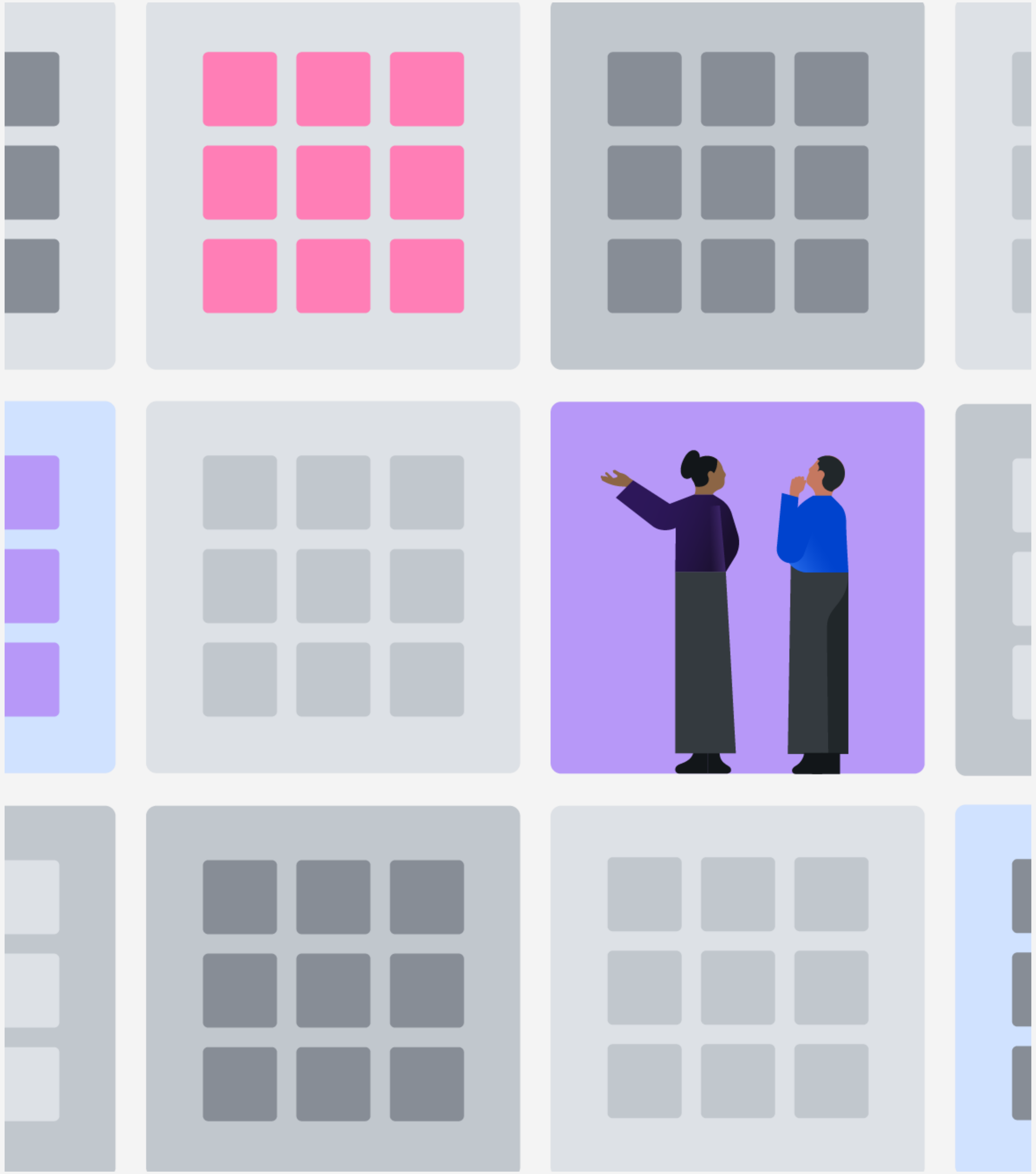
Understanding quantum information and computation

By John Watrous



Lesson 10

Quantum channels

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What are channels?

Channels describe *discrete-time changes* in systems that store quantum information.

This includes useful operations (such as unitary operations described by quantum gates and circuits) and changes we might wish to avoid (such as noise).

Common names for generic channels include the capital Greek letters Φ, Ψ, and Ξ.

If a channel Φ is applied to a system is a state represented by a density matrix ρ, then we obtain a system in the state $\Phi(\rho)$.

Requirements

1. Channels are linear mappings.
2. Channels transform density matrices into density matrices.*

* This includes the possibility that a channel is applied to part of a compound system.

Further details

Every channel Φ has an *input* system X and an *output* system Y.

- Conceptually speaking, Φ transforms X into Y. (The input and output systems are not considered to simultaneously co-exist.)
- The input and output systems can be the same — in which case we may simply view that Φ changes the state of this system.

Suppose Z is an additional system, Γ is the classical state set of Z, and the pair (Z, X) is in the state ρ. We can express ρ in the following form.

$$\rho = \sum_{a,b\in\Gamma} |a\rangle\langle b| \otimes \rho_{a,b}$$

Applying the channel Φ to X transforms it into Y, leaving (Z, Y) in this state:

$$\sum_{a,b\in\Gamma} |a\rangle\langle b| \otimes \Phi(\rho_{a,b})$$

Further details

Suppose Z is an additional system, Γ is the classical state set of Z, and the pair (Z, X) is in the state ρ. We can express ρ in the following form.

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$$\sum_{a,b\in\Gamma} |a\rangle\langle b| \otimes \Phi(\rho_{a,b})$$

If $\Gamma = \{0, \ldots, m-1\}$ we can describe this transformation in terms of *block matrices:*

$$\rho = \begin{pmatrix} \rho_{0,0} & \cdots & \rho_{0,m-1} \\ \vdots & \ddots & \vdots \\ \rho_{m-1,0} & \cdots & \rho_{m-1,m-1} \end{pmatrix} \mapsto \begin{pmatrix} \Phi(\rho_{0,0}) & \cdots & \Phi(\rho_{0,m-1}) \\ \vdots & \ddots & \vdots \\ \Phi(\rho_{m-1,0}) & \cdots & \Phi(\rho_{m-1,m-1}) \end{pmatrix}$$

This must be a *density matrix* (for every possible Z and every density matrix ρ).

Unitary channels

Suppose U is a unitary matrix representing a unitary operation on a system X.

The channel Φ corresponding to this operation is defined as follows.

$$\Phi(\rho) = \mathsf{U}\rho\mathsf{U}^\dagger$$

This is consistent with $|\psi\rangle\langle\psi|$ being the density matrix representation of the quantum state vector $|\psi\rangle$.

If U is performed on $|\psi\rangle$, we obtain the quantum state vector $\mathsf{U}|\psi\rangle$, whose density matrix representation is as follows.

$$(\mathsf{U}|\psi\rangle)(\mathsf{U}|\psi\rangle)^\dagger = \mathsf{U}|\psi\rangle\langle\psi|\mathsf{U}^\dagger$$

Fact

This is always a valid channel. If the state of (Z, X) is represented by a density matrix ρ and Φ is applied to X, then we necessarily obtain a density matrix:

$$(\mathbb{1}_\mathsf{Z} \otimes \mathsf{U})\rho(\mathbb{1}_\mathsf{Z} \otimes \mathsf{U})^\dagger$$

Unitary channels

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Fact

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$$(\mathbb{1}_{\mathsf{Z}} \otimes \mathrm{U})\rho(\mathbb{1}_{\mathsf{Z}} \otimes \mathrm{U})^{\dagger}$$

Example: the identity channel

The channel we obtain when we take $\mathrm{U} = \mathbb{1}$ is the *identity channel:*

$$\mathrm{Id}(\rho) = \rho$$

Convex combinations

Let Φ_0 and Φ_1 be channels from X to Y and let $p \in [0, 1]$.

Consider applying Φ_0 with probability p and Φ_1 with probability $1 - p$. We obtain a new channel:

$$\Psi = p\Phi_0 + (1 - p)\Phi_1$$

$$\Psi(\rho) = (p\Phi_0 + (1 - p)\Phi_1)(\rho) = p\Phi_0(\rho) + (1 - p)\Phi_1(\rho)$$

Convex combinations

Let Φ_0 and Φ_1 be channels from X to Y and let $p \in [0, 1]$.

Consider applying Φ_0 with probability p and Φ_1 with probability $1 - p$. We obtain a new channel:

$$\Psi = p\Phi_0 + (1 - p)\Phi_1$$

More generally, if $\Phi_0, \ldots, \Phi_{m-1}$ are channels and $(p_0, \ldots, p_{m-1})$ is a probability vector, then averaging in a similar way creates a new channel:

$$\Psi = \sum_{k=0}^{m-1} p_k \Phi_k$$

Example: mixed-unitary channels

Applying one of a collection of unitary operations to a system yields a *mixed unitary* channel.

$$\Psi(\rho) = \sum_{k=0}^{m-1} p_k U_k \rho U_k^{\dagger}$$

Qubit reset

Qubit reset channel

The ***qubit reset channel*** resets a qubit to the $|0\rangle$ state.

$$\Lambda(\rho) = \mathrm{Tr}(\rho)|0\rangle\langle 0|$$

Suppose A and B are qubits, (A, B) is in the state $|\phi^+\rangle$, and Λ is applied to A.

First we can express $|\phi^+\rangle$ as a density matrix (using Dirac notation):

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle$$

$$|\phi^+\rangle\langle\phi^+| = \frac{1}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| \otimes |0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| \otimes |1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

Then we can apply the channel to the first qubit, evaluate, and simplify:

$$\begin{aligned}
&\frac{1}{2}\Lambda(|0\rangle\langle 0|) \otimes |0\rangle\langle 0| + \frac{1}{2}\Lambda(|0\rangle\langle 1|) \otimes |0\rangle\langle 1| + \frac{1}{2}\Lambda(|1\rangle\langle 0|) \otimes |1\rangle\langle 0| + \frac{1}{2}\Lambda(|1\rangle\langle 1|) \otimes |1\rangle\langle 1| \\
&= \frac{1}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 0| \otimes |1\rangle\langle 1| = |0\rangle\langle 0| \otimes \frac{\mathbb{1}}{2}
\end{aligned}$$

Complete dephasing

Completely dephasing channel

The *completely dephasing channel* zeros-out off-diagonal matrix entries.

$$\Delta\begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} = \begin{pmatrix} \alpha_{00} & 0 \\ 0 & \alpha_{11} \end{pmatrix}$$

This channel can alternatively be expressed using Dirac notation.

$$\begin{aligned} \Delta(|0\rangle\langle 0|) &= |0\rangle\langle 0| \qquad & \Delta(|0\rangle\langle 1|) &= 0 \\ \Delta(|1\rangle\langle 0|) &= 0 \qquad & \Delta(|1\rangle\langle 1|) &= |1\rangle\langle 1| \end{aligned}$$

Suppose (A, B) is a pair of qubits in the state $|\phi^+\rangle$, and Δ is applied to A:

$$\begin{aligned} &\frac{1}{2}\Delta(|0\rangle\langle 0|) \otimes |0\rangle\langle 0| + \frac{1}{2}\Delta(|0\rangle\langle 1|) \otimes |0\rangle\langle 1| + \frac{1}{2}\Delta(|1\rangle\langle 0|) \otimes |1\rangle\langle 0| + \frac{1}{2}\Delta(|1\rangle\langle 1|) \otimes |1\rangle\langle 1| \\ &= \frac{1}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \otimes |1\rangle\langle 1| \end{aligned}$$

Complete depolarizing

Completely depolarizing channel

The *completely depolarizing channel* always outputs the completely mixed state.

$$\Omega(\rho) = \operatorname{Tr}(\rho)\frac{\mathbb{1}}{2}$$

The completely depolarizing channel represents an extreme form of *noise.*

We can describe a less extreme form of noise by averaging with the identity channel:

$$\Omega_\varepsilon = (1-\varepsilon)\,\mathrm{Id} + \varepsilon\Omega$$
$$\Omega_\varepsilon(\rho) = (1-\varepsilon)\rho + \frac{\varepsilon}{2}\mathbb{1}$$

Something similar can be done with the completely dephasing channel:

$$\Delta_\varepsilon = (1-\varepsilon)\,\mathrm{Id} + \varepsilon\Delta$$
$$\Delta_\varepsilon\begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} = \begin{pmatrix} \alpha_{00} & (1-\varepsilon)\alpha_{01} \\ (1-\varepsilon)\alpha_{10} & \alpha_{11} \end{pmatrix}$$

Three qubit channels

Qubit reset channel

The *qubit reset channel* resets a qubit to the $|0\rangle$ state.

$$\Lambda(\rho) = \mathrm{Tr}(\rho)|0\rangle\langle 0|$$

Completely dephasing channel

The *completely dephasing channel* zeros-out off-diagonal matrix entries.

$$\begin{aligned} \Delta(|0\rangle\langle 0|) &= |0\rangle\langle 0| & \qquad \Delta(|0\rangle\langle 1|) &= 0 \\ \Delta(|1\rangle\langle 0|) &= 0 & \qquad \Delta(|1\rangle\langle 1|) &= |1\rangle\langle 1| \end{aligned}$$

Completely depolarizing channel

The *completely depolarizing channel* always outputs the completely mixed state.

$$\Omega(\rho) = \mathrm{Tr}(\rho)\frac{\mathbb{1}}{2}$$

Channel representations

Question

Linear mapping from vectors to vectors can be represented by matrices in a familiar way…

… but channels are linear mappings *from matrices to matrices.*

How can we express *arbitrary channels* in mathematical terms?

Sometimes we may have a simple formula that expresses the action of a channel — such as $\Lambda(\rho) = \mathrm{Tr}(\rho)|0\rangle\langle 0|$ for the qubit reset channel — but this is not practical in general.

We'll discuss three different general ways to represent channels and how they relate:

1. Stinespring representations
2. Kraus representations
3. Choi representations

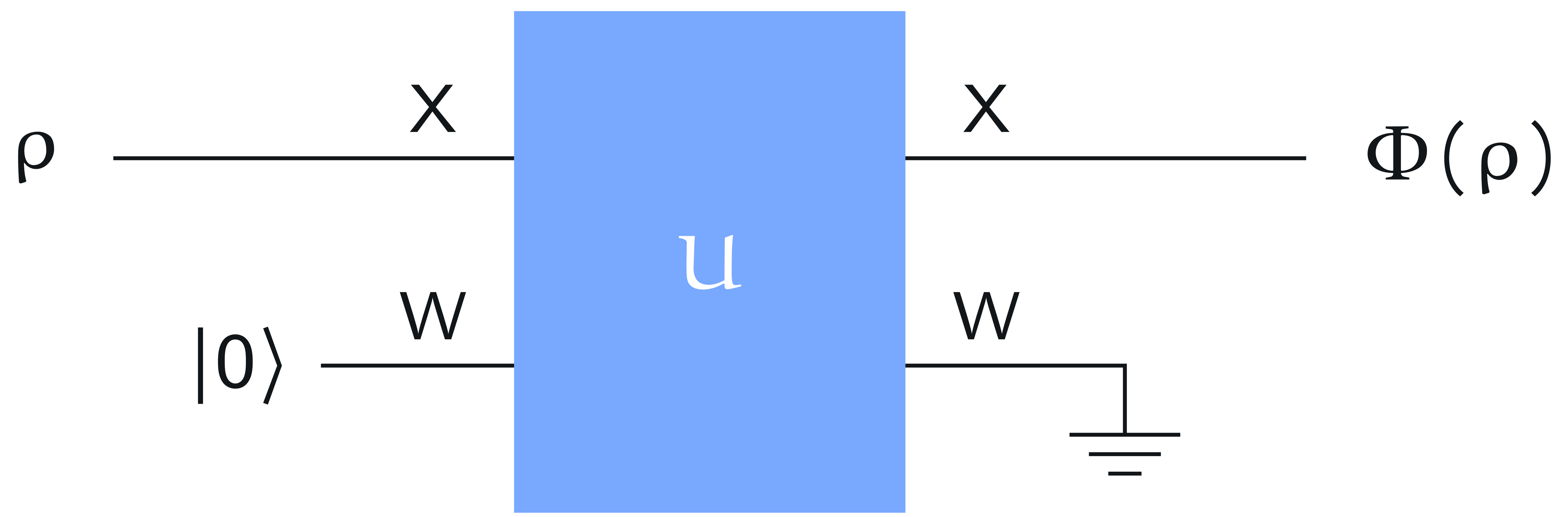
Stinespring representations

Every channel can be implemented in the following way:

1. Form a compound system including the *input system* and an initialized *workspace system.*
2. Perform a *unitary operation* on the compound systems.
3. *Discard* everything except the output system.

Special case

A channel Φ from a system X to itself can be implemented like this:



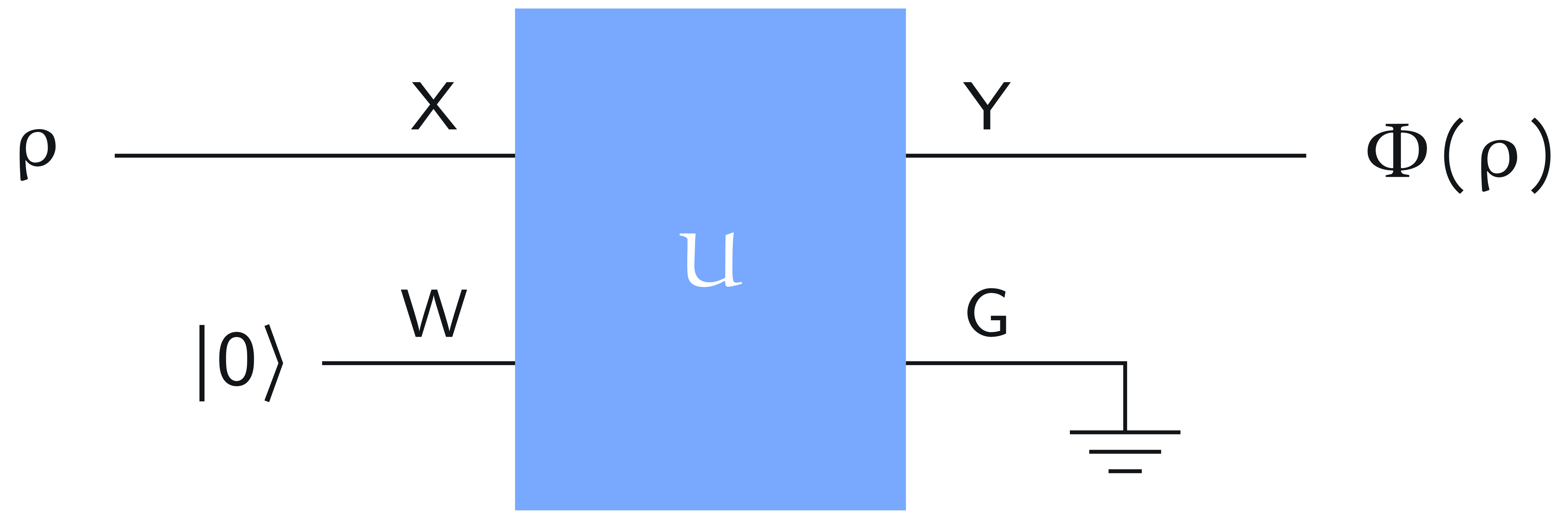
Stinespring representations

Every channel can be implemented in the following way:

1. Form a compound system including the *input system* and an initialized *workspace system.*
2. Perform a *unitary operation* on the compound systems.
3. *Discard* everything except the output system.

In general

A channel Φ from a system X to a system Y can be implemented as follows (for a suitable choice of systems W and G):

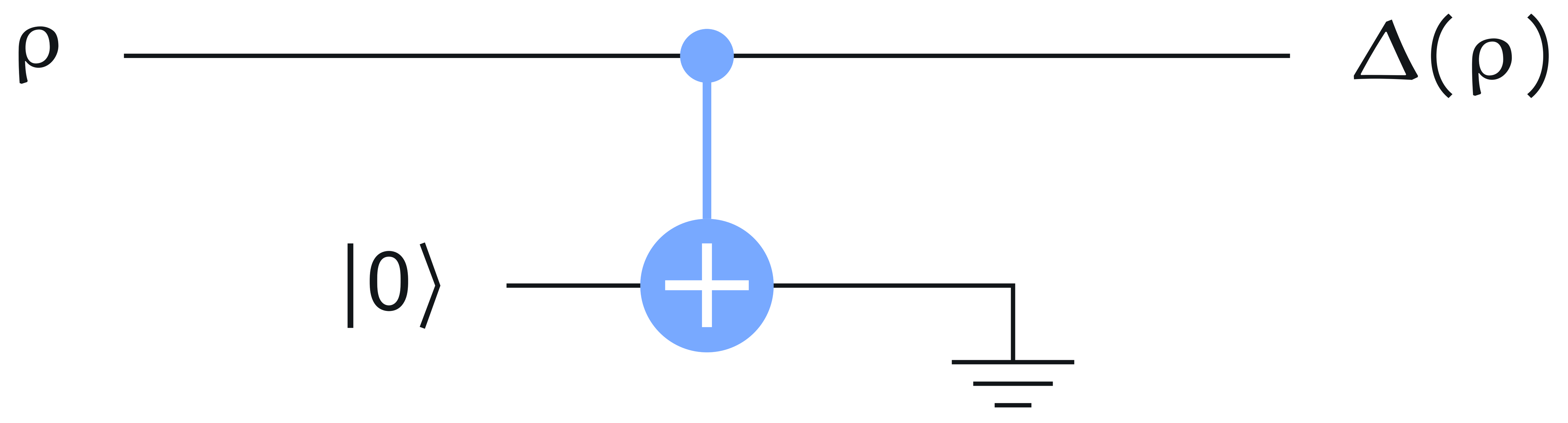


Such a description (consisting of the unitary operation and a specification of the input and output systems) is a *Stinespring representation* of the channel.

Example: dephasing

Example 1

An implementation of the completely dephasing channel:



$$|0\rangle\langle 0| \otimes \rho = \begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle & 0 & 0 \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example: dephasing

Example 1

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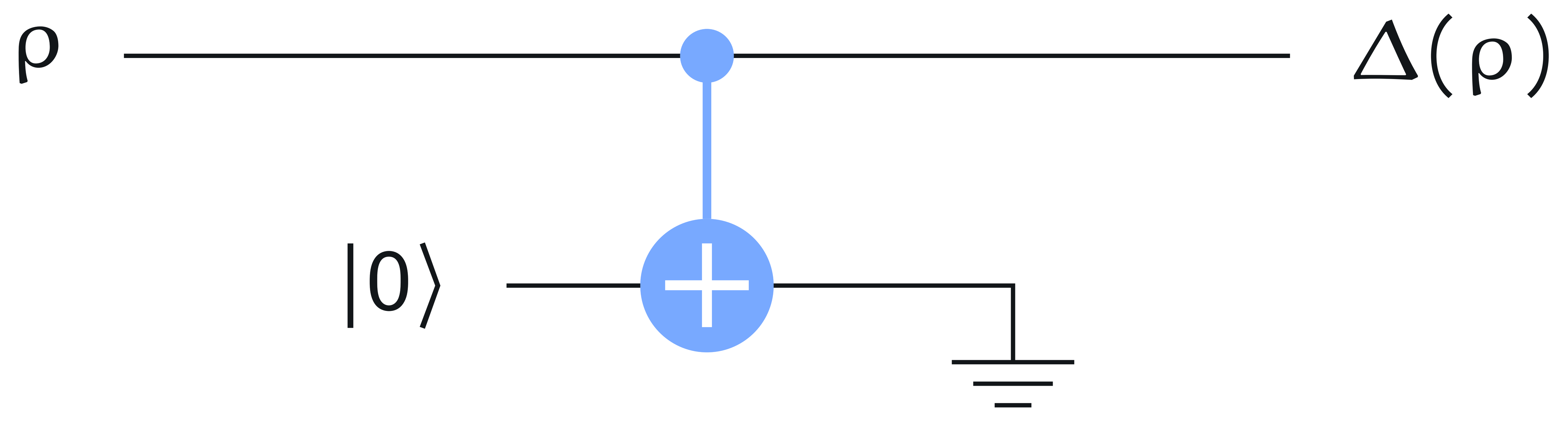

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle & 0 & 0 \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \langle 0|\rho|0\rangle & 0 & 0 & \langle 0|\rho|1\rangle \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \langle 1|\rho|0\rangle & 0 & 0 & \langle 1|\rho|1\rangle \end{pmatrix}$$

Example: dephasing

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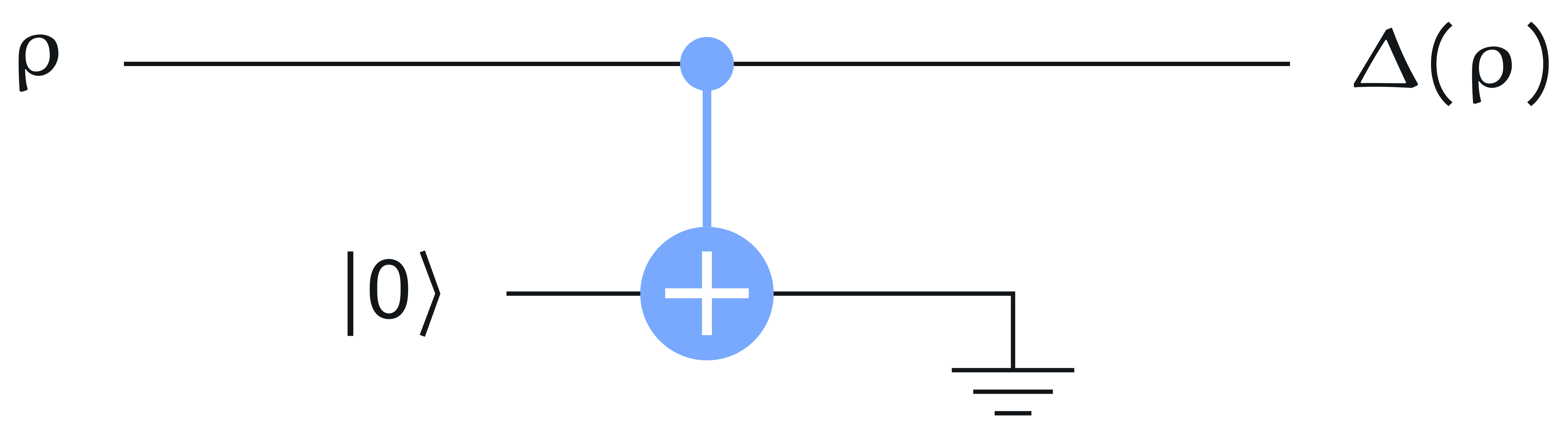


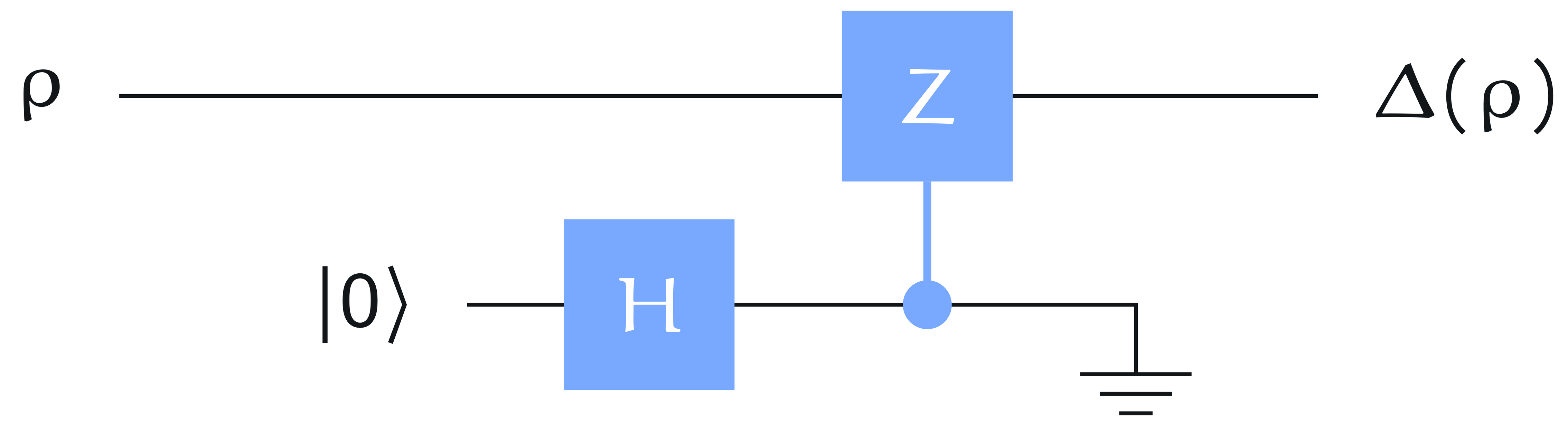
$$\begin{pmatrix} \langle 0|\rho|0\rangle & 0 & 0 & \langle 0|\rho|1\rangle \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \langle 1|\rho|0\rangle & 0 & 0 & \langle 1|\rho|1\rangle \end{pmatrix} = \begin{array}{l} \quad \langle 0|\rho|0\rangle\, |0\rangle\langle 0| \otimes |0\rangle\langle 0| \\ + \langle 0|\rho|1\rangle\, |0\rangle\langle 1| \otimes |0\rangle\langle 1| \\ + \langle 1|\rho|0\rangle\, |1\rangle\langle 0| \otimes |1\rangle\langle 0| \\ + \langle 1|\rho|1\rangle\, |1\rangle\langle 1| \otimes |1\rangle\langle 1| \end{array}$$

$$\xmapsto{\text{partial trace}} \langle 0|\rho|0\rangle\, |0\rangle\langle 0| + \langle 1|\rho|1\rangle\, |1\rangle\langle 1| = \Delta(\rho)$$

Example: dephasing

Example 2

An alternative implementation of the completely dephasing channel:

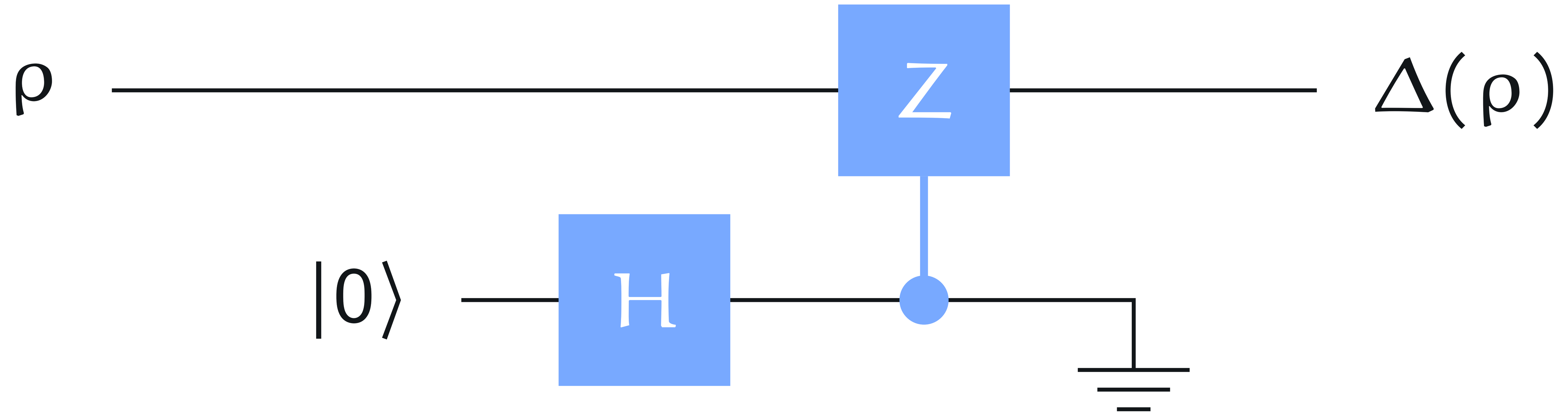


$$|+\rangle\langle+| \otimes \rho = \frac{1}{2}\begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle & \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle & \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \\ \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle & \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle & \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix}$$

Example: dephasing

Example 2

An alternative implementation of the completely dephasing channel:



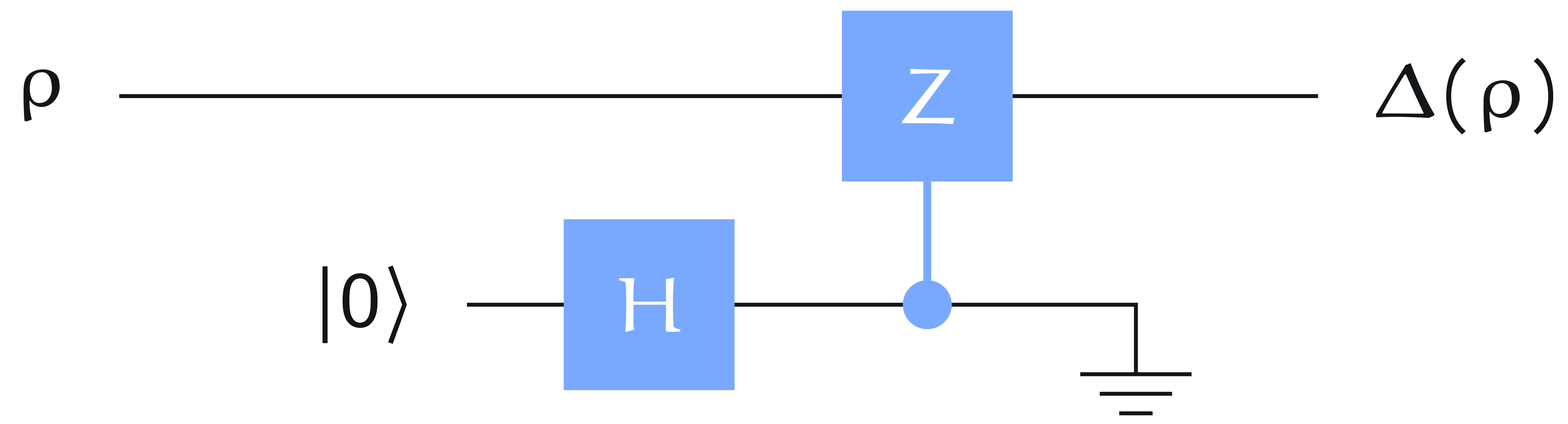
$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle & \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle & \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \\ \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle & \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle & \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle & \langle 0|\rho|0\rangle & -\langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle & \langle 1|\rho|0\rangle & -\langle 1|\rho|1\rangle \\ \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle & \langle 0|\rho|0\rangle & -\langle 0|\rho|1\rangle \\ -\langle 1|\rho|0\rangle & -\langle 1|\rho|1\rangle & -\langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix}$$

Example: dephasing

Example 2

An alternative implementation of the completely dephasing channel:



$$\frac{1}{2}\begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle & \langle 0|\rho|0\rangle & -\langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle & \langle 1|\rho|0\rangle & -\langle 1|\rho|1\rangle \\ \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle & \langle 0|\rho|0\rangle & -\langle 0|\rho|1\rangle \\ -\langle 1|\rho|0\rangle & -\langle 1|\rho|1\rangle & -\langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix}$$

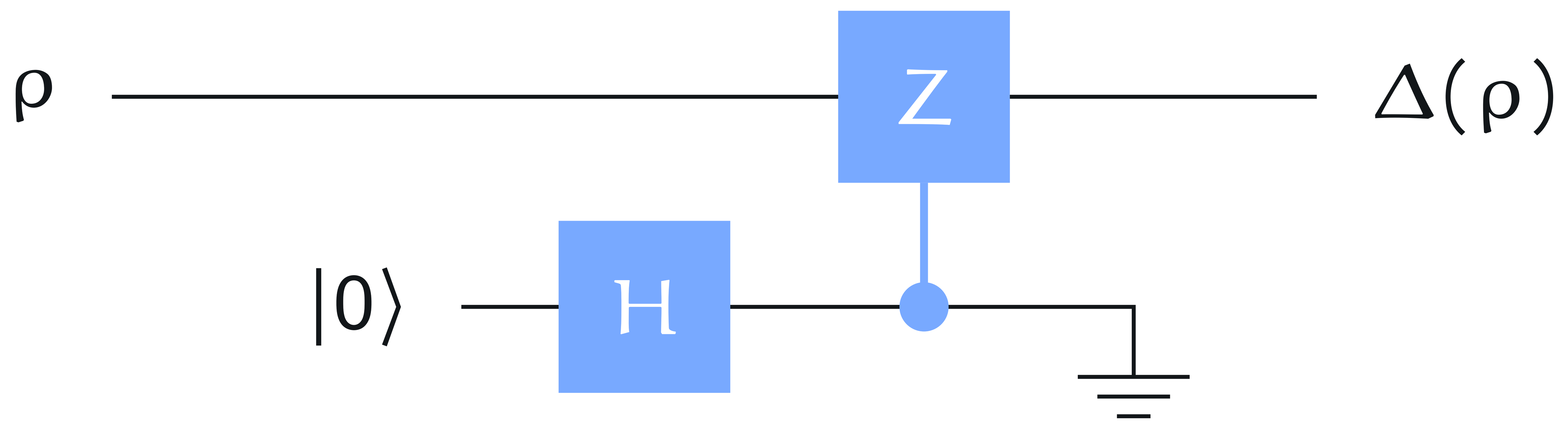
$$\xmapsto{\text{partial trace}} \frac{1}{2}\begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \langle 0|\rho|0\rangle & -\langle 0|\rho|1\rangle \\ -\langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix} = \begin{pmatrix} \langle 0|\rho|0\rangle & 0 \\ 0 & \langle 1|\rho|1\rangle \end{pmatrix}$$

$$= \Delta(\rho)$$

Example: dephasing

Example 2

An alternative implementation of the completely dephasing channel:



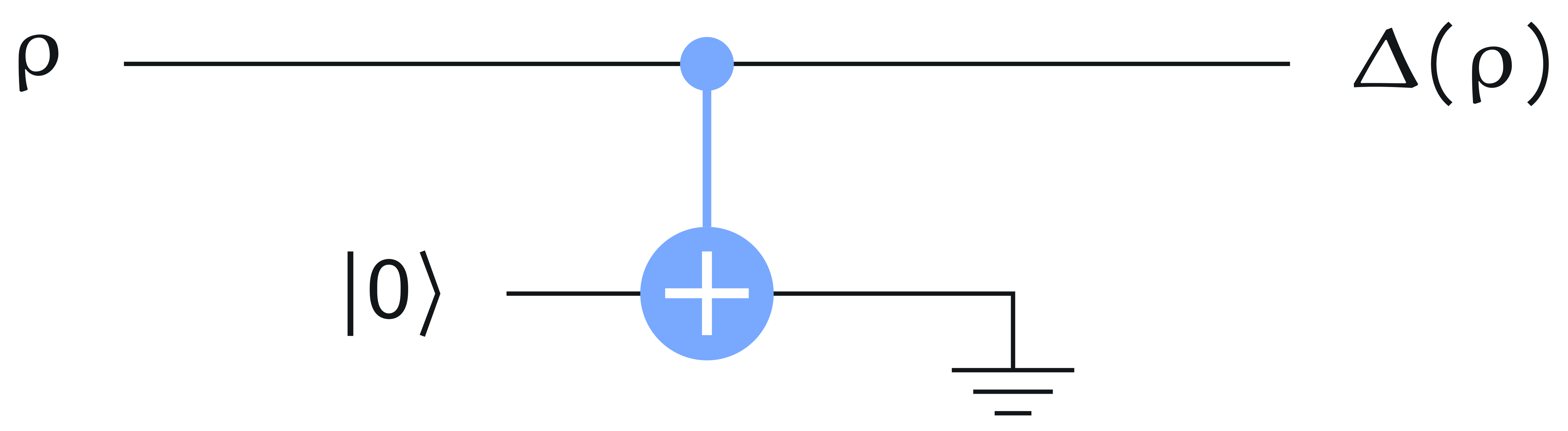
$$\frac{1}{2}\begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \langle 0|\rho|0\rangle & -\langle 0|\rho|1\rangle \\ -\langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix} = \begin{pmatrix} \langle 0|\rho|0\rangle & 0 \\ 0 & \langle 1|\rho|1\rangle \end{pmatrix}$$

$$\frac{1}{2}\rho + \frac{1}{2}\sigma_z\rho\sigma_z \;=\; \Delta(\rho)$$

Example: dephasing

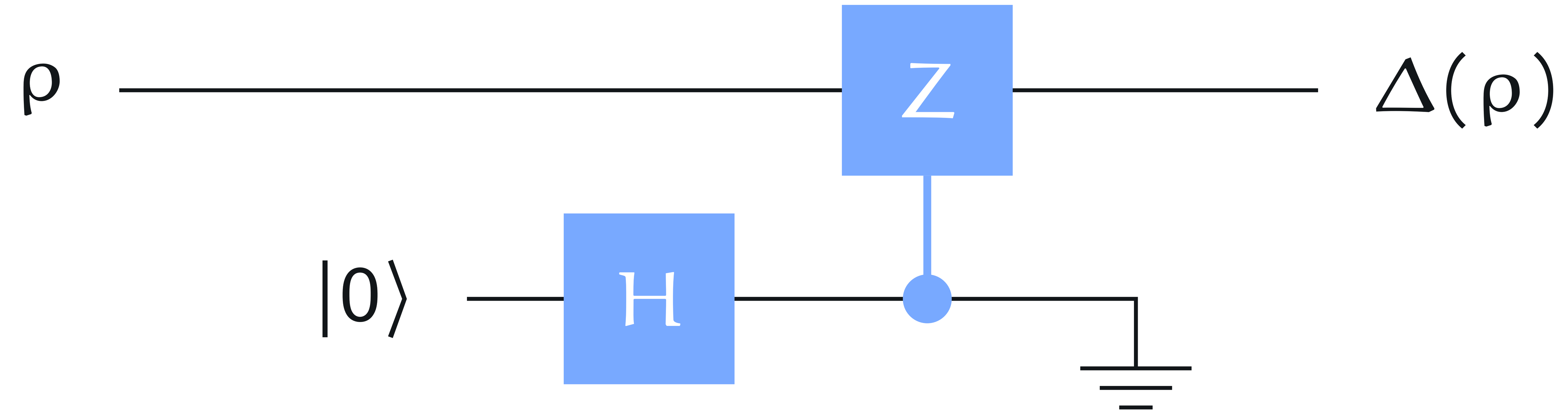
Example 1

An implementation of the completely dephasing channel:



Example 2

An alternative implementation of the completely dephasing channel:



Kraus representations

Kraus representations are a convenient formulaic way of expressing channels through matrix multiplication and addition.

In general, a Kraus representation of a channel Φ looks like this:

$$\Phi(\rho) = \sum_{k=0}^{N-1} A_k \rho A_k^{\dagger}$$

Here, $A_0, \ldots, A_{N-1}$ are matrices that all have the same dimensions:

- The columns of $A_0, \ldots, A_{N-1}$ correspond to the classical states of the input system.
- The rows of $A_0, \ldots, A_{N-1}$ correspond to the classical states of the output system.

These matrices must satisfy the following condition.

$$\sum_{k=0}^{N-1} A_k^{\dagger} A_k = \mathbb{1}$$

Examples

$$\Phi(\rho) = \sum_{k=0}^{N-1} A_k \rho A_k^\dagger \qquad \sum_{k=0}^{N-1} A_k^\dagger A_k = \mathbb{1}$$

Example: qubit reset channel

We can obtain a Kraus representation of the qubit reset channel by taking $A_0 = |0\rangle\langle 0|$ and $A_1 = |0\rangle\langle 1|$.

$$\begin{aligned} \sum_{k=0}^{1} A_k \rho A_k^\dagger &= |0\rangle\langle 0|\rho|0\rangle\langle 0| + |0\rangle\langle 1|\rho|1\rangle\langle 0| \\ &= \bigl(\langle 0|\rho|0\rangle + \langle 1|\rho|1\rangle\bigr)\,|0\rangle\langle 0| \\ &= \mathrm{Tr}(\rho)\,|0\rangle\langle 0| \end{aligned}$$

Here's a check that the required condition is met:

$$\sum_{k=0}^{1} A_k^\dagger A_k = |0\rangle\langle 0|0\rangle\langle 0| + |1\rangle\langle 0|0\rangle\langle 1| = |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$$

Examples

$$\Phi(\rho) = \sum_{k=0}^{N-1} A_k \rho A_k^\dagger \qquad \sum_{k=0}^{N-1} A_k^\dagger A_k = \mathbb{1}$$

Example: completely dephasing channel

We can obtain a Kraus representation of the completely dephasing channel by taking $A_0 = |0\rangle\langle 0|$ and $A_1 = |1\rangle\langle 1|$.

$$\begin{aligned}\sum_{k=0}^{1} A_k \rho A_k^\dagger &= |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1| \\ &= \langle 0|\rho|0\rangle\, |0\rangle\langle 0| + \langle 1|\rho|1\rangle\, |1\rangle\langle 1| \\ &= \Delta(\rho)\end{aligned}$$

Here's a check that the required condition is met:

$$\sum_{k=0}^{1} A_k^\dagger A_k = |0\rangle\langle 0|0\rangle\langle 0| + |1\rangle\langle 1|1\rangle\langle 1| = |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$$

Examples

$$\Phi(\rho) = \sum_{k=0}^{N-1} A_k \rho A_k^\dagger \qquad \sum_{k=0}^{N-1} A_k^\dagger A_k = \mathbb{1}$$

Example: completely dephasing channel (alternative Kraus representation)

We can obtain a different Kraus representation of the completely dephasing channel by taking $A_0 = \mathbb{1}/\sqrt{2}$ and $A_1 = \sigma_z/\sqrt{2}$.

$$\begin{aligned}\sum_{k=0}^{1} A_k \rho A_k^\dagger &= \frac{\mathbb{1}}{\sqrt{2}}\rho\frac{\mathbb{1}}{\sqrt{2}} + \frac{\sigma_z}{\sqrt{2}}\rho\frac{\sigma_z}{\sqrt{2}} \\ &= \frac{1}{2}\rho + \frac{1}{2}\sigma_z\rho\sigma_z \\ &= \Delta(\rho)\end{aligned}$$

Again we can check that the required condition is met:

$$\sum_{k=0}^{1} A_k^\dagger A_k = \frac{\mathbb{1}}{\sqrt{2}}\frac{\mathbb{1}}{\sqrt{2}} + \frac{\sigma_z}{\sqrt{2}}\frac{\sigma_z}{\sqrt{2}} = \frac{\mathbb{1}}{2} + \frac{\mathbb{1}}{2} = \mathbb{1}$$

Examples

$$\Phi(\rho) = \sum_{k=0}^{N-1} A_k \rho A_k^\dagger \qquad \sum_{k=0}^{N-1} A_k^\dagger A_k = \mathbb{1}$$

Exercise: completely depolarizing channel

For the completely depolarizing qubit channel Ω we need four (or more) Kraus matrices. Show that these two alternatives both work:

$$A_0 = \frac{|0\rangle\langle 0|}{\sqrt{2}} \qquad A_1 = \frac{|0\rangle\langle 1|}{\sqrt{2}} \qquad A_2 = \frac{|1\rangle\langle 0|}{\sqrt{2}} \qquad A_3 = \frac{|1\rangle\langle 1|}{\sqrt{2}}$$

$$A_0 = \frac{\mathbb{1}}{2} \qquad A_1 = \frac{\sigma_x}{2} \qquad A_2 = \frac{\sigma_y}{2} \qquad A_3 = \frac{\sigma_z}{2}$$

Notice (by the second alternative) that applying a *random Pauli operation* to a qubit completely depolarizes it.

$$\Omega(\rho) = \frac{\rho + \sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z}{4}$$

Choi representations

The *Choi representation* of a channel Φ is a single matrix denoted $J(\Phi)$.

If the input system has n classical states and the output system has m classical states, then $J(\Phi)$ is an $(nm) \times (nm)$ matrix.

Key properties of the Choi representation

1. The Choi representation is a *faithful* representation: for two channels Φ and Ψ we have $J(\Phi) = J(\Psi)$ if and only if $\Phi = \Psi$.
2. Simple-to-check conditions on $J(\Phi)$ are true if and only if Φ is a valid channel.

Remark: The matrix $J(\Phi)$ does not directly represent Φ as a linear mapping. (The action of Φ can, however, be recovered from $J(\Phi)$ by a simple formula.)

Choi representations

Let Φ be a channel from a system X to a system Y, and assume the classical state set of the input system X is Σ. The *Choi representation* of Φ is defined as follows.

$$J(\Phi) = \sum_{a,b\in\Sigma} |a\rangle\langle b| \otimes \Phi\bigl(|a\rangle\langle b|\bigr)$$

If we assume $\Sigma = \{0,\ldots,n-1\}$, then we can express $J(\Phi)$ as a block matrix:

$$J(\Phi) = \begin{pmatrix} \Phi\bigl(|0\rangle\langle 0|\bigr) & \Phi\bigl(|0\rangle\langle 1|\bigr) & \cdots & \Phi\bigl(|0\rangle\langle n-1|\bigr) \\ \Phi\bigl(|1\rangle\langle 0|\bigr) & \Phi\bigl(|1\rangle\langle 1|\bigr) & \cdots & \Phi\bigl(|1\rangle\langle n-1|\bigr) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi\bigl(|n-1\rangle\langle 0|\bigr) & \Phi\bigl(|n-1\rangle\langle 1|\bigr) & \cdots & \Phi\bigl(|n-1\rangle\langle n-1|\bigr) \end{pmatrix}$$

The set $\bigl\{|a\rangle\langle b| \,:\, 0 \leq a,b < n\bigr\}$ forms a basis for the space of all $n\times n$ complex matrices — so the blocks of $J(\Phi)$ determine the action of Φ on all $n \times n$ matrices.

The Choi state of a channel

Let Φ be a channel from a system X to a system Y, and assume the classical state set of the input system X is Σ. The *Choi representation* of Φ is defined as follows.

$$J(\Phi) = \sum_{a,b\in\Sigma} |a\rangle\langle b| \otimes \Phi\bigl(|a\rangle\langle b|\bigr)$$

If we normalize the Choi representation $J(\Phi)$ of a channel Φ by dividing by $n = |\Sigma|$, we obtain a density matrix. This state is called the *Choi state* of Φ.

$$|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{a\in\Sigma} |a\rangle \otimes |a\rangle$$

$$|\psi\rangle\langle\psi| = \frac{1}{n} \sum_{a,b\in\Sigma} |a\rangle\langle b| \otimes |a\rangle\langle b|$$

$$\frac{1}{n} J(\Phi) = \frac{1}{n} \sum_{a,b\in\Sigma} |a\rangle\langle b| \otimes \Phi\bigl(|a\rangle\langle b|\bigr)$$

The Choi state of a channel

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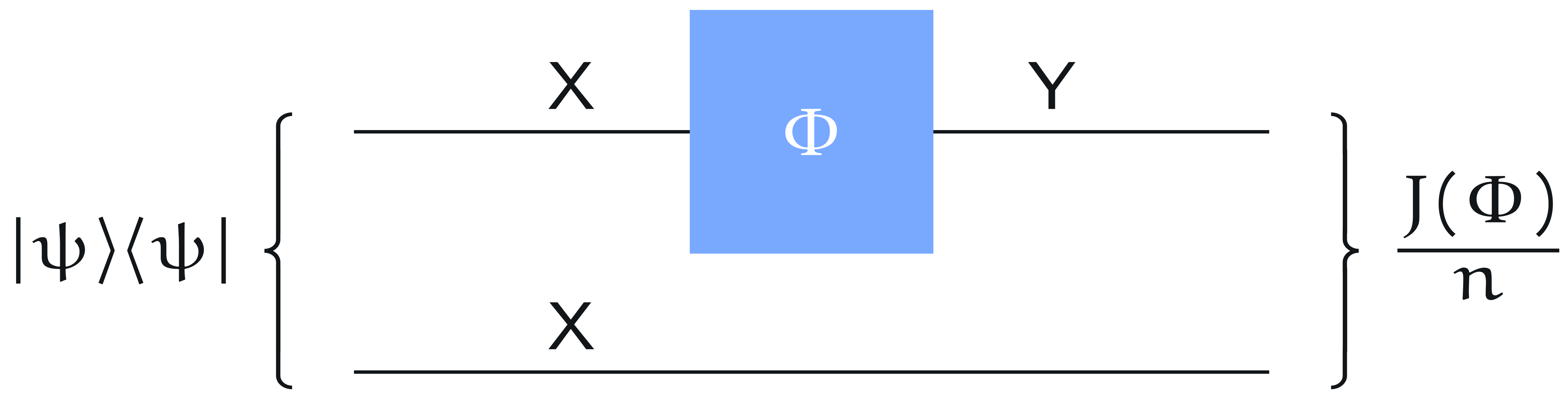

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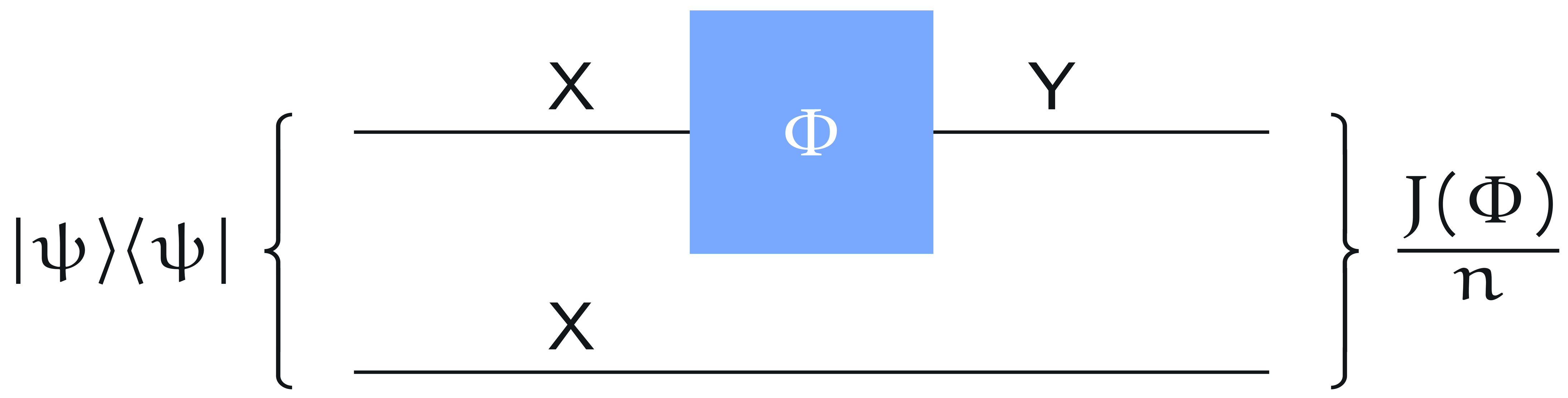


Implications:

1. $J(\Phi)/n$ is a density matrix — and therefore $J(\Phi) \geq 0$.
2. Tracing out the system Y from $J(\Phi)/n$ leaves the completely mixed state on X.

$$\mathrm{Tr}_Y\left(\frac{J(\Phi)}{n}\right) = \frac{\mathbb{1}_X}{n} \quad \Rightarrow \quad \mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X$$

These conditions on $J(\Phi)$ turn out to be *necessary and sufficient* for Φ to be a channel.

Examples

Example: the completely dephasing channel

The Choi representation of the (qubit) completely dephasing channel:

$$
\begin{aligned}
J(\Delta) &= \sum_{a,b=0}^{1} |a\rangle\langle b| \otimes \Delta\big(|a\rangle\langle b|\big) \\
&= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|
\end{aligned}
$$

As a block matrix:

$$
J(\Delta) = \begin{pmatrix} \Delta\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \Delta\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\[2ex] \Delta\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \Delta\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Examples

Example: the completely depolarizing channel

The Choi representation of the (qubit) completely depolarizing channel:

$$
\begin{aligned}
J(\Omega) &= \sum_{a,b=0}^{1} |a\rangle\langle b| \otimes \Omega\bigl(|a\rangle\langle b|\bigr) \\
&= |0\rangle\langle 0| \otimes \frac{\mathbb{1}}{2} + |1\rangle\langle 1| \otimes \frac{\mathbb{1}}{2} \\
&= \frac{1}{2}\mathbb{1} \otimes \mathbb{1}
\end{aligned}
$$

As a block matrix:

$$
J(\Omega) = \begin{pmatrix} \Omega\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \Omega\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\[2ex] \Omega\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \Omega\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}
$$

Examples

Example: the identity channel

The Choi representation of the (qubit) identity channel:

$$
\begin{aligned}
J(\mathrm{Id}) &= \sum_{a,b=0}^{1} |a\rangle\langle b| \otimes \mathrm{Id}\bigl(|a\rangle\langle b|\bigr) \\
&= \sum_{a,b=0}^{1} |a\rangle\langle b| \otimes |a\rangle\langle b| \\
&= 2\,|\phi^{+}\rangle\langle\phi^{+}|
\end{aligned}
$$

As a block matrix:

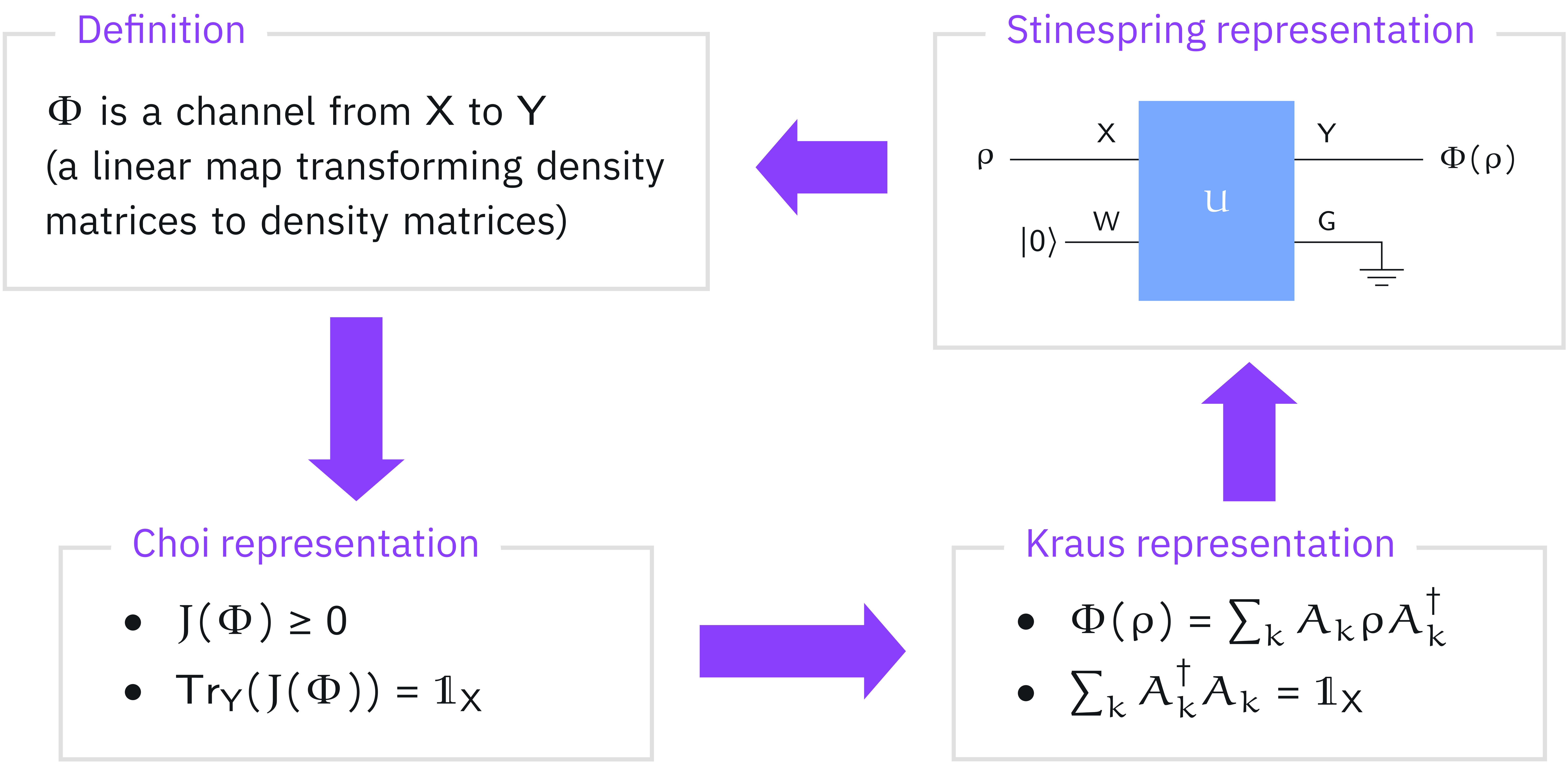
$$
J(\mathrm{Id}) = \begin{pmatrix} \mathrm{Id}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \mathrm{Id}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \mathrm{Id}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \mathrm{Id}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}
$$

Equivalence of representations

Three different ways to represent channels:

1. Stinespring representations
2. Kraus representations
3. Choi representations

The remainder of the lesson discusses how their equivalence can be reasoned.



First implication

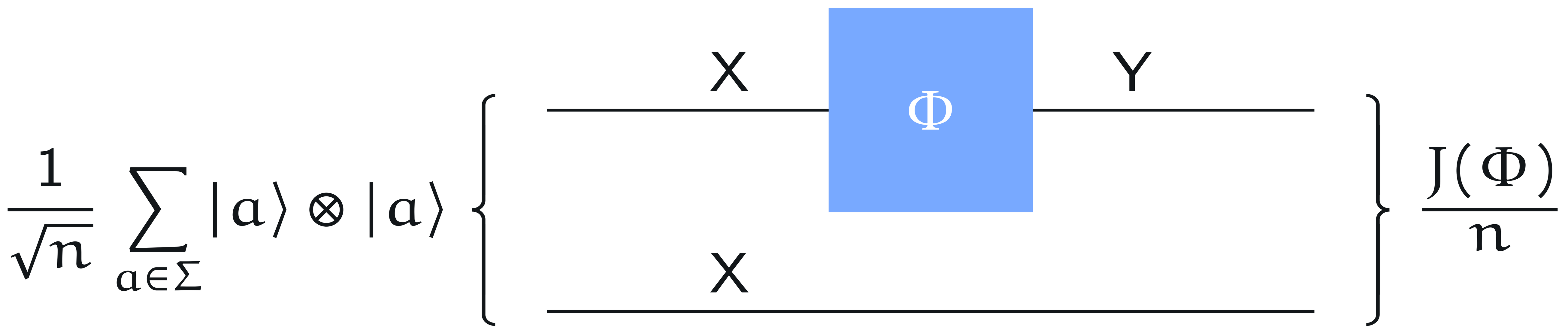
Definition

Φ is a channel from X to Y
(a linear map transforming density matrices to density matrices)

Choi representation

- $J(\Phi) \geq 0$
- $\mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X$

We already argued this implication the context of Choi states...



Implications:

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2. Tracing out the system Y from $J(\Phi)/n$ leaves the completely mixed state on X.

$$\mathrm{Tr}_Y\left(\frac{J(\Phi)}{n}\right) = \frac{\mathbb{1}_X}{n} \quad \Rightarrow \quad \mathrm{Tr}_Y\big(J(\Phi)\big) = \mathbb{1}_X$$

Second implication

Choi representation

- $J(\Phi) \geq 0$
- $\mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X$

$\Rightarrow$

Kraus representation

- $\Phi(\rho) = \sum_k A_k \rho A_k^\dagger$
- $\sum_k A_k^\dagger A_k = \mathbb{1}_X$

Because $J(\Phi)$ is positive semidefinite it can be expressed as follows.

$$J(\Phi) = \sum_{k=0}^{N-1} |\psi_k\rangle\langle\psi_k|$$

Each vector $|\psi_k\rangle$ can be further decomposed.

$$|\psi_k\rangle = \sum_{a\in\Sigma} |a\rangle \otimes |\phi_{k,a}\rangle$$

Second implication

Choi representation

- $J(\Phi) \geq 0$
- $\mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X$

Kraus representation

- $\Phi(\rho) = \sum_k A_k \rho A_k^\dagger$
- $\sum_k A_k^\dagger A_k = \mathbb{1}_X$

Because $J(\Phi)$ is positive semidefinite it can be expressed as follows.

$$J(\Phi) = \sum_{k=0}^{N-1} |\psi_k\rangle\langle\psi_k| \qquad |\psi_k\rangle = \sum_{a\in\Sigma} |a\rangle \otimes |\phi_{k,a}\rangle$$

We can now obtain a Kraus representation by choosing $A_0, \ldots, A_{N-1}$ like this:

$$A_k = \sum_{a\in\Sigma} |\phi_{k,a}\rangle\langle a|$$

Second implication

Choi representation

- $J(\Phi) \geq 0$
- $\mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X$

Kraus representation

- $\Phi(\rho) = \sum_k A_k \rho A_k^\dagger$
- $\sum_k A_k^\dagger A_k = \mathbb{1}_X$

$$J(\Phi) = \sum_{k=0}^{N-1} |\psi_k\rangle\langle\psi_k| \qquad |\psi_k\rangle = \sum_{a\in\Sigma} |a\rangle \otimes |\phi_{k,a}\rangle \qquad A_k = \sum_{a\in\Sigma} |\phi_{k,a}\rangle\langle a|$$

Why should this work? Consider the mapping that these matrices define.

$$\Psi(\rho) = \sum_{k=0}^{N-1} A_k \rho A_k^\dagger$$

The Choi representation of Ψ agrees with Φ — so we must have $\Psi = \Phi$.

$$J(\Psi) = \sum_{a,b\in\Sigma} |a\rangle\langle b| \otimes \Psi(|a\rangle\langle b|) = \sum_{k=0}^{N-1} |\psi_k\rangle\langle\psi_k| = J(\Phi)$$

Second implication

Choi representation

- $J(\Phi) \geq 0$
- $\mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X$

Kraus representation

- $\Phi(\rho) = \sum_k A_k \rho A_k^\dagger$
- $\sum_k A_k^\dagger A_k = \mathbb{1}_X$

$$J(\Phi) = \sum_{k=0}^{N-1} |\psi_k\rangle\langle\psi_k| \qquad |\psi_k\rangle = \sum_{a\in\Sigma} |a\rangle \otimes |\phi_{k,a}\rangle \qquad A_k = \sum_{a\in\Sigma} |\phi_{k,a}\rangle\langle a|$$

The two other conditions turn out to be equivalent:

$$\mathrm{Tr}_Y(J(\Phi)) = \mathbb{1}_X \quad \Longleftrightarrow \quad \sum_{k=0}^{N-1} A_k^\dagger A_k = \mathbb{1}_X$$

This follows from this equation:

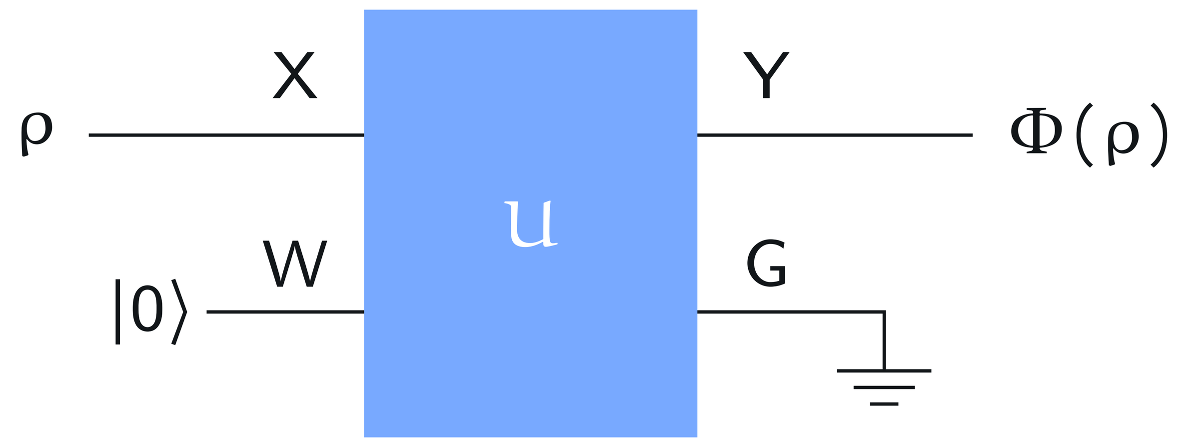
$$\left(\sum_{k=0}^{N-1} A_k^\dagger A_k\right)^T = \mathrm{Tr}_Y(J(\Phi))$$

Third implication

Kraus representation

- $\Phi(\rho) = \sum_k A_k \rho A_k^\dagger$
- $\sum_k A_k^\dagger A_k = \mathbb{1}_X$

Stinespring representation



$$U = \begin{pmatrix} A_0 & \\ A_1 & \\ \vdots & ? \\ A_{N-1} & \end{pmatrix}$$

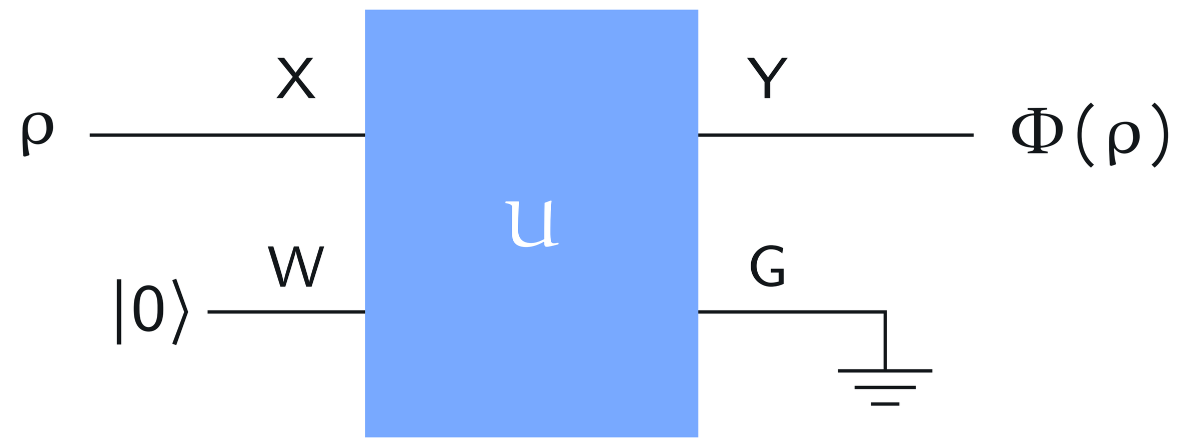
$$\mathrm{Tr}_G\big(U(|0\rangle\langle 0| \otimes \rho)U^\dagger\big) = \sum_{k=0}^{N-1} A_k \rho A_k^\dagger$$

The condition $\sum_{k=0}^{N-1} A_k^\dagger A_k = \mathbb{1}$ is equivalent to the n columns formed by $A_0, \ldots, A_{N-1}$ being orthonormal. Denote these first n columns by $|\gamma_0\rangle, \ldots, |\gamma_{n-1}\rangle$.

$$|\gamma_a\rangle = \sum_{k=0}^{N-1} |k\rangle \otimes A_k |a\rangle \qquad \langle \gamma_a | \gamma_b \rangle = \langle a | \left(\sum_{k=0}^{N-1} A_k^\dagger A_k \right) | b \rangle$$

Fourth implication

Stinespring representation



Definition

Φ is a channel from X to Y
(a linear map transforming density matrices to density matrices)

We already covered this implication in the context of Stinespring representations (and unitary channels).

In summary:

1. The introduction of an initialized workspace system is a channel.
2. Unitary operations are channels.
3. Tracing out a system is a channel.
4. Compositions of channels are channels.

Equivalence of representations

Definition

Φ is a channel from X to Y
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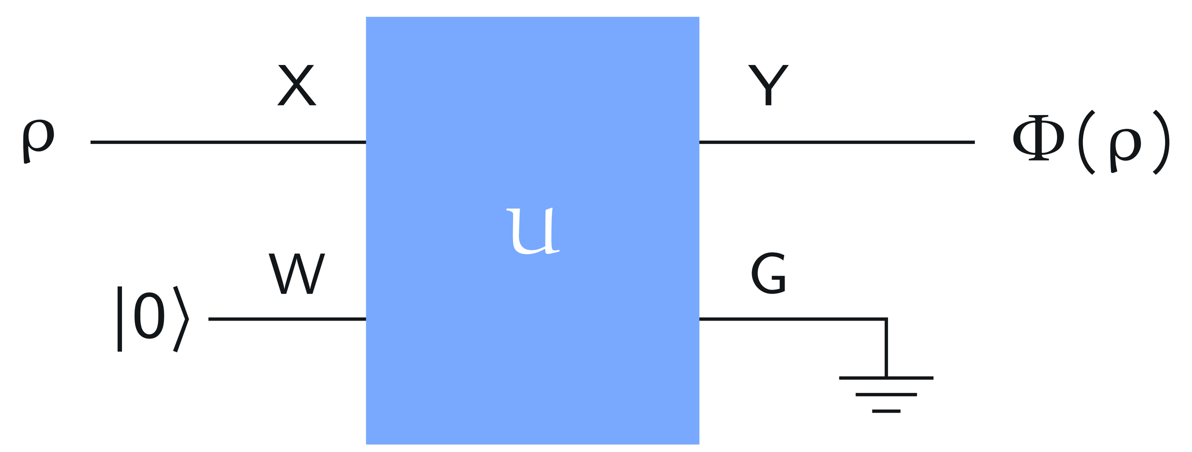
Stinespring representation



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