5. Quantum Algorithms: Phase estimation

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Kento Ueda
IBM Research – Tokyo

Overview

- Phase estimation problem
  - Warm-up: using the phase kickback
  - Iterating the unitary operation
  - Two control qubits
  - Two-qubit phase estimation
- Quantum Fourier transform
- Phase estimation procedure

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Phase estimation problem

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Input: A unitary quantum circuit for an n-qubit operation U and an n qubit quantum state $|\psi\rangle$

Promise: $|\psi\rangle$ is an eigenvector of U

Output: An approximation to the number $\theta \in [0, 1)$ satisfying

$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$$

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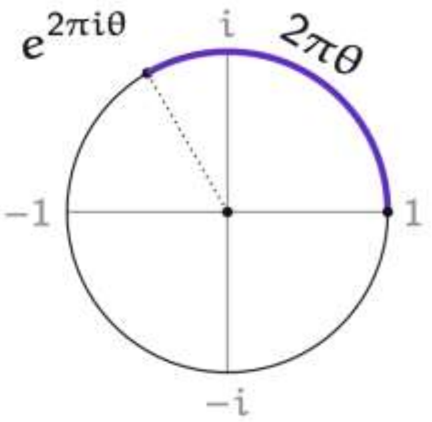

We can approximate θ by a fraction

$$\theta \approx \frac{y}{2^m}$$

for $y \in \{0, 1, \ldots, 2^m - 1\}$.

This approximation is taken “modulo 1.”

Phase estimation problem

Phase estimation problem

Input:	A unitary quantum circuit for an n-qubit operation U and an n qubit quantum state $\vert\psi\rangle$
Promise:	$\vert\psi\rangle$ is an eigenvector of U
Output:	An approximation to the number $\theta \in [0, 1)$ satisfying

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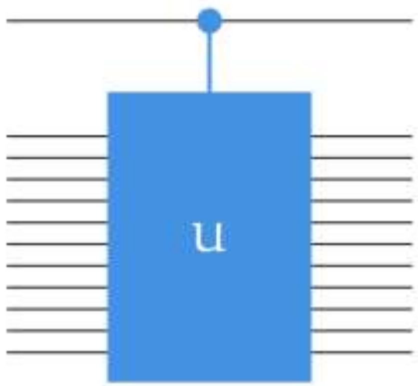
The Phase estimation problem is applied to the energy calculations for quantum many-body systems and Shor's algorithm.

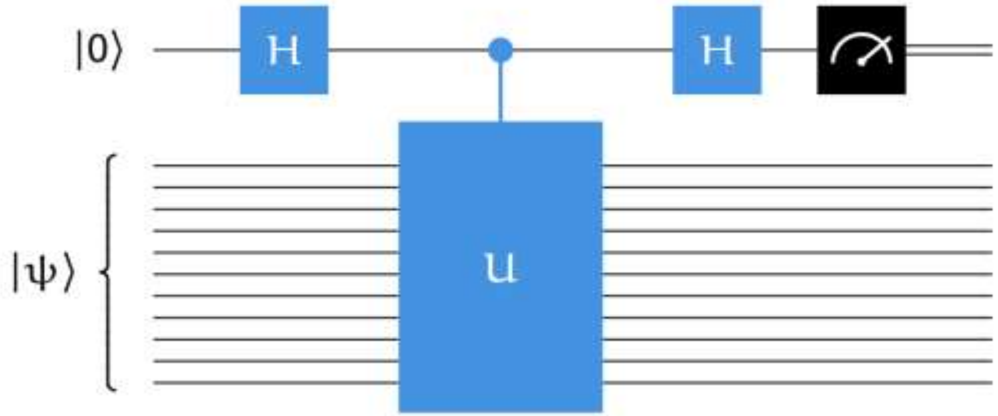
e.g. Shor’s algorithm : Acceleration from quasi-exponential to polynomial time.

Warm-up: using the phase kickback

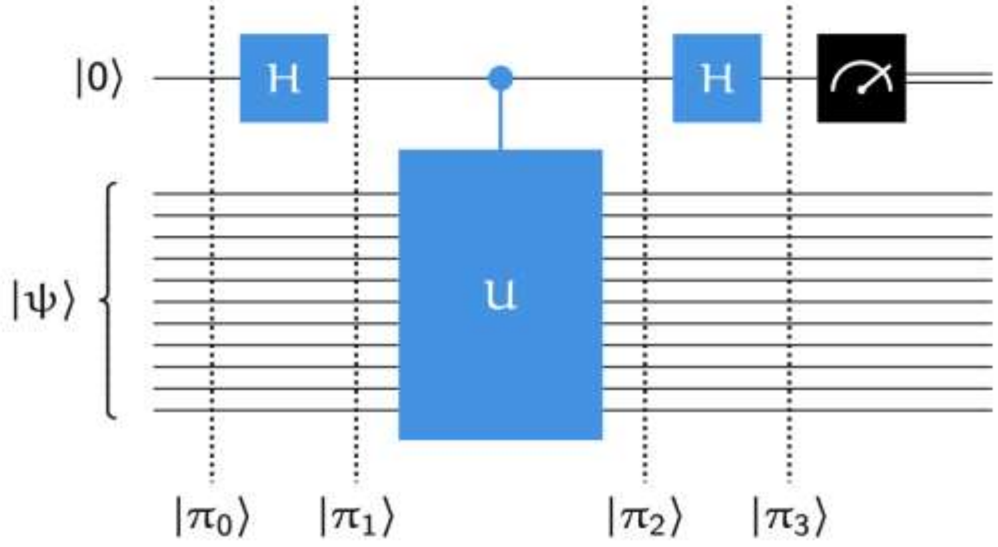
Given a circuit for U, we can create a circuit for a controlled-U operation:



Let's consider this circuit:



Warm-up: using the phase kickback

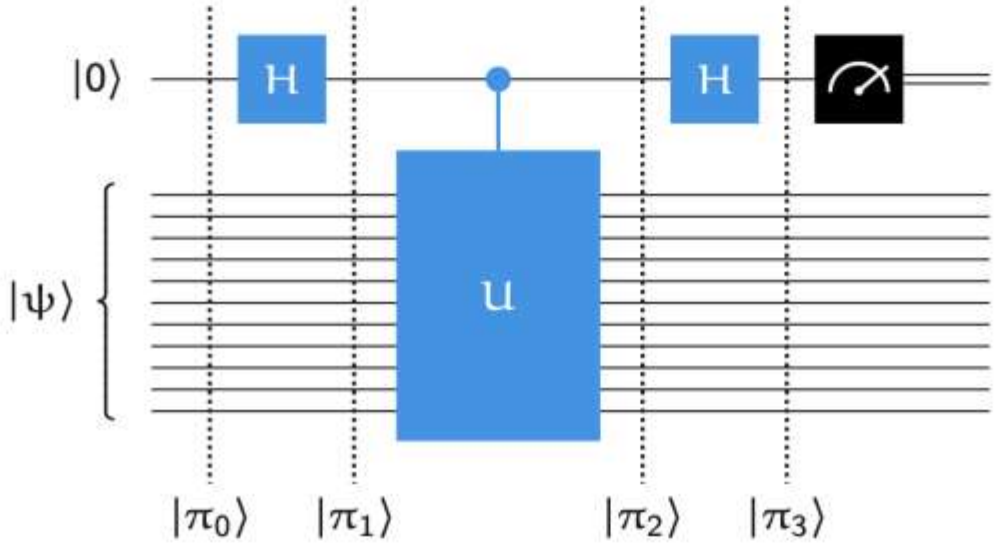


$$|\pi_0\rangle = |\psi\rangle|0\rangle$$

$$|\pi_1\rangle = \frac{1}{\sqrt{2}}|\psi\rangle|0\rangle + \frac{1}{\sqrt{2}}|\psi\rangle|1\rangle$$

$$|\pi_2\rangle = \frac{1}{\sqrt{2}}|\psi\rangle|0\rangle + \frac{1}{\sqrt{2}}(U|\psi\rangle)|1\rangle = |\psi\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{2\pi i\theta}}{\sqrt{2}}|1\rangle\right)$$

Warm-up: using the phase kickback



$$|\pi_2\rangle = |\psi\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{2\pi i\theta}}{\sqrt{2}}|1\rangle \right)$$

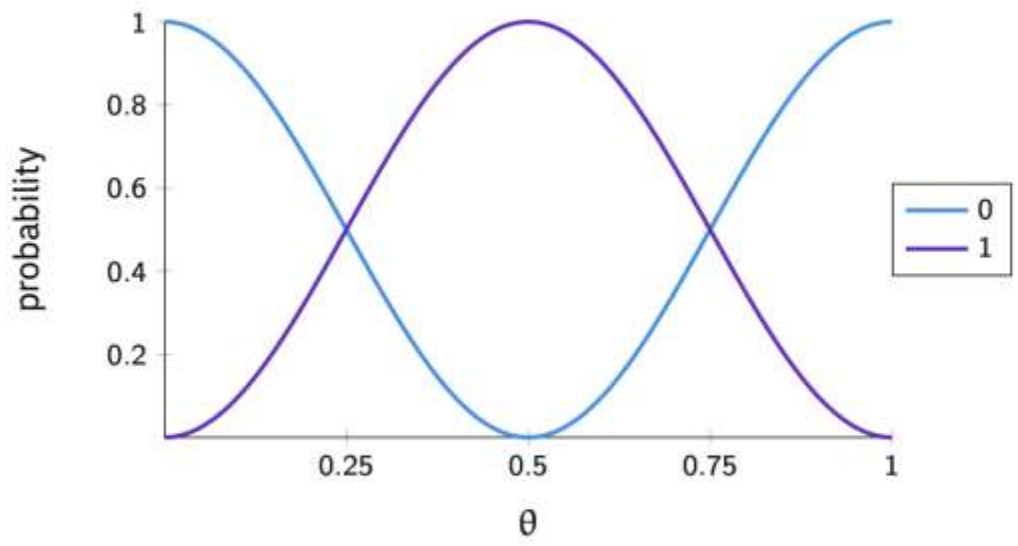
$$|\pi_3\rangle = |\psi\rangle \otimes \left(\frac{1 + e^{2\pi i\theta}}{2}|0\rangle + \frac{1 - e^{2\pi i\theta}}{2}|1\rangle \right)$$

Warm-up: using the phase kickback

$$|\psi\rangle \otimes \left(\frac{1+e^{2\pi i \theta}}{2} |0\rangle + \frac{1-e^{2\pi i \theta}}{2} |1\rangle \right)$$

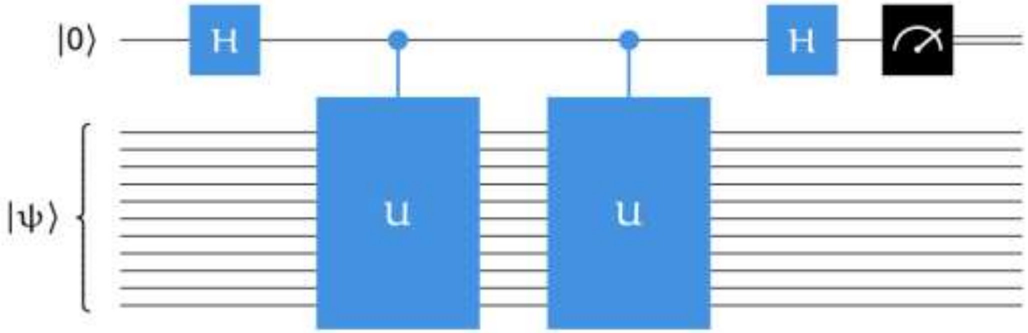
Measuring the top qubit yields the outcomes 0 and 1 with these probabilities:

$$p_0 = \left| \frac{1+e^{2\pi i \theta}}{2} \right|^2 = \cos^2(\pi\theta) \qquad p_1 = \left| \frac{1-e^{2\pi i \theta}}{2} \right|^2 = \sin^2(\pi\theta)$$

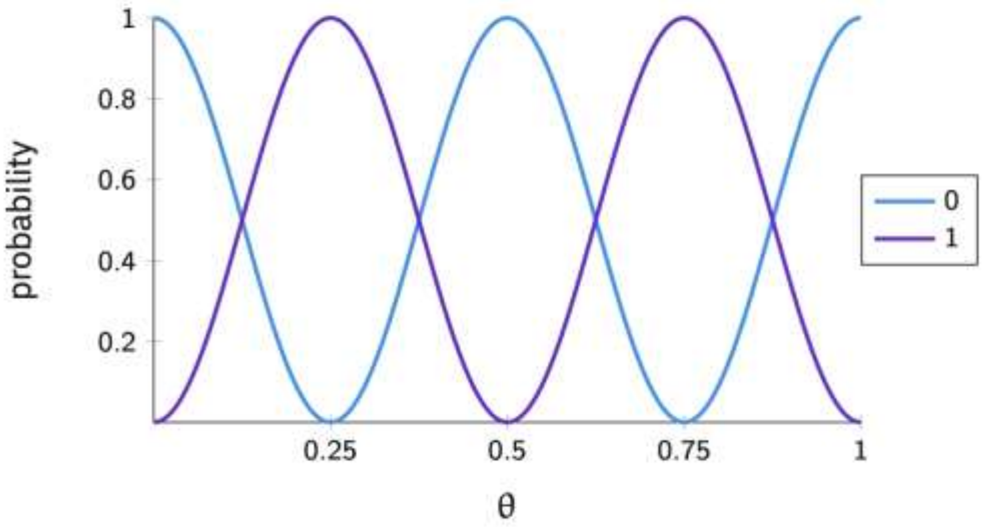


Iterating the unitary operation

How can we learn more about θ? One possibility is to apply the controlled-$\mathcal{U}$ operation twice (or multiple times):

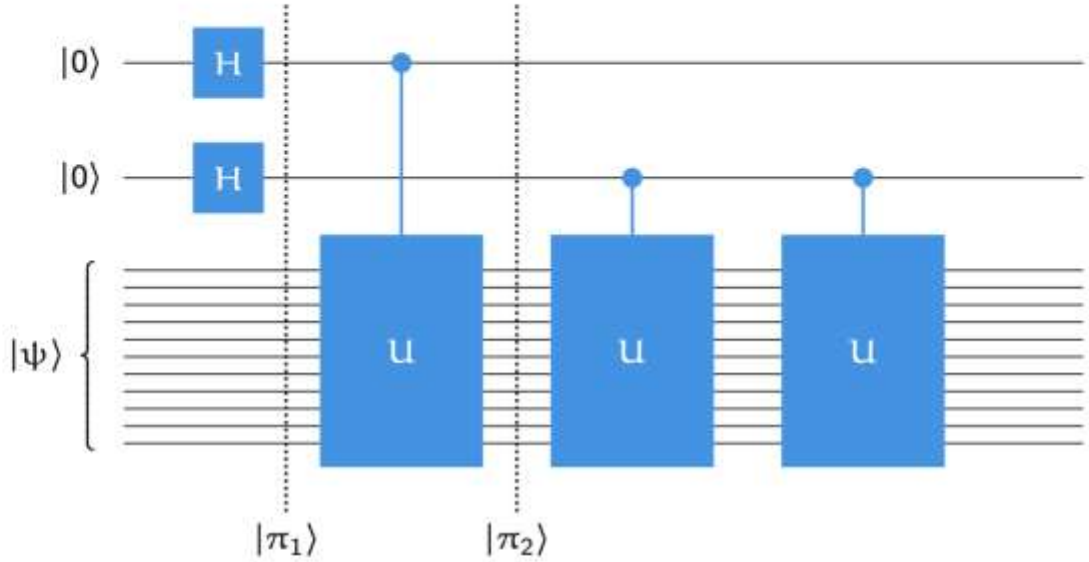


Performing the controlled-$\mathcal{U}$ operation twice has the effect of squaring the eigenvalue:



Two control qubits

Let's use two control qubits to perform the controlled-U operations — and then we'll see how best to proceed.



$$|\pi_1\rangle = |\psi\rangle \otimes \frac{1}{2} \sum_{a_0=0}^{1} \sum_{a_1=0}^{1} |a_1 a_0\rangle$$

$$|\pi_2\rangle = |\psi\rangle \otimes \frac{1}{2} \sum_{a_0=0}^{1} \sum_{a_1=0}^{1} e^{2\pi i a_0 \theta} |a_1 a_0\rangle$$

Two control qubits

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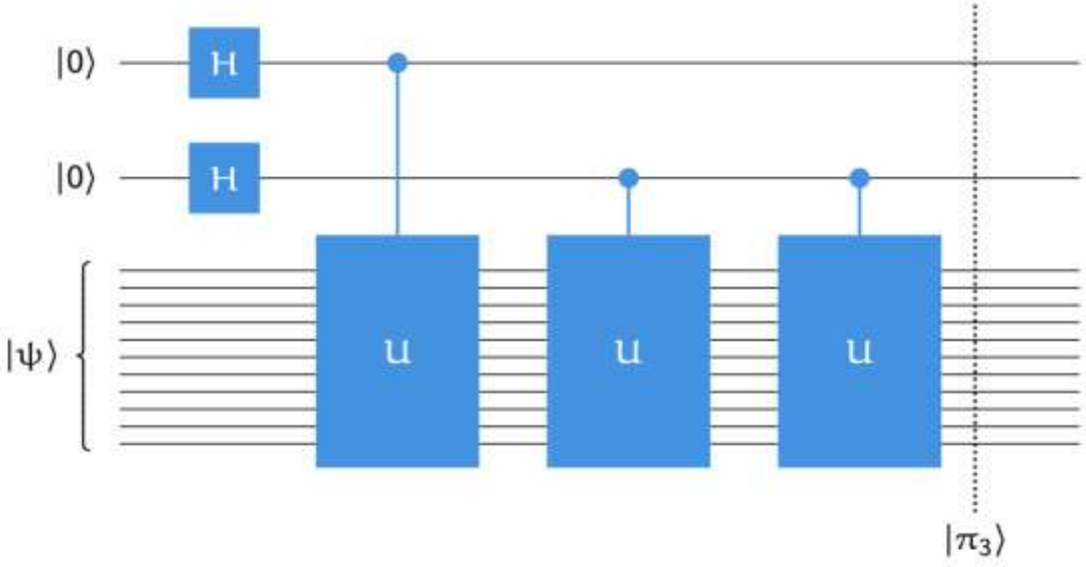

$$|\pi_3\rangle = |\psi\rangle \otimes \frac{1}{2} \sum_{a_0=0}^{1} \sum_{a_1=0}^{1} e^{2\pi i (2a_1 + a_0)\theta} |a_1 a_0\rangle$$

$$= |\psi\rangle \otimes \frac{1}{2} \sum_{x=0}^{3} e^{2\pi i x \theta} |x\rangle$$

$$\sum_{a_0=0}^{1} \sum_{a_1=0}^{1} e^{2\pi i (2a_1 + a_0)\theta} \, |a_1 a_0\rangle.$$

$$= |0\rangle + e^{2\pi i * 1\theta}|1\rangle + e^{2\pi i * 2\theta}|2\rangle + e^{2\pi i * 3\theta}|3\rangle$$

Two control qubits

$$\frac{1}{2}\sum_{x=0}^{3} e^{2\pi i x\theta}|x\rangle$$

What can we learn about θ from this state? Suppose we're promised that $\theta = \frac{y}{4}$ for $y \in \{0, 1, 2, 3\}$. Can we figure out which one it is?

Define a two-qubit state for each possibility:

$$|\phi_y\rangle = \frac{1}{2}\sum_{x=0}^{3} e^{2\pi i \frac{xy}{4}}|x\rangle$$

$$|\phi_0\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$$

$$|\phi_1\rangle = \frac{1}{2}|0\rangle + \frac{i}{2}|1\rangle - \frac{1}{2}|2\rangle - \frac{i}{2}|3\rangle$$

$$|\phi_2\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle - \frac{1}{2}|3\rangle$$

$$|\phi_3\rangle = \frac{1}{2}|0\rangle - \frac{i}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{i}{2}|3\rangle$$

These vectors are *orthonormal* — so they can be discriminated perfectly by a projective measurement.

Two control qubits

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$$|\phi_3\rangle = \frac{1}{2}|0\rangle - \frac{i}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{i}{2}|3\rangle$$

$$V = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

The unitary matrix V whose *columns* are $|\phi_0\rangle$, $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$ has this action:

$$V|y\rangle = |\phi_y\rangle \quad \text{(for every } y \in \{0, 1, 2, 3\}\text{)}$$

We can identify y by performing the inverse of V then a standard basis measurement.

$$V^\dagger|\phi_y\rangle = |y\rangle \quad \text{(for every } y \in \{0, 1, 2, 3\}\text{)}$$

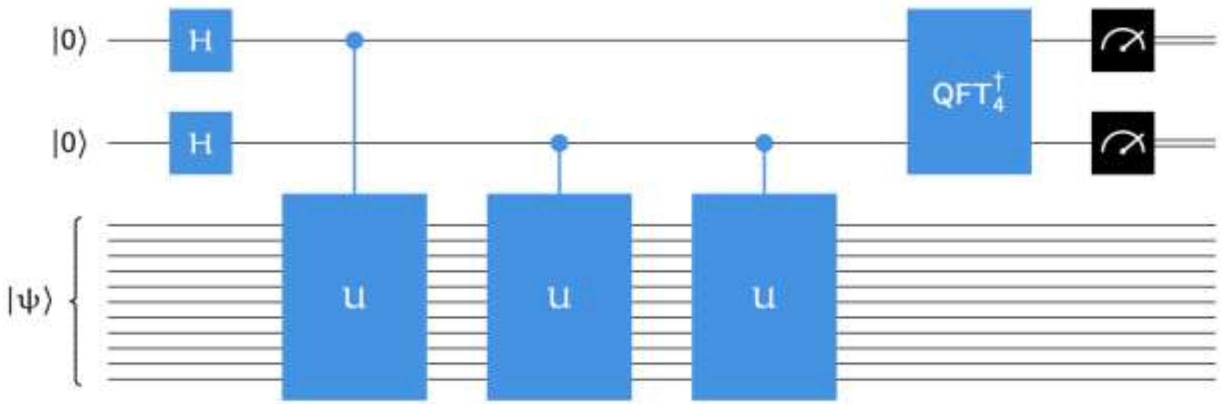
Two-qubit phase estimation

$$\mathrm{QFT}_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

This matrix is associated with the *discrete Fourier transform* (for 4 dimensions).

When we think about this matrix as a unitary operation, we call it the *quantum Fourier transform.*

The complete circuit for learning $y \in \{0, 1, 2, 3\}$ when $\theta = y/4$:



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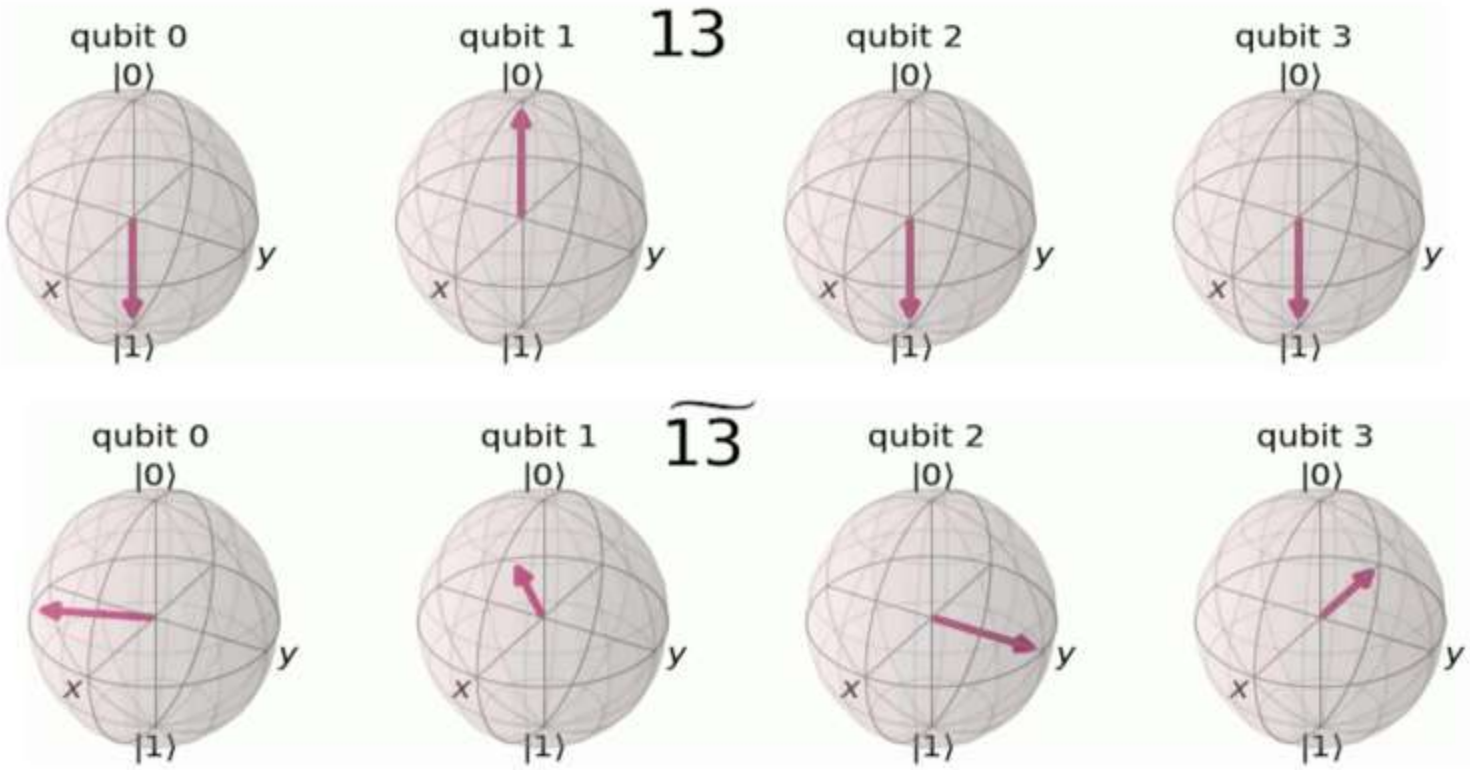
Quantum Fourier transform

The quantum Fourier transform transform between the computational basis and the Fourier basis.

$$|\text{State in Computational Basis}\rangle \xrightarrow{\text{QFT}} |\text{State in Fourier Basis}\rangle$$

$$\text{QFT}|x\rangle = |\widetilde{x}\rangle$$



Quantum Fourier transform

The quantum Fourier transform is defined for each positive integer N as follows.

$$\mathrm{QFT}_N = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} e^{2\pi i \frac{xy}{N}} |x\rangle\langle y|$$

$$\mathrm{QFT}_N |y\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{2\pi i \frac{xy}{N}} |x\rangle$$

Example

$$\mathrm{QFT}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

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Example

$$\mathrm{QFT}_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

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$$\mathrm{QFT}_N |y\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{2\pi i \frac{xy}{N}} |x\rangle$$

Example

$$\mathrm{QFT}_8 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} & -1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1-i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & \frac{-1+i}{\sqrt{2}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} & -1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

Quantum Fourier transform

$$\mathrm{QFT_N}\,|j\rangle = \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} exp\,(2\pi i jk/2^n)\,|k\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \dots \sum_{k_n=0}^{1} e^{2\pi i j(\sum_{l=1}^{n} k_l 2^{-l})}\,|k_1 k_2 \dots k_n\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \dots \sum_{k_n=0}^{1} \prod_{l=1}^{n} e^{2\pi i j k_l 2^{-l}} \bigotimes_{m=1}^{n} |k_m\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \dots \sum_{k_n=0}^{1} \bigotimes_{m=1}^{n} e^{2\pi i j k_m 2^{-m}}\,|k_m\rangle$$

$$= \frac{1}{2^{n/2}} \bigotimes_{m=1}^{n} \left(|0\rangle + e^{2\pi i j 2^{-m}}\,|1\rangle\right)$$

$$= \frac{\left(|0\rangle + e^{2\pi i 0.j_n}\,|1\rangle\right)}{\sqrt{2}} \otimes \frac{\left(|0\rangle + e^{2\pi i 0.j_{n-1}j_n}\,|1\rangle\right)}{\sqrt{2}} \otimes \dots \otimes \frac{\left(|0\rangle + e^{2\pi i 0.j_2\dots j_n}\,|1\rangle\right)}{\sqrt{2}} \otimes \frac{\left(|0\rangle + e^{2\pi i 0.j_1 j_2\dots j_n}\,|1\rangle\right)}{\sqrt{2}}$$

$$k = k_1 k_2 \dots k_n = k_1 2^{n-1} + k_2 2^{n-2} + \dots + k_n 2^0 = \sum_{l=1}^{n} k_l 2^{n-l}$$

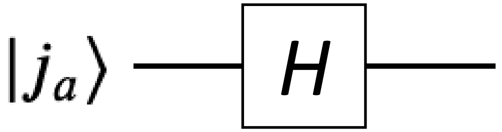
$$k/2^n = 0.k_1 k_2 \dots k_n = k_1 2^{-1} + k_2 2^{-2} + \dots + k_n 2^{-n} = \sum_{l=1}^{n} k_l 2^{-l}$$

Quantum Fourier transform

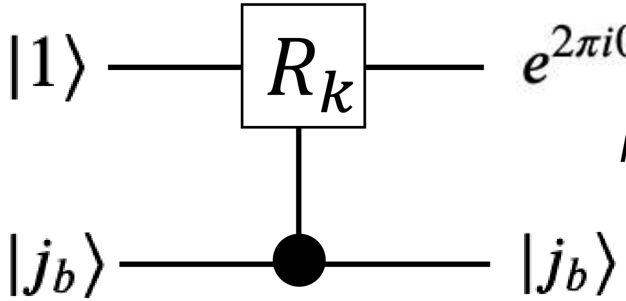
$$
\begin{aligned}
\mathrm{QFT_N}\,|j\rangle &= \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} exp\,(2\pi i\,jk/2^n)\,|k\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi i j\left(\sum_{l=1}^{n} k_l 2^{-l}\right)}\,|k_1 k_2 \ldots k_n\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} \prod_{l=1}^{n} e^{2\pi i j k_l 2^{-l}} \bigotimes_{m=1}^{n} |k_m\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} \bigotimes_{m=1}^{n} e^{2\pi i j k_m 2^{-m}}\,|k_m\rangle \\
&= \frac{1}{2^{n/2}} \bigotimes_{m=1}^{n} \left(|0\rangle + e^{2\pi i j 2^{-m}}\,|1\rangle\right) \\
&= \frac{\left(|0\rangle + e^{2\pi i 0.j_n}\,|1\rangle\right)}{\sqrt{2}} \otimes \frac{\left(|0\rangle + e^{2\pi i 0.j_{n-1}j_n}\,|1\rangle\right)}{\sqrt{2}} \otimes \ldots \otimes \frac{\left(|0\rangle + e^{2\pi i 0.j_2\ldots j_n}\,|1\rangle\right)}{\sqrt{2}} \otimes \frac{\left(|0\rangle + e^{2\pi i 0.j_1 j_2\ldots j_n}\,|1\rangle\right)}{\sqrt{2}}
\end{aligned}
$$

$$
\boxed{\begin{aligned} j2^{-m} &= j_1 \cdots j_{n-m} \cdot j_{n-m+1} \cdots j_n \\ e^{2\pi i j 2^{-m}} &= e^{2\pi i 0.j_{n-m+1}\cdots j_n} \end{aligned}}
$$

Quantum Fourier transform



$$\frac{\left(|0\rangle + e^{2\pi i 0.j_a}|1\rangle\right)}{\sqrt{2}}$$

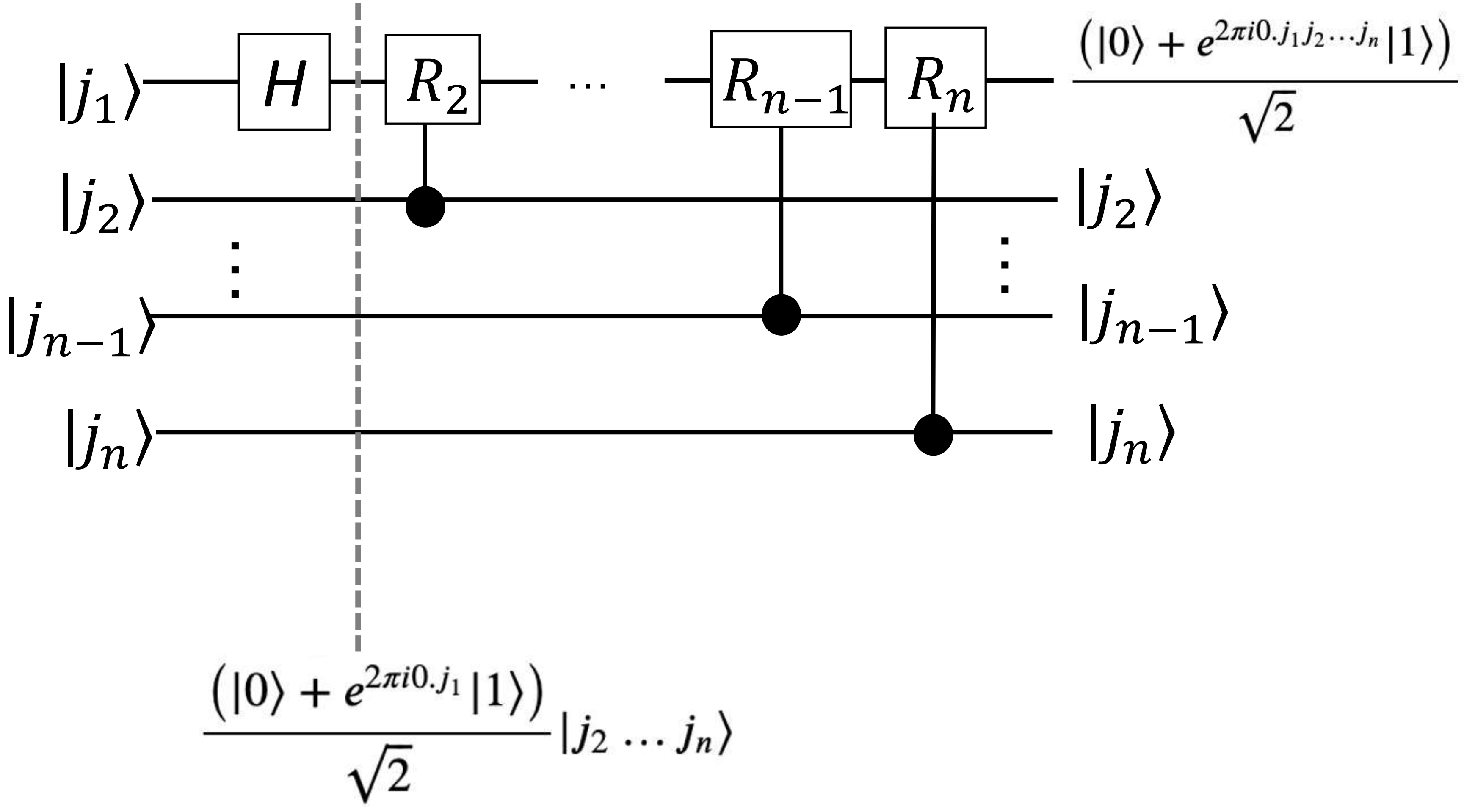


$$e^{2\pi i 0.0\ldots j_b}|1\rangle$$

k digits in fraction part

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & \exp\left(\frac{2\pi i}{2^k}\right) \end{bmatrix}$$

Quantum Fourier transform



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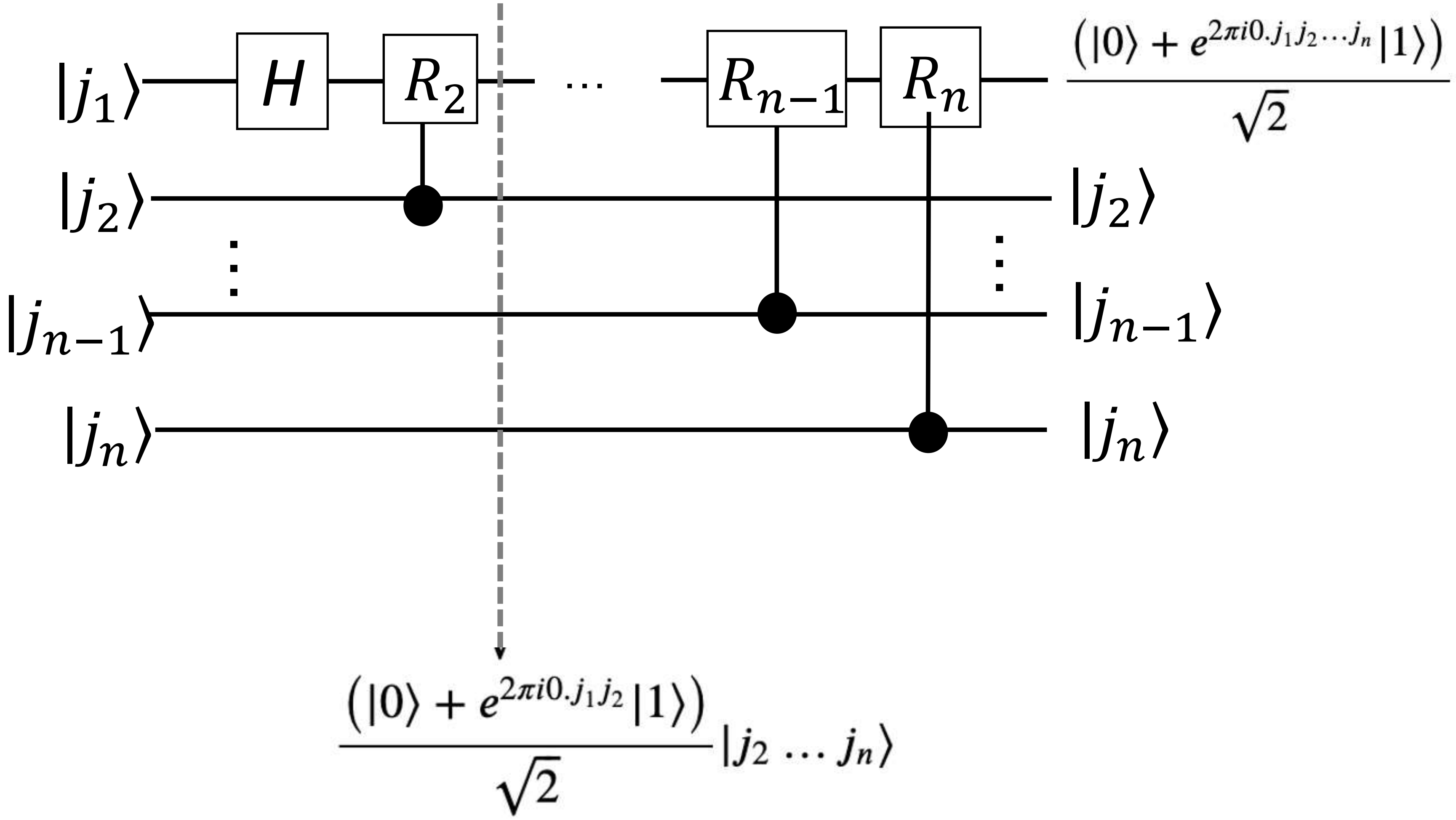
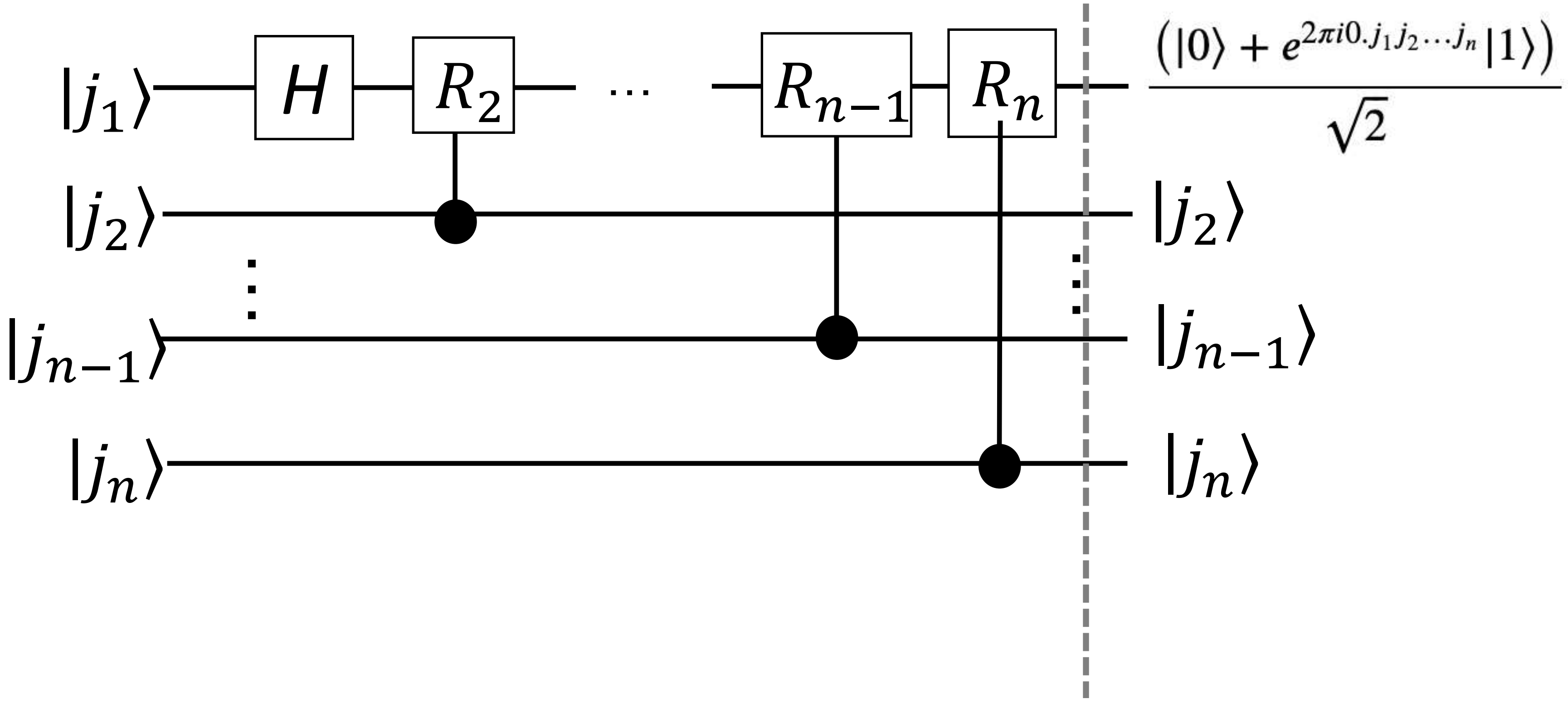


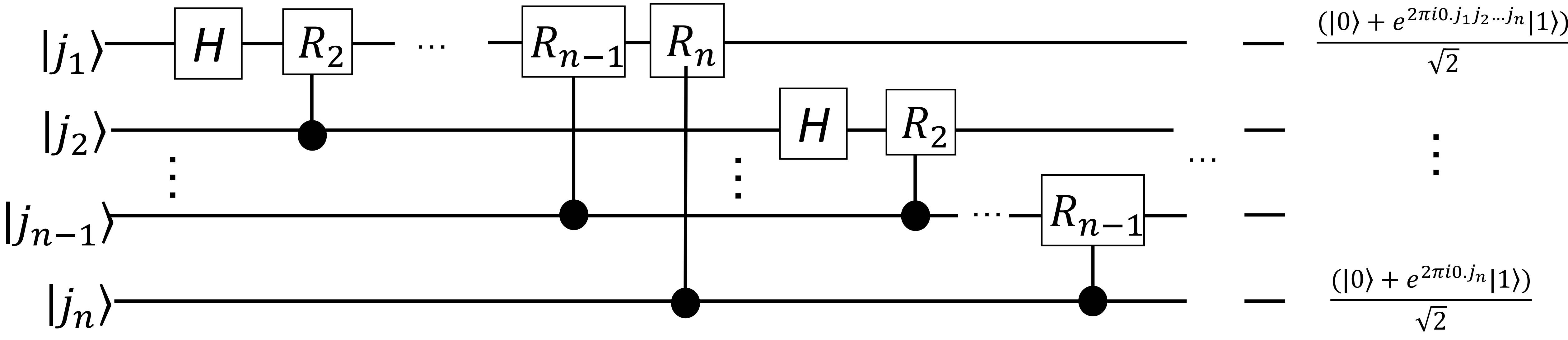
Quantum Fourier transform



$$\frac{\left(|0\rangle + e^{2\pi i 0.j_1 j_2 \ldots j_n}|1\rangle\right)}{\sqrt{2}} \; |j_2, \ldots j_n\rangle$$

Quantum Fourier transform

The implementation is recursive in nature.



Quantum Fourier transform

At last, Swap operations are executed.

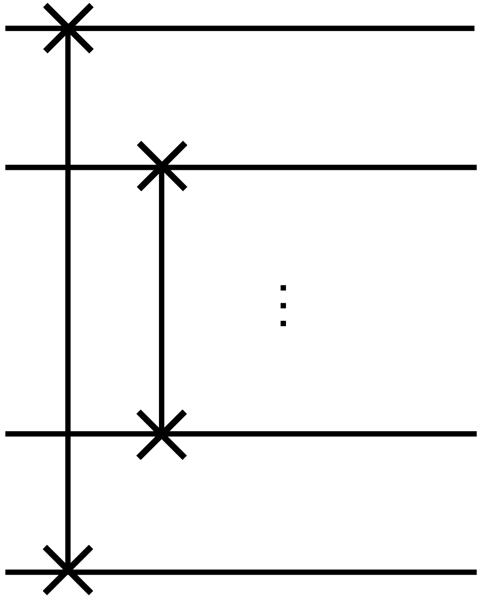
$$\frac{(|0\rangle + e^{2\pi i 0.j_1 j_2 \dots j_n}|1\rangle)}{\sqrt{2}}$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_2 \dots j_n}|1\rangle)}{\sqrt{2}}$$

$$\vdots$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_{n-1} j_n}|1\rangle)}{\sqrt{2}}$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_n}|1\rangle)}{\sqrt{2}}$$



$$\frac{(|0\rangle + e^{2\pi i 0.j_n}|1\rangle)}{\sqrt{2}}$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_{n-1} j_n}|1\rangle)}{\sqrt{2}}$$

$$\vdots$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_2 \dots j_n}|1\rangle)}{\sqrt{2}}$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_1 j_2 \dots j_n}|1\rangle)}{\sqrt{2}}$$

$$\mapsto \quad \frac{(|0\rangle + e^{2\pi i 0.j_n}|1\rangle)}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i 0.j_{n-1} j_n}|1\rangle)}{\sqrt{2}} \otimes \cdots \otimes \frac{(|0\rangle + e^{2\pi i 0.j_2 \dots j_n}|1\rangle)}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i 0.j_1 j_2 \dots j_n}|1\rangle)}{\sqrt{2}}$$

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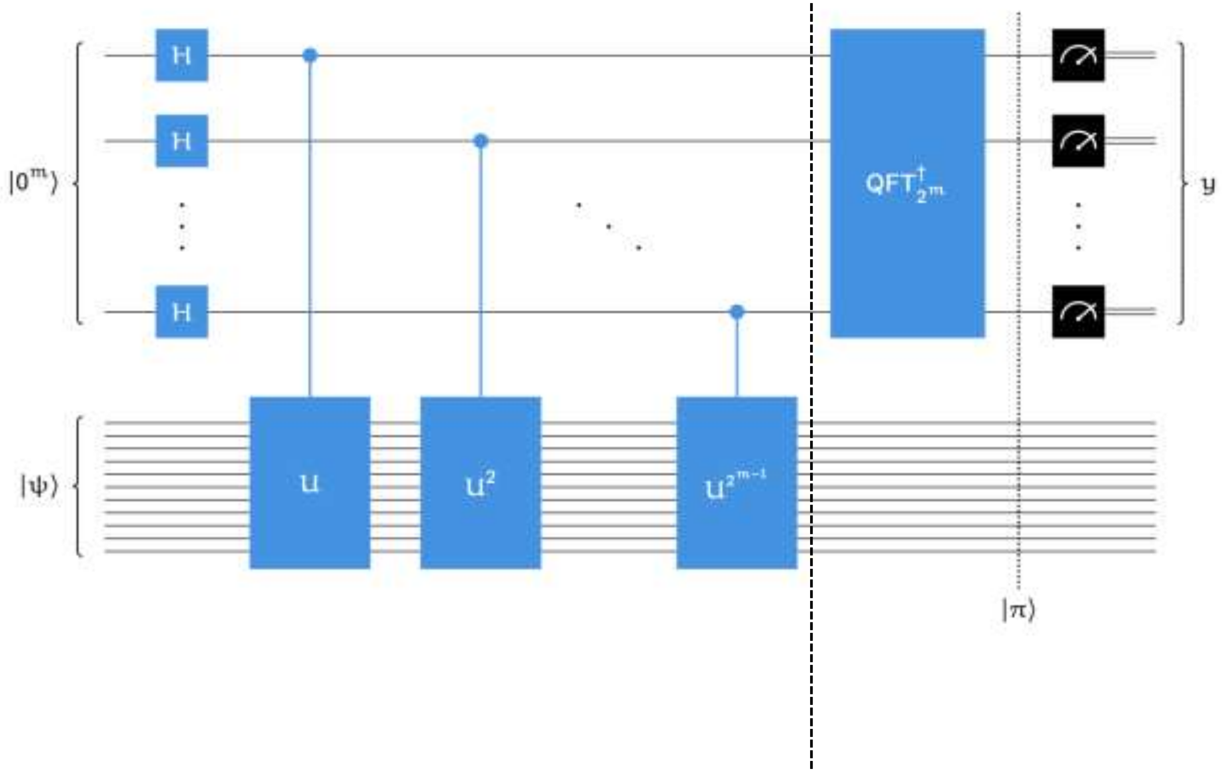
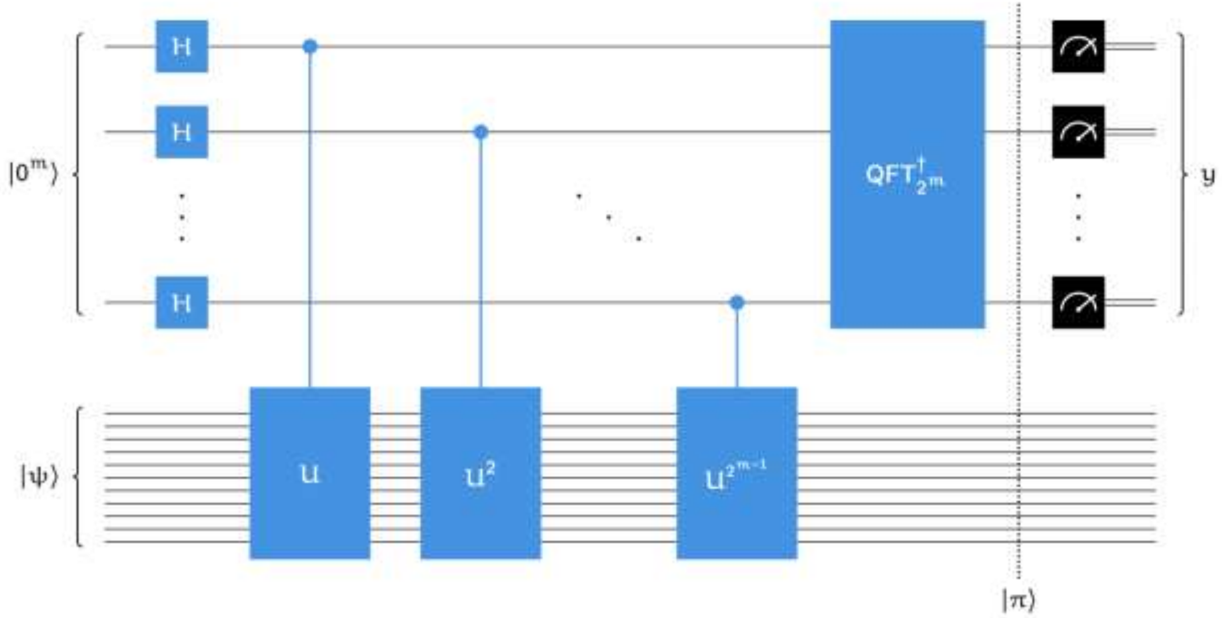
Phase estimation procedure



$$|\psi\rangle \otimes \frac{1}{2^{m/2}} \sum_{x=0}^{2^m-1} e^{2\pi i x\theta} |x\rangle$$

Phase estimation procedure



$$|\pi\rangle = |\psi\rangle \otimes \frac{1}{2^m}\sum_{y=0}^{2^m-1}\sum_{x=0}^{2^m-1} e^{2\pi i x(\theta - y/2^m)}|y\rangle$$

$$p_y = \left|\frac{1}{2^m}\sum_{x=0}^{2^m-1} e^{2\pi i x(\theta - y/2^m)}\right|^2$$

Phase estimation procedure

Best approximations

Suppose $y/2^m$ is the *best approximation* to θ:

$$\left|\theta - \frac{y}{2^m}\right|_1 \leq 2^{-(m+1)}$$

Then the probability to measure y will relatively high:

$$p_y \geq \frac{4}{\pi^2} \approx 0.405$$

Worse approximations

Suppose there is a *better approximation* to θ between $y/2^m$ and θ:

$$\left|\theta - \frac{y}{2^m}\right|_1 \geq 2^{-m}$$

Then the probability to measure y will be relatively low:

$$p_y \leq \frac{1}{4}$$

To obtain an approximation $y/2^m$ that is *very likely* to satisfy

$$\left|\theta - \frac{y}{2^m}\right|_1 < 2^{-m}$$

we can run the phase estimation procedure using m control qubits *several times* and take y to be the *mode* of the outcomes.

(The eigenvector $|\psi\rangle$ is unchanged by the procedure and can be reused as many times as needed.)

Summary

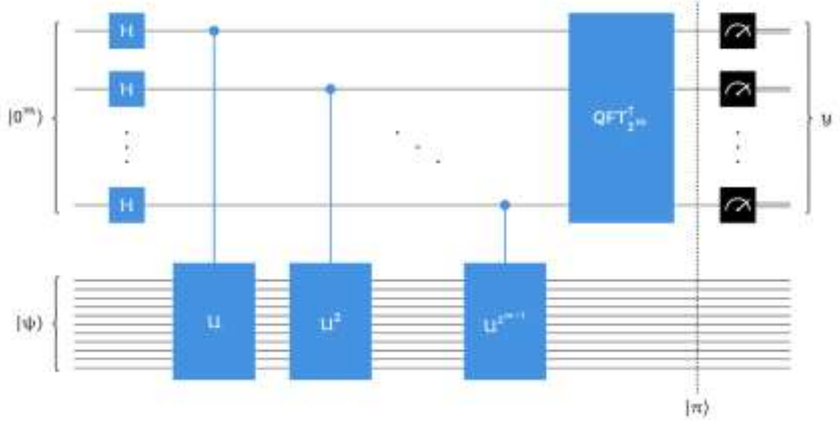
- The Phase estimation problem can find the approximation to the number $\theta \in [0,1)$ satisfying

$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$$

- QFT transform between the computational basis and the Fourier basis.

- QFT can be implemented in the quantum circuit.

- The Phase estimation problem is solved using the QFT.



Reference

- John Watrous, https://learning.quantum.ibm.com/course/fundamentals-of-quantum-algorithms/phase-estimation-and-factoring .

Thank you