Homework 4

Week 4

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1 Week 4: Expectation and Variance of Discrete RVs

Problem 1. A fair 10-sided die is rolled. Let X be the random variable representing the outcome of the roll.

a) What is the expectation $E[X]$?

b) What is the variance $Var(X)$?

c) Let $Y = 5X - 3$. Calculate $E[Y]$ and $Var(Y)$.

Solution.

a). The expectation $E[X]$ of a fair 10-sided die can be calculated using the formula for the expectation of a discrete random variable:

$$E(X) = \sum_{x \in \text{range}(X)} xP(X = x)$$

while range$(X) = \{1, 2, 3, \ldots, 10\}$, and $P(X = x) = \frac{1}{10}$. Thus,

$$\begin{aligned}
E(X) &= \sum_{x \in \text{range}(X)} xP(X = x) \\
&= \sum_{x \in \{1,2,3,\ldots,10\}} x \cdot \frac{1}{10} \\
&= \frac{1}{10} \sum_{x=1}^{10} x \\
&= \frac{1}{10} \cdot \frac{10(10+1)}{2} \\
&= \frac{11}{2} \\
E(X) &= \boxed{5.5}
\end{aligned}$$

b). The variance $Var(X)$ of a fair 10-sided die can be calculated using the formula for the variance of a discrete random variable:

$$Var(X) = E\left[(X - E[X])^2\right] = E(X^2) - (E(X))^2$$

while $E(X) = 5.5$, we need to calculate $E(X^2)$:

$$\begin{aligned}
E(x^2) &= \sum_{x=1}^{10} x^2 P(X = x) \\
&= \frac{(10)(10+1)(2 \times 10 + 1)}{6} \times \frac{1}{10} \\
&= \frac{(10)(11)(21)}{6} \times \frac{1}{10}
\end{aligned}$$

$$= 38.5$$

So that, $Var(X) = E(X^2) - (E(X))^2 = 38.5 - (()\,5.5)^2 = 38.5 - 30.25 = \boxed{8.25}$.

c). Becase,

$$E(aX + b) = aE(X) + b$$

and,

$$Var(aX + b) = a^2 Var(X)$$

while, $Y = 5X - 3$. Thus,

$$\begin{aligned} E(Y) &= E(5X - 3) \\ &= 5E(X) - 3 \\ &= 5 \times 5.5 - 3 \\ &= 27.5 - 3 \\ &= 24.5 \end{aligned}$$

and,

$$\begin{aligned} Var(Y) &= Var(5X - 3) \\ &= 5^2 \cdot Var(X) \\ &= 25 \times 8.25 \\ &= 206.25 \end{aligned}$$

Which means

$$\boxed{E(Y) = 24.5,\ Var(Y) = 206.25}$$

TO SUBMIT

Problem 3. A manufacturer produces computer chips, and 5% of them are defective. A sample of 20 chips is selected for testing. Let Y be the number of defective chips in the sample.

a) What is the probability of finding exactly 2 defective chips?

b) What is the expected number of defective chips, $E[Y]$?

c) What is the variance of the number of defective chips, $Var(Y)$?

Solution. According to the statement, $Y \sim Binomial(20, 0.05)$

a). The probability of finding exactly 2 defective chips can be calculated using the binomial probability formula:

$$P(Y = 2) = \binom{20}{2}(0.05)^2(0.95)^{18} \approx \boxed{0.3774}$$

b). The expected number of defective chips, $E[Y]$, is the expected value of a binomial distribution, $Y \sim Binomial(20, 0.05)$:

$$E(Y) = (20)(0.05) = \boxed{1}$$

c). The variance of number of defective chips, $Var[Y]$, is the variance of a binomial distribution, $Y \sim Binomial(20, 0.05)$:

$$E(Y) = (20)(0.05)(1 - 0.05) = \boxed{0.95}$$

TO SUBMIT

Problem 6. A number X is chosen at random from the set $\{1, 2, 3, \cdot, 12\}$. A prize is awarded based on the formula $P = 3X^2 - 5$.

a) Find the expected value of the prize, $E[P]$.

b) Find the variance of X.

$\left(\text{Hint: } \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ and } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\right)$

Solution.

$$E[P] = E[3X^2 - 5] = 3E[X^2] - 5$$

While $X \sim Uniform(12)$, the expectation of Uniform Distribution is:

$$E[X] = \frac{n+1}{2} = \frac{12+1}{2} = 6.5$$

and the variance of Uniform Distribution is:

$$Var(X) = \frac{n^2-1}{12} = \frac{12^2-1}{12} = \boxed{\frac{143}{12}} \text{ b)}.$$

We can calculate $E[X^2]$ as follows:

$$\begin{aligned}
Var(X) &= E[X^2] - (E[x])^2 \\
E[X^2] &= Var(X) + (E[x])^2 \\
&= \frac{143}{12} + (6.5)^2 \\
&= \frac{143}{12} + \frac{169}{4} \\
&= \frac{650}{12} \\
E[X^2] &= \frac{325}{6}
\end{aligned}$$

, we know that:

$$E[P] = 3E[X^2] - 5 = 3 \times \frac{325}{6} - 5 = \frac{325}{2} - 5 = \frac{315}{2} = \boxed{157.5} \text{ a)}.$$