Approximate Statistical Properties of the Sharpe Ratio

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The Error on the Sharpe Ratio for IID Innovations

The Sharpe ratio is usually estimated from the ratio of the estimated mean return to the estimated standard deviation, scaled to represent an annualized number. Consider a sequence of returns $\{r_t\}_{t=1}^{n}$. The estimated mean $\hat{\mu}$ and variance $\hat{\sigma}^2$ are given by the usual formulae

$$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} r_t \text{ and } \hat{\sigma}^2 = \frac{1}{n}\sum_{t=1}^{n}\left(r_t - \hat{\mu}\right)^2.$$

The estimated Sharpe ratio, $\hat{Z}$, is then

$$\hat{Z} = \hat{\mu}\left(\hat{\sigma}^2\right)^{-\frac{1}{2}}.$$

The error on the estimated Sharpe ratio, $\sigma_{\hat{Z}}$, may be computed from the standard propagation of errors formula. i.e.

$$Var(\hat{Z}) = \left|\frac{\partial \hat{Z}}{\partial \hat{\mu}}\right|^2 Var(\hat{\mu}) + \left|\frac{\partial \hat{Z}}{\partial \hat{\sigma}^2}\right|^2 Var(\hat{\sigma}^2) + 2\left|\frac{\partial \hat{Z}}{\partial \hat{\mu}} \cdot \frac{\partial \hat{Z}}{\partial \hat{\sigma}^2}\right| Cov(\hat{\mu}, \hat{\sigma}^2).$$

If we assume that $\hat{\mu}$ and $\hat{\sigma}^2$ are uncorrelated with asymptotic variances[1] σ^2/n and σ^4/n respectively (here σ is the *true* process variance), then this equation may be rewritten

$$\sigma_{\hat{Z}}^2 = \frac{\sigma^2}{n\hat{\sigma}^2} + \frac{\hat{\mu}^2\sigma^4}{4n\left(\hat{\sigma}^2\right)^3} \approx \frac{1}{n}\left(1 + \frac{1}{4}\hat{Z}^2\right)$$

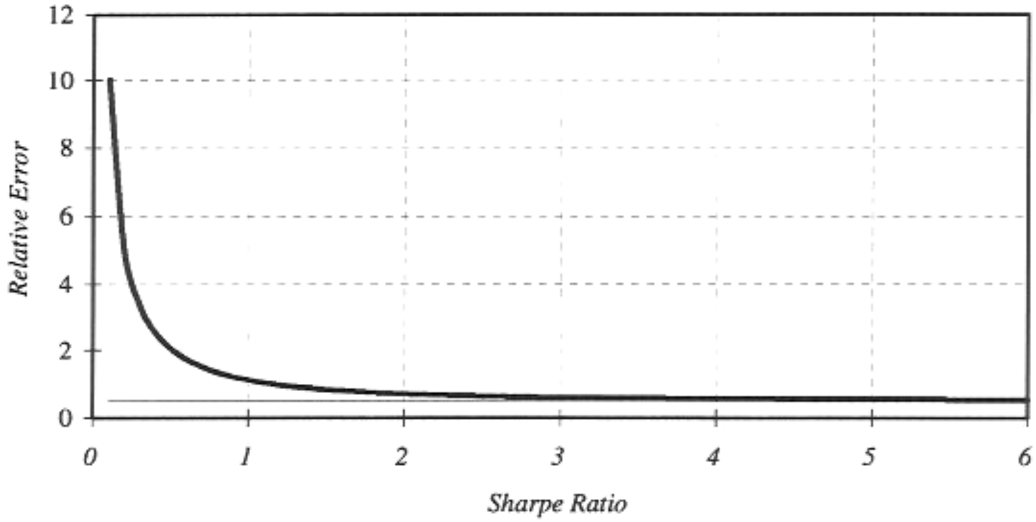
where the true variance has been replaced by the sample variance in the approximation. The relative error in the Sharpe ratio measurement, $\sigma_{\hat{Z}}/\hat{Z}$, is approximately given by

$$\frac{1}{\hat{Z}}\sqrt{\frac{1 + \hat{Z}^2/4}{n}},$$

[1] See Kendall and Stuart, *The Advanced Theory of Statistics*; or, Campbell, Lo and MacKinlay, *The Econometrics of Financial Markets*; for a discussion of the sampling distributions of the standard mean and variance estimators.

which is a *decreasing* function of $\hat{Z}$ that approaches the limit the limit $1/2\sqrt{n}$ for large $\hat{Z}$. That is, it is easier to measure the Sharpe ratio of a high Sharpe ratio process than that of a low Sharpe ratio process. We would expect the sampling distribution of $\hat{Z}$ to be skewed and biased due to Jensen's inequality effects[2]. The functional dependence of the relative error is illustrated in Figure 1 below (with the stochastic scale $1/\sqrt{n}$ removed).

Figure 1



Monte Carlo Simulation of the Measurement Error

Figure 2 shows the result of a Monte Carlo simulation of the measurement of the Sharpe ratio. It can be seen that although this analysis seems to represent the general trend in $\sigma_{\hat{Z}}$ the actual distribution appears to be wider than we would expect from the results here, and is biased upwards.

[Figure 2 overleaf.]

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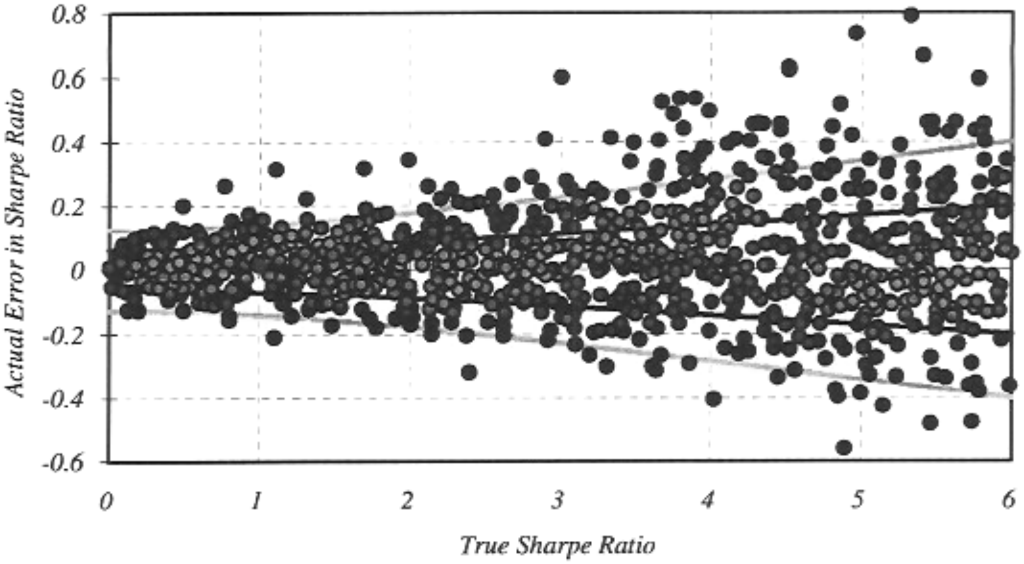
[2] The variance estimator $\hat{\sigma}^2$ is efficient (although it possesses a small bias due to the use of $\hat{\mu}$, rather than μ, in the formula). However, the transformation $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$ is non-linear and so skews the distribution of $\hat{\sigma}$. This skew means that $E\hat{\sigma} = \sigma + \varepsilon$, where ε is an error term.

Figure 2



0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
Actual Error in Sharpe Ratio
0
1
2
3
4
5
6
True Sharpe Ratio