

Homework Signal 3

Week 3

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Collaborators. ChatGPT (for L^AT_EX styling and grammar checking)

1 Fourier Series

Problem 2. Find the Fourier Series (FS) of the periodic function $x(t)$ which are provided as follows.

TO SUBMIT

2.2 $x(t) = \pi - t; -\pi \leq t \leq \pi$

Solution. To find the Fourier series of the function $x(t) = \pi - t$ for $-\pi \leq t \leq \pi$, where $T = 2\pi$ (the period of the function).

Calculating ω_0 :

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Using the Fourier series formula:

$$x(t) = \sum_k a_k e^{jk\omega_0 t} = a_0 + \sum_{k \neq 0} a_k e^{jk\omega_0 t}$$

where the Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt$$

Calculating a_0 :

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-j(0)\omega_0 t} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - t) dt \\ &= \frac{1}{2\pi} \left[\pi t - \frac{t^2}{2} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left(\pi^2 - \frac{\pi^2}{2} - (-\pi^2 + \frac{\pi^2}{2}) \right) \\ a_0 &= \pi \end{aligned}$$

Calculating a_k for $k \neq 0$:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - t) e^{-jkt} dt \\ a_k &= \frac{1}{2\pi} \left[\pi \int_{-\pi}^{\pi} e^{-jkt} dt - \int_{-\pi}^{\pi} t e^{-jkt} dt \right] \end{aligned}$$

To solve the integral, we can use integration by parts multiple times. Using tabular integration by parts, we find:

Sign	Derivative	Integral
+	t	$e^{-j\pi kt}$
-	1	$\frac{1}{-j\pi k} e^{-j\pi kt}$
+	0	$\frac{1}{(-j\pi k)^2} e^{-j\pi kt}$

Thus, we have:

$$\int t e^{-j\pi kt} dt = \frac{t}{-jk} e^{-j\pi kt} - \frac{1}{(-jk)^2} e^{-j\pi kt}$$

Evaluating this from $-\pi$ to π to find a_k :

$$\begin{aligned} a_k &= \frac{1}{2\pi} \left[\pi \left[\frac{e^{-j\pi kt}}{-jk} \right]_{-\pi}^{\pi} - \left[\frac{t}{-jk} e^{-j\pi kt} - \frac{1}{(-jk)^2} e^{-j\pi kt} \right]_{-\pi}^{\pi} \right] \\ &= \frac{1}{2} \left[\frac{e^{-j\pi k}}{-jk} \right]_{-\pi}^{\pi} - \frac{1}{2\pi} \left[\frac{t}{-jk} e^{-j\pi kt} - \frac{1}{(-jk)^2} e^{-j\pi kt} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2} \left[\frac{(e^{-j\pi k} - e^{j\pi k})}{-jk} \right] - \frac{1}{2\pi} \left[\frac{(\pi e^{-j\pi k} + \pi e^{j\pi k})}{-jk} - \frac{(e^{-j\pi k} - e^{j\pi k})}{(-jk)^2} \right] \\ &= \frac{1}{2} \left[\frac{(-2j \sin(\pi k))}{-jk} \right] - \frac{1}{2\pi} \left[\frac{(2\pi \cos(\pi k))}{-jk} - \frac{(-2j \sin(\pi k))}{(-jk)^2} \right] \\ a_k &= \frac{\sin(\pi k)}{k} + \frac{\cos(\pi k)}{jk} - \frac{\sin(\pi k)}{j\pi k^2} \end{aligned}$$

We can simplify a_k :

$$a_k = 0 + \frac{(-1)^k}{jk} - 0 = \frac{(-1)^k}{jk} \text{ for } k \neq 0$$

Thus, the Fourier series expansion of $x(t)$ is:

$$x(t) = \pi + \sum_{k \neq 0} \left(\operatorname{Re} \left\{ \frac{(-1)^k}{jk} e^{j\pi kt} \right\} \right)$$

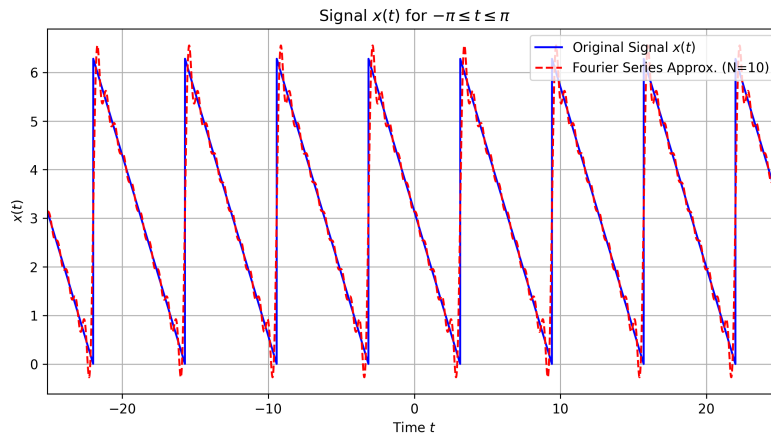
Because $e^{jx} - e^{-jx} = 2j \sin(x)$ for any real x , we have:

$$\begin{aligned} \sum_{k \neq 0} \frac{(-1)^k}{jk} e^{j\pi kt} &= \sum_{k=1}^{\infty} \frac{(-1)^k}{jk} e^{j\pi kt} + \sum_{k=-\infty}^{-1} \frac{(-1)^k}{jk} e^{j\pi kt} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{jk} e^{j\pi kt} + \sum_{k=1}^{\infty} \frac{(-1)^{-k}}{-jk} e^{-j\pi kt} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{jk} (e^{j\pi kt} - e^{-j\pi kt}) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{jk} (2j \sin(\pi kt)) \\ \sum_{k \neq 0} \frac{(-1)^k}{jk} e^{j\pi kt} &= \sum_{k=1}^{\infty} \frac{2(-1)^k}{k} \sin(\pi kt) \end{aligned}$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$x(t) = \pi + \sum_{k=1}^{\infty} \left(\frac{2(-1)^k}{k} \sin(kt) \right)$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows:



TO SUBMIT

2.3 $x(t) = t^2 + \sin^3(\pi t)$; $-1 \leq t \leq 1$

Solution. To find the Fourier series of the function $x(t) = t^2 + \sin^3(\pi t)$ for $-1 \leq t \leq 1$, where $T = 2$ (the period of the function).

Calculating ω_0 :

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

Using the Fourier series formula:

$$x(t) = \sum_k a_k e^{jk\omega_0 t} = a_0 + \sum_{k \neq 0} a_k e^{jk\omega_0 t}$$

where the Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt$$

Calculating a_0 :

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-j(0)\omega_0 t} dt \\ &= \frac{1}{2} \int_{-1}^1 (t^2 + \sin^3(\pi t)) dt \\ &= \frac{1}{2} \int_{-1}^1 t^2 dt + \frac{1}{2} \int_{-1}^1 \sin^3(\pi t) dt \\ &= \frac{1}{2} \left[\frac{t^3}{3} \right]_{-1}^1 + \frac{1}{2} (0) \quad \text{since } \sin^3(\pi t) \text{ is odd function} \\ &= \frac{1}{2} \left[\frac{1^3}{3} - \frac{(-1)^3}{3} \right] + 0 \end{aligned}$$

$$= \frac{1}{2} \cdot \frac{2}{3}$$

$$a_0 = \frac{1}{3}$$

Calculating a_k for $k \neq 0$:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{2} \int_{-1}^1 (t^2 + \sin^3(\pi t)) e^{-jk\pi t} dt$$

$$a_k = \frac{1}{2} \left[\int_{-1}^1 t^2 e^{-jk\pi t} dt + \int_{-1}^1 \sin^3(\pi t) e^{-jk\pi t} dt \right]$$

Define

$$I_1 = \int_{-1}^1 t^2 e^{-jk\pi t} dt \quad \text{and} \quad I_2 = \int_{-1}^1 \sin^3(\pi t) e^{-jk\pi t} dt$$

Hence,

$$a_k = \frac{1}{2} (I_1 + I_2)$$

To solve the integral I_1 , we can use integration by parts multiple times. Using tabular integration by parts, we find:

Sign	Derivative	Integral
+	t^2	$e^{-j\pi kt}$
-	$2t$	$\frac{1}{-j\pi k} e^{-j\pi kt}$
+	2	$\frac{1}{(-j\pi k)^2} e^{-j\pi kt}$
-	0	$\frac{1}{(-j\pi k)^3} e^{-j\pi kt}$

Thus, we have:

$$I_1 = \int t^2 e^{-jk\pi t} dt = \frac{t^2}{-j\pi k} e^{-jk\pi t} - \frac{2t}{(-j\pi k)^2} e^{-jk\pi t} + \frac{2}{(-j\pi k)^3} e^{-jk\pi t}$$

Evaluating this from -1 to 1 to find I_1 :

$$I_1 = \left[\frac{t^2}{-j\pi k} e^{-jk\pi t} - \frac{2t}{(-j\pi k)^2} e^{-jk\pi t} + \frac{2}{(-j\pi k)^3} e^{-jk\pi t} \right]_{-1}^1$$

$$= \left[\frac{-2j \sin(\pi k)}{-j\pi k} - \frac{4 \cos(\pi k)}{(-j\pi k)^2} + \frac{-4j \sin(\pi k)}{(-j\pi k)^3} \right]$$

$$= \frac{-2 \sin(\pi k)}{-\pi k} - \frac{4 \cos(\pi k)}{(-j\pi k)^2} + \frac{-4 \sin(\pi k)}{j^2 (-\pi k)^3}$$

$$I_1 = \frac{2 \sin(\pi k)}{\pi k} + \frac{4 \cos(\pi k)}{(\pi k)^2} - \frac{4 \sin(\pi k)}{(\pi k)^3}$$

Next, to solve the integral I_2 , we can use the euler identity:

$$\sin^3(x) = \left\{ \frac{1}{2j} (e^{jx} - e^{-jx}) \right\}^3 = -\frac{1}{8j} (e^{3jx} - 3e^{jx} + 3e^{-jx} - e^{-3jx})$$

Thus,

$$\begin{aligned}
 I_2 &= \int_{-1}^1 \sin^3(\pi t) e^{-jk\pi t} dt \\
 &= \int_{-1}^1 -\frac{1}{8j} (e^{3j\pi t} - 3e^{j\pi t} + 3e^{-j\pi t} - e^{-3j\pi t}) e^{-jk\pi t} dt \\
 &= -\frac{1}{8j} \int_{-1}^1 (e^{j\pi t(3-k)} - 3e^{j\pi t(1-k)} + 3e^{-j\pi t(1+k)} - e^{-j\pi t(3+k)}) dt \\
 &= -\frac{1}{8j} \left[\frac{e^{j\pi t(3-k)}}{j\pi(3-k)} - \frac{3e^{j\pi t(1-k)}}{j\pi(1-k)} + \frac{3e^{-j\pi t(1+k)}}{-j\pi(1+k)} - \frac{e^{-j\pi t(3+k)}}{-j\pi(3+k)} \right]_{-1}^1 \\
 &= -\frac{1}{8j} \left[\frac{2j \sin(\pi(3-k))}{\pi(3-k)} - \frac{3(2j \sin(\pi(1-k)))}{\pi(1-k)} + \frac{3(-2j \sin(\pi(1+k)))}{-\pi(1+k)} - \frac{-2j \sin(\pi(3+k))}{-\pi(3+k)} \right] \\
 I_2 &= -\frac{1}{4j} \left[\frac{\sin(\pi(3-k))}{\pi(3-k)} - \frac{3 \sin(\pi(1-k))}{\pi(1-k)} + \frac{3 \sin(\pi(1+k))}{\pi(1+k)} - \frac{\sin(\pi(3+k))}{\pi(3+k)} \right]
 \end{aligned}$$

Therefore, we have:

$$\begin{aligned}
 a_k &= \frac{1}{2} (I_1 + I_2) \\
 &= \frac{1}{2} \left[\frac{2 \sin(\pi k)}{\pi k} + \frac{4^2 \cos(\pi k)}{(\pi k)^2} - \frac{4^2 \sin(\pi k)}{(\pi k)^3} \right] \\
 &\quad - \frac{1}{2} \cdot \frac{1}{4j} \left[\frac{\sin(\pi(3-k))}{\pi(3-k)} - \frac{3 \sin(\pi(1-k))}{\pi(1-k)} + \frac{3 \sin(\pi(1+k))}{\pi(1+k)} - \frac{\sin(\pi(3+k))}{\pi(3+k)} \right] \\
 a_k &= \frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \\
 &\quad - \frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)}
 \end{aligned}$$

Consider the value of a_k , we can see that at $|k| = 1$ and $|k| = 3$, the terms will be undefined. Therefore, we need to calculate these four cases separately using limits.

For $k = 1$:

$$\begin{aligned}
 a_1 &= \lim_{k \rightarrow 1} \left[\frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \right] \\
 &\quad + \lim_{k \rightarrow 1} \left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)} \right] \\
 &= \left(0 + \frac{2(-1)}{\pi^2} - 0 \right) + \left(-\frac{0}{16j\pi} + \lim_{k \rightarrow 1} \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{0}{16j\pi} + \frac{0}{32j\pi} \right) \\
 a_1 &= -\frac{2}{\pi^2} - \frac{3j}{8}
 \end{aligned}$$

For $k = -1$:

$$\begin{aligned}
 a_{-1} &= \lim_{k \rightarrow -1} \left[\frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \right] \\
 &\quad + \lim_{k \rightarrow -1} \left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)} \right] \\
 &= \left(0 + \frac{2(-1)}{\pi^2} - 0 \right) + \left(-\frac{0}{16j\pi} + \frac{0}{6j\pi} - \lim_{k \rightarrow -1} \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{0}{32j\pi} \right) \\
 a_{-1} &= -\frac{2}{\pi^2} + \frac{3j}{8}
 \end{aligned}$$

Thus, we have:

$$a_k = -\frac{2}{\pi^2} - \frac{3jk}{8} \text{ for } |k| = 1$$

For $k = 3$:

$$\begin{aligned} a_3 &= \lim_{k \rightarrow 3} \left[\frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \right] \\ &\quad + \lim_{k \rightarrow 3} \left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)} \right] \\ &= \left(0 + \frac{2(-1)}{(3\pi)^2} - 0 \right) + \left(-\lim_{k \rightarrow 3} \frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{0}{-16j\pi} - \frac{0}{32j\pi} + \frac{0}{48j\pi} \right) \\ a_3 &= -\frac{2}{9\pi^2} + \frac{j}{8} \end{aligned}$$

For $k = -3$:

$$\begin{aligned} a_{-3} &= \lim_{k \rightarrow -3} \left[\frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \right] \\ &\quad + \lim_{k \rightarrow -3} \left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)} \right] \\ &= \left(0 + \frac{2(-1)}{(3\pi)^2} - 0 \right) + \left(-\frac{0}{16j\pi} + \frac{0}{6j\pi} - \lim_{k \rightarrow -3} \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{0}{48j\pi} \right) \\ a_{-3} &= -\frac{2}{9\pi^2} - \frac{j}{8} \end{aligned}$$

Thus, we have:

$$a_k = -\frac{2}{9\pi^2} + \frac{jk}{24} \text{ for } |k| = 3$$

For other values of k where $|k| \neq 0, 1, 3$:

$$\begin{aligned} a_k &= \left(\frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \right) \\ &\quad + \left(-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)} \right) \\ &= \left(0 + \frac{2(-1)^k}{(\pi k)^2} \right) + \left(-0 + \frac{0}{8j\pi(3-k)} + \frac{0}{8j\pi(1-k)} - \frac{0}{8j\pi(1+k)} + \frac{0}{8j\pi(3+k)} \right) \\ a_k &= \frac{2(-1)^k}{k^2\pi^2} \end{aligned}$$

Simplify a_k :

$$a_k = \begin{cases} \frac{1}{3} & k = 0 \\ -\frac{2}{\pi^2} - \frac{3jk}{8} & |k| = 1 \\ -\frac{2}{9\pi^2} + \frac{jk}{24} & |k| = 3 \\ \frac{2(-1)^k}{k^2\pi^2} & \text{otherwise} \end{cases}$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$\begin{aligned} x(t) &= \frac{1}{3} + \sum_{|k| \neq 0, 1, 3} \left(\operatorname{Re} \left\{ \frac{2(-1)^k}{k^2\pi^2} e^{j\pi kt} \right\} \right) \\ &\quad + \sum_{|k|=1} \left(\operatorname{Re} \left\{ \left(-\frac{2}{\pi^2} - \frac{3jk}{8} \right) e^{j\pi kt} \right\} \right) + \sum_{|k|=3} \left(\operatorname{Re} \left\{ \left(-\frac{2}{9\pi^2} - \frac{jk}{24} \right) e^{j\pi kt} \right\} \right) \end{aligned}$$

Because $e^{jx} + e^{-jx} = 2 \cos(x)$ and $e^{jx} - e^{-jx} = 2j \sin(x)$ for any real x , we can simplify the Fourier series expansion further.

Consider the sums separately:

1. For $|k| \neq 0, 1, 3$:

$$\begin{aligned}
 \sum_{|k| \neq 0, 1, 3} \left(\frac{2(-1)^k}{k^2 \pi^2} e^{j\pi kt} \right) &= \frac{2}{\pi^2} \sum_{|k| \neq 0, 1, 3} \left(\frac{(-1)^k}{k^2} e^{j\pi kt} \right) \\
 &= \frac{2}{\pi^2} \left(\sum_{\substack{k=-\infty \\ |k| \neq 0, 1, 3}}^{-1} \left(\frac{(-1)^k}{k^2} e^{j\pi kt} \right) + \sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^k}{k^2} e^{j\pi kt} \right) \right) \\
 &= \frac{2}{\pi^2} \left(\sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^{-k}}{(-k)^2} e^{-j\pi kt} \right) + \sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^k}{k^2} e^{j\pi kt} \right) \right) \\
 &= \frac{2}{\pi^2} \sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^k}{k^2} (e^{-j\pi kt} + e^{j\pi kt}) \right) \\
 &= \frac{2}{\pi^2} \sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^k}{k^2} (2 \cos(\pi kt)) \right) \\
 \sum_{|k| \neq 0, 1, 3} \left(\frac{2(-1)^k}{k^2 \pi^2} e^{j\pi kt} \right) &= \frac{4}{\pi^2} \sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^k}{k^2} \cos(\pi kt) \right)
 \end{aligned}$$

2. For $|k| = 1$:

$$\begin{aligned}
 \sum_{|k|=1} \left(\left(-\frac{2}{\pi^2} - \frac{3jk}{8} \right) e^{j\pi kt} \right) &= \sum_{|k|=1} \left(-\frac{2}{\pi^2} e^{j\pi kt} \right) + \sum_{|k|=1} \left(-\frac{3jk}{8} e^{j\pi kt} \right) \\
 &= -\frac{2}{\pi^2} (e^{-j\pi t} + e^{j\pi t}) - \frac{3j}{8} (-e^{-j\pi t} + e^{j\pi t}) \\
 &= -\frac{2}{\pi^2} (2 \cos(\pi t)) - \frac{3j}{8} (2j \sin(\pi t)) \\
 \sum_{|k|=1} \left(\left(-\frac{2}{\pi^2} - \frac{3jk}{8} \right) e^{j\pi kt} \right) &= -\frac{4}{\pi^2} \cos(\pi t) + \frac{3}{4} \sin(\pi t)
 \end{aligned}$$

3. For $|k| = 3$:

$$\begin{aligned}
 \sum_{|k|=3} \left(\left(-\frac{2}{9\pi^2} + \frac{jk}{24} \right) e^{j\pi kt} \right) &= \sum_{|k|=3} \left(-\frac{2}{9\pi^2} e^{j\pi kt} \right) + \sum_{|k|=3} \left(\frac{jk}{24} e^{j\pi kt} \right) \\
 &= -\frac{2}{9\pi^2} (e^{-j3\pi t} + e^{j3\pi t}) + \frac{j(3)}{24} (-e^{-j3\pi t} + e^{j3\pi t}) \\
 &= -\frac{2}{9\pi^2} (2 \cos(3\pi t)) + \frac{j}{8} (2j \sin(3\pi t)) \\
 \sum_{|k|=3} \left(\left(-\frac{2}{9\pi^2} + \frac{jk}{24} \right) e^{j\pi kt} \right) &= -\frac{4}{9\pi^2} \cos(3\pi t) - \frac{1}{4} \sin(3\pi t)
 \end{aligned}$$

Consider $|k| = 1$ and $|k| = 3$ together.

$$\begin{aligned} &= -\frac{4}{\pi^2} \cos(\pi t) + \frac{3}{4} \sin(\pi t) - \frac{4}{9\pi^2} \cos(3\pi t) - \frac{1}{4} \sin(3\pi t) \\ &= -\frac{4}{\pi^2} \left(\cos(\pi t) + \frac{1}{9} \cos(3\pi t) \right) + \frac{3}{4} \left(\sin(\pi t) - \frac{1}{3} \sin(3\pi t) \right) \end{aligned}$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$x(t) = \frac{1}{3} - \frac{4}{\pi^2} \left(\cos(\pi t) + \frac{1}{9} \cos(3\pi t) \right) + \frac{3}{4} \left(\sin(\pi t) - \frac{1}{3} \sin(3\pi t) \right) + \sum_{\substack{k=1 \\ |k| \neq 1,3}}^{\infty} \left(\frac{4(-1)^k}{\pi^2 k^2} \cos(\pi k t) \right)$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows:

