Homework 3

Week 3

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1 Week 3: Intro to Discrete Random Variable

Problem 1. A complex bet on a race has a payout value, X, determined by which tier of horse wins. The probability mass function (PMF) is given below:

Payout Value, k ($)	0	10	50	100
pmf $P(X = k)$	0.65	?	0.10	0.05

a) What is the probability of winning a $10 payout, $P(X = 10)$?

b) Calculate the cumulative distribution function value at 50, $F(50)$.

Solution. Because, the total PMF value for all possible outcomes is 1,

$$\sum_{\text{all} k} P(X = k) = 1$$

Thus, we can find the missing PMF value:

$$\begin{aligned}
1 &= P(X = 0) + P(X = 10) + P(X = 50) + P(X = 100) \\
1 &= 0.65 + P(X = 10) + 0.10 + 0.05 \\
P(X = 10) &= 1 - 0.65 - 0.10 - 0.05 \\
P(X = 10) &= \boxed{0.20} \ \textbf{a).}
\end{aligned}$$

The cumulative distribution function (CDF) value at 50, $F(50)$, is the sum of the PMF values for all outcomes less than or equal to 50:

$$\begin{aligned}
F(50) &= P(X = 0) + P(X = 10) + P(X = 50) \\
F(50) &= 0.65 + 0.20 + 0.10 \\
F(50) &= \boxed{0.95} \ \textbf{b).}
\end{aligned}$$

Problem 2. Are the following statements true or false?

a) The number of a horse's wins in its next 10 races can be modeled by a Geometric distribution.

b) The PMF of a bet's outcome (a numerical value for win/loss) cannot be negative.

c) The number of races until a specific long-shot horse wins for the first time is a random variable that can, in theory, take on an infinitely large value.

Solution.
a). $\boxed{\textbf{False.}}$ The number of wins in a fixed number of races (like 10) follows a Binomial distribution, not a Geometric distribution.
b). $\boxed{\textbf{True.}}$ The PMF of a bet's outcome cannot be negative, as it represents probabilities.
c). $\boxed{\textbf{True.}}$ The number of races until the first win can be infinitely large, as it includes the possibility of never winning.

Problem 3. For each scenario, what is the most appropriate discrete probability distribution? (Choose from Bernoulli, Binomial, Geometric, Poisson, or Discrete Uniform).

a) The number of wins for the champion horse *Sure Bet* in his next 8 races, given he has a fixed probability of winning each race.
b) The number of false starts during a full day of 12 races at the track.
c) The starting gate number (from 1 to 8) assigned to a horse, assuming all gates are assigned randomly.

Solution.

a). $\boxed{\textbf{Binomial Distribution}}$ Each race can be seen as a Bernoulli trial (win or not), with the same fixed probability of winning, and the outcomes across 8 independent races are counted.

b). $\boxed{\textbf{Poisson Distribution}}$ False starts are relatively rare, occur independently, and can happen multiple times across races. When counting such events over a fixed period (the 12 races), the Poisson distribution provides the natural model.

c). $\boxed{\textbf{Discrete Uniform Distribution}}$ Each gate is equally likely to be assigned, with no bias toward any specific number, so the assignment follows a discrete uniform distribution over the 8 possible gates.

TO SUBMIT

Problem 5. A long-shot horse named "Hopeful" has only a 5% chance of winning any given race. Let X be the number of races he runs until he achieves his first win.

a) What is the probability that Hopeful wins on his 4th race?
b) What is the probability that his first win occurs after the 2nd race?

Solution. The probability of "Hopeful" winning the 4th race is the same with the probability of winning–$\boxed{0.05}$. **a).**
The probability that his first win occurs after the 2nd race is the same with the probability of losing the first 2 races,

$$P(\text{first win occurs after 2nd race}) = (0.95 \times 0.95) = \boxed{0.9025}\ \textbf{b).}$$

TO SUBMIT

Problem 7. A star jockey has a 30% chance of winning any given race. Let X be his number of wins in the 3 morning races, and Y be his number of wins in the 2 afternoon races. Assume his performance is independent between races. Let $Z = X + Y$ be his total wins for the day.

a) What is the distribution of Z and what are its parameters?

b) What is the probability that he wins exactly 2 races all day, $P(Z = 2)$?

Solution. a). The distribution of Z is a binomial distribution with parameters $n = 5$ (the total number of races) and $p = 0.3$ (the probability of winning any given race).
The probability mass function (PMF) of a binomial distribution is given by

$$P(Z = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Thus, the probability that he wins exactly 2 races all day is

$$P(Z = 2) = \binom{5}{2}(0.3)^2(0.7)^3 = 10 \cdot 0.09 \cdot 0.343 = \boxed{0.3087}\ \textbf{b).}$$

TO SUBMIT

Problem 12. A trainer has two horses. Horse A is in a race where its probability of winning is 0.1. Horse B is in a separate race where its probability of winning is 0.2. Let Z be the total number of wins for the trainer. Find the probability mass function (PMF) for Z.

Solution. Let $X_A \sim \text{Bernoulli}(0.1)$ for Horse A, $X_B \sim \text{Bernoulli}(0.2)$ for Horse B.
Thus, $Z = X_A + X_B$ with possible values: $z \in \{0, 1, 2\}$.

- $P(Z = 0) = P(A \text{ loses})P(B \text{ loses}) = (1 - 0.1)(1 - 0.2) = 0.9 \cdot 0.8 = 0.72$
- $P(Z = 2) = P(A \text{ wins})P(B \text{ wins}) = 0.1 \cdot 0.2 = 0.02$
- $P(Z = 1) = 1 - P(Z = 0) - P(Z = 2) = 1 - 0.72 - 0.02 = 0.26$

So the PMF is:

$$\boxed{P(Z = z) = \begin{cases} 0.72, & z = 0 \\ 0.26, & z = 1 \\ 0.02, & z = 2 \\ 0, & \text{otherwise.} \end{cases}}$$