2. Quantum Bits, Gates, and Circuits

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Lecture 2: Quantum Bits, Gates, and Circuits

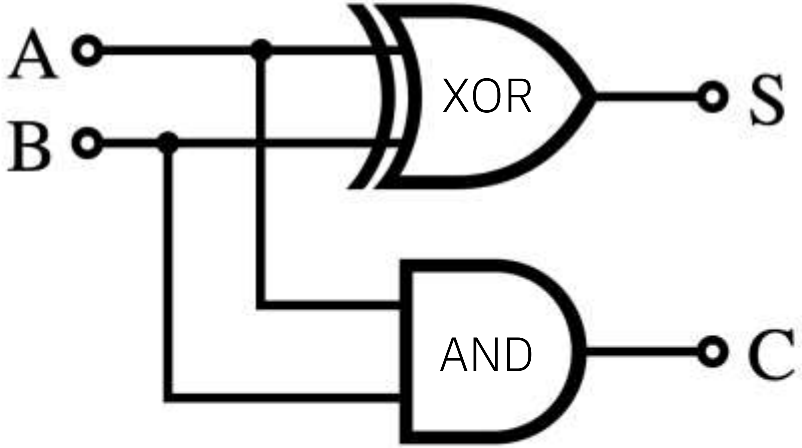
- Understanding Quantum Computation with Circuit Models using quantum bits and gates.

- Hands on using Qiskit
  - If you didn’t install Qiskit in your laptop, please install it.

  https://docs.quantum.ibm.com/guides/install-qiskit

Circuits for addition in classical computing

A classical logic circuit is a set of gate operations on bits and is the unit of computation.



Half adders circuit

Truth table

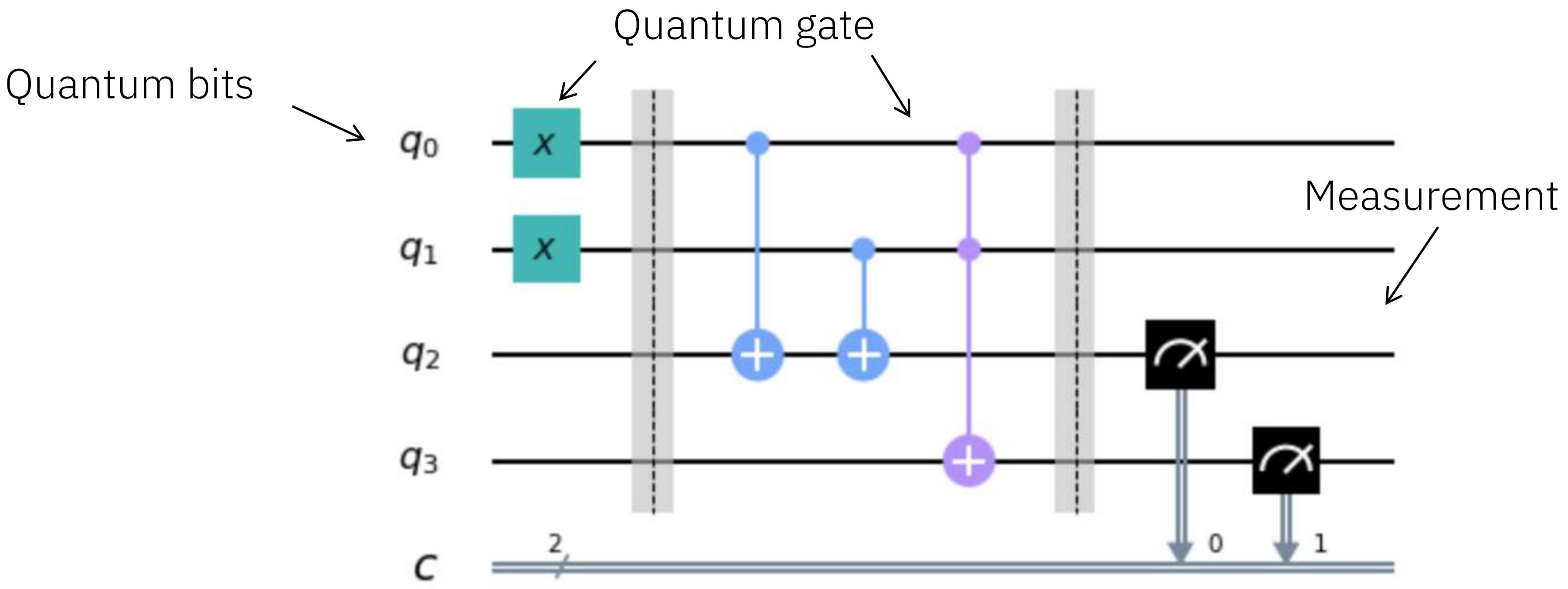
A (input)	B (input)	S (sum)	C (carry out)
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Inputs are on the left, outputs are on the right, and operations are represented by symbols between them.

Models of Quantum Computing

For quantum computers, we use the same basic idea but have different conventions for how to represent inputs, outputs, and the symbols used for operations.

- A sequence of basic quantum gates are applied on quantum bits.



Half adders circuit in quantum computing

Typical single-qubit gates

NOT gate

$$|0\rangle \Rightarrow \boxed{X} \Rightarrow |1\rangle$$

Hadamard gate

$$|0\rangle \Rightarrow \boxed{H} \Rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

Superposition

Single-qubit quantum state

$|0\rangle$ and $|1\rangle$ are vectors in the two-dimensional complex vector space $\mathbb{C}^2$:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For example, X gate is

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$X\,|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

Single-qubit quantum state and unitary evolution

The arbitrary quantum state can be represented as a linear combination of $|0\rangle$ and $|1\rangle$.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$.

The quantum state is evolved by Unitary operator U.

$$|\psi'\rangle = U|\psi\rangle$$

$$U^\dagger U = UU^\dagger = I, \qquad U^\dagger = U^{-1}$$

The quantum operation is reversible.

$$U^{-1}U|\psi\rangle = |\psi\rangle$$

Bloch Sphere

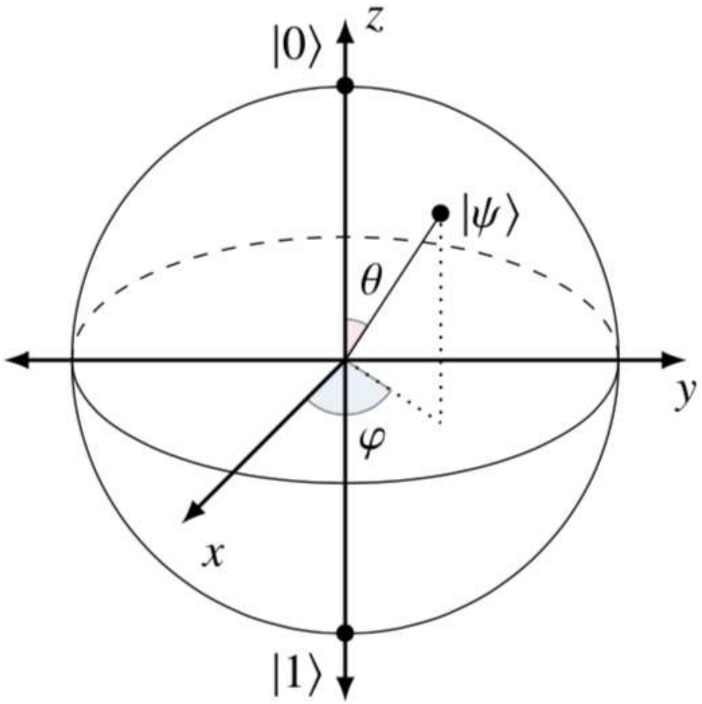
- A quantum state of single-qubit is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad s.t. |\alpha|^2 + |\beta|^2 = 1$$

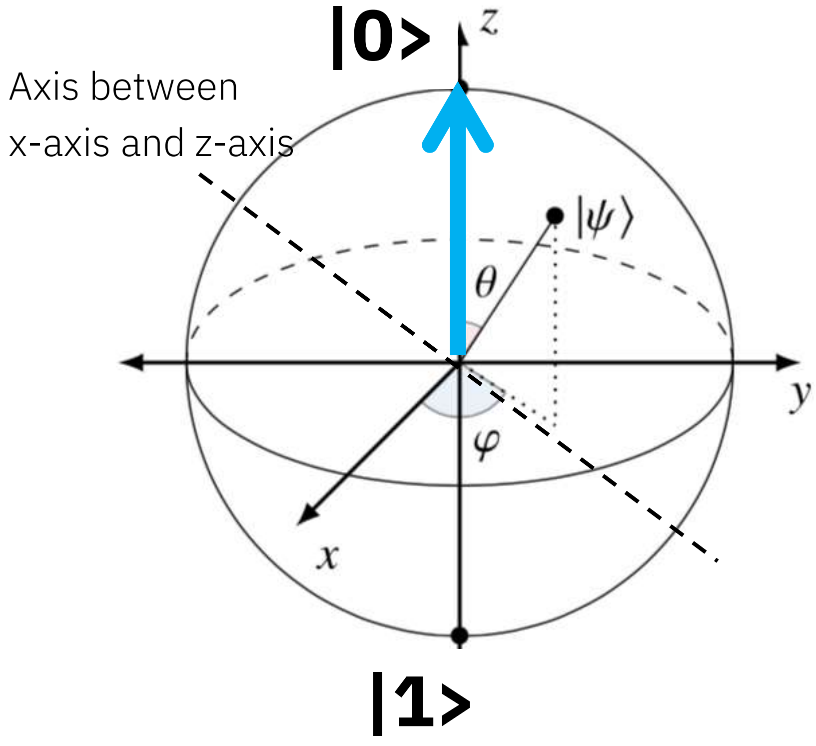
- This allows us to write the quantum state as

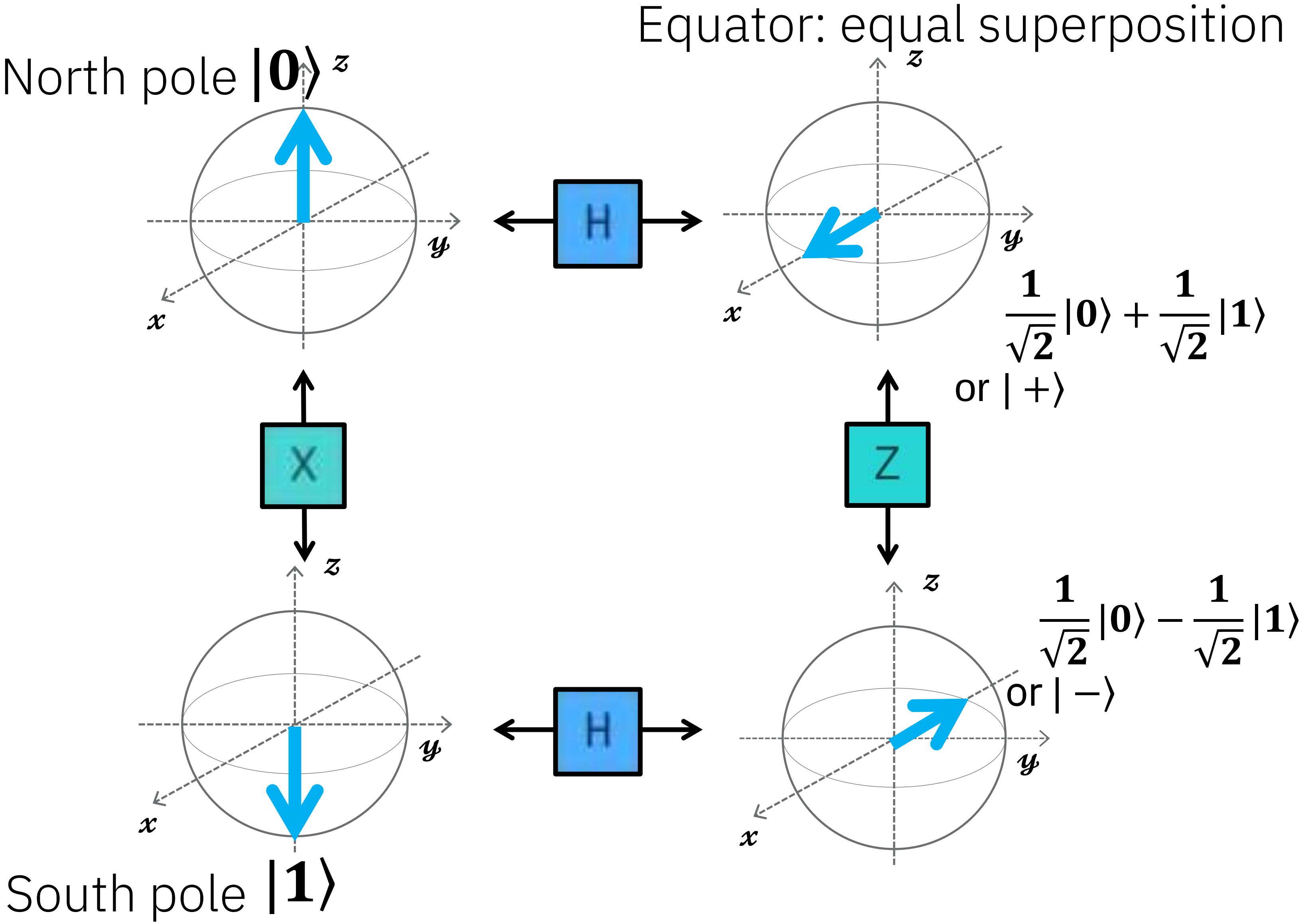
$$|\psi\rangle = cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\, sin\left(\frac{\theta}{2}\right)|1\rangle = \begin{pmatrix} cos\frac{\theta}{2} \\ e^{i\varphi} sin\frac{\theta}{2} \end{pmatrix}$$

- The single qubit quantum state can be mapped to the Bloch sphere.



Bloch sphere





A pure quantum state is a vector pointing from the center to a point on the sphere of radius 1.

Typical single-qubit gates

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}; \quad T = \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\pi/4) \end{bmatrix}.$$

$$R_x(\theta) \equiv e^{-i\theta X/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_y(\theta) \equiv e^{-i\theta Y/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_z(\theta) \equiv e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}.$$

Reference: https://javafxpert.github.io/grok-bloch/

Superposition

Superposition is creating a quantum state that is a combination of $|0\rangle$ and $|1\rangle$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad s.t. |\alpha|^2 + |\beta|^2 = 1$$

Note that if α and β are non-zero, then the qubit's state contains both $|0\rangle$ and $|1\rangle$.

This is what people mean when they say that a qubit can be "0 and 1 at the same time."

Measurement

Measurement is forcing the qubit's state

$$\alpha|0\rangle + \beta|1\rangle \qquad s.t. |\alpha|^2 + |\beta|^2 = 1$$

to $|0\rangle$ or $|1\rangle$ by observing it, where

$|\alpha|^2$ is the probability we will get $|0\rangle$ when we measure.

$|\beta|^2$ is the probability we will get $|1\rangle$ when we measure. (Born rule)

So, α and β are called probability amplitudes.

For example,

$\frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle$ has an equal probability of becoming $|0\rangle$ or $|1\rangle$, and

$\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}i|1\rangle$ has a 75% chance of becoming $|0\rangle$.

Measurement operators

In case of standard basis measurements, the measurement operators are

$$M_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; M_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Standard basis is $|0\rangle$ and $|1\rangle$.

If the state of the quantum system is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, then the probabilities of observing the outcome are

$$p_0(outcome\ is\ 0) = \langle\psi|M_0^\dagger M_0|\psi\rangle = (\alpha^*, \beta^*) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \binom{\alpha}{\beta} = |\alpha|^2$$

$$p_1(outcome\ is\ 1) = \langle\psi|M_1^\dagger M_1|\psi\rangle = (\alpha^*, \beta^*) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \binom{\alpha}{\beta} = |\beta|^2$$

and the quantum states after the measurement are

$$\frac{M_0|\psi\rangle}{\sqrt{\langle\psi|M_0^\dagger M_0|\psi\rangle}} = \frac{\alpha}{|\alpha|}|0\rangle \cong |0\rangle, \quad \frac{M_1|\psi\rangle}{\sqrt{\langle\psi|M_1^\dagger M_1|\psi\rangle}} = \frac{\beta}{|\beta|}|1\rangle \cong |1\rangle$$

Global phase

Suppose that $|\psi\rangle$ and $|\phi\rangle$ are unit vectors representing quantum states, and assume that there exists a complex number α on the unit circle (meaning that $|\alpha| = 1$, or alternatively $\alpha = e^{i\theta}$ for some real number θ) such that

$$|\phi\rangle \ = \alpha|\psi\rangle.$$

Then, the vectors $|\psi\rangle$ and $|\phi\rangle$ are said to differ by a **global phase.**
We also refer to α as a **global phase**.

The two states are considered to be equivalent, because when we measure them, we got the same result:

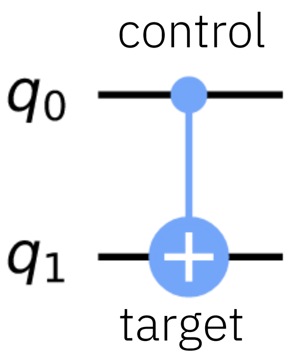
$$\langle\phi|M_j^\dagger\, M_j|\phi\rangle = \alpha^*\alpha\langle\psi|M_j^\dagger\, M_j|\psi\rangle = \langle\psi|M_j^\dagger\, M_j|\psi\rangle$$

For example,

- Different state: $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$
- Same state: $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ and $-|-\rangle = -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

Typical two-qubit gates

CNOT gate is a conditional gate that performs an X-gate on the target qubit, if the state of the control qubit is |1>.



Truth table

Input (t,c)	Output (t,c)
00	00
01	11
10	10
11	01

Note: Qiskit uses **Little Endian**, $|q_1 q_0\rangle$

Acting on the 4D-statevector, it has one of the two matrices, depending on which qubit is the control and which is the target.

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Different books, simulators and papers order their qubits differently. In Qiskit, the left matrix corresponds to the CNOT in the circuit above.

Superposition of multiple systems

- A one-qubit system can be in the superposition of two states:

$$|0\rangle\,, |1\rangle$$

- A two-qubit system can be in the superposition of 2^2 states:

$$|0\rangle \otimes |0\rangle, |1\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle\,, |1\rangle \otimes |1\rangle$$

- An n-qubit system can be in the superposition of 2^n states:

$$|0\rangle_{n-1} \otimes \cdots \otimes |0\rangle_0, |0\rangle_{n-1} \otimes \cdots \otimes |0\rangle_1 \otimes |1\rangle_0, \cdots, |1\rangle_{n-1} \otimes \cdots \otimes |1\rangle_0$$

*Important Notations in Quantum Computing

- Tensor products

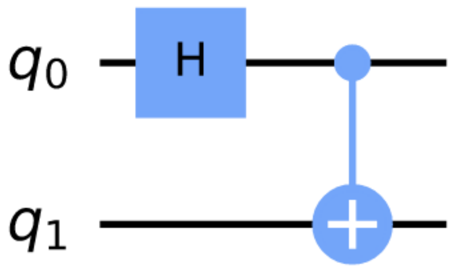
$$|0\rangle \otimes |0\rangle \equiv |0\rangle\,|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$$

More generally,

$$\begin{pmatrix} \alpha_1 \\ \dots \\ \alpha_m \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \dots \\ \beta_n \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \dots \\ \alpha_1\beta_n \\ \dots \\ \alpha_m\beta_n \end{pmatrix}$$

Entangled state

An entangled state is a state $|\psi\rangle_{AB}$ consisting of quantum states $|\psi\rangle_A$ and $|\psi\rangle_{\mathrm{B}}$ that cannot be represented by a tensor product of individual quantum states.



$$
\begin{aligned}
|0\rangle \otimes |0\rangle \rightarrow H \otimes I \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \\
= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \\
\rightarrow CNOT \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
\end{aligned}
$$

- $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a unit vector.

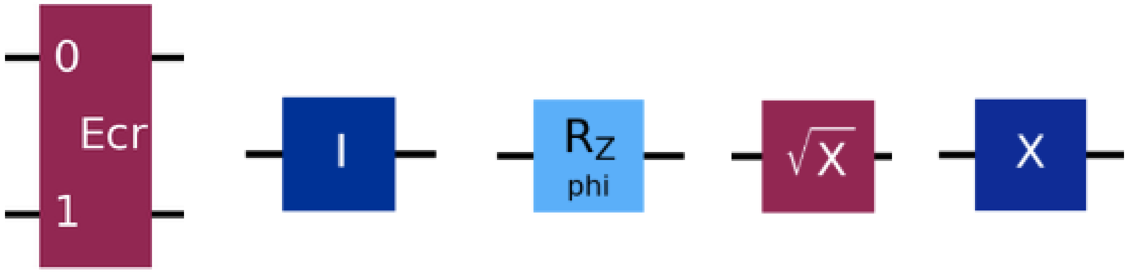
- However, $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle)$

There is no coefficient which satisfies this equation.

Basis gate set

Only a limited set of gates can be executed directly on the hardware.

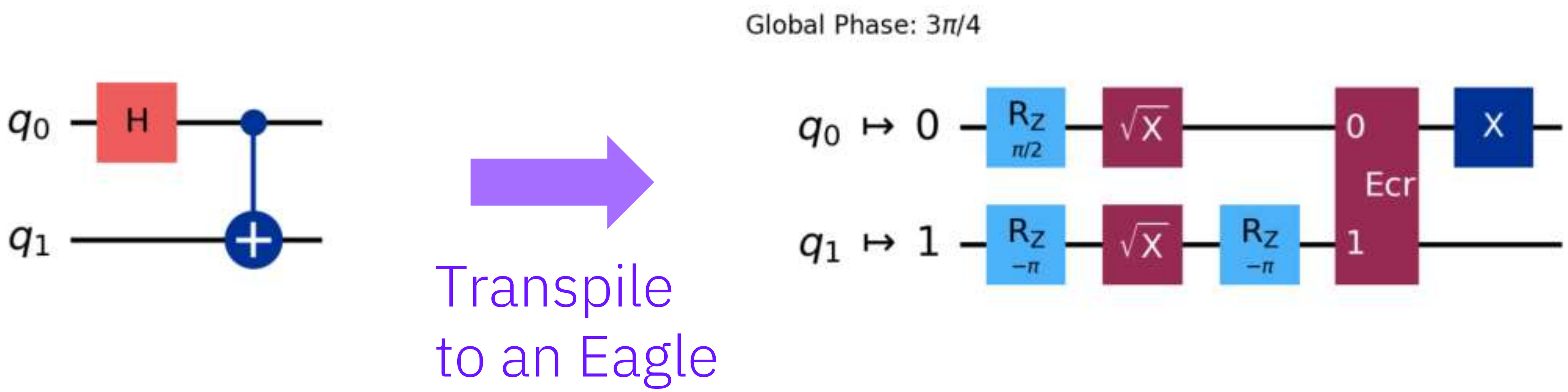
The basis gate set of an IBM Quantum Eagle processor is {ECR, ID, RZ, SX, X}.



- ECR (Echoed Cross Resonance) $= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \\ 1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \end{pmatrix}$
- SX (sqrt X) $= \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$

Basis gate set

Only a limited set of gates can be executed directly on the hardware.
Other gates can be transpiled into these basis gates.



Hands on

Lecture 2: Quantum Bits, Gates, and Circuits

- Understanding Quantum Computation with Circuit Models using quantum bits and gates.

- Hands on using Qiskit
  1. Single-qubit quantum gates
     - State vector simulator, Bloch sphere
  2. Multi-qubit quantum gates
     - Aer simulator, Real device, Qiskit Patterns
     - GHZ state of 8 qubits with the shallowest depth

Install and set up Qiskit 1.x (macOS)

- Reference URL : https://docs.quantum.ibm.com/guides/install-qiskit (For non-macOS users, please refer this.)
- Caution: You must start a new virtual environment to install Qiskit 1.x. It is very tricky and error-prone to upgrade an existing installation of Qiskit 0.x in-place to Qiskit 1.x.

1. Create a new virtual environment, using Python 3.8 or later.

```
python3 -m venv qiskit-1.x-venv
```

2. Activate the environment.

```
source qiskit-1.x-venv/bin/activate
```

3. Install Qiskit.

```
pip install qiskit
```

4. Install the necessary packages.

```
pip install qiskit-ibm-runtime
pip install qiskit[visualization]
pip install jupyter
pip install qiskit-aer
```

zsh users need to put 'qiskit[visualization]' in single quotes.

5. With the following command, you can launch Jupyter notebook and start using Qiskit.

```
jupyter notebook
```

6. Try the first cell of Hello world by copy and paste, and execute it by “Shift”+”Enter”.

6. If you are not planning to use the environment immediately, use the deactivate command to leave it.

```
deactivate
```

Thank you