3. Quantum Teleportation / Superdense Coding

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Quantum Teleportation / Superdense Coding

Understanding typical quantum advantages with **small number of qubits.**

- **Quantum State Tomography**
- **Quantum Teleportation**
- **Quantum Superdense Coding**
- **Qiskit Hands-on.**
  - If you didn't install Qiskit in your laptop, please install it.
  - https://docs.quantum.ibm.com/start/install

Density matrix

Arbitrary quantum state :

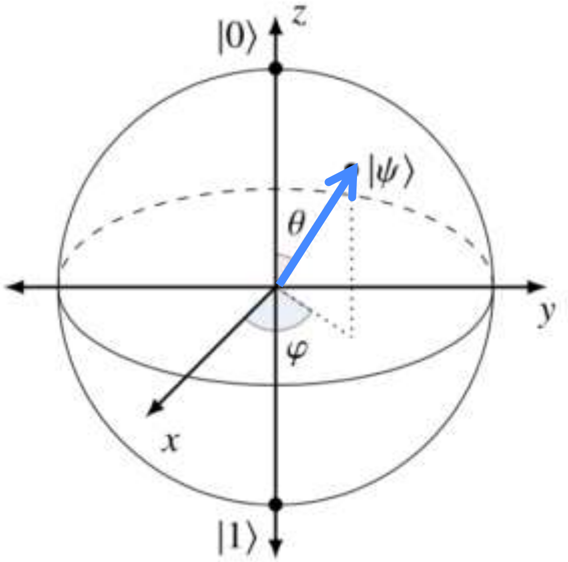
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1$$

(α, β :Probability amplitude)

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + exp(i\varphi)\, sin\left(\frac{\theta}{2}\right)|1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \mathrm{e}^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

Statevector



Bloch sphere

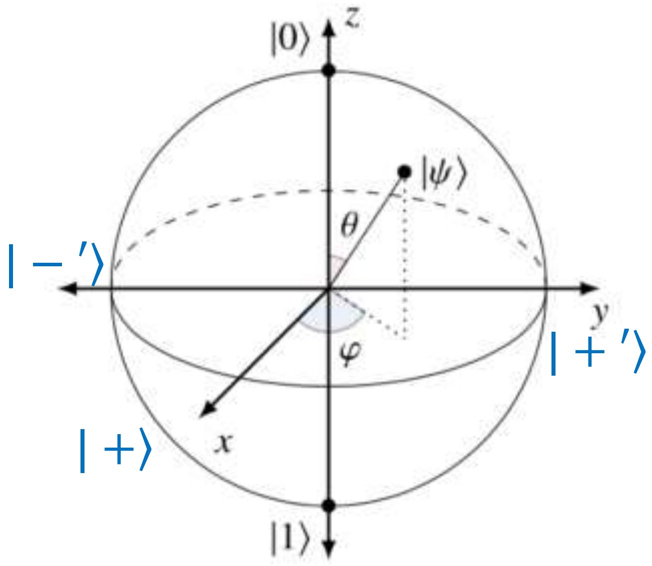
We can also denote the quantum state with its **density matrix ρ.**

$$\rho \equiv |\psi\rangle\langle\psi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ \mathrm{e}^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \mathrm{e}^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 + cos\theta & \mathrm{e}^{-i\varphi}\sin\theta \\ \mathrm{e}^{i\varphi}\sin\theta & 1 - cos\theta \end{pmatrix}$$

(Use the double angle formula)

Density Matrix and Bloch vector

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \cdot \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & 1-cos\theta \end{pmatrix}$$



Density matrix ρ is a linear summation of Pauli matrices X, Y, Z :

$$\rho = \frac{1}{2}(\mathrm{I} + (sin\theta\ cos\varphi)\mathrm{X} + (sin\theta\ sin\varphi)\mathrm{Y} + (cos\theta)\mathrm{Z})$$
$$= \frac{1}{2}\left(\mathrm{I} + r_x X + r_y Y + r_z Z\right)$$

Note : $\mathrm{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\mathrm{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\mathrm{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\mathrm{X} = |+\rangle\langle+| - |-\rangle\langle-|$ $\mathrm{Y} = |+'\rangle\langle+'| - |-'\rangle\langle-'|$ $\mathrm{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$|+'\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$$
$$|-'\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$$

Bloch vector: $\boldsymbol{r} = \left(r_x, r_y, r_z\right)$

In a pure quantum state, this Bloch vector is mapped to a point on the Bloch sphere.

Density matrix and Global phase

Suppose that $|\psi\rangle$ and $|\phi\rangle$ are unit vectors representing quantum states:

$$|\phi\rangle = \alpha|\psi\rangle \quad s.t. |\alpha| = 1$$

Then, $|\psi\rangle$ and $|\phi\rangle$ are equal up to a global phase.

Example: $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ and $-\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

In the description of the density matrix, ρ_ψ and ρ_ϕ are the same:

$$\rho_\phi = |\phi\rangle\langle\phi| = \alpha\alpha^*|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho_\psi$$

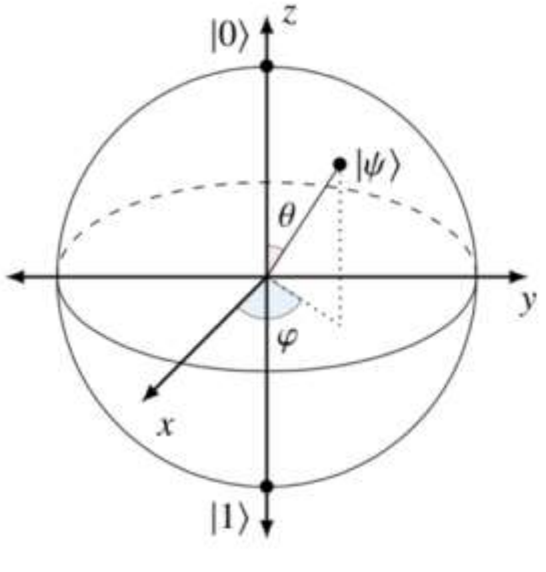
Density matrix can uniquely represent quantum states.

Quantum State Tomography

If you just measure the quantum state in the computational basis ($|0\rangle$, $|1\rangle$), the phase information (the complex number information) will be lost.

But if we have many copies of $|\psi\rangle$ by repeating the experiment, we can estimate the density matrix ρ by estimating the components of Bloch vector $\left(r_x, r_y, r_z\right)$.

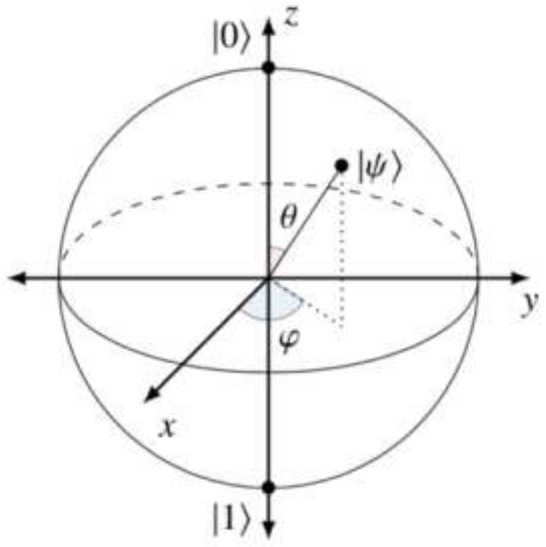
$$\rho = \frac{1}{2}(\mathbf{I} + (sin\theta\ cos\varphi)\mathbf{X} + (sin\theta\ sin\varphi)\mathbf{Y} + (cos\theta)\mathbf{Z})$$
$$= \frac{1}{2}\left(\mathbf{I} + r_x\mathbf{X} + r_y\mathbf{Y} + r_z\mathbf{Z}\right)$$



$$\mathrm{Tr}(\mathbf{X}\,\rho) = r_x, \quad \mathrm{Tr}(\mathbf{Y}\,\rho) = r_y, \quad \mathrm{Tr}(\mathbf{Z}\,\rho) = r_z$$

Quantum State Tomography

$$\rho = \frac{1}{2}(\mathbf{I} + (sin\theta\ cos\varphi)\mathbf{X} + (sin\theta\ sin\varphi)\mathbf{Y} + (cos\theta)\mathbf{Z})$$
$$= \frac{1}{2}\left(\mathbf{I} + r_x\mathbf{X} + r_y\mathbf{Y} + r_z\mathbf{Z}\right)$$



$$\mathrm{Tr}(\mathbf{X}\,\rho) = r_x, \quad \mathrm{Tr}(\mathbf{Y}\,\rho) = r_y, \quad \mathrm{Tr}(\mathbf{Z}\,\rho) = r_z$$

$$\begin{aligned}\mathrm{Tr}(\mathbf{Z}\,\rho) &= \langle 0|\mathbf{Z}\rho|0\rangle + \langle 1|\mathbf{Z}\rho|1\rangle \\ &= \langle 0|(|0\rangle\langle 0| - |1\rangle\langle 1|)\rho|0\rangle + \langle 1|(|0\rangle\langle 0| - |1\rangle\langle 1|)\rho|1\rangle \\ &= \langle 0|\rho|0\rangle - \langle 1|\rho|1\rangle \\ &= \langle 0|\psi\rangle\langle\psi|0\rangle - \langle 1|\psi\rangle\langle\psi|1\rangle \\ &= |\alpha|^2 - |\beta|^2\end{aligned}$$

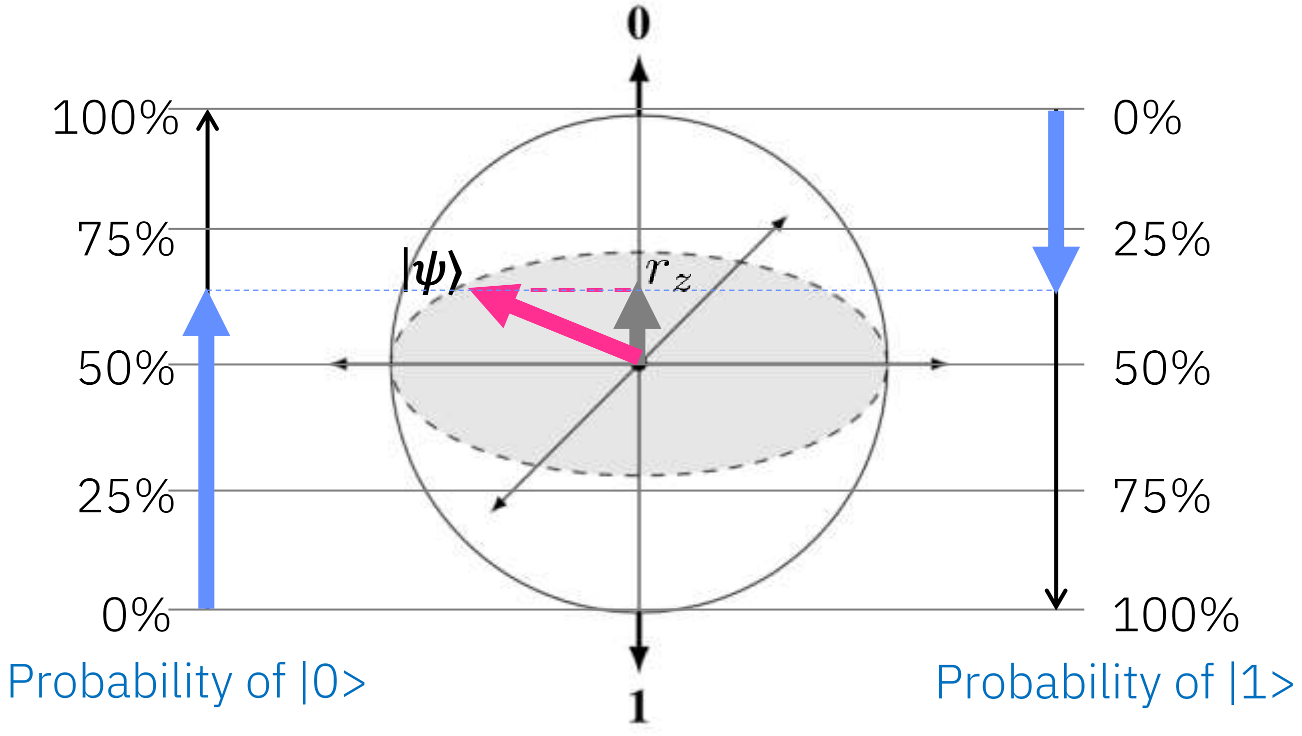
in case of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

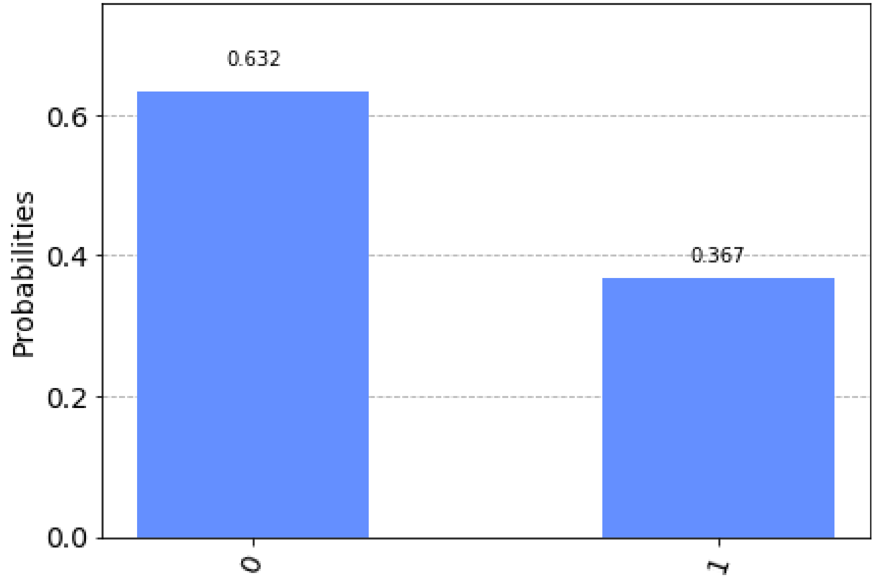
Note :
$\mathbf{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$

r_z = Probability of |0> – Probability of |1>

Quantum State Tomography

To estimate r_z, we create a quantum state and measure it and then repeat it many times, and then we will take the statistics of the measurement.





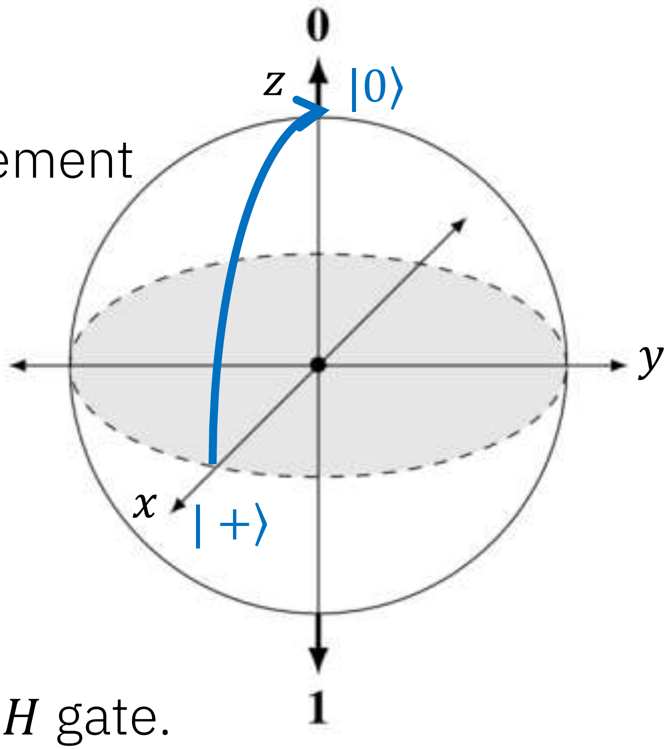
r_z = Probability of |0> – Probability of |1>

$r_z = 0.632 - 0.367 = 0.265$

Measurement on X-axis and Y-axis

IBM Quantum Systems support only the computational basis measurement (Z-axis measurement).
We can realize X-axis measurement and Y-axis measurement by rotation of axis.

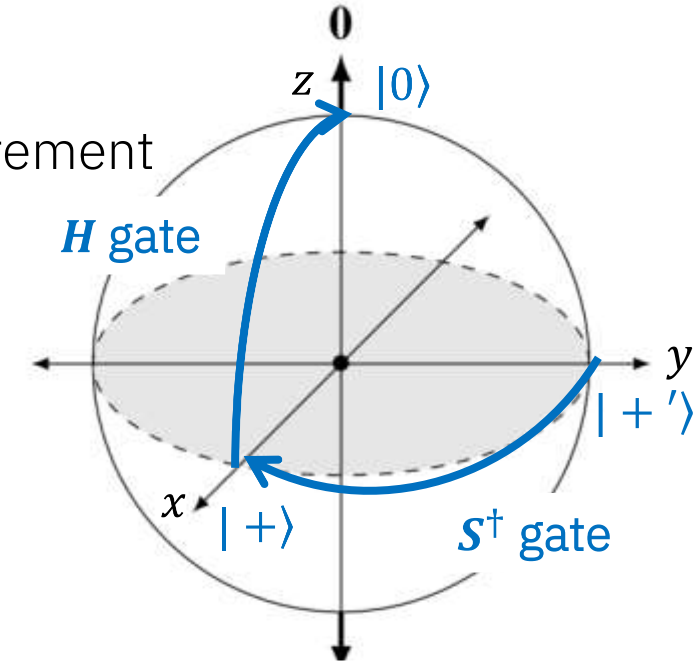
X-axis measurement



Applying H gate.

$$H|+\rangle = |0\rangle$$
$$H|-\rangle = |1\rangle$$

Y-axis measurement



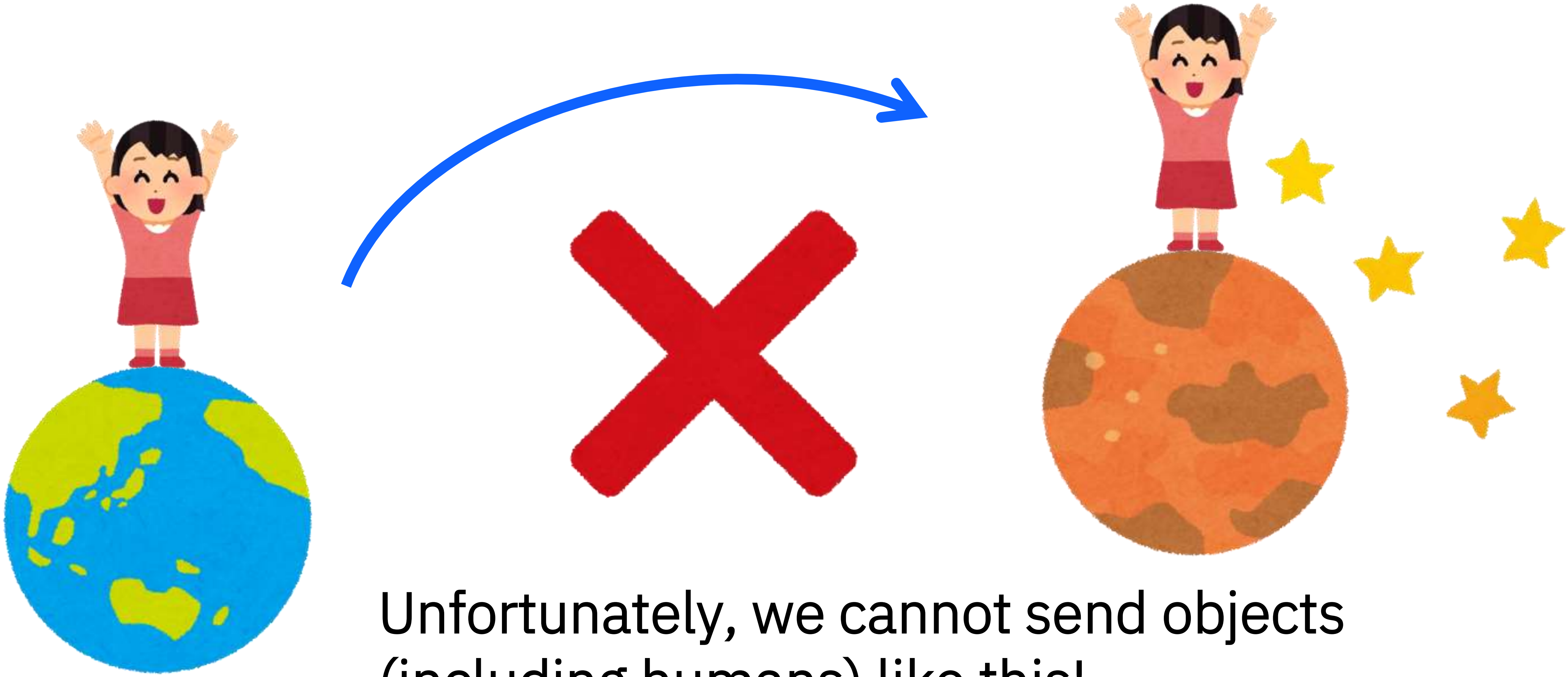
Applying $S^\dagger$ gate and H gate.

$$HS^\dagger|+'\rangle = H|+\rangle = |0\rangle$$
$$HS^\dagger|-'\rangle = H|-\rangle = |1\rangle$$

What do you think quantum teleportation is?

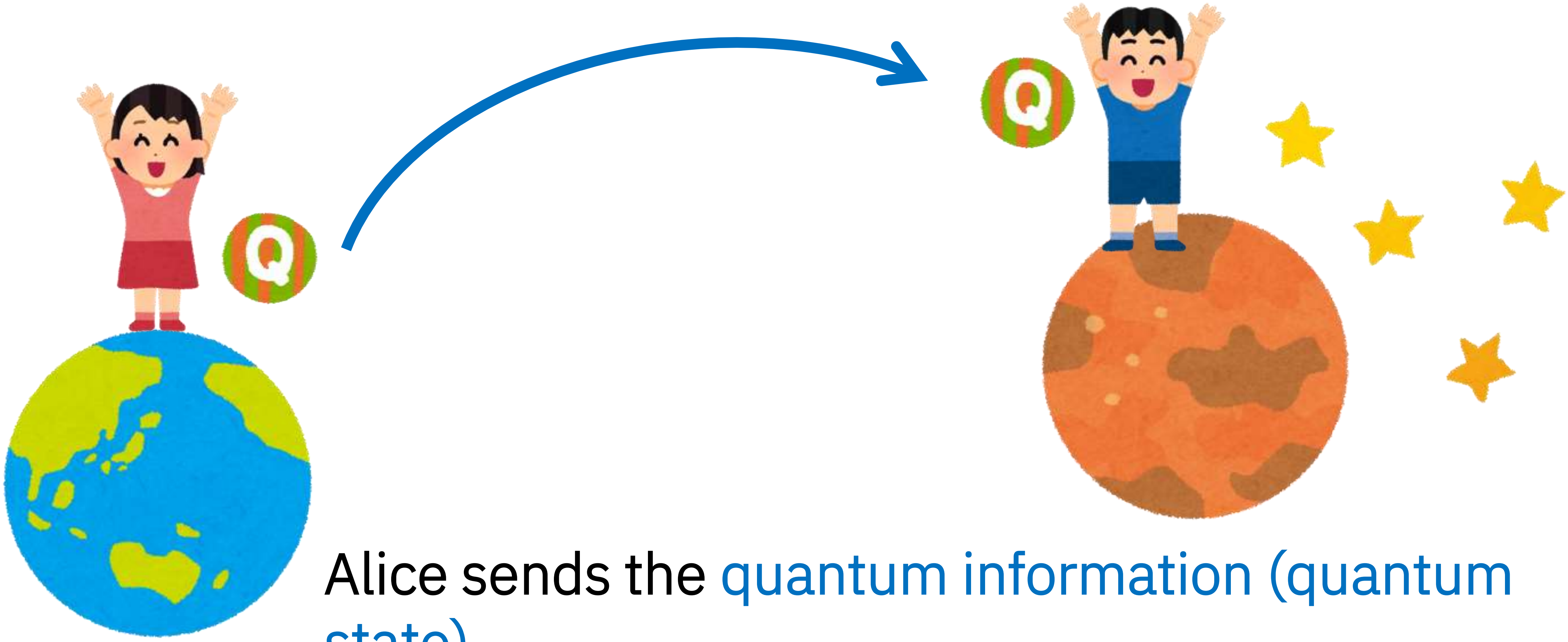
What is Quantum Teleportation? (Expected Example)

What is Quantum Teleportation?



Unfortunately, we cannot send objects (including humans) like this!

What is Quantum Teleportation?



Alice sends the quantum information (quantum state).

Quantum Teleportation

Let us consider the situation when Alice wants to send **an unknown quantum state $|\boldsymbol{\psi}\rangle$** to Bob who is far away, but they can only communicate with classical communication (emails or phone).

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

If Alice has a lot of $|\psi\rangle$, she can know α, β by quantum state tomography and she can inform Bob.

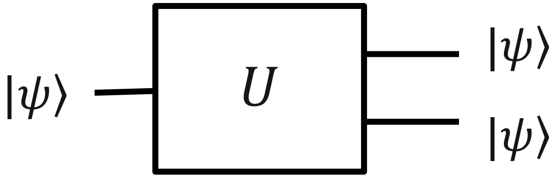
However, this method cannot be used in general because of No-cloning theorem that unknown quantum states cannot be duplicated.

No-cloning theorem

It is not possible to make a copy of an unknown quantum state.

Suppose that Unitary operation U can make a copy of the quantum state $|\psi\rangle$.

Input:
unknown quantum state
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



Output
$|\psi\psi\rangle$

This U works for not only an unknown quantum state $|\psi\rangle$ but also $|0\rangle$ and $|1\rangle$, so

$$U|0\rangle = |00\rangle, \quad U|1\rangle = |11\rangle$$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, therefore $U|\psi\rangle = \alpha U|0\rangle + \beta U|1\rangle = \alpha|00\rangle + \beta|11\rangle$

U can also copy $|\psi\rangle$, therefore $U|\psi\rangle = |\psi\rangle|\psi\rangle = \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$

However, there is no α, β that satisfies these two conditions.

Quantum Teleportation

Alice wants to send an unknown quantum state $|\psi\rangle$ to Bob who is far away, but they can only communicate with classical communication.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

If Alice has a lot of $|\psi\rangle$, she can get α, β by **quantum state tomography** and she can inform Bob. According to the **non-cloning theorem**, this cannot be used for general quantum states.

But, when they share an EPR pair, Alice can teleport the unknown quantum state to Bob by local operations and classical communication.

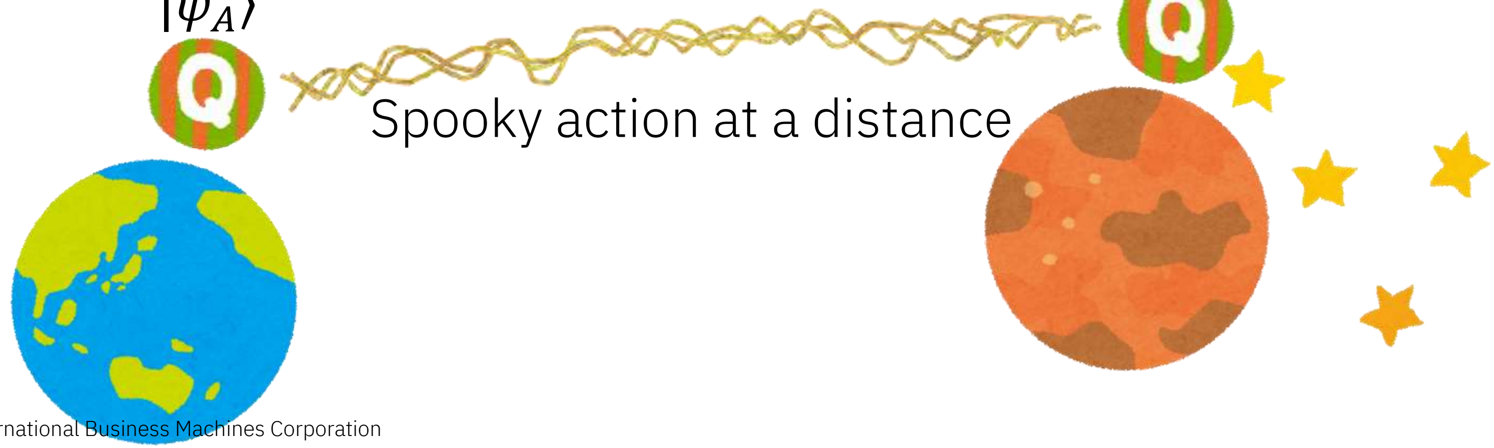
EPR Pair

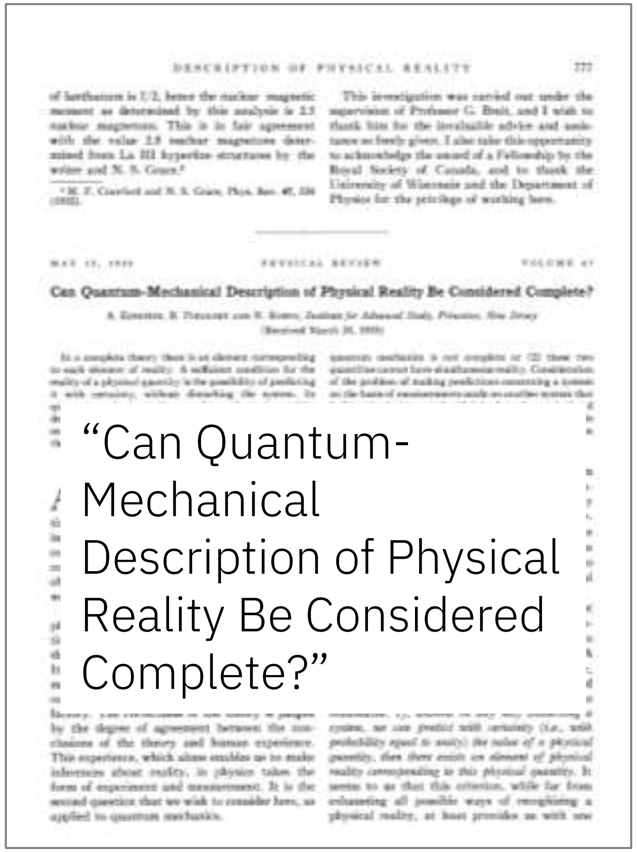
$$\text{EPR pair: } |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

More generally, two quantum states that are entangled are called EPR (Einstein-Podolsky-Rosen) pair after the EPR paradox.

EPR paradox: Suppose that quantum entangled pair are separated by a distance and one of them is observed. What happens?

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$$

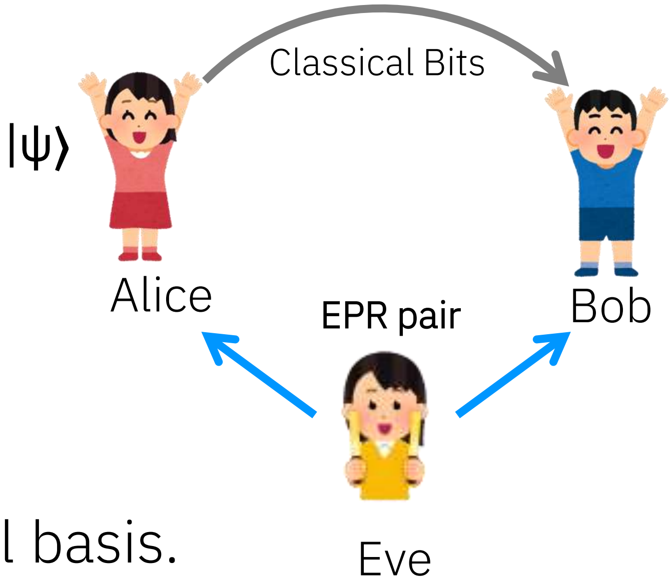




"Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?"

doi:10.1103/PhysRev.47.777 (1935)

The protocol of Quantum Teleportation



Assumption:

Alice has an **unknown quantum state $|\psi\rangle$** to be sent to Bob.

Eve makes **EPR pair,** and gives one to Alice and one to Bob.

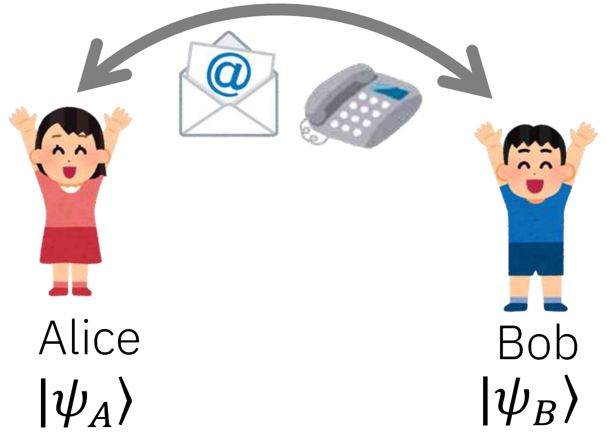
1. Alice entangles $|\psi\rangle$ with her **EPR part** using the CNOT gate.
2. Alice applies H gate to $|\psi\rangle$, and measures in the computational basis.
3. Alice sends Bob her **measurement results** (either “00”, “01”, “10”, or “11”).
4. Bob performs a correction operator based on Alice’s 2-bits of information on his part of **EPR pair**.
   - If “00”, does nothing
   - If “01”, applies X gate
   - If “10”, applies Z gate
   - If “11”, applies ZX gate
5. The **EPR part** of Bob becomes $|\psi\rangle$.

LOCC (Local Operations and Classical Communication)

LOCC is a class of quantum operations where parties can perform local operations on their individual quantum systems and communicate classically.

- Local Operations (LO):
  - Operations applied independently to each quantum system.
  - There is no direct exchange of information between the quantum systems.

- Classical Communication (CC):
  - Sharing the information using classical ways between the parties who have their individual quantum systems.
  - Based on this shared information, decisions about the next operations can be made.



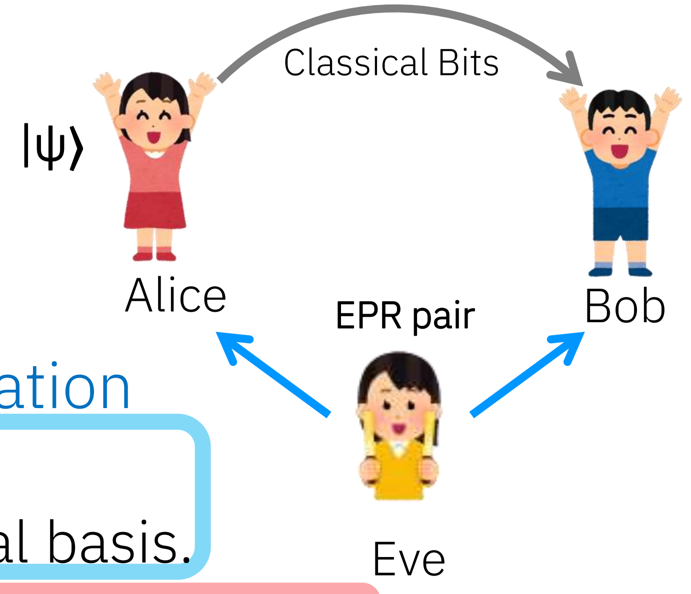
If systems A and B are not originally quantum entangled, they cannot be converted to a quantum entangled state by LOCC alone.

The protocol of Quantum Teleportation

Assumption:

Alice has an **unknown quantum state $|\psi\rangle$** to be sent to Bob.

Eve makes **EPR pair,** and gives one to Alice and one to Bob.



Local operation

1. Alice entangles $|\psi\rangle$ with her **EPR part** using the CNOT gate.
2. Alice applies H gate to $|\psi\rangle$, and measures in the computational basis.
3. Alice sends Bob her **measurement results** (either "00", "01", "10", or "11").

Classical communication

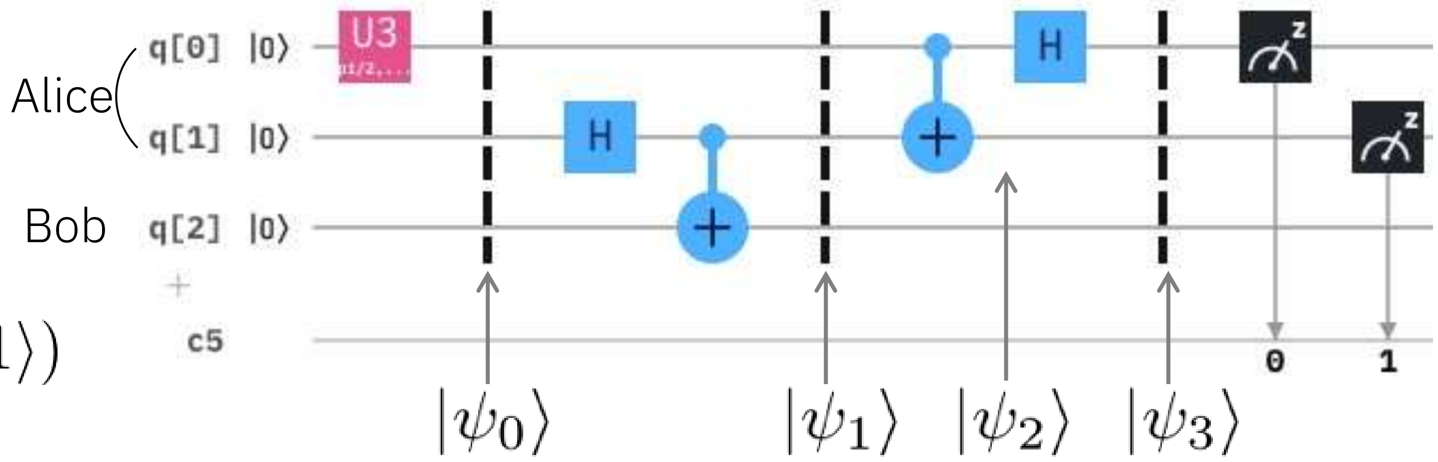
4. Bob performs a correction operator based on Alice's 2-bits of information on his part of **EPR pair**.
   - If "00", does nothing
   - If "01", applies X gate
   - If "10", applies Z gate
   - If "11", applies iY = ZX gate

Local operation

5. The **EPR part** of Bob becomes $|\psi\rangle$.

Detail of the protocol of Quantum Teleportation

Note : Bit ordering in Qiskit is |q2 q1 q0>



$$|\psi_0\rangle = |00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|110\rangle + \beta|0\underline{1}1\rangle + \beta|1\underline{0}1\rangle)$$

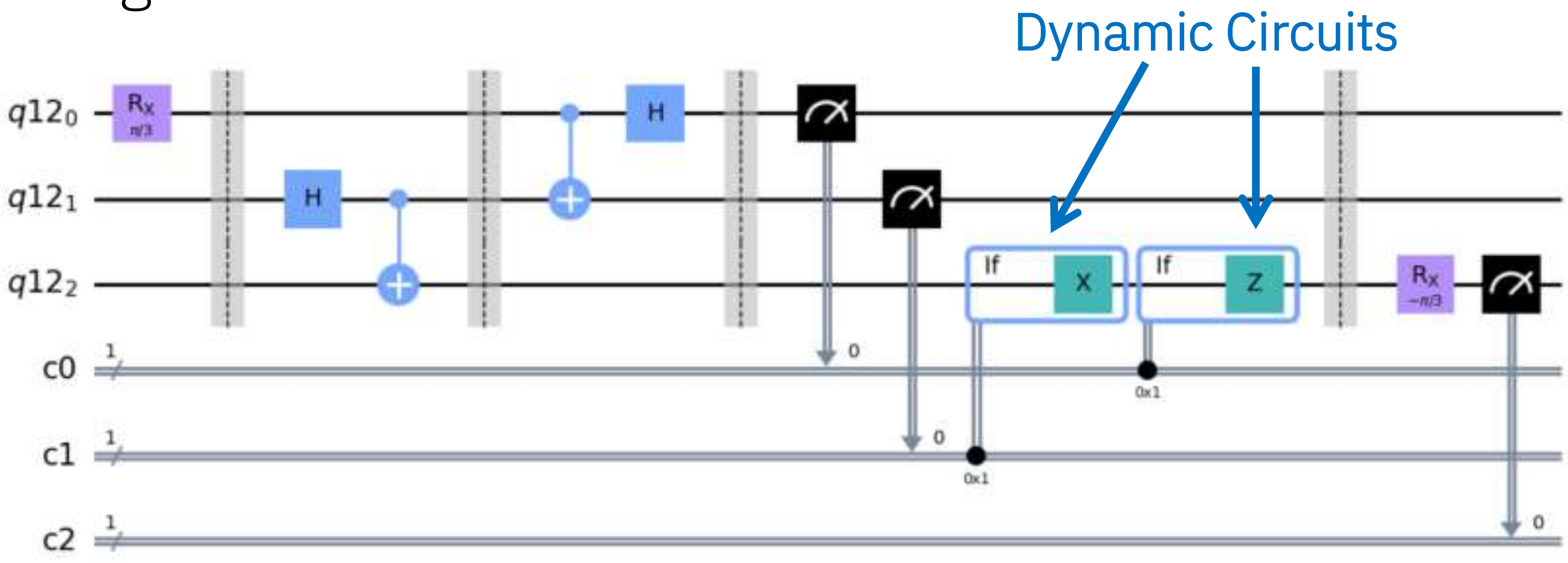
$$= \frac{1}{\sqrt{2}}(\alpha(|00\rangle + |11\rangle)|0\rangle + \beta(|01\rangle + |10\rangle)|1\rangle)$$

$$|\psi_3\rangle = \frac{1}{2}(\alpha(|00\rangle + |11\rangle)(|0\rangle + |1\rangle) + \beta(|01\rangle + |10\rangle)(|0\rangle - |1\rangle))$$

$$= \frac{1}{2}((\underline{\alpha|0\rangle + \beta|1\rangle})|00\rangle + (\underbrace{\alpha|1\rangle + \beta|0\rangle}_{\text{applies X gate}})|10\rangle + (\underbrace{\alpha|0\rangle - \beta|1\rangle}_{\text{applies Z gate}})|01\rangle + (\underbrace{\alpha|1\rangle - \beta|0\rangle}_{\text{applies ZX gate}})|11\rangle)$$

Dynamic Circuits

Dynamic Circuits is a feature of Qiskit that makes a measurement in the middle of a quantum circuit and dynamically changes the quantum gate that is subsequently applied according to the results.



Support function:

- If statement
- Switch statement
- For loop
- While loop

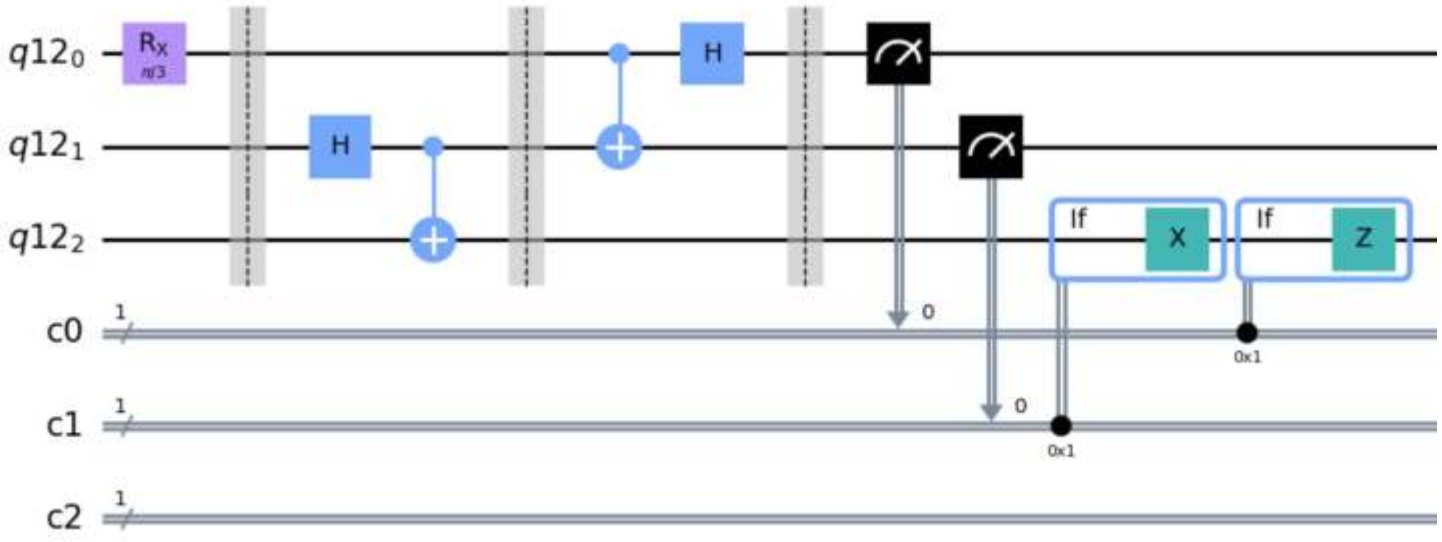
It is relatively new function made available since 2023.
It can be used for the quantum error correction, etc.

https://docs.quantum.ibm.com/build/classical-feedforward-and-control-flow

Hands-on

Remarks of Quantum Teleportation

We can transport the quantum state to the distant friend by sharing an EPR pair.



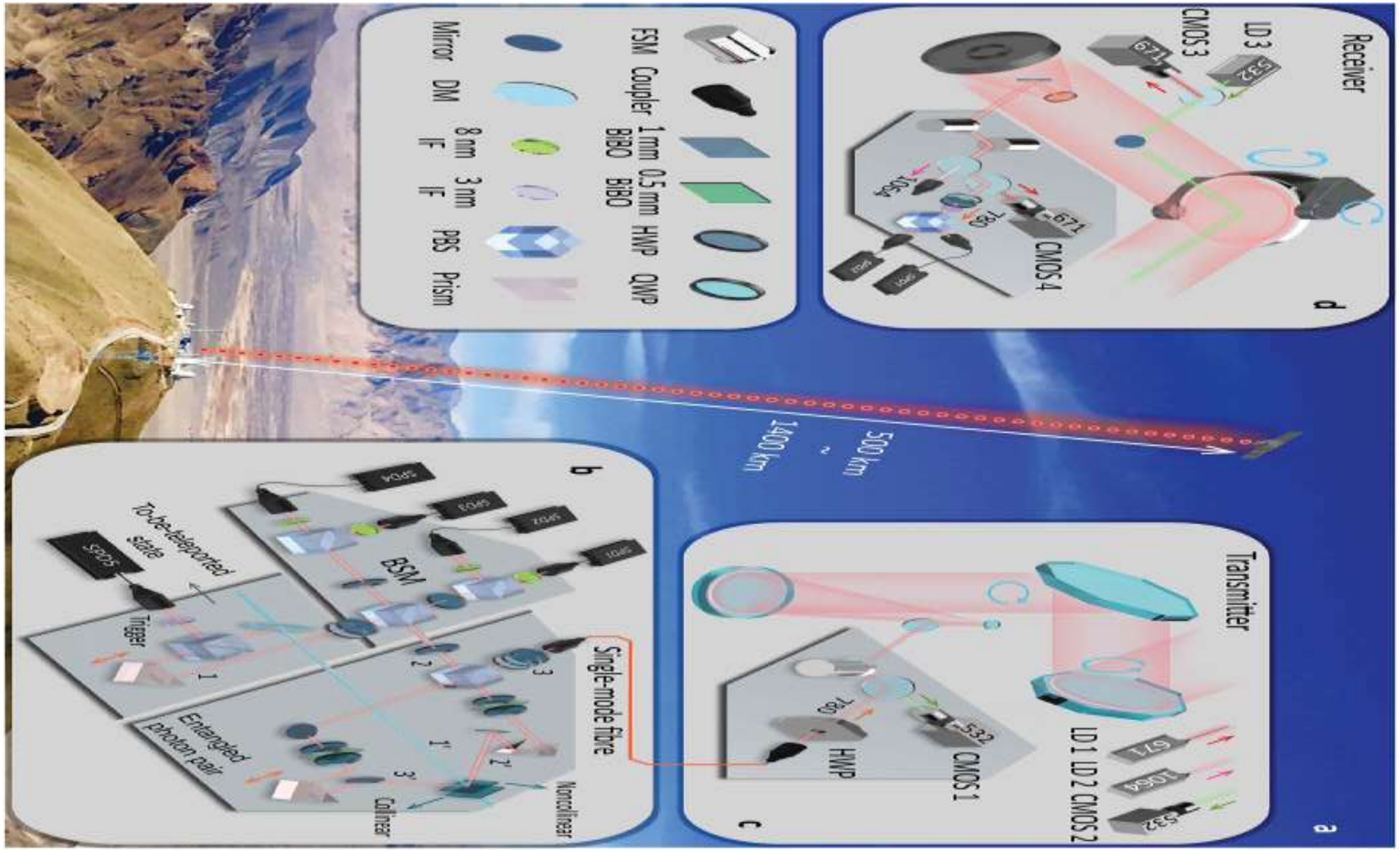
1. Can Quantum Teleportation send the quantum state **faster than light**?

   No. Because Alice has to tell Bob the measurement results in a classical way.

2. Would Quantum Teleportation break the **No Cloning theorem**, which forbids copying of unknown quantum states?

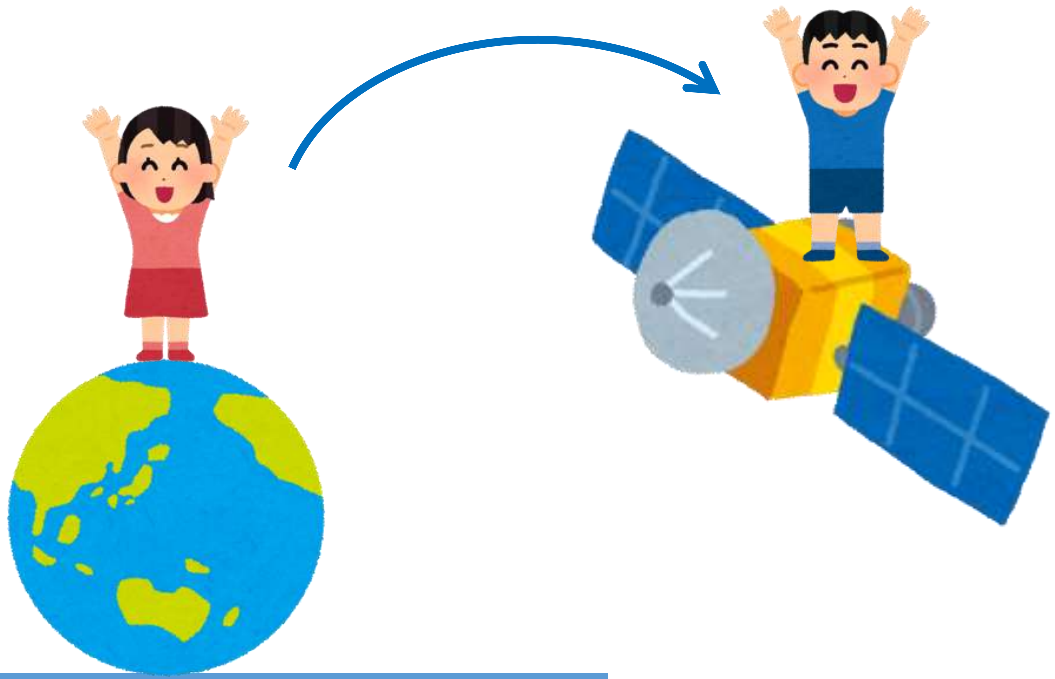
   No. Because the original quantum state in Alice lost to 0 or 1 by observation.

Ground-to-satellite quantum teleportation



Source： Ji-Gang Ren, etc. https://arxiv.org/ftp/arxiv/papers/1707/1707.00934.pdf

Quantum Superdense Coding is a kind of opposite algorithm of quantum teleportation.

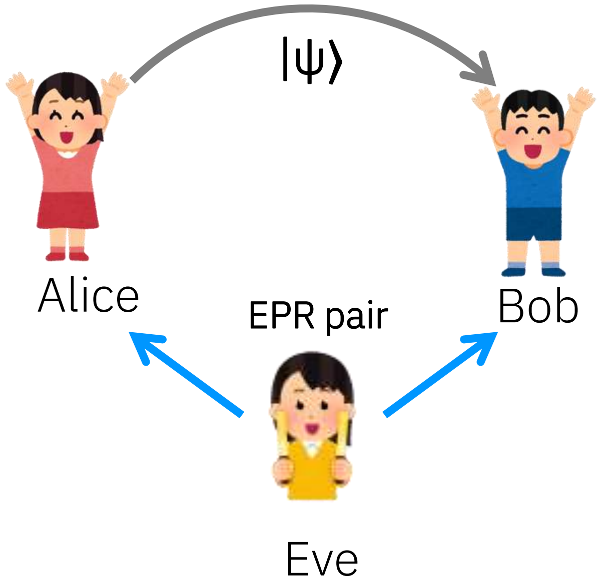


	Teleportation	Superdense Coding
Transmit	one qubit	two classical bits
Using	two classical bits	one qubit

with prior shared entangled qubits

Quantum Superdense Coding

Suppose Alice can only communicate with Bob using a qubit while she wants to send him two bits of classical information.
When Alice and Bob share an **EPR pair**, like the **quantum teleportation** case, Alice can inform Bob her two bits of information.



	Teleportation	Superdense Coding
Transmit	one qubit	two classical bits
Using	two classical bits	one qubit

The superdense coding protocol can be thought of as a flipped version of the teleportation protocol, in the sense that Alice and Bob only "swap their equipment."

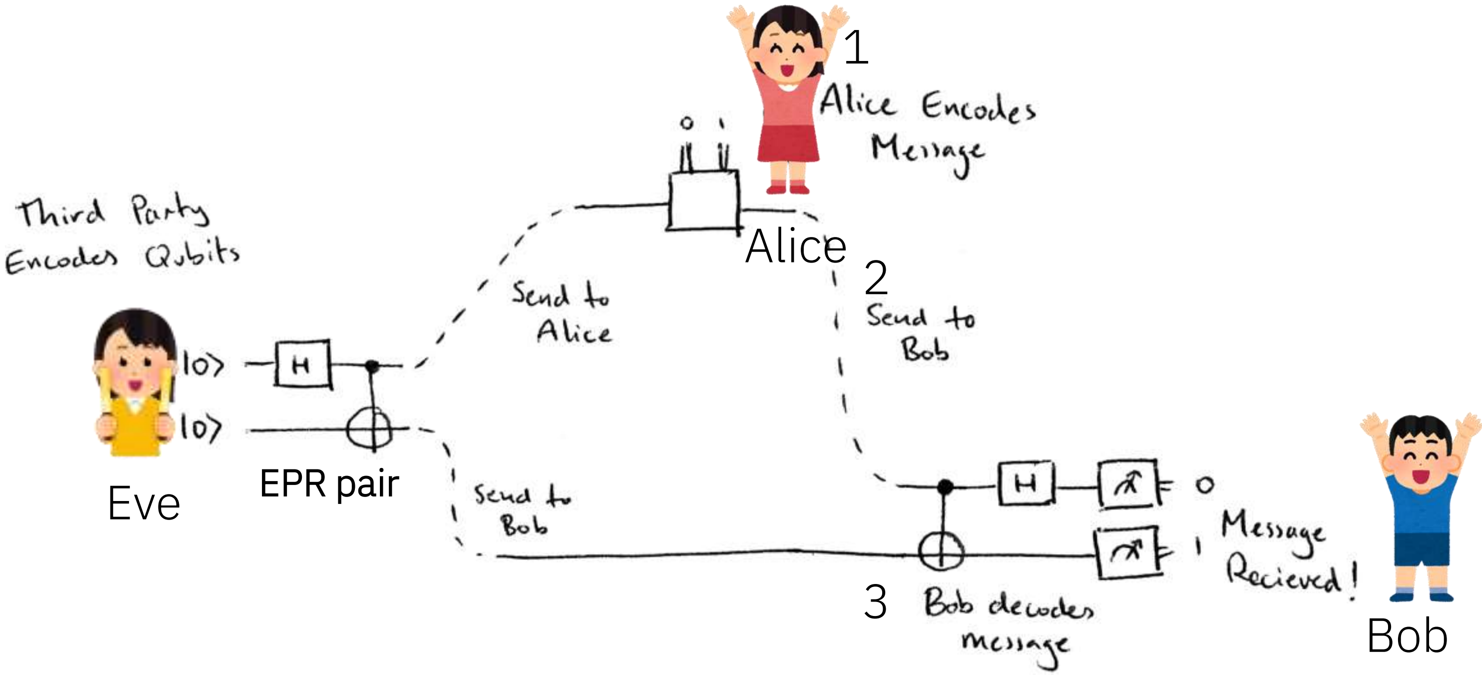
The protocol of superdense coding

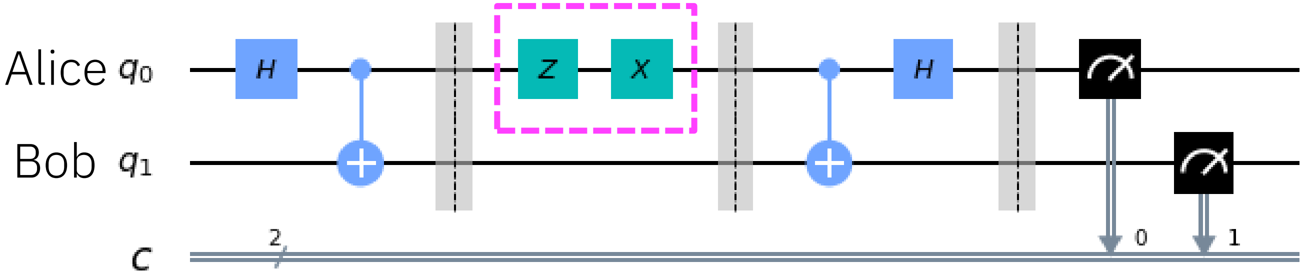
Assumption: Alice has two bits of information, $a_1a_2 \in \{00, 01, 10, 11\}$, which she wants to transmit to Bob. Alice can only send one qubit of information to Bob, but they share an **EPR pair**.

1. Alice performs one of the following operations on her part of **EPR pair**.
   - If $a_1a_2 = 00$, she does nothing
   - If $a_1a_2 = 10$, she applies Z gate
   - If $a_1a_2 = 01$, she applies X gate
   - If $a_1a_2 = 11$, she applies Z gate and X gate.

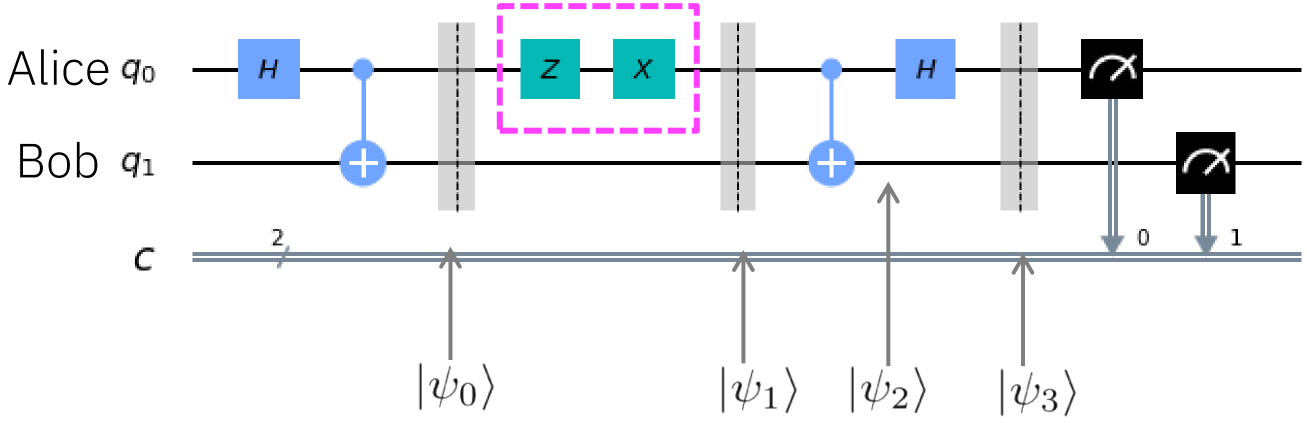
2. Alice sends her part of **EPR pair** to Bob.

3. Bob applies **CNOT** gate with the qubit from Alice as control and his qubit as target, then applies **H** gate to the qubit from Alice, and measures the two qubits.





Detail of the protocol of superdense coding



Note :
Bit ordering in Qiskit is |q1 q0>

They share an EPR pair:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Intended Message	$\|\psi_0\rangle\ (\cdot\sqrt{2})$	Applied Gate	$\|\psi_1\rangle\ (\cdot\sqrt{2})$	$\|\psi_2\rangle\ (\cdot\sqrt{2})$	$\|\psi_3\rangle$
00	|00> + |11>	I	|00> + |11>	|00> + |01>	|00>
10	|00> + |11>	X	|01> + |10>	|11> + |10>	|10>
01	|00> + |11>	Z	|00> - |11>	|00> - |01>	|01>
11	|00> + |11>	ZX	|01> - |10>	|11> - |10>	-|11>

Note that a negative sign of $-|11\rangle$ is global phase, so it is not measurable.

Hands-on

Summary

We have experienced the amazing quantum advantages using a very small number of qubits.

LOCC (Local Operations and Classical Communication):

LOCC is a class of quantum operations where parties can perform local operations on their individual quantum systems and communicate classically.

Quantum Teleportation:

Although we cannot copy quantum states, we can teleport unknown quantum state by having shared entanglement.

Quantum Superdense coding:

An EPR pair with 1 qubit of communication enable communicating 2 bits of information.

	Teleportation	Superdense Coding
Transmit	one qubit	two classical bits
Using	two classical bits	one qubit
Sharing	EPR pair	EPR pair

Thank you