Homework 6

Week 7

6733172621 Patthadon Phengpinij

Collaborators. ChatGPT (for LaTeX styling and grammar checking)

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1 Law of Large Number (LLN)

Problem 1. Let X be a Gaussian distribution with mean 5 and variance 4. Y be a uniform distribution with mean 5 and variance 4. $Z = X + Y$. What should be the expected value and variance of Z?

Solution. Coming Soon...

2 Central Limit Theorem (CLT) and Joint Distribution

Problem 8. We have two coins A and B. A has a 0.7 probability to land heads. B has a 0.5 probability to land heads. We use the two coins to play a game of coin tossing. Both coins are tossed at the same time, if they both land heads, we win. Let W be the number of wins after we play the game 100,000 times. What is the distribution of W? Answer both the distribution name and its parameters. Compute $P(W > 36,000)$.

Solution. Coming Soon...

TO SUBMIT

Problem 16.

$$P(X,Y) = \begin{cases} c \cdot (x+y) & \text{if } y \geq x, 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the constant c, $P(X)$, $P(Y)$, $E[X]$, and $E[Y]$.

Solution. Given the joint probability density function (pdf) $P(X,Y) = f_{X,Y}(x,y)$, the integral over the entire support must equal 1:

$$\begin{aligned}
1 &= \int_0^1 \int_x^1 f_{X,Y}(x,y)\, dy\, dx \\
&= \int_0^1 \int_x^1 c(x+y)\, dy\, dx \\
&= \int_0^1 c \left[xy + \frac{y^2}{2} \right]_{y=x}^{y=1} dx \\
&= \int_0^1 c \left[x + \frac{1}{2} - x^2 - \frac{x^2}{2} \right] dx \\
&= c \int_0^1 \left[\frac{1}{2} + x - \frac{3}{2}x^2 \right] dx \\
&= c \left[\frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{2}x^3 \right]_0^1 \\
1 &= c \cdot \frac{1}{2} \\
\implies c &= 2
\end{aligned}$$

To find the marginal pdfs $f_X(x)$ and $f_Y(y)$:
For $0 \leq x \leq 1$:

$$\begin{aligned}
f_X(x) &= \int_x^1 f_{X,Y}(x,y)\, dy \\
&= \int_x^1 2(x+y)\, dy \\
&= 2 \left[xy + \frac{y^2}{2} \right]_{y=x}^{y=1} \\
&= 2 \left[x + \frac{1}{2} - x^2 - \frac{x^2}{2} \right] \\
f_X(x) &= 1 + 2x - 3x^2
\end{aligned}$$

For $0 \leq y \leq 1$:

$$\begin{aligned}
f_Y(y) &= \int_0^y f_{X,Y}(x,y)\, dx \\
&= \int_0^y 2(x+y)\, dx \\
&= 2 \left[\frac{x^2}{2} + yx \right]_0^y \\
&= 2 \left[\frac{y^2}{2} + y^2 \right] \\
f_Y(y) &= 3y^2
\end{aligned}$$

To find the expected values $E[X]$ and $E[Y]$:

$$\begin{aligned}
E[X] &= \int_0^1 x f_X(x)\,dx \\
&= \int_0^1 x(1+2x-3x^2)\,dx \\
&= \int_0^1 (x+2x^2-3x^3)\,dx \\
&= \left[\frac{x^2}{2}+\frac{2x^3}{3}-\frac{3x^4}{4}\right]_0^1 \\
E[X] &= \frac{1}{2}+\frac{2}{3}-\frac{3}{4}=\frac{5}{12}
\end{aligned}$$

And,

$$\begin{aligned}
E[Y] &= \int_0^1 y f_Y(y)\,dy \\
&= \int_0^1 y(3y^2)\,dy \\
&= \int_0^1 3y^3\,dy \\
&= \left[\frac{3y^4}{4}\right]_0^1 \\
E[Y] &= \frac{3}{4}
\end{aligned}$$

Thus, the results are:

$$\boxed{c=2},$$

$$\boxed{f_X(x)=\begin{cases}1+2x-3x^2 & \text{for } 0\le x\le 1\\ 0 & \text{otherwise}\end{cases}},$$

$$\boxed{f_Y(y)=\begin{cases}3y^2 & \text{for } 0\le y\le 1\\ 0 & \text{otherwise}\end{cases}},$$

$$\boxed{E[X]=\frac{5}{12}},\quad \boxed{E[Y]=\frac{3}{4}}$$