Homework 4

Week 4

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1 Week 4: Expectation and Variance of Discrete RVs

Problem 1. A fair 10-sided die is rolled. Let X be the random variable representing the outcome of the roll.

a) What is the expectation $E[X]$?
b) What is the variance $\mathrm{Var}(X)$?
c) Let $Y = 5X - 3$. Calculate $E[Y]$ and $\mathrm{Var}(Y)$.

Solution. In this situation, we have a discrete uniform distribution–$X \sim \mathrm{Uniform}(10)$. For uniform distribution $Z \sim \mathrm{Uniform}(n)$:

$$E[Z] = \frac{n+1}{2}, \quad \mathrm{Var}(Z) = \frac{n^2-1}{12}$$

Therefore,

$$E[X] = \frac{10+1}{2} = \frac{11}{2} = \boxed{5.5} \ \textbf{a).}$$

and,

$$\mathrm{Var}[X] = \frac{10^2-1}{12} = \frac{99}{12} = \frac{33}{4} = \boxed{8.25} \ \textbf{b).}$$

while $\mathrm{range}(X) = \{1, 2, 3, \ldots, 10\}$, and $P(X = x) = \frac{1}{10}$.

Because,

$$E(aX + b) = aE(X) + b$$

and,

$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

while, $Y = 5X - 3$. Thus,

$$\begin{aligned} E(Y) &= E(5X-3) \\ &= 5E(X) - 3 \\ &= 5 \times 5.5 - 3 \\ &= 27.5 - 3 \\ &= 24.5 \end{aligned}$$

and,

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{Var}(5X-3) \\ &= 5^2 \cdot \mathrm{Var}(X) \\ &= 25 \times 8.25 \\ &= 206.25 \end{aligned}$$

Which means

$$\boxed{E(Y) = 24.5,\ \mathrm{Var}(Y) = 206.25} \ \textbf{c).}$$

Problem 2. A student takes a multiple-choice quiz. For a single question, the probability of answering correctly is 0.8. Let $X = 1$ if the answer is correct and $X = 0$ otherwise.

a) Find the expectation $E[X]$.
b) Find the variance $\mathrm{Var}(X)$.
c) Calculate $E[X^2]$.

Solution. This distribution is a Bernoulli distribution–$X \sim \text{Bernoulli}(0.8)$. The expected value and variance of $Y \sim \text{Bernoulli}(p)$ are:

$$E[Y] = p, \quad \mathrm{Var}(Y) = p(1-p)$$

Thus,

$$E[X] = \boxed{0.8}\ \textbf{a).}$$

and,

$$\mathrm{Var}(X) = 0.8(1-0.8) = 0.8 \cdot 0.2 = \boxed{0.16}\ \textbf{b).}$$

Because, $\mathrm{Var}(X) = E[X^2] - E[X]^2$, so that:

$$\begin{aligned} E[X^2] &= \mathrm{Var}(X) + E[X]^2 \\ &= 0.16 + (0.8)^2 \\ &= 0.16 + 0.64 \\ &= \boxed{0.8}\ \textbf{c).} \end{aligned}$$

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Problem 3. A manufacturer produces computer chips, and 5% of them are defective. A sample of 20 chips is selected for testing. Let Y be the number of defective chips in the sample.

a) What is the probability of finding exactly 2 defective chips?
b) What is the expected number of defective chips, $E[Y]$?
c) What is the variance of the number of defective chips, $\mathrm{Var}(Y)$?

Solution. According to the statement, $Y \sim \text{Binomial}(20, 0.05)$

a). The probability of finding exactly 2 defective chips can be calculated using the binomial probability formula:

$$P(Y=2) = \binom{20}{2}(0.05)^2(0.95)^{18} \approx \boxed{0.1887}$$

b). The expected number of defective chips, $E[Y]$, is the expected value of a binomial distribution, $Y \sim \text{Binomial}(20, 0.05)$:

$$E(Y) = (20)(0.05) = \boxed{1}$$

c). The variance of number of defective chips, $\mathrm{Var}[Y]$, is the variance of a binomial distribution, $Y \sim \text{Binomial}(20, 0.05)$:

$$E(Y) = (20)(0.05)(1-0.05) = \boxed{0.95}$$

Problem 4. The average number of calls arriving at a customer service center is 8 calls per hour. Let X be the number of calls in a given hour.

a) What is the probability that exactly 5 calls are received in one hour?

b) What is the expectation $E[X]$ and variance $\text{Var}(X)$.

c) If the call center is open for 8 hours, what is the expected total number of calls?

Solution. This distribution is a Poisson distribution–$X \sim \text{Poisson}(8)$. The probability mass function of a Poisson distribution is given by:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

for this problem, $\lambda = 8$ and $k = 5$:

$$P(X = 5) = \frac{8^5 e^{-8}}{5!} \approx \boxed{0.0916} \textbf{ a).}$$

The expected value and variance of $Y \sim \text{Poisson}(\lambda)$ are:

$$E[Y] = \lambda, \quad \text{Var}(Y) = \lambda$$

Thus,

$$\boxed{E[X] = \text{Var}(X) = 8} \textbf{ b).}$$

The expected value of the total number of calls in one hour is $E[X] = 8$, so that

$$\text{Total number of calls in 8 hours} = 8\text{hour} \times \frac{8\text{call}}{1\text{hour}} = \boxed{64\,\text{calls}} \textbf{ c).}$$

Problem 5. Let X be a random variable with $E[X] = 6$ and $\text{Var}(X) = 2$. Let $Y = -2X + 7$.

a) Find $E[Y]$.

b) Find $\text{Var}(Y)$.

c) Calculate $E[X^2]$.

Solution. Because $Y = -2X + 7$:

$$\begin{aligned} E[Y] &= -2E[X] + 7 \\ &= -2(6) + 7 \\ &= -12 + 7 \\ &= \boxed{-5} \textbf{ a).} \end{aligned}$$

and,

$$\begin{aligned} \text{Var}(Y) &= (-2)^2 \text{Var}(X) \\ &= 4(2) \\ &= \boxed{8} \textbf{ b).} \end{aligned}$$

From the relation, $\text{Var}(X) = E[X^2] - E[X]^2$, we can compute $E[X^2]$:

$$\begin{aligned} E[X^2] &= \text{Var}(X) + E[X]^2 \\ &= 2 + 6^2 \end{aligned}$$

$$
\begin{aligned}
&= 2 + 36 \\
&= \boxed{38} \textbf{ c).}
\end{aligned}
$$

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Problem 6. A number X is chosen at random from the set $\{1, 2, 3, \dots, 12\}$. A prize is awarded based on the formula $P = 3X^2 - 5$.

a) Find the expected value of the prize, $E[P]$.

b) Find the variance of X.

$\Big($Hint: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\Big)$

Solution.

$$E[P] = E[3X^2 - 5] = 3E[X^2] - 5$$

While $X \sim \text{Uniform}(12)$, the expectation of Uniform Distribution is:

$$E[X] = \frac{n+1}{2} = \frac{12+1}{2} = 6.5$$

and the variance of Uniform Distribution is:

$$\text{Var}(X) = \frac{n^2-1}{12} = \frac{12^2-1}{12} = \boxed{\frac{143}{12}} \text{ b).}$$

We can calculate $E[X^2]$ as follows:

$$
\begin{aligned}
\text{Var}(X) &= E[X^2] - (E[x])^2 \\
E[X^2] &= \text{Var}(X) + (E[x])^2 \\
&= \frac{143}{12} + (6.5)^2 \\
&= \frac{143}{12} + \frac{169}{4} \\
&= \frac{650}{12} \\
E[X^2] &= \frac{325}{6}
\end{aligned}
$$

we know that:

$$
\begin{aligned}
E[P] &= 3E[X^2] - 5 \\
&= 3 \times \frac{325}{6} - 5 \\
&= \frac{325}{2} - 5 \\
&= \frac{315}{2} \\
&= \boxed{157.5} \text{ a).}
\end{aligned}
$$

Problem 7. A scientist runs an experiment that succeeds with probability $p = 0.4$. The experiments are run until one is successful. The cost of each experiment is \$100. Let N be the number of experiments run.

a) What is the expected total cost, $E[100N]$?

b) What is the variance of the total cost, $\text{Var}(100N)$?

Solution. The random variable N follows a geometric distribution with parameter $p = 0.4$. The expected value of a geometrically distributed random variable is given by:

$$E[N] = \frac{1}{p} = \frac{1}{0.4} = 2.5$$

Thus, the expected total cost is:

$$E[100N] = 100E[N] = 100 \times 2.5 = \boxed{250}\ \textbf{a).}$$

The variance of a geometrically distributed random variable is given by:

$$\text{Var}(N) = \frac{1-p}{p^2} = \frac{0.6}{(0.4)^2} = \frac{0.6}{0.16} = 3.75$$

Thus, the variance of the total cost is:

$$\text{Var}(100N) = 100^2\text{Var}(N) = 10000 \times 3.75 = \boxed{37500}\ \textbf{b).}$$

Problem 8. A factory has two machines. Machine 1 produces items with defects at a Poisson rate of 0.8 defects per item. Machine 2 produces items with defects at a Poisson rate of 1.2 defects per item. Machine 1 produces 60% of the factory's output, and Machine 2 produces 40%. An item is selected at random and found to have exactly 1 defect.

a) What is the probability that this item was produced by Machine 1?

Solution. Let $D_1 \sim \textit{Poisson}(0.8)$ be the event that the item was produced by Machine 1, and $D_2 \sim \textit{Poisson}(1.2)$ be the event that it was produced by Machine 2.

We want to find $P(D_1|1 \text{ defect})$.
Using Bayes' theorem:

$$P(D_1|1 \text{ defect}) = \frac{P(1 \text{ defect}|D_1)P(D_1)}{P(1 \text{ defect})}$$

We know:

$$P(D_1) = 0.6$$
$$P(D_2) = 0.4$$

Next, we need to find $P(1 \text{ defect}|D_1)$ and $P(1 \text{ defect}|D_2)$.
For Machine 1:

$$P(1 \text{ defect}|D_1) = \frac{(0.8)^1 e^{-0.8}}{1!} = 0.8e^{-0.8}$$

For Machine 2:

$$P(1 \text{ defect}|D_2) = \frac{(1.2)^1 e^{-1.2}}{1!} = 1.2e^{-1.2}$$

Now we can find $P(1 \text{ defect})$:

$$P(1 \text{ defect}) = P(1 \text{ defect}|D_1)P(D_1) + P(1 \text{ defect}|D_2)P(D_2)$$

$$= (0.8e^{-0.8})(0.6) + (1.2e^{-1.2})(0.4)$$

Substituting these values back into Bayes' theorem:

$$P(D_1|1 \text{ defect}) = \frac{(0.8e^{-0.8})(0.6)}{(0.8e^{-0.8})(0.6) + (1.2e^{-1.2})(0.4)}$$

This simplifies to:

$$P(D_1|1 \text{ defect}) = \frac{0.48e^{-0.8}}{0.48e^{-0.8} + 0.48e^{-1.2}} = \frac{e^{-0.8}}{e^{-0.8} + e^{-1.2}} \approx \boxed{0.599} \textbf{ a).}$$

Problem 9. There are two bags of marbles. Bag A contains 3 red and 7 blue marbles. Bag B contains 6 red and 4 blue marbles. You choose a bag at random and draw one marble. The marble is red.

a) What is the probability that the marble came from Bag A?

Solution. Let A be the event that the marble was drawn from Bag A, and B be the event that the marble was drawn from Bag B.

We want to find $P(A|\text{red})$.
Using Bayes' theorem:

$$P(A|\text{red}) = \frac{P(\text{red}|A)P(A)}{P(\text{red})}$$

We know:

$$P(A) = 0.5$$
$$P(B) = 0.5$$

Next, we need to find $P(\text{red}|A)$ and $P(\text{red}|B)$.
For Bag A:

$$P(\text{red}|A) = \frac{3}{10}$$

For Bag B:

$$P(\text{red}|B) = \frac{6}{10}$$

Now we can find $P(\text{red})$:

$$\begin{aligned} P(\text{red}) &= P(\text{red}|A)P(A) + P(\text{red}|B)P(B) \\ &= \left(\frac{3}{10}\right)(0.5) + \left(\frac{6}{10}\right)(0.5) \\ &= \frac{3}{20} + \frac{6}{20} = \frac{9}{20} \end{aligned}$$

Substituting these values back into Bayes' theorem:

$$P(A|\text{red}) = \frac{\left(\frac{3}{10}\right)(0.5)}{\frac{9}{20}} = \frac{\frac{3}{20}}{\frac{9}{20}} = \frac{3}{9} = \frac{1}{3}$$

Thus, the probability that the marble came from Bag A is $\boxed{\frac{1}{3}}$.

Problem 10. You roll a fair six-sided die. Let the outcome be K. You then flip a biased coin K times. The coin has a probability of heads of 0.7. Let H be the number of heads obtained.

a) Find the expected number of heads, $E[H]$.

b) Find the variance of the number of heads, $\mathrm{Var}(H)$.

Solution. We can use the information given by the statement to conclude that:

$$K \sim \text{Uniform}(6), \ H|K \sim \text{Binomial}(K, 0.7)$$

a). From the law of total expectation:

$$E[H] = E[E[H|K]]$$

Thus,

$$\begin{aligned}
E[H] &= E[E[H|K]] \\
&= E[K \times 0.7] \\
&= 0.7 \times E[K] \\
&= 0.7 \times \frac{6+1}{2} \\
&= 0.7 \times \frac{7}{2} \\
&= 0.7 \times 3.5 \\
&= \boxed{2.45}
\end{aligned}$$

b). From the law of total variance:

$$\mathrm{Var}(H) = E[\mathrm{Var}(H|K)] + \mathrm{Var}(E[H|K])$$

Thus,

$$\begin{aligned}
\mathrm{Var}(H) &= E[\mathrm{Var}(H|K)] + \mathrm{Var}(E[H|K]) \\
&= E[K \times 0.7 \times (1 - 0.7)] + \mathrm{Var}(K \times 0.7) \\
&= E[K \times 0.21] + \left((0.7)^2 \times \mathrm{Var}(K)\right) \\
&= (0.21 \times E[K]) + (0.49 \times \mathrm{Var}(K)) \\
&= \left(0.21 \times \frac{6+1}{2}\right) + \left(0.49 \times \frac{6^2-1}{12}\right) \\
&= \left(0.21 \times \frac{7}{2}\right) + \left(0.49 \times \frac{35}{12}\right) \\
&= 0.735 + (0.49 \times 2.9167) \\
&= 0.735 + 1.429 \\
&= \boxed{2.164}
\end{aligned}$$