Catalytic quantum teleportation

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Quantum catalysis is an intricate feature of quantum entanglement. It demonstrates that in certain situations the very presence of entanglement can improve one's abilities of manipulating other entangled states. At the same time, however, it is not clear if using entanglement catalytically can provide additional power for any of the existing quantum protocols. Here we show, for the first time, that catalysis of entanglement can provide a genuine advantage in the task of quantum teleportation. More specifically, we show that extending the standard teleportation protocol by giving Alice and Bob the ability to use entanglement catalytically, allows them to achieve fidelity of teleportation at least as large as the regularisation of the standard teleportation quantifier, the so-called average fidelity of teleportation. Consequently, we show that this regularised quantifier surpasses the standard benchmark for a variety of quantum states, therefore demonstrating that there are quantum states whose ability to teleport can be further improved when assisted with entanglement in a catalytic way. This hints that entanglement catalysis can be a promising new avenue for exploring novel advantages in the quantum domain.

I. INTRODUCTION

The existence of quantum entanglement is one of the most striking consequences implied by the laws of quantum mechanics [1–3]. It is manifested when correlations between different particles are strong enough so that the action on one of them affects the other, in a way so subtle that it cannot be explained by any classical mechanism. Although entanglement was initially recognized as a bizarre property separating quantum from classical physics, it is nowadays viewed as an indispensable resource with an enormous number of modern applications.

Arguably one of the most important applications of quantum entanglement is the protocol of quantum teleportation [4]. In its standard form it is a communication task which involves transferring an unknown quantum state to a remote recipient using classical communication and shared entanglement. This is also perhaps the best evidence for the resource nature of quantum entanglement, as the laws of quantum mechanics rule out transferring quantum states without its presence. The significance of the protocol can be best evidenced by its wide applicability in various areas of quantum information [5, 6], computation [7–9] and even the theory of relativity [10, 11]. Quantum teleportation is also a building block for many quantum information processing tasks and has been realised in laboratories using variety of different technologies, including photonic qubits [12–17], optical modes [18–20], nuclear magnetic resonance (NMR) [21], atomic ensembles [22–24], trapped atoms [25–27] or solid-state systems [28–30].

It is also a folklore knowledge that quantum entanglement is a resource which must be consumed in order to outperform classical systems in information processing tasks. What is less known, however, is that in certain cases the very presence of entanglement can provide advantage, without it being consumed or degraded. This surprising and yet not clearly understood phenomenon is called *quantum catalysis* and was introduced in [31], further analysed in [32–38] and subsequently adapted to other physical settings like quantum thermodynamics [39–49], resource theory of coherence [50, 51], purity [52], asymmetry [53] or to the study of quantum reference frames [54]. In the particular case of entanglement theory, quantum catalysis demonstrates that access to a special entangled state (the catalyst) can sometimes allow two distant parties to manipulate their entanglement in a way that would otherwise be impossible. Importantly, the ancillary entangled state used in this process is not consumed, so that the parties can repeat their task or use the catalyst for some other task. This makes catalysis a particularly interesting extension of the standard paradigm of local operations and classical communication (LOCC). However, despite its fundamental significance for the theory of quantum entanglement, it is still not clear whether the subtle improvement of manipulation abilities provided by catalysis can ever lead to any advantages for genuine quantum task. Finding protocols whose performance can be improved by utilising catalysis is therefore an important open problem, both from a theoretical and experimental point of view.

In this work we solve this problem in positive by extending the task of quantum teleportation by allowing Alice and Bob use an arbitrary amount of entanglement that has to be returned intact. We then show that this extension of the standard protocol can sometimes significantly improve their optimal fidelity of teleportation. In particular, we show that in a catalytic setting the optimal fidelity of teleportation is lower-bounded by a regularisation of the standard fidelity of teleportation. This new quantity is then shown to be strictly larger than the standard fidelity of teleportation for a wide range of pure states, meaning that catalysis allows for a generic improvement over the standard teleportation protocol. To the best of our knowledge, this is the first time when catalysis is used to provide a quantitative advantage in a quantum information processing task. Our proof is based on a technique which uses a catalyst to increase entanglement fraction of another state, a realisation which we believe to be of independent interest. Since entanglement fraction determines performance is many relevant information-processing tasks, we believe that this could shed more light on catalytic advantages that could be observed in other information-processing tasks.

II. FRAMEWORK

In what follows we denote a discrete and finite-dimensional Hilbert space associated with a quantum system S with $\mathcal{H}_S$. We also denote the space of all density operators in $\mathcal{H}_S$ with $\mathcal{D}(\mathcal{H}_S)$. We will be interested in scenarios involving two distant parties (Alice and Bob) who are allowed to use local operations and classical communication (LOCC). A quantum channel $\mathcal{E}$ is a completely positive and trace-preserving linear map acting between spaces of density operators. We say that $\mathcal{E} \in \text{LOCC}(A:B)$ if it can be written as a sequence of quantum channels applied locally by A and B, intertwined with classical communication between the two parties. An important entanglement quantifier which we use extensively here is the so-called entanglement fraction of a state, defined as the best overlap with a maximally entangled state [55]. More precisely,

$$\begin{aligned} f(\rho) := \quad & \max_{\mathcal{E}} \quad \langle \phi^+_{AB} | \mathcal{E}(\rho_{AB}) | \phi^+_{AB} \rangle \\ & \text{s.t.} \quad \mathcal{E} \in \text{LOCC}(A:B), \end{aligned} \tag{1}$$

where $|\phi^+_{AB}\rangle = \sum_{i=1}^d |i\rangle_A |i\rangle_B / \sqrt{d}$ denotes a maximally-entangled state shared between A and B.

A. Standard quantum teleportation

Before presenting our main results let us briefly recall the task of quantum teleportation [4]. In its most general form the protocol involves two spatially separated parties, Alice and Bob, who share an arbitrary quantum state ρ_{AB} of local dimension d. A third party, often called the Referee, provides Alice with a quantum state φ_R of dimension d_R which is unknown to both parties. The goal set before Alice and Bob is to transfer the unknown state from one party to another, using only local operations and classical communications, i.e. quantum channels $\mathcal{T} \in \text{LOCC}(RA:B)$, and shared entanglement. Under this conditions all possible states which can be achieved in Bob's lab can be written as:

$$\rho'_B = \text{tr}_{RA}\, \mathcal{T}(\varphi_R \otimes \rho_{AB}), \tag{2}$$

where tr_S denotes a partial trace over subsystem S. The above protocol can be equivalently viewed as a process of establishing a quantum channel between Alice and Bob that maps the input state φ_R to the output ρ'_B. The quality of a teleportation protocol or, equivalently, the fidelity of the resulting teleportation channel, can be quantified using the optimal fidelity of teleportation [56], which for a density operator ρ is defined as:

$$\begin{aligned} \langle F \rangle_\rho := \max_{\mathcal{T}} & \int \langle \varphi | \, \text{tr}_{RA}\, \mathcal{T}(\varphi_R \otimes \rho_{AB}) | \varphi \rangle \, \mathrm{d}\psi \\ \text{s.t.} & \quad \mathcal{T} \in \text{LOCC}(RA:B). \end{aligned} \tag{3}$$

In what follows we will refer to the above quantity as simply "fidelity of teleportation". The integral in (3) is computed over a uniform distribution of all possible input states $\varphi = |\varphi\rangle\langle\varphi|$ according to a normalised Haar measure $\int \mathrm{d}\psi = 1$. It can be easily verified that $0 \leq \langle F \rangle_\rho \leq 1$ for all density operators $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Furthermore, the case $\langle F \rangle_\rho = 1$ corresponds to perfect teleportation and is possible if and only if ρ is a maximally-entangled state. In practice, teleportation can never be perfect and the optimal fidelity of teleportation will generally be less than one. If Alice and Bob do not share an entangled state, then the corresponding teleportation protocol is said to be "classical". In that case the optimal fidelity can never exceed the threshold value $\langle F \rangle_c := 2/(d+1)$. Therefore, if $\langle F \rangle_\rho > \langle F \rangle_c$ can be demonstrated, then the associated state ρ is necessarily entangled. Computing the raw expression (3) is generally a difficult task. However, it was shown in Ref. [55] that fidelity of teleportation (3) is related with entanglement fraction (1) via:

$$\langle F \rangle_\rho = \frac{f(\rho)d + 1}{d+1}. \tag{4}$$

This important realisation allows to easily compute fidelity of teleportation for many relevant cases. In the next section we focus on this quantity and show that using catalysts in a proper way allows to increase entanglement fraction of the state, without consuming additional entanglement.

III. RESULTS

A. Catalytic quantum teleportation

Let us now describe the catalytic extension of the standard quantum teleportation protocol. We assume that Alice and Bob, in addition to their shared state ρ_{AB}, have also access to a quantum system CC' which is prepared in some state $\omega_{CC'}$. This additional system is then distributed to both parties, so that Alice has access only to C, and Bob only to its C' part. Alice is then given an unknown quantum state φ_R and the parties perform a protocol $\mathcal{T} \in \text{LOCC}(RAC:BC')$ which now acts on both systems they share and the input system. Moreover, for the protocol to be catalytic we demand that the action of $\mathcal{T}$ does not modify the state of the catalyst. Notably, we do allow the catalyst to become correlated with other systems during this protocol. Later in the Appendix we show that these correlations can be made arbitrarily small, at the expanse of using larger catalysts. The final state of Bob's subsystem at the end of the catalytic teleportation protocol reads:

$$\rho'_B = \text{tr}_{RACC'} \left[\mathcal{T}(\varphi_R \otimes \rho_{AB} \otimes \omega_{CC'}) \right] \tag{5}$$

The quality of this protocol can be quantified similarly as in the case of standard teleportation, i.e. using the fidelity of teleportation (3). Notice, however, that now we also have freedom to choose any entangled state to be used as the catalyst. In order to benchmark the quality of the arising teleportation protocol, we now define a new teleportation quantifier that captures this additional freedom. Therefore, we define the fidelity of catalytic teleportation $\langle F_{\text{cat}} \rangle_\rho$ as the the solution of the following optimisation problem:

$$\begin{aligned}\langle F_{\text{cat}}\rangle_\rho = \max_{\mathcal{T},\,\omega} \int \langle\psi|\, \mathrm{tr}_{RACC'}\, \mathcal{T}(\varphi_R \otimes \rho_{AB} \otimes \omega_{CC'})|\psi\rangle\, \mathrm{d}\psi \\ \text{s.t.}\quad \mathrm{tr}_{RAB}\, \mathcal{T}(\rho_{RAB} \otimes \omega_{CC'}) = \omega_{CC'}, \\ \mathcal{T} \in \mathrm{LOCC}(RAC : BC') \\ \omega_{CC'} \in \mathcal{D}(\mathcal{H}_C \otimes \mathcal{H}_{C'}).\end{aligned} \tag{6}$$

By definition we again have $0 \le \langle F_{\text{cat}}\rangle_\rho \le 1$ and the reduced state of the catalyst $\omega_{CC'}$ is left unchanged, as we discussed in the Introduction. Before we present our main result let us first define a regularisation of the entanglement fraction from Eq. (1), a quantity whose significance will soon become evident. Therefore, the *regularised entanglement fraction* $f_{\text{reg}}(\rho)$ will be defined as:

$$f_{\text{reg}}(\rho) := \lim_{n\to\infty} \frac{f_n(\rho^{\otimes n})}{n}, \tag{7}$$

where $f_n(\rho)$ is the solution to:

$$\begin{aligned} f_n(\rho) := \max_{\mathcal{E}} \sum_{i=1}^n \langle\phi^+|\, \mathrm{tr}_{/i}\, \mathcal{E}(\rho_{AB})|\phi^+\rangle, \\ \text{s.t. } \mathcal{E} \in \mathrm{LOCC}(A_1 \ldots A_n : B_1 \ldots B_n), \end{aligned} \tag{8}$$

where $\mathrm{tr}_{/i}(\cdot)$ is the partial trace performed over particles $1 \ldots i-1, i+1 \ldots n$. Notice that by taking a sub-optimal guess $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 = \ldots \mathcal{E}_n$ with $\mathcal{E}_1 = \mathcal{E}_2 = \ldots = \mathcal{E}_n$ we can infer that $f_{\text{reg}}(\rho) \ge f(\rho)$ for all density operators ρ. With the above definitions we are now ready to present our main result.

Theorem 1. *Let $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The fidelity of catalytic teleportation of ρ satisfies:*

$$\langle F_{\text{cat}}\rangle_\rho \ge \frac{f_{\text{reg}}(\rho)d + 1}{d+1} \tag{9}$$

In other words, there is a protocol $\mathcal{T} \in \mathrm{LOCC}(RAC : BC')$ and a catalyst $\omega_{CC'} \in \mathcal{D}(\mathcal{H}_C \otimes \mathcal{H}_{C'})$ which achieves the bound in (9).

Proof. Here we will only sketch the proof of Theorem 1 and postpone its formal derivation to the Appendix. The structure of the proof is as follows: we start by constructing the catalyst and a special protocol $\mathcal{T}_1$ which increases entanglement fraction of the shared state and then use this state to perform an optimal teleportation protocol $\mathcal{T}_2$.

Let $n \ge 2$ be a finite natural number and let $C := C_2 \ldots C_n M$ and $C' := C'_2 \ldots C'_n M$, where M is a classical register. Moreover, let $\mathcal{E} \in \mathrm{LOCC}(AC : BC')$ be a channel performed by Alice and Bob (yet to be determined) and denote $\sigma^{n-i} := \mathrm{tr}_{1\ldots i}\, \mathcal{E}(\rho^{\otimes n})$, where $\mathrm{tr}_{1\ldots i}(\cdot)$ denotes the trace over the first i copies of $\rho^{\otimes n}$. Consider the following catalyst:

$$\omega_{CC'} = \frac{1}{n} \sum_{i=1}^n \underbrace{\rho^{\otimes i} \otimes \sigma_{n-i}}_{C_2 C'_2 \ldots C_n C'_n} \otimes |i\rangle\langle i|_M\,. \tag{10}$$

To the best of our knowledge, this family of states was introduced for the first time in the context of entanglement transformations by Duan in [57]. Therefore, in what follows, we will refer to the family of states from Eq. (10) as *Duan states*.

Let us label for clarity $A_1 \equiv A$ and $A_i \equiv C_i$ for $2 \le i \le n$ and similarly for B_i and BC'. The joint state of the shared system and the catalyst, $\rho_{AB} \otimes \omega_{CC'}$, is presented in Fig. 1a for the exemplary case when $n = 5$. The initial protocol $\mathcal{T}_1$ can be summarised as follows:

1. Apply $\mathcal{E} \in \mathrm{LOCC}(AC : BC')$ to the n-th pair using the classical register as the control (see Fig. 1b).
2. Relabel the register M using $|i\rangle_M \to |i+1\rangle_M$ for $1 \le i < n$ and $|n\rangle_M \to |1\rangle_M$ (see Fig. 1c).
3. Relabel quantum systems according to the value in the register: $A_1 B_1 \to A_i B_i$ for $2 \le i \le n$ (see Fig. 1d).
4. Discard the catalyst CC'

This results in the system and the catalyst being transformed as:

$$\begin{aligned} \rho_{AB} \to \rho_{AB}^{(n)} &= \mathrm{tr}_{CC'}\, \mathcal{T}_1(\rho_{AB} \otimes \omega_{CC'}) \\ &= \frac{1}{n} \sum_{i=1}^n \mathrm{tr}_{/i}\, \mathcal{E}(\rho_{AB}^{\otimes n}), \end{aligned} \tag{11}$$

$$\omega_{CC'} \to \omega'_{CC'} = \mathrm{tr}_{AB}\, \mathcal{T}_1(\rho_{AB} \otimes \omega_{CC'}) = \omega_{CC'}. \tag{12}$$

Let us now describe the protocol $\mathcal{T}_2$, which is a standard teleportation scheme for noisy states [58]. Let $\{U_a^{\mathrm{A}}\}$ for $a \in \{1, \ldots, d^2\}$ be a set of generalised Pauli operators with respect to the basis $\{|i\rangle^{\mathrm{A}}\}$. The protocol $\mathcal{T}_2$ reads as follows:

1. Twirl the shared state into an isotropic state:

$$\textsc{Twirl}(\rho_{AB}^{(n)}) = f(\rho_{AB}^{(n)})\phi_{AB}^+ + (1 - f(\rho_{AB}^{(n)}))\phi_{AB}^\perp, \tag{13}$$

   where $\phi^\perp = (\mathbb{1} - \phi^+)/(d^2-1)$ and $\mathrm{tr}\big(\phi^+\phi^\perp\big) = 0$.

2. Perform standard teleportation on $RA \to B$:
   (a) Alice measures RA using a POVM with elements:
   $$M_a^{RA} = (\mathbb{1} \otimes U_a)\phi_{RA}^+(\mathbb{1} \otimes U_a^\dagger), \tag{14}$$
   (b) Alice communicates outcome a to Bob,
   (c) Bob applies $U_a^\dagger(\cdot)U_a$ to his share of the state.

The optimal fidelity of teleportation that can be achieved in the above process reads:

$$\frac{f(\rho_{AB}^{(n)})d + 1}{d+1}. \tag{15}$$

Notice that so far the channel $\mathcal{E}$ was arbitrary. Let us now optimize the protocol $\mathcal{T} = \mathcal{T}_2 \circ \mathcal{T}_1$ over all feasible channels $\mathcal{E} \in \mathrm{LOCC}(AC : BC')$. Taking the limit $n \to \infty$ and using $\lim_{n\to\infty} f(\rho_{AB}^{(n)}) = f_{\text{reg}}(\rho_{AB})$ leads to Eq. (9). □

The regularised entanglement fraction can be, in general, difficult to compute, both because of the limit $n \to \infty$ and the optimisation over all LOCC protocols in (7). However, it turns out that in the limit of large n we can make use of typicality arguments to determine a wide range of states for which the inequality in Eq. (9) is *strict*. In other words, there are many entangled states whose performance in teleportation can be improved by utilising the effect of catalysis.

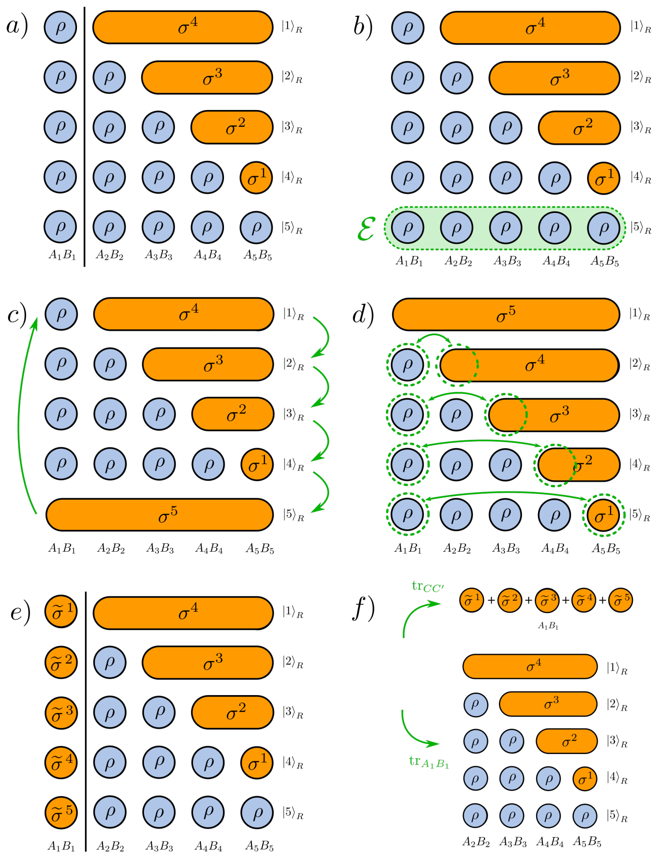

FIG. 1. The main part of the protocol which uses a noisy entangled state as a catalyst to improve the fidelity of teleportation. Subplots $(a)-(e)$ describe different steps of the protocol and (f) describes the final state of the main system and the catalyst. In particular, the catalyst remains unchanged by the protocol as the system is transformed into a state with a higher entanglement fraction. This state is then used as a basis for the standard teleportation protocol.

B. Demonstrating catalytic advantage in teleportation

Our reasoning so far was valid for arbitrary bipartite density operators $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. In this section we will restrict our attention to pure states $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ and use typicality arguments to infer that the presented protocol for catalytic teleportation leads to a generic advantage over the standard teleportation protocol. Interestingly, this is a consequence of an essential property of catalysis: that certain catalysts (Duan states) "amplify" typical properties of states, even at the level of a single copy. This remarkable property of catalysis was first used in the resource-theoretic formulation of thermodynamics [39, 44] and more recently in several other contexts [45, 46, 59]. Interestingly, all of these works reveal some interesting properties of catalysis and suggest that catalysts of the form (10) allow to access the power of multi-copy transformations, while effectively consuming only a single copy of the state. A similar behavior can be observed in the setting of catalytic teleportation. To see this, consider the following lemma:

Lemma 1. *The regularised entanglement fraction $f_{reg}(\psi_{AB})$ for pure states $\psi_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be bounded as:*

$$f_{reg}(\psi_{AB}) \geq \max_{\psi'} \; f(\psi'_{AB}) \tag{16}$$

$$\text{s.t.} \quad S(\rho_A) \geq S(\rho'_A), \tag{17}$$

where $\rho_A = \operatorname{tr}_B \psi_{AB}$ and $\rho'_A = \operatorname{tr}_B \psi'_{AB}$ and $S(\rho) = -\operatorname{tr}\rho\log\rho$ is the Shannon entropy.

The state ψ'_{AB} from the above lemma can be interpreted as produced in the first part of the catalytic teleportation protocol $\mathcal{T}_1$, that is $\psi'_{AB} = \operatorname{tr}_{CC'} \mathcal{T}_1(\psi_{AB} \otimes \omega_{CC'})$, where $\omega_{CC'}$ is the Duan state (10) for a sufficiently large n. Importantly, Theorem 1 along with Lemma 1 tells us that, at least for pure states, the fidelity of catalytic teleportation can be strictly larger the standard fidelity of teleportation. To see this more clearly, consider the following example.

Example. Let us consider teleporting a three-dimensional quantum system ($d_R = 3$) using the singlet state. In this case the state shared between Alice and Bob can be written as $\psi_{AB} = \sum_{i=1}^{3} \sqrt{\lambda_i}\,|i\rangle_A |i\rangle_B$, with Schmidt coefficients $\lambda_1 = 1/2$, $\lambda_2 = 1/2$ and $\lambda_3 = 0$. Its entanglement fraction is equal to $f(\psi_{AB}) = (\sum_{i=1}^{3}\sqrt{\lambda_i})^2/3 = 2/3$ and therefore its fidelity of teleportation is given by:

$$\langle F\rangle_\psi = 0.75, \tag{18}$$

which is also larger than the classical threshold $\langle F_c\rangle = 1/2$.

Let us now analyse the analogous protocol for catalytic teleportation. In this case the relevant benchmark is the fidelity of catalytic teleportation (6) whose lower bound can be found using Lemma 1. To compute it, let us choose the optimizer in (16) to be the state ψ^*_{AB} with Schmidt coefficients $\lambda_1^* = x$ and $\lambda_2^* = \lambda_3^* = (1-x)/2$, where x is the unique solution to $h(x) = x\log 2$ (which is approximately $x \approx 0.77$) and $h(x) = -x\log x - (1-x)\log(1-x)$ is the binary entropy. It can be easily verified that this is a feasible choice since the entropy of marginals of both states ψ_{AB} and ψ^*_{AB} is equal to $\log 2$. According to Lemma 1 the regularised entanglement fidelity can be lower-bounded by the entanglement fidelity of ψ^*_{AB}, therefore $f_{\text{reg}}(\psi_{AB}) \geq f(\psi^*_{AB}) \approx 4/5$. Using Theorem 1 we can then lower-bound the fidelity of catalytic teleportation as:

$$\langle F_{\text{cat}}\rangle \geq 0.85, \tag{19}$$

which is roughly 13% larger than the best fidelity that could ever be obtained when using the state ψ_{AB} alone. Interestingly, this simple example is not a singular case: there are in fact many entangled states whose performance in teleportation can be improved using the assistance of catalysts. To show this in Fig. 2 we used Lemma 1 and numerically computed the lower-bound on the catalytic advantage $\eta(\psi)$ defined as $\eta(\psi) := (\langle F_{\text{cat}}\rangle - \langle F\rangle)/\langle F\rangle$. It is easy to show that a similar improvement can be demonstrated for higher dimensional entangled states as well, e.g. using existing numerical optimisation packages to solve (16) .

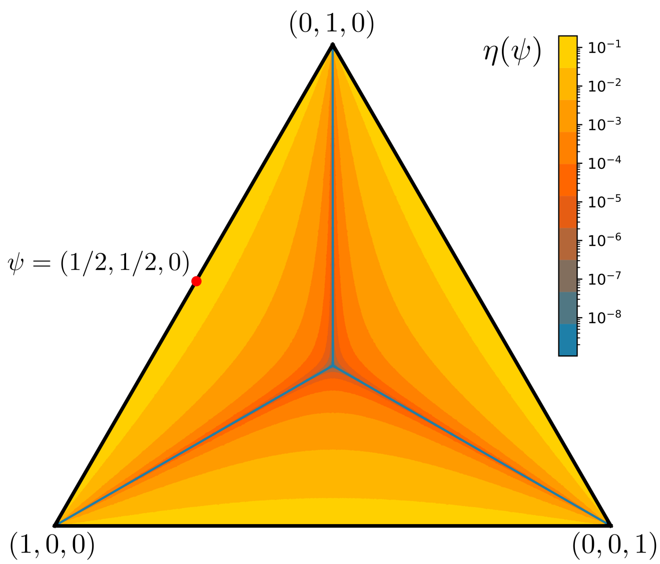

FIG. 2. The advantage $\eta(\psi)$ achieved by using entanglement catalysis in the protocol of quantum teleportation. The triangle corresponds to the space of all three-dimensional pure biparitite quantum states. In particular, each point $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ in the plot corresponds to a unique (up to local unitaries) pure bipartite state with Shmidt coefficients $\{\lambda_i\}$ for $1 \leq i \leq 3$. The red point corresponds to the explicit example presented in the main text.

IV. DISCUSSION

We have studied an extension of the standard teleportation protocol to the case when Alice and Bob are allowed to use ancillary entangled states in a catalytic way. We have shown that when arbitrary catalysts are allowed the fidelity of teleportation is replaced with a more general quantity which we referred to as the fidelity of catalytic teleportation. We then derived a lower bound for this quantifier which can be viewed as a regularisation of the standard fidelity of teleportation. By employing typicality arguments we then showed a simple lower bound of this regularised quantifier valid for pure states. Finally, we have shown that this lower bound is tight enough to demonstrate a genuine catalytic advantage for a wide range of quantum states.

We emphasise that quantum teleportation is one of many quantum protocols whose performance depends directly on the entanglement fraction of the shared state. Our protocol for catalytic teleportation involves a preprocessing step whose sole purpose is to increase the entanglement fraction of the shared state. Therefore this preprocessing can be readily applied to other scenarios and we expect that a similar type of catalytic advantage can be demonstrated for other relevant information-processing tasks as well.

Furthermore, our definition of catalytic fidelity of teleportation (6) a priori allows for using arbitrary large-dimensional states as catalysts. It would be interesting to see the effect of constraining the dimension of the catalyst and, perhaps, quantifying the trade-off between the size of the catalyst and the improvement it can offer in quantum teleportation.

NOTE ADDED

After completing this work, an interesting and independent work of Kondra et. al. appeared on the arXiv [59]. In that work, the authors propose an extension of the task of quantum state merging and show that using catalysts, also in that setting, allows for an improved performance.

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Appendix A: Proof of Theorem 1

Consider two spatially separated parties $A = A_1 \dots A_n$ and $B = B_1 \dots B_n$ composed of n identical and independently distributed (i.i.d) entangled states. Let $\rho_{A_1B_1}$ be a state with $E_{D,\Omega}(\rho_{A_1B_1}) > 0$ and let us denote $A = A_1A_2 \dots A_n$ and $B = B_1B_2 \dots B_n$. We will also use the shorthand $1\!:\!i := A_1B_1 \dots A_iB_i$. Le us consider a CPTP map $\mathcal{E} \in \Omega$ and denote:

$$\mathcal{E}(\rho_{AB}^{\otimes n}) = \sigma_{AB}^n \qquad \text{st.} \qquad \operatorname{tr}_{1:n-m}(\sigma_{AB}^n) = (\phi_+^{\otimes m})_{A'B'}, \tag{A1}$$

Consider the following state of the catalyst:

$$\omega_{CC'} := \sum_{i=1}^n \frac{1}{n} \underbrace{\rho^{\otimes i-1} \otimes \sigma^{n-i}}_{A_2B_2\dots A_nB_n} \otimes |i\rangle\langle i|_R \,. \tag{A2}$$

In what follows A_1B_1 is the shared state between Alice and Bob and $C = A_2 \dots A_n R$ and $C' = B_2 \dots B_n R$ correspond to the catalyst they share, with R being a classical register. We also denote $\sigma^{n-i} := \operatorname{tr}_{1:i}(\sigma_{AB}^n)$. The initial state shared between Alice and Bob takes the form:

$$\rho_{A_1B_1} \otimes \omega_{\mathrm{CC}'} = \frac{1}{n}\left(\rho_{A_1B_1} \ \otimes \sigma_{2:n}^{n-1} \otimes |1\rangle\langle 1|_R + \ldots + \rho_{1:n-1}^{\otimes n-1} \otimes \sigma^1_{A_nB_n} \otimes |n-1\rangle\langle n-1|_R + \rho_{1:n}^{\otimes n} \otimes |n\rangle\langle n|_R\right) \tag{A3}$$

$$= \sum_{i=1}^{n} \frac{1}{n} \underbrace{\rho^{\otimes i} \otimes \sigma^{n-i}}_{A_1B_1\ldots A_nB_n} \otimes |i\rangle\langle i|_R \tag{A4}$$

The catalytic distillation protocol $\mathcal{P}$ can be summarised as follows:

1. Alice and Bob apply $\mathcal{E} \in \Omega$ to the state in the n-th register, conditioned on the classical register R. Therefore, the map they apply takes the form:

$$\mathrm{id}(\cdot) \otimes \sum_{i=1}^{n-1} \langle i| \cdot |i\rangle + \mathcal{E}(\cdot) \otimes \langle n| \cdot |n\rangle. \tag{A5}$$

2. Alice and Bob relabel their shared classical register R in the following way:

$$|i\rangle\langle i|_R \to |i+1\rangle\langle i+1|_R \qquad \text{for} \quad i < n, \tag{A6}$$

$$|n\rangle\langle n|_R \to |1\rangle\langle 1|_R\,. \tag{A7}$$

3. Alice and Bob relabel their quantum systems conditioned on the classical register in the following way:

$$\rho_{A_1B_1A_2B_2\ldots A_iB_iA_{i+1}B_{i+1}\ldots A_nB_n} \otimes |i\rangle\langle i|_R \to \rho_{A_{i+1}B_{i+1}A_2B_2\ldots A_iB_iA_1B_1\ldots A_nB_n} \otimes |i\rangle\langle i|_R \qquad \text{for} \quad 0 \le i \le n \tag{A8}$$

The state shared between Alice and Bob during the steps of the protocol $\mathcal{P}$ can be written as:

$$\rho_{A_1B_1} \otimes \omega_{\mathrm{CC}'} \xrightarrow{\ 1\ } \sum_{i=1}^{n-1} \frac{1}{n} \rho^{\otimes i} \otimes \sigma^{n-i} \otimes |i\rangle\langle i|_R + \frac{1}{n}\mathcal{E}(\rho^{\otimes n}) \otimes |n\rangle\langle n|_R \tag{A9}$$

$$= \frac{1}{n}\left(\rho \otimes \sigma^{n-1} \otimes |1\rangle\langle 1|_R + \ldots + \sigma^n \otimes |n\rangle\langle n|_R\right) \tag{A10}$$

$$\xrightarrow{\ 2\ } \sum_{i=0}^{n-1} \frac{1}{n} \rho^{\otimes i} \otimes \sigma^{n-i} \otimes |i+1\rangle\langle i+1|_R \tag{A11}$$

$$= \frac{1}{n}\left(\sigma^n \otimes |1\rangle\langle 1|_R + \ldots + \rho^{\otimes(n-1)} \otimes \sigma^1 \otimes |n\rangle\langle n|_R\right) \tag{A12}$$

$$\xrightarrow{\ 3\ } \frac{1}{n}\sigma^n \otimes |1\rangle\langle 1|_R + \frac{1}{n}\sum_{i=2}^{n} \widetilde{\sigma}^{\,i-1} \otimes \rho^{\otimes i-1} \otimes \sigma^{n-i} \otimes |i\rangle\langle i|_R \tag{A13}$$

$$= \frac{1}{n}\left(\sigma^n \otimes |1\rangle\langle 1|_R + \widetilde{\sigma}^{\,1} \otimes \rho \otimes \sigma^{n-2} \otimes |2\rangle\langle 2|_R + \ldots + \widetilde{\sigma}^{\,n-1} \otimes \rho^{\otimes n-1} \otimes |n\rangle\langle n|_R\right), \tag{A14}$$

where we labelled a single-particle state $\widetilde{\sigma}^{\,i} := \mathrm{tr}_{1\ldots i-1,i+1\ldots n}(\sigma^n)$. The reduced state of the catalyst ($CC' = A_2B_2\ldots A_nB_nR$) after applying the above protocol and noting that $\mathrm{Tr}_{A_1B_1}(\sigma^n) = \sigma^{n-1}$, reads:

$$\mathrm{tr}_{A_1B_1}\, \mathcal{P}\left(\rho_{A_1B_1} \otimes \omega_{CC'}\right) = \sum_{i=1}^{n} \frac{1}{n} \rho^{\otimes i-1} \otimes \sigma^{n-1} \otimes |i\rangle\langle i|_R = \omega_{CC'} \tag{A15}$$

The reduced state of the main system (A_1B_1) now reads:

$$\rho'_{A_1B_1} = \mathrm{tr}_{CC'}\, \mathcal{P}\left(\rho_{A_1B_1} \otimes \omega_{CC'}\right) = \frac{1}{n}\left(\widetilde{\sigma}^{\,0} + \widetilde{\sigma}^{\,1} + \ldots + \widetilde{\sigma}^{\,n}\right) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{tr}_{1\ldots i-1,i+1\ldots n}\, \mathcal{E}(\rho^n). \tag{A16}$$

In particular, we see that the main system ends up in a certain averaged state, while the reduced state of the catalyst remains unchanged. The fully entangled fraction of the main system becomes:

$$f(\rho'_{A_1B_1}) = \frac{1}{n}\sum_{i=1}^{n} \langle \phi^+|\, \mathrm{tr}_{1\ldots i-1,i+1\ldots n}\, \mathcal{E}(\rho^{\otimes n})|\phi^+\rangle \tag{A17}$$

$$= \mathrm{tr}\left(\mathcal{E}(\rho^{\otimes n})\Omega\right), \tag{A18}$$

where Ω is a positive semidefinite operator defined as: $\Omega := \frac{1}{n}\sum_{i=1}^{n} \phi^+_{A_iB_i} \otimes \mathbb{1}_{\overline{A_iB_i}}$.

Appendix B: Proof of Lemma 1

Let us begin the proof by recalling the following well-known fact [60]:

Lemma 2. *For any two pure states ψ_{AB} and ψ'_{AB} there exists $\mathcal{E} \in \mathrm{LOCC}(A:B)$ such that for all $\epsilon > 0$ and sufficiently large n:*

$$\mathcal{E}(\psi_{AB}^{\otimes n}) = \widehat{\psi}_{AB}^{n} \quad \textit{s.t.} \quad \left\| \widehat{\psi}_{AB}^{n} - (\psi'_{AB})^{\otimes n} \right\|_1 \leq \epsilon, \tag{B1}$$

if and only if:

$$S(\rho_A) \geq S(\rho'_A), \tag{B2}$$

where $\rho_A = \mathrm{tr}_B \, |\psi_{AB}\rangle\langle\psi_{AB}|$ and $\rho'_A = \mathrm{tr}_B \, |\psi'_{AB}\rangle\langle\psi'_{AB}|$.

Let n be the smallest possible number of copies such that there exists a transformation $\mathcal{E} = \mathcal{E}^*$ from Lemma 2. Using this as our educated guess for the optimisation in the definition of $f_{\text{reg}}(\rho_{AB})$ we have that for all sufficiently large n:

$$\begin{aligned} f_n(\psi_{AB}^{\otimes n}) &\geq \sum_{i=1}^{n} \langle \phi^+ | \, \mathrm{tr}_{/i} \, \mathcal{E}^*(\psi_{AB}^{\otimes n}) | \phi^+ \rangle && \text{(B3)} \\ &= \sum_{i=1}^{n} \langle \phi^+ | \, \mathrm{tr}_{/i} \, \widehat{\psi}_{AB}^{n} | \phi^+ \rangle && \text{(B4)} \\ &\geq \sum_{i=1}^{n} \big(\langle \phi^+ | \psi'_{AB} | \phi^+ \rangle - \epsilon \big) && \text{(B5)} \\ &= n f(\psi'_{AB}) - n\epsilon. && \text{(B6)} \end{aligned}$$

Using this and the fact that ϵ can be made arbitrarily small we can infer that the regularised entanglement fraction can be lower-bounded by:

$$f_{\text{reg}}(\psi_{AB}) \geq f(\psi'_{AB}) \tag{B7}$$

for all ψ'_{AB} such that $S(\rho_A) \geq S(\rho'_A)$.