Homework 5

Week 6

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1 Week 6: Continuous Random Variables

Problem 1. For each statement below, determine if it is True or False.

a) The value of a Probability Density Function (PDF), $f(x)$, can be greater than 1.
b) The value of a Cumulative Distribution Function (CDF), $F(x)$, can be greater than 1.
c) The integral of a valid PDF, $\int_{-\infty}^{\infty} f(x)\,dx$, over its entire range must equal 1.
d) For any continuous random variable X, the probability $P(X = c)$ is always 0.

Solution.

a) $\boxed{\textbf{True.}}$ A PDF can take values greater than 1, as long as the total area under the curve equals 1.
b) $\boxed{\textbf{False.}}$ A CDF must always be between 0 and 1, inclusive.
c) $\boxed{\textbf{True.}}$ The integral of a valid PDF over its entire range must equal 1, representing the total probability.
d) $\boxed{\textbf{True.}}$ For continuous random variables, the probability of taking any specific value is 0, since there are infinitely many possible values.

Problem 2. A continuous random variable X has the following piecewise PDF:

$$f(x) = \begin{cases} cx & 0 \le x \le 2 \\ c\,(4-x) & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of c that makes $f(x)$ a valid PDF.
b) Sketch the graph of the PDF.

Solution. For a valid PDF, the total area under the curve must equal 1. We can find c by integrating $f(x)$ over its entire range and setting the integral equal to 1.

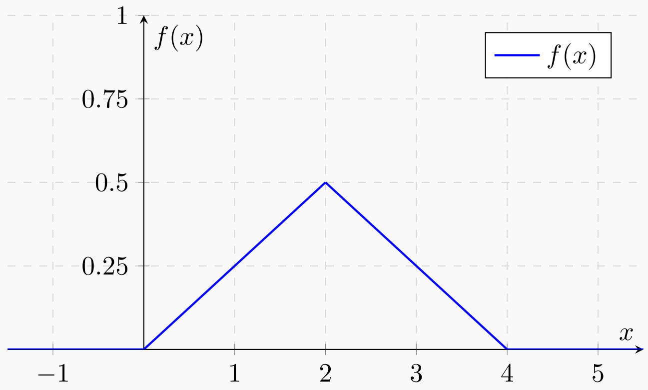
$$\begin{aligned} 1 = \int_{-\infty}^{\infty} f(x)\,dx &= \int_0^2 cx\,dx + \int_2^4 c(4-x)\,dx \\ &= c\left[\frac{x^2}{2}\right]_0^2 + c\left[4x - \frac{x^2}{2}\right]_2^4 \\ &= c\left[\frac{2^2}{2} - 0 + \left(4(4) - \frac{4^2}{2}\right) - \left(4(2) - \frac{2^2}{2}\right)\right] \\ &= c\,[2 + (16-8) - (8-2)] \\ 1 &= c\,[2 + 8 - 6] = 4c \end{aligned}$$

Thus, the value of c that would make this PDF valid is: $\boxed{c = \frac{1}{4}}$ **a).**

This PDF would be:

$$f(x) = \begin{cases} \frac{1}{4}x & 0 \leq x \leq 2 \\ \frac{1}{4}(4-x) & 2 \leq x \leq 4 \\ 0 & otherwise \end{cases}$$

b). The graph of the PDF is as follows:



Problem 3. Using the PDF from Problem 2:

a) Calculate the probability $P(X > 2.5)$.

b) Derive the Cumulative Distribution Function (CDF), $F(x)$.

Solution. From the previous problem, we have the PDF:

$$f(x) = \begin{cases} \frac{1}{4}x & 0 \leq x \leq 2 \\ \frac{1}{4}(4-x) & 2 \leq x \leq 4 \\ 0 & otherwise \end{cases}$$

To calculate $P(X > 2.5)$, we integrate the PDF from 2.5 to 4:

$$\begin{aligned} P(X > 2.5) &= \int_{2.5}^{\infty} f(x)\,dx \\ &= \int_{2.5}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx \\ &= \int_{2.5}^{4} \frac{1}{4}(4-x)\,dx + \int_{4}^{\infty} 0\,dx \\ &= \frac{1}{4}\left[4x - \frac{x^2}{2}\right]_{2.5}^{4} + 0 \\ &= \frac{1}{4}\left[(16-8) - (10 - 3.125)\right] \\ &= \frac{1}{4}\left[8 - 6.875\right] \\ P(X > 2.5) &= \frac{1}{4} \cdot 1.125 = 0.28125 \end{aligned}$$

Thus, $P(X > 2.5) = \boxed{0.28125}$ **a).**

To derive the CDF, $F(x)$, we integrate the PDF from the lower limit to x. Considering the piecewise nature of the PDF, we have:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{4}x & 0 \leq x \leq 2 \\ \frac{1}{4}(4-x) & 2 \leq x \leq 4 \\ 0 & x \geq 4 \end{cases}$$

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x \frac{1}{4}t\,dt & 0 \leq x \leq 2 \\ \int_0^2 \frac{1}{4}t\,dt + \int_2^x \frac{1}{4}(4-t)\,dt & 2 \leq x \leq 4 \\ 1 & x \geq 4 \end{cases}$$

Calculating the integrals:

$$\begin{aligned} \int_0^x \frac{1}{4}t\,dt &= \frac{1}{4} \cdot \int_0^x t\,dt \\ &= \frac{1}{4} \cdot \left[\frac{t^2}{2}\right]_0^x \\ &= \frac{1}{4} \cdot \frac{x^2}{2} \\ \int_0^x \frac{1}{4}t\,dt &= \frac{x^2}{8} \end{aligned}$$

And:

$$\begin{aligned} \int_2^x \frac{1}{4}(4-t)\,dt &= \frac{1}{4} \cdot \left[4t - \frac{t^2}{2}\right]_2^x \\ &= \frac{1}{4} \cdot \left((4x - \frac{x^2}{2}) - (8-2)\right) \\ &= \frac{1}{4} \cdot (4x - \frac{x^2}{2} - 6) \\ \int_2^x \frac{1}{4}(4-t)\,dt &= x - \frac{x^2}{8} - \frac{3}{2} \end{aligned}$$

Thus, the CDF $F(x)$ is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{8} & 0 \leq x \leq 2 \\ \left(\frac{x^2}{8}\right)_{x=2} + \left(x - \frac{x^2}{8} - \frac{3}{2}\right) & 2 \leq x \leq 4 \\ 1 & x \geq 4 \end{cases}$$

Therefore, the CDF is:

$$\boxed{F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{8} & 0 \leq x < 2 \\ x - \frac{x^2}{8} - \frac{3}{2} & 2 \leq x < 4 \\ 1 & x \geq 4 \end{cases} \textbf{ b).}}$$

Problem 4. Using the CDF from Problem 3, find the median of the random variable X.

Solution. From the previous problem, we have the CDF:

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{8} & 0 \leq x \leq 2 \\ x - \frac{3}{2} & 2 \leq x \leq 4 \\ 1 & x \geq 4 \end{cases}$$

For a random variable X, the median is the value m such that $F(m) = 0.5$. To find the median, we need to solve for m in the equation $F(m) = 0.5$. We consider the piecewise nature of the CDF:

1. For $0 \leq m \leq 2$:
$$F(m) = \frac{m^2}{8} = 0.5$$
Solving for m:
$$m^2 = 4 \implies m = 2 \implies m \in [0, 2], \underline{\textbf{valid.}}$$
2. For $2 \leq m \leq 4$:
$$F(m) = m - \frac{3}{2} = 0.5$$
Solving for m:
$$m = 2 \implies m \in [2, 4], \underline{\textbf{valid.}}$$

Thus, the median of the random variable X is $\boxed{2}$.

Problem 5. A shuttle bus arrives at a student dormitory at a random time within a 15-minute window, from 8:00 AM to 8:15 AM. Let T be the student's waiting time in minutes if they arrive at exactly 8:00 AM.

a) What is the distribution of T? (Specify type and parameters).

b) What is the probability that a student waits for more than 10 minutes?

Solution. **a).** The waiting time T follows a uniform distribution over the interval [0, 15] minutes. Thus, we can denote this as: $T \sim \text{Uniform}(0, 15)$

b). To find the probability that a student waits for more than 10 minutes: $P(T > 10)$. The PDF of a uniform distribution over the interval [a, b] is given by:

$$f_T(t) = \begin{cases} \frac{1}{b-a} & a \leq t \leq b \\ 0 & \text{otherwise} \end{cases}$$

For our case, $a = 0$ and $b = 15$, so the PDF is:

$$f_T(t) = \begin{cases} \frac{1}{15} & 0 \leq t \leq 15 \\ 0 & \text{otherwise} \end{cases}$$

To find $P(T > 10)$, we can integrate the PDF from 10 to 15:

$$P(T > 10) = \int_{10}^{15} f_T(t)\, dt = \int_{10}^{15} \frac{1}{15}\, dt = \frac{1}{15} \cdot (15 - 10) = \frac{5}{15} = \frac{1}{3}$$

Thus, the probability that a student waits for more than 10 minutes is $\boxed{\frac{1}{3}}$.

Problem 6. For the shuttle bus scenario in Problem 5:

a) What is the expected waiting time, $\mathrm{E}[T]$?
b) What is the variance of the waiting time, $\mathrm{Var}(T)$?

Solution. From Problem 5, we know that $T \sim \mathrm{Uniform}(0, 15)$.

For a uniform distribution $\mathrm{Uniform}(a, b)$:

- The expected value $\mathrm{E}[X] = \frac{a+b}{2}$
- The variance $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$

The expected waiting time $\mathrm{E}[T]$ is:

$$\mathrm{E}[T] = \frac{0+15}{2} = \frac{15}{2} = \boxed{7.5 \text{ minutes}} \textbf{ a).}$$

The variance of the waiting time $\mathrm{Var}(T)$ is:

$$\mathrm{Var}(T) = \frac{(15-0)^2}{12} = \frac{225}{12} = \boxed{18.75 \text{ minutes}^2} \textbf{ b).}$$

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Problem 7. For the shuttle bus scenario in Problem 5, what is the 80^{th} percentile of the waiting time? In other words, find the time t (in minutes) such that 80% of waiting times are less than t.

Solution. From Problem 5, we know that $T \sim \mathrm{Uniform}(0, 15)$.
The CDF of a uniform distribution $\mathrm{Uniform}(a, b)$ is given by:

$$F_T(t) = \begin{cases} 0 & t < a \\ \frac{t-a}{b-a} & a \le t \le b \\ 1 & t > b \end{cases}$$

For our case, $a = 0$ and $b = 15$, so the CDF is:

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{15} & 0 \le t \le 15 \\ 1 & t > 15 \end{cases}$$

To find the 80th percentile, we need to solve for t in the equation $F_T(t) = 0.8$:

$$\frac{t}{15} = 0.8$$

Solving for t:

$$t = 0.8 \times 15 = 12$$

Thus, the 80th percentile of the waiting time is $\boxed{12 \text{ minutes}}$.