Homework Signal 4

Week 4

6733172621 Patthadon Phengpinij

Collaborators. ChatGPT (for LaTeX styling and grammar checking)

---

1 Continuous-Time Fourier Transform (CTFT)

Problem 1. Find the Fourier transform of the following signals in terms of $X(j\omega)$, the Fourier transform of $x(t)$ ($\mathscr{F}\{x(t)\} = X(j\omega)$)

1.1 $x(-t)$

Solution. Using the Time-scaling property of the Fourier transform, we have:

$$\textbf{Time-scaling: } \mathscr{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Substituting $a = -1$ into the time-scaling property, we get:

$$\mathscr{F}\{x(-t)\} = \frac{1}{|-1|} X\left(\frac{j\omega}{-1}\right)$$

Therefore, we can express the Fourier transform of $x(-t)$ as:

$$\boxed{\mathscr{F}\{x(-t)\} = X(-j\omega)}$$

1.2 $x_e(t) = \frac{x(t)+x(-t)}{2}$

Solution. Using the Time-scaling and Linearity properties of the Fourier transform, we have:

$$\textbf{Time-scaling: } \mathscr{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$\textbf{Linearity: } \mathscr{F}\{ax_1(t) + bx_2(t)\} = aX_1(j\omega) + bX_2(j\omega)$$

Considering the signal $x_e(t) = \frac{x(t)+x(-t)}{2}$, we can find its Fourier transform as follows:

$$\begin{aligned}
\mathscr{F}\{x_e(t)\} &= \mathscr{F}\left\{\frac{x(t)+x(-t)}{2}\right\} \\
&= \frac{1}{2}\left(\mathscr{F}\{x(t)\} + \mathscr{F}\{x(-t)\}\right) \\
\mathscr{F}\{x_e(t)\} &= \frac{1}{2}\left(X(j\omega) + \frac{1}{|-1|} X\left(\frac{j\omega}{-1}\right)\right)
\end{aligned}$$

Therefore, we can express the Fourier transform of $x_e(t)$ as:

$$\boxed{\mathscr{F}\{x_e(t)\} = \frac{X(j\omega) + X(-j\omega)}{2}}$$

1.3 $x_o(t) = \frac{x(t)-x(-t)}{2}$

Solution. Using the Time-scaling and Linearity properties of the Fourier transform, we have:

$$\textbf{Time-scaling: } \mathscr{F}\{x(at)\} = \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

$$\textbf{Linearity: } \mathscr{F}\{ax_1(t)+bx_2(t)\} = aX_1(j\omega)+bX_2(j\omega)$$

Considering the signal $x_o(t) = \frac{x(t)-x(-t)}{2}$, we can find its Fourier transform as follows:

$$\begin{aligned}
\mathscr{F}\{x_o(t)\} &= \mathscr{F}\left\{\frac{x(t)-x(-t)}{2}\right\} \\
&= \frac{1}{2}\left(\mathscr{F}\{x(t)\} - \mathscr{F}\{x(-t)\}\right) \\
\mathscr{F}\{x_o(t)\} &= \frac{1}{2}\left(X(j\omega) - \frac{1}{|-1|}X\left(\frac{j\omega}{-1}\right)\right)
\end{aligned}$$

Therefore, we can express the Fourier transform of $x_o(t)$ as:

$$\boxed{\mathscr{F}\{x_o(t)\} = \frac{X(j\omega)-X(-j\omega)}{2}}$$

Problem 2. Let $\mathscr{F}\{x(t)\} = X(j\omega) = \text{rect}\left((\omega-1)/2\right)$. Find Fourier tranform of

TO SUBMIT

2.1 $x(-2t+4)$

Solution. From the Time-scaling and Time-shifting properties of the Fourier transform, we have:

$$\textbf{Time-scaling: } \mathscr{F}\{x(at)\} = \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

$$\textbf{Time-shifting: } \mathscr{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega)$$

Combining these two properties, we can find the Fourier transform of $x(at-b)$.

$$\mathscr{F}\{x(at-b)\} = \mathscr{F}\left\{x\left(a\left(t-\frac{b}{a}\right)\right)\right\} = \frac{1}{|a|}e^{-j\omega\frac{b}{a}}X\left(\frac{j\omega}{a}\right)$$

To find the Fourier transform of $x(-2t+4)$, we have $a=-2$ and $b=-4$. Applying the combined properties and substituting $\mathscr{F}\{x(t)\}$, we get:

$$\begin{aligned}
\mathscr{F}\{x(-2t+4)\} &= \frac{1}{|a|}e^{-j\omega\frac{b}{a}}X\left(\frac{j\omega}{a}\right) \\
&= \frac{1}{|-2|}e^{-j\omega\frac{-4}{-2}}X\left(\frac{j\omega}{-2}\right) \\
&= \frac{1}{2}e^{-2j\omega}\text{rect}\left(\frac{\frac{\omega}{-2}-1}{2}\right) \\
\mathscr{F}\{x(-2t+4)\} &= \frac{1}{2}e^{-2j\omega}\text{rect}\left(\frac{-\omega-2}{4}\right)
\end{aligned}$$

Because, rect is an even function, we can express the Fourier transform of $x(-2t+4)$ as:

$$\boxed{\mathscr{F}\{x(-2t+4)\} = \frac{1}{2}e^{-2j\omega}\text{rect}\left(\frac{\omega+2}{4}\right)}$$

TO SUBMIT

2.2 $(t-1)x(t-1)$

Solution. Using the Time-shifting property of the Fourier transform, we have:

$$\textbf{Time-shifting: } \mathscr{F}\left\{x(t-t_0)\right\} = e^{-j\omega t_0}X(j\omega)$$

First, define a new signal $y(t) = tx(t)$. Then, we can express $(t-1)x(t-1)$ as:

$$(t-1)x(t-1) = y(t-1)$$

Now, applying the Time-shifting property to $y(t-1)$, we get:

$$\begin{aligned}
\mathscr{F}\left\{y(t-1)\right\} &= e^{-j\omega\cdot 1}Y(j\omega) \\
&= e^{-j\omega}\mathscr{F}\left\{y(t)\right\} \\
\mathscr{F}\left\{y(t-1)\right\} &= e^{-j\omega}\mathscr{F}\left\{tx(t)\right\}
\end{aligned}$$

Next, consider the differentiation of $\mathscr{F}\left\{x(t)\right\}$:

$$\begin{aligned}
\frac{d}{d\omega}\mathscr{F}\left\{x(t)\right\} &= \frac{d}{d\omega}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} x(t)\frac{d}{d\omega}\left(e^{-j\omega t}\right)\,dt \\
&= \int_{-\infty}^{\infty} x(t)\left(-jte^{-j\omega t}\right)\,dt \\
&= -j\int_{-\infty}^{\infty} tx(t)e^{-j\omega t}\,dt \\
\frac{d}{d\omega}\mathscr{F}\left\{x(t)\right\} &= -j\mathscr{F}\left\{tx(t)\right\} \\
\mathscr{F}\left\{tx(t)\right\} &= j\frac{d}{d\omega}X(j\omega)
\end{aligned}$$

Substituting $\mathscr{F}\left\{x(t)\right\}$ into the equation, we get:

$$\begin{aligned}
\mathscr{F}\left\{tx(t)\right\} &= j\frac{d}{d\omega}X(j\omega) \\
&= j\frac{d}{d\omega}\left(\text{rect}\left(\frac{\omega-1}{2}\right)\right) \\
&= j\frac{d}{d\omega}\left(u\left(\omega\right)-u\left(\omega-2\right)\right) \\
\mathscr{F}\left\{tx(t)\right\} &= j\left(\delta(\omega)-\delta(\omega-2)\right)
\end{aligned}$$

Therefore, we can express the Fourier transform of $(t-1)x(t-1)$ as:

$$\boxed{\mathscr{F}\left\{(t-1)x(t-1)\right\} = je^{-j\omega}\left(\delta(\omega)-\delta(\omega-2)\right)}$$

TO SUBMIT

2.3 $t\frac{dx(t)}{dt}$

Solution. Using the Differentiation in time property of the Fourier transform, we have:

$$\textbf{Differentiation in time: } \mathscr{F}\left\{\frac{dx(t)}{dt}\right\} = j\omega X(j\omega)$$

And, using the Differentiation in frequency property (proved in the previous problem) of the Fourier transform, we have:

$$\textbf{Differentiation in frequency: } \mathscr{F}\left\{tx(t)\right\} = j\frac{d}{d\omega}X(j\omega)$$

First, define a new signal $y(t) = \frac{dx(t)}{dt}$. Then, we can express $t\frac{dx(t)}{dt}$ as:

$$t\frac{dx(t)}{dt} = ty(t)$$

Now, applying the Differentiation in frequency property to $ty(t)$, we get:

$$\mathscr{F}\left\{ty(t)\right\} = j\frac{d}{d\omega}Y(j\omega) = j\frac{d}{d\omega}\mathscr{F}\left\{y(t)\right\} = j\frac{d}{d\omega}\mathscr{F}\left\{\frac{dx(t)}{dt}\right\}$$

Next, substituting the Differentiation in time property into the equation, we get:

$$\begin{aligned}
\frac{d}{d\omega}\mathscr{F}\left\{\frac{dx(t)}{dt}\right\} &= \frac{d}{d\omega}\left(j\omega X(j\omega)\right) \\
&= \frac{d}{d\omega}\left(j\omega\cdot\text{rect}\left(\frac{\omega-1}{2}\right)\right) \\
&= \frac{d}{d\omega}\left(j\omega\left(u(\omega)-u(\omega-2)\right)\right) \\
&= j\omega\frac{d}{d\omega}\left(u(\omega)-u(\omega-2)\right) + \left(u(\omega)-u(\omega-2)\right)\frac{d}{d\omega}(j\omega) \\
&= j\omega\left(\delta(\omega)-\delta(\omega-2)\right) + \text{rect}\left(\frac{\omega-1}{2}\right)\cdot(j) \\
&= j\omega\delta(\omega) - j\omega\delta(\omega-2) + j\text{rect}\left(\frac{\omega-1}{2}\right) \\
&= 0 - j(2)\delta(\omega-2) + j\text{rect}\left(\frac{\omega-1}{2}\right) \\
\frac{d}{d\omega}\mathscr{F}\left\{\frac{dx(t)}{dt}\right\} &= -j(2)\delta(\omega-2) + j\text{rect}\left(\frac{\omega-1}{2}\right)
\end{aligned}$$

Thus, substituting back, we have:

$$\begin{aligned}
\mathscr{F}\left\{ty(t)\right\} &= j\frac{d}{d\omega}\mathscr{F}\left\{\frac{dx(t)}{dt}\right\} \\
&= j\cdot\left[-j(2)\delta(\omega-2) + j\text{rect}\left(\frac{\omega-1}{2}\right)\right] \\
\mathscr{F}\left\{ty(t)\right\} &= 2\delta(\omega-2) - \text{rect}\left(\frac{\omega-1}{2}\right)
\end{aligned}$$

Therefore, we can express the Fourier transform of $t\frac{dx(t)}{dt}$ as:

$$\boxed{\mathscr{F}\left\{t\frac{dx(t)}{dt}\right\} = 2\delta(\omega-2) - \text{rect}\left(\frac{\omega-1}{2}\right)}$$

TO SUBMIT

2.4 $x(2t-1)e^{-j2t}$

Solution. Using the Time-scaling, Time-shifting, and Frequency-shifting properties of the Fourier transform, we have:

$$\textbf{Time-scaling + Time-shifting:}\ \mathscr{F}\{x(at-b)\} = \frac{1}{|a|}e^{-j\omega\frac{b}{a}}X\left(\frac{j\omega}{a}\right)$$

$$\textbf{Frequency-shifting:}\ \mathscr{F}\left\{x(t)e^{j\omega_0 t}\right\} = X(j(\omega-\omega_0))$$

Define a new signal $y(t) = x(2t-1)$. Then, we can express $x(2t-1)e^{-j2t}$ as:

$$\begin{aligned}
Y(j\omega) &= \mathscr{F}\{y(t)\} \\
&= \mathscr{F}\{x(2t-1)\} \\
&= \frac{1}{|2|}e^{-j\omega\frac{1}{2}}X\left(\frac{j\omega}{2}\right) \\
Y(j\omega) &= \frac{1}{2}e^{-j\frac{\omega}{2}}X\left(\frac{j\omega}{2}\right)
\end{aligned}$$

Now, applying the Frequency-shifting property to $y(t)e^{-j2t}$, we get:

$$\begin{aligned}
\mathscr{F}\left\{y(t)e^{-j2t}\right\} &= \mathscr{F}\left\{y(t)e^{j(-2)t}\right\} \\
&= Y(j(\omega-(-2))) \\
&= Y(j(\omega+2)) \\
\mathscr{F}\left\{y(t)e^{-j2t}\right\} &= \frac{1}{2}e^{-j\frac{\omega+2}{2}}X\left(\frac{j(\omega+2)}{2}\right)
\end{aligned}$$

Then, substituting $\mathscr{F}\{x(t)\}$ into the equation, we get:

$$\begin{aligned}
\mathscr{F}\left\{y(t)e^{-j2t}\right\} &= \frac{1}{2}e^{-j\frac{\omega+2}{2}}X\left(\frac{j(\omega+2)}{2}\right) \\
&= \frac{1}{2}e^{-j\frac{\omega+2}{2}}\mathrm{rect}\left(\frac{\frac{\omega+2}{2}-1}{2}\right) \\
&= \frac{1}{2}e^{-j\frac{\omega+2}{2}}\mathrm{rect}\left(\frac{\omega+2-2}{4}\right) \\
\mathscr{F}\left\{y(t)e^{-j2t}\right\} &= \frac{1}{2}e^{-j\frac{\omega+2}{2}}\mathrm{rect}\left(\frac{\omega}{4}\right)
\end{aligned}$$

Therefore, we can express the Fourier transform of $x(2t-1)e^{-j2t}$ as:

$$\boxed{\mathscr{F}\left\{x(2t-1)e^{-j2t}\right\} = \frac{1}{2}e^{-j\frac{\omega+2}{2}}\mathrm{rect}\left(\frac{\omega}{4}\right)}$$

TO SUBMIT

2.5 $x(t) * x(t-1)$

Solution. Using the Convolution property and Time-shifting property of the Fourier transform, we have:

$$\textbf{Convolution: } \mathscr{F}\{x_1(t) * x_2(t)\} = X_1(j\omega) \cdot X_2(j\omega)$$

$$\textbf{Time-shifting: } \mathscr{F}\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega)$$

First, define a new signal $y(t) = x(t-1)$. Then, we can express $x(t) * x(t-1)$ as:

$$x(t) * x(t-1) = x(t) * y(t)$$

Next, substituting the Time-shifting property into the equation, we get:

$$\begin{aligned} Y(j\omega) &= \mathscr{F}\{y(t)\} \\ &= \mathscr{F}\{x(t-1)\} \\ &= e^{-j\omega(1)} X(j\omega) \\ Y(j\omega) &= e^{-j\omega} X(j\omega) \end{aligned}$$

Now, applying the Convolution property to $x(t) * y(t)$, we get:

$$\begin{aligned} \mathscr{F}\{x(t) * y(t)\} &= \mathscr{F}\{x(t)\} \cdot \mathscr{F}\{y(t)\} \\ &= X(j\omega) \cdot \mathscr{F}\{x(t-1)\} \\ \mathscr{F}\{x(t) * y(t)\} &= X(j\omega) \cdot e^{-j\omega(1)} X(j\omega) \end{aligned}$$

Lastly, substituting $\mathscr{F}\{x(t)\} = \text{rect}\left((\omega-1)/2\right)$ back into the equation, we get:
Therefore, we can express the Fourier transform of $x(t) * x(t-1)$ as:

$$\boxed{\mathscr{F}\{x(t) * x(t-1)\} = e^{-j\omega} \text{rect}^2\left(\frac{\omega-1}{2}\right)}$$

Problem 3.

3.1 Proof that $\mathscr{F}\left\{e^{-|t|}\right\} = \mathscr{F}\{exp(-|t|)\} = \frac{2}{\omega^2+1}$

Solution. Using the definition of the Continuous-Time Fourier Transform (CTFT), we have:

$$\mathscr{F}\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Substituting $x(t) = e^{-|t|}$ into the CTFT definition, we get:

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{0} e^{t}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-t}e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{0} e^{(1-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(1+j\omega)t}\,dt \\
&= \left[\frac{e^{(1-j\omega)t}}{1-j\omega}\right]_{-\infty}^{0} + \left[\frac{-e^{-(1+j\omega)t}}{1+j\omega}\right]_{0}^{\infty} \\
&= \frac{1}{1-j\omega} + \frac{1}{1+j\omega} \\
&= \frac{(1+j\omega)+(1-j\omega)}{(1-j\omega)(1+j\omega)} \\
X(j\omega) &= \frac{2}{1+\omega^2}
\end{aligned}$$

Therefore, we have proven that:

$$\boxed{\mathscr{F}\left\{e^{-|t|}\right\} = \frac{2}{\omega^2+1}} \quad \square.$$

3.2 Using the outcome obtained in Problem 3.1, Find the Fourier Transform of the given equation.

3.2.1 $\frac{d}{dt}(e^{-|t|})$

Solution. Using the Differentiation in time property of the Fourier transform, we have:

$$\textbf{Differentiation in time: } \mathscr{F}\left\{\frac{dx(t)}{dt}\right\} = j\omega X(j\omega)$$

Define a new signal $y(t) = e^{-|t|}$, applying the Differentiation in time property to $\frac{dy(t)}{dt}$, we get:

$$\begin{aligned}
\mathscr{F}\left\{\frac{dy(t)}{dt}\right\} &= j\omega Y(j\omega) \\
&= j\omega\mathscr{F}\{y(t)\} \\
\mathscr{F}\left\{\frac{dy(t)}{dt}\right\} &= j\omega\mathscr{F}\left\{e^{-|t|}\right\}
\end{aligned}$$

Substituting the result from Problem 3.1 into the equation, we get:

$$\boxed{\mathscr{F}\left\{\frac{d}{dt}(e^{-|t|})\right\} = \frac{2j\omega}{\omega^2+1}}$$

3.2.2 $\exp(3jt - |2t+2|)$

Solution. First, define a new signal $x(t) = e^{-|t|}$ and $y(t) = x(2t+2) = e^{-|2t+2|}$. Then, we can express $\exp(3jt - |2t+2|)$ as:

$$\exp(3jt - |2t+2|) = y(t)e^{j3t}$$

Using the Time-scaling and Time-shifting property, we have:

$$\textbf{Time-scaling + Time-shifting: } \mathscr{F}\left\{x(at-b)\right\} = \frac{1}{|a|}e^{-j\omega\frac{b}{a}}X\left(\frac{j\omega}{a}\right)$$

Substituting this property into the equation, we get:

$$\begin{aligned}
Y(j\omega) &= \mathscr{F}\left\{y(t)\right\} \\
&= \mathscr{F}\left\{x(2t+2)\right\} \\
&= \frac{1}{|2|}e^{-j\omega\frac{2}{2}}X\left(\frac{j\omega}{2}\right) \\
Y(j\omega) &= \frac{1}{2}e^{-j\omega}X\left(\frac{j\omega}{2}\right)
\end{aligned}$$

Now, applying the Frequency-shifting property

$$\textbf{Frequency-shifting: } \mathscr{F}\left\{x(t)e^{j\omega_0 t}\right\} = X(j(\omega-\omega_0))$$

to $y(t)e^{j3t}$, we get:

$$\mathscr{F}\left\{y(t)e^{j3t}\right\} = Y(j(\omega-3))$$

Then, substituting back and use the result from Problem 3.1, we have:

$$\begin{aligned}
\mathscr{F}\left\{y(t)e^{j3t}\right\} &= Y(j(\omega-3)) \\
&= \frac{1}{2}e^{-j(\omega-3)}X\left(\frac{j(\omega-3)}{2}\right) \\
&= \frac{1}{2}e^{-j(\omega-3)} \cdot \frac{2}{\left(\frac{\omega-3}{2}\right)^2+1} \\
&= \frac{e^{-j(\omega-3)}}{\frac{(\omega-3)^2}{4}+1} \\
\mathscr{F}\left\{y(t)e^{j3t}\right\} &= \frac{4e^{-j(\omega-3)}}{(\omega-3)^2+4}
\end{aligned}$$

Therefore, we can express the Fourier transform of $\exp(3jt - |2t+2|)$ as:

$$\boxed{\mathscr{F}\left\{\exp(3jt-|2t+2|)\right\} = \frac{4e^{-j(\omega-3)}}{(\omega-3)^2+4}}$$

3.2.3 $\frac{1}{2\pi t^2+1}$

Solution. Consider the CFTF of $e^{-|t|}$ obtained in Problem 3.1:

$$\mathscr{F}\left\{e^{-|t|}\right\} = \frac{2}{\omega^2+1}$$

Using the Duality property of the Fourier transform, we have:

$$\textbf{Duality: } \mathscr{F}\left\{X(t)\right\} = 2\pi x(-\omega)$$

Define a new signal $y(t) = \frac{2}{t^2+1}$. Then, applying the Duality property to $y(t)$, we get:

$$\begin{aligned}
Y(j\omega) &= \mathscr{F}\left\{y(t)\right\} \\
&= 2\pi x(-\omega) \\
&= 2\pi \mathscr{F}^{-1}\left\{X(t)\right\}\Big|_{t=-\omega} \\
&= 2\pi e^{-|-\omega|} \\
Y(j\omega) &= 2\pi e^{-|\omega|}
\end{aligned}$$

Using the Time-scaling property of the Fourier transform, we have:

$$\textbf{Time-scaling: } \mathscr{F}\left\{x(at)\right\} = \frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

We can rewrite the given signal as:

$$\frac{1}{2\pi t^2+1} = \frac{1}{2}\frac{2}{\left(\sqrt{2\pi}t\right)^2+1} = \frac{1}{2}y\left(\sqrt{2\pi}t\right)$$

Find $y\left(\sqrt{2\pi}t\right)$ by substituting $a=\sqrt{2\pi}$ into the Time-scaling property, we get:

$$\begin{aligned}
\mathscr{F}\left\{y\left(\sqrt{2\pi}t\right)\right\} &= \frac{1}{|\sqrt{2\pi}|}Y\left(\frac{j\omega}{\sqrt{2\pi}}\right) \\
&= \frac{1}{\sqrt{2\pi}}\cdot 2\pi e^{-\left|\frac{\omega}{\sqrt{2\pi}}\right|} \\
\mathscr{F}\left\{y\left(\sqrt{2\pi}t\right)\right\} &= \sqrt{2\pi}e^{-\frac{|\omega|}{\sqrt{2\pi}}}
\end{aligned}$$

Therefore, we can express the Fourier transform of $\frac{1}{2\pi t^2+1}$ as:

$$\boxed{\mathscr{F}\left\{\frac{1}{2\pi t^2+1}\right\} = \frac{\sqrt{2\pi}}{2}e^{-\frac{|\omega|}{\sqrt{2\pi}}}}$$