

Homework Signal 4

Week 4

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Collaborators. ChatGPT (for L^AT_EX styling and grammar checking)

1 Continuous-Time Fourier Transform (CTFT)

Problem 2. Let $\mathcal{F}\{x(t)\} = X(j\omega) = \text{rect}((\omega - 1)/2)$. Find Fourier transform of

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2.1 $x(-2t + 4)$

Solution. From the Time-scaling and Time-shifting properties of the Fourier transform, we have:

$$\text{Time-scaling: } \mathcal{F}\{x(at)\} = \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

$$\text{Time-shifting: } \mathcal{F}\{x(t - t_0)\} = e^{-j\omega t_0}X(j\omega)$$

Combining these two properties, we can find the Fourier transform of $x(at - b)$.

$$\mathcal{F}\{x(at - b)\} = \mathcal{F}\left\{x\left(a\left(t - \frac{b}{a}\right)\right)\right\} = \frac{1}{|a|}e^{-j\omega \frac{b}{a}}X\left(\frac{j\omega}{a}\right)$$

To find the Fourier transform of $x(-2t + 4)$, we have $a = -2$ and $b = -4$. Applying the combined properties and substituting $\mathcal{F}\{x(t)\}$, we get:

$$\begin{aligned}\mathcal{F}\{x(-2t + 4)\} &= \frac{1}{|-2|}e^{-j\omega \frac{-4}{-2}}X\left(\frac{j\omega}{-2}\right) \\ &= \frac{1}{|-2|}e^{-j\omega \frac{-4}{2}}X\left(\frac{j\omega}{-2}\right) \\ &= \frac{1}{2}e^{-2j\omega}\text{rect}\left(\frac{\frac{\omega}{-2} - 1}{2}\right) \\ \mathcal{F}\{x(-2t + 4)\} &= \frac{1}{2}e^{-2j\omega}\text{rect}\left(\frac{-\omega - 2}{4}\right)\end{aligned}$$

Because, rect is an even function, we can express the Fourier transform of $x(-2t + 4)$ as:

$$\mathcal{F}\{x(-2t + 4)\} = \frac{1}{2}e^{-2j\omega}\text{rect}\left(\frac{\omega + 2}{4}\right)$$

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2.2 $(t-1)x(t-1)$

Solution. First, define a new signal $y(t) = tx(t)$. Then, we can express $(t-1)x(t-1)$ as:

$$(t-1)x(t-1) = y(t-1)$$

Now, using the Time-shifting property of the Fourier transform, we have:

$$\text{Time-shifting: } \mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega)$$

Applying to $y(t-1)$, we get:

$$\begin{aligned}\mathcal{F}\{y(t-1)\} &= e^{-j\omega \cdot (1)} Y(j\omega) \\ &= e^{-j\omega} \mathcal{F}\{y(t)\} \\ \mathcal{F}\{y(t-1)\} &= e^{-j\omega} \mathcal{F}\{tx(t)\}\end{aligned}$$

Next, consider the differentiation of $\mathcal{F}\{x(t)\}$:

$$\begin{aligned}\frac{d}{d\omega} \mathcal{F}\{x(t)\} &= \frac{d}{d\omega} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(t) \frac{d}{d\omega} (e^{-j\omega t}) dt \\ &= \int_{-\infty}^{\infty} x(t) (-jte^{-j\omega t}) dt \\ &= -j \int_{-\infty}^{\infty} tx(t) e^{-j\omega t} dt \\ \frac{d}{d\omega} \mathcal{F}\{x(t)\} &= -j \mathcal{F}\{tx(t)\} \\ \mathcal{F}\{tx(t)\} &= j \frac{d}{d\omega} X(j\omega)\end{aligned}$$

Substituting $\mathcal{F}\{x(t)\}$ into the equation, we get:

$$\begin{aligned}\mathcal{F}\{tx(t)\} &= j \frac{d}{d\omega} X(j\omega) \\ &= j \frac{d}{d\omega} \left(\text{rect}\left(\frac{\omega-1}{2}\right) \right) \\ &= j \frac{d}{d\omega} (u(\omega) - u(\omega-2)) \\ \mathcal{F}\{tx(t)\} &= j (\delta(\omega) - \delta(\omega-2))\end{aligned}$$

Therefore, we can express the Fourier transform of $(t-1)x(t-1)$ as:

$$\mathcal{F}\{(t-1)x(t-1)\} = je^{-j\omega} (\delta(\omega) - \delta(\omega-2))$$

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$$2.3 \quad t \frac{dx(t)}{dt}$$

Solution. First, define a new signal $y(t) = \frac{dx(t)}{dt}$. Then, we can express $t \frac{dx(t)}{dt}$ as:

$$t \frac{dx(t)}{dt} = ty(t)$$

Now, using the Differentiation in frequency property (proved in the previous problem) of the Fourier transform, we have:

$$\text{Differentiation in frequency: } \mathcal{F}\{tx(t)\} = j \frac{d}{d\omega} X(j\omega)$$

Applying to $ty(t)$, we get:

$$\mathcal{F}\{ty(t)\} = j \frac{d}{d\omega} Y(j\omega) = j \frac{d}{d\omega} \mathcal{F}\{y(t)\} = j \frac{d}{d\omega} \mathcal{F}\left\{\frac{dx(t)}{dt}\right\}$$

Next, using the Differentiation in time property of the Fourier transform, we have:

$$\text{Differentiation in time: } \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j\omega X(j\omega)$$

Substituting into the equation, we get:

$$\begin{aligned} \frac{d}{d\omega} \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} &= \frac{d}{d\omega} (j\omega X(j\omega)) \\ &= \frac{d}{d\omega} \left(j\omega \cdot \text{rect}\left(\frac{\omega-1}{2}\right) \right) \\ &= \frac{d}{d\omega} (j\omega (u(\omega) - u(\omega-2))) \\ &= j\omega \frac{d}{d\omega} (u(\omega) - u(\omega-2)) + (u(\omega) - u(\omega-2)) \frac{d}{d\omega} (j\omega) \\ &= j\omega (\delta(\omega) - \delta(\omega-2)) + \text{rect}\left(\frac{\omega-1}{2}\right) \cdot (j) \\ &= j\omega \delta(\omega) - j\omega \delta(\omega-2) + j \text{rect}\left(\frac{\omega-1}{2}\right) \\ &= 0 - j(2)\delta(\omega-2) + j \text{rect}\left(\frac{\omega-1}{2}\right) \\ \frac{d}{d\omega} \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} &= -j(2)\delta(\omega-2) + j \text{rect}\left(\frac{\omega-1}{2}\right) \end{aligned}$$

Thus, substituting back, we have:

$$\begin{aligned} \mathcal{F}\{ty(t)\} &= j \frac{d}{d\omega} \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} \\ &= j \cdot \left[-j(2)\delta(\omega-2) + j \text{rect}\left(\frac{\omega-1}{2}\right) \right] \\ \mathcal{F}\{ty(t)\} &= 2\delta(\omega-2) - \text{rect}\left(\frac{\omega-1}{2}\right) \end{aligned}$$

Therefore, we can express the Fourier transform of $t \frac{dx(t)}{dt}$ as:

$$\boxed{\mathcal{F}\left\{t \frac{dx(t)}{dt}\right\} = 2\delta(\omega-2) - \text{rect}\left(\frac{\omega-1}{2}\right)}$$

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$$2.4 \quad x(2t - 1)e^{-j2t}$$

Solution. Using the Time-scaling, and Time-shifting properties of the Fourier transform, we have:

$$\text{Time-scaling + Time-shifting: } \mathcal{F}\{x(at - b)\} = \frac{1}{|a|} e^{-j\omega \frac{b}{a}} X\left(\frac{j\omega}{a}\right)$$

Define a new signal $y(t) = x(2t - 1)$. Then, we can express $x(2t - 1)e^{-j2t}$ as:

$$\begin{aligned} Y(j\omega) &= \mathcal{F}\{y(t)\} \\ &= \mathcal{F}\{x(2t - 1)\} \\ &= \frac{1}{|2|} e^{-j\omega \frac{1}{2}} X\left(\frac{j\omega}{2}\right) \\ Y(j\omega) &= \frac{1}{2} e^{-j\frac{\omega}{2}} X\left(\frac{j\omega}{2}\right) \end{aligned}$$

Now, using the Frequency-shifting property of the Fourier transform, we have:

$$\text{Frequency-shifting: } \mathcal{F}\{x(t)e^{j\omega_0 t}\} = X(j(\omega - \omega_0))$$

Applying to $y(t)e^{-j2t}$, we get:

$$\begin{aligned} \mathcal{F}\{y(t)e^{-j2t}\} &= \mathcal{F}\{y(t)e^{j(-2)t}\} \\ &= Y(j(\omega - (-2))) \\ &= Y(j(\omega + 2)) \\ \mathcal{F}\{y(t)e^{-j2t}\} &= \frac{1}{2} e^{-j\frac{\omega+2}{2}} X\left(\frac{j(\omega + 2)}{2}\right) \end{aligned}$$

Then, substituting $\mathcal{F}\{x(t)\}$ into the equation, we get:

$$\begin{aligned} \mathcal{F}\{y(t)e^{-j2t}\} &= \frac{1}{2} e^{-j\frac{\omega+2}{2}} X\left(\frac{j(\omega + 2)}{2}\right) \\ &= \frac{1}{2} e^{-j\frac{\omega+2}{2}} \text{rect}\left(\frac{\frac{\omega+2}{2} - 1}{2}\right) \\ &= \frac{1}{2} e^{-j\frac{\omega+2}{2}} \text{rect}\left(\frac{\omega + 2 - 2}{4}\right) \\ \mathcal{F}\{y(t)e^{-j2t}\} &= \frac{1}{2} e^{-j\frac{\omega+2}{2}} \text{rect}\left(\frac{\omega}{4}\right) \end{aligned}$$

Therefore, we can express the Fourier transform of $x(2t - 1)e^{-j2t}$ as:

$$\boxed{\mathcal{F}\{x(2t - 1)e^{-j2t}\} = \frac{1}{2} e^{-j\frac{\omega+2}{2}} \text{rect}\left(\frac{\omega}{4}\right)}$$

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2.5 $x(t) * x(t - 1)$

Solution. First, define a new signal $y(t) = x(t - 1)$. Then, we can express $x(t) * x(t - 1)$ as:

$$x(t) * x(t - 1) = x(t) * y(t)$$

Next, using the Time-shifting property of the Fourier transform, we have:

$$\text{Time-shifting: } \mathcal{F}\{x(t - t_0)\} = e^{-j\omega t_0} X(j\omega)$$

Substituting into the equation, we get:

$$\begin{aligned} Y(j\omega) &= \mathcal{F}\{y(t)\} \\ &= \mathcal{F}\{x(t - 1)\} \\ &= e^{-j\omega(1)} X(j\omega) \\ Y(j\omega) &= e^{-j\omega} X(j\omega) \end{aligned}$$

Now, using the Convolution property of the Fourier transform, we have:

$$\text{Convolution: } \mathcal{F}\{x_1(t) * x_2(t)\} = X_1(j\omega) \cdot X_2(j\omega)$$

Applying to $x(t) * y(t)$, we get:

$$\begin{aligned} \mathcal{F}\{x(t) * y(t)\} &= \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{y(t)\} \\ &= X(j\omega) \cdot \mathcal{F}\{x(t - 1)\} \\ &= X(j\omega) \cdot e^{-j\omega(1)} X(j\omega) \\ \mathcal{F}\{x(t) * y(t)\} &= (X(j\omega))^2 \cdot e^{-j\omega} \end{aligned}$$

Lastly, substituting $\mathcal{F}\{x(t)\} = \text{rect}((\omega - 1)/2)$ back into the equation, we get:

$$\mathcal{F}\{x(t) * y(t)\} = \text{rect}^2\left(\frac{\omega - 1}{2}\right) \cdot e^{-j\omega}$$

But, since $\text{rect}(\cdot)$ is equal to either 0 or 1, we have:

$$\text{rect}^2\left(\frac{\omega - 1}{2}\right) = \text{rect}\left(\frac{\omega - 1}{2}\right)$$

Therefore, we can express the Fourier transform of $x(t) * x(t - 1)$ as:

$$\mathcal{F}\{x(t) * x(t - 1)\} = e^{-j\omega} \text{rect}\left(\frac{\omega - 1}{2}\right)$$