Homework Optimize 2

Week 7

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1 Simplex Method & Two-Phase Method

Problem 1. Use the simplex method to solve the following LP.

$$\begin{array}{rrcl} \max & z & = & 3x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 & \le & 3 \\ & x_1 + 2x_2 & \le & 5 \\ & 2x_1 + x_2 & \le & 5 \\ & x_1, x_2 & \ge & 0 \end{array}$$

Solution. Re-write the LP into standard form by introducing slack variables s_1, s_2, s_3:

$$\begin{array}{rrcl} \max & z & = & 3x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 + s_1 & = & 3 \\ & x_1 + 2x_2 + s_2 & = & 5 \\ & 2x_1 + x_2 + s_3 & = & 5 \\ & x_1, x_2, s_1, s_2, s_3 & \ge & 0 \end{array}$$

Initial simplex tableau:

Basis	x_1	x_2	s_1	s_2	s_3	RHS
s_1	1	1	1	0	0	3
s_2	1	2	0	1	0	5
s_3	2	1	0	0	1	5
z	-3	-2	0	0	0	0

Starting at basic feasible solution $(x_1, x_2, s_1, s_2, s_3) = (0, 0, 3, 5, 5)$.

1. Entering variable: x_1 (most negative in z-row).
2. Leaving variable: s_1 (minimum ratio test: $3/1 = 3$).
3. Pivot on $(1, 1)$ to update tableau.

Basis	x_1	x_2	s_1	s_2	s_3	RHS
x_1	1	1	1	0	0	3
s_2	0	1	-1	1	0	2
s_3	0	-1	-2	0	1	-1
z	0	1	3	0	0	9

Since there are no negative coefficients in the z-row, the optimal solution is reached:

$$\boxed{(x_1, x_2) = (3, 0) \text{ with } z = 9}.$$

Problem 2. Determine whether each of the following LPs is degenerate or nondegenerate.

a)

$$\begin{array}{lrcl} \text{max} & z & = & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 & \leq & 4 \\ & x_1 + 2x_2 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Solution. Re-write the LP into standard form by introducing slack variables s_1, s_2:

$$\begin{array}{lrcl} \text{max} & z & = & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 + s_1 & = & 4 \\ & x_1 + 2x_2 + s_2 & = & 4 \\ & x_1, x_2, s_1, s_2 & \geq & 0 \end{array}$$

Initial simplex tableau:

Basis	x_1	x_2	s_1	s_2	RHS
s_1	1	1	1	0	4
s_2	1	2	0	1	4
z	-2	-1	0	0	0

Starting at basic feasible solution $(x_1, x_2, s_1, s_2) = (0, 0, 4, 4)$.

1. Entering variable: x_1 (most negative in z-row).
2. Leaving variable: s_1 (minimum ratio test: $4/1 = 4$).
3. Pivot on $(1, 1)$ to update tableau.

Basis	x_1	x_2	s_1	s_2	RHS
x_1	1	1	1	0	4
s_2	0	1	-1	1	0
z	0	1	2	0	8

Since the RHS of s_2 is zero, the basic feasible solution is $\boxed{\textbf{degenerate}}$.

b)

$$\begin{array}{lrcl} \text{max} & z & = & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 & \leq & 4 \\ & x_1 + 2x_2 & \leq & 6 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Solution. Re-write the LP into standard form by introducing slack variables s_1, s_2:

$$\begin{array}{lrcl} \text{max} & z & = & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 + s_1 & = & 4 \\ & x_1 + 2x_2 + s_2 & = & 6 \\ & x_1, x_2, s_1, s_2 & \geq & 0 \end{array}$$

Initial simplex tableau:

Basis	x_1	x_2	s_1	s_2	RHS
s_1	1	1	1	0	4
s_2	1	2	0	1	6
z	-2	-1	0	0	0

Starting at basic feasible solution $(x_1, x_2, s_1, s_2) = (0, 0, 4, 6)$.

1. Entering variable: x_1 (most negative in z-row).
2. Leaving variable: s_1 (minimum ratio test: $4/1 = 4$).
3. Pivot on $(1, 1)$ to update tableau.

Basis	x_1	x_2	s_1	s_2	RHS
x_1	1	1	1	0	4
s_2	0	1	-1	1	2
z	0	1	2	0	8

Since there is no zero in the RHS, the basic feasible solution is $\boxed{\textbf{nondegenerate}}$.

c)

$$\begin{array}{lrcl} \max & z & = & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 & \leq & 4 \\ & x_1 + 2x_2 & \leq & 8 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Solution. Re-write the LP into standard form by introducing slack variables s_1, s_2:

$$\begin{array}{lrcl} \max & z & = & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 + s_1 & = & 4 \\ & x_1 + 2x_2 + s_2 & = & 8 \\ & x_1, x_2, s_1, s_2 & \geq & 0 \end{array}$$

Initial simplex tableau:

Basis	x_1	x_2	s_1	s_2	RHS
s_1	1	1	1	0	4
s_2	1	2	0	1	8
z	-2	-1	0	0	0

Starting at basic feasible solution $(x_1, x_2, s_1, s_2) = (0, 0, 4, 8)$.

1. Entering variable: x_1 (most negative in z-row).
2. Leaving variable: s_1 (minimum ratio test: $4/1 = 4$).
3. Pivot on $(1, 1)$ to update tableau.

Basis	x_1	x_2	s_1	s_2	RHS
x_1	1	1	1	0	4
s_2	0	1	-1	1	4
z	0	1	2	0	8

Since there is no zero in the RHS, the basic feasible solution is $\boxed{\textbf{nondegenerate}}$.

Problem 3. Use the two-phase method to solve the following LP.

$$\begin{array}{rrcl} \text{max} & z & = & 2x_1 + 3x_2 \\ \text{subject to} & 3x_1 + x_2 & \leq & 14 \\ & x_1 - x_2 & \geq & 1 \\ & -x_1 + 3x_2 & \leq & 2 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Solution. Re-write the LP into standard form by introducing slack, excess, and artificial variables s_1, s_2, e_1, a_1:

$$\begin{array}{rrcl} \text{max} & z & = & 2x_1 + 3x_2 \\ \text{subject to} & 3x_1 + x_2 + s_1 & = & 14 \\ & x_1 - x_2 - e_1 + a_1 & = & 1 \\ & -x_1 + 3x_2 + s_2 & = & 2 \\ & x_1, x_2, s_1, s_2, e_1, a_1 & \geq & 0 \end{array}$$

Using Two-Phase Method, we first solve Phase 1 by minimizing the sum of artificial variables:

$$\min w = a_1$$

Initial simplex tableau for Phase 1:

Basis	x_1	x_2	s_1	e_1	a_1	s_2	RHS
s_1	3	1	1	0	0	0	14
a_1	1	-1	0	-1	1	0	1
s_2	-1	3	0	0	0	1	2
w	0	0	0	0	1	0	0

Becase we have to eliminate a_1 from the w-row, we perform the row operation:

$$\text{R}_w \leftarrow \text{R}_w - \text{R}_{a_1}$$

The updated tableau:

Basis	x_1	x_2	s_1	e_1	a_1	s_2	RHS
s_1	3	1	1	0	0	0	14
a_1	1	-1	0	-1	1	0	1
s_2	-1	3	0	0	0	1	2
w	-1	1	0	1	0	0	-1

Starting at basic feasible solution $(x_1, x_2, s_1, e_1, s_2, a_1) = (0, 0, 14, 0, 2, 1)$.

1. Entering variable: x_1 (most negative in w-row).
2. Leaving variable: a_1 (minimum ratio test: $1/1 = 1$).
3. Pivot on $(2, 1)$ to update tableau.

Basis	x_1	x_2	s_1	e_1	a_1	s_2	RHS
s_1	0	4	1	3	-3	0	11
x_1	1	-1	0	-1	1	0	1
s_2	0	2	0	-1	1	1	3
w	0	0	0	0	1	0	0

Since there are no negative coefficients in the w-row and $w = 0$, we proceed to Phase 2 by removing the artificial variable a_1 and solving the original objective function.

Initial simplex tableau for Phase 2:

Basis	x_1	x_2	s_1	e_1	s_2	RHS
s_1	0	4	1	3	0	11
x_1	1	-1	0	-1	0	1
s_2	0	2	0	-1	1	3
z	-2	-3	0	0	0	0

Becase we have to adjust the z-row according to the current basis, we perform the row operations:

$$\mathrm{R}_z \leftarrow \mathrm{R}_z + 2\mathrm{R}_{x_1}$$

The updated tableau:

Basis	x_1	x_2	s_1	e_1	s_2	RHS
s_1	0	4	1	3	0	11
x_1	1	-1	0	-1	0	1
s_2	0	2	0	-1	1	3
z	0	-5	0	-2	0	2

Starting at basic feasible solution $(x_1, x_2, s_1, e_1, s_2) = (1, 0, 11, 0, 3)$.

1. Entering variable: x_2 (most negative in z-row).
2. Leaving variable: s_2 (minimum ratio test: $3/2 = 1.5$).
3. Pivot on $(3, 2)$ to update tableau.

Basis	x_1	x_2	s_1	e_1	s_2	RHS
s_1	0	0	1	5	-2	5
x_1	1	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{5}{2}$
s_2	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
z	0	0	0	$-\frac{9}{2}$	$\frac{5}{2}$	$\frac{19}{2}$

Then,

1. Entering variable: e_1 (most negative in z-row).
2. Leaving variable: s_1 (minimum ratio test: $5/5 = 1$).
3. Pivot on $(1, 4)$ to update tableau.

Basis	x_1	x_2	s_1	e_1	s_2	RHS
s_1	0	0	$\frac{1}{5}$	1	$-\frac{2}{5}$	1
x_1	1	0	0	0	$-\frac{1}{10}$	4
s_2	0	1	0	0	$\frac{3}{10}$	2
z	0	0	0	0	$\frac{7}{10}$	14

Since there are no negative coefficients in the z-row, the optimal solution is reached:

$$\boxed{(x_1, x_2) = (4, 2) \text{ with } z = 14}.$$

TO SUBMIT

Problem 4. Formulate the following problem into LP and solve it. You can use any method you want.

Hamtaro likes to eat sunflower seed. He wants to eat it as much as possible. There are two types of sunflower seed: the regular one and the low-fat one. A gram of regular sunflower seed contains 0.5g of fat, 0.2g of protein, and 0.1g of fiber. A gram of low-fat sunflower seed contains 0.3g of fat, 0.3g of protein, and 0.15g of fiber.

Hamtaro should get at most 11 grams of fat and at most 8 grams of protein per day. To be healthy, he should get at least 3 grams of fiber per day. Find the maximum possible total amount of sunflower seed he can eat in a day.

Solution. Let x_1 be the grams of regular sunflower seed and x_2 be the grams of low-fat sunflower seed that Hamtaro eats in a day. The objective is to maximize the total amount of sunflower seed he can eat:

$$\max z = x_1 + x_2$$

The constraints based on fat, protein, and fiber intake are as follows:

$$\begin{aligned} \text{subject to} \quad 0.5x_1 + 0.3x_2 &\leq 11 \quad \text{(fat constraint)} \\ 0.2x_1 + 0.3x_2 &\leq 8 \quad \text{(protein constraint)} \\ 0.1x_1 + 0.15x_2 &\geq 3 \quad \text{(fiber constraint)} \\ x_1, x_2 &\geq 0 \quad \text{(non-negativity constraint)} \end{aligned}$$

Since, the third constraint is a 'greater than or equal to' type, we should use the two-phase method to solve this LP.

Re-write the standard form by introducing slack variables s_1, s_2, excess variable e_1, and artificial variable a_1:

$$\begin{aligned} \max \quad z &= x_1 + x_2 \\ \text{subject to} \quad 0.5x_1 + 0.3x_2 + s_1 &= 11 \\ 0.2x_1 + 0.3x_2 + s_2 &= 8 \\ 0.1x_1 + 0.15x_2 - e_1 + a_1 &= 3 \\ x_1, x_2, s_1, s_2, e_1, a_1 &\geq 0 \end{aligned}$$

Using Two-Phase Method, we first solve Phase 1 by minimizing the sum of artificial variables:

$$\min w = a_1$$

Initial simplex tableau for Phase 1:

Basis	x_1	x_2	s_1	s_2	e_1	a_1	RHS
s_1	0.5	0.3	1	0	0	0	11
s_2	0.2	0.3	0	1	0	0	8
a_1	0.1	0.15	0	0	-1	1	3
w	0	0	0	0	0	1	0

Becase we have to eliminate a_1 from the w-row, we perform the row operation:

$$\mathrm{R}_w \leftarrow \mathrm{R}_w - \mathrm{R}_{a_1}$$

The updated tableau:

Basis	x_1	x_2	s_1	s_2	e_1	a_1	RHS
s_1	0.5	0.3	1	0	0	0	11
s_2	0.2	0.3	0	1	0	0	8
a_1	0.1	0.15	0	0	-1	1	3
w	-0.1	-0.15	0	0	1	0	-3

Starting at basic feasible solution $(x_1, x_2, s_1, s_2, e_1, a_1) = (0, 0, 11, 8, 0, 3)$.

1. Entering variable: x_2 (most negative in w-row).
2. Leaving variable: a_1 (minimum ratio test: $3/0.15 = 20$).
3. Pivot on $(3, 2)$ to update tableau.

Basis	x_1	x_2	s_1	s_2	e_1	a_1	RHS
s_1	0.3	0	1	0	2	-2	5
s_2	0	0	0	1	2	-2	2
a_1	$\frac{2}{3}$	1	0	0	$-\frac{20}{3}$	1	20
w	0	0	0	0	0	$\frac{20}{3}$	0

Since there are no negative coefficients in the w-row and $w = 0$, we proceed to Phase 2 by removing the artificial variable a_1 and solving the original objective function.

Initial simplex tableau for Phase 2:

Basis	x_1	x_2	s_1	s_2	e_1	RHS
s_1	0.3	0	1	0	2	5
s_2	0	0	0	1	2	2
x_2	$\frac{2}{3}$	1	0	0	$-\frac{20}{3}$	20
z	-1	-1	0	0	0	0

Becase we have to adjust the z-row according to the current basis, we perform the row operation:

$$\mathrm{R}_z \leftarrow \mathrm{R}_z + \mathrm{R}_{x_2}$$

The updated tableau:

Basis	x_1	x_2	s_1	s_2	e_1	RHS
s_1	0.3	0	1	0	2	5
s_2	0	0	0	1	2	2
x_2	$\frac{2}{3}$	1	0	0	$-\frac{20}{3}$	20
z	$-\frac{1}{3}$	0	0	0	$-\frac{20}{3}$	20

Starting at basic feasible solution $(x_1, x_2, s_1, s_2, e_1) = (0, 20, 5, 2, 0)$.

1. Entering variable: e_1 (most negative in z-row).
2. Leaving variable: s_2 (minimum ratio test: $2/2 = 1$).
3. Pivot on $(2, 5)$ to update tableau.

Basis	x_1	x_2	s_1	s_2	e_1	RHS
s_1	0.3	0	1	0	0	3
s_2	0	0	0	$\frac{1}{2}$	1	1
x_2	$\frac{2}{3}$	1	0	$\frac{10}{3}$	0	$\frac{80}{3}$
z	$-\frac{1}{3}$	0	0	$\frac{10}{3}$	0	$\frac{80}{3}$

Then,

1. Entering variable: x_1 (most negative in z-row).
2. Leaving variable: s_1 (minimum ratio test: $3/0.3 = 10$).
3. Pivot on $(1,1)$ to update tableau.

Basis	x_1	x_2	s_1	s_2	e_1	RHS
s_1	1	0	$\frac{10}{3}$	0	0	10
s_2	0	0	0	$\frac{1}{2}$	1	1
x_2	0	1	$-\frac{20}{9}$	$\frac{10}{3}$	0	20
z	0	0	$\frac{10}{9}$	$\frac{10}{3}$	0	30

Since there are no negative coefficients in the z-row, the optimal solution is reached:

$$\boxed{(x_1, x_2) = (10, 20) \text{ with } z = 30}.$$