Volterra network modeling of the nonlinear finite-impulse reponse of the radiation belt flux

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Abstract. We show how a general class of spatio-temporal nonlinear impulse-response forecast networks (Volterra networks) can be constructed from a taxonomy of nonlinear autoregressive integrated moving average with exogenous inputs (NARMAX) input-output equations, and used to model the evolution of energetic particle f uxes in the Van Allen radiation belts. We present initial results for the nonlinear response of the radiation belts to conditions a month earlier. The essential features of spatio-temporal observations are recovered with the model echoing the results of state space models and linear f nite impulse-response models whereby the strongest coupling peak occurs in the preceding 1-2 days. It appears that such networks hold promise for the development of accurate and fully data-driven space weather modelling, monitoring and forecast tools.

Keywords: Radiation belts, input-output models, nonlinear neural networks
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1. INTRODUCTION

The chain of events leading to geospace magnetic storms begins with the ejection of solar plasma and plasma waves, followed by their propagation through the interplanetary medium and subsequent impact on the Earth's magnetosphere. Magnetic storms and substorms, two of the major complex dynamic phenomena in the terrestrial magnetosphere, have a number of distinct effects on the geospace environment such as electron and ion acceleration, auroral displays in the upper atmosphere at high latitudes, and geomagnetically-induced currents on the ground which may lead to electrical grid blackouts[5, 7, 6]. A highly prominent effect is the energization of the Van Allen radiation belts (see for example [3]) which is of particular interest to satellite operators, since radiation belt enhancements endanger spacecraft circuits and subsystems.

Many efforts in the last two decades have focused on the development of data-derived dynamical models of the energetic particle f ux in the radiation belts. Early linear prediction f lter studies focused on the temporal response of daily-averaged relativistic electrons at geostationary altitudes to solar plasma, interplanetary and magnetospheric drivers[10, 1]. Vassiliadis et al[14] extended this technique spatially by incorporating SAMPEX/PET electron f ux data over a broad range of L-shells from 1.1 to 10 Earth Radii (RE). The f rst self-consistent spatio-temporal state space models were then constructed[11], providing "one-step ahead" predictions for the nonlinear impulse-reponse of the radiation belts and, in particular, provided new and important clues to the timescales (typically days) involved in the dissipation of f ux.

In order to attempt to increase accuracy and, more importantly, to extend forecasts further forward in time in preparation for the development of a radiation belt storm warning index, we have adopted a methodology based on assuming an equivalence between NARMAX input-output equations and time-delay neural networks (which we will refer to as Volterra networks). Key to the success of this method are three fundamental theorems: the Wold Theorem[17] for general time series decomposition of data, Takens' Theorem[13] for time-delay embedding of nonlinear dynamics, and Hornik's Theorem[8] for universal function approximator neural networks. Takens' Theorem postulates that nonlinear input-output equations for decomposed time series data are fully capable of representing the nonlinear dynamics of the system under study provided that enough time lag variables are used. Equivalent Volterra networks can then be constructed by using time delays on the inputs and outputs and training on time series data to identify the nonlinear functional relation between input and output variables.

In this paper we brief y outline the method and present initial results of a 30^{th} order auto-regressive (feedback) 2D spatio-temporal model of the electric f ux in the radiation belts and show how the nonlinear f nite-impulse response transfer function as a function of lag time can be extracted from the network.

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2. METHODOLOGY

A schematic of the overall methodology adopted here is presented in Figure 1.

STEP 1: Construction of a taxonomy of NARMAX input-output equations

We began with a generalisation of the Wold time series decomposition [17] having the form,

$$J(t) = c + \Psi(t) + \varepsilon(t) \equiv \hat{J}(t) + \varepsilon(t) \quad (1)$$

where $J(t)$ is the electron f ux time series, $\hat{J}(t) = c + \Psi(t)$ are the model predictions, c is a constant (zero in the absence of trend), $\varepsilon(t) = J(t) - \hat{J}(t)$ are the prediction errors and $\Psi(t)$ is an "information function" constructed from lag series Φ_p, Θ_q, Ω_r

$$\Phi_p J(t) = \sum_{i=1}^{p} \varphi_i f_i \left[L^i J(t)\right] \quad (2)$$

$$\Theta_q \varepsilon(t) = \sum_{j=1}^{q} \theta_j g_j \left[L^j \varepsilon(t)\right] \quad (3)$$

$$\Omega_r I(t) = \sum_{k=1}^{r} \omega_k h_k \left[L^k I(t)\right] \quad (4)$$

with coeff cients φ_i, θ_j, ω_k, general functions f_i, g_j, h_k and lag operators,

$$L^i J(t) = J(t-i) \quad (5)$$

$$L^j \varepsilon(t) = \varepsilon(t-j) \quad (6)$$

$$L^k I(t) = I(t-k), \quad (7)$$

such that,

$$\begin{aligned} J(t) = c &+ \sum_{i=1}^{p} \varphi_i f_i \left[J(t-i)\right] + \sum_{j=1}^{q} \theta_j g_j \left[\varepsilon(t-j)\right] \\ &+ \sum_{k=1}^{r} \omega_k h_k \left[I(t-k)\right] + \varepsilon(t) \end{aligned} \quad (8)$$

in accordance with the Nonlinear AutoRegressive Moving-Average eXogenous input NARMAX(p,q,r) process. The introduction of general functions f, g and h into the lag series allows for a generalisation of the polynomial NARMAX models of Leontaritis and Billings (1985)[4]. Furthermore, in the case that the nonlinear system is driven by several inputs s then we will have a vector of inputs $I_l = \mathbf{I} = [I_1, I_2, \ldots, I_s]^\dagger$ and a corresponding vector of input lag series $\Omega_{r_l} = \Omega_{\mathbf{r}} = [\Omega_{r_1}, \Omega_{r_2}, \ldots, \Omega_{r_s}]$ where $r_l = \mathbf{r} = [r_1, r_2, \ldots, r_s]$ so that each I_l can have its own lag order r_l. In this general case, the nonlinear time series decomposition will be given by,

$$J(t) = c + \Phi_p J(t) + \Theta_q \varepsilon(t) + \Omega_{\mathbf{r}} \bullet \mathbf{I}(t) + \varepsilon(t) \quad (9)$$

and represents the general NARMAX$(p,q,\mathbf{r})$ process. Note that, our notation allows the NARMAX process to be written as an *explicit* sum of terms rather than the traditional presentation which would describe, for example, the NARMAX(p,q,r) process in terms of a general polynomial function F as follows, follows[4],

$$\begin{aligned} J(t) = c &+ F\{J(t-1), J(t-2)\ldots, J(t-p); \\ &\quad \varepsilon(t-1), \varepsilon(t-2), \ldots, \varepsilon(t-q); \\ &\quad I(t-1), I(t-2), \ldots, I(t-r)\} + \varepsilon(t). \end{aligned}$$

It is precisely the linear nature of the explicit sum introduced above that permits the development of a connectionist solution via neural network architectures. To recap, the information function contains linear combinations of (general nonlinear) operators acting on the autoregressive time-delayed (lagged) time series of radiation belt f ux $J(t-p)$, moving-average lagged equation errors $\varepsilon(t-q)$, and lagged exogenous inputs $\mathbf{I}(t-\mathbf{r})$. The particular class of model chosen depends on how exactly $\Psi(t)$ is def ned from the form of the functions f_i, g_j and h_k and the order of the autoregression, the moving average and the exogenous inputs. Table 1 below shows some key examples from the general taxonomy of NARMAX$(p,q,\mathbf{r})$ input-output equations.

STEP 2: Inclusion of nonlinear dynamics via time-delay embedding

Since Takens' Theorem[13] means that there is a 1-to-1 mapping between a time series and the underlying dynamical state space, then, provided that a nonlinear functional is used and enough lag variables are incorporated into the specif cation of the input-output model, equivalence will exist between any given NARMAX$(p,q,\mathbf{r})$ process and the nonlinear dynamical system it aims to represent. Assuming for now that the model orders p, q and $\mathbf{r}$ can be identif ed using phase space techniques (such as false nearest neighbours) then the problem at hand reduces to solving the follow equation for the nonlinear time series decomposition of the radiation belt f ux $J(t)$ in terms of the unknown coeff cients ϕ_i, ϕ_j, $\omega_{l,k}$ and the functions f_i, g_j and h_l, k,

$$\begin{aligned} J(t) = c &+ \varphi_1 f_1 \left[J(t-1)\right] + \cdots + \varphi_p f_p \left[J(t-p)\right] \\ &+ \theta_1 g_1 \left[\varepsilon(t-1)\right] + \cdots + \theta_q g_q \left[\varepsilon(t-q)\right] \\ &+ \omega_{1,1} h_{1,1} \left[I_1(t-1)\right] + \cdots \end{aligned}$$

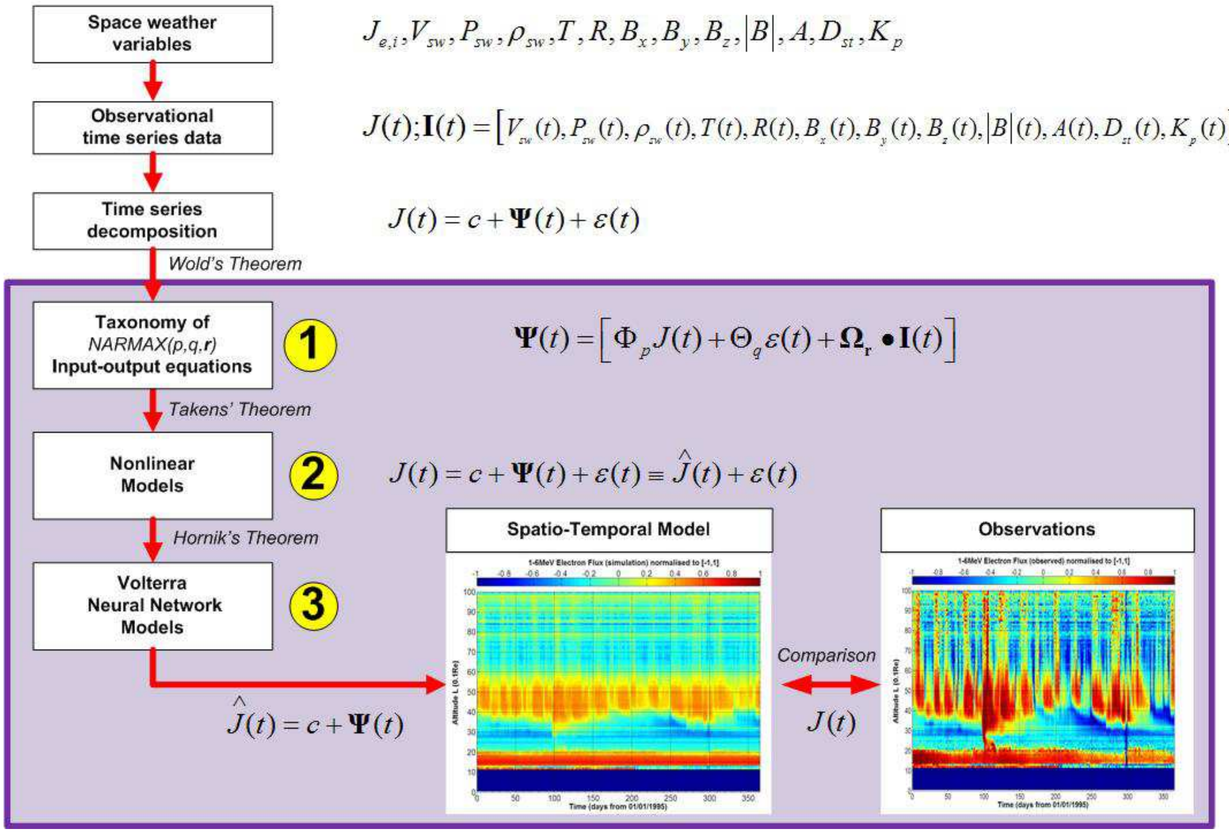

FIGURE 1. The overall methodology we have adopted together with a comparison of the observed SAMPEX/PET electron f ux observations normalised to the interval [-1,1] with initial results from our 2D spatio-temporal model based on the nonlinear autoregressive $NAR(30)$ Volterra network.

TABLE 1. A taxonomy of input-ouput models in the NARMAX$(p,q,\mathbf{r})$ class

Functions f, g, h	Autoregression Order	Moving-Avergae Order	Inputs	Model
1	1	0	0	AR(1)=Random walk
1	p	0	0	AR(p)
1	p	0	r	ARX(p,r)
1	p	0	$\mathbf{r}$	ARX$(p,\mathbf{r})$
1	0	q	0	MA(q)
1	0	∞	0	MA(∞)=Wold Decomposition
1	0	q	r	MAX(q,r)
1	0	q	$\mathbf{r}$	MAX$(q,\mathbf{r})$ (multivariate)
1	p	q	0	ARMA(p,q)
1	p	q	r	ARMAX(p,q,r)
1	p	q	$\mathbf{r}$	ARMAX$(p,q,\mathbf{r})$ (multivariate)
f	p	0	0	NAR(p)
f,h	p	0	r	NARX(p,r)
f,h	p	0	$\mathbf{r}$	NARX$(p,\mathbf{r})$ (multivariate)
g	0	q	0	NMA(q)
g,h	0	q	r	NMAX(q,r)
g,h	0	q	$\mathbf{r}$	NMAX$(q,\mathbf{r})$ (multivariate)
f,g	p	q	0	NARMA(p,q)
f,g,h	p	q	r	NARMAX(p,q,r)
f,g,h	p	q	$\mathbf{r}$	NARMAX$(p,q,\mathbf{r})$ (multivariate)

$$
\begin{aligned}
&+ \quad \omega_{1,r_1} h_{1,r_1} \left[I_1(t-r_1) \right] \\
&+ \quad \omega_{2,1} h_{2,1} \left[I_2(t-1) \right] + \cdots \\
&+ \quad \omega_{2,r_2} h_{2,r_2} \left[I_2(t-r_2) \right] \\
&\vdots \\
&+ \quad \omega_{s,1} h_{s,1} \left[I_s(t-1) \right] + \cdots \\
&+ \quad \omega_{s,r_s} h_{s,r_s} \left[I_s(t-r_s) \right] + \varepsilon(t). \qquad (10)
\end{aligned}
$$

STEP 3: Construction of nonlinear Volterra network models

Hornik's Theorem[8] tells us that nonlinear multilayer perceptrons (feed-forward neural networks) are universal and exact function approximators. Feedforward neural networks with lagged inputs create short-term memory and incorporate nonlinear dynamics into the network state space, and in the case that the activation functions of the artif cial neurons are linear, then they such networks operate as f nite impulse-response (FIR) f lters[16]. Such networks are connectionist analogs of the linear FIR integral models of Vassiliadis et al[14]. Here, we present results from *nonlinear* FIR networks (which we call Volterra networks as they redeem the mathematical properties of the nonlinear Volterra integral), constructed from NARMAX$(p,0,\mathbf{r})$ input-output equations, and whose general architecture is shown in Figure 2.

3. RESULTS

3.1. Identification of physical models

We trained Volterra networks on daily-averaged SAMPET/PET measurements of the radiation belt f ux from 01/01/1995 minus 30 days to day 335 using the Levenberg-Marquardt backpropagation algorithm[12] over 100 epochs and with 10 adaptive passes at each step in L-shell (0.1 RE). This allowed us to use the networks to predict data during the whole of 1995 for comparison with observed values. The year 1995 was selected so that the results could also be compared with those obtained by [11] with state space models. Figure 1 shows schematically the overall methodology we have adopted together with a comparison of the observed SAMPEX/PET electron f ux observations normalised to the interval [-1,1] with initial results from the a 30^{th}-order nonlinear autoregressive $NAR(30)$ Volterra network. In order to measure the degree of success in reproducing observed values $J(t)$ from the network model predictions $\hat{J}(t)$, we used the data-model correlation coeff cient C described in [14]:

$$
C = \frac{1}{T}\frac{1}{\sigma_J \sigma_{\hat{J}}} \int_0^T \left(\hat{J}(t) - \langle \hat{J}(t) \rangle \right) \left(J(t) - \langle J(t) \rangle \right) dt \qquad (11)
$$

where $\langle J(t) \rangle$ and σ_J are the mean and standard deviation of $J(t)$. Since the neuron activation function and neuron connection weights in Volterra networks are extractable and therefore explicit, the network architecture can be converted into equations with known AR(p), MA(q) and $X(\mathbf{r})$ coeff cients ϕ_p, θ_q, $\omega_{l,k}$ and functions f, g and h, specif ed through the form of the neuron activation functions (in the case of the Volterra networks used in this work, hyperbolic tangent functions).

3.2. Extraction of the nonlinear FIR transfer function

The spatio-temporal evolution of the radiation belt f ux calculated from the *linear* f nite-impulse response to a generalised input driver $I(t)$ over a range of lag intervals $\tau = T_s$ to $\tau = T_f$ and parameterised by L-shell is given by the solution of the integral,

$$
J(t;L) = \int_{\tau=T_s}^{T_f} H(\tau;L) I(t-\tau) d\tau, \qquad (12)
$$

for $H(\tau;L)$ [14]. In an $AR(p)$ model, the generalised input driver $I(t)$ is simply the auto-regressed lagged time series of the f ux $J(t-p)$. The corresponding integral equation is,

$$
J(t;L) = \int_{\tau=0}^{p} H(\tau;L) J(t-\tau) d\tau. \qquad (13)
$$

Since the time lags are taken to be integer multiples of the data sampling rate (i.e. $\tau = 1$ day), then $d\tau = 1$ and the integral can be written as the discrete sum,

$$
\begin{aligned}
J(t;L) &= \sum_{\tau=0}^{p} H(\tau;L) J(t-\tau;L) \qquad (14) \\
&= H(0;L)J(t) + H(1;L)J(t-1) \\
&+ H(2;L)J(t-2) + \ldots + H(p;L)J(t-p).
\end{aligned}
$$

In the case that the lagged time series in the FIR f lters of the Volterra network in Figure 2 are *linear* then for p lags and m neurons in the hidden layer of the FIR f lter, the network output constructed from the weights W_{pm} of the neurons connecting each time-lagged input is,

$$
\begin{aligned}
J(t;L) &= J(t-0;L)(W_{0,0} + W_{0,1} + \ldots + W_{0,m}) \\
&+ J(t-1;L)(W_{1,0} + W_{1,1} + \ldots + W_{1,m}) \\
&+ J(t-2;L)(W_{2,0} + W_{2,1} + \ldots + W_{2,m})
\end{aligned}
$$

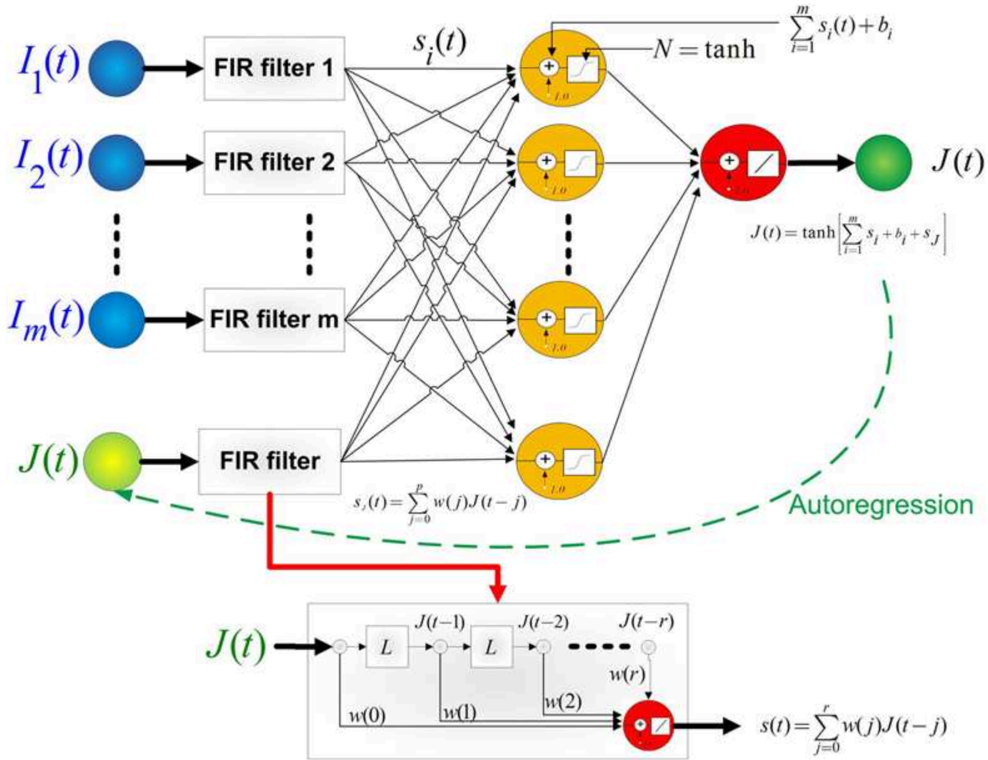

FIGURE 2. A schematic diagram of equivalent Volterra networks constructed from NARMAX$(p, q, \mathbf{r})$ input-output equations.

$$\begin{aligned} &\vdots \\ &+ \quad J(t-p;L)(W_{p,0} + W_{p,1} + \ldots + W_{p,m}) \\ &= \quad \sum_{\tau=0}^{p} J(t-\tau;L) \sum_{j=0}^{q} W_{\tau,j}. \end{aligned} \tag{15}$$

Comparing the equation for the linear impulse-response transfer function $H(\tau;L)$, then we see that,

$$H(\tau;L) = \sum_{j=0}^{q} W_{\tau,j} \tag{16}$$

i.e. the linear impulse-response transfer function for each τ is just the sum of the network weights in the hidden layer connected to the lagged time series $J(t-\tau)$. In the case of *nonlinear* f nite-impulse response, the transfer function will still be the sum of network weights but now the reconstructed transfer function will include a sum of weighted nonlinear activation functions (hyperbolic tangent functions).

In Figure 3a, a spatio-temporal model of the electron f ux calculated with a NAR(30) process is shown. Although the 30 lags used in the nonlinear FIR f lter makes it impractical to write down the full physical model obtained by the optimum network solution, in Figure 3b, we reproduce the nonlinear response obtained from the nonlinear FIR transfer function with the "one-step" ahead state-space results of [11] for the same data and L-shell for comparison. It can be seen that the Volterra network results echo the *geometrical* features of the nonlinear spatio-temporal response obtained by state space buts, but with higher resolution due to the increase in information content provided by the use of mutliple lags (in this case 30 time steps-ahead). Figure 3c shows a visual comparison of the raw data with the Volterra model calculated in the centre of the main response at $L = 3.6RE$. Figure 3d shows the value of the data-model correlation coeff cient C calculated over all L-shells peaking as expected in the centre of the main response.

4. DISCUSSION

These early results suggest that the construction of equivalent Volterra networks from NARMAX input-output equations has potential for improving the spatio-temporal modelling of the radiation belts and for recovering the nonlinear dynamics implicit in the data. We are currently in the process of including exogenous inputs for solar plasma, interplanetary and magnetospheric drivers into the Volterra network architecture so as to move toward identif cation of a 2D spatio-temporal *physical* model of the Van Allen belts.

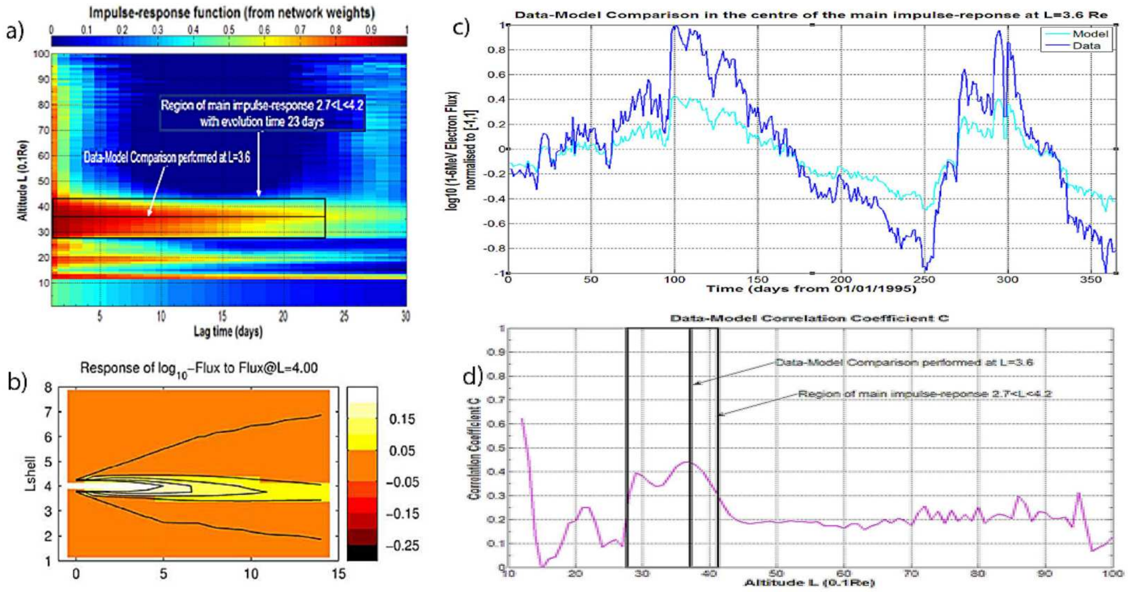

FIGURE 3. The nonlinear 2D spatio-temporal impulse-response during the year 1995. Figure 3a) shows the nonlinear FIR transfer function as calculated from the weights of the Volterra NAR(30) network. Figure 3b) reproduces the response from the "one-step" ahead state-space model of Rigler et al[11] (their Figure 4) for the same data, while Figure 3c) shows a visual data-model comparision at the centre of the main response at $L = 3.6$ while Figure 3d shows the value of the data-model correlation coeff cient C calculated over all L-shells.

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