \quad (55)$$

This means that the entanglement entropy of $|\phi\rangle^{AB}$ is bounded above by $\min\left\{H\left(\psi^{A}\right), H\left(\psi^{B}\right)\right\}$.

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