AGH

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Signals and Systems
Final report

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Index:

1. Project title:

Project #3: Hilbert transform and analytic signals.

Hilbert transform in time and frequency domain and use it for creating analytic signals.

2. Objective

Our project objective was to create algorithms that calculate the Hilbert transform in time and frequency domain by using different input signals. The results will be plotted in an application where we will be able to compare them.

3. Theoretical introduction:

The Hilbert transform plays an important role in a signal analysis because it can be used for a direct examination of instantaneous amplitude and frequency of signals. In the continuous time domain, the Hilbert transform H of the real signal x(t) is defined as:

$$H[x(t)] = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau, \qquad [1]$$

where the integral is considered as a Cauchy principal value because of the possible singularity at $\tau = t$.

Hilbert transform is convolution of signal x(t) with hyperbolic function $h(t) = \frac{1}{\pi t}$:

$$H[x(t)] = x(t) * h(t) = x(t) * \frac{1}{\pi t} \qquad [2]$$

Hilbert transform is an LTI- system and it is defined in the frequency-domain or in the time-domain. In contrast to the Laplace and Fourier transforms the Hilbert transform is not a transformation between the time-domain and frequency-domain.

4. Basic properties of the Hilbert transform:

Let $y[t] = H\{x[t]\}$, $y_1[t] = H\{x_1[t]\}$, $y_2[t] = H\{x_2[t]\}$ and let a, a_1, a_2 be arbitrary constant. Then the Hilbert transform satisfies the following basic properties:

(i) **Linearity:** $H\{ a_1 x_1[t] + a_2 x_2[t]\} = a_1 H\{x_1[t]\} + a_2 H\{x_2[t]\}$
(ii) **Time shift:** $H\{x[t-a]\} = y(t-a)$
(iii) **Scaling:** $H\{x[ta]\} = y(ta), \qquad a > 0$
(iv) **Time reversal:** $H\{x[-ta]\} = -y(-ta), \qquad a > 0$
(v) **Derivative:** $H\{x'[t]\} = y'(t)$

4.1 Other properties

A signal x(t) and its Hilbert transform have:

- The same amplitude spectrum
- The same autocorrelation function
- The same energy spectra density
- Hilbert transform of a constant is zero

5. Applications

The Hilbert transform has many applications in signal processing, imaging, modulation and demodulation, determination of instantaneous frequency and in cryptography.
A few specific uses:

- Sampling of narrowband signals in telecommunications (mostly using Hilbert filters).
- Medical imaging.
- Array processing for Direction of Arrival – for example this can be used in radio telescopes to send a signal in a certain direction.
- System response analysis.

6. Hilbert transform of fundamental signals:

$$H[cos\omega_0 t] = sin\omega_0 t, \qquad [3]$$

$$H[sin\omega_0 t] = -cos\omega_0 t, \qquad [4]$$

$$H[e^{\omega_0 t}] = je^{j\omega_0 t}. \qquad [5]$$

Knowing that Fourier transform of h(t) function is defined as:

$$F\{h(t)\}(\omega) = -j \cdot sgn(\omega) = \begin{cases} +j \; for \; \omega < 0 \\ 0 \; for \; \omega = 0 \\ -j \; for \; \omega > 0 \end{cases} \qquad [6]$$

and functions' convolution is a product of their Fourier transforms, Hilbert transform in frequency domain is equal to:

$$F\{H[x(t)]\}(\omega) = F\{x(t)\}(\omega) \cdot F\{h(t)\}(\omega) = -j \cdot sgn(\omega) \cdot F\{x(t)\}(\omega) = \begin{cases} +j \cdot X(\omega) for\ \omega < 0 \\ 0\ for\ \omega = 0 \\ -j \cdot X(\omega) for\ \omega > 0 \end{cases}, \quad [7]$$

$$where\ X(\omega) = F\{x(t)\}(\omega) - \text{Fourier transorm of x signal.}$$

Resulting Hilbert transform spectrum's phase is shifted by ±90° in comparison to x(t) signal spectrum. Its amplitude remains unchanged.

In practice, signals which can be analysed via the Hilbert transform are of finite length and digitally sampled, thus having the signal x(t) defined in the time interval [0,T] and using the uniform sampling with the period T_s, we obtain the discrete-time signal x[n]:

$$x[n] = x(nT_s), n = 0,1,\dots,N. \quad [8]$$

In discrete time case, the Hilbert transform H is replaced by the discrete Hilbert transform H_d which for non-periodic discrete signals is defined as discrete convolution - multiplication:

$$H_d\{x[n]\} = \sum_{m=-\infty}^{\infty} x[m] \cdot h[n-m] = \sum_{m=-\infty}^{\infty} x[m] \cdot \frac{1}{\pi\cdot(n-m)} = \frac{1}{\pi}\sum_{m=-\infty}^{\infty} \frac{x[m]}{n-m} \quad [9]$$

Discrete Hilbert transform H_d can also be calculated as:

$$H_d\{x[n]\} = \begin{cases} \frac{2}{\pi}\sum_{m=odd} \frac{x[m]}{n-m}\ \ for\ n\ even \\ \frac{2}{\pi}\sum_{m=even} \frac{x[m]}{n-m}\ \ for\ n\ odd \end{cases} \quad [10]$$

The signal $x[n]$ and its Hilbert transform $H_d\{x[n]\}$ are related to each other and together they create an analytic discrete-time signal $z[n]$ that is defined as:

$$z[n] = x[n] + jH_d\{x[n]\} = A[n]e^{\Phi[n]} \quad [11]$$

$$A[n] = \sqrt{x^2[n] + H_d^2\{x[n]\}} \quad [12]$$

$$\Phi[n] = tan^{-1}\left(\frac{H_d\{x[n]\}}{x[n]}\right) \quad [13]$$

$A[n]$ and $\Phi[n]$ are the instantaneous amplitude and phase of the signal $x[n]$.

7. Project aim:

The aim of this project is to perform Hilbert transform in time and frequency domain and use it for creating analytic signals. In order to achieve this, it is necessary to explore the theory related to the properties of the Hilbert and Fourier transforms. The acquired theory will be used to create an algorithm in MATLAB® that will compute Hilbert transform of any given signal.

8. Used Tools

MATLAB® will be the only tool used to write an algorithm and also to create a functional GUI.

9. Designed application:

Application designed in MATLAB® calculates Hilbert transform and analytical signal corresponding to given input signal. Hilbert transform is calculated based on:

- time-domain based approach:
  - 1st variant based on definition:

$$H_d\{x[n]\} = \sum_{m=-\infty}^{\infty} x[m] \cdot h[n-m] = \sum_{m=-\infty}^{\infty} x[m] \cdot \frac{1}{\pi \cdot (n-m)} = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \frac{x[m]}{n-m} \qquad [14]$$

  - 2nd variant based on definition:

$$H_d\{x[n]\} = \begin{cases} \frac{2}{\pi} \sum_{m=odd} \frac{x[m]}{n-m} & for\ n\ even \\ \frac{2}{\pi} \sum_{m=even} \frac{x[m]}{n-m} & for\ n\ odd \end{cases} \qquad [15]$$

- frequency-domain based approach:
  - 1st variant based on definition:

$$H\{x(t)\} = F^{-1}\{F\{x\} \cdot F\{h\}\} = F^{-1}\{F\{x\} \cdot (-j) \cdot sgn(\omega)\} = F^{-1} \begin{cases} j \cdot X(\omega)\ for\ \omega < 0 \\ 0\ for\ \omega = 0 \\ -j \cdot X(\omega)\ for\ \omega > 0 \end{cases}, \qquad [16]$$

$$where\ X(\omega) = F\{x(t)\}(\omega) - \text{Fourier transorm of x signal.}$$

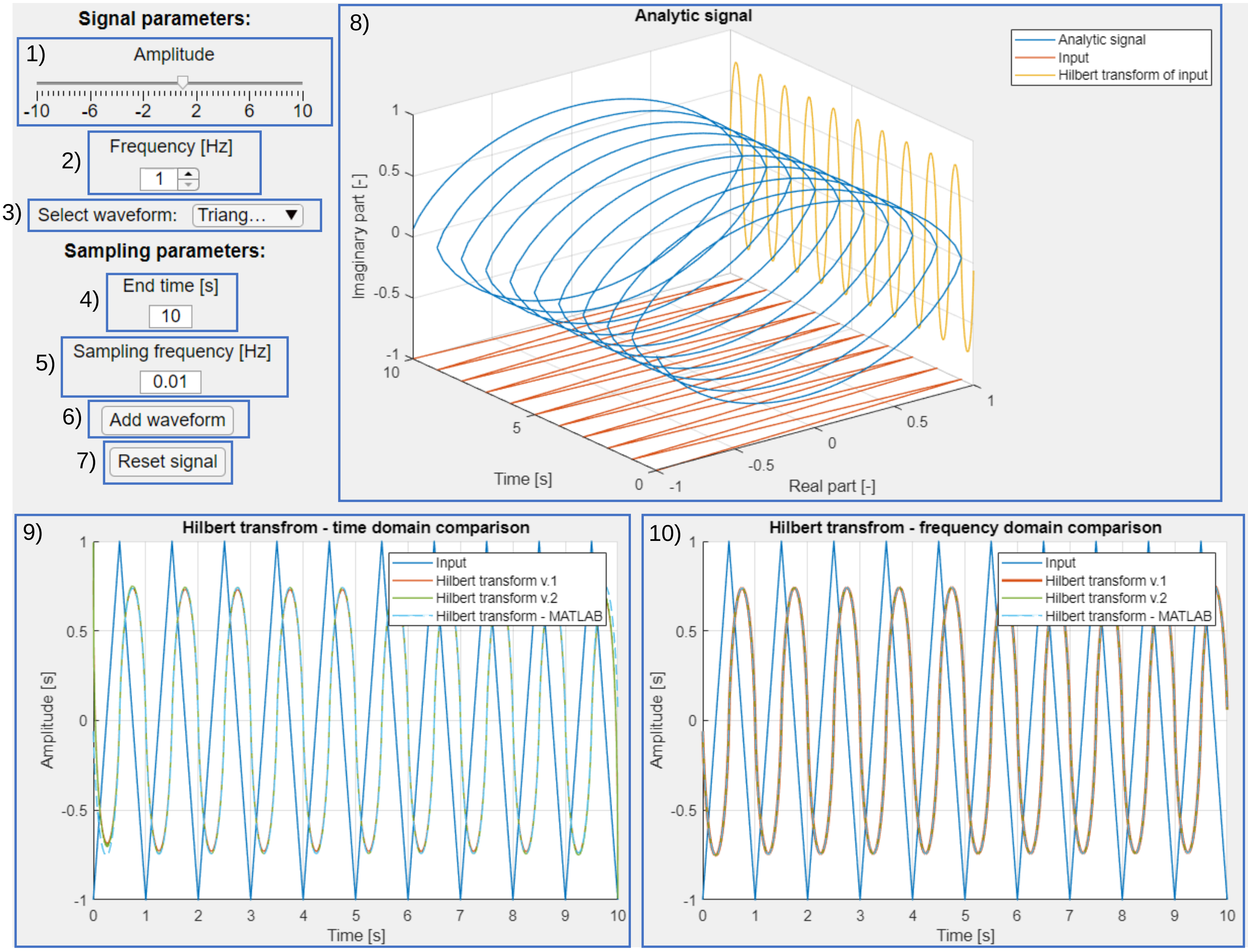
  - 2nd variant based on algorithm:

1. Create discrete frequency spectrum $X[n] = F\{x[n]\}$ as a result of discrete Fourier transform of input signal $x[n]$, where:
   n – sample number, $n = 0, 1, 2, \dots, N-1$,
   N – number of samples.
2. Multiply spectrum $X[n]$ by 2.
3. Set samples corresponding to negative frequencies to 0:
   $$X[n] = 0,\ \ for\ \ \frac{N}{2} + 1 \le n \le N - 1$$
4. Divide $X(0)$ and $X(N/2)$ by 2.
5. Calculate reverse Fourier transform of $X[n]$ spectrum to achieve $x_c[n]$ signal.
6. Signal $x_c[n]$ is analytic signal, its real part is equal to input signal $x[n]$, its imaginary part is equal to Hilbert transform of input signal $x[n]$.

Hilbert transform of input signals are plotted together – time-domain approach results on one figure and frequency-domain approach on another figure. Result of Hilbert transform implemented by built-in MATLAB® function is plotted as well, so that accuracy of created algorithms can be analysed.

Generated analytic signal is plotted and its real and imaginary parts projected onto real and imaginary planes.

Designed GUI is shown below:



GUI's elements breakdown:

1) Signal amplitude can be input using silder.
2) Signal frequency can be input using slider.
3) Waveform can be chosen from drop-down list (available waveforms: sine, cosine, square, sawtooth, traingular).
4) Signal end time can be input using textbox.
5) Sampling frequency can be input using textbox.
6) "Add waveform" button adds configured signal to currently defined input signal – therefore analyzed signal can have many components.
7) "Reset signal" sets input signal to 0.
8) Analytic signal is plottted on this figure, its real and imaginary parts are also projected onto real and imaginary planes to show analytic signal's real and imaginary component.
9) Results of both variants of time-domain approach Hilbert transform, result of built-in MATLAB® Hilbert transform, and input signal are plotted on this figure.
10) Results of both variants of frequency-domain approach Hilbert transform, result of built-in MATLAB® Hilbert transform, and input signal are plotted on this figure.

10. Conclusion

During this project, we managed to analyse and implement four algorithms which can calculate Hilbert transform in a MATLAB® GUI application. These algorithms can calculate the Hilbert transform in time domain and in frequency domain.

1st variant of time-domain approach Hilbert transform yielded better accuracy than 2nd variant, in comparison with MATLAB® built-in Hilbert transform function.

Results of both variants of frequency-domain Hilbert transform approach, and MATLAB® built-in Hilbert transform function are the same.

The GUI application was developed successfully, which can be used to visualize results of different methods, these methods can be easily compared with each other. We can sum different signals together and easily check if the Hilbert transform was calculated correctly due to the projections of the input signal and the signal after being transformed from the 3D plot to a 2D plane.

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