Quantum noise and error mitigation

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What you learn today

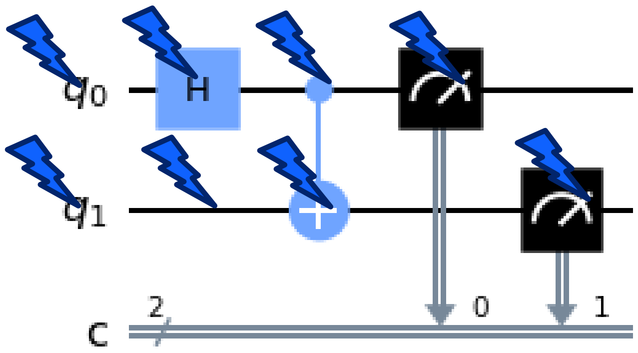
- Talk 1 (Basic, 35min)
  - What is quantum noise/error
  - Error suppression and mitigation techniques
    - TREX (Twirled Readout Error eXtinction)
    - ZNE (Zero Noise Extrapolation)
    - PEA (Probabilistic Error Amplification)

<Break>

- Hands-on (20 min)
- Talk 2 (Advanced, 30min)
  - Formalism of quantum errors
    - Standard error channels, e.g. Pauli error channel
    - Quantum channel
    - PTM (Pauli Transfer Matrix) representation

Fight noise after avoiding it as possible

- Noises everywhere:
  - Initialization
  - Gates (even in idling time)
  - Measurements



- **Noises** cause (computational) **errors**
- Errors prevent the realization of useful quantum computers

Quantum circuit optimization (last week) ➔ Reduce noise

Error mitigation (today) ➔ Fight noise

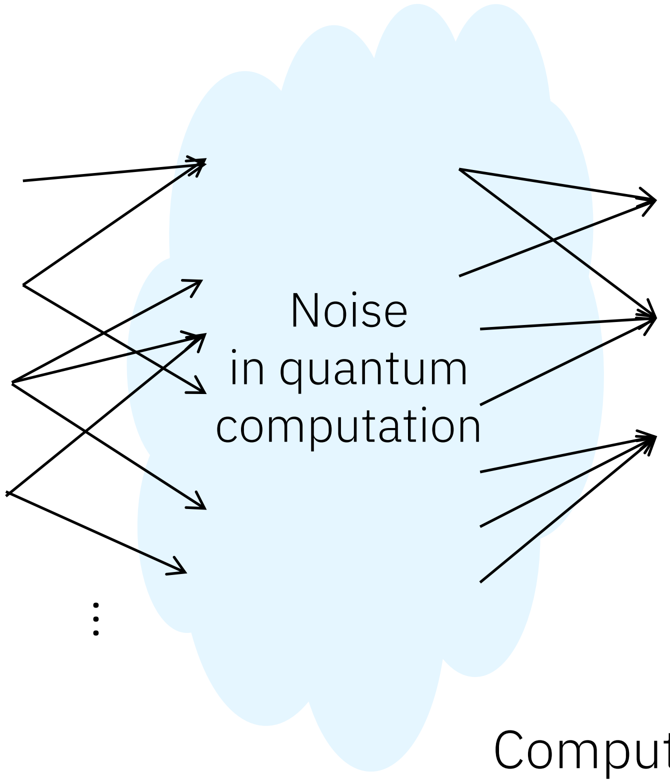
Approaches against quantum noise

We focus on **how noise affects computation (errors)**

Noise sources:

- Pulse distortion
- Material defects
- Heating
- Cosmic ray
- etc. (innumerable)

⋮



Classification of errors
(CS view)

Affected operations:

- **Coherent error** — Gates
- **Incoherent error** — Gates
- **SPAM error** — Resets & measurements
  (SPAM: State Preparation And Measurement)
- **Sampling error** — Measurements

Physics approach:

- Mechanism noise is produced
- How to protect from the noise

Computer science (CS) approach: **Today**

- Effects of noises in computation
- How to minimize the effect of noises (errors)

Coherent error (Unitary error)

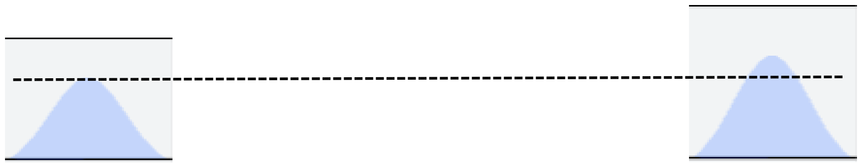
[Sources]
- Miscalibration (e.g. pulse amplitudes, qubit frequency)
- Unwanted interaction between qubits

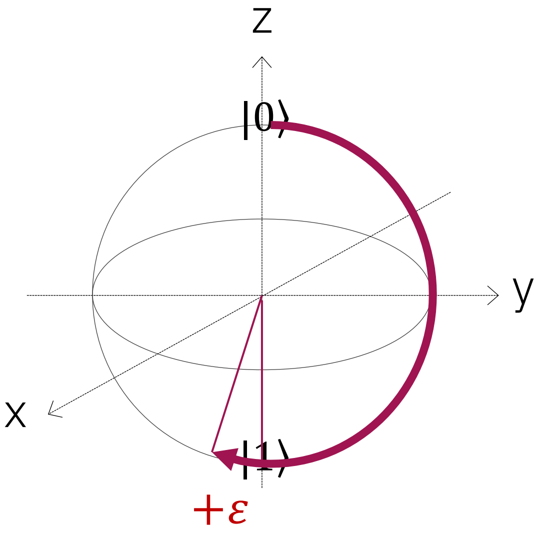
[Characters]
- Unitary evolution, No change in purity (pure state → pure state)

[Measures]
- (Better calibration), Error mitigation/suppression

Ex) Miscalibration of X gate

$$R_x(\pi) \xrightarrow{\text{Miscalibration}} R_x(\pi + \varepsilon)$$





Purity: $\mathrm{tr}(\rho^2)$

$$\frac{1}{d} \leq \mathrm{tr}(\rho^2) \leq 1$$

Completely mixed — pure

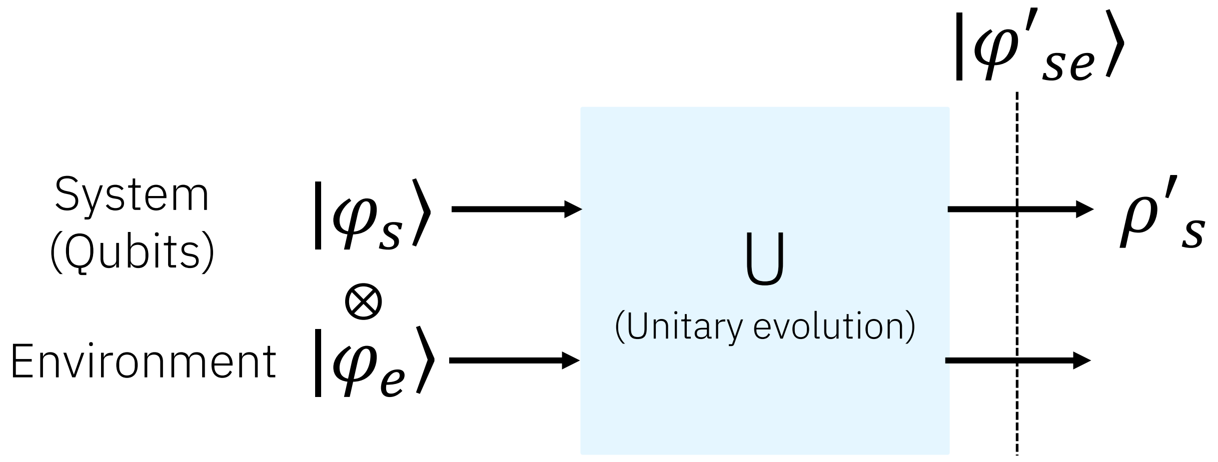
Miscalibration may cause over/under rotation errors

Incoherent error

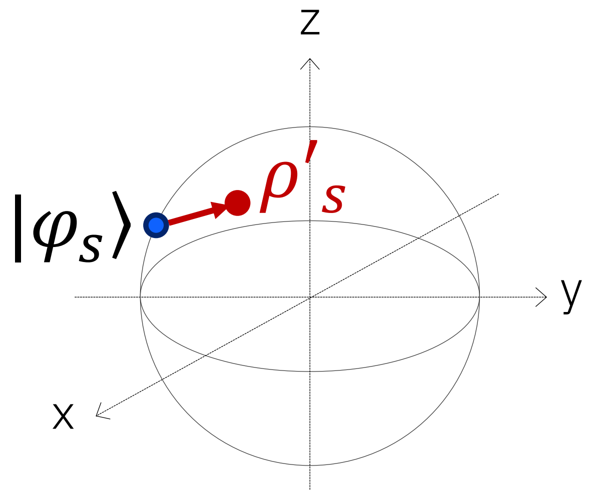
[Sources] • Entanglement (coupling) with environment (system is open)

[Characters] • Non-unitary, Loss of purity (pure state → mixed state)

[Measures] • Error mitigation



Quantum interaction
as a whole



$$\rho'_s = \mathrm{tr}_e(|\varphi'_{se}\rangle\langle\varphi'_{se}|)$$

Partial trace over e (discard environment e, leave system s)

(https://en.wikipedia.org/wiki/Partial_trace)

Extreme example: Subsystem of the Bell state

The Bell state is a pure state, but the reduced density operator of the first qubit is a mixed state (the completely mixed state)

Stronger entanglement (with env.) → More error (on the system)

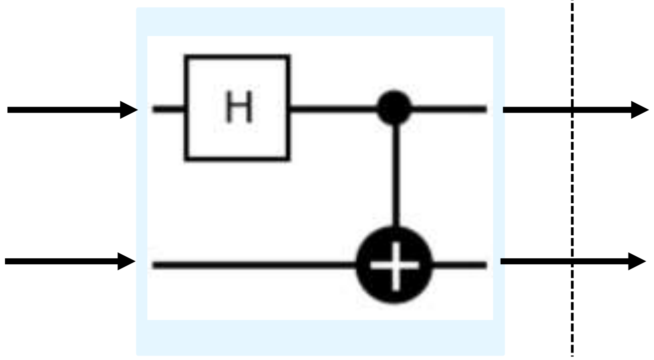
Bell state

$$|\varphi'_{se}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

System (Qubit) $|0\rangle$

$\otimes$

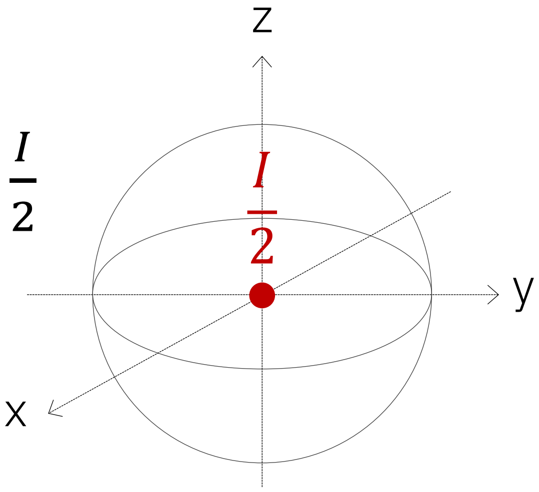
Environment (Qubit) $|0\rangle$



Quantum interaction as a whole

$$\rho'_s = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{I}{2}$$

Completely mixed state



$$\rho'_s = \mathrm{tr}_e(|\varphi'_{se}\rangle\langle\varphi'_{se}|)$$

Partial trace over e (discard environment e, leave system s)

Measurement or Readout error (SPA<u>M</u> error) (SPAM: State Preparation And **Measurement**)

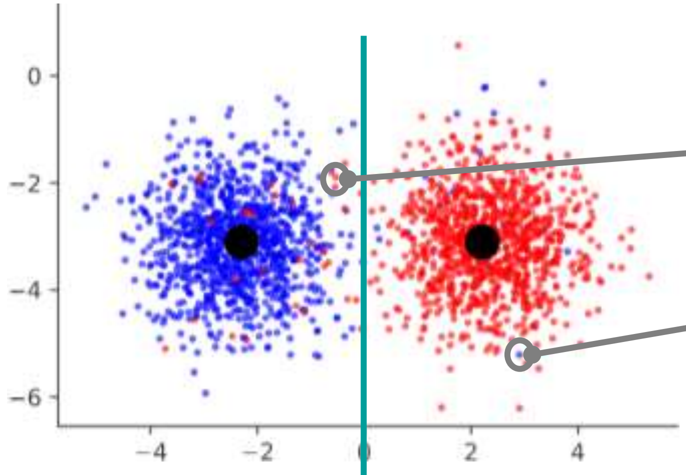
[Sources] • Mis-discrimination (in qubit state readout)

[Characters] • Classical errors (bit-flip errors)

[Measures] • (Better calibration), Error mitigation

Ex) IQ plot for qubit state readout

- Prepare $|0\rangle$
- Prepare $|1\rangle$



$|0\rangle$ $|1\rangle$

Discrimination

Two types of readout errors

$P(0|1)$: Prepare $|1\rangle$, Measure $|0\rangle$

$P(1|0)$: Prepare $|0\rangle$, Measure $|1\rangle$

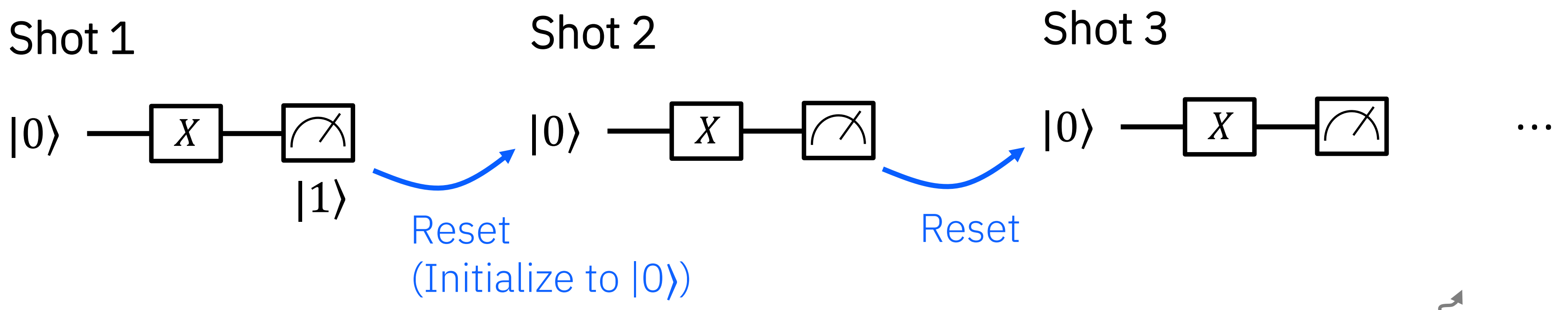
Initialization or Reset error (SPAM error)

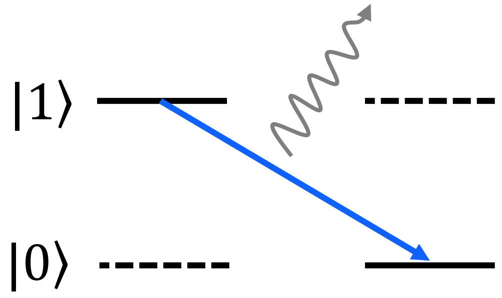
(SPAM: **State Preparation** And Measurement)

[Sources] • Imperfect reset (of previously measured state)

[Measures] • Long shot intervals

Shots: Run a circuit multiple times to sample results (bits)





Sampling error (Shot error)

[Sources] • Core nature of quantum physics

[Measures] • Increase the number of shots

Measure a qubit → Observe a bit 0 or 1, following

Bernoulli distribution

$$\begin{cases} 0 \text{ with probability } p \\ 1 \text{ with probability } 1-p \end{cases}$$

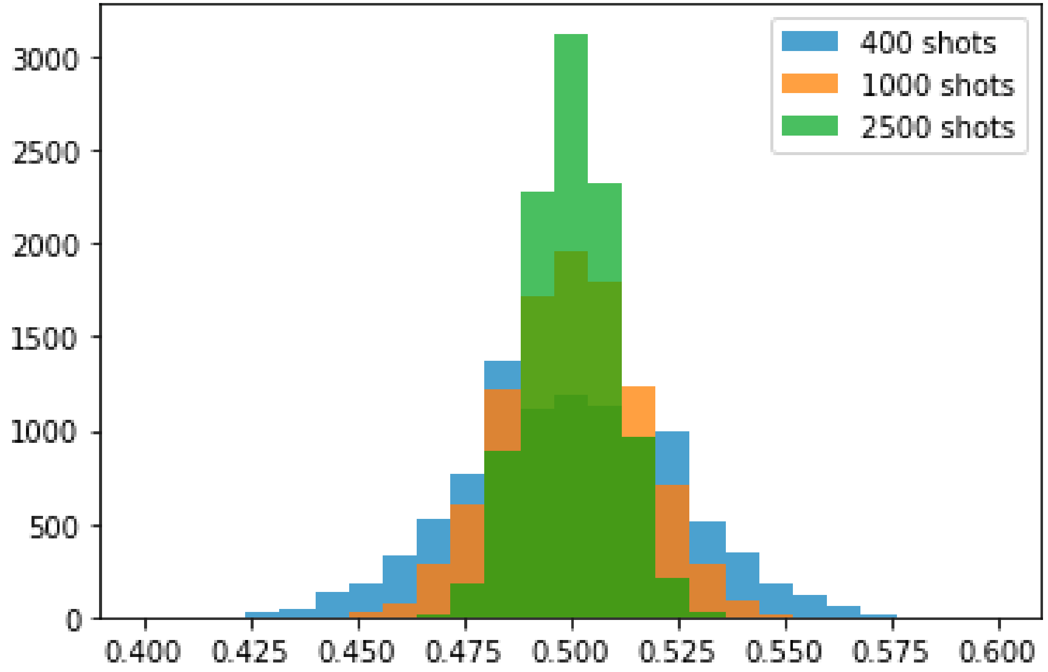
p depends on amplitude of $|0\rangle$ of the state

Measure multiple times to know p

→ Obtain sample mean of Bernoulli random variables $\hat{p}$

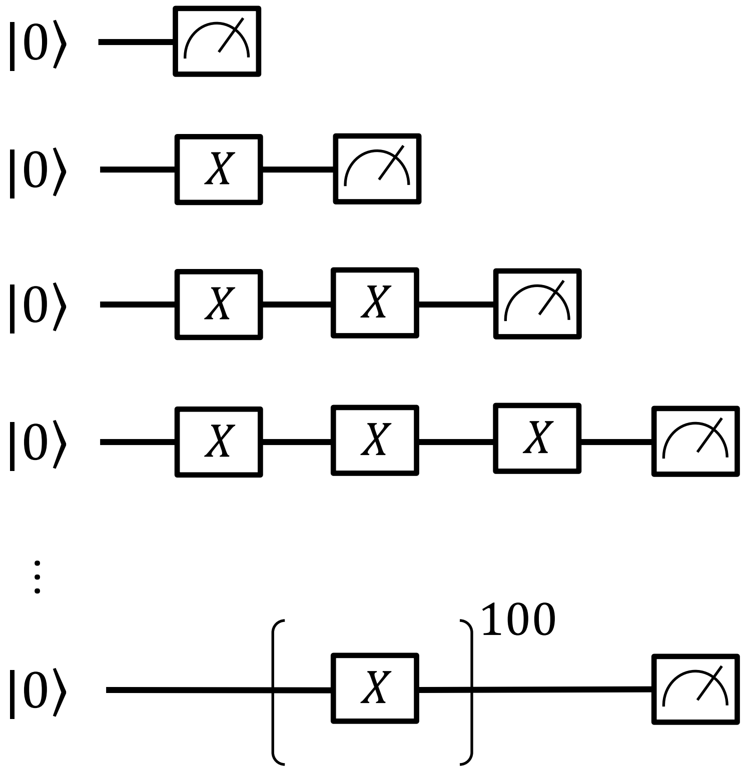
More shots ➔ Less variance (more precise $\hat{p}$)

Distribution of mean of Bernoulli random variables (p=0.5)



Quiz: What errors look like

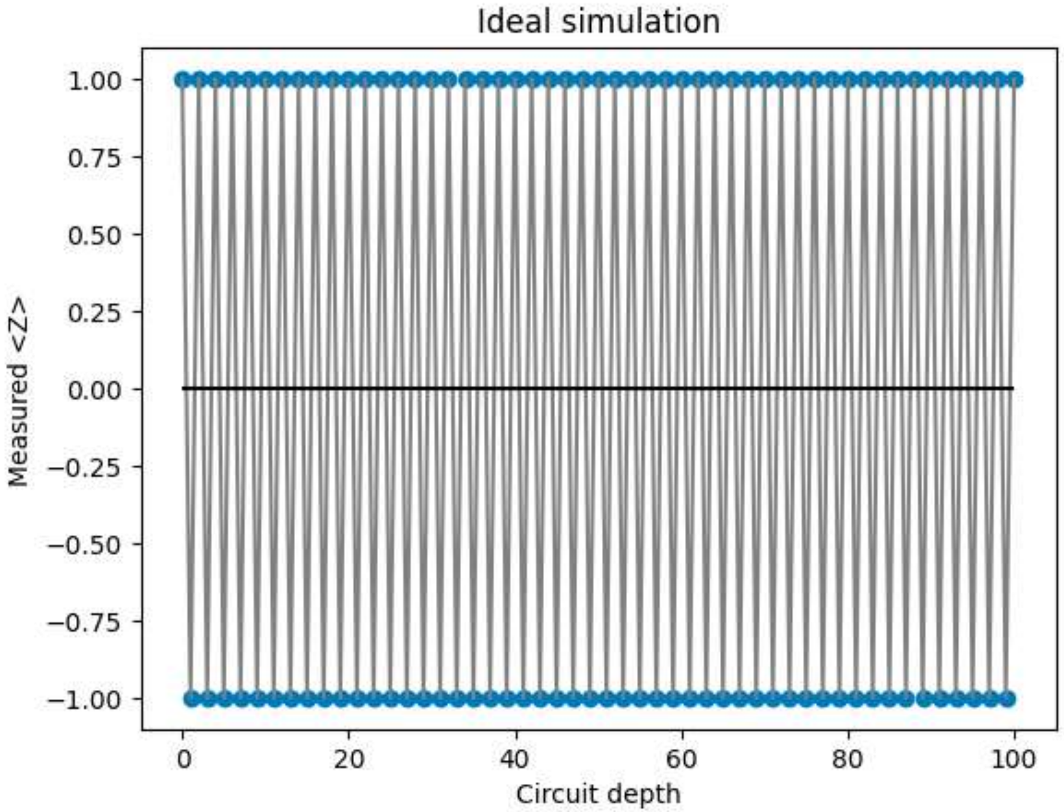
Run the following 101 circuits



400 shots for each circuit

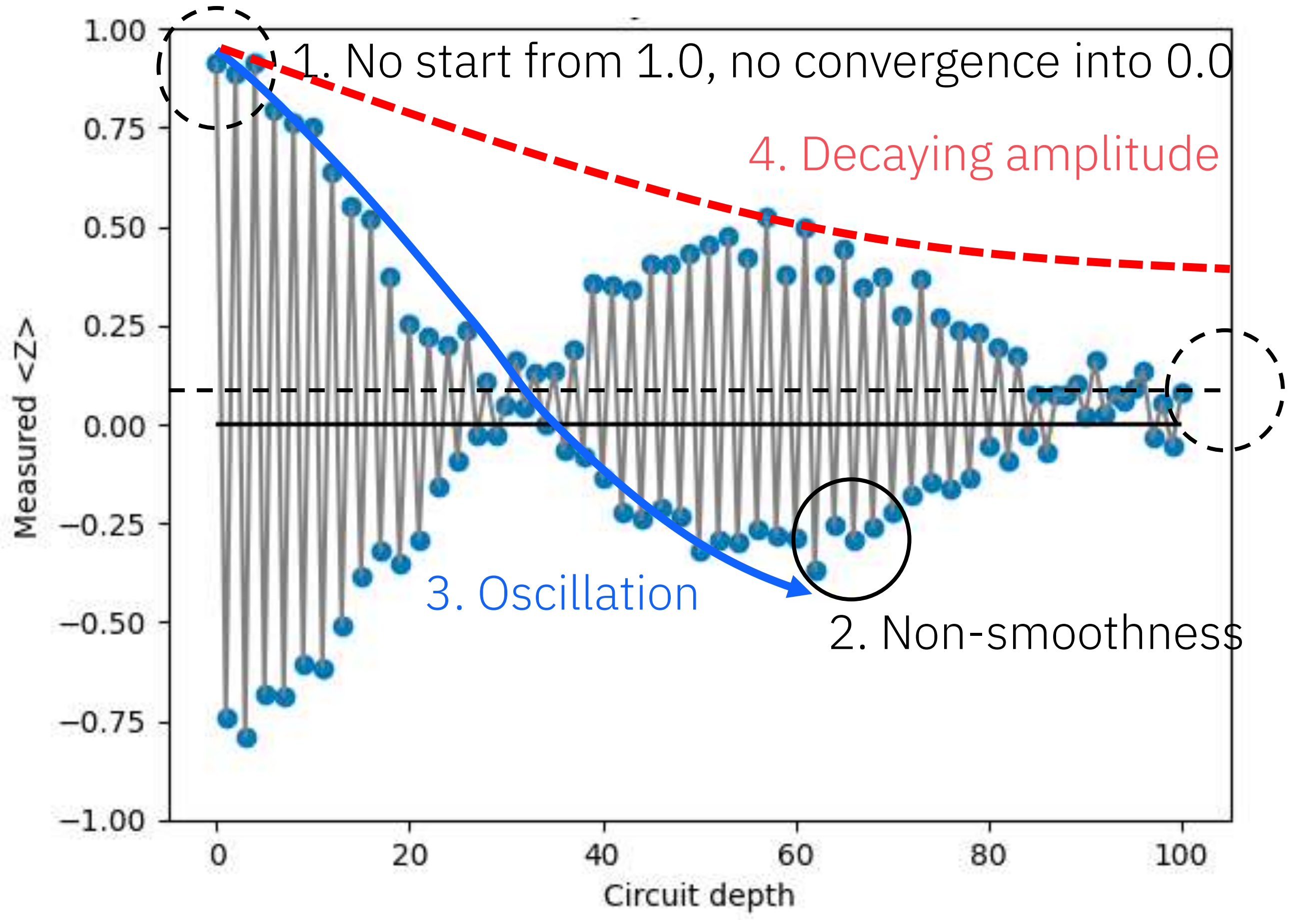
Plot $\langle Z \rangle = \langle \varphi | Z | \varphi \rangle = P(0) - P(1)$

Ideally, observe 1 and -1 alternatively



Quiz: What errors look like

Running on noisy quantum computer, we observe



Why?

Connect an observation with the error causing it by a line

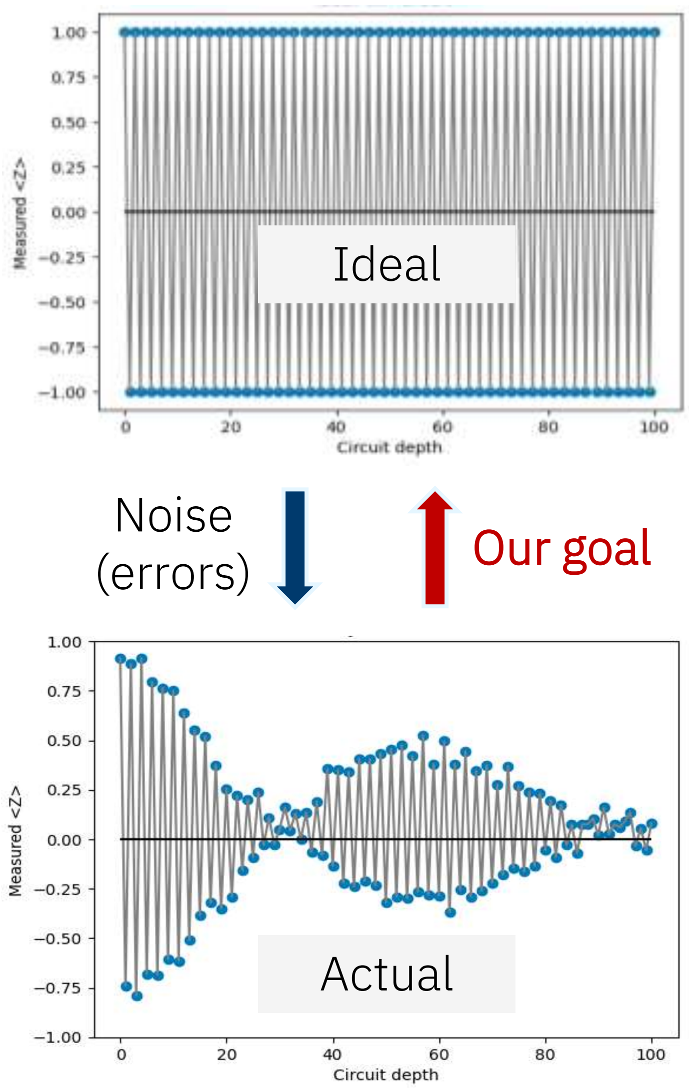
Observations:

1. Shrink/bias (d=0/∞)
2. Non-smoothness
3. Oscillation
4. Decay

Errors:

- Coherent
- Incoherent
- SPAM
- Sampling

Fight gate/measurement errors



- Gate and measurement errors are dominant in today's superconducting-qubit computers

Classification of errors (CS view)	Affected operations:
• Coherent error	Gates
• Incoherent error	Gates
• SPAM error	Resets & measurements
• Sampling error	Measurements

(SPAM: State Preparation And Measurement)

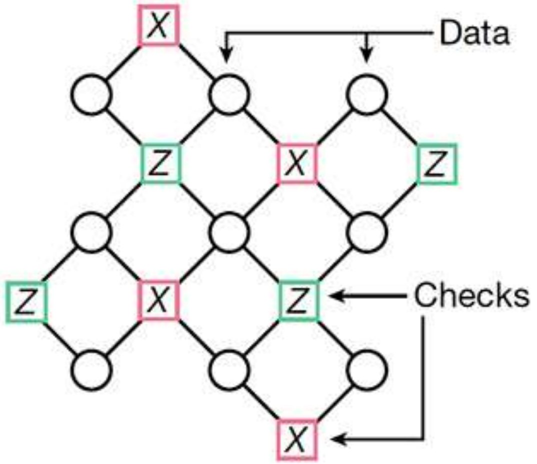
What you learn today

- Talk (30min)
  - What is quantum noise/error
  - **Error suppression and mitigation techniques**
    - TREX (Twirled Readout Error eXtinction)
    - ZNE (Zero Noise Extrapolation)
    - PEA (Probabilistic Error Amplification)
- Break
- Hands-on (20 min)
- Theory (Hard – 30min)
  - Formalism of quantum errors
    - Standard error channels, e.g. Pauli error channel
    - Quantum channel
    - PTM (Pauli Transfer Matrix) representation

Error correction or error mitigation?

How to deal with errors due to noise?

Quantum error correction (QEC)



Source: Fig. 1 in [1]

Monitor
Error occurs
Error detected

Correct in quantum computation (in real time)

Quantum error mitigation (QEM)

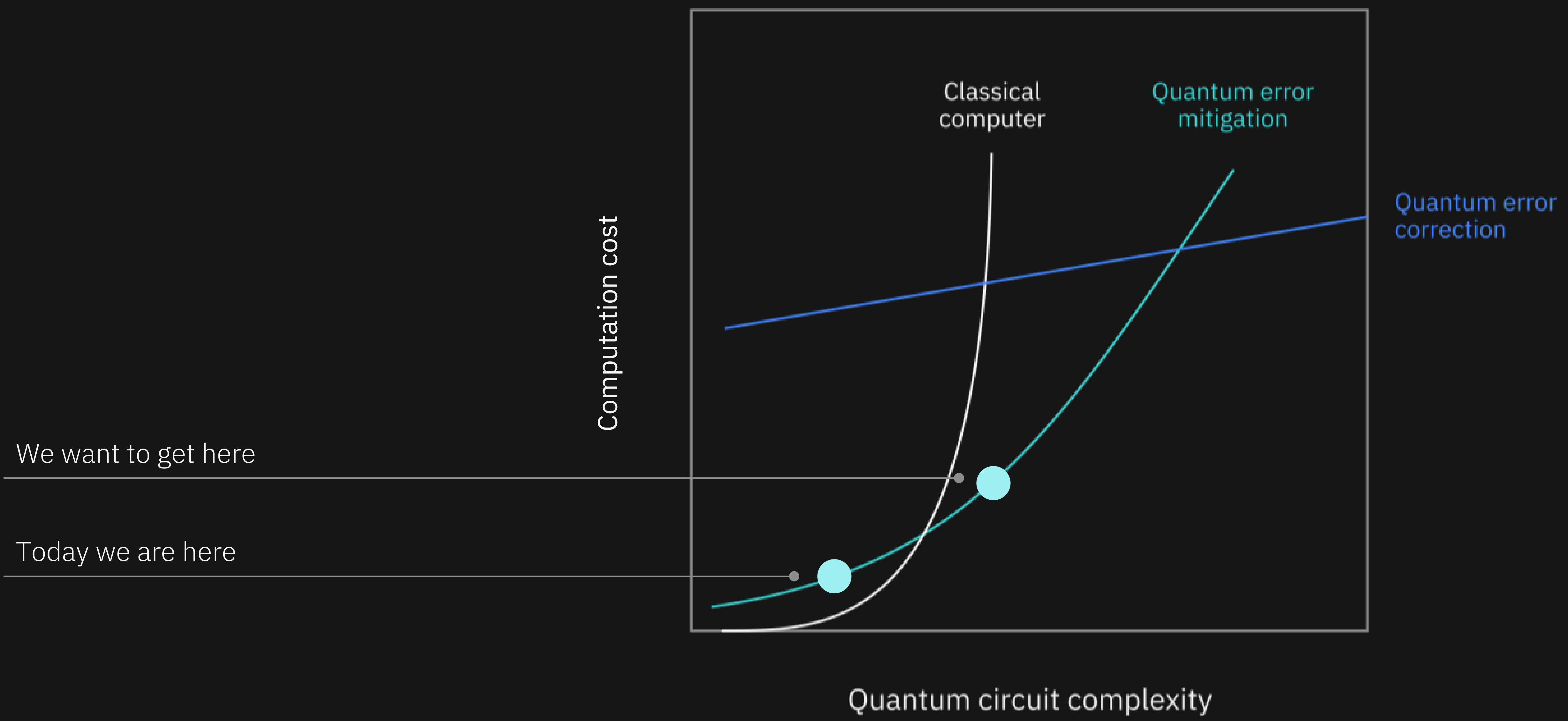
Source: [2]

No monitor
Error occurs
Error undetected

Estimate corrected with classical computation (by post processing)

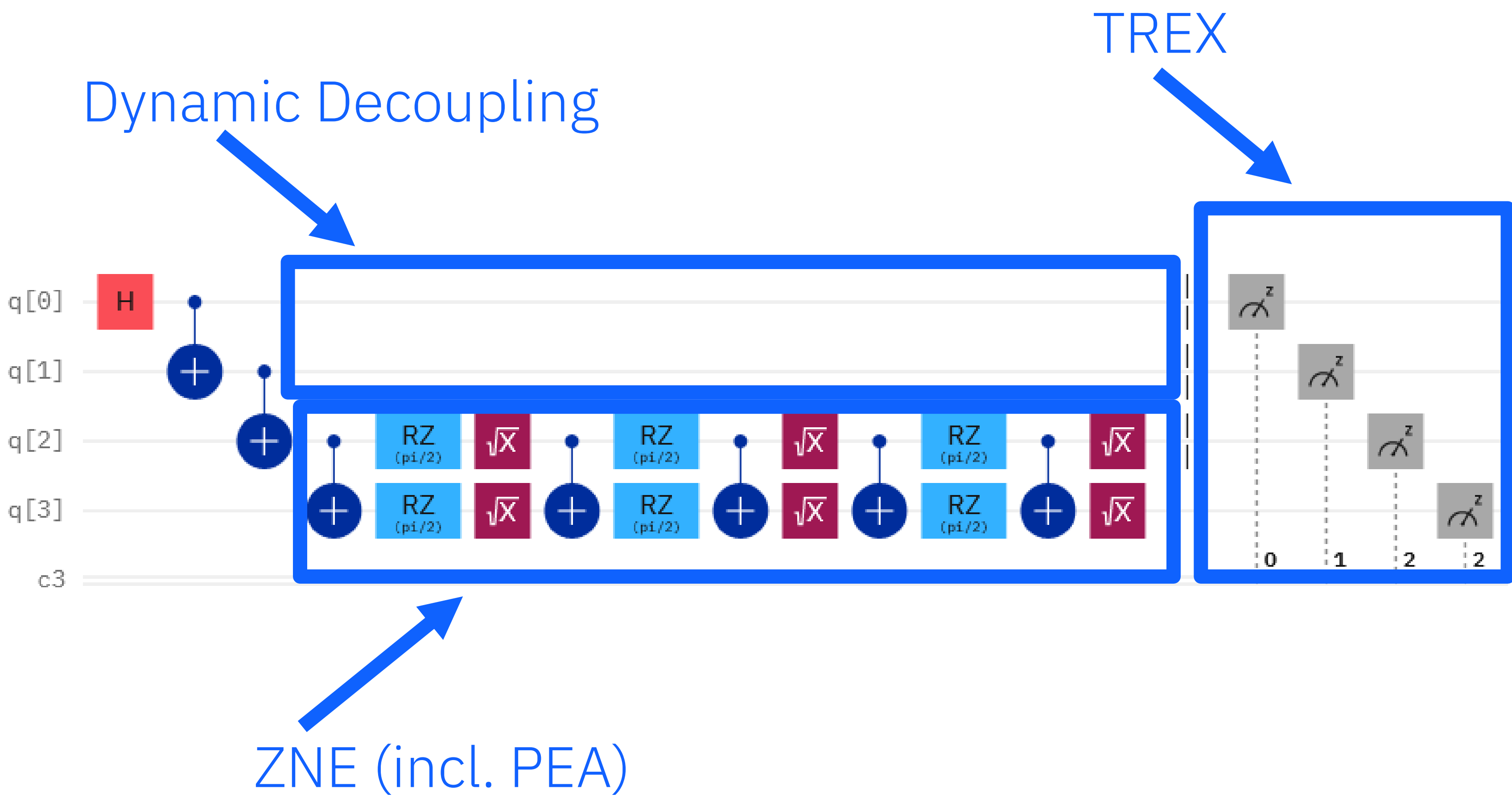
[1] Bravyi, S., Cross, A.W., Gambetta, J.M. *et al.* High-threshold and low-overhead fault-tolerant quantum memory. *Nature* **627**, 778–782 (2024).
[2] Minov Z., Probabilistic Error Cancellation with Sparse Pauli-Lindblad Models on Noisy Quantum Processors (https://www.youtube.com/watch?v=oPSBivh2rxQ)

Quantum Error Mitigation and Correction



Error suppression and mitigation techniques

- Different types of errors need different suppression and mitigation techniques.
- Different types of techniques can be combined!



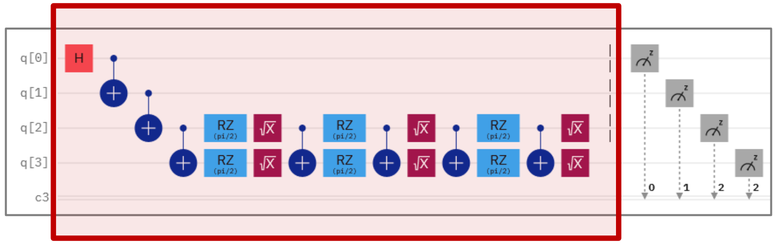
- TREX (Twirled Readout Error eXtinction)
- ZNE (Zero Noise Extrapolation)
- PEA (Probabilistic Error Amplification)

Error suppression and Error mitigation

Error suppression

Aim to reduce the error itself (in real time)

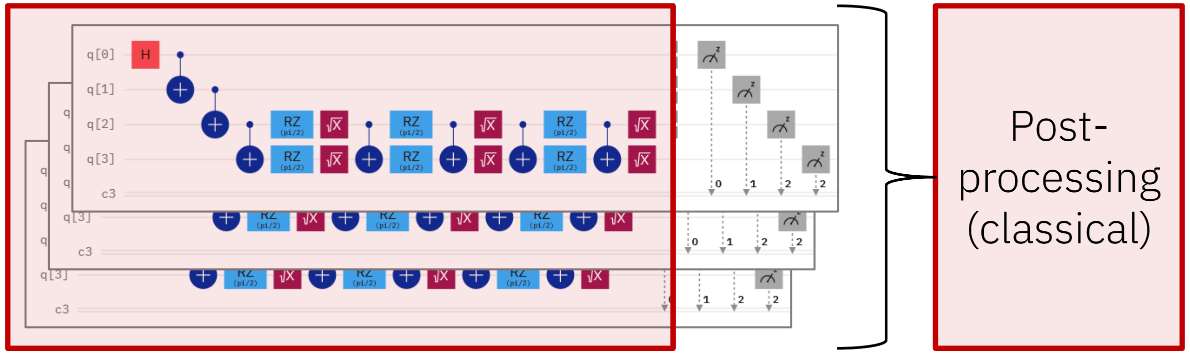
- Do something before measurement
- No change in the number of circuits
- Work even for a single shot

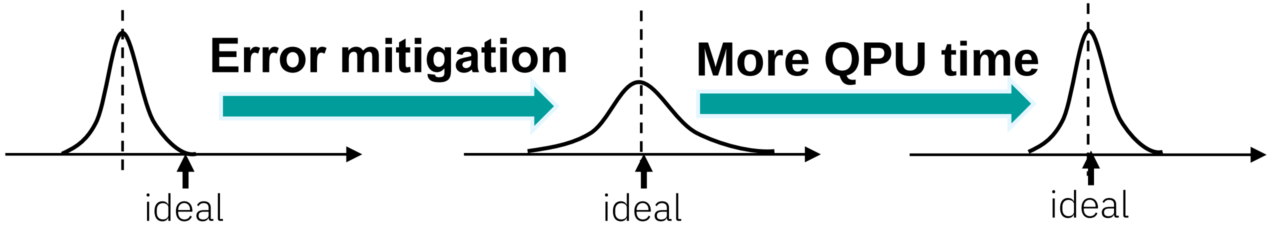


Error mitigation

Aim to recover the error-free result (with post-processing)

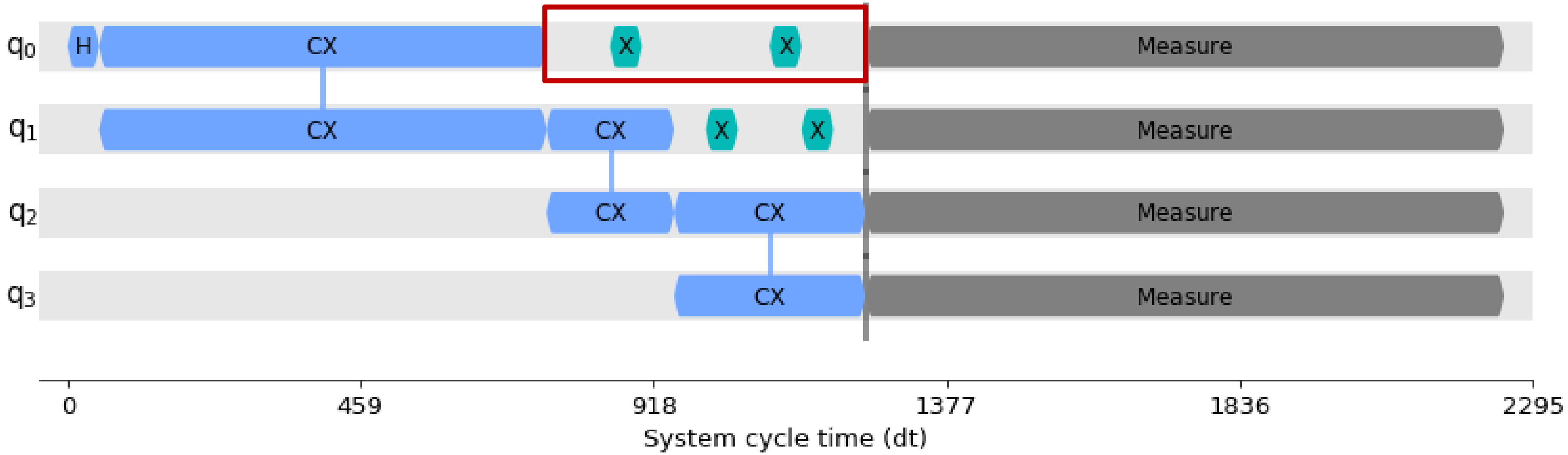
- Require classical post-processing
- Require more circuits to run
- Require multiple shots





Error suppression: Dynamical Decoupling (DD)

- **Suppress errors in qubit idling time** effectively
- Insert gates add up to the identity, e.g. X—X, X—Y—X—Y



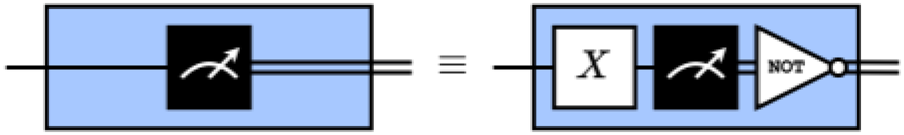
In this case, at least coherent Rz errors are cancelled out (assuming no errors on X gates):

$R_z(\theta)\, X\, R_z(2\theta)\, X\, R_z(\theta) = R_z(\theta)\, R_z(-2\theta)\, R_z(\theta) = I$ No DD → $R_z(4\theta)$ error

Twirled Readout Error eXtinction (TREX)

EV: Expectation Value

Idea:

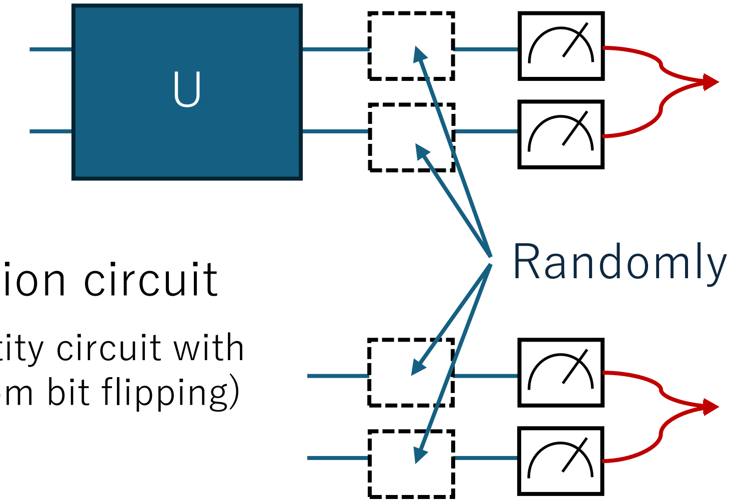


Focus on the computation of the EVs of Pauli observables composed only of Pauli I and Z for simplicity

Ex) EV of ZZ for $|\varphi\rangle = \mathrm{U}|00\rangle$, i.e. $\langle\varphi|ZZ|\varphi\rangle$

$$\langle ZZ\rangle = P(00) - P(01) - P(10) + P(11)$$

1. Original circuit (with random bit flipping)



Flip measured bit if X gate is inserted → $\langle\tilde{\varphi}|ZZ|\tilde{\varphi}\rangle$

Randomly insert X gate

2. Calibration circuit

(Identity circuit with random bit flipping)

Flip measured bit if X gate is inserted → $\langle\tilde{0}|ZZ|\tilde{0}\rangle$

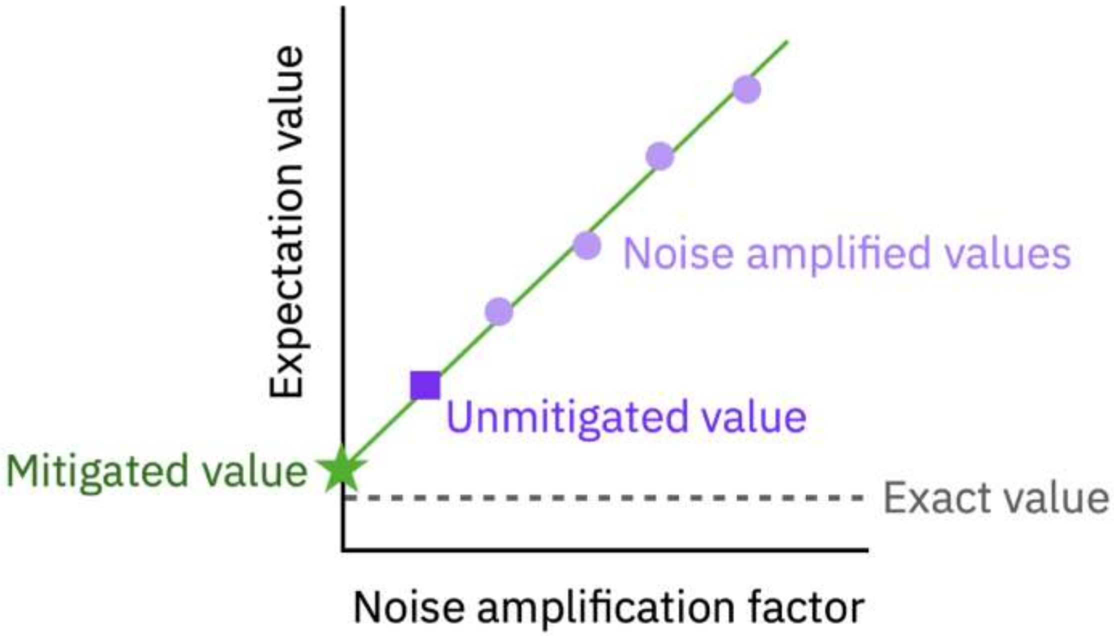
Mitigated EV = [EV from 1] / [EV from 2]

$\langle\tilde{\varphi}|ZZ|\tilde{\varphi}\rangle$ $\langle\tilde{0}|ZZ|\tilde{0}\rangle$

Van Den Berg, E., Minev, Z. K., & Temme, K. (2022). Model-free readout-error mitigation for quantum expectation values. *Physical Review A*, 105(3), 032620.

Zero Noise Extrapolation (ZNE)

Run multiple circuits with different gate error rates and extrapolate the expectation value at zero-noise point



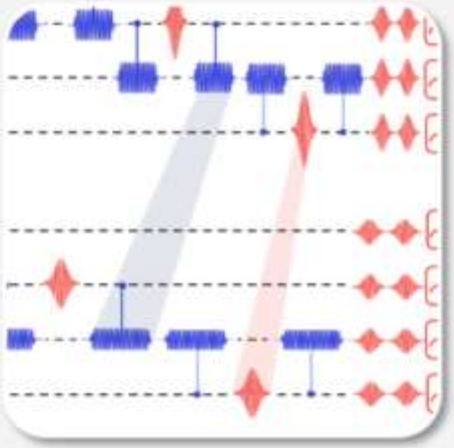
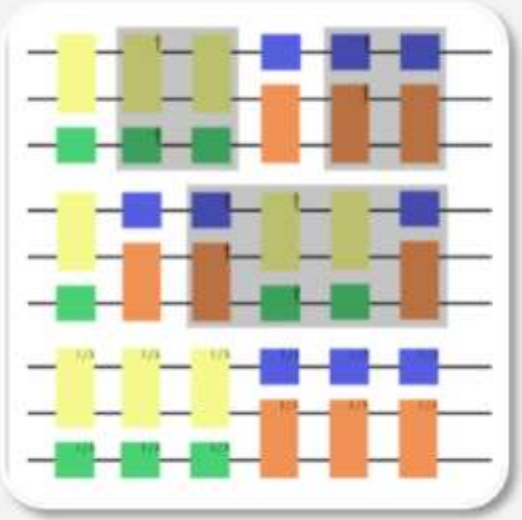
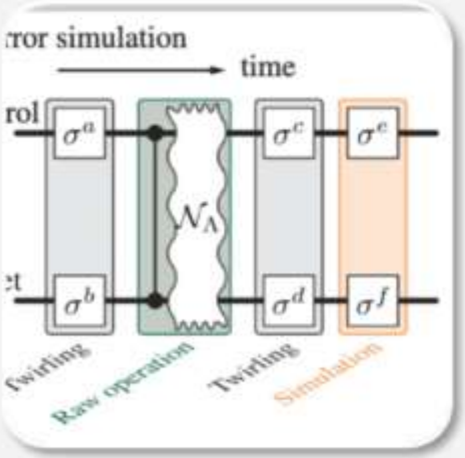
Options:

- Noise amplifier
- Noise factors e.g. [1.0, 1.2, 1.5], [1, 3, 5]...
- Extrapolator e.g. Linear, Quadratic, Exponential ...

https://github.com/Qiskit/qiskit-ibm-runtime/blob/stable/0.17/docs/tutorials/Error-Suppression-and-Error-Mitigation.ipynb

ZNE: Noise Amplification

- Pulse stretching assumes gate noise is proportional to duration, which is typically not true. Calibration is also costly.
- Gate folding requires large stretch factors that greatly limit the depth of circuits that can be run.
- PEA can be applied to any circuit that can be run with native noise factor (λ=1λ=1) but requires learning the noise model.
- You can write your own amplification!

Pulse Stretching	**Gate Folding**	**Probabilistic Error Amplification**
Scale pulse duration via calibration	Repeat gates in identity cycles $U \mapsto U(U^{-1}U)^{\lambda-1}/2$	Add noise via sampling Pauli channels
		
Kandala et al. Nature (2019)	Shultz et al. PRA (2022)	Li & Benjamin PRX (2017)

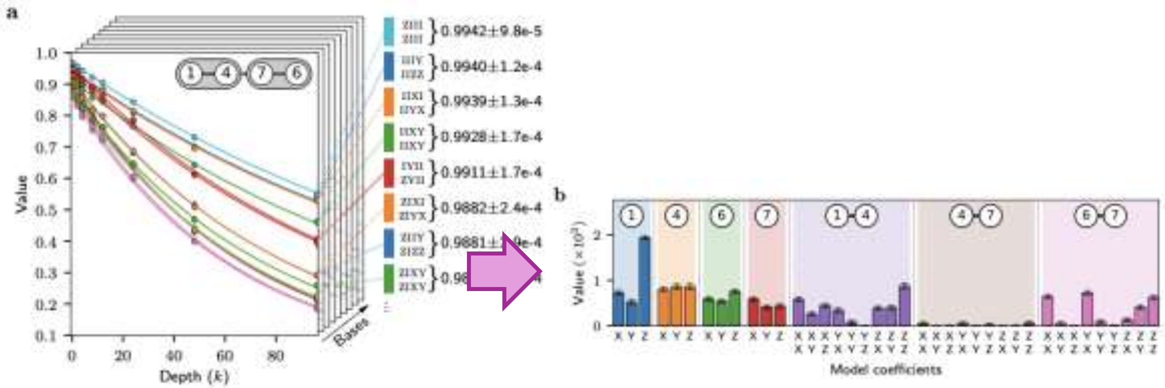
Probabilistic Error Amplification (PEA)

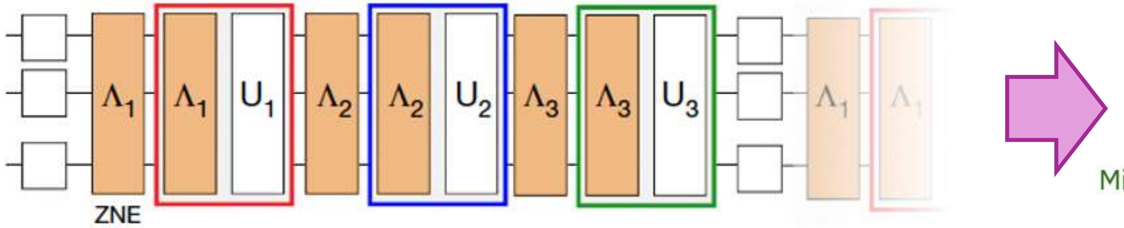
Pauli Twirling

1) Simplify noise: Gate noise → Pauli channel

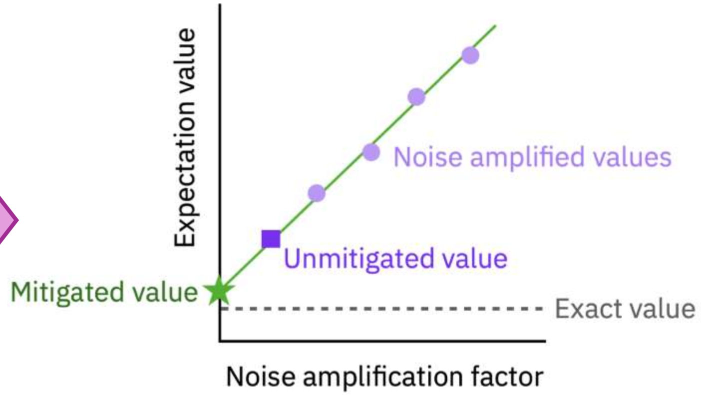
2) Learn noise (Estimate Pauli channel params)

3) Amplify noise + ZNE





Zero Noise Extrapolation (ZNE)



Kim, Y., Eddins, A., Anand, S., Wei, K. X., Van Den Berg, E., Rosenblatt, S., ... & Kandala, A. (2023). Evidence for the utility of quantum computing before fault tolerance. *Nature*, *618*(7965), 500-505.

Pauli Twirling

- Also called **randomized compiling**.
- Used to convert arbitrary noise channels into **Pauli channels**.
- Helps when dealing with coherent noise.
- Helps in the extrapolation stage of ZNE by making noise increase more or less monotonically.
- Often exclusively used on two qubit gates.

Clifford group Pauli group

$$\mathbf{C}_n = \{V \in U_{2^n} \mid V\mathbf{P}_n V^\dagger = \mathbf{P}_n\}$$

Clifford maps a Pauli to another Pauli by conjugation

$U\ P\ U^\dagger = P'$

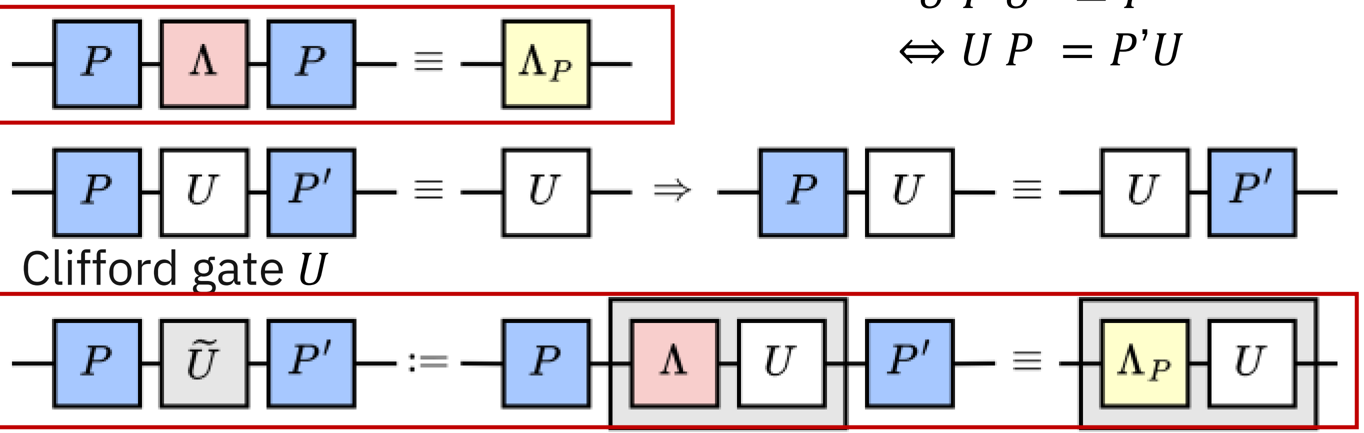
$\Leftrightarrow U\ P\ = P'U$

Random Pauli P

Goal:

Clifford gate U

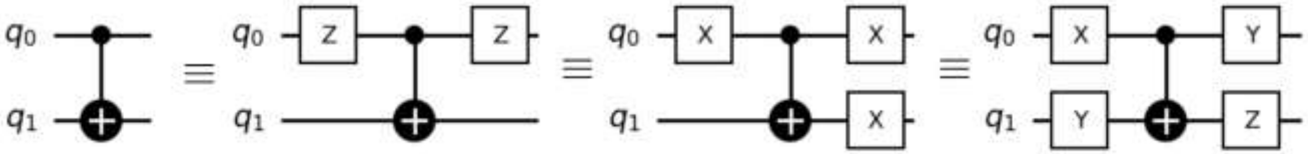
Implementation:



Partner Pauli P'

Ex) CNOT gate as U

P U P'



Break

We have a hands-on session next.
Please make sure to prepare your laptop.

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What you learn today

- Talk (35min)
  - What is quantum noise/error
  - Error suppression and mitigation techniques
    - TREX (Twirled Readout Error eXtinction)
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- Break
- Hands-on (20 min)
- **Theory (30min – Hard)**
  - **Formalism of quantum errors**
    - Standard error channels, e.g. Pauli error channel
    - Quantum channel
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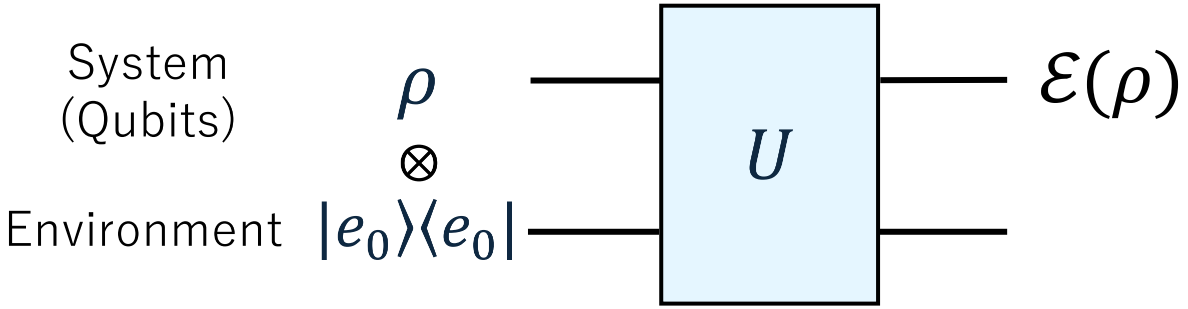
System-Environment representation of noise

- (Incoherent) error is from entanglement with environment



- Any quantum error on the system is fully characterized by U
  - Include coherent (unitary) error as a special case $U = U_s \otimes U_e$
- Difficult to describe the environment explicitly
  - Difficult to know U directly

Quantum Channel

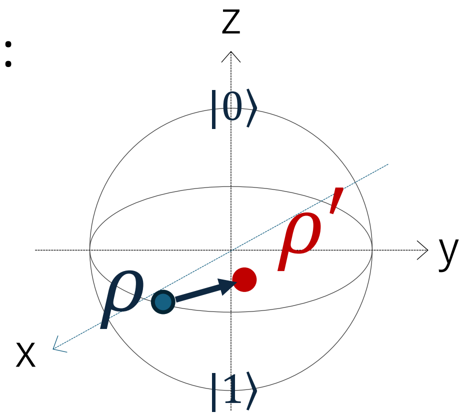
A linear map of a **quantum state** to another **quantum state**

Quantum Channel

A mixed state

Another mixed state

1-qubit case:



Recall we use density matrix to represent a mixed state

$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} = \rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| + \rho_{10}|1\rangle\langle 0| + \rho_{11}|1\rangle\langle 1|$$

$$= \frac{1}{2}\left(\mathrm{I} + r_x X + r_y Y + r_z Z\right)$$

N-qubit density matrix is 2^N by 2^N

Noisy gate is a quantum channel

Two ways to represent a noisy gate using quantum channel

Noisy gate

$$\rho \longrightarrow \boxed{\mathcal{E}_U} \longrightarrow \rho'' = \mathcal{E}_U(\rho)$$

$\Updownarrow$

Ideal gate + Noise

$$\rho \longrightarrow \boxed{U} \longrightarrow \boxed{\mathcal{E}} \longrightarrow \rho'' = \mathcal{E}(U\rho U^\dagger)$$

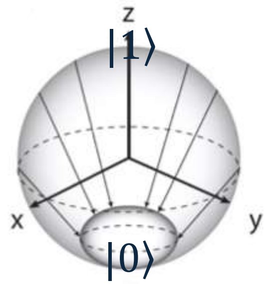
Ideal gate

$$\rho \longrightarrow \boxed{U} \longrightarrow \rho'$$

$$\rho' = U(\rho) = U\rho U^\dagger$$

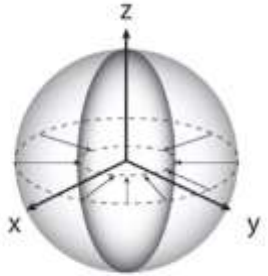
Common quantum errors

Incoherent errors



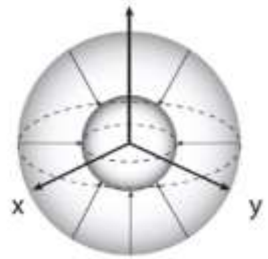
Amplitude damping error:

Relaxation error(|1> → |0>)



Phase damping error (dephasing):

Loss of phase information



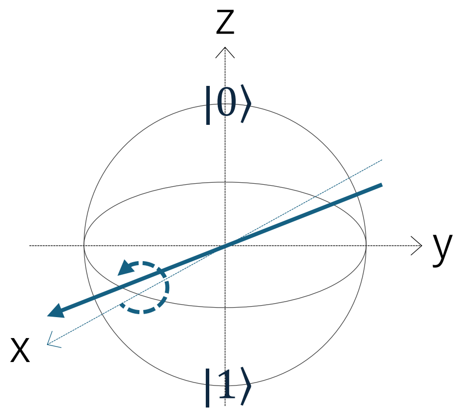
Depolarizing error:

Isotropic loss of purity (A special case of Pauli error)

Pauli error:

Different loss in X/Y/Z direction (see the next page for the details)

Coherent errors



Unitary error:

Miscalibration (over-/under-rotation)

$$\rho \mapsto U\rho U^\dagger$$

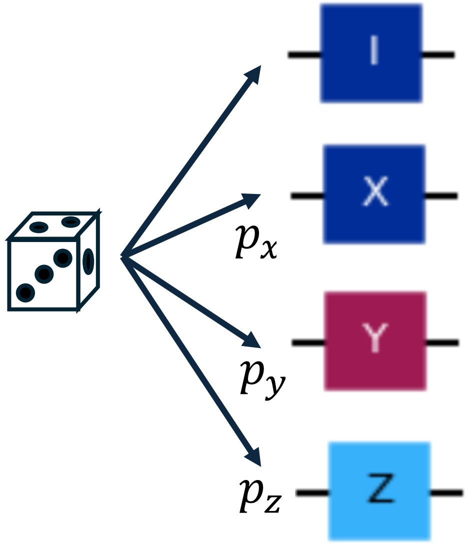
Figures cited from Quantum benchmarking - Petar Jurcevic - QCHS 2022 (https://www.youtube.com/watch?v=rM0iOVKqUFE)

Pauli error (channel)

1q-Paulis:

$$\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Random application of Pauli gates



1-qubit case:

- Apply X with probability p_x,
- Apply Y with probability p_y,
- Apply Z with probability p_z,
- Apply I with probability $1 - p_x - p_y - p_z$

2-qubit case:

- Apply XI, XX, XY, XZ with probability $p_{XI}, p_{XX}, p_{XY}, p_{XZ}$
- Apply YI, YX, YY, YZ with probability $p_{YI}, p_{YX}, p_{YY}, p_{YZ}$
- Apply ZI, ZX, ZY, ZZ with probability $p_{ZI}, p_{ZX}, p_{ZY}, p_{ZZ}$
- Apply IX, IY, IZ with probability p_{IX}, p_{IY}, p_{IZ}
- Apply II with the rest probability $1 - p_{IX} - p_{IY} - p_{IZ} - \cdots$

N-qubit case: 4^N random Pauli application (with $4^N - 1$ parameters)

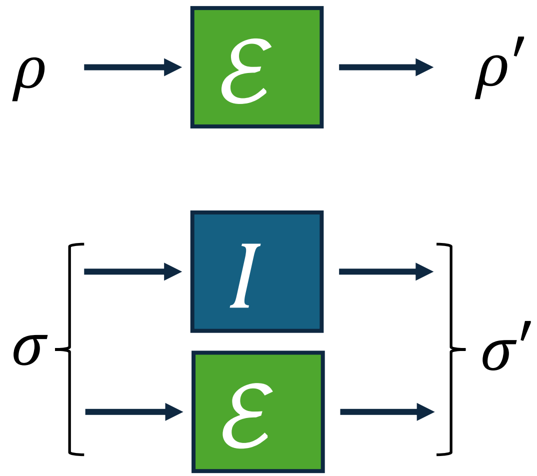
(Abbreviation for Pauli XY $= X \otimes Y$)

All $4^N - 1$ parameters are the same ➔ depolarizing error

[Theory] Quantum Channel := CPTP-map

Properties required for a mapping $\mathcal{E}$ between quantum states:

1. CP (Completely Positive): $\rho \succeq 0 \Rightarrow \mathcal{E}(\rho) \succeq 0 \quad \text{and} \quad \sigma \succeq 0 \Rightarrow (I \otimes \mathcal{E})(\sigma) \succeq 0$
2. TP (Trace Preserving): $\mathrm{tr}(\mathcal{E}(\rho)) = \mathrm{tr}(\rho) = 1$
3. Convex linear: $\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i)$



$\Updownarrow$ [Choi 1975]

$\mathcal{E}$ has a Kraus representation:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{s.t.} \quad \sum_k E_k^\dagger E_k = I$$

[Choi 1975] M. D. Choi. Completely Positive Linear Maps on Complex Matrices. Linear Algebrra Appl., 10(285), 1975.

Kraus (Operator-Sum) representation

Any quantum channel can be represented by

a set of operators (matrices) $\{E_k\}$ s.t. $\sum_k E_k^\dagger E_k = I$

$$\mathcal{E}^{\text{Kraus}}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{s.t.} \quad \sum_k E_k^\dagger E_k = I$$

$$\mathcal{E}^{\text{Kraus}} : \rho \mapsto \sum_k E_k \rho E_k^\dagger \quad \Leftrightarrow$$

(Physical interpretation)

Randomly take one of k states:

$\rho_k = \dfrac{E_k \rho E_k^\dagger}{\mathrm{tr}(E_k \rho E_k^\dagger)}$ with probability $\mathrm{tr}(E_k \rho E_k^\dagger)$

Ex. 1) Gate / Unitary evolution (special case: |k|=1)

$$\rho \mapsto U \rho U^\dagger \qquad U^\dagger U = I$$

Examples: Kraus (Operator-Sum) representation

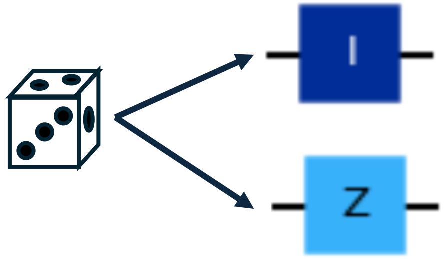
$$\mathcal{E}^{\text{Kraus}}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{s.t.} \quad \sum_k E_k^\dagger E_k = I$$

Ex. 2) Positive operator-valued measurement (POVM)

Projection onto computational basis (0 or 1)

$$E_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad E_2 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex. 3) (Probabilistic) Mixture of unitaries



50% Pauli I - 50% Pauli Z

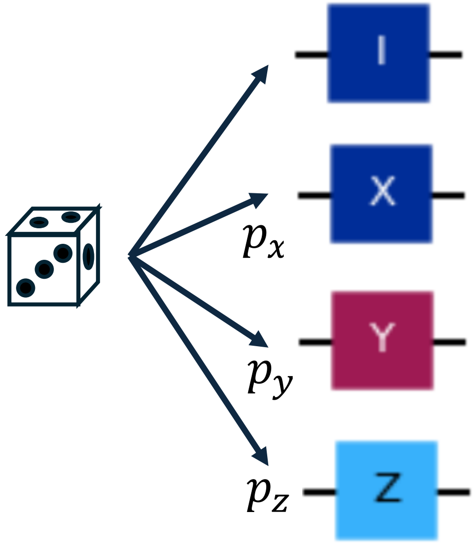
$$E_1 = \frac{1}{\sqrt{2}} I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_2 = \frac{1}{\sqrt{2}} Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Exercise: Kraus representation of 1q Pauli error?

Pauli error = Random application of Pauli gates

Ex) 1-qubit case

- Apply X with probability p_x,
- Apply Y with probability p_y,
- Apply Z with probability p_z,
- Apply I with probability $1 - p_x - p_y - p_z$



$$\mathcal{E}^{\mathrm{Kraus}}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{s.t.} \quad \sum_k E_k^\dagger E_k = I$$

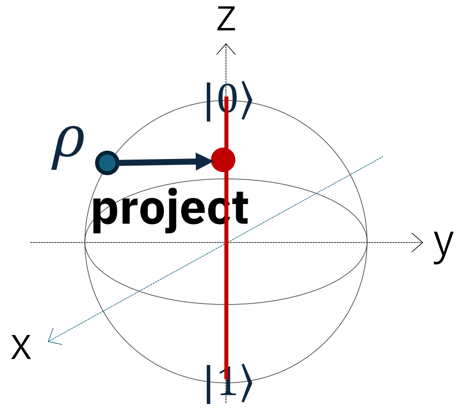
Describe Kraus operators representing 1q Pauli error above.

Answer: $\sqrt{p_x}\,X, \sqrt{p_y}\,Y, \sqrt{p_z}\,Z, \sqrt{1 - p_x - p_y - p_z}\,I$

Examples: Kraus (Operator-Sum) representation

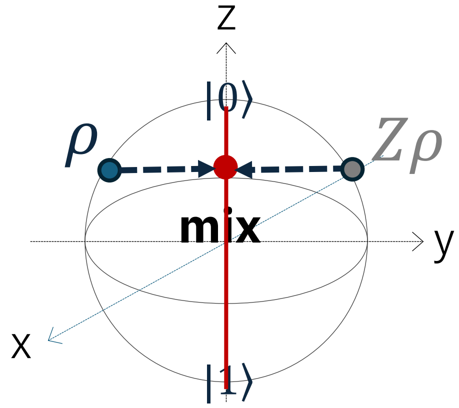
Ex. 2) Projection onto computational basis (0 or 1)

$$E_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad E_2 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



Ex. 3) Mixture of 50% Pauli I and 50% Pauli Z

$$F_1 = \frac{1}{\sqrt{2}} I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad F_2 = \frac{1}{\sqrt{2}} Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



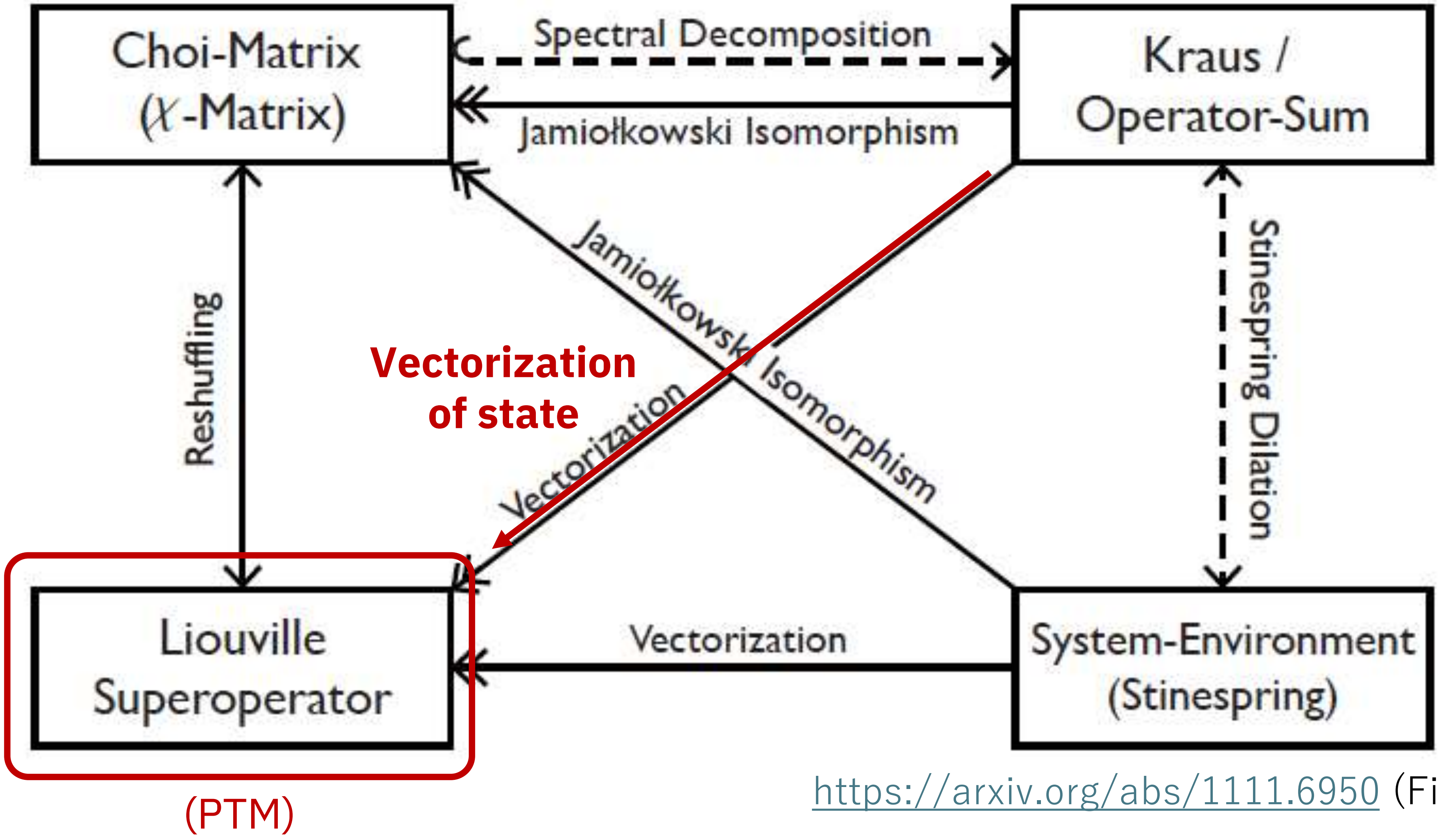
Those are the same channel! $E_1 \rho E_1^\dagger + E_2 \rho E_2^\dagger = F_1 \rho F_1^\dagger + F_2 \rho F_2^\dagger$

Projection to Z-axis

→ Kraus is not a unique representation!

Unique representation? ➔ Superoperator

Various representation of Quantum Channel



https://arxiv.org/abs/1111.6950 (Fig. 1)

See "Exploring Quantum Channels, Understanding Quantum Information & Computation: Lesson 10 by John Watrous https://www.youtube.com/watch?v=cMl-xIDSmXI" for the equivalence of Choi, Kraus and Stinespring representations)

Kraus to Superoperator transformation

Kraus (2^N x 2^N matrices)

$$\mathcal{E}^{\text{Kraus}}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{s.t.} \quad \sum_k E_k^\dagger E_k = I$$

$$\mathcal{E}^{\text{Kraus}} : \underline{\rho} \mapsto \sum_k E_k \rho E_k^\dagger$$

Matrix

Vectorize state (vec trick)

Liouville **Superoperator** (4^N x 4^N matrix)

$$\mathcal{E}^{\text{SuperOp}} = \sum_k \overline{E_k} \otimes E_k = \sum_k (E_k^\dagger)^T \otimes E_k$$

$$\mathcal{E}^{\text{SuperOp}} : \underline{\text{vec}(\rho)} \mapsto \boxed{\left\{ \sum_k \overline{E_k} \otimes E_k \right\} \text{vec}(\rho)}$$

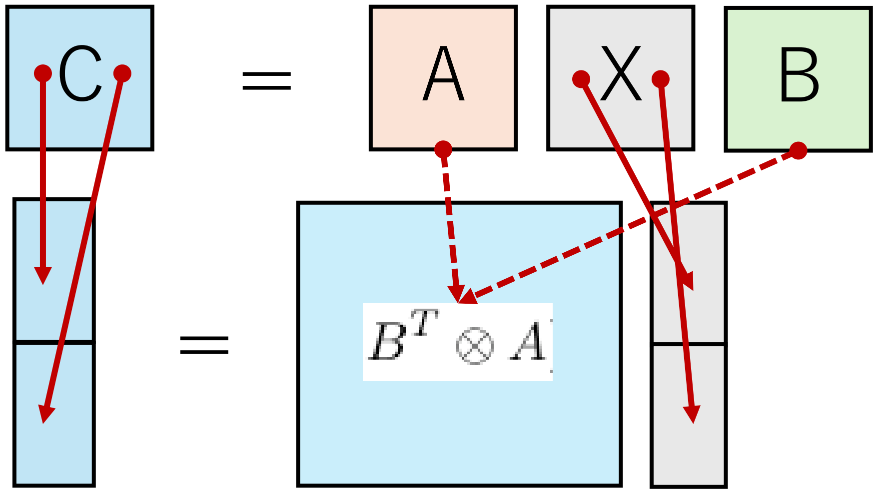
Vector

Vec trick:

$$C = A\,X\,B \Leftrightarrow \text{vec}(C) = (B^T \otimes A)\,\text{vec}(X)$$

$$\rho' = E_k\,\rho\,E_k^\dagger \Leftrightarrow \text{vec}(\rho') = ((E_k^\dagger)^T \otimes E_k)\,\text{vec}(\rho)$$

$\text{vec}(A)$ is also written as $|A\rangle\rangle$ in some literature



Equivalence check of two quantum channels

$$\mathcal{E}^{\mathrm{SuperOp}} = \sum_k \overline{E_k} \otimes E_k$$

Ex. 2) Projection onto computational basis (0 or 1)

$$E_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$E_2 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Kraus ⇒ SuperOp

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\parallel$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\parallel$$

Ex. 3) Mixture of 50% Pauli I and 50% Pauli Z

$$E_1 = \frac{1}{\sqrt{2}} I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_2 = \frac{1}{\sqrt{2}} Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Kraus ⇒ SuperOp

$$\frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Superoperator is a unique representation!

PTM: Pauli Transfer Matrix

PTM is a superoperator with different basis

$$\mathcal{E}^{SuperOp} \xrightarrow[\text{Change of basis } (c \to \sigma)]{} \mathcal{E}^{PTM}$$

c: Computational basis

$|0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1|$

$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} = \rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| \cdots$$

σ: Pauli basis

I X Y Z

$$\rho = \frac{1}{2}(\mathrm{I} + r_x X + r_y Y + r_z Z)$$

Ex) 1-qubit basis change unitary

$$T_{c\to\sigma} = \frac{1}{\sqrt{2}} \begin{array}{c|c} & \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} \\ \begin{matrix} \mathrm{I} \\ \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

- unique channel representation
- 4^N by 4^N matrix

Ex) 1-qubit Pauli channel

SuperOp

$$\begin{bmatrix} -p_x - p_y + 1 & 0 & 0 & p_x + p_y \\ 0 & -p_x - p_y - 2p_z + 1 & p_x - p_y & 0 \\ 0 & p_x - p_y & -p_x - p_y - 2p_z + 1 & 0 \\ p_x + p_y & 0 & 0 & -p_x - p_y + 1 \end{bmatrix}$$

PTM

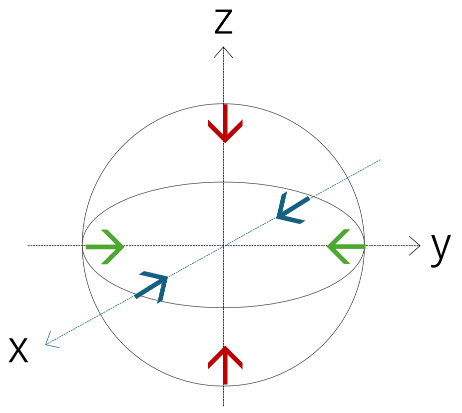
$$\begin{array}{c|c} & \begin{matrix} \mathrm{I} & \mathrm{X} & \mathrm{Y} & \mathrm{Z} \end{matrix} \\ \begin{matrix} \mathrm{I} \\ \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2(p_y+p_z) & 0 & 0 \\ 0 & 0 & 1-2(p_x+p_z) & 0 \\ 0 & 0 & 0 & 1-2(p_x+p_y) \end{bmatrix} \end{array}$$

PTM of Pauli channel

PTM of Pauli channel is a **diagonal** 4^N by 4^N matrix

Ex) PTM of 1-qubit Pauli error

$$
\begin{array}{c|c}
 & \begin{array}{cccc} \mathrm{I} & \mathrm{X} & \mathrm{Y} & \mathrm{Z} \end{array} \\
\begin{array}{c} \mathrm{I} \\ \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 \\
\boxed{0} & 1-2(p_y+p_z) & 0 & 0 \\
\boxed{0} & 0 & 1-2(p_x+p_z) & 0 \\
\boxed{0} & 0 & 0 & 1-2(p_x+p_y)
\end{bmatrix}
\end{array}
$$

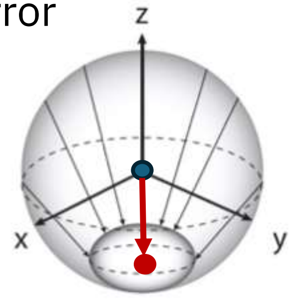


How much info on Z-axis will be kept
(1: Keep ←→0: Lost)

Where the origin shift
(No shift in Pauli channel)

Ref: PTM of Phase-amplitude damping (PAD) error

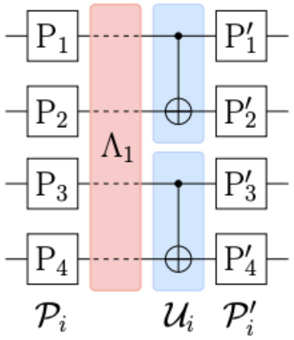
$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \sqrt{1-a-b} & 0 & 0 \\
0 & 0 & \sqrt{1-a-b} & 0 \\
a(1-2p_1) & 0 & 0 & 1-a
\end{bmatrix}
$$



a: amplitude damping parameter,
b: phase damping parameter,
p_1 : excited state population (ratio)

Cited from Quantum benchmarking - Petar Jurcevic - QCHS 2022

Pauli twirling



Source: Fig 1c in [1]

Used in the first step of PEA

Pauli Twirling

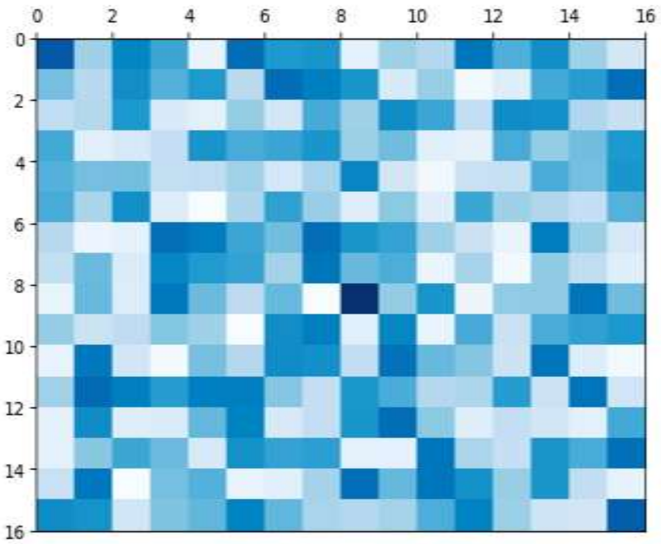
1) Simplify noise: Gate noise → Pauli channel

2) Learn noise

3) Amplify noise + ZNE

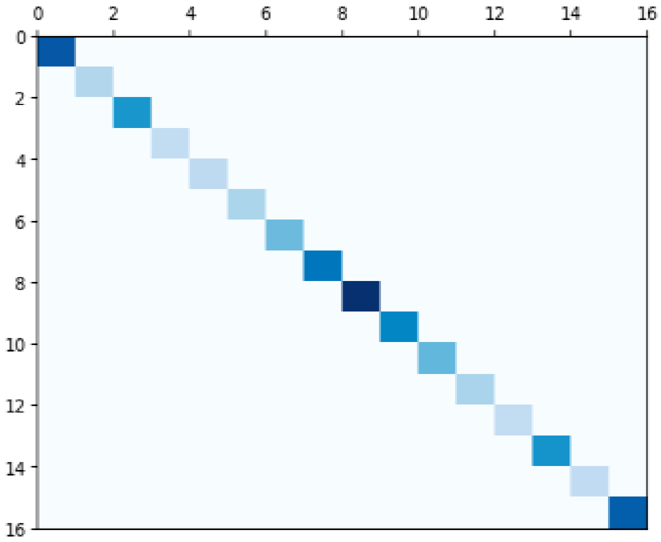
[1] Van Den Berg, E., Minev, Z. K., Kandala, A., & Temme, K. (2023). Probabilistic error cancellation with sparse Pauli–Lindblad models on noisy quantum processors. *Nature physics*, *19*(8), 1116-1121.

- Convert arbitrary error channels into Pauli channels
- PTM with off-diagonal elements → Diagonal PTM



Original gate error channel

Pauli twirling

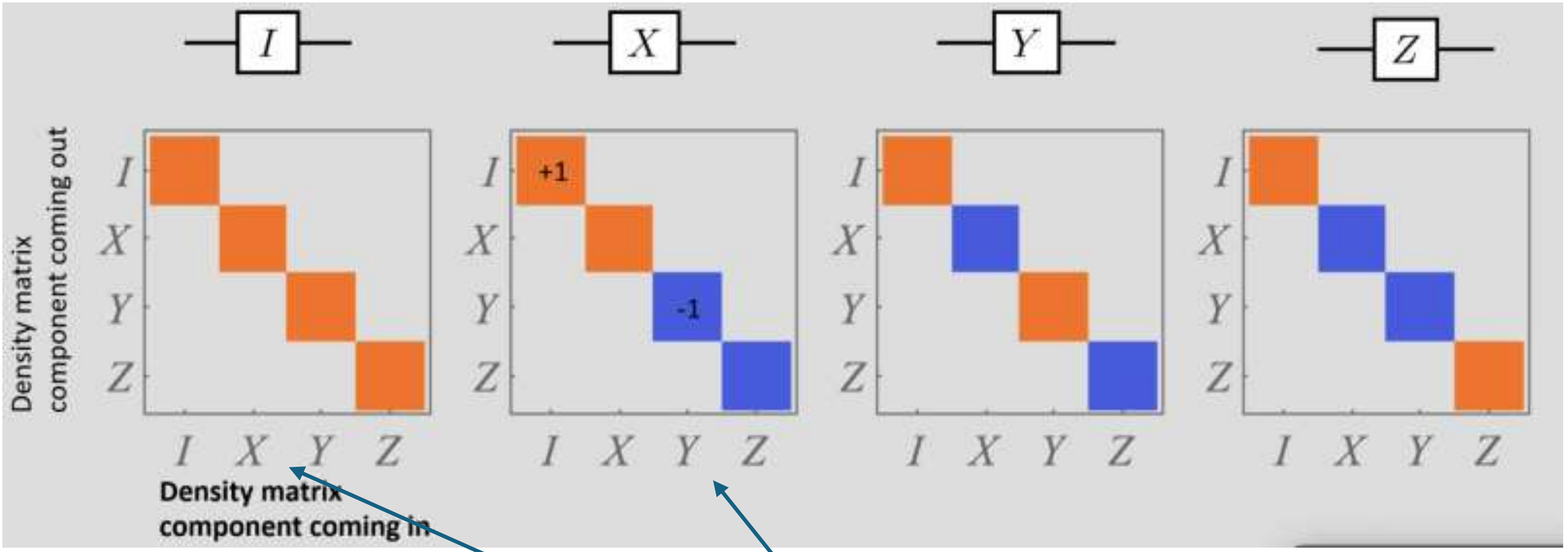


Pauli channel

Why Pauli twirling diagonalizes PTM? (1)

Zlatko Minov, A tutorial on tailoring quantum noise - Twirling 101 (https://www.zlatko-minev.com/blog/twirling)

Preparation: PTM of each Pauli gate



$p_x = p_y = p_z = 0$

$p_x = 1, p_y = p_z = 0$

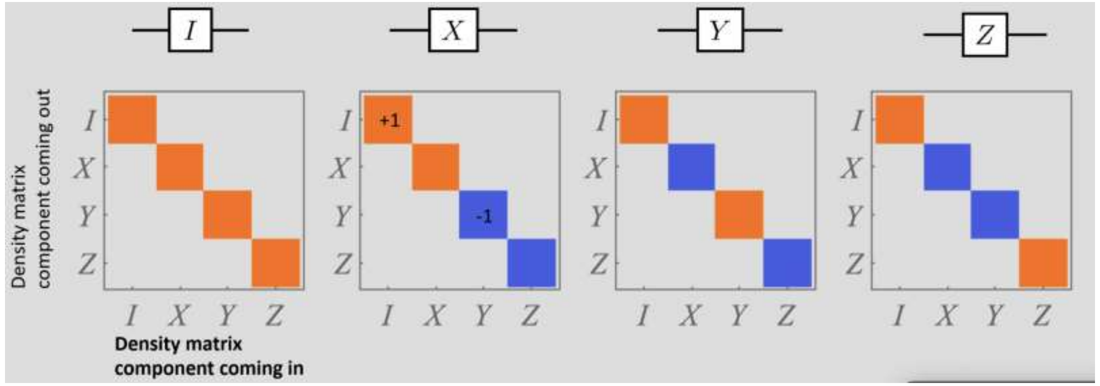
1q Pauli channel :

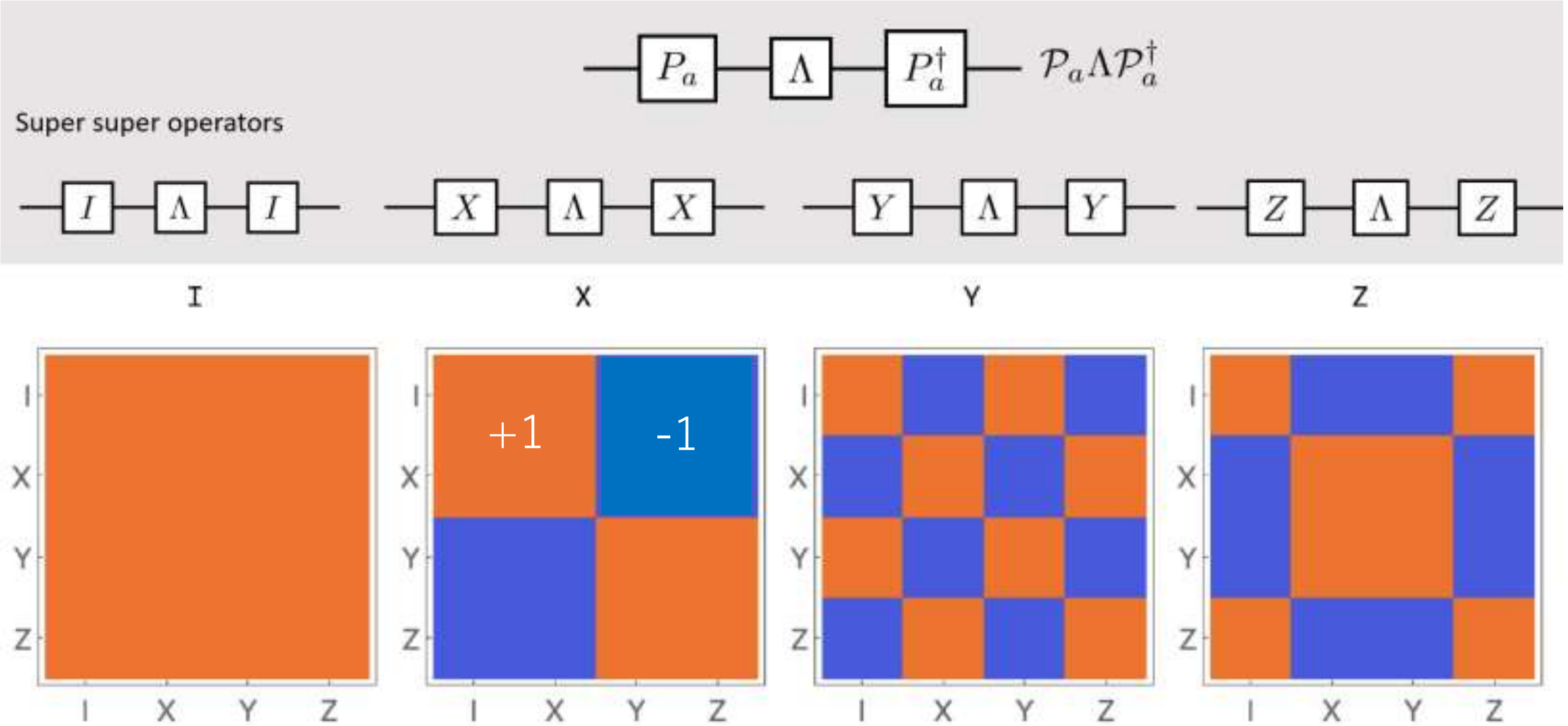
$$
\begin{array}{c c}
 & \begin{array}{cccc} I & X & Y & Z \end{array} \\
\begin{array}{c} I \\ X \\ Y \\ Z \end{array} &
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1-2(p_y+p_z) & 0 & 0 \\
0 & 0 & 1-2(p_x+p_z) & 0 \\
0 & 0 & 0 & 1-2(p_x+p_y)
\end{bmatrix}
\end{array}
$$

Why Pauli twirling diagonalizes PTM? (2)

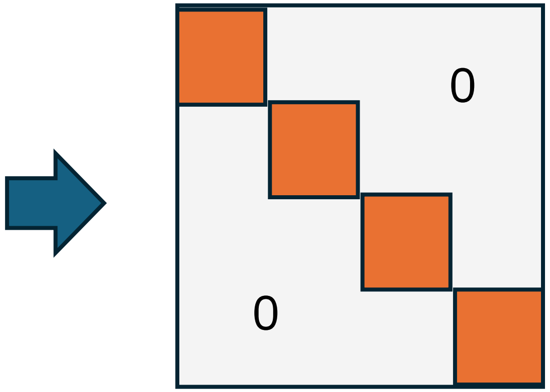
Zlatko Minov, A tutorial on tailoring quantum noise - Twirling 101
(https://www.zlatko-minev.com/blog/twirling)

PTM of each Pauli gate





Effective mask of Λ



Each Pauli pair works as a "mask" of error channel Λ (in terms of PTM)

References (Further reading)

- Introduction to Quantum Noise - Part 1 & 2 | Qiskit Global Summer School 2023
  Zlatko Minov
  - https://www.youtube.com/watch?v=3Ka11boCm1M
  - https://www.youtube.com/watch?v=gsKOx40gCUU

- Tensor networks and graphical calculus for open quantum systems
  Christopher J. Wood, Jacob D. Biamonte, David G. Cory
  - https://arxiv.org/abs/1111.6950

- A tutorial on tailoring quantum noise - Twirling 101
  Zlatko Minov
  - https://www.zlatko-minev.com/blog/twirling

- Exploring Quantum Channels | Understanding Quantum Information & Computation: Lesson 10
  John Watrous
  - https://www.youtube.com/watch?v=cMI-xIDSmXI **New** (Posted on June 6)

What you have learnt today

- What is quantum noise/error

- Error mitigation techniques
  - TREX (Twirled Readout Error eXtinction)
  - ZNE (Zero Noise Extrapolation)
  - PEA (Probabilistic Error Amplification)

- Formalism of quantum errors
  - Quantum channel
  - Standard error channels, e.g. Pauli error channel
  - PTM (Pauli Transfer Matrix) representation

Thank you

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