A GENERALIZED ERROR DISTRIBUTION

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ABSTRACT. We review the properties of a univariate probability distribution that is a possible candidate for the description of financial market price changes. This distribution is an "error" distribution that represents a generalized form of the Normal, possesses a natural multivariate form, has a parametric kurtosis that is unbounded above and possesses special cases that are identical to the Normal and the double exponential (Laplace) distributions.

1. THE UNIVARIATE GENERALIZED ERROR DISTRIBUTION

1.1. **Definition.** The Generalized Error Distribution[1] is a symmetrical unimodal member of the exponential family. The domain of the p.d.f. is $x \in [-\infty, \infty]$ and the distribution is defined by three parameters: $\mu \in (-\infty, \infty)$, which locates the mode of the distribution; $\sigma \in (0, \infty)$, which defines the dispersion of the distribution; and, $\kappa \in (0, \infty)$, which controls the skewness. We will use the notation $x \sim G(\mu, \sigma^2, \kappa)$ to define x as a variate drawn from this distribution. (A suitable reference for this distribution is [1].)

The probability distribution function, $F(x)$, is given by

$$dF(x|\mu,\sigma,\kappa) = \frac{e^{-\frac{1}{2}\left|\frac{x-\mu}{\sigma}\right|^{\frac{1}{\kappa}}}}{2^{\kappa+1}\sigma\Gamma(\kappa+1)}\,dx. \tag{1}$$

This function is represented in Figure 1, on the following page.

It is clear from this definition that the mode of the p.d.f. is μ and that it is unimodal and symmetrical about the mode. Therefore the median and the mean are also equal to μ.

If we choose $\kappa = \frac{1}{2}$ then Equation 1 is recognized as the p.d.f. for the univariate Normal Distribution, i.e. $G(\mu, \sigma^2, \frac{1}{2}) = N(\mu, \sigma^2)$. If we choose $\kappa = 1$ then Equation 1 is recognized as the p.d.f. for the Double Exponential, or Laplace, distribution, i.e. $G(\mu, \sigma^2, 1) = L(\mu, 4\sigma^2)$. In the limit $\kappa \to 0$ the p.d.f. tends to the uniform distribution $U(\mu - \sigma, \mu + \sigma)$.

1.2. **The Central Moments.** The central moments are defined by Equation 2.

$$\mu_r = E(x-\mu)^r = \frac{1}{2^{\kappa+1}\sigma\Gamma(\kappa+1)}\int_{x=-\infty}^{\infty}(x-\mu)^r e^{-\frac{1}{2}\left|\frac{x-\mu}{\sigma}\right|^{\frac{1}{k}}}\,dx. \tag{2}$$

The odd moments clearly all vanish by symmetry. For the even moments, Equation 2 may be written as Equation 3, in which we recognize that the integral is a representation of the gamma function.

$$\mu_r = \frac{2^{r\kappa}\sigma^r}{\Gamma(\kappa)}\int_0^{\infty} t^{\kappa(r+1)-1}e^{-t}\,dt = 2^{r\kappa}\sigma^r\frac{\Gamma\{\kappa(r+1)\}}{\Gamma(\kappa)} \tag{3}$$

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[1]Sometimes just called "The Error Distribution"

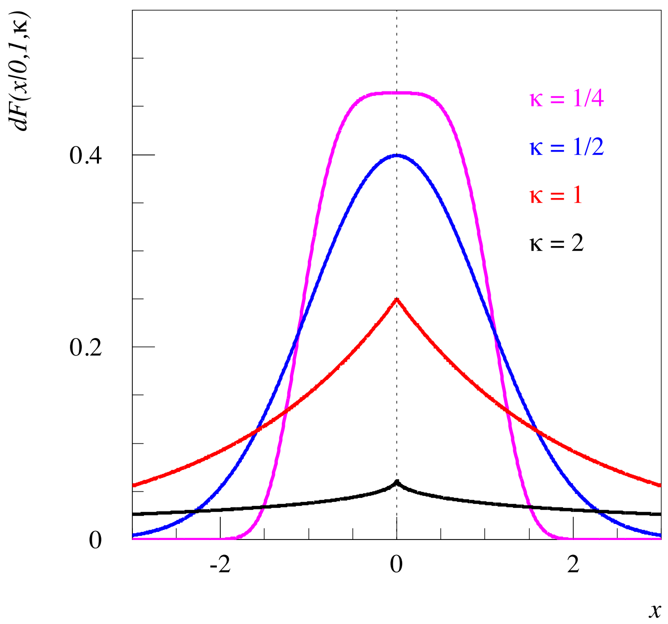

FIGURE 1. The Generalized Error Probability Density Function

Therefore, the distribution has the parameters:

$$\text{mean} = \mu; \tag{4}$$

$$\text{variance} = 2^{2\kappa}\sigma^2 \frac{\Gamma(3\kappa)}{\Gamma(\kappa)}; \tag{5}$$

$$\text{skew, } \beta_1 = 0; \text{and,} \tag{6}$$

$$\text{kurtosis, } \beta_2 = \frac{\Gamma(5\kappa)\Gamma(\kappa)}{\Gamma^2(3\kappa)}. \tag{7}$$

For $\kappa < \frac{1}{2}$ the distribution is platykurtotic and for $\kappa > \frac{1}{2}$ it is leptokurtotic [2]. The excess kurtosis, γ_2, is tends to $-6/5$ as $\kappa \to 0$ and is unbounded for $\kappa > \frac{1}{2}$. The leptokurtotic region is illustrated in Figure 2, on the next page. We may use Stirling's formula for $\Gamma(z)$ to obtain the following approximation for the kurtosis:

$$\gamma_2(\kappa) \simeq \frac{3}{\sqrt{5}} \left(\frac{3125}{729}\right)^{\kappa} \approx 1.3 \times 4.3^{\kappa}. \tag{8}$$

1.3. **A Standardized Generalized Error Distribution.** It is often convenient to work with the p.d.f. which is "standardized." By this it is meant that that population mean is zero and the population variance is unity. We see from Equation 5 that the variance of the G.E.D. p.d.f, as defined in Equation 1, is a very strong function of κ.

However, it is trivial to rescale the variance to transform Equation 1 into an equivalent p.d.f. with constant variance σ^2. Let us introduce the scaling parameter ξ and make the

[2] For this reason, some authors write $c/2$ for κ, parameterising the Normal distribution as $c = 1$.

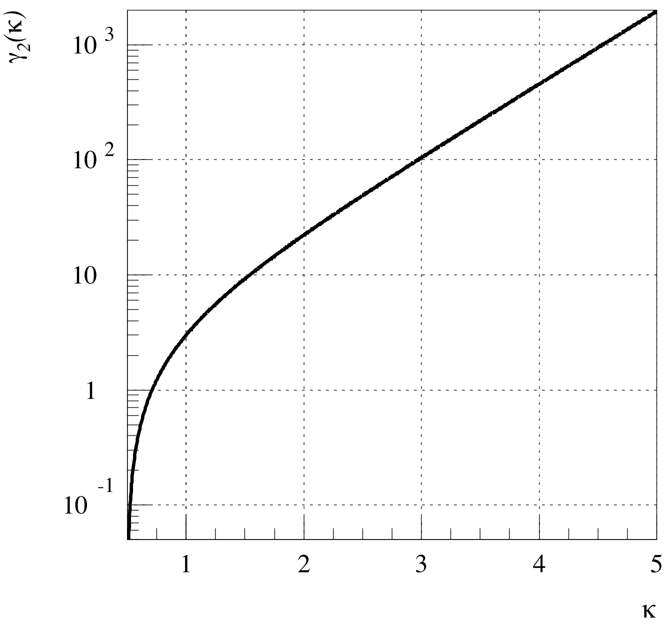

FIGURE 2. Excess Kurtosis Measure γ_2 for $\kappa > \frac{1}{2}$

substitution $\sigma \rightarrow \sigma\xi^\kappa$ in Equation 1. The normalized p.d.f. now has the form

$$dF(x|\mu,\sigma,\kappa;\xi) = \frac{e^{-\frac{1}{2\xi}\left|\frac{x-\mu}{\sigma}\right|^{\frac{1}{\kappa}}}}{2^{\kappa+1}\sigma\xi^\kappa\Gamma(\kappa+1)}\,dx. \tag{9}$$

This p.d.f. has the variance

$$2^{2\kappa}\sigma^2\xi^{2\kappa}\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}. \tag{10}$$

If we choose ξ to eliminate all dependence of the variance on κ, then we may define a homoskedastic p.d.f. as

$$dF_H(x|\mu,\sigma,\kappa) = \left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}\right\}^{\frac{1}{2}} \frac{e^{-\left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}\left(\frac{x-\mu}{\sigma}\right)^2\right\}^{\frac{1}{2\kappa}}}}{2\sigma\Gamma(\kappa+1)}\,dx. \tag{11}$$

A standardized p.d.f. is therefore trivially given by $dF_S(x|\kappa) = dF_H(x|0,1,\kappa)$.

With this formulation the extremely rapid increase in kurtosis, as κ is increased from the Normal reference value of $\frac{1}{2}$, is clearly demonstrated in Figure 3, on the following page.

2. A MULTIVARIATE GENERALIZATION

2.1. **Construction of a Multivariate Distribution.** The p.d.f. of Equation 1 is of the form suitable for the construction of a multivariate p.d.f. using the recipe of reference [2]. This

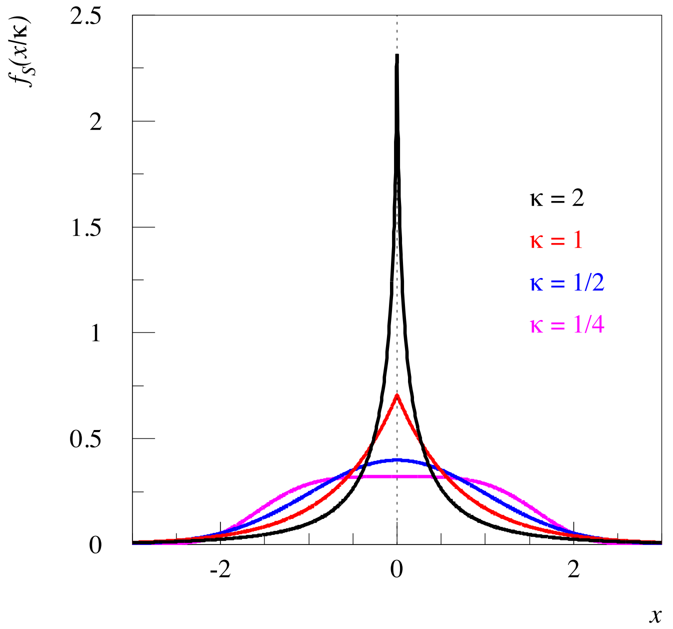

FIGURE 3. Univariate Standardized Generalized Error Distribution

procedure is applied to the standardized univariate p.d.f, $f(x^2)$, defined for our distribution as

(12)
$$f(x^2) = \frac{dF_S(x|\kappa)}{dx} = \frac{1}{2\Gamma(\kappa+1)} \left\{ \frac{\Gamma(3\kappa)}{\Gamma(\kappa)} \right\}^{\frac{1}{2}} e^{-\left\{ \frac{\Gamma(3\kappa)}{\Gamma(\kappa)} x^2 \right\}^{\frac{1}{2\kappa}}}.$$

Replacing x^2 in Equation 12 by the Mahanalobis distance $\Delta^2_\Sigma(\boldsymbol{x}, \boldsymbol{\mu})$, gives

(13)
$$dF(\boldsymbol{x}|\boldsymbol{\mu}, \Sigma, \kappa) = \frac{\mathcal{A}\, d^n\boldsymbol{x}}{2\Gamma(\kappa+1)} \left\{ \frac{\Gamma(3\kappa)}{\Gamma(\kappa)} \right\}^{\frac{1}{2}} \exp - \left\{ \frac{\Gamma(3\kappa)}{\Gamma(\kappa)} (\boldsymbol{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu}) \right\}^{\frac{1}{2\kappa}}.$$

The constant $\mathcal{A}$ is introduced to maintain the normalization of the new function. It is given by

$$\frac{1}{\mathcal{A}} = \frac{\sqrt{\pi^n|\Sigma|}}{\Gamma(\kappa+1)\Gamma(\frac{n}{2})} \left\{ \frac{\Gamma(3\kappa)}{\Gamma(\kappa)} \right\}^{\frac{1}{2}} \int_0^\infty e^{-\left\{ \frac{\Gamma(3\kappa)}{\Gamma(\kappa)} \right\}^{\frac{1}{2\kappa}} g^{\frac{1}{k}}} g^{n-1}\, dg$$

(14)
$$= \sqrt{\pi^n|\Sigma|} \frac{\Gamma(n\kappa)}{\Gamma(\kappa)\Gamma(\frac{n}{2})} \left\{ \frac{\Gamma(\kappa)}{\Gamma(3\kappa)} \right\}^{\frac{n-1}{2}}.$$

Substituting this result into Equation 13 gives

(15)
$$dF(\boldsymbol{x}|\boldsymbol{\mu}, \Sigma, \kappa) = \frac{d^n\boldsymbol{x}}{\sqrt{\pi^n|\Sigma|}} \frac{\Gamma(1+\frac{n}{2})}{\Gamma(1+n\kappa)} \left\{ \frac{\Gamma(3\kappa)}{\Gamma(\kappa)} \right\}^{\frac{n}{2}} \exp - \left\{ \frac{\Gamma(3\kappa)}{\Gamma(\kappa)} (\boldsymbol{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu}) \right\}^{\frac{1}{2\kappa}}.$$

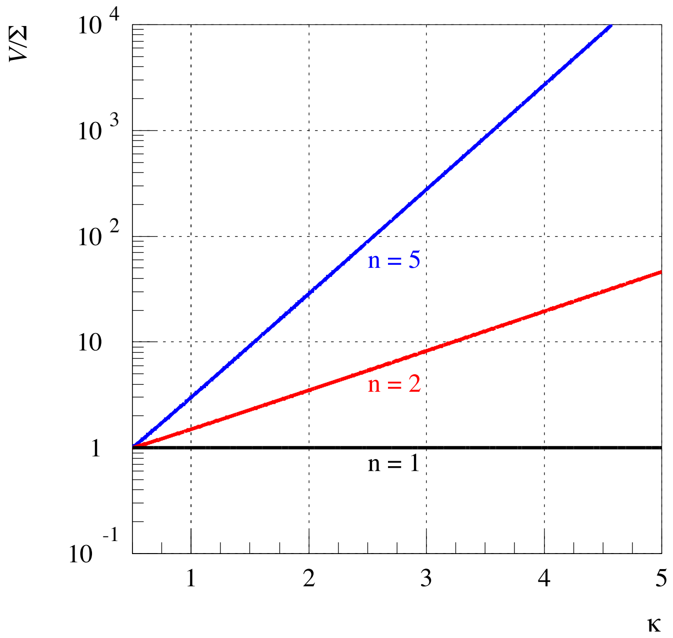

FIGURE 4. Variance Scale Factor for Constructed Multivariate Distributions

2.2. **Moments of the Constructed Distribution.** Using results of reference [2], we see that the p.d.f. of Equation 13 is unimodal with mode $\boldsymbol{\mu}$. This is also equal to the mean of the distribution. The covariance matrix, V, is equal to the matrix Σ multiplied by the scale factor

$$\frac{1}{n}\frac{\int_0^\infty e^{-\left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}\right\}^{\frac{1}{2\kappa}} g^{\frac{1}{\kappa}}} g^{n+1}\,dg}{\int_0^\infty e^{-\left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}\right\}^{\frac{1}{2\kappa}} g^{\frac{1}{\kappa}}} g^{n-1}\,dg} = \frac{\Gamma\{(n+2)\kappa\}\Gamma(1+\kappa)}{\Gamma(3\kappa)\Gamma(1+n\kappa)}. \tag{16}$$

Note that in the limit $\kappa \to 0$ this becomes $\frac{3}{n+2}$. The strong dependence of this factor on κ, for several values of n, is shown in Figure 4, above. The skew of the distribution is zero by construction ($\beta_{1,n} = 0$) and the multivariate kurtosis parameter is

$$\begin{aligned}\beta_{2,n} &= n^2 \frac{\int_0^\infty e^{-\left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}\right\}^{\frac{1}{2\kappa}} g^{\frac{1}{\kappa}}} g^{n+3}\,dg \int_0^\infty e^{-\left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}\right\}^{\frac{1}{2\kappa}} g^{\frac{1}{\kappa}}} g^{n-1}\,dg}{\left[\int_0^\infty e^{-\left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}\right\}^{\frac{1}{2\kappa}} g^{\frac{1}{\kappa}}} g^{n+1}\,dg\right]^2} \\ &= n^2 \frac{\Gamma\{(n+4)\kappa\}\Gamma(n\kappa)}{\Gamma^2\{(n+2)\kappa\}}. \end{aligned} \tag{17}$$

The leptokurtotic region is illustrated in Figure 5, on the following page.

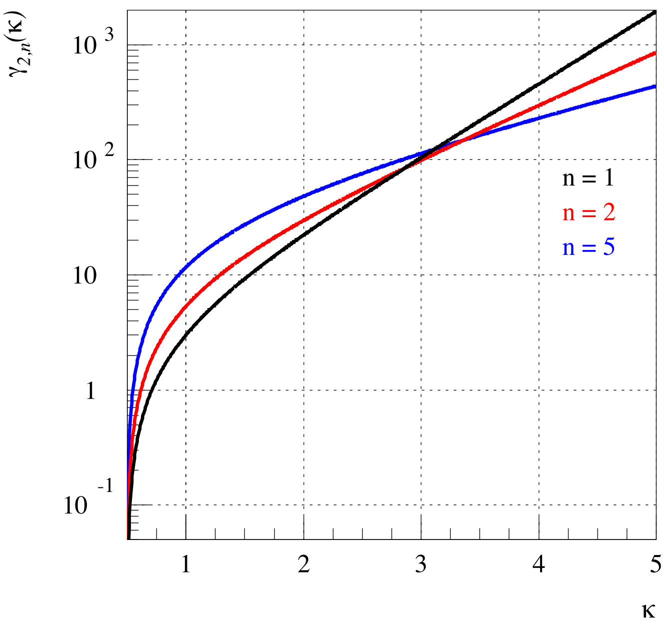

FIGURE 5. Excess Kurtosis Measure $\gamma_{2,n}$ for $\kappa > \frac{1}{2}$

2.3. **The Multivariate Kolmogorov Test Statistic.** Let $\{G_i^2\}_{i=1}^N$ represent an ordered set of sample values of $\Delta^2_\Sigma(\boldsymbol{x}, \boldsymbol{\mu})$. From reference [2], we know that

$$\Pr(g^2 < G^2) = F'(G^2) = \frac{\int_{g=0}^{G} f(g^2)\, g^{n-1}\, dg}{\int_{g=0}^{\infty} f(g^2)\, g^{n-1}\, dg}. \tag{18}$$

Substituting our expression for $f(\cdot)$, Equation 12, gives

$$F'(G^2) = \frac{\gamma\left[n\kappa, \left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)} G^2\right\}^{\frac{1}{2\kappa}}\right]}{\Gamma(n\kappa)}, \tag{19}$$

where $\gamma(\cdot)$ is the lower incomplete gamma function[3]. We may use the Kolmogorov statistic

$$d_N = \max_i |S_i - F'(G_i^2)|, \tag{20}$$

where $\{S_i\}_{i=1}^N$ are the order statistics associated with the sample, to test the null hypothesis that a given dataset is represented by Equation 15.

3. Maximum Likelihood Regression

Given a set of N i.i.d. random vectors, $\{\boldsymbol{X}_i\}_{i=1}^N$, each drawn from the Generalized Error Distribution, the joint probability, or likelihood, of a particular realization, $\{\boldsymbol{x}_i\}_{i=1}^N$, is given by

$$\mathcal{L}(\boldsymbol{\mu}, \Sigma, \kappa) = \prod_{i=1}^{N} dF(\boldsymbol{x}_i|\boldsymbol{\mu}, \Sigma, \kappa). \tag{21}$$

The commonly used likelihood function, $L = -\ln \mathcal{L}$, is therefore given by

$$\begin{aligned} L(\boldsymbol{\mu}, \Sigma, \kappa) &= \sum_{i=1}^{N} \left\{ \frac{\Gamma(3\kappa)}{\Gamma(\kappa)} (\boldsymbol{x}_i - \boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{x}_i - \boldsymbol{\mu}) \right\}^{\frac{1}{2\kappa}} + \frac{N}{2} \ln |\Sigma| \\ &+ \frac{Nn}{2} \ln \frac{\pi\Gamma(\kappa)}{\Gamma(3\kappa)} + N \ln \frac{\Gamma(1 + n\kappa)}{\Gamma(1 + \frac{n}{2})}. \end{aligned} \tag{22}$$

We may also write this expression in terms of the covariance matrix, V, as below.

$$\begin{aligned} L(\boldsymbol{\mu}, V, \kappa) &= \sum_{i=1}^{N} \left[\kappa \frac{\Gamma\{(n+2)\kappa)\}}{\Gamma(1 + n\kappa)} (\boldsymbol{x}_i - \boldsymbol{\mu})^T V^{-1} (\boldsymbol{x}_i - \boldsymbol{\mu}) \right]^{\frac{1}{2\kappa}} + \frac{N}{2} \ln |V| \\ &+ \frac{Nn}{2} \ln \frac{\pi\Gamma(1 + n\kappa)}{\kappa\Gamma\{(n+2)\kappa\}} + N \ln \frac{\Gamma(1 + n\kappa)}{\Gamma(1 + \frac{n}{2})}. \end{aligned} \tag{23}$$

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[2] Giller, G.L., “On the Construction, Integration and Identification of Multivariate Probability Density Functions,” p. 8, *Giller Investments Research Note 20030320/1*,

[3] Gradshteyn, I.S. & Ryzhik, I.M., “Table of Integrals, Series, and Products,” 6th. Edn., p. 890, Academic Press, 2000.