

Homework Signal 3

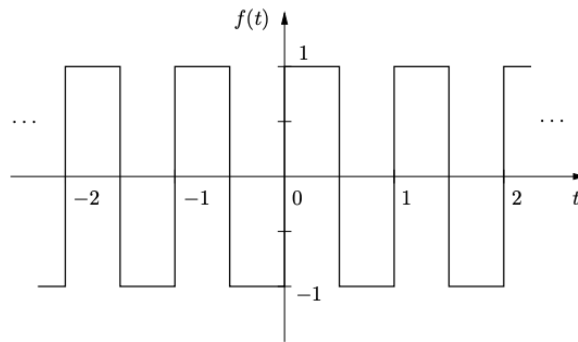
Week 3

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Collaborators. ChatGPT (for L^AT_EX styling and grammar checking)

1 Fourier Series

Problem 1. Find the Fourier series of the following periodic function:



Solution. From the graph, we can see that the function $x(t)$ can be defined piecewise as follows:

$$x(t) = \begin{cases} -1, & -\frac{1}{2} \leq t < 0 \\ 1, & 0 \leq t < \frac{1}{2} \end{cases}$$

with a period $T = 1$.

Calculating ω_0 :

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$$

Using the Fourier series formula:

$$x(t) = \sum_k a_k e^{jk\omega_0 t} = a_0 + \sum_{k \neq 0} a_k e^{jk\omega_0 t}$$

where the Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt$$

Calculating a_0 :

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-j(0)\omega_0 t} dt \\ &= \frac{1}{1} \int_{-0.5}^{0.5} x(t) dt \\ &= \int_{-0.5}^0 (-1) dt + \int_0^{0.5} (1) dt \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ a_0 &= 0 \end{aligned}$$

Calculating a_k for $k \neq 0$:

$$\begin{aligned}
 a_k &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt \\
 &= \frac{1}{1} \int_{-0.5}^{0.5} x(t) e^{-jk(2\pi/T)t} dt \\
 &= \frac{1}{1} \int_{-0.5}^{0.5} x(t) e^{-jk(2\pi/1)t} dt \\
 &= \int_{-0.5}^0 (-1) \cdot e^{-j2\pi kt} dt + \int_0^{0.5} (1) \cdot e^{-j2\pi kt} dt \\
 &= \left[-\frac{e^{-j2\pi kt}}{-j2\pi k} \right]_{-0.5}^0 + \left[\frac{e^{-j2\pi kt}}{-j2\pi k} \right]_0^{0.5} \\
 &= \frac{1}{-j2\pi k} (e^{j\pi k} - 1) + \frac{1}{j2\pi k} (1 - e^{-j\pi k}) \\
 &= \frac{j}{2\pi k} (e^{j\pi k} - 1 - 1 + e^{-j\pi k}) \\
 &= \frac{j}{2\pi k} (2\cos(\pi k) - 2) \\
 a_k &= \frac{j}{\pi k} (\cos(\pi k) - 1)
 \end{aligned}$$

We can simplify a_k :

$$a_k = \begin{cases} 0, & \text{if } k \text{ is even} \\ \frac{-2j}{\pi k}, & \text{if } k \text{ is odd} \end{cases}$$

Thus, the Fourier series expansion of $x(t)$ is:

$$x(t) = \sum_{k \text{ odd}} \left(\operatorname{Re} \left\{ \frac{-2j}{\pi k} e^{j2\pi kt} \right\} \right)$$

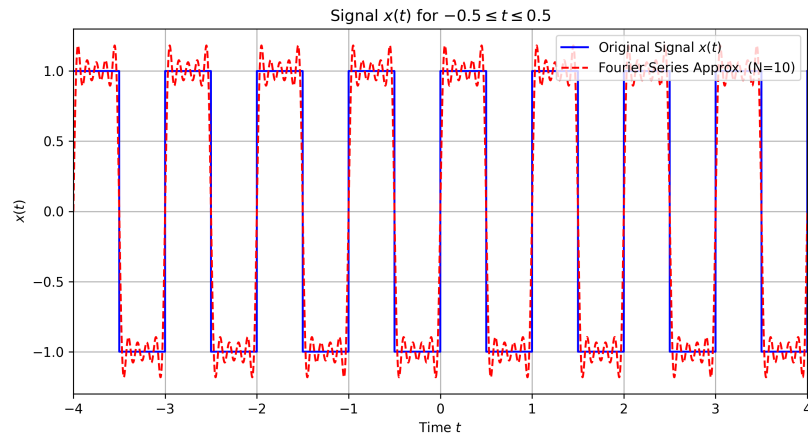
Because $e^{jx} - e^{-jx} = 2j \sin(x)$ for any real x , we have:

$$\begin{aligned}
 \sum_{k \text{ odd}} \frac{-2j}{\pi k} e^{j2\pi kt} &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{-2j}{\pi k} e^{j2\pi kt} + \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{-1} \frac{-2j}{\pi k} e^{j2\pi kt} \\
 &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{-2j}{\pi k} (e^{j2\pi kt} - e^{-j2\pi kt}) \\
 &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{-2j}{\pi k} (2j \sin(2\pi kt)) \\
 \sum_{k \text{ odd}} \frac{-2j}{\pi k} e^{j2\pi kt} &= \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi k} \sin(2\pi kt)
 \end{aligned}$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$x(t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \left(\frac{4}{\pi k} \sin(2\pi kt) \right)$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows:



Problem 2. Find the Fourier Series (FS) of the periodic function $x(t)$ which are provided as follows.

2.1 $x(t) = \frac{\pi t^3}{2}; -1 < t < 1$

Solution. To find the Fourier series of the function $x(t) = \frac{\pi t^3}{2}$ for $-1 < t < 1$, where $T = 2$ (the period of the function).

Calculating ω_0 :

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

Using the Fourier series formula:

$$x(t) = \sum_k a_k e^{jk\omega_0 t} = a_0 + \sum_{k \neq 0} a_k e^{jk\omega_0 t}$$

where the Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt$$

Calculating a_0 :

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-j(0)\omega_0 t} dt \\ &= \frac{1}{2} \int_{-1}^1 \frac{\pi t^3}{2} dt \\ &= \frac{\pi}{4} \left[\frac{t^4}{4} \right]_{-1}^1 \\ &= \frac{\pi}{4} \left(\frac{1}{4} - \frac{1}{4} \right) \\ a_0 &= 0 \end{aligned}$$

Calculating a_k for $k \neq 0$:

$$\begin{aligned} a_k &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{2} \int_{-1}^1 \frac{\pi t^3}{2} e^{-j\pi k t} dt \\ a_k &= \frac{\pi}{4} \int_{-1}^1 t^3 e^{-j\pi k t} dt \end{aligned}$$

To solve the integral, we can use integration by parts multiple times. Using tabular integration by parts, we find:

Sign	Derivative	Integral
+	t^3	$e^{-j\pi k t}$
-	$3t^2$	$\frac{1}{-j\pi k} e^{-j\pi k t}$
+	$6t$	$\frac{1}{(-j\pi k)^2} e^{-j\pi k t}$
-	6	$\frac{1}{(-j\pi k)^3} e^{-j\pi k t}$
+	0	$\frac{1}{(-j\pi k)^4} e^{-j\pi k t}$

Thus, we have:

$$\int t^3 e^{-j\pi k t} dt = \frac{t^3}{-j\pi k} e^{-j\pi k t} - \frac{3t^2}{(-j\pi k)^2} e^{-j\pi k t} + \frac{6t}{(-j\pi k)^3} e^{-j\pi k t} - \frac{6}{(-j\pi k)^4} e^{-j\pi k t}$$

Evaluating this from -1 to 1 to find a_k :

$$\begin{aligned} a_k &= \frac{\pi}{4} \left[\frac{t^3}{-j\pi k} e^{-j\pi k t} - \frac{3t^2}{(-j\pi k)^2} e^{-j\pi k t} + \frac{6t}{(-j\pi k)^3} e^{-j\pi k t} - \frac{6}{(-j\pi k)^4} e^{-j\pi k t} \right]_{-1}^1 \\ &= \frac{\pi}{4} \left[\frac{2j \cos(\pi k)}{\pi k} - \frac{6j \sin(\pi k)}{(\pi k)^2} + \frac{-12j \cos(\pi k)}{(\pi k)^3} - \frac{-12j \sin(\pi k)}{(\pi k)^4} \right] \\ &= \frac{\pi}{4^2} \left[\frac{2j \cos(\pi k)}{\pi k} - \frac{6j \sin(\pi k)}{(\pi k)^2} - \frac{12j \cos(\pi k)}{(\pi k)^3} + \frac{12j \sin(\pi k)}{(\pi k)^4} \right] \\ a_k &= \frac{\pi}{2} \left[\frac{j \cos(\pi k)}{\pi k} - \frac{3j \sin(\pi k)}{(\pi k)^2} - \frac{6j \cos(\pi k)}{(\pi k)^3} + \frac{6j \sin(\pi k)}{(\pi k)^4} \right] \end{aligned}$$

We can simplify a_k :

$$a_k = \frac{\pi}{2} \left[\frac{j(-1)^k}{\pi k} - 0 - \frac{6j(-1)^k}{(\pi k)^3} + 0 \right] = -\frac{(-1)^k}{2jk} + \frac{3(-1)^k}{j\pi^2 k^3}$$

Thus, the Fourier series expansion of $x(t)$ is:

$$x(t) = \sum_{k \neq 0} \left(\operatorname{Re} \left\{ \left(-\frac{(-1)^k}{2jk} + \frac{3(-1)^k}{j\pi^2 k^3} \right) e^{j\pi k t} \right\} \right)$$

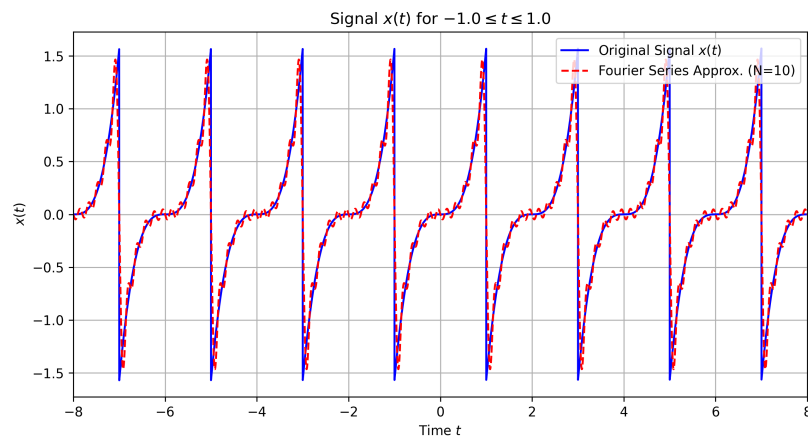
Because $e^{jx} - e^{-jx} = 2j \sin(x)$ for any real x , we have:

$$\begin{aligned}
 \sum_{k \neq 0} \left(-\frac{(-1)^k}{2jk} + \frac{3(-1)^k}{j\pi^2 k^3} \right) e^{j\pi kt} &= \sum_{k=1}^{\infty} \left(-\frac{(-1)^k}{2jk} + \frac{3(-1)^k}{j\pi^2 k^3} \right) e^{j\pi kt} \\
 &\quad + \sum_{k=-\infty}^{-1} \left(-\frac{(-1)^k}{2jk} + \frac{3(-1)^k}{j\pi^2 k^3} \right) e^{j\pi kt} \\
 &= \sum_{k=1}^{\infty} \left(-\frac{(-1)^k}{2jk} + \frac{3(-1)^k}{j\pi^2 k^3} \right) e^{j\pi kt} \\
 &\quad + \sum_{k=1}^{\infty} \left(-\frac{(-1)^{-k}}{-2jk} + \frac{3(-1)^{-k}}{-j\pi^2 k^3} \right) e^{-j\pi kt} \\
 &= \sum_{k=1}^{\infty} \left(-\frac{(-1)^k}{2jk} + \frac{3(-1)^k}{j\pi^2 k^3} \right) (e^{j\pi kt} - e^{-j\pi kt}) \\
 &= \sum_{k=1}^{\infty} \left(-\frac{(-1)^k}{2jk} + \frac{3(-1)^k}{j\pi^2 k^3} \right) (2j \sin(\pi kt)) \\
 &= \sum_{k=1}^{\infty} \left(-\frac{(-1)^k}{k} + \frac{6(-1)^k}{\pi^2 k^3} \right) \sin(\pi kt) \\
 \sum_{k \neq 0} \left(-\frac{(-1)^k}{2jk} + \frac{3(-1)^k}{j\pi^2 k^3} \right) e^{j\pi kt} &= \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{6}{\pi^2 k^2} - 1 \right) \sin(\pi kt)
 \end{aligned}$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$x(t) = \sum_{k=1}^{\infty} \left(\frac{(-1)^k}{k} \left(\frac{6}{\pi^2 k^2} - 1 \right) \sin(\pi kt) \right)$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows:



TO SUBMIT

2.2 $x(t) = \pi - t$; $-\pi \leq t \leq \pi$

Solution. To find the Fourier series of the function $x(t) = \pi - t$ for $-\pi \leq t \leq \pi$, where $T = 2\pi$ (the period of the function).

Calculating ω_0 :

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Using the Fourier series formula:

$$x(t) = \sum_k a_k e^{jk\omega_0 t} = a_0 + \sum_{k \neq 0} a_k e^{jk\omega_0 t}$$

where the Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt$$

Calculating a_0 :

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-j(0)\omega_0 t} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - t) dt \\ &= \frac{1}{2\pi} \left[\pi t - \frac{t^2}{2} \right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \left(\pi^2 - \frac{\pi^2}{2} - \left(-\pi^2 - \frac{\pi^2}{2} \right) \right) \\ a_0 &= \pi \end{aligned}$$

Calculating a_k for $k \neq 0$:

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - t) e^{-jkt} dt \\ a_k &= \frac{1}{2\pi} \left[\pi \int_{-\pi}^{\pi} e^{-jkt} dt - \int_{-\pi}^{\pi} t e^{-jkt} dt \right] \end{aligned}$$

To solve the integral, we can use integration by parts multiple times. Using tabular integration by parts, we find:

Sign	Derivative	Integral
+	t	$e^{-j\pi kt}$
-	1	$\frac{1}{-j\pi k} e^{-j\pi kt}$
+	0	$\frac{1}{(-j\pi k)^2} e^{-j\pi kt}$

Thus, we have:

$$\int t e^{-jkt} dt = \frac{t}{-jk} e^{-jkt} - \frac{1}{(-jk)^2} e^{-jkt}$$

Evaluating this from $-\pi$ to π to find a_k :

$$\begin{aligned}
 a_k &= \frac{1}{2\pi} \left[\pi \left[\frac{e^{-jkt}}{-jk} \right]_{-\pi}^{\pi} - \left[\frac{t}{-jk} e^{-jkt} - \frac{1}{(-jk)^2} e^{-jkt} \right]_{-\pi}^{\pi} \right] \\
 &= \frac{1}{2} \left[\frac{e^{-jkt}}{-jk} \right]_{-\pi}^{\pi} - \frac{1}{2\pi} \left[\frac{t}{-jk} e^{-jkt} - \frac{1}{(-jk)^2} e^{-jkt} \right]_{-\pi}^{\pi} \\
 &= \frac{1}{2} \left[\frac{(e^{-j\pi k} - e^{j\pi k})}{-jk} \right] - \frac{1}{2\pi} \left[\frac{(\pi e^{-j\pi k} + \pi e^{j\pi k})}{-jk} - \frac{(e^{-j\pi k} - e^{j\pi k})}{(-jk)^2} \right] \\
 &= \frac{1}{2} \left[\frac{(-2j \sin(\pi k))}{-jk} \right] - \frac{1}{2\pi} \left[\frac{(2\pi \cos(\pi k))}{-jk} - \frac{(-2j \sin(\pi k))}{(-jk)^2} \right] \\
 a_k &= \frac{\sin(\pi k)}{k} + \frac{\cos(\pi k)}{jk} - \frac{\sin(\pi k)}{j\pi k^2}
 \end{aligned}$$

We can simplify a_k :

$$a_k = 0 + \frac{(-1)^k}{jk} - 0 = \frac{(-1)^k}{jk} \text{ for } k \neq 0$$

Thus, the Fourier series expansion of $x(t)$ is:

$$x(t) = \pi + \sum_{k \neq 0} \left(\operatorname{Re} \left\{ \frac{(-1)^k}{jk} e^{jkt} \right\} \right)$$

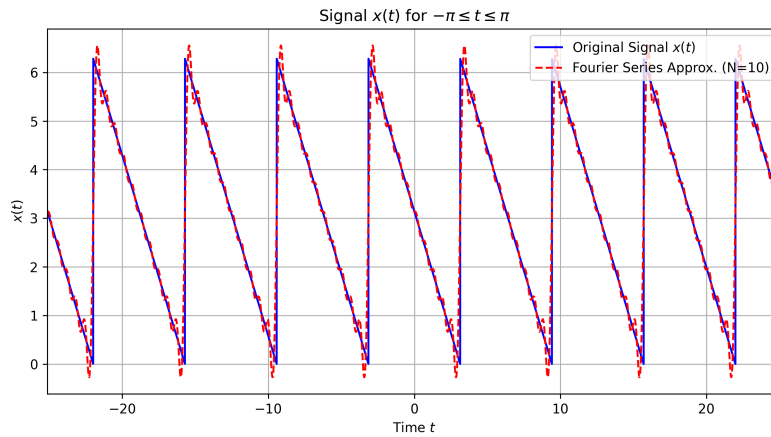
Because $e^{jx} - e^{-jx} = 2j \sin(x)$ for any real x , we have:

$$\begin{aligned}
 \sum_{k \neq 0} \frac{(-1)^k}{jk} e^{jkt} &= \sum_{k=1}^{\infty} \frac{(-1)^k}{jk} e^{jkt} + \sum_{k=-\infty}^{-1} \frac{(-1)^k}{jk} e^{jkt} \\
 &= \sum_{k=1}^{\infty} \frac{(-1)^k}{jk} e^{jkt} + \sum_{k=1}^{\infty} \frac{(-1)^{-k}}{-jk} e^{-jkt} \\
 &= \sum_{k=1}^{\infty} \frac{(-1)^k}{jk} (e^{jkt} - e^{-jkt}) \\
 &= \sum_{k=1}^{\infty} \frac{(-1)^k}{jk} (2j \sin(kt)) \\
 \sum_{k \neq 0} \frac{(-1)^k}{jk} e^{jkt} &= \sum_{k=1}^{\infty} \frac{2(-1)^k}{k} \sin(kt)
 \end{aligned}$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$x(t) = \pi + \sum_{k=1}^{\infty} \left(\frac{2(-1)^k}{k} \sin(kt) \right)$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows:



TO SUBMIT

2.3 $x(t) = t^2 + \sin^3(\pi t)$; $-1 \leq t \leq 1$

Solution. To find the Fourier series of the function $x(t) = t^2 + \sin^3(\pi t)$ for $-1 \leq t \leq 1$, where $T = 2$ (the period of the function).

Calculating ω_0 :

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

Using the Fourier series formula:

$$x(t) = \sum_k a_k e^{jk\omega_0 t} = a_0 + \sum_{k \neq 0} a_k e^{jk\omega_0 t}$$

where the Fourier coefficients a_k are given by:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt$$

Calculating a_0 :

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-j(0)\omega_0 t} dt \\ &= \frac{1}{2} \int_{-1}^1 (t^2 + \sin^3(\pi t)) dt \\ &= \frac{1}{2} \int_{-1}^1 t^2 dt + \frac{1}{2} \int_{-1}^1 \sin^3(\pi t) dt \\ &= \frac{1}{2} \left[\frac{t^3}{3} \right]_{-1}^1 + \frac{1}{2} (0) \quad \text{since } \sin^3(\pi t) \text{ is odd function} \\ &= \frac{1}{2} \left[\frac{1^3}{3} - \frac{(-1)^3}{3} \right] + 0 \\ &= \frac{1}{2} \cdot \frac{2}{3} \\ a_0 &= \frac{1}{3} \end{aligned}$$

Calculating a_k for $k \neq 0$:

$$\begin{aligned} a_k &= \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{2} \int_{-1}^1 (t^2 + \sin^3(\pi t)) e^{-jk\pi t} dt \\ a_k &= \frac{1}{2} \left[\int_{-1}^1 t^2 e^{-jk\pi t} dt + \int_{-1}^1 \sin^3(\pi t) e^{-jk\pi t} dt \right] \end{aligned}$$

Define

$$I_1 = \int_{-1}^1 t^2 e^{-jk\pi t} dt \quad \text{and} \quad I_2 = \int_{-1}^1 \sin^3(\pi t) e^{-jk\pi t} dt$$

Hence,

$$a_k = \frac{1}{2}(I_1 + I_2)$$

To solve the integral I_1 , we can use integration by parts multiple times. Using tabular integration by parts, we find:

Sign	Derivative	Integral
+	t^2	$e^{-jk\pi t}$
-	$2t$	$\frac{1}{-jk\pi} e^{-jk\pi t}$
+	2	$\frac{1}{(-jk\pi)^2} e^{-jk\pi t}$
-	0	$\frac{1}{(-jk\pi)^3} e^{-jk\pi t}$

Thus, we have:

$$I_1 = \int t^2 e^{-jk\pi t} dt = \frac{t^2}{-jk\pi} e^{-jk\pi t} - \frac{2t}{(-jk\pi)^2} e^{-jk\pi t} + \frac{2}{(-jk\pi)^3} e^{-jk\pi t}$$

Evaluating this from -1 to 1 to find I_1 :

$$\begin{aligned} I_1 &= \left[\frac{t^2}{-jk\pi} e^{-jk\pi t} - \frac{2t}{(-jk\pi)^2} e^{-jk\pi t} + \frac{2}{(-jk\pi)^3} e^{-jk\pi t} \right]_{-1}^1 \\ &= \left[\frac{-2j \sin(\pi k)}{-jk\pi} - \frac{4 \cos(\pi k)}{(-jk\pi)^2} + \frac{-4j \sin(\pi k)}{(-jk\pi)^3} \right] \\ &= \frac{-2 \sin(\pi k)}{-\pi k} - \frac{4 \cos(\pi k)}{(-jk\pi)^2} + \frac{-4 \sin(\pi k)}{j^2(-\pi k)^3} \\ I_1 &= \frac{2 \sin(\pi k)}{\pi k} + \frac{4 \cos(\pi k)}{(\pi k)^2} - \frac{4 \sin(\pi k)}{(\pi k)^3} \end{aligned}$$

Next, to solve the integral I_2 , we can use the euler identity:

$$\sin^3(x) = \left\{ \frac{1}{2j} (e^{jx} - e^{-jx}) \right\}^3 = -\frac{1}{8j} (e^{3jx} - 3e^{jx} + 3e^{-jx} - e^{-3jx})$$

Thus,

$$\begin{aligned}
 I_2 &= \int_{-1}^1 \sin^3(\pi t) e^{-jk\pi t} dt \\
 &= \int_{-1}^1 -\frac{1}{8j} (e^{3j\pi t} - 3e^{j\pi t} + 3e^{-j\pi t} - e^{-3j\pi t}) e^{-jk\pi t} dt \\
 &= -\frac{1}{8j} \int_{-1}^1 (e^{j\pi t(3-k)} - 3e^{j\pi t(1-k)} + 3e^{-j\pi t(1+k)} - e^{-j\pi t(3+k)}) dt \\
 &= -\frac{1}{8j} \left[\frac{e^{j\pi t(3-k)}}{j\pi(3-k)} - \frac{3e^{j\pi t(1-k)}}{j\pi(1-k)} + \frac{3e^{-j\pi t(1+k)}}{-j\pi(1+k)} - \frac{e^{-j\pi t(3+k)}}{-j\pi(3+k)} \right]_{-1}^1 \\
 &= -\frac{1}{8j} \left[\frac{2j \sin(\pi(3-k))}{\pi(3-k)} - \frac{3(2j \sin(\pi(1-k)))}{\pi(1-k)} + \frac{3(-2j \sin(\pi(1+k)))}{-\pi(1+k)} - \frac{-2j \sin(\pi(3+k))}{-\pi(3+k)} \right] \\
 I_2 &= -\frac{1}{4j} \left[\frac{\sin(\pi(3-k))}{\pi(3-k)} - \frac{3 \sin(\pi(1-k))}{\pi(1-k)} + \frac{3 \sin(\pi(1+k))}{\pi(1+k)} - \frac{\sin(\pi(3+k))}{\pi(3+k)} \right]
 \end{aligned}$$

Therefore, we have:

$$\begin{aligned}
 a_k &= \frac{1}{2} (I_1 + I_2) \\
 &= \frac{1}{2} \left[\frac{2 \sin(\pi k)}{\pi k} + \frac{4^2 \cos(\pi k)}{(\pi k)^2} - \frac{4^2 \sin(\pi k)}{(\pi k)^3} \right] \\
 &\quad - \frac{1}{2} \cdot \frac{1}{4j} \left[\frac{\sin(\pi(3-k))}{\pi(3-k)} - \frac{3 \sin(\pi(1-k))}{\pi(1-k)} + \frac{3 \sin(\pi(1+k))}{\pi(1+k)} - \frac{\sin(\pi(3+k))}{\pi(3+k)} \right] \\
 a_k &= \frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \\
 &\quad - \frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)}
 \end{aligned}$$

Consider the value of a_k , we can see that at $|k| = 1$ and $|k| = 3$, the terms will be undefined. Therefore, we need to calculate these four cases separately using limits.

For $k = 1$:

$$\begin{aligned}
 a_1 &= \lim_{k \rightarrow 1} \left[\frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \right] \\
 &\quad + \lim_{k \rightarrow 1} \left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)} \right] \\
 &= \left(0 + \frac{2(-1)}{\pi^2} - 0 \right) + \left(-\frac{0}{16j\pi} + \lim_{k \rightarrow 1} \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{0}{16j\pi} + \frac{0}{32j\pi} \right) \\
 a_1 &= -\frac{2}{\pi^2} - \frac{3j}{8}
 \end{aligned}$$

For $k = -1$:

$$\begin{aligned}
 a_{-1} &= \lim_{k \rightarrow -1} \left[\frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \right] \\
 &\quad + \lim_{k \rightarrow -1} \left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)} \right] \\
 &= \left(0 + \frac{2(-1)}{\pi^2} - 0 \right) + \left(-\frac{0}{16j\pi} + \frac{0}{6j\pi} - \lim_{k \rightarrow -1} \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{0}{32j\pi} \right) \\
 a_{-1} &= -\frac{2}{\pi^2} + \frac{3j}{8}
 \end{aligned}$$

Thus, we have:

$$a_k = -\frac{2}{\pi^2} - \frac{3jk}{8} \text{ for } |k| = 1$$

For $k = 3$:

$$\begin{aligned} a_3 &= \lim_{k \rightarrow 3} \left[\frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \right] \\ &\quad + \lim_{k \rightarrow 3} \left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)} \right] \\ &= \left(0 + \frac{2(-1)}{(3\pi)^2} - 0 \right) + \left(-\lim_{k \rightarrow 3} \frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{0}{-16j\pi} - \frac{0}{32j\pi} + \frac{0}{48j\pi} \right) \\ a_3 &= -\frac{2}{9\pi^2} + \frac{j}{8} \end{aligned}$$

For $k = -3$:

$$\begin{aligned} a_{-3} &= \lim_{k \rightarrow -3} \left[\frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \right] \\ &\quad + \lim_{k \rightarrow -3} \left[-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)} \right] \\ &= \left(0 + \frac{2(-1)}{(3\pi)^2} - 0 \right) + \left(-\frac{0}{16j\pi} + \frac{0}{6j\pi} - \lim_{k \rightarrow -3} \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{0}{48j\pi} \right) \\ a_{-3} &= -\frac{2}{9\pi^2} - \frac{j}{8} \end{aligned}$$

Thus, we have:

$$a_k = -\frac{2}{9\pi^2} + \frac{jk}{24} \text{ for } |k| = 3$$

For other values of k where $|k| \neq 0, 1, 3$:

$$\begin{aligned} a_k &= \left(\frac{\sin(\pi k)}{\pi k} + \frac{2 \cos(\pi k)}{(\pi k)^2} - \frac{2 \sin(\pi k)}{(\pi k)^3} \right) \\ &\quad + \left(-\frac{\sin(\pi(3-k))}{8j\pi(3-k)} + \frac{3 \sin(\pi(1-k))}{8j\pi(1-k)} - \frac{3 \sin(\pi(1+k))}{8j\pi(1+k)} + \frac{\sin(\pi(3+k))}{8j\pi(3+k)} \right) \\ &= \left(0 + \frac{2(-1)^k}{(\pi k)^2} \right) + \left(-0 + \frac{0}{8j\pi(3-k)} + \frac{0}{8j\pi(1-k)} - \frac{0}{8j\pi(1+k)} + \frac{0}{8j\pi(3+k)} \right) \\ a_k &= \frac{2(-1)^k}{k^2\pi^2} \end{aligned}$$

Simplify a_k :

$$a_k = \begin{cases} \frac{1}{3} & k = 0 \\ -\frac{2}{\pi^2} - \frac{3jk}{8} & |k| = 1 \\ -\frac{2}{9\pi^2} + \frac{jk}{24} & |k| = 3 \\ \frac{2(-1)^k}{k^2\pi^2} & \text{otherwise} \end{cases}$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$\begin{aligned} x(t) &= \frac{1}{3} + \sum_{|k| \neq 0, 1, 3} \left(\operatorname{Re} \left\{ \frac{2(-1)^k}{k^2\pi^2} e^{j\pi kt} \right\} \right) \\ &\quad + \sum_{|k|=1} \left(\operatorname{Re} \left\{ \left(-\frac{2}{\pi^2} - \frac{3jk}{8} \right) e^{j\pi kt} \right\} \right) + \sum_{|k|=3} \left(\operatorname{Re} \left\{ \left(-\frac{2}{9\pi^2} - \frac{jk}{24} \right) e^{j\pi kt} \right\} \right) \end{aligned}$$

Because $e^{jx} + e^{-jx} = 2 \cos(x)$ and $e^{jx} - e^{-jx} = 2j \sin(x)$ for any real x , we can simplify the Fourier series expansion further.

Consider the sums separately:

1. For $|k| \neq 0, 1, 3$:

$$\begin{aligned}
 \sum_{|k| \neq 0, 1, 3} \left(\frac{2(-1)^k}{k^2 \pi^2} e^{j\pi kt} \right) &= \frac{2}{\pi^2} \sum_{|k| \neq 0, 1, 3} \left(\frac{(-1)^k}{k^2} e^{j\pi kt} \right) \\
 &= \frac{2}{\pi^2} \left(\sum_{\substack{k=-\infty \\ |k| \neq 0, 1, 3}}^{-1} \left(\frac{(-1)^k}{k^2} e^{j\pi kt} \right) + \sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^k}{k^2} e^{j\pi kt} \right) \right) \\
 &= \frac{2}{\pi^2} \left(\sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^{-k}}{(-k)^2} e^{-j\pi kt} \right) + \sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^k}{k^2} e^{j\pi kt} \right) \right) \\
 &= \frac{2}{\pi^2} \sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^k}{k^2} (e^{-j\pi kt} + e^{j\pi kt}) \right) \\
 &= \frac{2}{\pi^2} \sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^k}{k^2} (2 \cos(\pi kt)) \right) \\
 \sum_{|k| \neq 0, 1, 3} \left(\frac{2(-1)^k}{k^2 \pi^2} e^{j\pi kt} \right) &= \frac{4}{\pi^2} \sum_{\substack{k=1 \\ |k| \neq 0, 1, 3}}^{\infty} \left(\frac{(-1)^k}{k^2} \cos(\pi kt) \right)
 \end{aligned}$$

2. For $|k| = 1$:

$$\begin{aligned}
 \sum_{|k|=1} \left(\left(-\frac{2}{\pi^2} - \frac{3jk}{8} \right) e^{j\pi kt} \right) &= \sum_{|k|=1} \left(-\frac{2}{\pi^2} e^{j\pi kt} \right) + \sum_{|k|=1} \left(-\frac{3jk}{8} e^{j\pi kt} \right) \\
 &= -\frac{2}{\pi^2} (e^{-j\pi t} + e^{j\pi t}) - \frac{3j}{8} (-e^{-j\pi t} + e^{j\pi t}) \\
 &= -\frac{2}{\pi^2} (2 \cos(\pi t)) - \frac{3j}{8} (2j \sin(\pi t)) \\
 \sum_{|k|=1} \left(\left(-\frac{2}{\pi^2} - \frac{3jk}{8} \right) e^{j\pi kt} \right) &= -\frac{4}{\pi^2} \cos(\pi t) + \frac{3}{4} \sin(\pi t)
 \end{aligned}$$

3. For $|k| = 3$:

$$\begin{aligned}
 \sum_{|k|=3} \left(\left(-\frac{2}{9\pi^2} + \frac{jk}{24} \right) e^{j\pi kt} \right) &= \sum_{|k|=3} \left(-\frac{2}{9\pi^2} e^{j\pi kt} \right) + \sum_{|k|=3} \left(\frac{jk}{24} e^{j\pi kt} \right) \\
 &= -\frac{2}{9\pi^2} (e^{-j3\pi t} + e^{j3\pi t}) + \frac{j(3)}{24} (-e^{-j3\pi t} + e^{j3\pi t}) \\
 &= -\frac{2}{9\pi^2} (2 \cos(3\pi t)) + \frac{j}{8} (2j \sin(3\pi t)) \\
 \sum_{|k|=3} \left(\left(-\frac{2}{9\pi^2} + \frac{jk}{24} \right) e^{j\pi kt} \right) &= -\frac{4}{9\pi^2} \cos(3\pi t) - \frac{1}{4} \sin(3\pi t)
 \end{aligned}$$

Consider $|k| = 1$ and $|k| = 3$ together.

$$\begin{aligned} &= -\frac{4}{\pi^2} \cos(\pi t) + \frac{3}{4} \sin(\pi t) - \frac{4}{9\pi^2} \cos(3\pi t) - \frac{1}{4} \sin(3\pi t) \\ &= -\frac{4}{\pi^2} \left(\cos(\pi t) + \frac{1}{9} \cos(3\pi t) \right) + \frac{3}{4} \left(\sin(\pi t) - \frac{1}{3} \sin(3\pi t) \right) \end{aligned}$$

Therefore, the Fourier series expansion of $x(t)$ is:

$$x(t) = \frac{1}{3} - \frac{4}{\pi^2} \left(\cos(\pi t) + \frac{1}{9} \cos(3\pi t) \right) + \frac{3}{4} \left(\sin(\pi t) - \frac{1}{3} \sin(3\pi t) \right) + \sum_{\substack{k=1 \\ |k| \neq 1,3}}^{\infty} \left(\frac{4(-1)^k}{\pi^2 k^2} \cos(\pi k t) \right)$$

By using Fourier series and Python approximation with $N = 10$ harmonics, we can approximate the signal as follows:

