Mutual catalysis of entanglement transformations for pure entangled states

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(Received 11 May 2001; published 4 January 2002)

We study the mutual catalysis of entanglement transformations, which refers to the fact that each of the two entanglement transformations cannot occur with certainty by local operations and classical communication independently, but both can occur jointly if collective local operations are performed on both of the initial entangled states. We surprisingly find that mutually catalyzed transformations can be realized even in the case when the standard entanglement catalyzed transformations [D. Jonathan and M. B. Plenio, Phys. Rev. Lett. **83**, 3566 (1999)] are forbidden.

DOI: 10.1103/PhysRevA.65.022307 PACS number(s): 03.67.Hk, 03.65.Ta

I. INTRODUCTION

The study of quantum entanglement between spatially separated quantum systems is of great importance in the understanding of fundamental issues of quantum mechanics and in the practical applications of quantum information theory. Quantum entanglement is often viewed as a basic resource in achieving the tasks of quantum information processing [1]. In many practical applications of this nonlocal resource, one is often allowed only to perform local operations on respective subsystems assisted with classical communication (LOCC). But even restricted to LOCC one can still modify the entanglement properties of the system and, in particular, one can transform one entangled state into another. This class of transformations has been studied in considerable detail in Refs. [2–7]. The necessary and sufficient conditions for the local transformation of pure bipartite entangled states were presented by Nielsen (Nielsen's theorem) [3]. Inspired by Nielsen's theory, Jonathan and Plenio [8] and Morikoshi [9] investigated, respectively, the entanglement catalysis and the recovery of entanglement lost in the entanglement transformation.

The purpose of this paper is to study mutual catalysis of entanglement transformations for pure bipartite entangled states [10]. The mutual catalysis involves two pairs of entangled states in which the transformations for each pair from the initial state to the target one cannot be realized with certainty under LOCC, but both can be realized jointly if the collective local operations are performed on both of the initial entangled states, thus each catalyzes the other. We surprisingly find that the mutual catalysis can occur even in the case when the standard entanglement catalysis [8] is forbidden. Therefore mutually catalyzed transformation can be regarded as a new kind of entanglement-assisted transformation, which is more powerful than the standard entanglement catalyzed transformation.

The paper is organized as follows. In Sec. II we review the relevant background material. Section III gives our main results about mutual catalysis in the entanglement transformations. In particular, we show that the mutual catalysis can occur in the case when the standard catalysis is forbidden. Section IV summarizes our results.

II. BACKGROUND

Let us begin with Nielsen's theory [3]. Suppose Alice and Bob wish to convert a pure bipartite entangled state $|\psi\rangle$ to a target state $|\phi\rangle$ under LOCC. Both $|\psi\rangle$ and $|\phi\rangle$ can be expressed in the Schmidt form

$$|\psi\rangle=\sum_{i=1}^{n}\sqrt{a_i}|i_A i_B\rangle, \tag{1}$$

$$|\phi\rangle=\sum_{i=1}^{m}\sqrt{b_i}|i_A i_B\rangle, \tag{2}$$

where $a_i\geqslant a_{i+1}\geqslant 0$ and $\Sigma_{i+1}^{n}\, a_i=1$, $b_i\geqslant b_{i+1}\geqslant 0$, and $\Sigma_{i=1}^{m}\, b_i=1$ and where the states $|i_{A,B}\rangle$ are orthonormal.

The necessary and sufficient conditions for this local transformation are as follows.

Theorem (Nielsen). The transformation from $|\psi\rangle$ to $|\phi\rangle$ with certainty can be realized using LOCC if and only if the vector $\vec{a}=(a_1,\ldots,a_n)$ is majorized by the vector $\vec{b}=(b_1,\ldots,b_m)$ $(m\leqslant n)$, written as $\vec{a}\prec\vec{b}$, that is,

$$|\psi\rangle\rightarrow|\phi\rangle \quad \text{iff} \quad \sum_{i=1}^{j} a_i\leqslant\sum_{i=1}^{j} b_i, \quad j=1,\ldots,n \tag{3}$$

with equality holding when $j=n$.

Nielsen's theorem was proved to be a useful tool in dealing with the local transformation of pure bipartite states. One consequence of Nielsen's theorem is that there exist pairs $|\psi\rangle$ and $|\phi\rangle$ where neither state is convertible into the other with certainty under LOCC. Such pairs are called incomparable [3] and can be indicated by $|\psi\rangle\leftrightarrow|\phi\rangle$. It was also shown in Ref. [3] that the quantity of entanglement decreases during the transformation. In other words, in order to realize the transformation $|\psi\rangle\rightarrow|\phi\rangle$ with certainty under LOCC, the initial state $|\psi\rangle$ must possess more quantity of entanglement

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than $|\phi\rangle$. Otherwise, the transformation from $|\psi\rangle$ to $|\phi\rangle$ can only be realized in a probabilistic manner [4].

By the application of Nielsen's theory, Jonathan and Plenio [8] and Morikoshi [9] investigated, respectively, the entanglement-assisted local transformation of pure entangled states and the recovery of entanglement lost in the entanglement transformation. In Ref. [8], a remarkable phenomenon serving as entanglement catalysis was demonstrated. Suppose two states $|\psi\rangle$ and $|\phi\rangle$ are incomparable, parties cannot convert the initial state $|\psi\rangle$ to the target state $|\phi\rangle$ with certainty under LOCC. But if the parties share a particular entangled state, say, $|\alpha\rangle$, they may realize the following transformation with certainty under LOCC:

$$|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\alpha\rangle, \tag{4}$$

while retaining $|\alpha\rangle$ unchanged at the end of the process. Here the auxiliary state $|\alpha\rangle$ plays the role of a catalyst as in some chemical reaction. It was shown in Ref. [8] that a necessary condition for standard entanglement catalysis is that the first and the penultimate conditions of Nielsen's theorem expressed in Eq. (3) should be met for $|\psi\rangle$ and $|\phi\rangle$, that is, $a_1 \leqslant b_1$ and $a_n \geqslant b_n$.

The phenomenon of entanglement catalysis was also proved to exist in the transformation of mixed states [11]. The entanglement catalysis, together with entanglement pumping [12] and the activation of bound entanglement [13], shows that entanglement collective operations are more powerful than the individual operations on a single entangled state.

The recovery of entanglement lost in the entanglement transformation [9] is another example of entanglement collective operations. Reference [9] investigated the transformation from a two-dimensional bipartite pure state $|\psi\rangle$ to $|\phi\rangle$. It was found that the entanglement lost during the transformation can be partially recovered by an auxiliary entangled state $|\alpha\rangle$. Alice and Bob perform collective operations on $|\psi\rangle$ and $|\alpha\rangle$ instead of individual operations on $|\psi\rangle$ and $|\alpha\rangle$, respectively, and can realize the following joint transformation under LOCC in a deterministic manner:

$$|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle, \tag{5}$$

with the quantity of entanglement of $|\beta\rangle$ larger than that of $|\alpha\rangle$. This transformation enables a part of entanglement lost in $|\psi\rangle$ to be transferred to the auxiliary state $|\alpha\rangle$. In the two-dimensional case, the above joint transformation requires that the two states $|\psi\rangle$ and $|\phi\rangle$ should satisfy Nielsen's conditions (3) (i.e., $|\psi\rangle\rightarrow|\phi\rangle$), which was just used in [9].

It is natural to ask, if $|\psi\rangle$ and $|\phi\rangle$ are incomparable ($|\psi\rangle\not\leftrightarrow|\phi\rangle$), whether one can still realize the joint transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$ under LOCC in a deterministic manner with the quantity of entanglement of $|\beta\rangle$ larger than that of $|\alpha\rangle$. In the next section we will give a positive answer to this question. Considering $|\psi\rangle\leftrightarrow|\phi\rangle$ and $|\alpha\rangle\rightarrow|\beta\rangle$, the above joint transformation demonstrates practically a new kind of entanglement-assisted transformation in which both $|\psi\rangle$ and $|\alpha\rangle$ mutually assist the transformations to their target states.

III. MUTUAL CATALYSIS OF ENTANGLEMENT TRANSFORMATIONS

Before clarifying our main results, let us review the quantity of entanglement—an important concept utilized in this paper. For a pure bipartite entangled state, say, $|\psi\rangle$, the quantity of entanglement can be defined as the von Neumann entropy of the reduced density operator of either subsystem A or B and takes the following form [14]:

$$E(|\psi\rangle)=-\sum_{i=1}^{n} a_i \log_2 a_I. \tag{6}$$

Using the quantity of entanglement, let us examine the following example.

Example 1. Suppose there are two pairs of entangled states, one pair is of the form

$$\begin{gathered}|\psi\rangle=\sqrt{0.4}|11\rangle+\sqrt{0.36}|22\rangle+\sqrt{0.14}|33\rangle+\sqrt{0.1}|44\rangle,\\ |\phi\rangle=\sqrt{0.5}|11\rangle+\sqrt{0.25}|22\rangle+\sqrt{0.25}|33\rangle,\end{gathered} \tag{7}$$

and the other pair is

$$\begin{gathered}|\alpha\rangle=\sqrt{0.6}|55\rangle+\sqrt{0.4}|66\rangle,\\ |\beta\rangle=\sqrt{0.55}|55\rangle+\sqrt{0.45}|66\rangle.\end{gathered} \tag{8}$$

It can be easily checked that $E(|\psi\rangle)\approx 1.788 > E(|\phi\rangle)=1.5$ and $E(|\beta\rangle)\approx 0.993 > E(|\alpha\rangle)\approx 0.971$ and that, for the transformation from $|\psi\rangle$ to $|\phi\rangle$, the second condition in Nielsen's theorem does not get satisfied, i.e., $a_1+a_2>b_1+b_2$, so $|\psi\rangle\not\leftrightarrow|\phi\rangle$ and $|\alpha\rangle\not\leftrightarrow|\beta\rangle$. This reflects a fact that one can convert neither $|\psi\rangle$ to $|\phi\rangle$ nor $|\alpha\rangle$ to $|\beta\rangle$ independently with certainty under LOCC. However, one can easily prove that if the collective local operations are performed on $|\psi\rangle$ and $|\alpha\rangle$, one can realize the joint transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$ with certainty under LOCC according to Nielsen's theorem. This example gives a positive answer to the question mentioned at the end of the preceding section and shows that there really exists mutual catalysis in the entanglement transformations.

All the joint transformations like $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$ belong to the scope of mutually catalyzed transformations if $|\psi\rangle\not\leftrightarrow|\phi\rangle$ and $|\alpha\rangle\not\leftrightarrow|\beta\rangle$, but it is trivial if $|\psi\rangle\rightarrow|\beta\rangle$ and $|\alpha\rangle\rightarrow|\phi\rangle$ since one needs no collective operations on the two initial states. In this paper, we will focus on a special kind of mutually catalyzed transformation in which one pair of entangled states $|\psi\rangle$ and $|\phi\rangle$ are incomparable ($|\psi\rangle\not\leftrightarrow|\phi\rangle$), and the other pair $|\alpha\rangle$ and $|\beta\rangle$ meet the relation $E(|\beta\rangle)>E(|\alpha\rangle)$, that is, the second pair can partially recover the entanglement lost during the transformation $|\psi\rangle\rightarrow|\phi\rangle$.

It would be desirable to find the necessary and sufficient conditions for such a kind of mutual catalysis of entanglement transformations. Unfortunately, we find that it is difficult to solve this problem. Here we may provide a few interesting partial results.

In the light of the nonincrease of the quantity of entanglement under LOCC [15] and additivity of entropy [16], the following lemma is obvious.

Lemma 1. A necessary condition for realizing the joint transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$ with $E(|\beta\rangle)>E(|\alpha\rangle)$ in a deterministic manner under LOCC is $E(|\psi\rangle)>E(|\phi\rangle)$.

This condition is essential either for the case $|\psi\rangle\rightarrow|\phi\rangle$ or for the case $|\psi\rangle\nrightarrow|\phi\rangle$, but it is not the case of mutual catalysis if $|\psi\rangle\rightarrow|\phi\rangle$.

For simplicity, we assume that the states $|\psi\rangle$ and $|\phi\rangle$ have the same dimension n $(m=n)$ and the states $|\alpha\rangle$ and $|\beta\rangle$ have the same dimension k. In fact, if the states mentioned above have different dimensions, we can let the corresponding Schmidt coefficients be zero. The Schmidt decomposition of the states $|\alpha\rangle$ and $|\beta\rangle$ can be expressed as

$$|\alpha\rangle=\sum_{i=1}^{k}\sqrt{c_i}|i_a i_B\rangle, \tag{9}$$

$$|\beta\rangle=\sum_{i=1}^{k}\sqrt{d_i}|i_A i_B\rangle, \tag{10}$$

where $c_i\geqslant c_{i+1}\geqslant 0$, $\Sigma_{i=1}^{k}c_i=1$ and $d_i\geqslant d_{i+1}\geqslant 0$, $\Sigma_{i=1}^{k}d_i=1$.

Comparing the mutual catalysis of entanglement transformations with the standard entanglement catalysis [8], one might regard the conditions for the mutually catalyzed transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$ [with $E(|\beta\rangle)>E(|\alpha\rangle)$] as being surely stricter than that for the standard entanglement-catalyzed transformation (4). However, we surprisingly find that this guess is not completely correct. If the state $|\beta\rangle$ is required not only to possess more quantity of entanglement than the state $|\alpha\rangle$, but also to be converted to $|\alpha\rangle$ with certainty under LOCC, i.e., $E(|\beta\rangle)>E(|\alpha\rangle)$ and $|\beta\rangle\rightarrow|\alpha\rangle$, the conditions for the mutually catalyzed transformation are indeed stricter than that for the standard entanglement-catalyzed transformation (4). But if $|\beta\rangle$ is required only to possess more quantity of entanglement than $|\alpha\rangle$, the conditions required for the mutually catalyzed transformation may be looser than that for the standard entanglement-catalyzed transformation. In other words, the mutual catalysis may occur even in the case when the standard entanglement catalysis is forbidden. In the following we will discuss the above two situations, respectively.

For the first case we have the following necessary conditions.

Lemma 2. The necessary conditions for realizing the transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$ with $E(|\beta\rangle)>E(|\alpha\rangle)$ and $|\beta\rangle\rightarrow|\alpha\rangle$ in a deterministic manner under LOCC are

$$a_1\leqslant b_1,\quad a_n\geqslant b_n. \tag{11}$$

Especially, if the entangled states $|\alpha\rangle$ and $|\beta\rangle$ are two-dimensional bipartite states, only the sign of inequality in the above conditions holds.

Proof. From Nielsen's theorem, the transformation $|\beta\rangle\rightarrow|\alpha\rangle$ with certainty under LOCC requires that $d_1\leqslant c_1$ and $d_k\geqslant c_k$. For the joint state $|\psi\rangle\otimes|\alpha\rangle$, the largest and smallest Schmidt coefficients are, respectively, $\gamma_1=a_1c_1$ and $\gamma_{nk}=a_nc_k$, while for the joint state $|\phi\rangle\otimes|\beta\rangle$, the largest and smallest Schmidt coefficients are, respectively, $\delta_1=b_1d_1$ and $\delta_{nk}=b_nd_k$. Also from Nielsen's theorem, the first condition and the penultimate condition for the transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$ under LOCC with certainty are, respectively, $\gamma_1\leqslant\delta_1$ and $\Sigma_{j=1}^{nk-1}\gamma_j=1-\gamma_{nk}\leqslant\Sigma_{j=1}^{nk-1}\delta_j=1-\delta_{nk}$. Considering $d_1\leqslant c_1$ and $d_k\geqslant c_k$, we immediately have $a_1\leqslant b_1$, $a_n\geqslant b_n$.

If the entangled states $|\alpha\rangle$ and $|\beta\rangle$ are two-dimensional bipartite states, $E(|\beta\rangle)>E(|\alpha\rangle)$ demands that $\frac{1}{2}\leqslant d_1<c_1\leqslant 1$ and $\frac{1}{2}\geqslant d_2>c_2\geqslant 0$. Hence from $\gamma_1\leqslant\delta_1$ (i.e., $a_1c_1\leqslant b_1d_1$) and $\Sigma_{j=1}^{n2-1}\gamma_j=1-\gamma_{n2}\leqslant\Sigma_{j=1}^{n2-1}\delta_j=1-\delta_{n2}$ (i.e., $1-a_nc_2\leqslant 1-b_nd_2$), we have $a_1<b_1$, $a_n>b_n$. ■

Lemma 2 tells us that the conditions for the mutually catalyzed transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$ with $E(|\beta\rangle)>E(|\alpha\rangle)$ and $|\beta\rangle\rightarrow|\alpha\rangle$ are very similar to the necessary condition for the standard entanglement-catalyzed transformation [8]; both of them require $|\psi\rangle$ and $|\phi\rangle$ to meet the first and the penultimate conditions of Nielsen's theorem. In fact, the standard entanglement-catalyzed transformation can be taken as a special situation for the mutually catalyzed transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$ with $E(|\beta\rangle)>E(|\alpha\rangle)$ and $|\beta\rangle\rightarrow|\alpha\rangle$. If the transformation $|\beta\rangle\rightarrow|\alpha\rangle$ is performed under LOCC after the realization of the transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$, as a whole, one can immediately get $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\alpha\rangle$, which is just the standard entanglement-catalyzed transformation. It is not difficult to check that example 1 belongs to the above situation. In example 1, the entangled states $|\alpha\rangle$ and $|\beta\rangle$ are two dimensional and satisfy the relations $E(|\beta\rangle)>E(|\alpha\rangle)$ and $|\beta\rangle\rightarrow|\alpha\rangle$. Therefore the initial state $|\psi\rangle$ and the target state $|\phi\rangle$ are required to meet relation (11) with only the sign of inequality holding, which is stricter than the requirement for the standard entanglement-catalyzed transformation.

Next we discuss the second case when $|\alpha\rangle$ and $|\beta\rangle$ are required to satisfy the relation $E(|\beta\rangle)>E(|\alpha\rangle)$ only. Let us examine the following two examples.

Example 2. The initial state $|\psi\rangle$ and the target state $|\phi\rangle$ are assumed to be

$$\begin{aligned}|\psi\rangle&=\sqrt{0.42}|11\rangle+\sqrt{0.36}|22\rangle+\sqrt{0.15}|33\rangle+\sqrt{0.07}|44\rangle,\\ |\phi\rangle&=\sqrt{0.4}|11\rangle+\sqrt{0.4}|22\rangle+\sqrt{0.2}|33\rangle.\end{aligned} \tag{12}$$

Since $a_1=0.42>b_1=0.4$, the first condition of Nielsen's theorem for the transformation $|\psi\rangle\rightarrow|\phi\rangle$ is violated, therefore the standard entanglement catalysis cannot be realized for $|\psi\rangle$ and $|\phi\rangle$ according to Ref. [8], that is, one cannot find any catalyst state $|\alpha\rangle$ to realize the transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\alpha\rangle$ with certainty under LOCC. But if we choose the following states $|\alpha\rangle$ and $|\beta\rangle$ as initial and final states, respectively:

$$\begin{aligned}|\alpha\rangle&=\sqrt{0.4}|55\rangle+\sqrt{0.3}|66\rangle+\sqrt{0.3}|77\rangle,\\ |\beta\rangle&=\sqrt{0.42}|55\rangle+\sqrt{0.4}|66\rangle+\sqrt{0.1}|77\rangle+\sqrt{0.08}|88\rangle,\end{aligned} \tag{13}$$

we can realize the joint transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$ with $E(|\beta\rangle)\approx 1.678>E(|\alpha\rangle)\approx 1.571$. Note that $|\beta\rangle\nrightarrow|\alpha\rangle$ under LOCC with certainty according to Nielsen's theorem although $E(|\beta\rangle)>E(|\alpha\rangle)$.

Example 3. The initial state $|\psi\rangle$ and the target state $|\phi\rangle$ are assumed to be

$$
\begin{aligned}
|\psi\rangle &= \sqrt{0.33}|11\rangle + \sqrt{0.32}|22\rangle + \sqrt{0.3}|33\rangle + \sqrt{0.05}|44\rangle,\\
|\phi\rangle &= \sqrt{0.6}|11\rangle + \sqrt{0.2}|22\rangle + \sqrt{0.14}|33\rangle + \sqrt{0.06}|44\rangle.
\end{aligned} \tag{14}
$$

It is not difficult to prove that the penultimate condition of Nielsen's theorem for the transformation from $|\psi\rangle$ to $|\phi\rangle$ is violated since $a_1+a_2+a_3=1-a_4=1-0.05>b_1+b_2+b_3=1-b_4=1-0.06$, so the transformation from $|\psi\rangle$ to $|\phi\rangle$ in a deterministic manner under LOCC cannot be realized even under the assistance of any catalyst state. But if we choose the following states:

$$
\begin{aligned}
|\alpha\rangle &= \sqrt{0.6}|55\rangle + \sqrt{0.3}|66\rangle + \sqrt{0.1}|77\rangle,\\
|\beta\rangle &= \sqrt{0.46}|55\rangle + \sqrt{0.46}|66\rangle + \sqrt{0.08}|77\rangle,
\end{aligned} \tag{15}
$$

we can realize the mutually catalyzed transformation $|\psi\rangle\otimes|\alpha\rangle\rightarrow|\phi\rangle\otimes|\beta\rangle$ under certainty in a deterministic manner with $E(|\beta\rangle)\approx1.322>E(|\alpha\rangle)\approx1.295$. Nielsen's theorem tells us that $|\beta\rangle\nrightarrow|\alpha\rangle$ although $E(|\beta\rangle)>E(|\alpha\rangle)$.

The above two examples show that the mutual catalysis of entanglement transformations may occur even in the case when the standard entanglement catalysis is forbidden. From this point of view, the mutual catalysis can be regarded to be more powerful than the standard entanglement catalysis and the mutually catalyzed entanglement transformation can be taken as a different kind of entanglement-assisted transformation.

IV. CONCLUSION AND OPEN QUESTIONS

In conclusion, we have studied the mutual catalysis of entanglement transformations and examined a particular class of mutually catalyzed transformations in which one pair of entangled states are incomparable and the other pair has the property that the final state $|\beta\rangle$ possesses more quantity of entanglement than the initial one $|\alpha\rangle$. The extra entanglement is the partial recovery of the entanglement lost in $|\psi\rangle$. We have shown that this kind of mutually catalyzed transformation can be realized even in the case when the standard entanglement catalyzed transformation is forbidden. Our results reveal a new property of entanglement-assisted transformations and raise many interesting questions: What are the sufficient and necessary conditions for the general mutual catalysis? Whether such a phenomenon exists in the mixed-state transformations or in the transformations of multiple pairs of entangled states? We believe that the research of mutual catalysis of entanglement transformations will help us in the deeper understanding of the property of entanglement.

ACKNOWLEDGMENTS

This project was supported by the Natural Science Foundation of China (Grant No. 60178014) and the Natural Science Foundation of Shanghai, People's Republic of China (Grant No. 99ZA14052).

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[10] During the review process we were informed that a number of our results had previously been derived by Jonathan, Nielsen, Plenio, and Smolin in their unpublished work, and that the phenomenon we investigate here had been given the name "mutual catalysis" by these authors. Accordingly, we have adopted this terminology in the present version.

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