Homework Signal 3

Week 3

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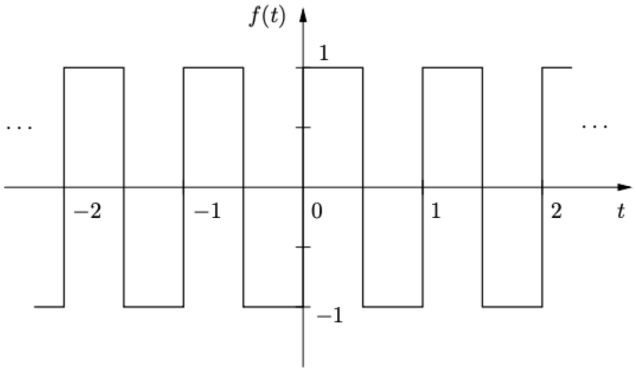
Collaborators. ChatGPT (for LaTeX styling and grammar checking)

1 Fourier Series

Problem 1. Find the Fourier series of the following periodic function:



Solution. From the graph, we can see that the function $x(t)$ can be defined piecewise as follows:

$$x(t) = \begin{cases} 1, & -\frac{1}{2} \leq t < 0 \\ -1, & 0 \leq t < \frac{1}{2} \end{cases}$$

with a period $T = 1$.

Calculating ω_0:

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$$

Using the Fourier series formula:

$$x(t) = \sum_k a_k e^{jk\omega_0 t} = a_0 + \sum_{k \neq 0} a_k e^{jk\omega_0 t}$$

where the Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} \, dt$$

Calculating a_0:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-j(0)\omega_0 t} \, dt \\ &= \frac{1}{1} \int_{-0.5}^{0.5} x(t) \, dt \\ &= \int_{-0.5}^{0} (1) \, dt + \int_{0}^{0.5} (-1) \, dt \\ &= \frac{1}{2} - \frac{1}{2} \\ a_0 &= 0 \end{aligned}$$

Calculating a_k for $k \neq 0$:

$$\begin{aligned}
a_k &= \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jk\omega_0 t}\,dt \\
&= \frac{1}{1}\int_{-0.5}^{0.5} x(t)e^{-jk(2\pi/T)t}\,dt \\
&= \frac{1}{1}\int_{-0.5}^{0.5} x(t)e^{-jk(2\pi/1)t}\,dt \\
&= \int_{-0.5}^{0} (1)\cdot e^{-j2\pi kt}\,dt + \int_{0}^{0.5} (-1)\cdot e^{-j2\pi kt}\,dt \\
&= \left[\frac{e^{-j2\pi kt}}{-j2\pi k}\right]_{-0.5}^{0} + \left[\frac{e^{-j2\pi kt}}{j2\pi k}\right]_{0}^{0.5} \\
&= \frac{1}{-j2\pi k}\left(e^{j\pi k}-1\right) + \frac{1}{j2\pi k}\left(1-e^{-j\pi k}\right) \\
&= \frac{j}{2\pi k}\left(e^{j\pi k}-1-1+e^{-j\pi k}\right) \\
&= \frac{j}{\cancel{2}\pi k}\left(\cancel{2}\cos(\pi k)-\cancel{2}\right) \\
a_k &= \frac{j}{\pi k}\left(\cos(\pi k)-1\right)
\end{aligned}$$

We can simplify a_k:

$$a_k = \begin{cases} 0, & \text{if } k \text{ is even} \\ \frac{-2j}{\pi k}, & \text{if } k \text{ is odd} \end{cases}$$

Thus, the Fourier series expansion of $x(t)$ is:

$$x(t) = \sum_{k \text{ odd}} \left(\operatorname{Re}\left\{\frac{-2j}{\pi k}e^{j2\pi kt}\right\}\right)$$

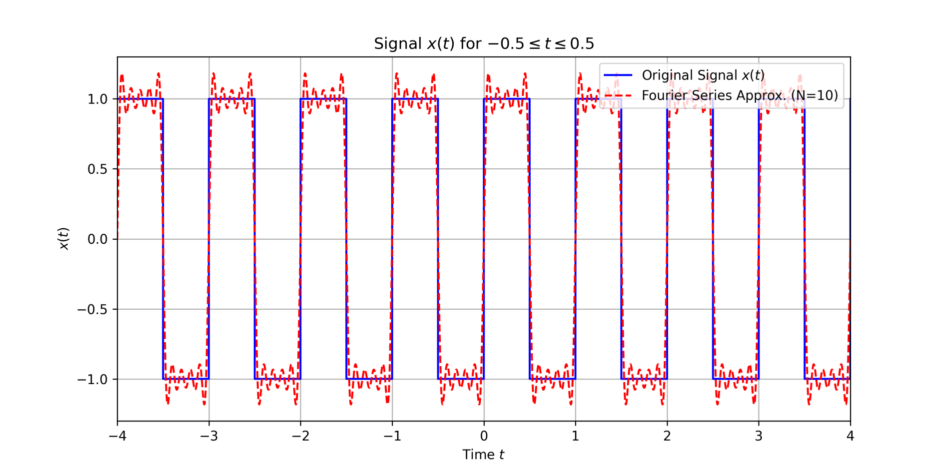
Because $e^{jx} - e^{-jx} = 2j\sin(x)$ for any real x, we have:

$$\begin{aligned}
\sum_{k \text{ odd}} \frac{-2j}{\pi k}e^{j2\pi kt} &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{-2j}{\pi k}e^{j2\pi kt} + \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{-1} \frac{-2j}{\pi k}e^{j2\pi kt} \\
&= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{-2j}{\pi k}\left(e^{j2\pi kt} - e^{-j2\pi kt}\right) \\
&= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{-2j}{\pi k}\left(2j\sin(2\pi kt)\right) \\
\sum_{k \text{ odd}} \frac{-2j}{\pi k}e^{j2\pi kt} &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi k}\sin(2\pi kt)
\end{aligned}$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$\boxed{x(t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \left(\frac{4}{\pi k}\sin(2\pi kt)\right)}$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows:



Problem 2. Find the Fourier Series (FS) of the periodic function $x(t)$ which are provided as follows.

2.1 $x(t) = \frac{\pi t^3}{2}$; $-1 < t < 1$

Solution. To find the Fourier series of the function $x(t) = \frac{\pi t^3}{2}$ for $-1 < t < 1$, where $T = 2$ (the period of the function).
Calculating ω_0:

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

Using the Fourier series formula:

$$x(t) = \sum_k a_k e^{jk\omega_0 t} = a_0 + \sum_{k \neq 0} a_k e^{jk\omega_0 t}$$

where the Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t}\, dt$$

Calculating a_0:

$$\begin{aligned}
a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-j(0)\omega_0 t}\, dt \\
&= \frac{1}{2} \int_{-1}^{1} \frac{\pi t^3}{2}\, dt \\
&= \frac{\pi}{4} \left[\frac{t^4}{4} \right]_{-1}^{1} \\
&= \frac{\pi}{4} \left(\frac{1}{4} - \frac{1}{4} \right) \\
a_0 &= 0
\end{aligned}$$

Calculating a_k for $k \neq 0$:

$$\begin{aligned}
a_k &= \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jk\omega_0 t}\,dt \\
&= \frac{1}{2}\int_{-1}^{1} \frac{\pi t^3}{2}e^{-j\pi kt}\,dt \\
a_k &= \frac{\pi}{4}\int_{-1}^{1} t^3 e^{-j\pi kt}dt
\end{aligned}$$

To solve the integral, we can use integration by parts multiple times. Using tabular integration by parts, we find:

Sign	Derivative	Integral
$+$	t^3	$e^{-j\pi kt}$
$-$	$3t^2$	$\frac{1}{-j\pi k}e^{-j\pi kt}$
$+$	$6t$	$\frac{1}{(-j\pi k)^2}e^{-j\pi kt}$
$-$	6	$\frac{1}{(-j\pi k)^3}e^{-j\pi kt}$
$+$	0	$\frac{1}{(-j\pi k)^4}e^{-j\pi kt}$

Thus, we have:

$$\int t^3 e^{-j\pi kt}\,dt = \frac{t^3}{-j\pi k}e^{-j\pi kt} - \frac{3t^2}{(-j\pi k)^2}e^{-j\pi kt} + \frac{6t}{(-j\pi k)^3}e^{-j\pi kt} - \frac{6}{(-j\pi k)^4}e^{-j\pi kt}$$

Evaluating this from -1 to 1 to find a_k:

$$\begin{aligned}
a_k &= \frac{\pi}{4}\left[\frac{t^3}{-j\pi k}e^{-j\pi kt} - \frac{3t^2}{(-j\pi k)^2}e^{-j\pi kt} + \frac{6t}{(-j\pi k)^3}e^{-j\pi kt} - \frac{6}{(-j\pi k)^4}e^{-j\pi kt}\right]_{-1}^{1} \\
&= \frac{\pi}{4}\left[\frac{2j\cos(\pi k)}{\pi k} - \frac{6\sin(\pi k)}{(\pi k)^2} + \frac{-12j\cos(\pi k)}{(\pi k)^3} - \frac{-12\sin(\pi k)}{(\pi k)^4}\right] \\
&= \frac{\pi}{\cancel{4}^2}\left[\frac{\cancel{2}j\cos(\pi k)}{\pi k} - \frac{\cancel{6}^3\sin(\pi k)}{(\pi k)^2} - \frac{\cancel{12}^6 j\cos(\pi k)}{(\pi k)^3} + \frac{\cancel{12}^6\sin(\pi k)}{(\pi k)^4}\right] \\
a_k &= \frac{\pi}{2}\left[\frac{j\cos(\pi k)}{\pi k} - \frac{3\sin(\pi k)}{(\pi k)^2} - \frac{6j\cos(\pi k)}{(\pi k)^3} + \frac{6\sin(\pi k)}{(\pi k)^4}\right]
\end{aligned}$$

We can simplify a_k:

$$a_k = \frac{\pi}{2}\left[\frac{j(-1)^k}{\pi k} - 0 - \frac{6j(-1)^k}{(\pi k)^3} + 0\right] = \frac{(-1)^k}{2jk} - \frac{3(-1)^k}{j\pi^2 k^3}$$

Thus, the Fourier series expansion of $x(t)$ is:

$$x(t) = \sum_{k\neq 0}\left(\mathrm{Re}\left\{\left(\frac{(-1)^k}{2jk} - \frac{3(-1)^k}{j\pi^2 k^3}\right)e^{j\pi kt}\right\}\right)$$

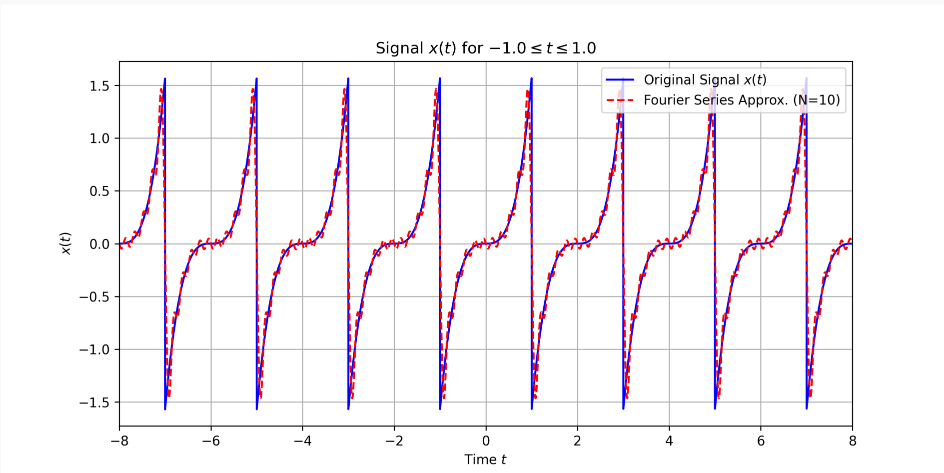
Because $e^{jx} - e^{-jx} = 2j\sin(x)$ for any real x, we have:

$$
\begin{aligned}
\sum_{k\neq 0}\left(\frac{(-1)^k}{2jk} - \frac{3(-1)^k}{j\pi^2k^3}\right)e^{j\pi kt} &= \sum_{k=1}^{\infty}\left(\frac{(-1)^k}{2jk} - \frac{3(-1)^k}{j\pi^2k^3}\right)e^{j\pi kt} \\
&\quad + \sum_{k=-\infty}^{-1}\left(\frac{(-1)^k}{2jk} - \frac{3(-1)^k}{j\pi^2k^3}\right)e^{j\pi kt} \\
&= \sum_{k=1}^{\infty}\left(\frac{(-1)^k}{2jk} - \frac{3(-1)^k}{j\pi^2k^3}\right)e^{j\pi kt} \\
&\quad + \sum_{k=1}^{\infty}\left(\frac{(-1)^{-k}}{-2jk} - \frac{3(-1)^{-k}}{-j\pi^2k^3}\right)e^{-j\pi kt} \\
&= \sum_{k=1}^{\infty}\left(\frac{(-1)^k}{2jk} - \frac{3(-1)^k}{j\pi^2k^3}\right)\left(e^{j\pi kt} - e^{-j\pi kt}\right) \\
&= \sum_{k=1}^{\infty}\left(\frac{(-1)^k}{2\cancel{j}k} - \frac{3(-1)^k}{\cancel{j}\pi^2k^3}\right)\left(2\cancel{j}\sin(\pi kt)\right) \\
&= \sum_{k=1}^{\infty}\left(\frac{(-1)^k}{k} - \frac{6(-1)^k}{\pi^2k^3}\right)\sin(\pi kt) \\
\sum_{k\neq 0}\left(\frac{(-1)^k}{2jk} - \frac{3(-1)^k}{j\pi^2k^3}\right)e^{j\pi kt} &= \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(1 - \frac{6}{\pi^2k^2}\right)\sin(\pi kt)
\end{aligned}
$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$
\boxed{x(t) = \sum_{k=1}^{\infty}\left(\frac{(-1)^k}{k}\left(1 - \frac{6}{\pi^2k^2}\right)\sin(\pi kt)\right)}
$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows:



TO SUBMIT

2.2 $x(t) = \pi - t;\ -\pi \leq t \leq \pi$

Solution. To find the Fourier series of the function $x(t) = \pi - t$ for $-\pi \leq t \leq \pi$, where $T = 2\pi$ (the period of the function).
Calculating ω_0:

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Using the Fourier series formula:

$$x(t) = \sum_k a_k e^{jk\omega_0 t} = a_0 + \sum_{k\neq 0} a_k e^{jk\omega_0 t}$$

where the Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jk\omega_0 t}\,dt$$

Calculating a_0:

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-j(0)\omega_0 t}\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - t)\,dt \\
&= \frac{1}{2\pi}\left[\pi t - \frac{t^2}{2}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2\pi}\left(\pi^2 - \frac{\pi^2}{2} - (-\pi^2 - \frac{\pi^2}{2})\right) \\
a_0 &= \pi
\end{aligned}$$

Calculating a_k for $k \neq 0$:

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jk\omega_0 t}\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi - t)e^{-jkt}\,dt \\
a_k &= \frac{1}{2\pi}\left[\pi\int_{-\pi}^{\pi} e^{-jkt}\,dt - \int_{-\pi}^{\pi} te^{-jkt}\,dt\right]
\end{aligned}$$

To solve the integral, we can use integration by parts multiple times. Using tabular integration by parts, we find:

Sign	Derivative	Integral
$+$	t	$e^{-j\pi kt}$
$-$	1	$\frac{1}{-j\pi k}e^{-j\pi kt}$
$+$	0	$\frac{1}{(-j\pi k)^2}e^{-j\pi kt}$

Thus, we have:

$$\int te^{-jkt}\,dt = \frac{t}{-jk}e^{-jkt} - \frac{1}{(-jk)^2}e^{-jkt}$$

Evaluating this from $-\pi$ to π to find a_k:

$$\begin{aligned}
a_k &= \frac{1}{2\pi}\left[\pi\left[\frac{e^{-jkt}}{-jk}\right]_{-\pi}^{\pi} - \left[\frac{t}{-jk}e^{-jkt} - \frac{1}{(-jk)^2}e^{-jkt}\right]_{-\pi}^{\pi}\right] \\
&= \frac{1}{2}\left[\frac{e^{-jkt}}{-jk}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\left[\frac{t}{-jk}e^{-jkt} - \frac{1}{(-jk)^2}e^{-jkt}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}\left[\frac{\left(e^{-j\pi k} - e^{j\pi k}\right)}{-jk}\right] - \frac{1}{2\pi}\left[\frac{\left(\pi e^{-j\pi k} + \pi e^{j\pi k}\right)}{-jk} - \frac{\left(e^{-j\pi k} - e^{j\pi k}\right)}{(-jk)^2}\right] \\
&= \frac{1}{\cancel{2}}\left[\frac{\left(-\cancel{2}\cancel{j}\sin(\pi k)\right)}{-\cancel{j}k}\right] - \frac{1}{\cancel{2}\pi}\left[\frac{\left(\cancel{2}\pi\cos(\pi k)\right)}{-jk} - \frac{\left(-\cancel{2}j\sin(\pi k)\right)}{(-jk)^2}\right] \\
a_k &= \frac{\sin(\pi k)}{n} + \frac{\cos(\pi k)}{jn} - \frac{\sin(\pi k)}{j\pi k^2}
\end{aligned}$$

We can simplify a_k:

$$a_k = 0 + \frac{(-1)^k}{jk} - 0 = \frac{(-1)^k}{jk} \text{ for } k \neq 0$$

Thus, the Fourier series expansion of $x(t)$ is:

$$x(t) = \pi + \sum_{k\neq 0}\left(\mathrm{Re}\left\{\frac{(-1)^k}{jk}e^{jkt}\right\}\right)$$

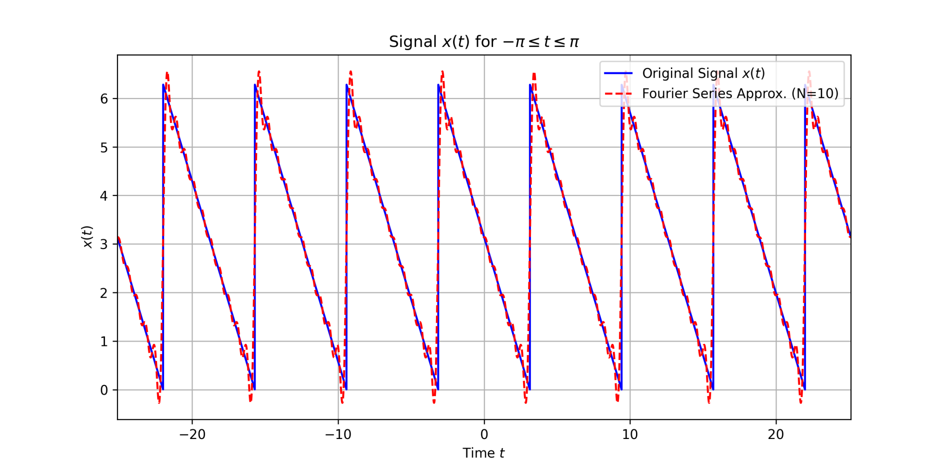
Because $e^{jx} - e^{-jx} = 2j\sin(x)$ for any real x, we have:

$$\begin{aligned}
\sum_{k\neq 0}\frac{(-1)^k}{jk}e^{jkt} &= \sum_{k=1}^{\infty}\frac{(-1)^k}{jk}e^{jkt} + \sum_{k=-\infty}^{-1}\frac{(-1)^k}{jk}e^{jkt} \\
&= \sum_{k=1}^{\infty}\frac{(-1)^k}{jk}e^{jkt} + \sum_{k=1}^{\infty}\frac{(-1)^{-k}}{-jk}e^{-jkt} \\
&= \sum_{k=1}^{\infty}\frac{(-1)^k}{jk}\left(e^{jkt} - e^{-jkt}\right) \\
&= \sum_{k=1}^{\infty}\frac{(-1)^k}{jk}\left(2j\sin(kt)\right) \\
\sum_{k\neq 0}\frac{(-1)^k}{jk}e^{jkt} &= \sum_{k=1}^{\infty}\frac{2(-1)^k}{k}\sin(kt)
\end{aligned}$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$\boxed{x(t) = \pi + \sum_{k=1}^{\infty}\left(\frac{2(-1)^k}{k}\sin(kt)\right)}$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows:



TO SUBMIT

2.3 $x(t) = t^2 + \sin^3(\pi t);\ -1 \leq t \leq 1$

Solution. To find the Fourier series of the function $x(t) = t^2 + \sin^3(\pi t)$ for $-1 \leq t \leq 1$, where $T = 2$ (the period of the function).
Calculating ω_0:

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

Using the Fourier series formula:

$$x(t) = \sum_k a_k e^{jk\omega_0 t} = a_0 + \sum_{k \neq 0} a_k e^{jk\omega_0 t}$$

where the Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t}\, dt$$

Calculating a_0:

$$\begin{aligned}
a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-j(0)\omega_0 t}\, dt \\
&= \frac{1}{2} \int_{-1}^{1} (t^2 + \sin^3(\pi t))\, dt \\
&= \frac{1}{2} \int_{-1}^{1} t^2\, dt + \frac{1}{2} \int_{-1}^{1} \sin^3(\pi t)\, dt \\
&= \frac{1}{2} \left[\frac{t^3}{3} \right]_{-1}^{1} + \frac{1}{2}(0) \qquad \text{since } \sin^3(\pi t) \text{ is odd function} \\
&= \frac{1}{2} \left[\frac{1^3}{3} - \frac{(-1)^3}{3} \right] + 0 \\
&= \frac{1}{2} \cdot \frac{2}{3} \\
a_0 &= \frac{1}{3}
\end{aligned}$$

Calculating a_k for $k \neq 0$:

$$\begin{aligned} a_k &= \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jk\omega_0 t}\,dt \\ &= \frac{1}{2}\int_{-1}^{1}(t^2+\sin^3(\pi t))e^{-jk\pi t}\,dt \\ a_k &= \frac{1}{2}\left[\int_{-1}^{1} t^2 e^{-jk\pi t}\,dt + \int_{-1}^{1}\sin^3(\pi t)e^{-jk\pi t}\,dt\right] \end{aligned}$$

Define

$$I_1 = \int_{-1}^{1} t^2 e^{-jk\pi t}\,dt \quad \text{and} \quad I_2 = \int_{-1}^{1}\sin^3(\pi t)e^{-jk\pi t}\,dt$$

Hence,

$$a_k = \frac{1}{2}(I_1 + I_2)$$

.
To solve the integral I_1, we can use integration by parts multiple times. Using tabular integration by parts, we find:

Sign	Derivative	Integral
$+$	t^2	$e^{-j\pi kt}$
$-$	$2t$	$\frac{1}{-j\pi k}e^{-j\pi kt}$
$+$	2	$\frac{1}{(-j\pi k)^2}e^{-j\pi kt}$
$-$	0	$\frac{1}{(-j\pi k)^3}e^{-j\pi kt}$

Thus, we have:

$$I_1 = \int t^2 e^{-jk\pi t}\,dt = \frac{t^2}{-j\pi k}e^{-jk\pi t} - \frac{2t}{(-j\pi k)^2}e^{-jk\pi t} + \frac{2}{(-j\pi k)^3}e^{-jk\pi t}$$

Evaluating this from -1 to 1 to find I_1:

$$\begin{aligned} I_1 &= \left[\frac{t^2}{-j\pi k}e^{-jk\pi t} - \frac{2t}{(-j\pi k)^2}e^{-jk\pi t} + \frac{2}{(-j\pi k)^3}e^{-jk\pi t}\right]_{-1}^{1} \\ &= \left[\frac{-2\cancel{j}\sin(\pi k)}{-\cancel{j}\pi k} - \frac{4\cos(\pi k)}{(-j\pi k)^2} + \frac{-4\cancel{j}\sin(\pi k)}{(-\cancel{j}\pi k)^3}\right] \\ &= \frac{-2\sin(\pi k)}{-\pi k} - \frac{4\cos(\pi k)}{(-j\pi k)^2} + \frac{-4\sin(\pi k)}{j^2(-\pi k)^3} \\ I_1 &= \frac{2\sin(\pi k)}{\pi k} + \frac{4\cos(\pi k)}{(\pi k)^2} - \frac{4\sin(\pi k)}{(\pi k)^3} \end{aligned}$$

Next, to solve the integral I_2, we can use the euler identity:

$$\sin^3(x) = \left\{\frac{1}{2j}\left(e^{jx} - e^{-jx}\right)\right\}^3 = -\frac{1}{8j}\left(e^{3jx} - 3e^{jx} + 3e^{-jx} - e^{-3jx}\right)$$

Thus,

$$
\begin{aligned}
I_2 &= \int_{-1}^{1} \sin^3(\pi t) e^{-jk\pi t}\, dt \\
&= \int_{-1}^{1} -\frac{1}{8j}\left(e^{3j\pi t} - 3e^{j\pi t} + 3e^{-j\pi t} - e^{-3j\pi t}\right) e^{-jk\pi t}\, dt \\
&= -\frac{1}{8j}\int_{-1}^{1} \left(e^{j\pi t(3-k)} - 3e^{j\pi t(1-k)} + 3e^{-j\pi t(1+k)} - e^{-j\pi t(3+k)}\right) dt \\
&= -\frac{1}{8j}\left[\frac{e^{j\pi t(3-k)}}{j\pi(3-k)} - \frac{3e^{j\pi t(1-k)}}{j\pi(1-k)} + \frac{3e^{-j\pi t(1+k)}}{-j\pi(1+k)} - \frac{e^{-j\pi t(3+k)}}{-j\pi(3+k)}\right]_{-1}^{1} \\
&= -\frac{1}{8j}\left[\frac{2\cancel{j}\sin(\pi(3-k))}{\cancel{j}\pi(3-k)} - \frac{3\left(2\cancel{j}\sin(\pi(1-k))\right)}{\cancel{j}\pi(1-k)} + \frac{3\left(-2\cancel{j}\sin(\pi(1+k))\right)}{-\cancel{j}\pi(1+k)} - \frac{-2\cancel{j}\sin(\pi(3+k))}{-\cancel{j}\pi(3+k)}\right] \\
I_2 &= -\frac{1}{4j}\left[\frac{\sin(\pi(3-k))}{\pi(3-k)} - \frac{3\sin(\pi(1-k))}{\pi(1-k)} + \frac{3\sin(\pi(1+k))}{\pi(1+k)} - \frac{\sin(\pi(3+k))}{\pi(3+k)}\right]
\end{aligned}
$$

Therefore, we have:

$$
\begin{aligned}
a_k &= \frac{1}{2}(I_1 + I_2) \\
&= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}\sin(\pi k)}{\pi k} + \frac{\cancel{4}^2\cos(\pi k)}{(\pi k)^2} - \frac{\cancel{4}^2\sin(\pi k)}{(\pi k)^3}\right] \\
&\quad - \frac{1}{2}\cdot\frac{1}{4j}\left[\frac{\sin(\pi(3-k))}{\pi(3-k)} - \frac{3\sin(\pi(1-k))}{\pi(1-k)} + \frac{3\sin(\pi(1+k))}{\pi(1+k)} - \frac{\sin(\pi(3+k))}{\pi(3+k)}\right] \\
a_k &= \frac{\sin(\pi k)}{\pi k} + \frac{2\cos(\pi k)}{(\pi k)^2} - \frac{2\sin(\pi k)}{(\pi k)^3} \\
&\quad - \frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3\sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3\sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)}
\end{aligned}
$$

Consider the value of a_k, we can see that at $|k| = 1$ and $|k| = 3$, the terms will be undefined. Therefore, we need to calculate these four cases separately using limits.

For $k = 1$:

$$
\begin{aligned}
a_1 &= \lim_{k\to 1}\left[\frac{\sin(\pi k)}{\pi k} + \frac{2\cos(\pi k)}{(\pi k)^2} - \frac{2\sin(\pi k)}{(\pi k)^3}\right] \\
&\quad + \lim_{k\to 1}\left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3\sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3\sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)}\right] \\
&= \left(0 + \frac{2(-1)}{\pi^2} - 0\right) + \left(-\frac{0}{16j\pi} + \lim_{k\to 1}\frac{3\sin(\pi(1-k))}{8j\pi(1-k)} - \frac{0}{16j\pi} + \frac{0}{32j\pi}\right) \\
a_1 &= -\frac{2}{\pi^2} - \frac{3j}{8}
\end{aligned}
$$

For $k = -1$:

$$
\begin{aligned}
a_{-1} &= \lim_{k\to -1}\left[\frac{\sin(\pi k)}{\pi k} + \frac{2\cos(\pi k)}{(\pi k)^2} - \frac{2\sin(\pi k)}{(\pi k)^3}\right] \\
&\quad + \lim_{k\to -1}\left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3\sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3\sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)}\right] \\
&= \left(0 + \frac{2(-1)}{\pi^2} - 0\right) + \left(-\frac{0}{16j\pi} + \frac{0}{6j\pi} - \lim_{k\to -1}\frac{3\sin(\pi(1+k))}{8j\pi(1+k)} + \frac{0}{32j\pi}\right) \\
a_{-1} &= -\frac{2}{\pi^2} + \frac{3j}{8}
\end{aligned}
$$

Thus, we have:

$$a_k = -\frac{2}{\pi^2} - \frac{3jk}{8} \text{ for } |k| = 1$$

For $k = 3$:

$$\begin{aligned}
a_3 &= \lim_{k\to 3}\left[\frac{\sin(\pi k)}{\pi k} + \frac{2\cos(\pi k)}{(\pi k)^2} - \frac{2\sin(\pi k)}{(\pi k)^3}\right] \\
&\quad + \lim_{k\to 3}\left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3\sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3\sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)}\right] \\
&= \left(0 + \frac{2(-1)}{(3\pi)^2} - 0\right) + \left(-\lim_{k\to 3}\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{0}{-16j\pi} - \frac{0}{32j\pi} + \frac{0}{48j\pi}\right) \\
a_3 &= -\frac{2}{9\pi^2} + \frac{j}{8}
\end{aligned}$$

For $k = -3$:

$$\begin{aligned}
a_{-3} &= \lim_{k\to -3}\left[\frac{\sin(\pi k)}{\pi k} + \frac{2\cos(\pi k)}{(\pi k)^2} - \frac{2\sin(\pi k)}{(\pi k)^3}\right] \\
&\quad + \lim_{k\to -3}\left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3\sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3\sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)}\right] \\
&= \left(0 + \frac{2(-1)}{(3\pi)^2} - 0\right) + \left(-\frac{0}{16j\pi} + \frac{0}{6j\pi} - \lim_{k\to -3}\frac{3\sin(\pi(1+k))}{8j\pi(1+k)} + \frac{0}{48j\pi}\right) \\
a_{-3} &= -\frac{2}{9\pi^2} - \frac{j}{8}
\end{aligned}$$

Thus, we have:

$$a_k = -\frac{2}{9\pi^2} + \frac{jk}{24} \text{ for } |k| = 3$$

For other values of k where $|k| \neq 0, 1, 3$:

$$\begin{aligned}
a_k &= \left(\frac{\sin(\pi k)}{\pi k} + \frac{2\cos(\pi k)}{(\pi k)^2} - \frac{2\sin(\pi k)}{(\pi k)^3}\right) \\
&\quad + \left(-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3\sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3\sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)}\right) \\
&= \left(0 + \frac{2(-1)^k}{(\pi k)^2}\right) + \left(-0 + \frac{0}{8j\pi(3-k)} + \frac{0}{8j\pi(1-k)} - \frac{0}{8j\pi(1+k)} + \frac{0}{8j\pi(3+k)}\right) \\
a_k &= \frac{2(-1)^k}{k^2\pi^2}
\end{aligned}$$

Simplify a_k:

$$a_k = \begin{cases} \frac{1}{3} & k = 0 \\ -\frac{2}{\pi^2} - \frac{3jk}{8} & |k| = 1 \\ -\frac{2}{9\pi^2} + \frac{jk}{24} & |k| = 3 \\ \frac{2(-1)^k}{k^2\pi^2} & \text{otherwise} \end{cases}$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$\begin{aligned}
x(t) &= \frac{1}{3} + \sum_{|k|\neq 0,1,3}\left(\operatorname{Re}\left\{\frac{2(-1)^k}{k^2\pi^2}e^{j\pi kt}\right\}\right) \\
&\quad + \sum_{|k|=1}\left(\operatorname{Re}\left\{\left(-\frac{2}{\pi^2} - \frac{3jk}{8}\right)e^{j\pi kt}\right\}\right) + \sum_{|k|=3}\left(\operatorname{Re}\left\{\left(-\frac{2}{9\pi^2} - \frac{jk}{24}\right)e^{j\pi kt}\right\}\right)
\end{aligned}$$

Because $e^{jx} + e^{-jx} = 2\cos(x)$ and $e^{jx} - e^{-jx} = 2j\sin(x)$ for any real x, we can simplify the Fourier series expansion further.

Consider the sums separately:

1. For $|k| \neq 0, 1, 3$:

$$\begin{aligned}
\sum_{|k|\neq 0,1,3} \left(\frac{2(-1)^k}{k^2\pi^2} e^{j\pi kt}\right) &= \frac{2}{\pi^2} \sum_{|k|\neq 0,1,3} \left(\frac{(-1)^k}{k^2} e^{j\pi kt}\right) \\
&= \frac{2}{\pi^2} \left(\sum_{\substack{k=-\infty \\ |k|\neq 0,1,3}}^{-1} \left(\frac{(-1)^k}{k^2} e^{j\pi kt}\right) + \sum_{\substack{k=1 \\ |k|\neq 0,1,3}}^{\infty} \left(\frac{(-1)^k}{k^2} e^{j\pi kt}\right) \right) \\
&= \frac{2}{\pi^2} \left(\sum_{\substack{k=1 \\ |k|\neq 0,1,3}}^{\infty} \left(\frac{(-1)^{-k}}{(-k)^2} e^{-j\pi kt}\right) + \sum_{\substack{k=1 \\ |k|\neq 0,1,3}}^{\infty} \left(\frac{(-1)^k}{k^2} e^{j\pi kt}\right) \right) \\
&= \frac{2}{\pi^2} \sum_{\substack{k=1 \\ |k|\neq 0,1,3}}^{\infty} \left(\frac{(-1)^k}{k^2} \left(e^{-j\pi kt} + e^{j\pi kt}\right)\right) \\
&= \frac{2}{\pi^2} \sum_{\substack{k=1 \\ |k|\neq 0,1,3}}^{\infty} \left(\frac{(-1)^k}{k^2} \left(2\cos(\pi kt)\right)\right) \\
\sum_{|k|\neq 0,1,3} \left(\frac{2(-1)^k}{k^2\pi^2} e^{j\pi kt}\right) &= \frac{4}{\pi^2} \sum_{\substack{k=1 \\ |k|\neq 0,1,3}}^{\infty} \left(\frac{(-1)^k}{k^2} \cos(\pi kt)\right)
\end{aligned}$$

2. For $|k| = 1$:

$$\begin{aligned}
\sum_{|k|=1} \left(\left(-\frac{2}{\pi^2} - \frac{3jk}{8}\right) e^{j\pi kt}\right) &= \sum_{|k|=1} \left(-\frac{2}{\pi^2} e^{j\pi kt}\right) + \sum_{|k|=1} \left(-\frac{3jk}{8} e^{j\pi kt}\right) \\
&= -\frac{2}{\pi^2}\left(e^{-j\pi t} + e^{j\pi t}\right) - \frac{3j}{8}\left(-e^{-j\pi t} + e^{j\pi t}\right) \\
&= -\frac{2}{\pi^2}\left(2\cos(\pi t)\right) - \frac{3j}{8}\left(2j\sin(\pi t)\right) \\
\sum_{|k|=1} \left(\left(-\frac{2}{\pi^2} - \frac{3jk}{8}\right) e^{j\pi kt}\right) &= -\frac{4}{\pi^2}\cos(\pi t) + \frac{3}{4}\sin(\pi t)
\end{aligned}$$

3. For $|k| = 3$:

$$\begin{aligned}
\sum_{|k|=3} \left(\left(-\frac{2}{9\pi^2} + \frac{jk}{24}\right) e^{j\pi kt}\right) &= \sum_{|k|=3} \left(-\frac{2}{9\pi^2} e^{j\pi kt}\right) + \sum_{|k|=3} \left(\frac{jk}{24} e^{j\pi kt}\right) \\
&= -\frac{2}{9\pi^2}\left(e^{-j3\pi t} + e^{j3\pi t}\right) + \frac{j}{24}\left(-e^{-j3\pi t} + e^{j3\pi t}\right) \\
&= -\frac{2}{9\pi^2}\left(2\cos(3\pi t)\right) + \frac{j}{24}\left(2j\sin(3\pi t)\right) \\
\sum_{|k|=3} \left(\left(-\frac{2}{9\pi^2} + \frac{jk}{24}\right) e^{j\pi kt}\right) &= -\frac{4}{9\pi^2}\cos(3\pi t) - \frac{1}{12}\sin(3\pi t)
\end{aligned}$$

Consider $|k| = 1$ and $|k| = 3$ together.

$$= -\frac{4}{\pi^2}\cos(\pi t) + \frac{3}{4}\sin(\pi t) - \frac{4}{9\pi^2}\cos(3\pi t) - \frac{1}{12}\sin(3\pi t)$$
$$= -\frac{4}{\pi^2}\left(\cos(\pi t) + \frac{1}{9}\cos(3\pi t)\right) + \frac{3}{4}\left(\sin(\pi t) - \frac{1}{9}\sin(3\pi t)\right)$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$\boxed{x(t) = \frac{1}{3} - \frac{4}{\pi^2}\left(\cos(\pi t) + \frac{1}{9}\cos(3\pi t)\right) + \frac{3}{4}\left(\sin(\pi t) - \frac{1}{9}\sin(3\pi t)\right) + \sum_{\substack{k=1 \\ |k|\neq 0,1,3}}^{\infty}\left(\frac{4(-1)^k}{\pi^2 k^2}\cos(\pi k t)\right)}$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows: