Homework Signal 3

Week 3

6733172621 Patthadon Phengpinij

Collaborators. ChatGPT (for LaTeX styling and grammar checking)

1 Fourier Series

Problem 2. Find the Fourier Series (FS) of the periodic function $x(t)$ which are provided as follows.

TO SUBMIT

2.2 $x(t) = \pi - t;\ -\pi \leq t \leq \pi$

Solution. To find the Fourier series of the function $x(t) = \pi - t$ for $-\pi \leq t \leq \pi$, we first need to compute the Fourier coefficients. The Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t}\, dt$$

where $T = 2\pi$ (the period of the function).
Calculating ω_0:

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Calculating a_0:

$$\begin{aligned}
a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-j(0)\omega_0 t}\, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - t)\, dt \\
&= \frac{1}{2\pi} \left[\pi t - \frac{t^2}{2} \right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi} \left(\pi^2 - \frac{\pi^2}{2} - (-\pi^2 - \frac{\pi^2}{2}) \right) \\
a_0 &= \pi
\end{aligned}$$

Calculating a_k for $k \neq 0$:

$$\begin{aligned}
a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t}\, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - t) e^{-jkt}\, dt \\
a_k &= \frac{1}{2\pi} \left[\pi \int_{-\pi}^{\pi} e^{-jkt}\, dt - \int_{-\pi}^{\pi} t e^{-jkt}\, dt \right]
\end{aligned}$$

To solve the integral, we can use integration by parts multiple times. Using tabular integration by parts, we find:

Sign	Derivative	Integral
$+$	t	$e^{-j\pi kt}$
$-$	1	$\frac{1}{-j\pi k}e^{-j\pi kt}$
$+$	0	$\frac{1}{(-j\pi k)^2}e^{-j\pi kt}$

Thus, we have:

$$\int te^{-jkt}\,dt = \frac{t}{-jk}e^{-jkt} - \frac{1}{(-jk)^2}e^{-jkt}$$

Evaluating this from $-\pi$ to π to find a_k:

$$\begin{aligned}
a_k &= \frac{1}{2\pi}\left[\pi\left[\frac{e^{-jkt}}{-jk}\right]_{-\pi}^{\pi} - \left[\frac{t}{-jk}e^{-jkt} - \frac{1}{(-jk)^2}e^{-jkt}\right]_{-\pi}^{\pi}\right] \\
&= \frac{1}{2}\left[\frac{e^{-jkt}}{-jk}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\left[\frac{t}{-jk}e^{-jkt} - \frac{1}{(-jk)^2}e^{-jkt}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}\left[\frac{\left(e^{-j\pi k} - e^{j\pi k}\right)}{-jk}\right] - \frac{1}{2\pi}\left[\frac{\left(\pi e^{-j\pi k} + \pi e^{j\pi k}\right)}{-jk} - \frac{\left(e^{-j\pi k} - e^{j\pi k}\right)}{(-jk)^2}\right] \\
&= \frac{1}{\cancel{2}}\left[\frac{\left(-\cancel{2}\cancel{j}\sin\pi k\right)}{-\cancel{j}k}\right] - \frac{1}{\cancel{2}\pi}\left[\frac{\left(\cancel{2}\pi\cos\pi k\right)}{-jk} - \frac{\left(-\cancel{2}j\sin\pi k\right)}{(-jk)^2}\right] \\
a_k &= \frac{\sin\pi k}{n} + \frac{\cos\pi k}{jn} - \frac{\sin\pi k}{j\pi k^2}
\end{aligned}$$

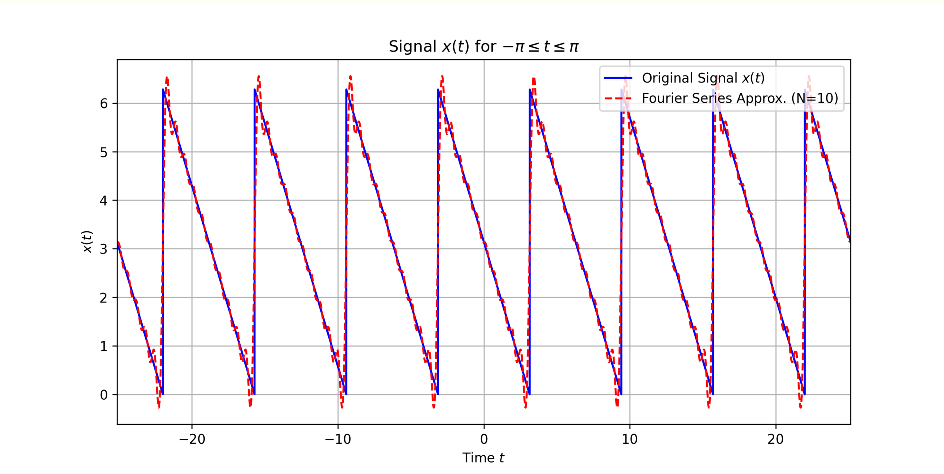
We can simplify a_k:

$$a_k = 0 + \frac{(-1)^k}{jk} - 0 = \frac{(-1)^k}{jk} \quad \text{for } k \neq 0$$

Thus, the Fourier series expansion of $x(t)$ is:

$$\boxed{x(t) = \pi + \sum_{k\neq 0}\frac{(-1)^k}{jk}e^{jkt}}$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows:



TO SUBMIT

2.3 $x(t) = t^2 + \sin^3(\pi t);\ -1 \leq t \leq 1$

Solution. To find the Fourier series of the function $x(t) = t^2 + \sin^3(\pi t)$ for $-1 \leq t \leq 1$, we first need to compute the Fourier coefficients. The Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jk\omega_0 t}\,dt$$

where $T = 2$ (the period of the function).
Calculating ω_0:

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

Calculating a_0:

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-j(0)\omega_0 t}\,dt \\
&= \frac{1}{2}\int_{-1}^{1} (t^2 + \sin^3(\pi t))\,dt \\
&= \frac{1}{2}\int_{-1}^{1} t^2\,dt + \frac{1}{2}\int_{-1}^{1} \sin^3(\pi t)\,dt \\
&= \frac{1}{2}\left[\frac{t^3}{3}\right]_{-1}^{1} + \frac{1}{2}(0) \qquad \text{since } \sin^3(\pi t) \text{ is odd function} \\
&= \frac{1}{2}\left[\frac{1^3}{3} - \frac{(-1)^3}{3}\right] + 0 \\
&= \frac{1}{2}\cdot\frac{2}{3} \\
a_0 &= \frac{1}{3}
\end{aligned}$$

Calculating a_k for $k \neq 0$:

$$\begin{aligned}
a_k &= \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jk\omega_0 t}\,dt \\
&= \frac{1}{2}\int_{-1}^{1} (t^2 + \sin^3(\pi t))e^{-jk\pi t}\,dt \\
a_k &= \frac{1}{2}\left[\int_{-1}^{1} t^2 e^{-jk\pi t}\,dt + \int_{-1}^{1} \sin^3(\pi t)e^{-jk\pi t}\,dt\right]
\end{aligned}$$

Define

$$I_1 = \int_{-1}^{1} t^2 e^{-jk\pi t}\,dt \quad \text{and} \quad I_2 = \int_{-1}^{1} \sin^3(\pi t)e^{-jk\pi t}\,dt$$

Hence,

$$a_k = \frac{1}{2}(I_1 + I_2)$$

.

To solve the integral I_1, we can use integration by parts multiple times. Using tabular integration by parts, we find:

Sign	Derivative	Integral
$+$	t^2	$e^{-j\pi kt}$
$-$	$2t$	$\frac{1}{-j\pi k}e^{-j\pi kt}$
$+$	2	$\frac{1}{(-j\pi k)^2}e^{-j\pi kt}$
$-$	0	$\frac{1}{(-j\pi k)^3}e^{-j\pi kt}$

Thus, we have:

$$I_1 = \int t^2 e^{-jk\pi t}\,dt = \frac{t^2}{-j\pi k}e^{-jk\pi t} - \frac{2t}{(-j\pi k)^2}e^{-jk\pi t} + \frac{2}{(-j\pi k)^3}e^{-jk\pi t}$$

Evaluating this from -1 to 1 to find I_1:

$$\begin{aligned}
I_1 &= \left[\frac{t^2}{-j\pi k}e^{-jk\pi t} - \frac{2t}{(-j\pi k)^2}e^{-jk\pi t} + \frac{2}{(-j\pi k)^3}e^{-jk\pi t}\right]_{-1}^{1} \\
&= \left[\frac{-2\cancel{j}\sin\pi k}{-\cancel{j}\pi k} - \frac{4\cos\pi k}{(-j\pi k)^2} + \frac{-4\cancel{j}\sin\pi k}{(-\cancel{j}\pi k)^3}\right] \\
&= \frac{-2\sin\pi k}{-\pi k} - \frac{4\cos\pi k}{(-j\pi k)^2} + \frac{-4\sin\pi k}{j^2(-\pi k)^3} \\
I_1 &= \frac{2\sin\pi k}{\pi k} + \frac{4\cos\pi k}{(\pi k)^2} - \frac{4\sin\pi k}{(\pi k)^3}
\end{aligned}$$

Next, to solve the integral I_2, we can use the euler identity:

$$\sin^3(x) = \left\{\frac{1}{2j}\left(e^{jx} - e^{-jx}\right)\right\}^3 = -\frac{1}{8j}\left(e^{3jx} - 3e^{jx} + 3e^{-jx} - e^{-3jx}\right)$$

Thus,

$$\begin{aligned}
I_2 &= \int_{-1}^{1} \sin^3(\pi t)e^{-jk\pi t}\,dt \\
&= \int_{-1}^{1} -\frac{1}{8j}\left(e^{3j\pi t} - 3e^{j\pi t} + 3e^{-j\pi t} - e^{-3j\pi t}\right)e^{-jk\pi t}\,dt \\
&= -\frac{1}{8j}\int_{-1}^{1}\left(e^{j\pi t(3-k)} - 3e^{j\pi t(1-k)} + 3e^{-j\pi t(1+k)} - e^{-j\pi t(3+k)}\right)dt \\
&= -\frac{1}{8j}\left[\frac{e^{j\pi t(3-k)}}{j\pi(3-k)} - \frac{3e^{j\pi t(1-k)}}{j\pi(1-k)} + \frac{3e^{-j\pi t(1+k)}}{-j\pi(1+k)} - \frac{e^{-j\pi t(3+k)}}{-j\pi(3+k)}\right]_{-1}^{1} \\
&= -\frac{1}{\cancel{8}4j}\left[\frac{\cancel{2}\cancel{j}\sin\pi(3-k)}{\cancel{j}\pi(3-k)} - \frac{3\left(\cancel{2}\cancel{j}\sin\pi(1-k)\right)}{\cancel{j}\pi(1-k)} + \frac{3\left(-\cancel{2}\cancel{j}\sin\pi(1+k)\right)}{-\cancel{j}\pi(1+k)} - \frac{-\cancel{2}\cancel{j}\sin\pi(3+k)}{-\cancel{j}\pi(3+k)}\right] \\
I_2 &= -\frac{1}{4j}\left[\frac{\sin\pi(3-k)}{\pi(3-k)} - \frac{3\sin\pi(1-k)}{\pi(1-k)} + \frac{3\sin\pi(1+k)}{\pi(1+k)} - \frac{\sin\pi(3+k)}{\pi(3+k)}\right]
\end{aligned}$$

Therefore, we have:

$$
\begin{aligned}
a_k &= \frac{1}{2}(I_1 + I_2) \\
&= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}\sin \pi k}{\pi k} + \frac{\cancel{4}^2 \cos \pi k}{(\pi k)^2} - \frac{\cancel{4}^2 \sin \pi k}{(\pi k)^3}\right] \\
&\quad - \frac{1}{2}\cdot\frac{1}{4j}\left[\frac{\sin \pi(3-k)}{\pi(3-k)} - \frac{3\sin \pi(1-k)}{\pi(1-k)} + \frac{3\sin \pi(1+k)}{\pi(1+k)} - \frac{\sin \pi(3+k)}{\pi(3+k)}\right] \\
a_k &= \frac{\sin \pi k}{\pi k} + \frac{2\cos \pi k}{(\pi k)^2} - \frac{2\sin \pi k}{(\pi k)^3} \\
&\quad - \frac{\sin \pi(3-k)}{8j\pi(3-k)} + \frac{3\sin \pi(1-k)}{8j\pi(1-k)} - \frac{3\sin \pi(1+k)}{8j\pi(1+k)} + \frac{\sin \pi(3+k)}{8j\pi(3+k)}
\end{aligned}
$$

Consider the value of a_k, we can see that at $|k| = 1$ and $|k| = 3$, the terms will be undefined. Therefore, we need to calculate these four cases separately using limits.

For $k = 1$:

$$
\begin{aligned}
a_1 &= \lim_{k\to 1}\left[\frac{\sin \pi k}{\pi k} + \frac{2\cos \pi k}{(\pi k)^2} - \frac{2\sin \pi k}{(\pi k)^3}\right] \\
&\quad + \lim_{k\to 1}\left[-\frac{\sin \pi(3-k)}{8j\pi(3-k)} + \frac{3\sin \pi(1-k)}{8j\pi(1-k)} - \frac{3\sin \pi(1+k)}{8j\pi(1+k)} + \frac{\sin \pi(3+k)}{8j\pi(3+k)}\right] \\
&= \left(0 + \frac{2(-1)}{\pi^2} - 0\right) + \left(-\frac{0}{16j\pi} + \lim_{k\to 1}\frac{3\sin \pi(1-k)}{8j\pi(1-k)} - \frac{0}{16j\pi} + \frac{0}{32j\pi}\right) \\
a_1 &= -\frac{2}{\pi^2} - \frac{3j}{8}
\end{aligned}
$$

For $k = -1$:

$$
\begin{aligned}
a_{-1} &= \lim_{k\to -1}\left[\frac{\sin \pi k}{\pi k} + \frac{2\cos \pi k}{(\pi k)^2} - \frac{2\sin \pi k}{(\pi k)^3}\right] \\
&\quad + \lim_{k\to -1}\left[-\frac{\sin \pi(3-k)}{8j\pi(3-k)} + \frac{3\sin \pi(1-k)}{8j\pi(1-k)} - \frac{3\sin \pi(1+k)}{8j\pi(1+k)} + \frac{\sin \pi(3+k)}{8j\pi(3+k)}\right] \\
&= \left(0 + \frac{2(-1)}{\pi^2} - 0\right) + \left(-\frac{0}{16j\pi} + \frac{0}{6j\pi} - \lim_{k\to -1}\frac{3\sin \pi(1+k)}{8j\pi(1+k)} + \frac{0}{32j\pi}\right) \\
a_{-1} &= -\frac{2}{\pi^2} + \frac{3j}{8}
\end{aligned}
$$

Thus, we have:

$$a_k = -\frac{2}{\pi^2} - \frac{3jk}{8} \text{ for } |k| = 1$$

For $k = 3$:

$$
\begin{aligned}
a_3 &= \lim_{k\to 3}\left[\frac{\sin \pi k}{\pi k} + \frac{2\cos \pi k}{(\pi k)^2} - \frac{2\sin \pi k}{(\pi k)^3}\right] \\
&\quad + \lim_{k\to 3}\left[-\frac{\sin \pi(3-k)}{8j\pi(3-k)} + \frac{3\sin \pi(1-k)}{8j\pi(1-k)} - \frac{3\sin \pi(1+k)}{8j\pi(1+k)} + \frac{\sin \pi(3+k)}{8j\pi(3+k)}\right] \\
&= \left(0 + \frac{2(-1)}{(3\pi)^2} - 0\right) + \left(-\lim_{k\to 3}\frac{\sin \pi(3-k)}{8j\pi(3-k)} + \frac{0}{-16j\pi} - \frac{0}{32j\pi} + \frac{0}{48j\pi}\right) \\
a_3 &= -\frac{2}{9\pi^2} + \frac{j}{8}
\end{aligned}
$$

For $k = -3$:

$$
\begin{aligned}
a_{-3} &= \lim_{k\to -3}\left[\frac{\sin \pi k}{\pi k} + \frac{2\cos \pi k}{(\pi k)^2} - \frac{2\sin \pi k}{(\pi k)^3}\right] \\
&\quad + \lim_{k\to -3}\left[-\frac{\sin \pi(3-k)}{8j\pi(3-k)} + \frac{3\sin \pi(1-k)}{8j\pi(1-k)} - \frac{3\sin \pi(1+k)}{8j\pi(1+k)} + \frac{\sin \pi(3+k)}{8j\pi(3+k)}\right] \\
&= \left(0 + \frac{2(-1)}{(3\pi)^2} - 0\right) + \left(-\frac{0}{16j\pi} + \frac{0}{6j\pi} - \lim_{k\to -3}\frac{3\sin \pi(1+k)}{8j\pi(1+k)} + \frac{0}{48j\pi}\right) \\
a_{-3} &= -\frac{2}{9\pi^2} - \frac{j}{8}
\end{aligned}
$$

Thus, we have:

$$
a_k = -\frac{2}{9\pi^2} + \frac{jk}{24} \text{ for } |k| = 3
$$

For other values of k where $|k| \neq 0, 1, 3$:

$$
\begin{aligned}
a_k &= \left(\frac{\sin \pi k}{\pi k} + \frac{2\cos \pi k}{(\pi k)^2} - \frac{2\sin \pi k}{(\pi k)^3}\right) \\
&\quad + \left(-\frac{\sin \pi(3-k)}{8j\pi(3-k)} + \frac{3\sin \pi(1-k)}{8j\pi(1-k)} - \frac{3\sin \pi(1+k)}{8j\pi(1+k)} + \frac{\sin \pi(3+k)}{8j\pi(3+k)}\right) \\
&= \left(0 + \frac{2(-1)^k}{(\pi k)^2}\right) + \left(-0 + \frac{0}{8j\pi(3-k)} + \frac{0}{8j\pi(1-k)} - \frac{0}{8j\pi(1+k)} + \frac{0}{8j\pi(3+k)}\right) \\
a_k &= \frac{2(-1)^k}{k^2\pi^2}
\end{aligned}
$$

Simplify a_k:

$$
a_k = \begin{cases} \frac{1}{3} & k = 0 \\ -\frac{2}{\pi^2} - \frac{3jk}{8} & |k| = 1 \\ -\frac{2}{9\pi^2} + \frac{jk}{24} & |k| = 3 \\ \frac{2(-1)^k}{k^2\pi^2} & \text{otherwise} \end{cases}
$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$
\boxed{x(t) = \left(\frac{1}{3} - \frac{40}{9\pi^2}\right)e^{j\pi t} + \sum_{|k|\neq 0,1,3} \frac{2(-1)^k}{k^2\pi^2} e^{j\pi t}}
$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows: