

# Homework Signal 4

Week 4

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## 1 Continuous-Time Fourier Transform (CTFT)

**Problem 1.** Find the Fourier transform of the following signals in terms of  $X(j\omega)$ , the Fourier transform of  $x(t)$  ( $\mathcal{F}\{x(t)\} = X(j\omega)$ )

1.1  $x(-t)$

**Solution.** Using the Time-scaling property of the Fourier transform, we have:

$$\textbf{Time-scaling: } \mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Substituting  $a = -1$  into the time-scaling property, we get:

$$\mathcal{F}\{x(-t)\} = \frac{1}{|-1|} X\left(\frac{j\omega}{-1}\right)$$

Therefore, we can express the Fourier transform of  $x(-t)$  as:

$$\boxed{\mathcal{F}\{x(-t)\} = X(-j\omega)}$$

1.2  $x_e(t) = \frac{x(t)+x(-t)}{2}$

**Solution.** Using the Time-scaling and Linearity properties of the Fourier transform, we have:

$$\textbf{Time-scaling: } \mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$\textbf{Linearity: } \mathcal{F}\{ax_1(t) + bx_2(t)\} = aX_1(j\omega) + bX_2(j\omega)$$

Considering the signal  $x_e(t) = \frac{x(t)+x(-t)}{2}$ , we can find its Fourier transform as follows:

$$\begin{aligned} \mathcal{F}\{x_e(t)\} &= \mathcal{F}\left\{\frac{x(t) + x(-t)}{2}\right\} \\ &= \frac{1}{2} (\mathcal{F}\{x(t)\} + \mathcal{F}\{x(-t)\}) \\ \mathcal{F}\{x_e(t)\} &= \frac{1}{2} \left( X(j\omega) + \frac{1}{|-1|} X\left(\frac{j\omega}{-1}\right) \right) \end{aligned}$$

Therefore, we can express the Fourier transform of  $x_e(t)$  as:

$$\boxed{\mathcal{F}\{x_e(t)\} = \frac{X(j\omega) + X(-j\omega)}{2}}$$

$$1.3 \ x_o(t) = \frac{x(t) - x(-t)}{2}$$

**Solution.** Using the Time-scaling and Linearity properties of the Fourier transform, we have:

$$\textbf{Time-scaling: } \mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$\textbf{Linearity: } \mathcal{F}\{ax_1(t) + bx_2(t)\} = aX_1(j\omega) + bX_2(j\omega)$$

Considering the signal  $x_o(t) = \frac{x(t) - x(-t)}{2}$ , we can find its Fourier transform as follows:

$$\begin{aligned} \mathcal{F}\{x_o(t)\} &= \mathcal{F}\left\{\frac{x(t) - x(-t)}{2}\right\} \\ &= \frac{1}{2} (\mathcal{F}\{x(t)\} - \mathcal{F}\{x(-t)\}) \\ \mathcal{F}\{x_o(t)\} &= \frac{1}{2} \left( X(j\omega) - \frac{1}{|-1|} X\left(\frac{j\omega}{-1}\right) \right) \end{aligned}$$

Therefore, we can express the Fourier transform of  $x_o(t)$  as:

$$\mathcal{F}\{x_o(t)\} = \frac{X(j\omega) - X(-j\omega)}{2}$$

**Problem 2.** Let  $\mathcal{F}\{x(t)\} = X(j\omega) = \text{rect}((\omega - 1)/2)$ . Find Fourier transform of

### TO SUBMIT

$$2.1 \ x(-2t + 4)$$

**Solution.** From the Time-scaling and Time-shifting properties of the Fourier transform, we have:

$$\textbf{Time-scaling: } \mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$\textbf{Time-shifting: } \mathcal{F}\{x(t - t_0)\} = e^{-j\omega t_0} X(j\omega)$$

Combining these two properties, we can find the Fourier transform of  $x(at - b)$ .

$$\mathcal{F}\{x(at - b)\} = \mathcal{F}\left\{x\left(a\left(t - \frac{b}{a}\right)\right)\right\} = \frac{1}{|a|} e^{-j\omega \frac{b}{a}} X\left(\frac{j\omega}{a}\right)$$

To find the Fourier transform of  $x(-2t + 4)$ , we have  $a = -2$  and  $b = -4$ . Applying the combined properties and substituting  $\mathcal{F}\{x(t)\}$ , we get:

$$\begin{aligned} \mathcal{F}\{x(-2t + 4)\} &= \frac{1}{|a|} e^{-j\omega \frac{b}{a}} X\left(\frac{j\omega}{a}\right) \\ &= \frac{1}{|-2|} e^{-j\omega \frac{-4}{-2}} X\left(\frac{j\omega}{-2}\right) \\ &= \frac{1}{2} e^{-2j\omega} \text{rect}\left(\frac{\frac{\omega}{-2} - 1}{2}\right) \\ \mathcal{F}\{x(-2t + 4)\} &= \frac{1}{2} e^{-2j\omega} \text{rect}\left(\frac{-\omega - 2}{4}\right) \end{aligned}$$

Because,  $\text{rect}$  is an even function, we can express the Fourier transform of  $x(-2t + 4)$  as:

$$\mathcal{F}\{x(-2t + 4)\} = \frac{1}{2} e^{-2j\omega} \text{rect}\left(\frac{\omega + 2}{4}\right)$$

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2.2  $(t-1)x(t-1)$

**Solution.** First, define a new signal  $y(t) = tx(t)$ . Then, we can express  $(t-1)x(t-1)$  as:

$$(t-1)x(t-1) = y(t-1)$$

Now, using the Time-shifting property of the Fourier transform, we have:

$$\textbf{Time-shifting: } \mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega)$$

Applying to  $y(t-1)$ , we get:

$$\begin{aligned} \mathcal{F}\{y(t-1)\} &= e^{-j\omega \cdot (1)} Y(j\omega) \\ &= e^{-j\omega} \mathcal{F}\{y(t)\} \\ \mathcal{F}\{y(t-1)\} &= e^{-j\omega} \mathcal{F}\{tx(t)\} \end{aligned}$$

Next, consider the differentiation of  $\mathcal{F}\{x(t)\}$ :

$$\begin{aligned} \frac{d}{d\omega} \mathcal{F}\{x(t)\} &= \frac{d}{d\omega} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(t) \frac{d}{d\omega} (e^{-j\omega t}) dt \\ &= \int_{-\infty}^{\infty} x(t) (-jte^{-j\omega t}) dt \\ &= -j \int_{-\infty}^{\infty} tx(t) e^{-j\omega t} dt \\ \frac{d}{d\omega} \mathcal{F}\{x(t)\} &= -j \mathcal{F}\{tx(t)\} \\ \mathcal{F}\{tx(t)\} &= j \frac{d}{d\omega} X(j\omega) \end{aligned}$$

Substituting  $\mathcal{F}\{x(t)\}$  into the equation, we get:

$$\begin{aligned} \mathcal{F}\{tx(t)\} &= j \frac{d}{d\omega} X(j\omega) \\ &= j \frac{d}{d\omega} \left( \text{rect} \left( \frac{\omega-1}{2} \right) \right) \\ &= j \frac{d}{d\omega} (u(\omega) - u(\omega-2)) \\ \mathcal{F}\{tx(t)\} &= j (\delta(\omega) - \delta(\omega-2)) \end{aligned}$$

Therefore, we can express the Fourier transform of  $(t-1)x(t-1)$  as:

$$\boxed{\mathcal{F}\{(t-1)x(t-1)\} = je^{-j\omega} (\delta(\omega) - \delta(\omega-2))}$$

## TO SUBMIT

2.3  $t \frac{dx(t)}{dt}$

**Solution.** First, define a new signal  $y(t) = \frac{dx(t)}{dt}$ . Then, we can express  $t \frac{dx(t)}{dt}$  as:

$$t \frac{dx(t)}{dt} = ty(t)$$

Now, using the Differentiation in frequency property (proved in the previous problem) of the Fourier transform, we have:

$$\textbf{Differentiation in frequency: } \mathcal{F}\{tx(t)\} = j \frac{d}{d\omega} X(j\omega)$$

Applying to  $ty(t)$ , we get:

$$\mathcal{F}\{ty(t)\} = j \frac{d}{d\omega} Y(j\omega) = j \frac{d}{d\omega} \mathcal{F}\{y(t)\} = j \frac{d}{d\omega} \mathcal{F}\left\{\frac{dx(t)}{dt}\right\}$$

Next, using the Differentiation in time property of the Fourier transform, we have:

$$\textbf{Differentiation in time: } \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j\omega X(j\omega)$$

Substituting into the equation, we get:

$$\begin{aligned} \frac{d}{d\omega} \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} &= \frac{d}{d\omega} (j\omega X(j\omega)) \\ &= \frac{d}{d\omega} \left( j\omega \cdot \text{rect}\left(\frac{\omega-1}{2}\right) \right) \\ &= \frac{d}{d\omega} (j\omega (u(\omega) - u(\omega-2))) \\ &= j\omega \frac{d}{d\omega} (u(\omega) - u(\omega-2)) + (u(\omega) - u(\omega-2)) \frac{d}{d\omega} (j\omega) \\ &= j\omega (\delta(\omega) - \delta(\omega-2)) + \text{rect}\left(\frac{\omega-1}{2}\right) \cdot (j) \\ &= j\omega\delta(\omega) - j\omega\delta(\omega-2) + j\text{rect}\left(\frac{\omega-1}{2}\right) \\ &= 0 - j(2)\delta(\omega-2) + j\text{rect}\left(\frac{\omega-1}{2}\right) \\ \frac{d}{d\omega} \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} &= -j(2)\delta(\omega-2) + j\text{rect}\left(\frac{\omega-1}{2}\right) \end{aligned}$$

Thus, substituting back, we have:

$$\begin{aligned} \mathcal{F}\{ty(t)\} &= j \frac{d}{d\omega} \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} \\ &= j \cdot \left[ -j(2)\delta(\omega-2) + j\text{rect}\left(\frac{\omega-1}{2}\right) \right] \\ \mathcal{F}\{ty(t)\} &= 2\delta(\omega-2) - \text{rect}\left(\frac{\omega-1}{2}\right) \end{aligned}$$

Therefore, we can express the Fourier transform of  $t \frac{dx(t)}{dt}$  as:

$$\mathcal{F}\left\{t \frac{dx(t)}{dt}\right\} = 2\delta(\omega-2) - \text{rect}\left(\frac{\omega-1}{2}\right)$$

## TO SUBMIT

2.4  $x(2t - 1)e^{-j2t}$

**Solution.** Using the Time-scaling, and Time-shifting properties of the Fourier transform, we have:

$$\text{Time-scaling} + \text{Time-shifting: } \mathcal{F}\{x(at - b)\} = \frac{1}{|a|} e^{-j\omega \frac{b}{a}} X\left(\frac{j\omega}{a}\right)$$

Define a new signal  $y(t) = x(2t - 1)$ . Then, we can express  $x(2t - 1)e^{-j2t}$  as:

$$\begin{aligned} Y(j\omega) &= \mathcal{F}\{y(t)\} \\ &= \mathcal{F}\{x(2t - 1)\} \\ &= \frac{1}{|2|} e^{-j\omega \frac{1}{2}} X\left(\frac{j\omega}{2}\right) \\ Y(j\omega) &= \frac{1}{2} e^{-j\frac{\omega}{2}} X\left(\frac{j\omega}{2}\right) \end{aligned}$$

Now, using the Frequency-shifting property of the Fourier transform, we have:

$$\text{Frequency-shifting: } \mathcal{F}\{x(t)e^{j\omega_0 t}\} = X(j(\omega - \omega_0))$$

Applying to  $y(t)e^{-j2t}$ , we get:

$$\begin{aligned} \mathcal{F}\{y(t)e^{-j2t}\} &= \mathcal{F}\{y(t)e^{j(-2)t}\} \\ &= Y(j(\omega - (-2))) \\ &= Y(j(\omega + 2)) \\ \mathcal{F}\{y(t)e^{-j2t}\} &= \frac{1}{2} e^{-j\frac{\omega+2}{2}} X\left(\frac{j(\omega+2)}{2}\right) \end{aligned}$$

Then, substituting  $\mathcal{F}\{x(t)\}$  into the equation, we get:

$$\begin{aligned} \mathcal{F}\{y(t)e^{-j2t}\} &= \frac{1}{2} e^{-j\frac{\omega+2}{2}} X\left(\frac{j(\omega+2)}{2}\right) \\ &= \frac{1}{2} e^{-j\frac{\omega+2}{2}} \text{rect}\left(\frac{\frac{\omega+2}{2} - 1}{2}\right) \\ &= \frac{1}{2} e^{-j\frac{\omega+2}{2}} \text{rect}\left(\frac{\omega+2-2}{4}\right) \\ \mathcal{F}\{y(t)e^{-j2t}\} &= \frac{1}{2} e^{-j\frac{\omega+2}{2}} \text{rect}\left(\frac{\omega}{4}\right) \end{aligned}$$

Therefore, we can express the Fourier transform of  $x(2t - 1)e^{-j2t}$  as:

$$\mathcal{F}\{x(2t - 1)e^{-j2t}\} = \frac{1}{2} e^{-j\frac{\omega+2}{2}} \text{rect}\left(\frac{\omega}{4}\right)$$

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2.5  $x(t) * x(t - 1)$

**Solution.** First, define a new signal  $y(t) = x(t - 1)$ . Then, we can express  $x(t) * x(t - 1)$  as:

$$x(t) * x(t - 1) = x(t) * y(t)$$

Next, using the Time-shifting property of the Fourier transform, we have:

$$\textbf{Time-shifting: } \mathcal{F}\{x(t - t_0)\} = e^{-j\omega t_0} X(j\omega)$$

Substituting into the equation, we get:

$$\begin{aligned} Y(j\omega) &= \mathcal{F}\{y(t)\} \\ &= \mathcal{F}\{x(t - 1)\} \\ &= e^{-j\omega(1)} X(j\omega) \\ Y(j\omega) &= e^{-j\omega} X(j\omega) \end{aligned}$$

Now, using the Convolution property of the Fourier transform, we have:

$$\textbf{Convolution: } \mathcal{F}\{x_1(t) * x_2(t)\} = X_1(j\omega) \cdot X_2(j\omega)$$

Applying to  $x(t) * y(t)$ , we get:

$$\begin{aligned} \mathcal{F}\{x(t) * y(t)\} &= \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{y(t)\} \\ &= X(j\omega) \cdot \mathcal{F}\{x(t - 1)\} \\ &= X(j\omega) \cdot e^{-j\omega(1)} X(j\omega) \\ \mathcal{F}\{x(t) * y(t)\} &= (X(j\omega))^2 \cdot e^{-j\omega} \end{aligned}$$

Lastly, substituting  $\mathcal{F}\{x(t)\} = \text{rect}((\omega - 1)/2)$  back into the equation, we get:

$$\mathcal{F}\{x(t) * y(t)\} = \text{rect}^2\left(\frac{\omega - 1}{2}\right) \cdot e^{-j\omega}$$

But, since  $\text{rect}(\cdot)$  is equal to either 0 or 1, we have:

$$\text{rect}^2\left(\frac{\omega - 1}{2}\right) = \text{rect}\left(\frac{\omega - 1}{2}\right)$$

Therefore, we can express the Fourier transform of  $x(t) * x(t - 1)$  as:

$$\mathcal{F}\{x(t) * x(t - 1)\} = e^{-j\omega} \text{rect}\left(\frac{\omega - 1}{2}\right)$$

**Problem 3.**

3.1 Proof that  $\mathcal{F}\{e^{-|t|}\} = \mathcal{F}\{\exp(-|t|)\} = \frac{2}{\omega^2 + 1}$

**Solution.** Using the definition of the Continuous-Time Fourier Transform (CTFT), we have:

$$\mathcal{F}\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Substituting  $x(t) = e^{-|t|}$  into the CTFT definition, we get:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-|t|} e^{-j\omega t} dt \\ &= \int_{-\infty}^0 e^t e^{-j\omega t} dt + \int_0^{\infty} e^{-t} e^{-j\omega t} dt \\ &= \int_{-\infty}^0 e^{(1-j\omega)t} dt + \int_0^{\infty} e^{-(1+j\omega)t} dt \\ &= \left[ \frac{e^{(1-j\omega)t}}{1-j\omega} \right]_{-\infty}^0 + \left[ \frac{-e^{-(1+j\omega)t}}{1+j\omega} \right]_0^{\infty} \\ &= \frac{1}{1-j\omega} + \frac{1}{1+j\omega} \\ &= \frac{(1+j\omega) + (1-j\omega)}{(1-j\omega)(1+j\omega)} \\ X(j\omega) &= \frac{2}{1+\omega^2} \end{aligned}$$

Therefore, we have proven that:

$$\boxed{\mathcal{F}\{e^{-|t|}\} = \frac{2}{\omega^2 + 1}} \quad \square.$$

3.2 Using the outcome obtained in Problem 3.1, Find the Fourier Transform of the given equation.

3.2.1  $\frac{d}{dt}(e^{-|t|})$

**Solution.** Using the Differentiation in time property of the Fourier transform, we have:

$$\text{Differentiation in time: } \mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j\omega X(j\omega)$$

Define a new signal  $y(t) = e^{-|t|}$ , applying the Differentiation in time property to  $\frac{dy(t)}{dt}$ , we get:

$$\begin{aligned} \mathcal{F}\left\{\frac{dy(t)}{dt}\right\} &= j\omega Y(j\omega) \\ &= j\omega \mathcal{F}\{y(t)\} \\ \mathcal{F}\left\{\frac{dy(t)}{dt}\right\} &= j\omega \mathcal{F}\{e^{-|t|}\} \end{aligned}$$

Substituting the result from Problem 3.1 into the equation, we get:

$$\boxed{\mathcal{F}\left\{\frac{d}{dt}(e^{-|t|})\right\} = \frac{2j\omega}{\omega^2 + 1}}$$

3.2.2  $\exp(3jt - |2t + 2|)$ 

**Solution.** First, define a new signal  $x(t) = e^{-|t|}$  and  $y(t) = x(2t + 2) = e^{-|2t+2|}$ . Then, we can express  $\exp(3jt - |2t + 2|)$  as:

$$\exp(3jt - |2t + 2|) = y(t)e^{j3t}$$

Using the Time-scaling and Time-shifting property, we have:

$$\textbf{Time-scaling + Time-shifting: } \mathcal{F}\{x(at - b)\} = \frac{1}{|a|}e^{-j\omega\frac{b}{a}}X\left(\frac{j\omega}{a}\right)$$

Substituting this property into the equation, we get:

$$\begin{aligned} Y(j\omega) &= \mathcal{F}\{y(t)\} \\ &= \mathcal{F}\{x(2t + 2)\} \\ &= \frac{1}{|2|}e^{-j\omega\frac{2}{2}}X\left(\frac{j\omega}{2}\right) \\ Y(j\omega) &= \frac{1}{2}e^{-j\omega}X\left(\frac{j\omega}{2}\right) \end{aligned}$$

Now, applying the Frequency-shifting property

$$\textbf{Frequency-shifting: } \mathcal{F}\{x(t)e^{j\omega_0 t}\} = X(j(\omega - \omega_0))$$

to  $y(t)e^{j3t}$ , we get:

$$\mathcal{F}\{y(t)e^{j3t}\} = Y(j(\omega - 3))$$

Then, substituting back and use the result from Problem 3.1, we have:

$$\begin{aligned} \mathcal{F}\{y(t)e^{j3t}\} &= Y(j(\omega - 3)) \\ &= \frac{1}{2}e^{-j(\omega-3)}X\left(\frac{j(\omega-3)}{2}\right) \\ &= \frac{1}{2}e^{-j(\omega-3)} \cdot \frac{2}{\left(\frac{\omega-3}{2}\right)^2 + 1} \\ &= \frac{e^{-j(\omega-3)}}{\frac{(\omega-3)^2}{4} + 1} \\ \mathcal{F}\{y(t)e^{j3t}\} &= \frac{4e^{-j(\omega-3)}}{(\omega-3)^2 + 4} \end{aligned}$$

Therefore, we can express the Fourier transform of  $\exp(3jt - |2t + 2|)$  as:

$$\boxed{\mathcal{F}\{\exp(3jt - |2t + 2|)\} = \frac{4e^{-j(\omega-3)}}{(\omega-3)^2 + 4}}$$



3.2.3  $\frac{1}{2\pi t^2 + 1}$ 

**Solution.** Consider the CFTF of  $e^{-|t|}$  obtained in Problem 3.1:

$$\mathcal{F}\{e^{-|t|}\} = \frac{2}{\omega^2 + 1}$$

Using the Duality property of the Fourier transform, we have:

$$\textbf{Duality: } \mathcal{F}\{X(t)\} = 2\pi x(-\omega)$$

Define a new signal  $y(t) = \frac{2}{t^2 + 1}$ . Then, applying the Duality property to  $y(t)$ , we get:

$$\begin{aligned} Y(j\omega) &= \mathcal{F}\{y(t)\} \\ &= 2\pi x(-\omega) \\ &= 2\pi \mathcal{F}^{-1}\{X(t)\} \Big|_{t=-\omega} \\ &= 2\pi e^{-|-\omega|} \\ Y(j\omega) &= 2\pi e^{-|\omega|} \end{aligned}$$

Using the Time-scaling property of the Fourier transform, we have:

$$\textbf{Time-scaling: } \mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

We can rewrite the given signal as:

$$\frac{1}{2\pi t^2 + 1} = \frac{1}{2} \frac{2}{(\sqrt{2\pi}t)^2 + 1} = \frac{1}{2} y(\sqrt{2\pi}t)$$

Find  $y(\sqrt{2\pi}t)$  by substituting  $a = \sqrt{2\pi}$  into the Time-scaling property, we get:

$$\begin{aligned} \mathcal{F}\{y(\sqrt{2\pi}t)\} &= \frac{1}{|\sqrt{2\pi}|} Y\left(\frac{j\omega}{\sqrt{2\pi}}\right) \\ &= \frac{1}{\sqrt{2\pi}} \cdot 2\pi e^{-\left|\frac{\omega}{\sqrt{2\pi}}\right|} \\ \mathcal{F}\{y(\sqrt{2\pi}t)\} &= \sqrt{2\pi} e^{-\frac{|\omega|}{\sqrt{2\pi}}} \end{aligned}$$

Therefore, we can express the Fourier transform of  $\frac{1}{2\pi t^2 + 1}$  as:

$$\boxed{\mathcal{F}\left\{\frac{1}{2\pi t^2 + 1}\right\} = \frac{\sqrt{2\pi}}{2} e^{-\frac{|\omega|}{\sqrt{2\pi}}}}$$