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Wave propagation through cyclotron resonance in the presence of large Larmor radius particles

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Absorption of waves propagating across an inhomogenous magnetic field is of crucial importance for cyclotron resonance heating. When the Larmor radius of the resonant particles is small compared to the wavelength then the propagation is described by differential equations, a comparatively simple method for obtaining which has recently been given by Cairns *et al.* [Phys. Fluids B **3**, 2953 (1991)]. In a fusion plasma there may, however, be a significant population of ions whose Larmor radius is not small compared to the wavelength. In this case the system is described by integro-differential equations, reflecting the fact that the plasma response at a given position is determined by the wave field over a region of width of the order of the Larmor radius. The simplified method referred to above is adapted to this case and used to obtain various forms of the equations. Methods of simplifying the equations while still retaining information from the non-local response, are discussed and some illustrated numerical results presented.

I. INTRODUCTION

Cyclotron heating of either ions or electrons is of vital importance in various schemes for heating magnetically confined plasmas. The theory of cyclotron absorption requires, as its starting point, the derivation of equations to describe the propagation of waves through a region of cyclotron resonance treating, in the simplest case, a slab geometry in which the gradient of the magnetic field strength is perpendicular to the field. A considerable number of authors have studied this problem for the case when the Larmor radius of a thermal particle is much less than the wavelength, in which case there is a local response of the plasma to the waves, in the sense that the current at a point depends only on the fields and their derivatives at that point, and the system is described by differential equations.[1–5] Some recent work by the present authors[6] has shown how these equations may be obtained in a comparatively simple way. Earlier work using a somewhat similar approach was carried out by Antonsen and Manheimer,[7] though they worked in Fourier transform space and could only obtain a tractable approximation by making an expansion which is equivalent to taking the asymptotic expansion of the plasma dispersion function. We work in real space, where it is possible to obtain much more general results. Our approach begins with the uniform plasma dielectric tensor and then recognises that, in the presence of a magnetic field gradient (with the gradient perpendicular to the direction of the field), the non-uniform response is obtained by evaluating the cyclotron frequency in the resonant denominators at the position of the particle guiding centre. This condition arises automatically in gyrokinetic theory where its importance for cyclotron resonance has been emphasised in Ref. 8. The technique has also been applied to the weakly relativistic problem, which is relevant to electron cyclotron heating.[9]

In the case of ion cyclotron heating, particularly when minority heating is being used or when hot fusion products are present, the assumption of small Larmor radius may not be valid. In this case the response of the plasma to the waves is non-local and the system is described by integro-differential equations. These have been derived by Sauter and Vaclavik[10,11] and by Brambilla.[12] Our purpose here is to show how the simple method referred to above can be used to obtain the governing equations for the large Larmor radius case more easily. The method also provides a convenient way of generating different forms of the equations. The results obtained are completely equivalent to those obtained rigorously by taking a Fourier transform of the wave problem in an inhomogeneous medium.

We then develop Wentzel–Kramers–Brillouin (WKB) and fast wave[13,14] approximations to these equations, which include the full finite Larmor radius effects in a non-uniform magnetic field, but which are computationally much simpler than the full integro-differential equations. In particular, the fast wave approximation, which reduces the problem to a second order ordinary differential equation, should be valuable in allowing simple and rapid numerical modelling of experiments in which fusion plasmas are heated by waves in the ion cyclotron range of frequencies. Some illustrative examples are given of the use of the fast wave approximation for the case of minority ion cyclotron heating.

II. DERIVATION OF THE EQUATIONS FOR A LINEAR FIELD GRADIENT

Initially we shall treat the case of a linear field gradient with $B=B_o(1-x/L)$, since this relates to our previous work on the small Larmor radius case and gives rather simpler equations than the more general case in which we allow arbitrary variations, in the direction perpendicular to the field, of the field strength, density and temperature. In the next section we shall discuss this general case, allowing for an arbitrary density, temperature in addition to magnetic field variation. For simplicity we shall discuss only the z-z element of the conductivity tensor, since it serves to illustrate the method. All other elements can be obtained in a similar way. We use the usual coordinate system in which the magnetic field is along the z-direction. Also, we shall consider resonance at the fundamental of the ion cyclotron frequency. Again, the basic method is easily adaptable to any harmonic.

We begin with a standard integration along orbits, for a uniform plasma, which gives

$$\sigma_{zz}=i\epsilon_o \frac{\omega_p^2}{\omega}\int uv_\perp du dv_\perp d\theta \frac{\partial f_0}{\partial u} J_1(b)e^{i(b\sin\theta-\theta)} \times \int_{-\infty}^{0} d\tau \exp\{-i\tau(\omega-k_\| u-\omega_c)\} \tag{1}$$

where the usual cylindrical coordinates in velocity space are being used, with u the parallel velocity, and $b=k_\perp v_\perp/\omega_c$. Now, we recognise that the part of Eq. (1) where the spatial dependence of ω_c is important is in the final resonant integral, and that we can take this into account by putting

$$\omega-\omega_c=\omega_c\left(\frac{x}{L}+\frac{v_\perp}{L\omega_c}\sin\theta\right). \tag{2}$$

Elsewhere we can simply put $\omega\approx\omega_c$. The second term in the bracket in Eq. (2) arises because, as pointed out above, we must evaluate the field at the guiding centre of the particle, not at its final position. This is the gyrokinetic effect discussed by Lashmore-Davies and Dendy.[8]

Since the variable x has already been Fourier transformed in obtaining Eq. (1), the introduction of x here should be regarded as being part of a separation into different length scales, with the $k_\perp$ corresponding to the short scale length of the waves and the x to the long scale length of the equilibrium gradient. This simple procedure gives the same result as orbit integration carried out to first order in x/L in a non-uniform field. We shall take $k_y=0$, but if $k_y\neq 0$ then the drift velocity due to the magnetic field gradient should be taken into account, since it can introduce a term of the same order as the gyrokinetic effect when $k_y\rho\gtrsim 1$ where ρ is the Larmor radius of a resonant particle.

If Eq. (2) is substituted into Eq. (1) and the variable in the τ integral changed to $k=-\omega_c\tau/L$ we obtain

$$\sigma_{zz}=-i\frac{L\omega_p^2}{\omega^2}\int uv_\perp du\, dv_\perp d\theta \frac{\partial f_o}{\partial u} J_1(b)e^{i(b\sin\theta-\theta)} \times\int_o^\infty dk \exp\left\{ikx-\frac{ikk_\| Lu}{\omega_c}+i\frac{kv_\perp}{\omega_c}\sin\theta\right\}.$$

Using

$$\exp\left\{i\frac{(k_\perp+k)}{\omega_c}v_\perp\sin\theta\right\}=\sum_n J_n\left(\frac{(k_\perp+k)}{\omega}v_\perp\right)e^{in\theta},$$

the integrals over velocity can be carried out in the usual way. In terms of $\sigma_{zz}(k_\perp)$, the z-component of the current coming from the z-z component of the conductivity tensor, is, in a uniform plasma,

$$J(x)=\int_{-\infty}^{\infty} dk' E(k')\sigma_{zz}(k')e^{ik'x}.$$

In the non-uniform case, we substitute the expression obtained above for σ_{zz}, depending both on $k_\perp$ and explicitly on x, in this integral to obtain

$$J(x)=\epsilon_o L\frac{\omega_p^2}{\omega^2}\int_{-\infty}^{\infty} dk' E(k')\int_0^\infty dk \times e^{i(k+k')x-(k^2\rho^2/4)-(k_\|^2L^2k^2\rho^2/4)} \times I_1\left(\frac{k'(k+k')\rho^2}{2}\right) \times e^{-k'(k+k')(\rho^2/2)}(1-\tfrac{1}{2}k_\|^2L^2k^2\rho^2). \tag{3}$$

In this equation ρ is the Larmor radius of a thermal particle, i.e. v_{th}/ω_c where the distribution function has been taken to be proportional to $\exp(-v^2/v_{th}^2)$, and E is the z-component of the electric field. If the Larmor radius is small we may expand the Bessel function and the final exponential function in power series and use the fact that

$$\int_0^\infty dk e^{ikx-(k^2\rho^2/4)-(k_\|^2L^2k^2\rho^2/4)} = \frac{1}{i\rho(1+k_\|^2L^2)^{1/2}}Z\left(\frac{x}{\rho(1+k_\|^2L^2)^{1/2}}\right).$$

Powers of k' then produce derivatives of the electric field and powers of k derivatives of the Z function and the integral in Eq. (3) becomes a differential operator acting on E, as discussed in Ref. 6. The procedure described here is a somewhat streamlined version of that given in the earlier paper. Now, however, we wish to consider the large Larmor radius regime where such an expansion is not valid. In this regime we might expect the response of the plasma to the field to be non-local and the current to be given by a term of the form

$$J(x)=\epsilon_0 L\frac{\omega_p^2}{\omega^2}\int_{-\infty}^{\infty} E(x')G(x,x')dx' = \epsilon_0 L\frac{\omega_p^2}{\omega^2}\int_{-\infty}^{\infty} dk' E(k')\int_{-\infty}^{\infty} dx' e^{ik'x'}G(x,x'). \tag{4}$$

We now note that Eqs. (3) and (4) will be identical if

$$\int_{-\infty}^{\infty} G(x,x')e^{ik'x'}dx' = \int_0^\infty dk\, e^{i(k+k')x-k^2\rho^2/4-k^2k_\|^2L^2\rho^2/4} I_1\left(\frac{k'(k+k')\rho^2}{2}\right) \times e^{-k'(k+k')\rho^2/2}(1-\tfrac{1}{2}k^2k_\|^2L^2\rho^2).$$

Since the left hand side of this equation is the Fourier transform of G with respect to x', we can use the Fourier inversion theorem to obtain

$$G(x,x')=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk'e^{-ik'x'}\int_0^{\infty}dk \times e^{i(k+k')x-k^2\rho^2/4-k^2k_\parallel^2L^2\rho^2/4} \times I_1\left(\frac{k'(k+k')\rho^2}{2}\right) \times e^{-k'(k+k')\rho^2/2}(1-\tfrac{1}{2}k^2k_\parallel^2L^2\rho^2). \quad (5)$$

Equation (5) gives an explicit expression for $G(x,x')$, but as a double integral over an infinite half-plane it is not a very suitable form for numerical calculation or further analysis.

One way of simplifying Eq. (5) to some extent is to use the expression

$$I_1(z)=\frac{1}{\pi}\int_0^{\pi}e^{z\cos\theta}\cos\theta\, d\theta. \quad (6)$$

The integrand then involves the exponential of a quadratic in k and k'. The transformation $k'=K-\frac{1}{2}k$ switches to the principal axes of this quadratic and separates the K and k integrals. Using

$$Z(\zeta)=i\int_0^{\infty}\exp\left(ik\zeta-\frac{k^2}{4}\right)dk,$$
$$Z''(\zeta)=-i\int_0^{\infty}k^2\exp\left(ik\zeta-\frac{k^2}{4}\right)dk \quad (7)$$

we obtain

$$G(x,x')=\frac{\rho}{2^{1/2}\pi^{3/2}i}\int_0^{\pi}d\theta\cos\theta \times e^{-[(x-x')^2/2\rho^2(1-\cos\theta)]}(1-\cos\theta)^{-1/2} \times\left[\frac{1}{a}Z\left(\frac{x+x'}{a\rho}\right)+\frac{k_\parallel^2L^2}{a^3}Z''\left(\frac{x+x'}{a\rho}\right)\right] \quad (8)$$

with

$$a=(2+2\cos\theta+4k_\parallel^2L^2)^{1/2}.$$

This reduces G to a single integral over a finite range rather than a double integral over an infinite half-plane. An alternative derivation avoiding the use of Fourier transforms is given in the Appendix.

III. GENERAL GRADIENTS

The previous section deals with linear magnetic field gradients and neglects gradients in density or temperature. Since the resonance condition is determined by the magnetic field, this approximation may be adequate for many purposes. It is, however, of interest to consider the more general case where we show that a comparatively simple calculation can give the results of Brambilla and of Vaclavik and Sauter.[10–12] Again, for the sake of illustration, we restrict our attention to the z-z element of the dielectric tensor, and begin with it in the form

$$\sigma_{zz}=-\epsilon_0\frac{\omega_p^2}{\omega^2}\int uv_\perp du\, dv_\perp d\theta\frac{\partial f_0}{\partial u} \times\frac{J_1(kv_\perp/\omega_c)e^{(ikv_\perp\sin\theta/\omega_c)-i\theta}}{\omega-\omega_c-k_\parallel u}. \quad (9)$$

This is just the standard homogeneous plasma expression, with k the perpendicular wave number. As before we have separated out the resonant contribution for the fundamental resonance.

If we now suppose that the parameters have a slow x-dependence, we can regard this as a dependence on a slowly varying variable x, despite the fact that we have already Fourier transformed over the rapid x variation corresponding to the oscillations of the fields in the wave. However, we must recognise that, as before, the dependence should be on the values of the parameters at the guiding centre, not at the final position of the particle. Thus the spatial dependence comes through the magnetic field, density and temperature being evaluated at

$$x+\frac{v_\perp\sin\theta}{\omega_c}.$$

This can be done by writing

$$\sigma_{zz}=-i\epsilon_0\int dx''\frac{\omega_p^2}{\omega}\int uv_\perp\, du\, dv_\perp\, d\theta\frac{\partial f_0}{\partial u} \times\frac{J_1(kv_\perp/\omega_c)e^{ikv_\perp\sin\theta/\omega_c-i\theta}}{\omega-\omega_c-k_\parallel u}\delta\left(x''-x-\frac{v_\perp\sin\theta}{\omega_c}\right). \quad (10)$$

In this integral the density, temperature and magnetic field, in the distribution function or elsewhere are to be taken as functions of x''. For convenience, we have taken the distribution function normalised so that its integral over velocity is one, the density variation being in the plasma frequency.

The contribution to the current from this tensor element is

$$J(x)=\int_{-\infty}^{\infty}dke^{ikx}\sigma_{zz}E(k), \quad (11)$$

where E is the Fourier transform of the z-component of the field. Again we suppose that this current is given by a nonlocal response of the form

$$J(x)=\int_{-\infty}^{\infty}G(x,x')E(x')dx' =\int_{-\infty}^{\infty}dk\, E(k)\int_{-\infty}^{\infty}dx'\, G(x,x')e^{ikx'}. \quad (12)$$

Comparing (11) and (12) gives

$$\int_{-\infty}^{\infty}dx'\, G(x,x')e^{ikx'}=e^{ikx}\sigma_{zz}$$

and, inverting the Fourier transform, we obtain

$$G(x,x')=-\frac{i\epsilon_0}{2\pi}\int_{-\infty}^{\infty}dk e^{ik(x-x')}\int dx''\frac{\omega_p^2}{\omega^2}$$

$$\times\int uv_\perp du dv_\perp d\theta$$

$$\times\frac{\partial f_0}{\partial u}\frac{J_1(kv_\perp/\omega_c)e^{ikv_\perp\sin\theta/\omega_c-i\theta}}{\omega-\omega_c-k_\parallel u}$$

$$\times\delta\left(x''-x-\frac{v_\perp\sin\theta}{\omega_c}\right). \quad (13)$$

Assuming the velocity distribution to be Maxwellian, we can carry out the integral over u, using

$$\int_{-\infty}^{\infty}du\frac{u^2}{\omega-\omega_c-k_\parallel u}e^{-u^2/v_{th}^2}=\frac{v_{th}^2}{k_\parallel}\xi(1+\xi Z(\xi))$$

where Z is the plasma dispersion function and

$$\xi=\frac{\omega-\omega_c}{k_\parallel v_{th}}.$$

If we now use the formula

$$\delta\left(x'-x-v_\perp\frac{\sin\theta}{\omega_c}\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ik'[x'-x-(v_\perp\sin\theta/\omega_c)]}dk'$$

then Eq. (13) is found to contain a factor

$$\int_0^\infty dv_\perp\int_0^{2\pi}d\theta\, v_\perp e^{-v_\perp^2/v_{th}^2}J_1\left(\frac{kv_\perp}{\omega_c}\right)$$

$$\times\exp\left[\frac{i(k-k')v_\perp}{\omega_c}\sin\theta-i\theta+ik'(x''-x)\right]$$

which can be treated by methods familiar from the derivation of the dielectric tensor in a hot uniform plasma to give

$$e^{ik'(x''-x)}I_1\left(\frac{k(k+k')v_{th}^2}{2\omega_c^2}\right)\exp\left[-\frac{k^2+(k+k')^2}{4\omega_c^2}v_{th}^2\right].$$

This still leaves infinite integrals over k and k' in the expression for G. As in the previous section it is possible to reduce these to a single integral over a finite range by again using the identity of Eq. (6).

The integrals over k and k' then become

$$\int_{-\infty}^{\infty}dk\int_{-\infty}^{\infty}dk'\exp\left\{ik'(x''-x)+ik(x-x')\right.$$

$$\left.+\frac{k(k+k')\rho^2}{2}\cos\theta-\frac{k^2\rho^2}{4}-\frac{(k+k')^2\rho^2}{4}\right\}$$

which can be integrated using standard techniques to give the final result

$$G(x,x')=\frac{i\epsilon_0}{\pi}\int dx''\frac{\omega_p^2}{k_\parallel v_{th}\rho^2}\xi(1+\xi Z(\xi))\int_0^\pi d\theta\cot\theta\exp\left[-\frac{(x-x')^2(1-\cos\theta)+4(x''-\frac{1}{2}x-\frac{1}{2}x')^2(1+\cos\theta)}{4\rho^2\sin^2\theta}\right]. \quad (14)$$

In this formula the spatial dependence of the plasma frequency, the Larmor radius ρ, etc., is to be taken into account by regarding them as the appropriate functions of x''. This result is in a form identical to that derived by Sauter and Vaclavik.[11] It is clear from the presence of the final exponential term in Eq. (14) that there is only a significant contribution from values of x, x' and x'' within a few Larmor radii of each other. It is unlikely that smoothing out density and temperature variations over such a scale length, as opposed to taking the local value, will make much difference to absorption calculations. The magnetic field, however, appears in the argument of the Z function which can vary rapidly in the vicinity of a cyclotron resonance. It is in the evaluation of ξ as a function of x'' that the important effects of inhomogeneity occur rather than in ω_p^2, v_{th} or ρ. For the linear magnetic field strength gradient study in Section II we have $\xi=\omega_c x''/Lk_\parallel v_{th}$. If the integral representation of the plasma dispersion function is used once again, the integral over x'' in Eq. (14) can then be carried out analytically and we recover the results of Section II, though it is more straightforward, as there, to introduce the linear magnetic field gradient at an earlier stage in the calculation. This calculation also demonstrates that in the limit as $k_\parallel\to 0$ there is not, as might appear from the form of Eq. (14), any singularity and that the absorption profile remains of finite width as would be expected since the gyrokinetic correction is included.[8]

IV. WKB SOLUTIONS AND REDUCTION TO DIFFERENTIAL EQUATIONS

A WKB approximation, using the integral response calculated in Section II can be obtained as follows. A convenient starting point is provided by combining Eqs. (4) and (5), showing that the plasma current is

$$J(x)=\frac{\varepsilon_o L}{2\pi}\frac{\omega_p^2}{\omega^2}\int_{-\infty}^{\infty}dx' E(x')\int_{-\infty}^{\infty}dk' e^{-ik'x'}$$

$$\times\int_0^\infty dk\exp\left\{i(k+k')x-\frac{k^2\rho^2}{4}-\frac{k_\parallel^2L^2k^2\rho^2}{4}\right\}$$

$$\times(1-\tfrac{1}{2}k_\parallel^2L^2k^2\rho^2)I_1\left(\frac{k'(k+k')}{2}\right)$$

$$\times\exp\left\{-\frac{k'(k+k')\rho^2}{2}\right\}. \quad (15)$$

If we take $E(x)=E_0e^{ik_o x}$ then the integral over x' in Eq. (15) just involves

$$\int_{-\infty}^{\infty} dx' e^{i(k_o-k')}x' = 2\pi\delta(k_o-k')$$

which, in turn allows us to evaluate the k' integral and leaves us with

$$J(x)=\varepsilon_o L \frac{\omega_p^2}{\omega^2} E_o e^{ik_o x} \int_0^{\infty} dk \times \exp\left\{ ikx - \frac{k^2\rho^2}{4} - \frac{k_\parallel^2 L^2 k^2 \rho^2}{4} \right\} \times (1-\tfrac{1}{2}k_\parallel^2 L^2 k^2 \rho^2) I_1\left(\frac{k_o(k+k_o)}{2} \right) \times \exp\left\{ -\frac{k_o(k+k_o)\rho^2}{2} \right\}. \quad (16)$$

All the dielectric tensor elements behave similarly, so we can obtain a local dispersion relation in which the coefficients are integrals, which retain the non-local response to the field, rather than the simple polynomials in k_o which would result from a differential equation. The integral of the imaginary part of k_o through the resonance region generally yields a good approximation to the wave transmission coefficient though it does not, of course, give any information on reflection or mode conversion.

A related approximation which can give the reflection coefficient, but does not separate mode conversion from cyclotron damping, is the fast wave approximation,[13,14] which is very similar to the widely used Born approximation in the theory of atomic collisions.[15] This is a perturbative method in which the unknown electric field, which occurs in the kernels of the integrals describing the resonant non-local response (the scattering terms), is approximated by a plane wave $E(x)=E_0 e^{ik_0 x}$ where the wave number k_0 is obtained from the cold plasma dispersion relation. A term of the form

$$H(k_0,x)E_0 e^{ik_0 x} \simeq H(k_0,x)E(x) \quad (17)$$

is obtained in exactly the same way as Eq. (16). The fast wave approximation consists of replacing the full integral by the terms of the form given in Eq. (17), while retaining the derivatives of E which come from the $\nabla\times(\nabla\times\mathbf{E})$ term in the wave equation. In this way a simple differential equation for the electric field is obtained, with the large Larmor radius effect included through the coefficients which are of the form of Eq. (16) and the corresponding terms of a similar nature for the other dielectric tensor elements.

Some preliminary work has been carried out on the application of this technique to minority cyclotron damping. With the usual neglect of the z-component of the electric field, the equations for the other two components become

$$\left(\frac{\omega^2}{c^2} - \frac{\omega}{c^2}\frac{\omega_{pa}^2}{\Omega_a}\left[\frac{r_1}{(r_1^2-1)} + \frac{r_2}{4} \right] \right) E_x(x) - \frac{i\omega}{c^2}\frac{L}{\rho_b}\frac{2\omega_{pb}^2}{v_{Tb}} \times K_{xx}(x)E_x(x) \frac{i\omega}{c^2}\frac{\omega_{pa}^2}{\Omega_a}\left[\frac{r_1^2}{(r_1^2-1)} + \frac{3}{4} r_2 \right] E_y(x) + \frac{\omega}{c^2}\frac{L}{\Omega_b}\omega_{pb}^2 K_{xy}(x)E_y(x)=0 \quad (18)$$

where the subscripts "a" and "b" denote the majority and minority ion species respectively,

$$K_{xx}(k_o,x) = \int_0^{\infty} e^{i\lambda x} e^{-\lambda^2\rho^2/4} e^{-(k_0+\lambda)k_0\rho^2/2} G_{xx}(k_0+\lambda,k_0)\,d\lambda, \quad (19)$$

$$K_{xy}(k_o,x) = \int_0^{\infty} e^{i\lambda x} e^{-\lambda^2\rho^2/4} e^{-(k_0+\lambda)k_0\rho^2/2} \times G_{xy}(k_0+\lambda,k_0)\,d\lambda, \quad (20)$$

and

$$G_{xx}(k,k') = \frac{1}{kk'} I_1(kk'\rho^2/2), \quad (21)$$

$$G_{xy}(k,k') = I_1'\left(\frac{kk'\rho^2}{2} \right) - \frac{k'}{k} I_1\left(\frac{kk'\rho^2}{2} \right). \quad (22)$$

We also obtain

$$-\frac{i\omega}{c^2}\frac{\omega_{pa}^2}{\Omega_a}\left[\frac{r_1^2}{(r_1^2-1)} + \frac{3}{4} r_2 \right] E_x(x) + \frac{L}{\Omega_b}\frac{\omega}{c^2}\omega_{pb}^2 K_{yx}(k_o,x)E_x(x) - \left(\frac{d^2}{dx^2} + \frac{\omega}{c^2} - \frac{\omega}{c^2}\frac{\omega_{pa}^2}{\Omega_a}\left[\frac{r_1}{(r_1^2-1)} + \frac{r_2}{4} \right] \right) E_y(x) + \frac{iL}{\Omega_b}\frac{\omega}{c^2}\omega_{pb}^2 K_{yy}(k_o,x)E_y(x)=0 \quad (23)$$

where

$$K_{yx}(k_o,x) = \int_0^{\infty} e^{i\lambda x} e^{-\lambda^2\rho^2/4} e^{-(k_0+\lambda)k_0\rho^2/2} \times G_{yx}(k_0+\lambda,k_0)\,d\lambda, \quad (24)$$

$$K_{yy}(k_o,x) = \int_0^{\infty} e^{i\lambda x} e^{-\lambda^2\rho^2/4} e^{-(k_0+\lambda)k_0\rho^2/4} \times G_{yy}(k_0+\lambda,k_0)\,d\lambda, \quad (25)$$

and

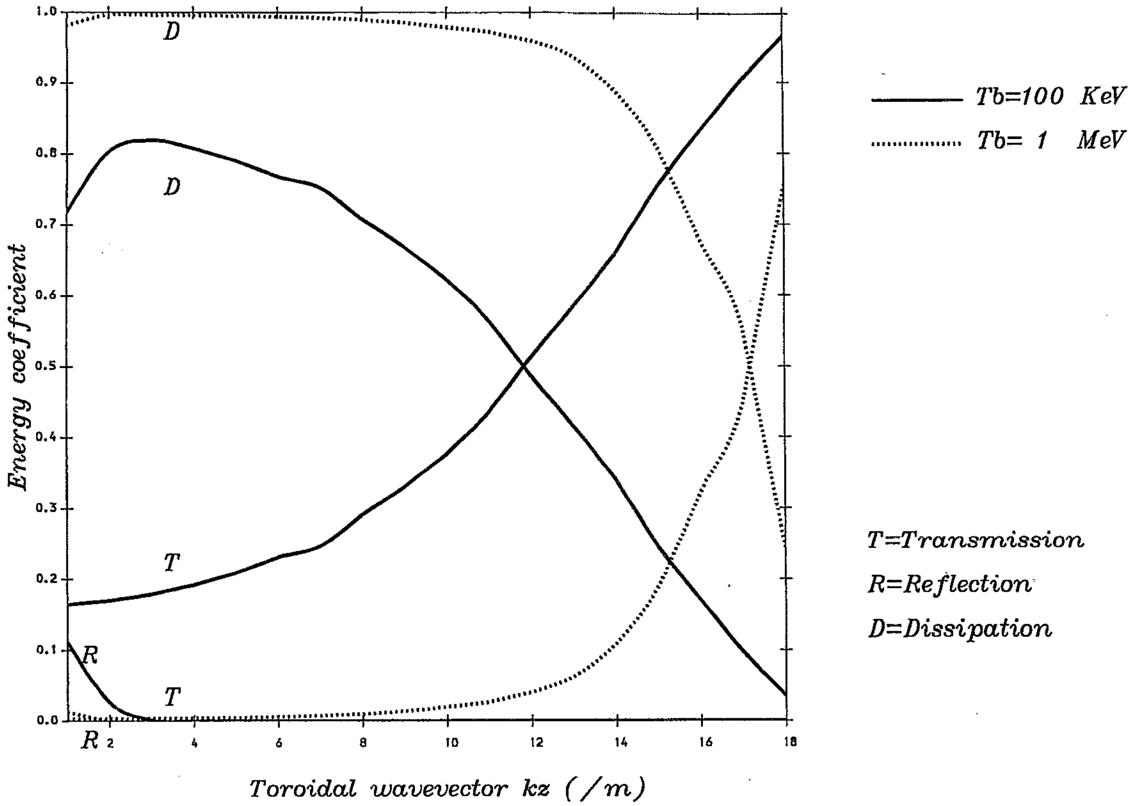

FIG. 1. Non-uniform, large Larmor radius calculation of the transmission (T), absorption (D) and reflection (R) coefficients as a function of the toroidal (parallel) wave number k_z for a fast wave incident on the helium-3 fundamental resonance from the low field side in a plasma where the majority ion species is deuterium. The plasma parameters are $n_e = 5\times 10^{19}\ \mathrm{m}^{-3}$, $n_{3_{\mathrm{He}}}/n_e = 0.05$, $B_o = 3.4$ T, $L = 3.1$ m for helium-3 temperatures of 100 keV (solid line) and 1 MeV (dotted line).

$$G_{yx}(k,k') = I_1'\left(\frac{kk'\rho^2}{2}\right) - \frac{k}{k'} I_1\left(\frac{kk'\rho^2}{2}\right), \tag{26}$$

$$G_{yy}(k,k') = \left(2 + \frac{4}{k^2 k'^2 \rho^4}\right)\frac{kk'\rho^2}{2} I_1\left(\frac{kk'\rho^2}{2}\right) - (k^2 + k'^2)\frac{\rho^2}{2} I_1'\left(\frac{kk'\rho^2}{2}\right). \tag{27}$$

The quantities r_1 and r_2 are given by $r_1 = \Omega_b/\Omega_a$, $r_2 = n_{0b}Z_b/n_{0a}Z_a$ where Z_a, Z_b are the charges of the two ion species and n_{oa} and n_{ob} their equilibrium densities. The fast wave equation can now be obtained from Eqs. (18) and (23) by eliminating $E_x(x)$ in favour of $E_y(x)$, giving

$$\frac{d^2}{dx^2} E_y(x) + V(x)E_y(x) = 0 \tag{28}$$

where

$$\begin{aligned} V(x) = &\left[\left\{\frac{\omega^2}{c^2} - \frac{\omega}{c^2}\frac{\omega_{pa}^2}{\Omega_a}\left[\frac{r_1}{(r_1^2-1)} + \frac{r_2}{4}\right] - \frac{iL}{\Omega_b}\frac{\omega}{c^2}\,\omega_{pb}^2 K_{yy}(k_o,x)\right\}\left\{\frac{\omega^2}{c^2} - \frac{\omega}{c^2}\frac{\omega_{pa}^2}{\Omega_a}\left[\frac{r_1}{(r_1^2-1)} + \frac{r_2}{4}\right] - \frac{i\omega}{c^2}\frac{L}{\rho_b}\frac{2\omega_{pb}^2}{v_{Tb}} K_{xx}(k_o,x)\right\}\right. \\ &\left. + \left\{\frac{i\omega}{c^2}\frac{\omega_{pa}^2}{\Omega_a}\left[\frac{r_1^2}{(r_1^2-1)} + \frac{3}{4} r_2\right] - \frac{L}{\Omega_b}\frac{\omega}{c^2}\,\omega_{pb}^2 K_{yx}(k_o,x)\right\}\left\{\frac{i\omega}{c^2}\frac{\omega_{pa}^2}{\Omega_a}\left[\frac{r_1^2}{(r_1^2-1)} + \frac{3}{4} r_2\right] - \frac{\omega}{c^2}\frac{L}{\Omega_b}\,\omega_{pb}^2 K_{xy}(k_o,x)\right\}\right] \\ &\times\left[\frac{\omega^2}{c^2} - \frac{\omega}{c^2}\frac{\omega_{pa}^2}{\Omega_a}\left[\frac{r_1}{(r_1^2-1)} + \frac{r_2}{4}\right] - \frac{i\omega}{c^2}\frac{L}{\rho_b}\frac{2\omega_{pb}^2}{v_{Tb}} K_{xx}(k_o,x)\right]^{-1}. \end{aligned} \tag{29}$$

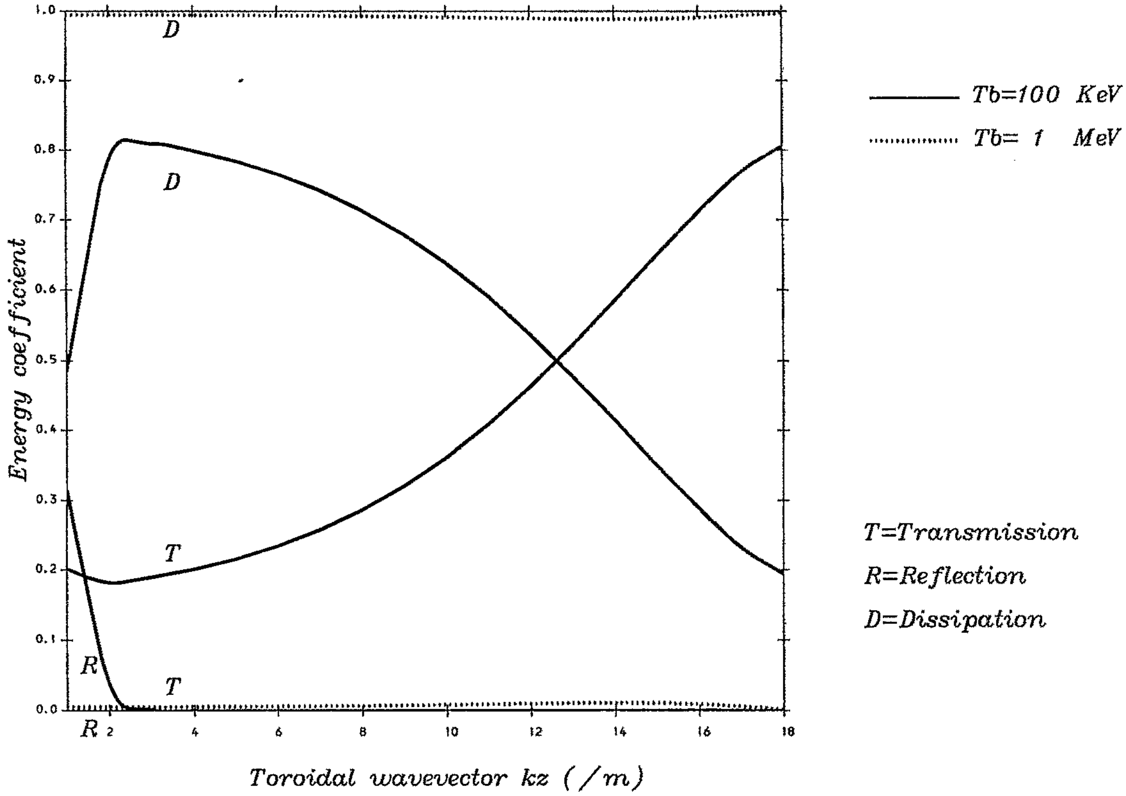

FIG. 2. Locally uniform, large Larmor radius calculation of the transmission, absorption and reflection coefficients for the same parameters as Fig. 1.

The fast wave approximation has therefore allowed us to reduce two coupled integro-differential equations to a second order differential equation. The response of the large Larmor orbit ions in the non-uniform magnetic field is contained in the fast wave potential given in Eq. (29).

We have obtained some preliminary results from a numerical solution of Eq. (28). These results are shown in Fig. 1 which refer to the case of a fast wave incident on a minority, helium-3 fundamental resonance from the low magnetic field side. The majority ion species is deuterium. Two sets of curves are shown in Fig. 1 which correspond to helium-3 temperatures of 100 keV (full line) and 1 MeV (dotted line). The other parameters specified in the calculation are an electron density of $5\times10^{19}\ \mathrm{m}^{-3}$, a minority ion to electron density ratio of 0.05, a magnetic field of 3.4 T and a magnetic field scale length of 3.1 m.

In the case of the 100 keV minority ions, $k_\perp\rho_b\simeq0.32$ and for the 1 MeV ions, $k_\perp\rho_b\simeq1.02$ where we have taken $k_\perp\simeq\Omega_b/c_A$ giving $k_\perp\rho_b=v_{Tb}/c_A$ with b denoting helium-3. The transmission coefficient for a minority fundamental cyclotron resonance obtained from a locally uniform model with the small Larmor radius approximation yields a value which is independent of the minority temperature.[14] However, Fig. 1 shows a pronounced change in the transmission coefficient between 100 keV and 1 MeV minority ions. Also shown in Fig. 1 is the total absorption which is the sum of the energy dissipated by minority ion cyclotron damping and the energy mode converted to an ion Bernstein wave[14]. The reflection coefficient can be seen to be completely negligible for the higher temperature case and only noticeable for the lower temperature for values of $k_\parallel$ below $2\ \mathrm{m}^{-1}$.

A comparison has been made between these results, obtained from the non-uniform, large Larmor radius theory and the corresponding results obtained from a locally uniform, large Larmor radius model. The results from the locally uniform model are given in Fig. 2. The curves for 100 keV are in reasonable agreement with those obtained from the non-uniform model. The main discrepancies occur for the reflection coefficient for the smaller values of $k_\parallel$ and the transmission and absorption coefficients at the larger values of $k_\parallel$. The locally uniform model predicts more reflection at the lower values of $k_\parallel$ and more absorption for the larger values of $k_\parallel$. The difference between the non-uniform and locally uniform theories is more pronounced at the higher minority ion temperature but only for values of $k_\parallel$ larger than $12\ \mathrm{m}^{-1}$. Notice that the dependence of the transmission coefficient on the minority ion temperature predicted by the non-uniform theory is also given by the locally uniform model. This dependence is evidently due to the inclusion of large Larmor radius effects.

V. CONCLUSIONS

We have shown how the response of an inhomogeneous plasma, with gradients in magnetic field strength, temperature and density perpendicular to the field direction, can be obtained using a comparatively simple technique. This technique is, in fact, fully equivalent to the Fourier transform of the inhomogeneous problem. For an inhomogeneity described by a linear spatial dependence the Fourier transform can be carried out exactly. The general results of earlier workers[10–12] can be reproduced but, in Section II, we have derived equations for the special case in which the strength of the magnetic field is assumed to have a linear gradient, while other quantities are constant. Since the effect of large Larmor radius ions extends only over a few Larmor radii, we have pointed out that gradients in temperature and density are not likely to be important, but that the magnetic field gradient in the vicinity of cyclotron resonance does lead to rapid variation in the plasma response. The terms which we have calculated for this case have not, so far as we are aware, been given previously in this form. Since they involve one fewer integration than the general forms they are likely to be of some advantage for numerical computations.

We have also shown how a local dispersion relation, which still retains features of the non-local response, can be obtained and how an approach analogous to the Born approximation of scattering theory can yield differential equations in which the coefficients are modified by the non-local response. Some preliminary results of the use of this approximation to describe minority heating are described. Fuller development of the numerical work and comparison of the solutions of the differential equation with those of the full integral equation are planned for the future.

Clearly many different representations of the non-local response of a plasma containing high energy ions are possible. The methods given have provided a relatively easy way of exploring the possibilities, with a view to obtaining forms amenable to numerical calculation. The forms given in Section II for a linear magnetic field gradient include, in our view, the major physical effects of importance and are simpler than the general form used in the numerical analysis of Sauter and Vaclavik.[10] We have also suggested ways in which the problem can be further simplified, at the cost of losing some information on the division between absorbed and mode-converted power. If further study verifies that these techniques, which have been successful in the small Larmor radius regime, are of use here, then a considerable simplification will result. This will make analysis of the important problems of ion cyclotron heating in the presence of a high energy minority tail or α-particle distribution much easier. It is also of relevance to ion cyclotron emission from fusion products and other energetic ions.

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APPENDIX: CONFIGURATION-SPACE CALCULATION OF NON-LOCAL REPONSE

For a detailed solution of propagation across a resonance, the equations must be solved numerically in x-space. We have already shown, in Section II, how such equations can be derived from the k-space expressions for the conductivity tensor. Here we offer an alternative approach in which the response is calculated directly in x-space without the need to Fourier transform forwards and backwards.

As usual, we look at the simplest case—that of the ordinary wave propagating perpendicularly through the fundamental—to illustrate the technique. Solving the linearised Vlasov equation by the method of characteristics, gives us the following standard expression for the perturbed current density:

$$J_1(x)=-\frac{q^2}{m}\int d^3v\int_{-\infty}^{0}d\tau$$

$$\times E_1\left(x+\frac{v_\perp}{\Omega}\{\sin(\Omega\tau-\theta)+\sin\theta\}\right)u\,\frac{\partial f_0}{\partial u}\,e^{-i\omega\tau};$$

E_1 is clearly oscillatory in $\Omega\tau-\theta$ and so we may express it in terms of a Fourier series,

$$E_1(x')=\sum_{n=-\infty}^{\infty}\langle E_1\rangle_n e^{in(\Omega\tau-\theta)}$$

where

$$\langle E_1\rangle_n=\frac{1}{2\pi}\int_0^{2\pi}d\alpha\;E_1\left(x+\frac{v_\perp}{\Omega}\{\sin\,\alpha+\sin\,\theta\}\right)e^{-in\alpha},$$

thus enabling us to express the perturbed current density, J_1, in terms of harmonics of $\Omega\tau$. For the fundamental resonance, we need only consider the first ($n=1$) harmonic of E_1, giving,

$$J_1(x)=-\frac{q^2}{2\pi m}\int d^3v\int_{-\infty}^{0}d\tau\int_0^{2\pi}d\alpha$$

$$\times E_1\left(x+\frac{v_\perp}{\Omega}\{\sin\,\alpha+\sin\,\theta\}\right)$$

$$\times u\,\frac{\partial f_0}{\partial u}\,e^{-i(\alpha+\theta)}e^{-i(\omega-\Omega)\tau}.$$

The gyrokinetic correction is now included by inserting,

$$\omega-\Omega=\frac{\Omega x}{L}+\frac{v_\perp}{L}\,\sin\theta,$$

in the final exponential, to give,

$$J_1(x)=-\frac{q^2}{2\pi m}\int d^3v\int_{-\infty}^{0}d\tau\int_0^{2\pi}d\alpha$$

$$\times E_1\left(x+\frac{v_\perp}{\Omega}\{\sin\,\alpha+\sin\,\theta\}\right)$$

$$\times u\,\frac{\partial f_0}{\partial u}\,e^{-i(\alpha+\theta)}e^{-i[(\Omega x/L)+(v_\perp/L)\sin\theta]\tau}.$$

The expression for J_1 now contains five integrals, two of which, u and τ, are reasonably straightforward. The v_x and v_y integrals, however, cannot be performed as they are both arguments of E_1. However, one of these integrals can be made tractable by linearly transforming the velocity coordinates so that only one occurs in E_1. Noting that, $\sin\alpha+\sin\theta=2\sin\frac{1}{2}(\alpha+\theta)\cos\frac{1}{2}(\alpha-\theta)$, we may make the substitution $\alpha'=\frac{1}{2}(\theta+\alpha)$ and $\theta'=\frac{1}{2}(\theta-\alpha)$ (which in Cartesian velocity space gives us the linearly transformed velocities $V_x=v_\perp\cos\theta'$, $V_y=v_\perp\sin\theta'$ and we also take $U=u$), to give,

$$J_1(x)=-\frac{2\varepsilon_0\omega_p^2}{\pi^{5/2}v_t^5}\int d^3V\int_{-\infty}^{0}d\tau\int_0^{2\pi}d\alpha' E_1\left(x+\frac{2V_x}{\Omega}\sin\alpha'\right)$$
$$\times U^2 e^{-V^2/v_t^2}e^{-2i\theta'}$$
$$\times e^{-i[(\Omega x/L)+(V_x\sin\alpha'+V_y\cos\alpha')/L]\tau},$$

with θ' now having the range $[-\pi,\pi]$. It should also be noted that we have taken a Maxwellian distribution, of the form $f_0=n_0\pi^{-3/2}v_t^{-3}e^{-v^2/v_t^2}$. The U integral is in the form of a gamma function and can be evaluated. The integrand of the V_y integral is quadratic, and can also be evaluated in the form of a gamma function, by completing the square with the substitution $V_y'=V_y/v_t+i\tau v_t\cos\alpha'/2L$. Performing both of these integrals, in this fashion, gives us,

$$J_1(x)=-\frac{\varepsilon_0\omega_p^2}{2\pi^{3/2}v_t}\int_{-\infty}^{\infty}dV_x\int_{-\infty}^{0}d\tau\int_0^{2\pi}d\alpha'$$
$$\times E_1\left(x+\frac{2V_x}{\Omega}\sin\alpha'\right)e^{-V_x^2/v_t^2}e^{-2i\alpha'}$$
$$\times e^{-i[(\Omega x/L)+(V_x\sin\alpha'/L)]\tau-(1/4)(v_t\cos\alpha'/L)^2\tau^2}.$$

The τ integral can also be performed by noting the identity,

$$\int_0^{\infty}dt\; e^{ixt-(1/4)a^2t^2}=\frac{1}{ia}Z(x/a),\quad \text{where } a>0,$$

giving,

$$J_1(x)=-\frac{i\varepsilon_0\omega_p^2L}{2\pi^{3/2}v_t^2}\int_{-\infty}^{\infty}dV_x\int_0^{2\pi}d\alpha' E_1\left(x+\frac{2V_x}{\Omega}\sin\alpha'\right)$$
$$\times e^{-V_x^2/v_t^2}e^{-2i\alpha'}|\sec\alpha'|Z\left(\frac{x+(V_x/\Omega)\sin\alpha'}{\rho|\cos\alpha'|}\right).$$

Finally, by making the change of variable $V_x=\frac{1}{2}(x-x')\Omega$ cosec α' the expression for the perturbed current density becomes,

$$J_1(x)=-\frac{i\varepsilon_0\omega_p^2L}{4\pi^{3/2}\rho v_t}\int_{-\infty}^{\infty}dx'\int_0^{2\pi}d\alpha'|\mathrm{cosec}\alpha'\sec\alpha'|e^{-2i\alpha'}$$
$$\times Z\left(\frac{x+x'}{2\rho|\cos\alpha'|}\right)\exp\left\{-\left(\frac{x-x'}{2\rho\sin\alpha'}\right)^2\right\}E(x').$$

If the change of variable $\theta=2\alpha$ is made, this becomes equivalent to the result of Eq. (7) for the case when $k_\parallel=0$.

[1]P. J. Colestock and R. J. Kashuba, Nucl. Fusion **23**, 763 (1983).
[2]D. G. Swanson, Phys. Fluids **28**, 2645 (1985).
[3]M. Brambilla, Plasma Phys. Controlled Fusion **31**, 723 (1987).
[4]H. A. Romero and J. Scharer, Nucl. Fusion **27**, 363 (1987).
[5]H. A. Romero and G. J. Morales, Phys. Fluids B **1**, 1805 (1989).
[6]R. A. Cairns, C. N. Lashmore-Davies, R. O. Dendy, B. M. Harvey, R. J. Hastie, and H. Holt, Phys. Fluids B **3**, 2953 (1991).
[7]T. M. Antonsen and W. M. Manheimer, Phys. Fluids **21**, 2295 (1978).
[8]C. N. Lashmore-Davies and R. O. Dendy, Phys Fluids B **1**, 1565 (1989).
[9]D. C. McDonald, R. A. Cairns, and C. N. Lashmore-Davies, Phys. Plasmas **1**, 842 (1994).
[10]O. Sauter and J. Vaclavik, in *Theory of Fusion Plasmas,* edited by E. Sindoni, F. Troyon, and J. Vaclavik, *Proceedings of the Joint Varenna–Lausanne International Workshop*, Varenna, 1990 (Editrice Compositori, Societá Italiana di Fisica, Bologna, 1990), p. 403.
[11]O. Sauter and J. Vaclavik, Nucl. Fusion **32**, 1455 (1992).
[12]M. Brambilla, Plasma Phys. Controlled Fusion **33**, 1029 (1991).
[13]A. Kay, R. A. Cairns, and C. N. Lashmore-Davies, Plasma Phys. Controlled Fusion **30**, 471 (1988).
[14]C. N. Lashmore-Davies, V. Fuchs, G. Francis, A. K. Ram, A. Bers, and L. Gauthier, Phys. Fluids **31**, 1614 (1988).
[15]N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions*, 3rd ed. (Oxford University Press, Oxford, 1965), p. 86.