Homework Signal 2

Week 2

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Collaborators. ChatGPT (for LaTeX styling and grammar checking)

1 Convolution

Problem 1. Evaluate the convolution of the following signals

1.1 $\text{rect}\left(\frac{t-a}{a}\right) * \delta(t-b)$

Solution. From the sifting property of the delta function, we have:

$$f(t) * \delta(t-b) = f(t-b)$$

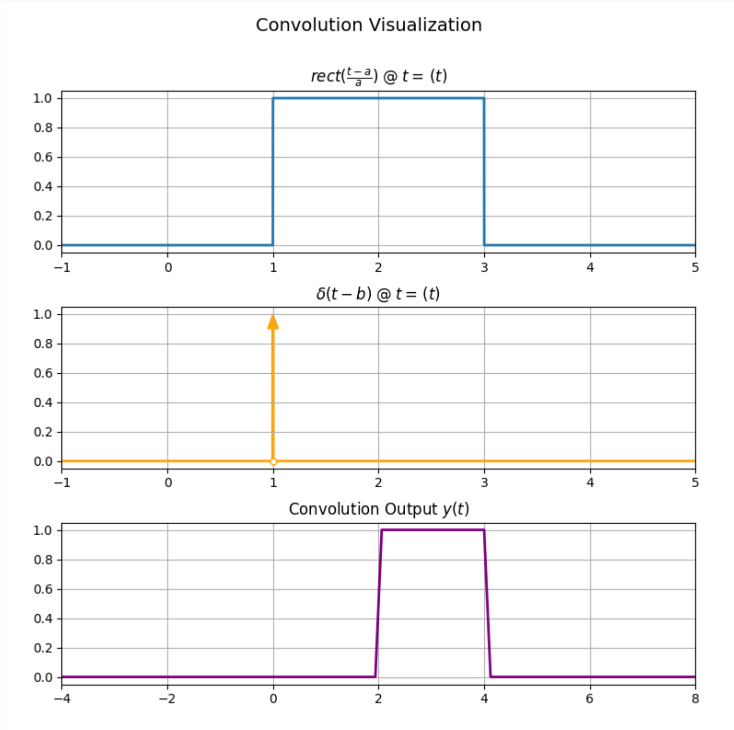
Applying this property to our problem, we get:

$$\text{rect}\left(\frac{t-a}{a}\right) * \delta(t-b) = \text{rect}\left(\frac{(t-b)-a}{a}\right) = \text{rect}\left(\frac{t-(a+b)}{a}\right)$$

Thus, the result of the convolution is:

$$\boxed{\text{rect}\left(\frac{t-(a+b)}{a}\right)}$$

Using Python to verify this result, we can implement the convolution and plot the results. The plot of the signal is shown below:



1.2 $\text{rect}\left(\frac{t}{a}\right) * \text{rect}\left(\frac{t}{a}\right)$

Solution. To evaluate the convolution of two rectangular functions, we start with the definition of the rectangular function:

$$\text{rect}\left(\frac{t}{a}\right) = \begin{cases} 1 & \text{if } |t| \leq \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

The convolution of two functions $f(t)$ and $g(t)$ is defined as:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$$

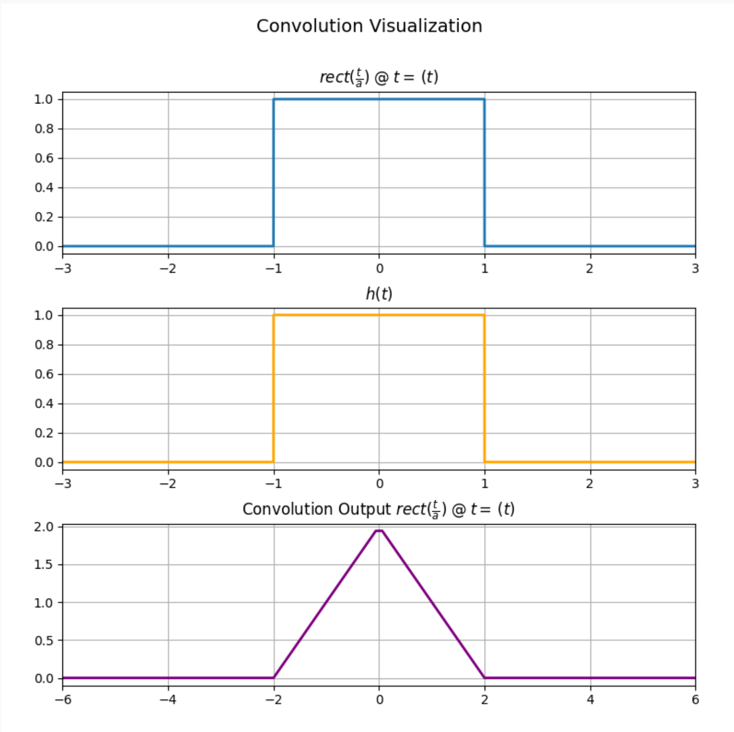
Applying this to our rectangular functions, we have:

$$\begin{aligned}
(\text{rect}\left(\frac{t}{a}\right) * \text{rect}\left(\frac{t}{a}\right))(t) &= \int_{-\infty}^{\infty} \text{rect}\left(\frac{\tau}{a}\right) \text{rect}\left(\frac{t-\tau}{a}\right)\,d\tau \\
&= \int_{-\frac{a}{2}}^{\frac{a}{2}} \text{rect}\left(\frac{t-\tau}{a}\right)\,d\tau \\
&= \int_{\max(-\frac{a}{2},t-\frac{a}{2})}^{\min(\frac{a}{2},t+\frac{a}{2})} 1\,d\tau \\
(\text{rect}\left(\frac{t}{a}\right) * \text{rect}\left(\frac{t}{a}\right))(t) &= \min\left(\frac{a}{2}, t+\frac{a}{2}\right) - \max\left(-\frac{a}{2}, t-\frac{a}{2}\right)
\end{aligned}$$

Evaluating the limits, we find that the result is a triangular function:

$$\boxed{\text{rect}\left(\frac{t}{a}\right) * \text{rect}\left(\frac{t}{a}\right) = \begin{cases} 0 & |t| > a \\ t+a & -a \leq t < 0 \\ a-t & 0 \leq t \leq a \end{cases}}$$

Using Python to verify this result, we can implement the convolution and plot the results. The plot of the signal is shown below:



1.3 $t[u(t) - u(t-1)] * u(t)$

Solution. First, we define the functions involved in the convolution:

$$x(t) = t[u(t) - u(t-1)] = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}$$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

The convolution $y(t) = x(t) * u(t)$ is given by:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)\, d\tau$$

Evaluating the convolution integral, we find:

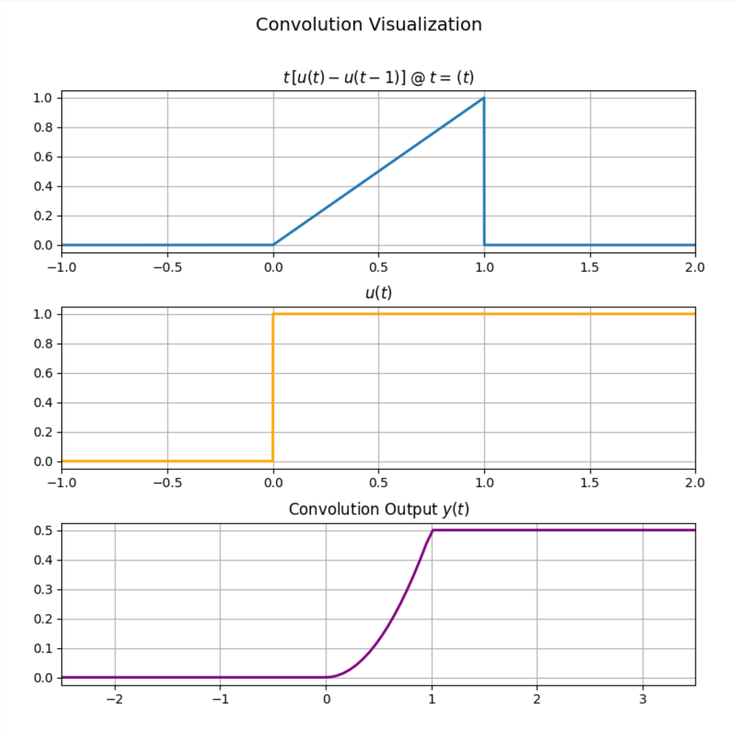
$$y(t) = \int_{0}^{1} \tau \cdot u(t-\tau)\, d\tau$$

$$y(t) = \int_{0}^{\min(t,1)} \tau\, d\tau$$

Thus,

$$\boxed{y(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t < 1 \\ \frac{1}{2} & t \geq 1 \end{cases}}$$

Using Python to verify this result, we can implement the convolution and plot the results. The plot of the signal is shown below:



Problem 2. Determine the convolution $y(t) = h(t) * x(t)$ using Graphical Interpretation of the pairs of the signals shown

Solution. The convolution $y(t) = h(t) * x(t)$ can be determined graphically by following these steps:

1. Flip one of the signals, typically $h(t)$, to get $h(-\tau)$.
2. Shift the flipped signal by t to get $h(t-\tau)$.
3. For each value of t, calculate the area of overlap between $x(\tau)$ and $h(t-\tau)$.
4. The value of the convolution $y(t)$ at each t is the area of overlap calculated in the previous step.

Using Python to visualize and compute the convolution graphically, we can create an animation that demonstrates the convolution process step-by-step.

The resulting convolution $y(t)$ is shown in the gif files in my GitHub repository for this homework.

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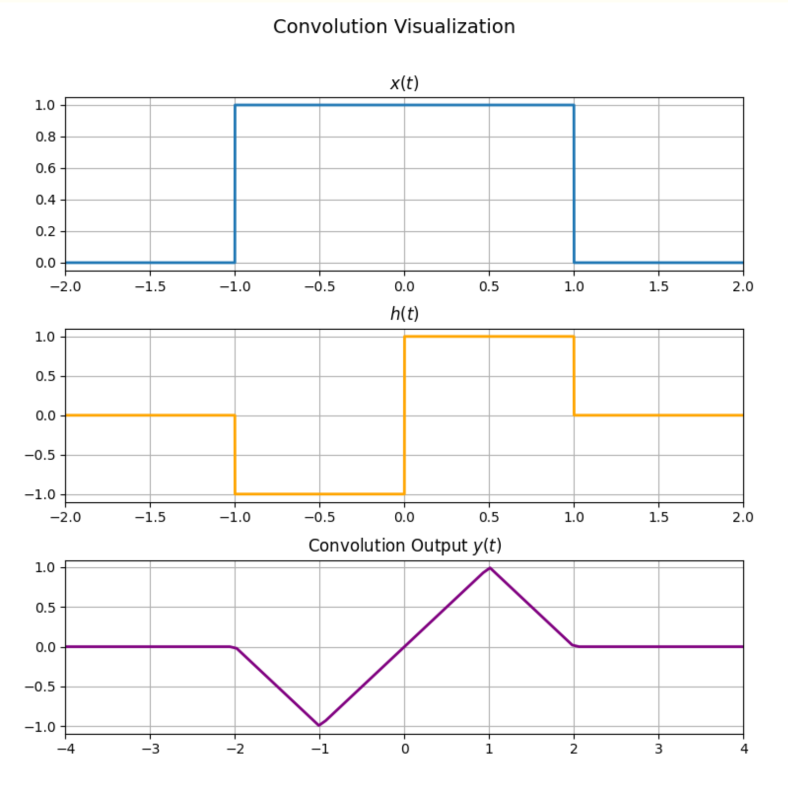
2.1 **Solution.**

Using Python to visualize and compute the convolution graphically, we can create an animation that demonstrates the convolution process step-by-step as shown in the gif files in Problem 2.1 Animation.

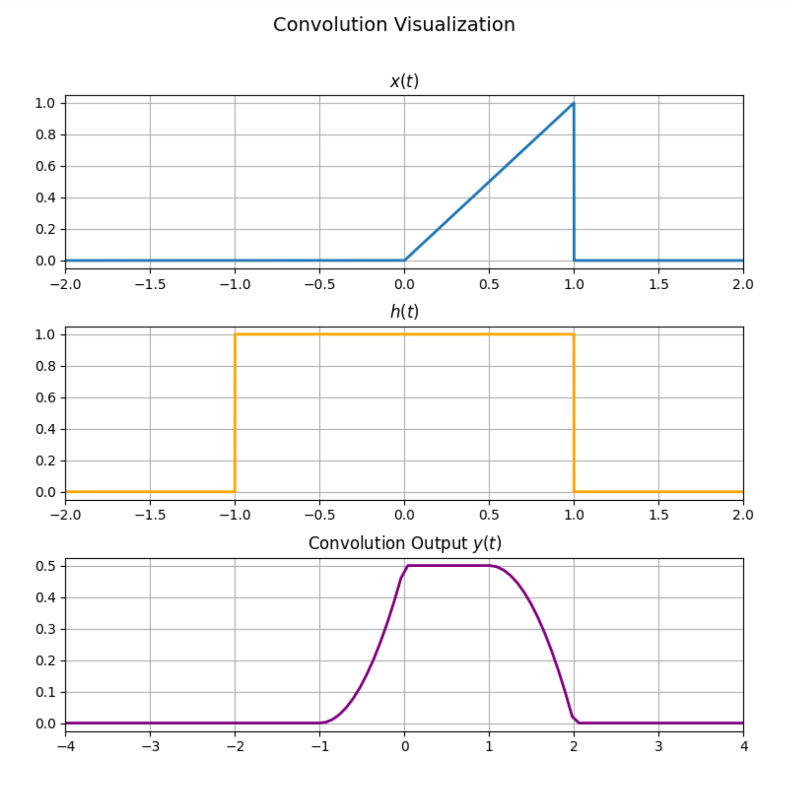
The plot of the signal is shown below:



2.2

Solution. Using Python to visualize and compute the convolution graphically, we can create an animation that demonstrates the convolution process step-by-step as shown in the gif files in Problem 2.2 Animation.

The plot of the signal is shown below:



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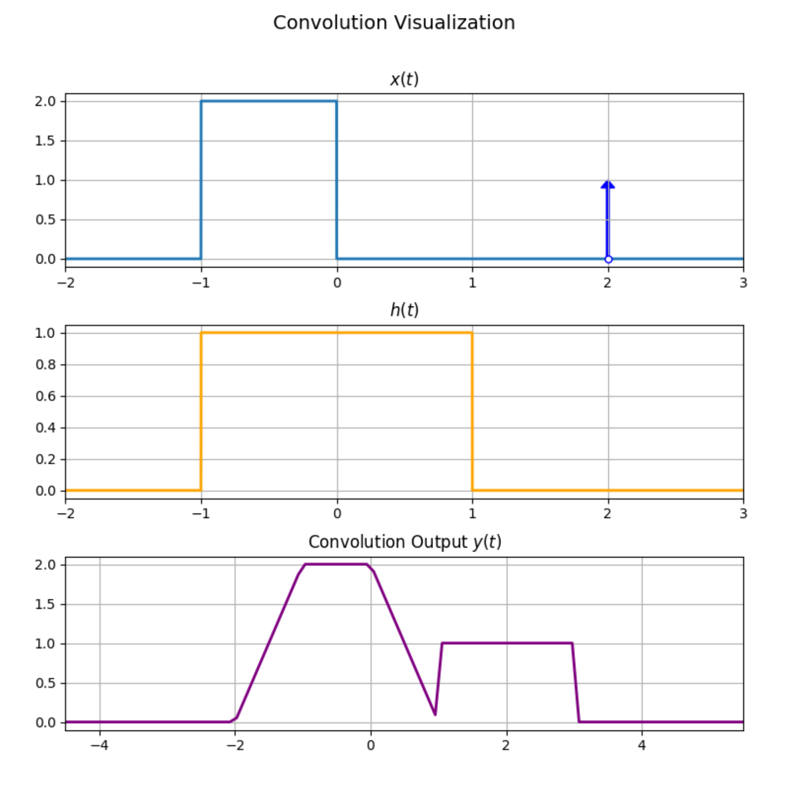
2.3 **Solution.**

Using Python to visualize and compute the convolution graphically, we can create an animation that demonstrates the convolution process step-by-step as shown in the gif files in Problem 2.3 Animation.

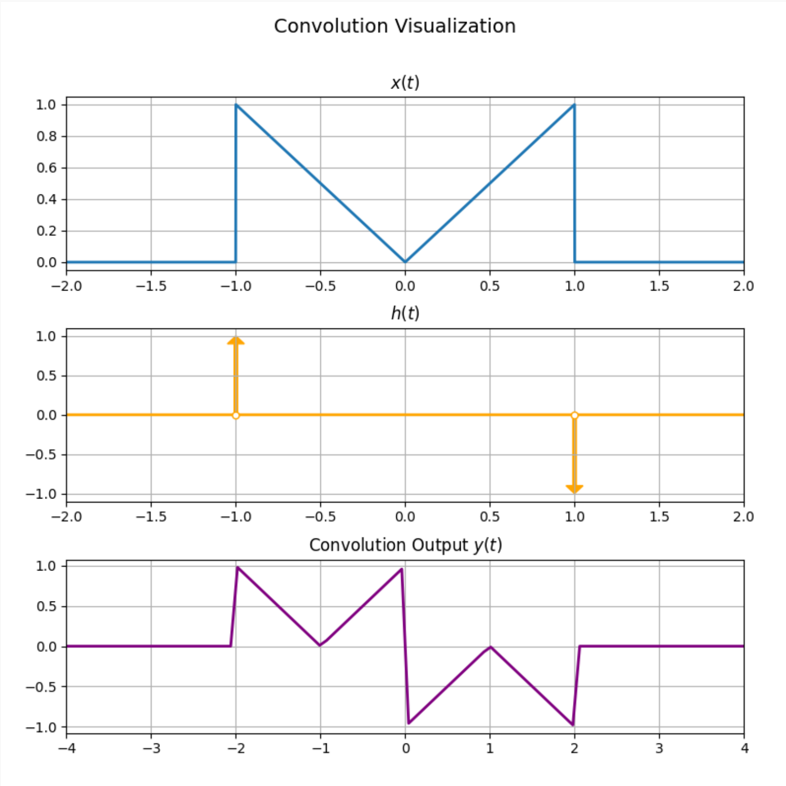
The plot of the signal is shown below:



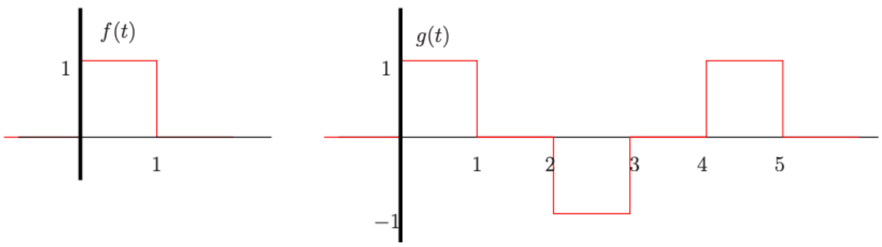
2.4

Solution. Using Python to visualize and compute the convolution graphically, we can create an animation that demonstrates the convolution process step-by-step as shown in the gif files in Problem 2.4 Animation.

The plot of the signal is shown below:



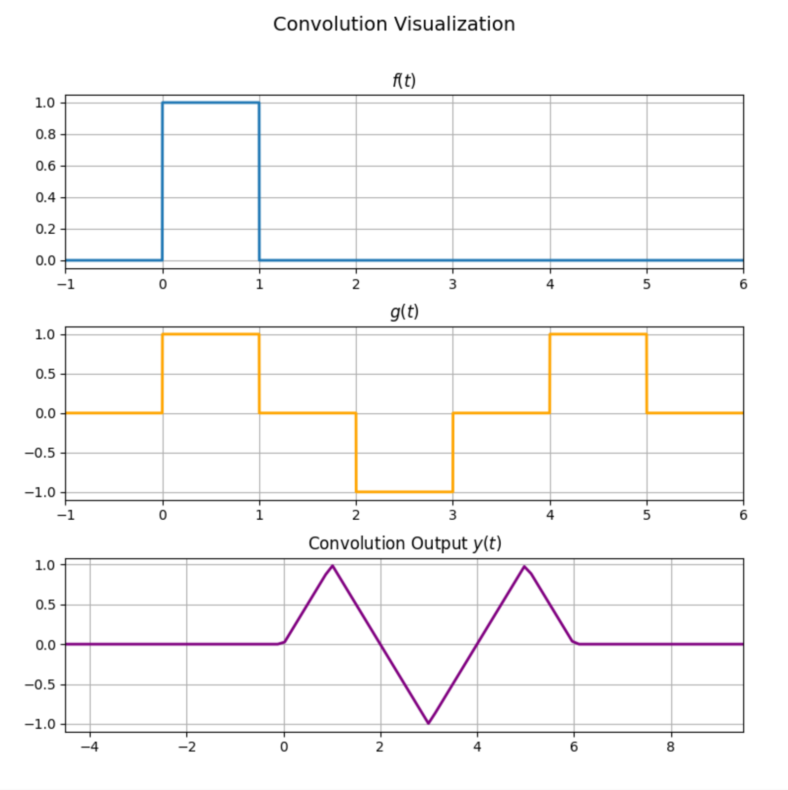
Problem 3. Let $f(t)$ and $g(t)$ be given as follows:



3.1 Sketch the function : $x(t) = f(t) * g(t)$

Solution. Using Python to visualize and compute the convolution graphically, we can create an animation that demonstrates the convolution process step-by-step as shown in the gif files in Problem 3.1 Animation.

The plot of the signal is shown below:



3.2 Show that if $a(t) = b(t) * c(t)$, then $(Mb(t)) * c(t) = Ma(t)$, for any real number M (hint: use the convolution integral formula)

Solution. Given that $a(t) = b(t) * c(t)$, we can express this using the convolution integral:

$$a(t) = \int_{-\infty}^{\infty} b(\tau)c(t - \tau)\, d\tau$$

Now, we want to show that $(Mb(t)) * c(t) = Ma(t)$. We start by writing the convolution of $Mb(t)$ with $c(t)$:

$$(Mb(t)) * c(t) = \int_{-\infty}^{\infty} Mb(\tau)c(t - \tau)\, d\tau$$

Factoring out the constant M from the integral, we have:

$$(Mb(t)) * c(t) = M \int_{-\infty}^{\infty} b(\tau)c(t - \tau)\, d\tau$$

$$(Mb(t)) * c(t) = Ma(t)$$

Thus, we have shown that:

$$\boxed{(Mb(t)) * c(t) = Ma(t)}$$

Problem 4. Find the convolution $y[n] = h[n] * x[n]$ of the following signals:

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4.1 $x[n] = \begin{cases} -1, -5 \leq n \leq -1 \\ 1, 0 \leq n \leq 4 \end{cases}$, $h[n] = 2u[n]$

Solution. To find the convolution $y[n] = h[n] * x[n]$, we use the discrete convolution formula:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Consider the value of $y[n]$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-5}^{-1} x[k]h[n-k] + \sum_{k=0}^{4} x[k]h[n-k] \\ &= \sum_{k=-5}^{-1} (-1) \cdot 2u[n-k] + \sum_{k=0}^{4} (1) \cdot 2u[n-k] \\ &= -2\left[\sum_{k=-5}^{-1} u[n-k] - \sum_{k=0}^{4} u[n-k]\right] \\ y[n] &= -2\left[\sum_{j=n+1}^{n+5} u[j] - \sum_{j=n-4}^{n} u[j]\right] \end{aligned}$$

Calculating the convolution for different ranges of n:

- For $-5 \leq n < 0$:

$$\begin{aligned} y[n] &= -2\left[\sum_{j=n+1}^{n+5} u[j] - \sum_{j=n-4}^{n} u[j]\right] \\ &= -2\left[n+6\right] \\ y[n] &= -2n - 12 \end{aligned}$$

- For $0 \leq n < 5$:

$$\begin{aligned} y[n] &= -2\left[\sum_{j=n+1}^{n+5} u[j] - \sum_{j=n-4}^{n} u[j]\right] \\ &= -2\left[5 - (n-3)\right] \\ y[n] &= 2n - 8 \end{aligned}$$

Thus, the final result of the convolution is:

$$\boxed{y[n] = \begin{cases} -2n - 12 & -5 \leq n < 0 \\ 2n - 8 & 0 \leq n < 5 \\ 0 & \text{otherwise} \end{cases}}$$

4.2 $x[n] = u[n]$, $h[n] = 1$; $0 \leq n \leq 9$

Solution. To find the convolution $y[n] = h[n] * x[n]$, we use the discrete convolution formula:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Consider the value of $y[n]$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=0}^{\infty} u[k] \cdot h[n-k] \\ &= \sum_{k=0}^{\infty} 1 \cdot h[n-k] \\ y[n] &= \sum_{j=-\infty}^{n} h[j] \end{aligned}$$

Calculating the convolution for different ranges of n:

- For $0 \leq n < 9$:

$$\begin{aligned} y[n] &= \sum_{j=-\infty}^{n} h[j] \\ &= \sum_{j=0}^{n} 1 \\ y[n] &= n+1 \end{aligned}$$

- For $n \geq 9$:

$$\begin{aligned} y[n] &= \sum_{j=-\infty}^{n} h[j] \\ &= \sum_{j=0}^{9} 1 \\ y[n] &= 10 \end{aligned}$$

Thus, the final result of the convolution is:

$$\boxed{y[n] = \begin{cases} n+1 & 0 \leq n < 9 \\ 10 & n \geq 9 \\ 0 & \text{otherwise} \end{cases}}$$

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4.3 $x[n] = \left(\frac{1}{2}\right)^n u[n]$, $h[n] = \delta[n] + \delta[n-1] + \left(\frac{1}{3}\right)^n u[n]$

Solution. To find the convolution $y[n] = h[n] * x[n]$, we use the discrete convolution formula:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Consider the value of $y[n]$:

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k u[k] \cdot \left(\delta[n-k] + \delta[n-k-1] + \left(\frac{1}{3}\right)^{n-k} u[n-k]\right) \\
y[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \delta[n-k] + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \delta[n-k-1] + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right)^{n-k} u[n-k]
\end{aligned}$$

Calculating the convolution for different ranges of n:

- For $n \geq 0$:

$$\begin{aligned}
y[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \delta[n-k] + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \delta[n-k-1] + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right)^{n-k} u[n-k] \\
&= \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right)^{n-k} \\
&= \left(\frac{1}{2}\right)^n + 2\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n \sum_{k=0}^{n} \left(\frac{3}{2}\right)^k \\
&= 3\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n \frac{1 - \left(\frac{3}{2}\right)^{n+1}}{1 - \frac{3}{2}} \\
&= 3\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n (-2)\left(1 - \left(\frac{3}{2}\right)^{n+1}\right) \\
&= 3\left(\frac{1}{2}\right)^n + (-2)\left(\frac{1}{3}\right)^n + 3\left(\frac{1}{2}\right)^n \\
y[n] &= 6\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n
\end{aligned}$$

Thus, the final result of the convolution is:

$$\boxed{y[n] = \begin{cases} 6\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n & n \geq 0 \\ 0 & \text{otherwise} \end{cases}}$$

4.4 $x[n] = \left(\frac{1}{3}\right)^n u[n]$, $h[n] = \delta[n] + \left(\frac{1}{2}\right)^n u[n]$

Solution. To find the convolution $y[n] = h[n] * x[n]$, we use the discrete convolution formula:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Consider the value of $y[n]$:

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k u[k] \cdot \left(\delta[n-k] + \left(\frac{1}{2}\right)^{n-k} u[n-k]\right) \\
y[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \delta[n-k] + \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \left(\frac{1}{2}\right)^{n-k} u[n-k]
\end{aligned}$$

Calculating the convolution for different ranges of n:

- For $n \geq 0$:

$$\begin{aligned}
y[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \delta[n-k] + \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k \left(\frac{1}{2}\right)^{n-k} u[n-k] \\
&= \left(\frac{1}{3}\right)^n + \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k \left(\frac{1}{2}\right)^{n-k} \\
&= \left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k \\
&= \left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n \cdot \frac{1 - \left(\frac{2}{3}\right)^{n+1}}{1 - \frac{2}{3}} \\
&= \left(\frac{1}{3}\right)^n + 3\left(\frac{1}{2}\right)^n \left[1 - \left(\frac{2}{3}\right)^{n+1}\right] \\
&= \left(\frac{1}{3}\right)^n + 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n \\
y[n] &= 3\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n
\end{aligned}$$

Thus, the final result of the convolution is:

$$\boxed{y[n] = \begin{cases} 3\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n & n \geq 0 \\ 0 & \text{otherwise} \end{cases}}$$