Homework 3

Week 3

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1 Week 3: Intro to Discrete Random Variable

Problem 1. A complex bet on a race has a payout value, X, determined by which tier of horse wins. The probability mass function (PMF) is given below:

Payout Value, k ($)	0	10	50	100
pmf $P(X = k)$	0.65	?	0.10	0.05

a) What is the probability of winning a $10 payout, $P(X = 10)$?

b) Calculate the cumulative distribution function value at 50, $F(50)$.

Solution. Because, the total PMF value for all possible outcomes is 1,

$$\sum_{\text{all}k} P(X = k) = 1$$

Thus, we can find the missing PMF value:

$$\begin{aligned} 1 &= P(X = 0) + P(X = 10) + P(X = 50) + P(X = 100) \\ 1 &= 0.65 + P(X = 10) + 0.10 + 0.05 \\ P(X = 10) &= 1 - 0.65 - 0.10 - 0.05 \\ P(X = 10) &= \boxed{0.20}\ \textbf{a).} \end{aligned}$$

The cumulative distribution function (CDF) value at 50, $F(50)$, is the sum of the PMF values for all outcomes less than or equal to 50:

$$\begin{aligned} F(50) &= P(X = 0) + P(X = 10) + P(X = 50) \\ F(50) &= 0.65 + 0.20 + 0.10 \\ F(50) &= \boxed{0.95}\ \textbf{b).} \end{aligned}$$

Problem 2. Are the following statements true or false?

a) The number of a horse's wins in its next 10 races can be modeled by a Geometric distribution.
b) The PMF of a bet's outcome (a numerical value for win/loss) cannot be negative.
c) The number of races until a specific long-shot horse wins for the first time is a random variable that can, in theory, take on an infinitely large value.

Solution.
a). $\boxed{\textbf{False.}}$ The number of wins in a fixed number of races (like 10) follows a Binomial distribution, not a Geometric distribution.
b). $\boxed{\textbf{True.}}$ The PMF of a bet's outcome cannot be negative, as it represents probabilities.
c). $\boxed{\textbf{True.}}$ The number of races until the first win can be infinitely large, as it includes the possibility of never winning.

Problem 3. For each scenario, what is the most appropriate discrete probability distribution? (Choose from Bernoulli, Binomial, Geometric, Poisson, or Discrete Uniform).

a) The number of wins for the champion horse *Sure Bet* in his next 8 races, given he has a fixed probability of winning each race.
b) The number of false starts during a full day of 12 races at the track.
c) The starting gate number (from 1 to 8) assigned to a horse, assuming all gates are assigned randomly.

Solution.

a). $\boxed{\textbf{Binomial Distribution}}$ Each race can be seen as a Bernoulli trial (win or not), with the same fixed probability of winning, and the outcomes across 8 independent races are counted.

b). $\boxed{\textbf{Poisson Distribution}}$ False starts are relatively rare, occur independently, and can happen multiple times across races. When counting such events over a fixed period (the 12 races), the Poisson distribution provides the natural model.

c). $\boxed{\textbf{Discrete Uniform Distribution}}$ Each gate is equally likely to be assigned, with no bias toward any specific number, so the assignment follows a discrete uniform distribution over the 8 possible gates.

Problem 4. The horse "Gallant Prince" wins 20% of the races it enters. Let X be the number of wins for Gallant Prince in his next 4 races. Assume the outcomes are independent.

a) Calculate the probability that he wins exactly one race, $P(X = 1)$.

b) Calculate the probability that he wins no races, $P(X = 0)$.

Solution. The probability that Gallant Prince wins exactly k races out of n races can be calculated using the Binomial distribution formula:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

where $n = 4$ (the number of races), and $p = 0.2$ (the probability of winning a race).

a). The probability that Gallant Prince wins exactly **one race** out of 4 is given by

$$\begin{aligned} P(X = 1) &= \binom{4}{1} (0.2)^1 (0.8)^{4-1} \\ &= 4 \cdot 0.2 \cdot (0.8)^3 \\ &= 4 \cdot 0.2 \cdot 0.512 \\ &= 0.4096 \\ &= \boxed{0.4096} \ \textbf{a).} \end{aligned}$$

b). The probability that Gallant Prince wins **no races** out of 4 is given by

$$\begin{aligned} P(X = 0) &= \binom{4}{0} (0.2)^0 (0.8)^{4-0} \\ &= 1 \cdot 1 \cdot (0.8)^4 \\ &= (0.8)^4 \\ &= 0.4096 \\ &= \boxed{0.4096} \ \textbf{b).} \end{aligned}$$

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Problem 5. A long-shot horse named "Hopeful" has only a 5% chance of winning any given race. Let X be the number of races he runs until he achieves his first win.

a) What is the probability that Hopeful wins on his 4th race?

b) What is the probability that his first win occurs after the 2nd race?

Solution. The probability of "Hopeful" winning the 4th race is the same with the probability of winning– $\boxed{0.05}$. **a).**
The probability that his first win occurs after the 2nd race is the same with the probability of losing the first 2 races,

$$P(\text{first win occurs after 2nd race}) = (0.95 \times 0.95) = \boxed{0.9025} \ \textbf{b).}$$

Problem 6. The number of jockey suspensions for misconduct at a large racetrack follows a Poisson distribution with an average of 2 suspensions per week.

a) What is the probability of having exactly one suspension in a given week?

b) What is the probability of having no suspensions in a given week?

Solution. The number of jockey suspensions for misconduct at a large racetrack follows a Poisson distribution with parameter $\lambda = 2$ (the average number of suspensions per week). The probability of having exactly k events (suspensions) in a Poisson distribution is given by the formula:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

a). For exactly one suspension ($k = 1$):

$$\begin{aligned} P(X = 1) &= \frac{2^1 e^{-2}}{1!} \\ &= 2e^{-2} \\ &\approx 2 \cdot 0.1353 \\ &\approx \boxed{0.2706}\ \textbf{a).} \end{aligned}$$

b). For no suspensions ($k = 0$):

$$\begin{aligned} P(X = 0) &= \frac{2^0 e^{-2}}{0!} \\ &= e^{-2} \\ &\approx \boxed{0.1353}\ \textbf{b).} \end{aligned}$$

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Problem 7. A star jockey has a 30% chance of winning any given race. Let X be his number of wins in the 3 morning races, and Y be his number of wins in the 2 afternoon races. Assume his performance is independent between races. Let $Z = X + Y$ be his total wins for the day.

a) What is the distribution of Z and what are its parameters?

b) What is the probability that he wins exactly 2 races all day, $P(Z = 2)$?

Solution. The distribution of Z is a binomial distribution with parameters $n = 5$ (the total number of races) and $p = 0.3$ (the probability of winning any given race).
Thus, $\boxed{Z \sim \text{Binomial}(5, 0.3)}$. **a).**

The probability mass function (PMF) of a binomial distribution is given by

$$P(Z = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Thus, the probability that he wins exactly 2 races all day is

$$P(Z = 2) = \binom{5}{2} (0.3)^2 (0.7)^3 = 10 \cdot 0.09 \cdot 0.343 = \boxed{0.3087}\ \textbf{b).}$$

Problem 8. In an 8-horse race, a gambler places a simple win bet on a single horse, "Lucky Number 7". Assume all horses have an equal chance of winning. Let the random variable $X = 1$ if Lucky Number 7 wins and $X = 0$ if it loses.

a) What is the name of the probability distribution for X?

b) Find the probability mass function (PMF) of X.

Solution.
a). The probability distribution for X is a **Bernoulli distribution**. Because X can take on only two possible outcomes: winning (1) or losing (0).

b). The probability mass function (PMF) of X is given by:

$$\boxed{P(X = x) = \begin{cases} \frac{1}{8}, & x = 1 \\ \frac{7}{8}, & x = 0 \\ 0, & \text{otherwise.} \end{cases}}$$

Problem 9. A 'Daily Double' bet requires picking the winners of two consecutive races. A gambler chooses one horse in the first race (8 horses, all equal chance) and one horse in the second race (10 horses, all equal chance). Let $Z = 1$ if the gambler wins the bet and $Z = 0$ if they lose. What is $P(Z = 1)$?

Solution. To win the 'Daily Double' bet, the gambler must correctly pick the winners of both:
1. **The first race**: There are 8 horses, so the probability of picking the winning horse is $\frac{1}{8}$.
2. **The second race**: There are 10 horses, so the probability of picking the winning horse is $\frac{1}{10}$.

Thus, the probability of winning the 'Daily Double' bet is:

$$P(Z = 1) = P(\text{win first}) \times P(\text{win second}) = \frac{1}{8} \times \frac{1}{10} = \frac{1}{80} = 0.0125.$$

So that, $P(Z = 0) = 1 - P(Z = 1) = 1 - 0.0125 = 0.9875$.

$$\boxed{P(Z = z) = \begin{cases} 0.0125, & z = 1 \\ 0.9875, & z = 0 \\ 0, & \text{otherwise.} \end{cases}}$$

Problem 10. The number of photo-finishes on a clear day is a Poisson random variable with mean 1.5. The number on a rainy day is an independent Poisson random variable with mean 3. Let W be the total number of photo-finishes from one clear day and one rainy day.

a) What is the distribution of W and its mean?

b) What is the probability of observing a total of exactly 4 photo-finishes?

Solution. Let X be the number of photo-finishes on a clear day, and Y be the number on a rainy day. Since X and Y are independent Poisson random variables, their sum $W = X+Y$ is also a Poisson random variable. The mean of W is the sum of the means of X and Y:

$$\lambda_W = \lambda_X + \lambda_Y = 1.5 + 3 = 4.5.$$

Thus, $\boxed{W \sim \text{Poisson}(4.5)}$. **a).**

The probability of observing exactly k events in a Poisson distribution is given by:

$$P(W = k) = \frac{\lambda_W^k e^{-\lambda_W}}{k!}$$

For exactly 4 photo-finishes ($k = 4$):

$$\begin{aligned} P(W = 4) &= \frac{(4.5)^4 e^{-4.5}}{4!} \\ &= \frac{410.0625 e^{-4.5}}{24} \\ &\approx \frac{410.0625 \times 0.0111}{24} \\ &\approx \frac{4.5517}{24} \\ &\approx \boxed{0.1897} \ \textbf{b).} \end{aligned}$$

Problem 11. The CDF for the number of wins (X) for a young horse in its first season is given below:

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.5, & 0 \le x < 1 \\ 0.8, & 1 \le x < 2 \\ 0.95, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

a) Find the probability that the horse wins exactly 2 races, $P(X = 2)$.

b) Find the probability that the horse wins more than 1 race, $P(X > 1)$.

Solution.

a). The probability that the horse wins exactly 2 races, $P(X = 2)$, can be found using the CDF:

$$P(X = 2) = F(2) - F(1) = 0.95 - 0.8 = \boxed{0.15} \ \textbf{a).}$$

b). The probability that the horse wins more than 1 race, $P(X > 1)$, is given by:

$$P(X > 1) = 1 - P(X \le 1) = 1 - F(1) = 1 - 0.8 = \boxed{0.2} \ \textbf{b).}$$

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Problem 12. A trainer has two horses. Horse A is in a race where its probability of winning is 0.1. Horse B is in a separate race where its probability of winning is 0.2. Let Z be the total number of wins for the trainer. Find the probability mass function (PMF) for Z.

Solution. Let $X_A \sim \text{Bernoulli}(0.1)$ for Horse A, $X_B \sim \text{Bernoulli}(0.2)$ for Horse B. Thus, $Z = X_A + X_B$ with possible values: $z \in \{0, 1, 2\}$.

- $P(Z = 0) = P(A \text{ loses})P(B \text{ loses}) = (1 - 0.1)(1 - 0.2) = 0.9 \cdot 0.8 = 0.72$
- $P(Z = 2) = P(A \text{ wins})P(B \text{ wins}) = 0.1 \cdot 0.2 = 0.02$
- $P(Z = 1) = 1 - P(Z = 0) - P(Z = 2) = 1 - 0.72 - 0.02 = 0.26$

So the PMF is:

$$\boxed{P(Z = z) = \begin{cases} 0.72, & z = 0 \\ 0.26, & z = 1 \\ 0.02, & z = 2 \\ 0, & \text{otherwise.} \end{cases}}$$

Problem 13. Calculate the following probabilities for a racing season:

a) A jockey has a 10% win rate. Find the probability he wins at least once in his next 5 races. $P(X \geq 1)$ for $X \sim Binomial(5, 0.1)$.

b) The average number of scratches per day is 2.5. Find the probability there are fewer than 2 scratches on a given day. $P(Y < 2)$ for $Y \sim Poisson(2.5)$.

Solution.

a). $X \sim Binomial(5, 0.1)$

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \left(\binom{5}{0}(0.1)^0(0.9)^5\right) \\ &= 1 - \left(1 \cdot 1 \cdot (0.9)^5\right) \\ &= 1 - (0.59049) \\ &= \boxed{0.40951}\ \textbf{a).} \end{aligned}$$

b). $Y \sim Poisson(2.5)$

$$\begin{aligned} P(Y < 2) &= P(Y = 0) + P(Y = 1) \\ &= \frac{(2.5)^0 e^{-2.5}}{0!} + \frac{(2.5)^1 e^{-2.5}}{1!} \\ &= e^{-2.5} + 2.5e^{-2.5} \\ &= (1 + 2.5)e^{-2.5} \\ &= 3.5e^{-2.5} \\ &\approx 3.5 \cdot 0.0821 \\ &\approx \boxed{0.28735}\ \textbf{b).} \end{aligned}$$

Problem 14. A jockey is scheduled for 7 races in a festival. His probability of winning any single race is 0.15. What is the probability he wins at least two races during the festival?

Solution. The probability that the jockey wins n races out of k races can be calculated using the Binomial distribution formula:

$$P(X = n) = \binom{k}{n} p^n (1-p)^{k-n}$$

In this case, $k = 7$ (the number scheduled races), and $p = 0.15$ (the probability of winning). The probability that he wins at least 2 races is:

$$\begin{aligned}
P(X \geq 2) &= 1 - P(X < 2) \\
&= 1 - (P(X = 0) + P(X = 1)) \\
&= 1 - \left(\binom{7}{0} (0.15)^0 (0.85)^7 + \binom{7}{1} (0.15)^1 (0.85)^6 \right) \\
&= 1 - \left(1 \cdot 1 \cdot (0.85)^7 + 7 \cdot 0.15 \cdot (0.85)^6\right) \\
&= 1 - \left((0.85)^7 + 1.05 \cdot (0.85)^6\right) \\
&\approx 1 - (0.3235 + 1.05 \cdot 0.3807) \\
&\approx 1 - (0.3235 + 0.3997) \\
&\approx 1 - 0.7232 \\
&\approx \boxed{0.2768}
\end{aligned}$$

Problem 15. In a 4-horse race, the probabilities of winning for each horse are: Horse 1: 0.4, Horse 2: 0.3, Horse 3: 0.2, Horse 4: 0.1. Let the random variable X be the number of the winning horse.

a) Write out the probability mass function (PMF) for X.

b) Find the probability that the winning horse's number is odd.

Solution.

a). The probability mass function (PMF) for X is:

$$\boxed{P(X = x) = \begin{cases} 0.4, & x = 1 \\ 0.3, & x = 2 \\ 0.2, & x = 3 \\ 0.1, & x = 4 \\ 0, & \text{otherwise.} \end{cases}}$$

b). The probability that the winning horse's number is odd (Horse 1 or Horse 3) is:

$$P(X \text{ is odd}) = P(X = 1) + P(X = 3) = 0.4 + 0.2 = \boxed{0.6}$$