CS 97SI: INTRODUCTION TO PROGRAMMING CONTESTS

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Computational Geometry

- Cross Product
  - Segment-Segment Intersection
- Convex Hull Problem
  - Graham Scan
- Sweep Line Algorithm
- Intersecting Half-planes
- A Useful Note on Binary/Ternary Search

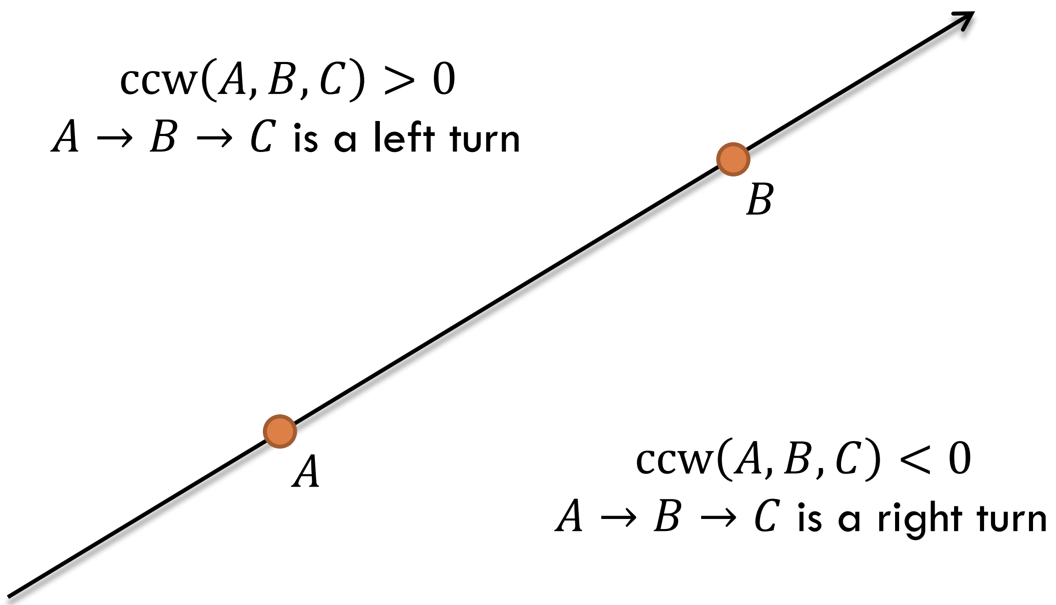
Cross Product

- Arguably the most important operation in 2D geometry
  - We'll use it all the time

- Applications:
  - Determining the (signed) area of a triangle
  - Testing if three points are collinear
  - Determining the orientation of three points
  - Testing if two line segments intersect

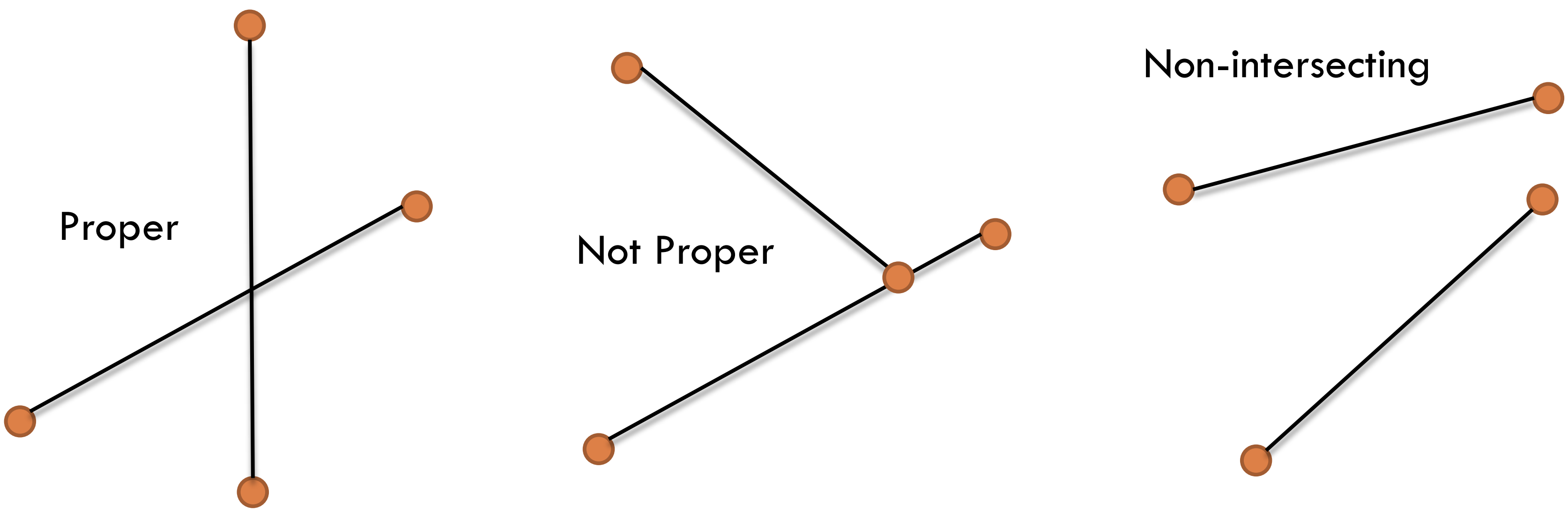
Cross Product

- Define $\mathrm{ccw}(A, B, C) = (B - A) \times (C - A)$



Segment-Segment Intersection Test

- Given two segments AB and CD
- Want to determine if they intersect properly: two segments meet at a single point that are strictly inside both segments



Segment-Segment Intersection Test

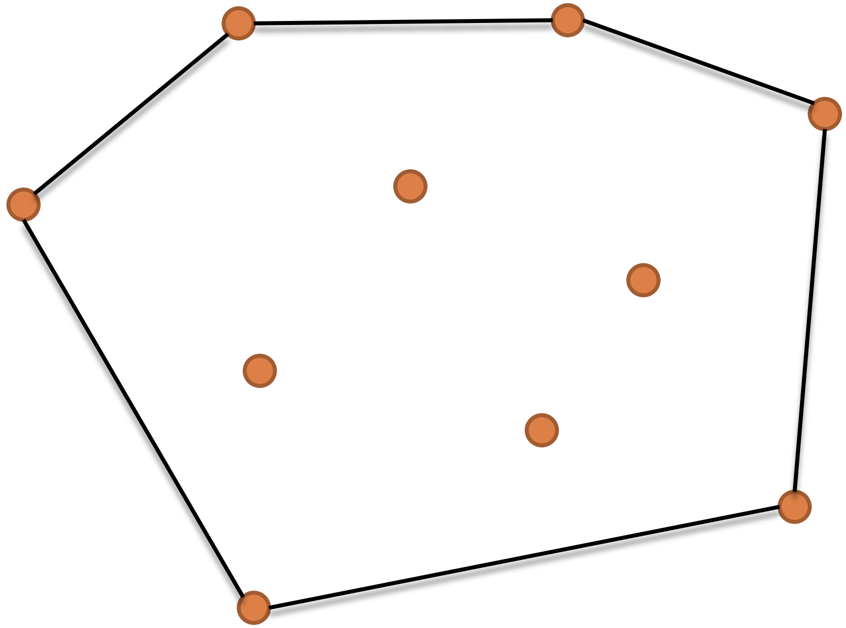
- Assume that the segments intersect
  - From A's point of view, looking straight to B, C and D must lie on different sides
  - Holds true for the other segment as well
- The intersection exists and is proper if:
  - $\text{ccw}(A, B, C) \times \text{ccw}(A, B, D) < 0$
  - AND $\text{ccw}(C, D, A) \times \text{ccw}(C, D, B) < 0$

Segment-Segment Intersection Test

- Determining non-proper intersections
  - We need more special cases to consider!
  - e.g. If $\text{ccw}(A, B, C)$, $\text{ccw}(A, B, D)$, $\text{ccw}(C, D, A)$, $\text{ccw}(C, D, B)$ are all zeros, then two segments are collinear
  - Very careful implementation is required

Convex Hull Problem

- Given n points on the plane, find the smallest convex polygon that contains all the given points
  - For simplicity, assume that no three points are collinear



Simple $O(n^3)$ algorithm

- AB is an edge of the convex hull iff $\mathrm{ccw}(A, B, C)$ have the same sign for all other given points C
  - This gives us a simple algorithm

- For each A and B:
  - If $\mathrm{ccw}(A, B, C) > 0$ for all $C \neq A, B$:
    - Record the edge $A \to B$
- Walk along the recorded edges to recover the convex hull

Faster Algorithm: Graham Scan

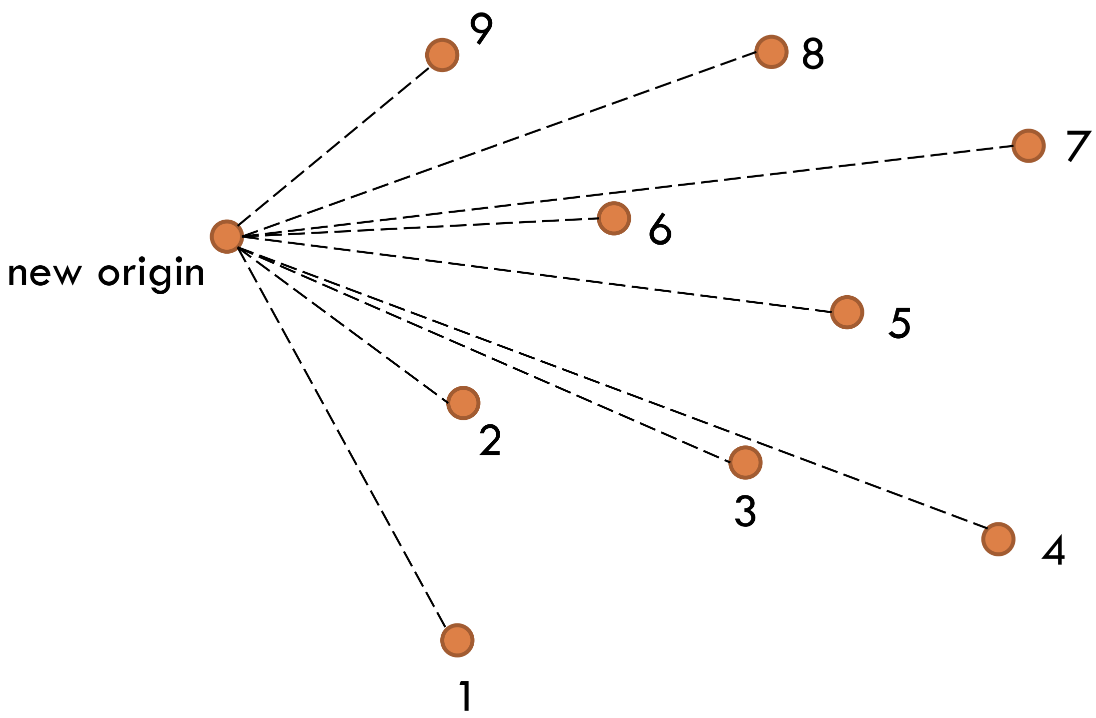
- We know that the leftmost given point has to be in the convex hull
  - We assume that there is a unique leftmost point
- Make the leftmost point the origin
  - So that all other points have positive x coordinates
- Sort the points in increasing order of y/x
  - Increasing order of angle, whatever you like to call it
- Incrementally construct the convex hull using a stack

Incremental Construction

- We maintain a *convex chain* of the given points
- For each i, we do the following:
  - Append point i to the current chain
  - If the new point causes a concave corner, remove the bad vertex from the chain that causes it
  - Repeat until the new chain becomes convex

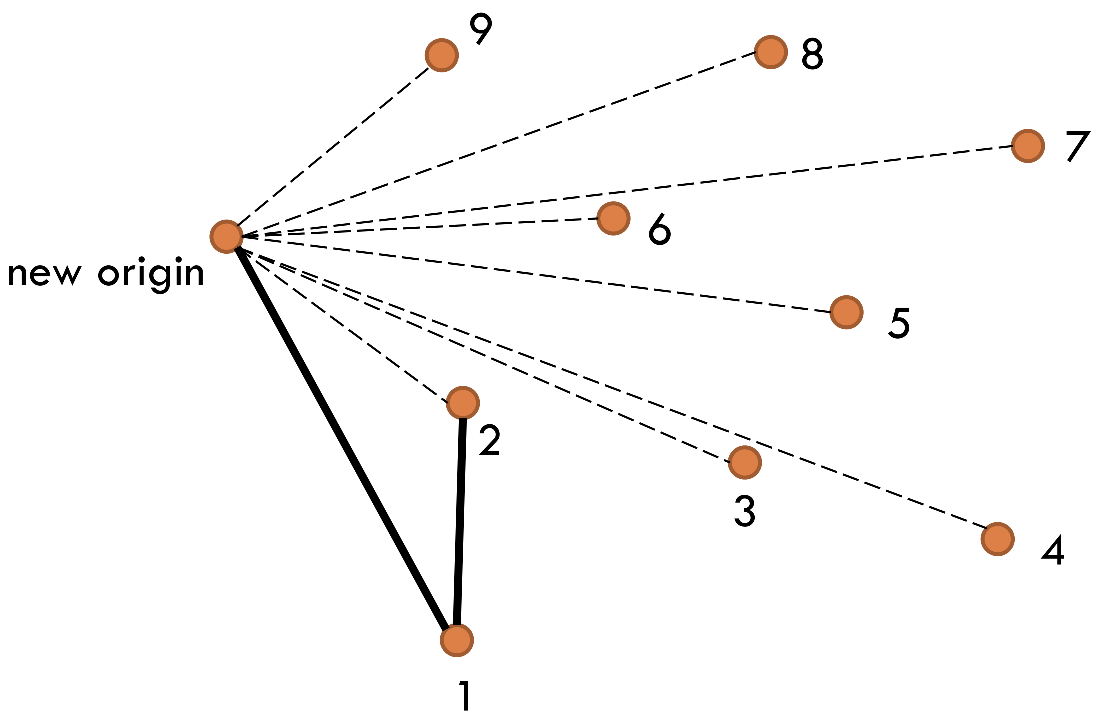
Example

- Points are numbered in increasing order of y/x



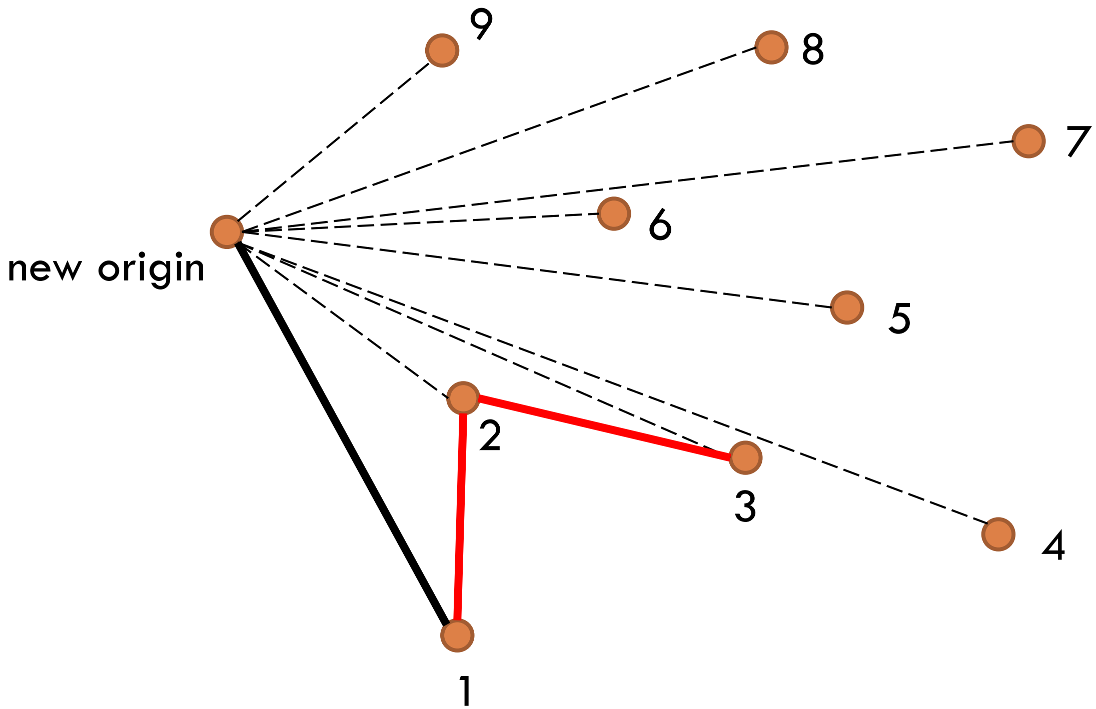
Example

- Add the first two points in the chain



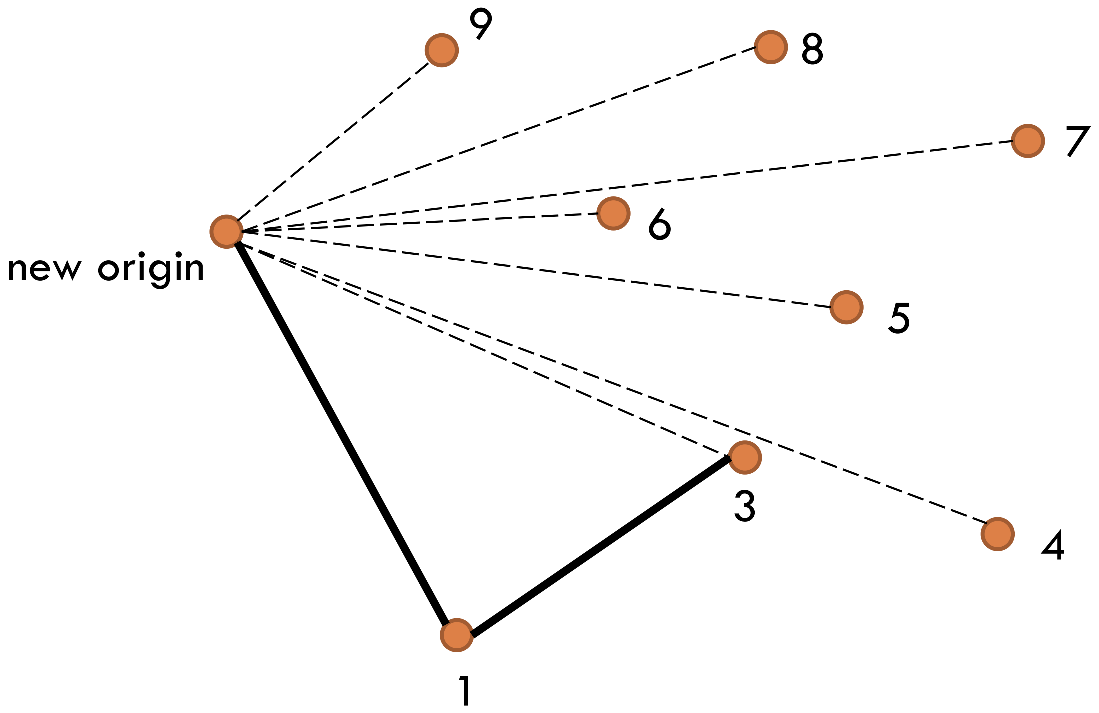
Example

- Adding point 3 causes a concave corner 1-2-3
  - Remove 2



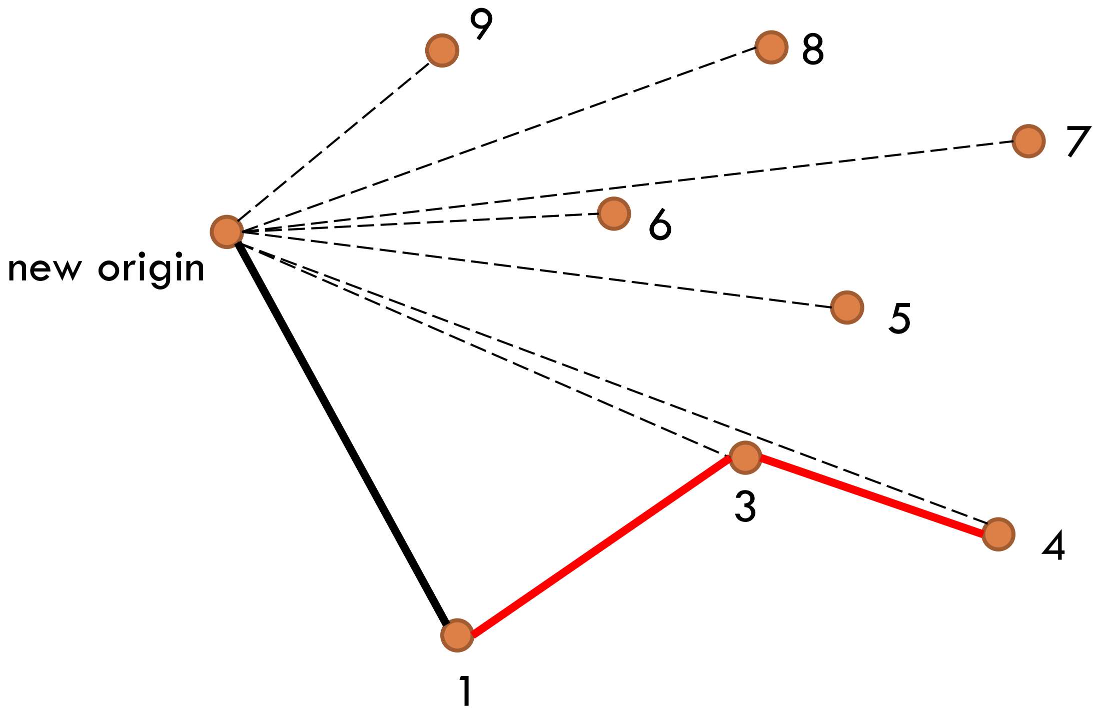
Example

- That's better…



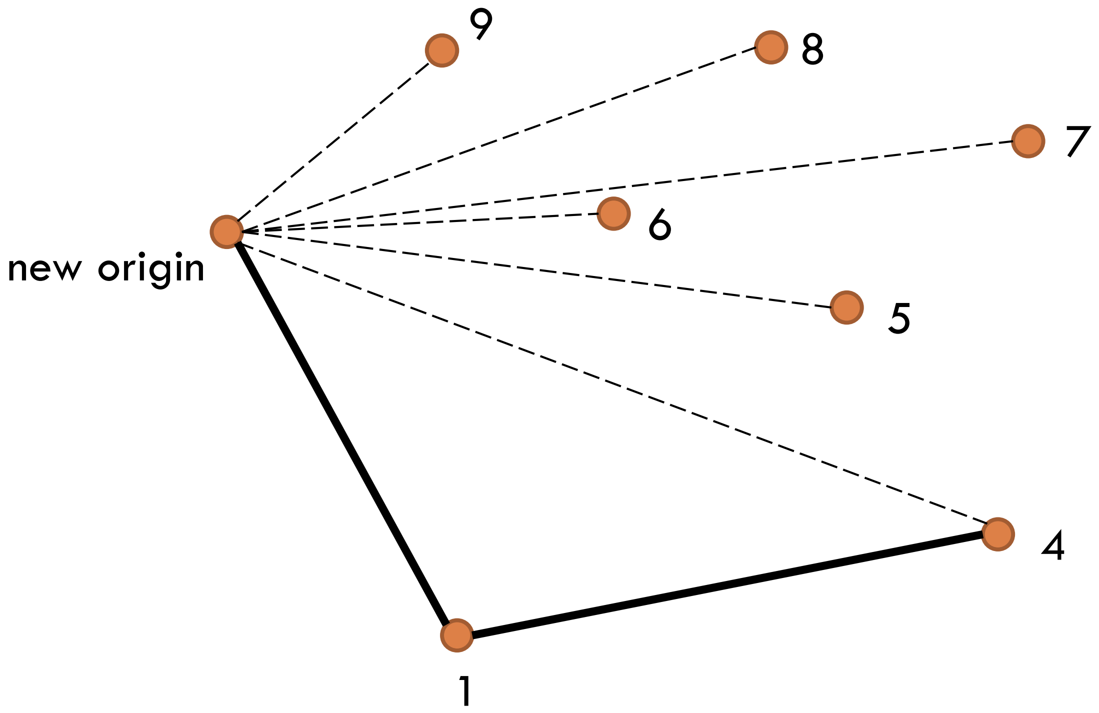
Example

- Adding 4 to the chain causes a problem
  - Remove 3



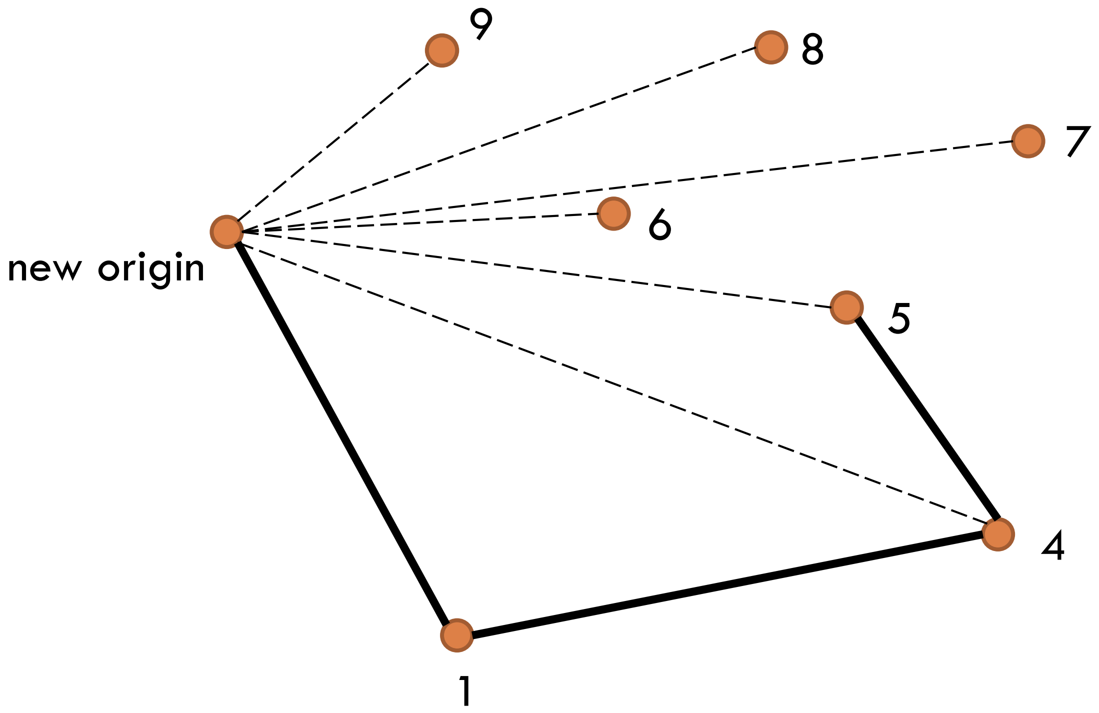
Example

- Continue adding points…



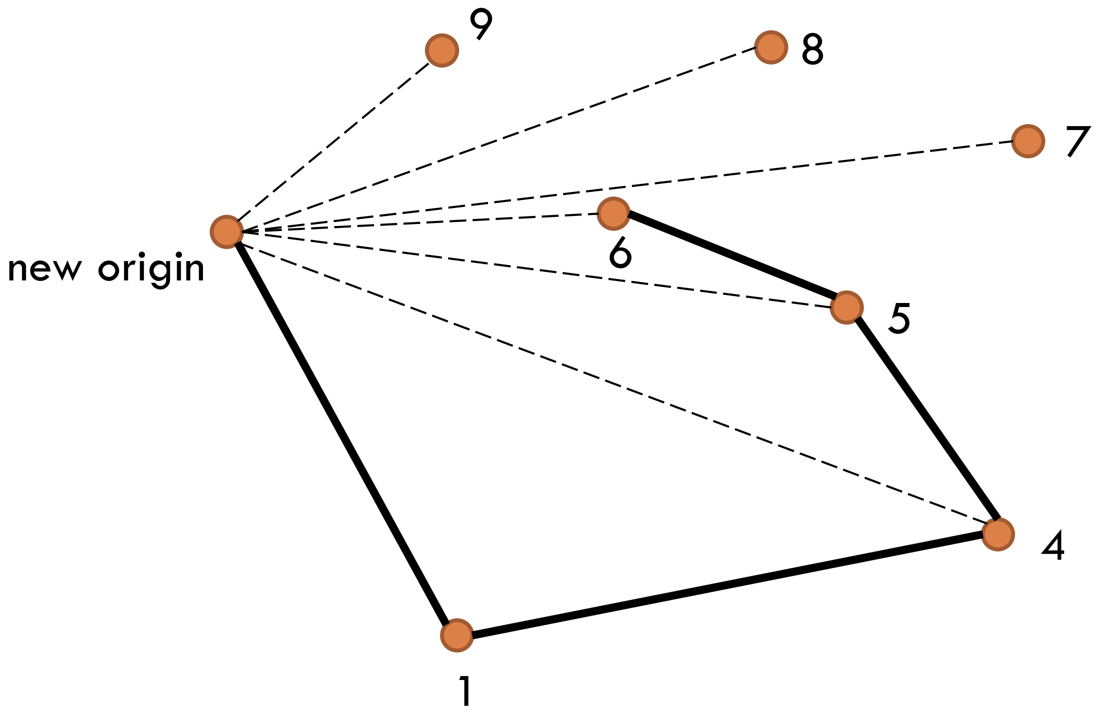
Example

- Continue adding points…



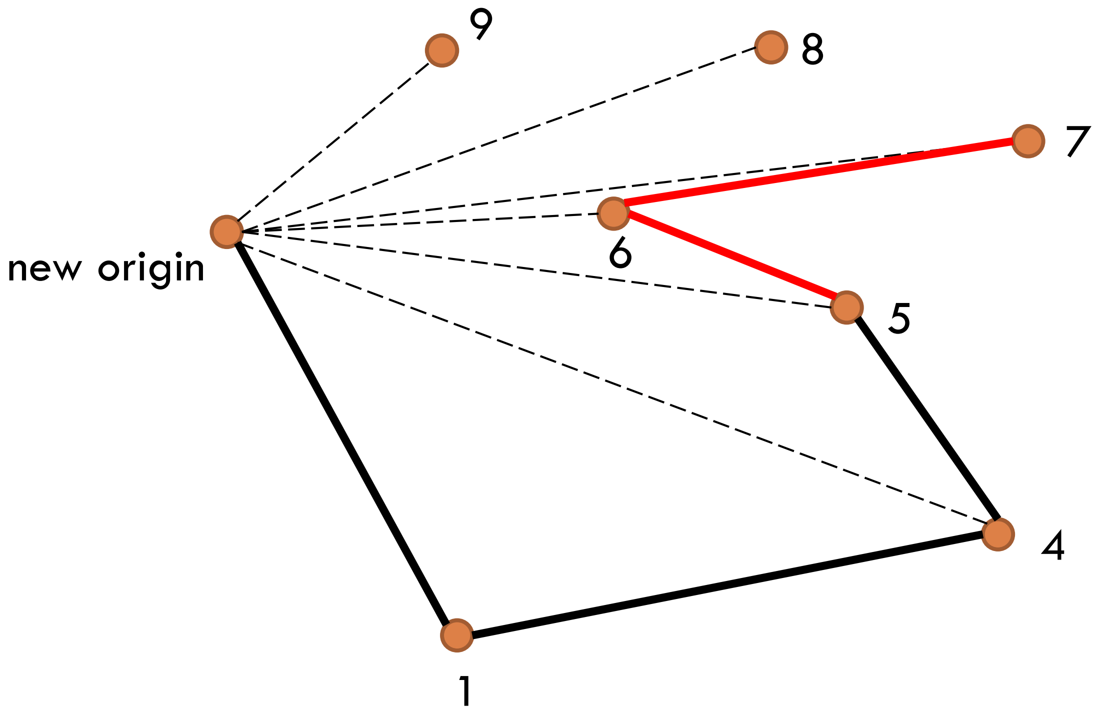
Example

- Continue adding points…



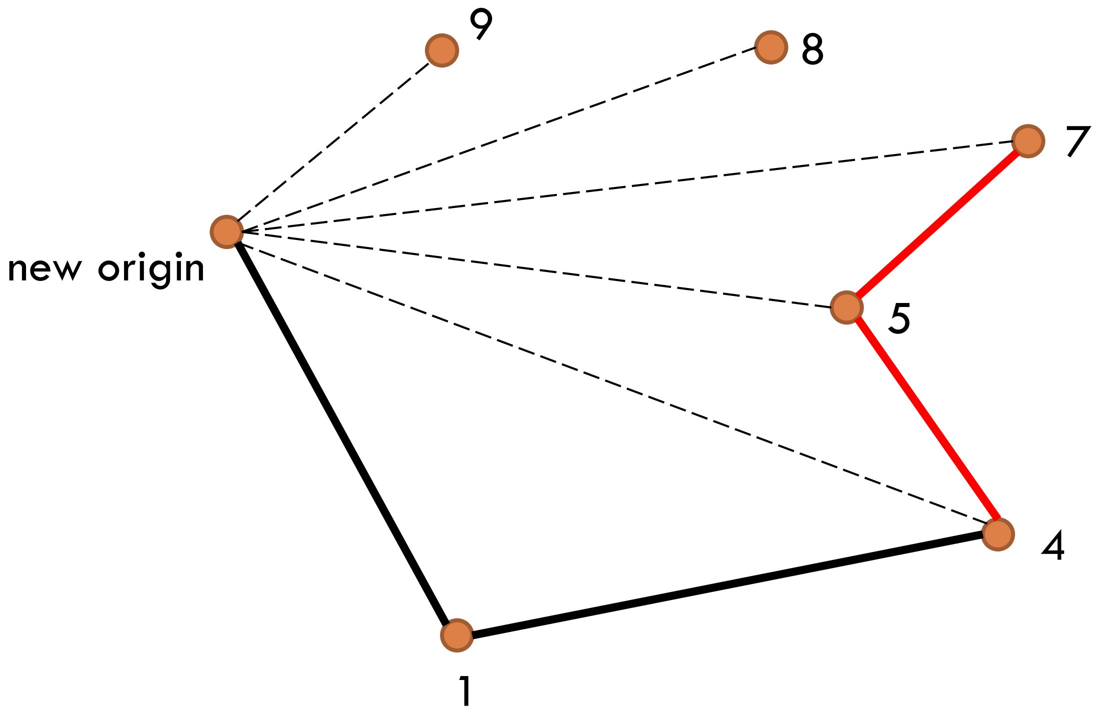
Example

- Bad corner!



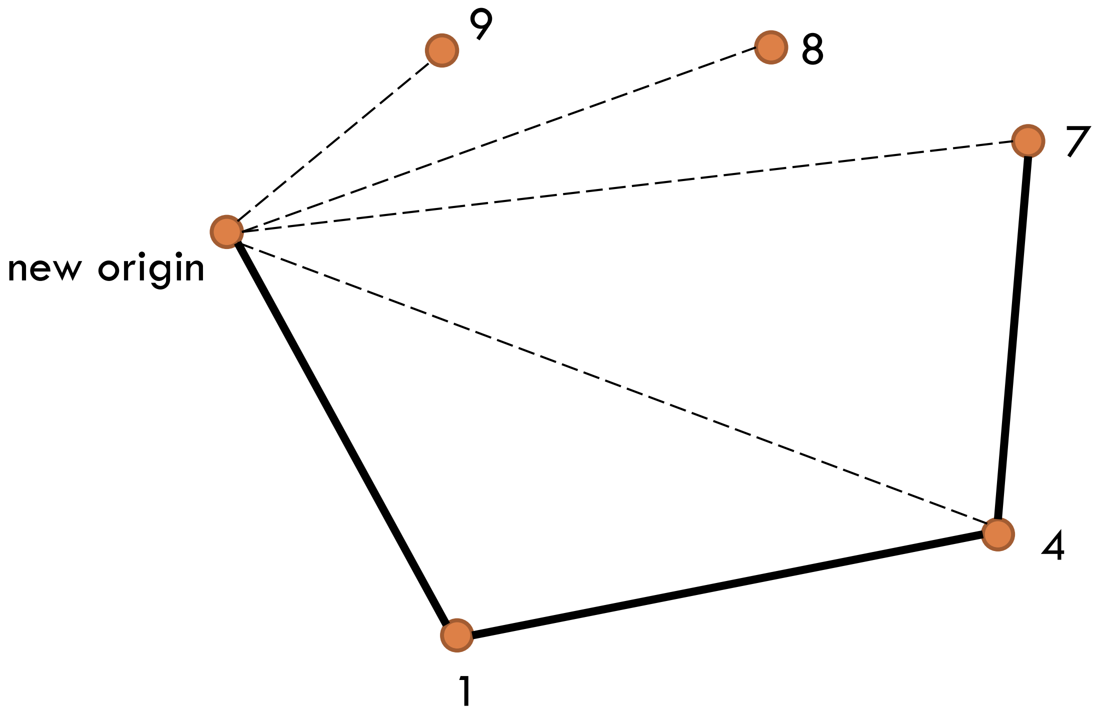
Example

- Bad corner again!



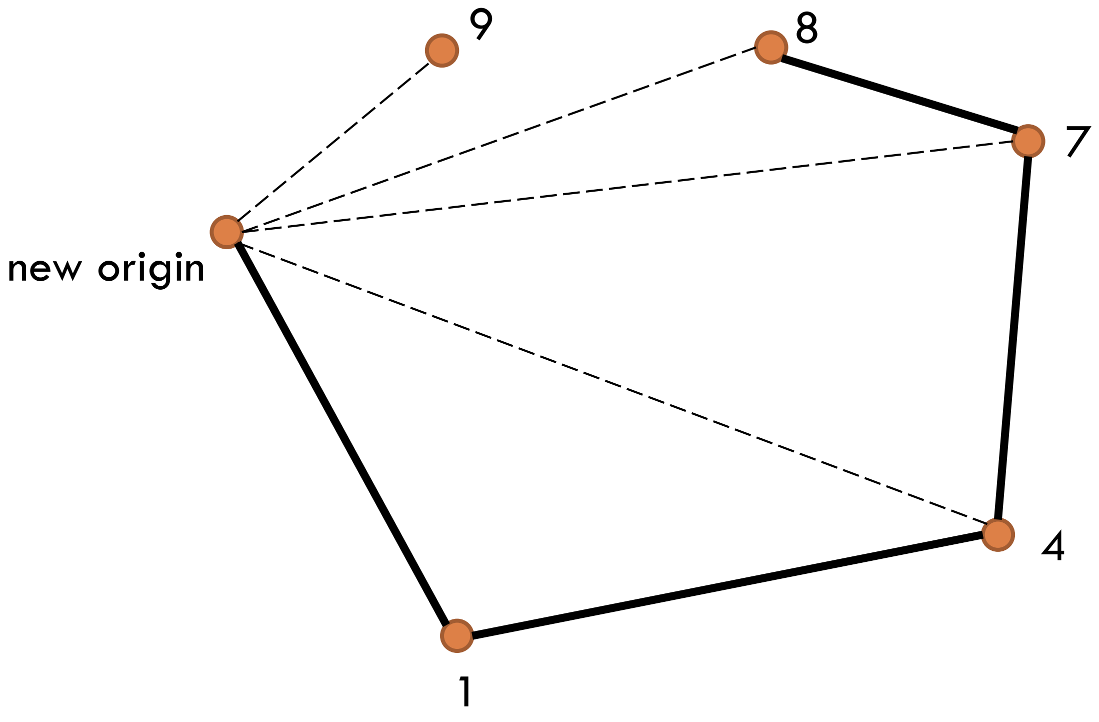
Example

- Continue adding points…



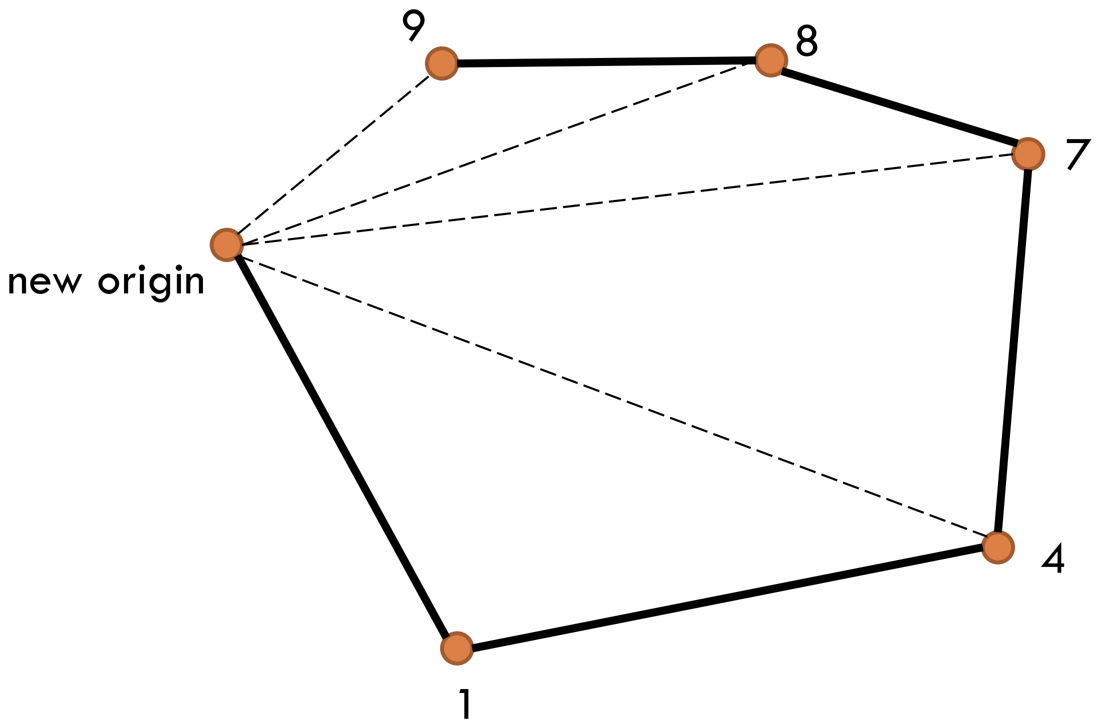
Example

- Continue adding points…



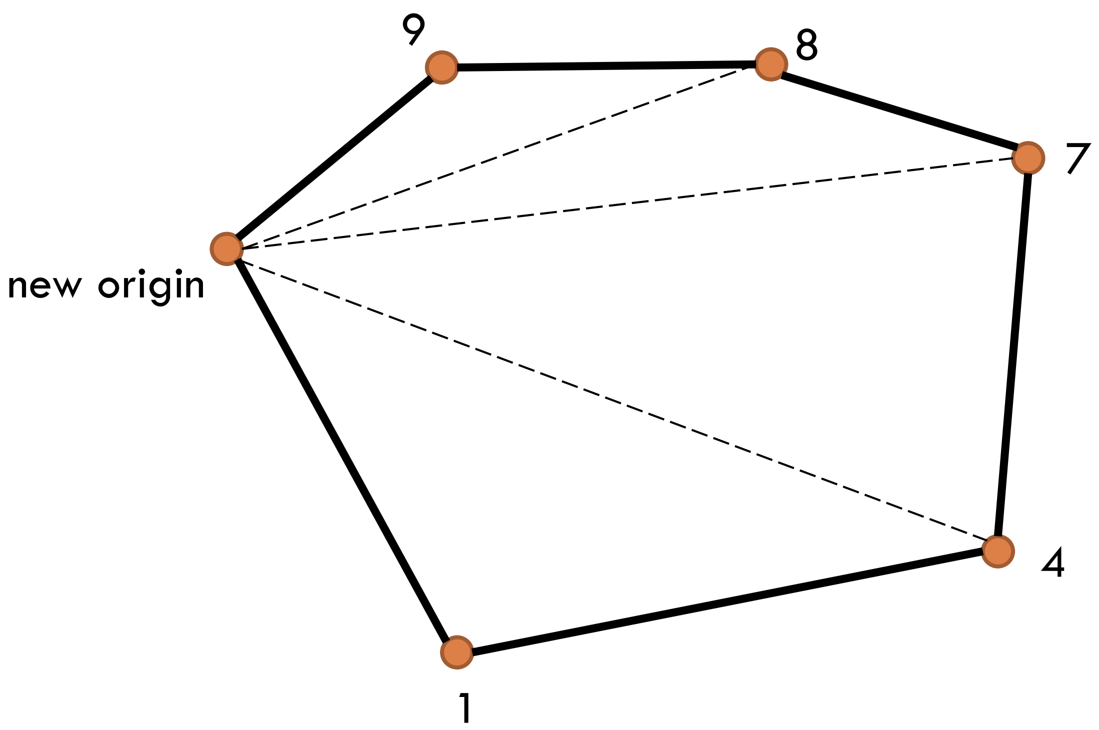
Example

- Continue adding points…



Example

- Done!



Pseudocode

- Set the leftmost point $(0,0)$, and sort the rest of the points in increasing order of y/x
- Initialize stack S
- For $i = 1 \dots n$:
  - Let A be the second topmost element of S, B be the topmost element of S, C be the ith point
  - If $\mathrm{ccw}(A, B, C) < 0$, pop S and go back
  - Push C to S
- Points in S form the convex hull

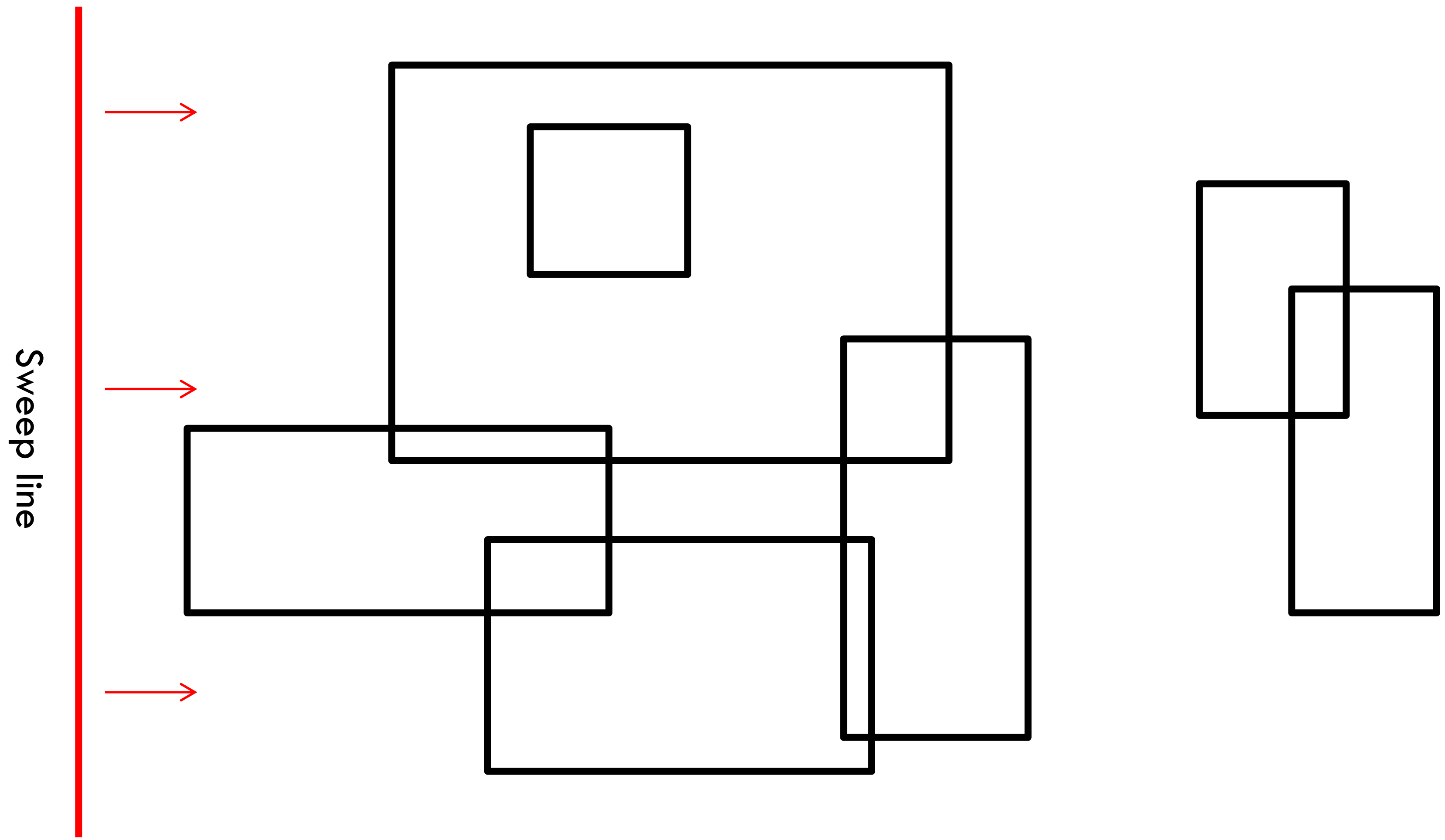
Sweep Line Algorithm

- A problem solving strategy for geometry problems
- The main idea is to maintain a line (with some auxiliary data structure) that sweeps through the entire plane and solve the problem locally
- We can't simulate a continuous process, (e.g. sweeping a line) so we define *events* that causes certain changes in our data structure
  - And process the events in the order of occurrence
- We'll cover one sweep line algorithm

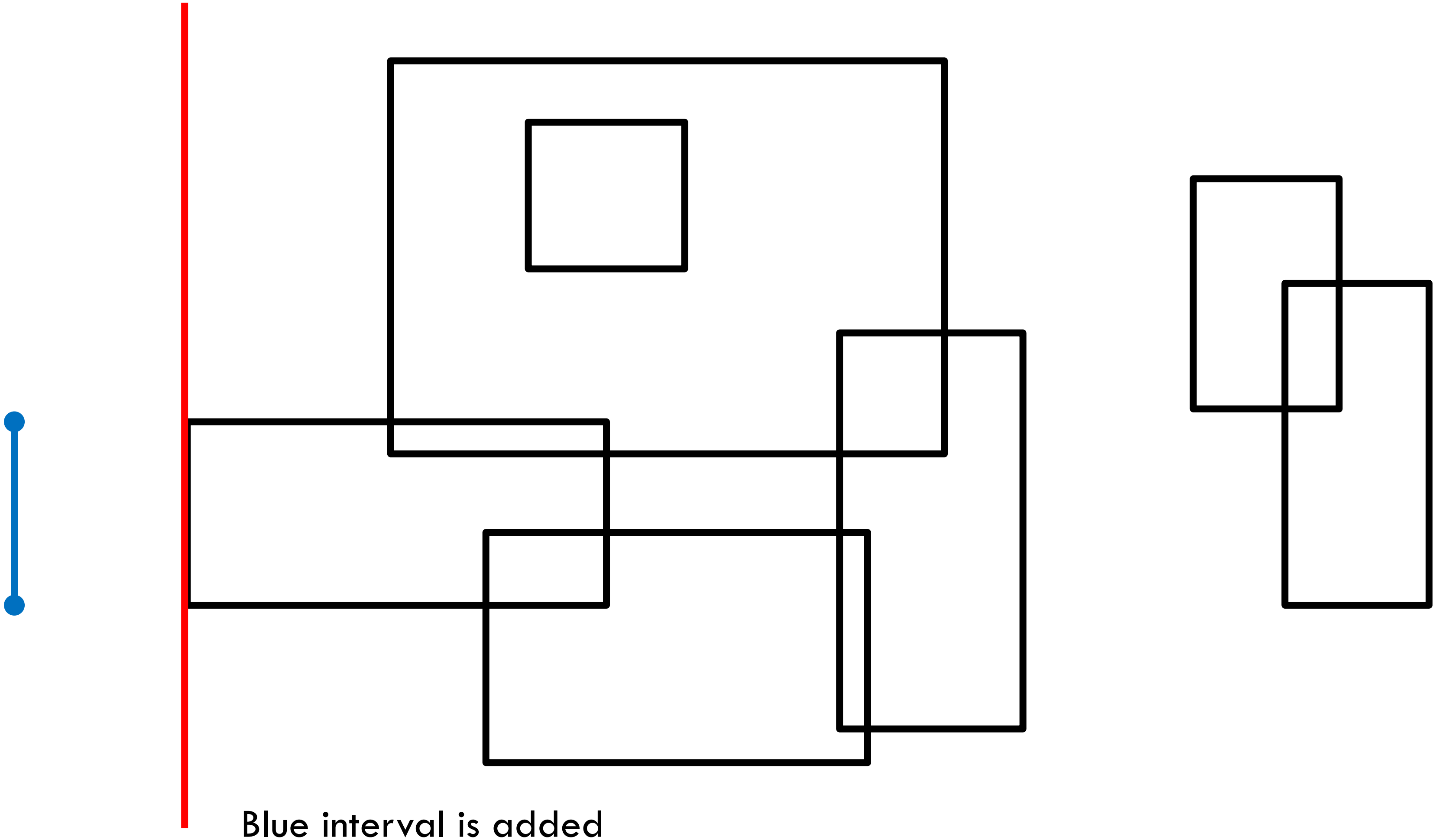
Sweep Line Algorithm

- Problem: Given n axis-aligned rectangles, find the area of the union of them
- We will sweep the plane from left to right
- Events: left and right edges of the rectangles
- The main idea is to maintain the set of "active" rectangles in order
  - It suffices to store the y-coordinates of the rectangles

Example

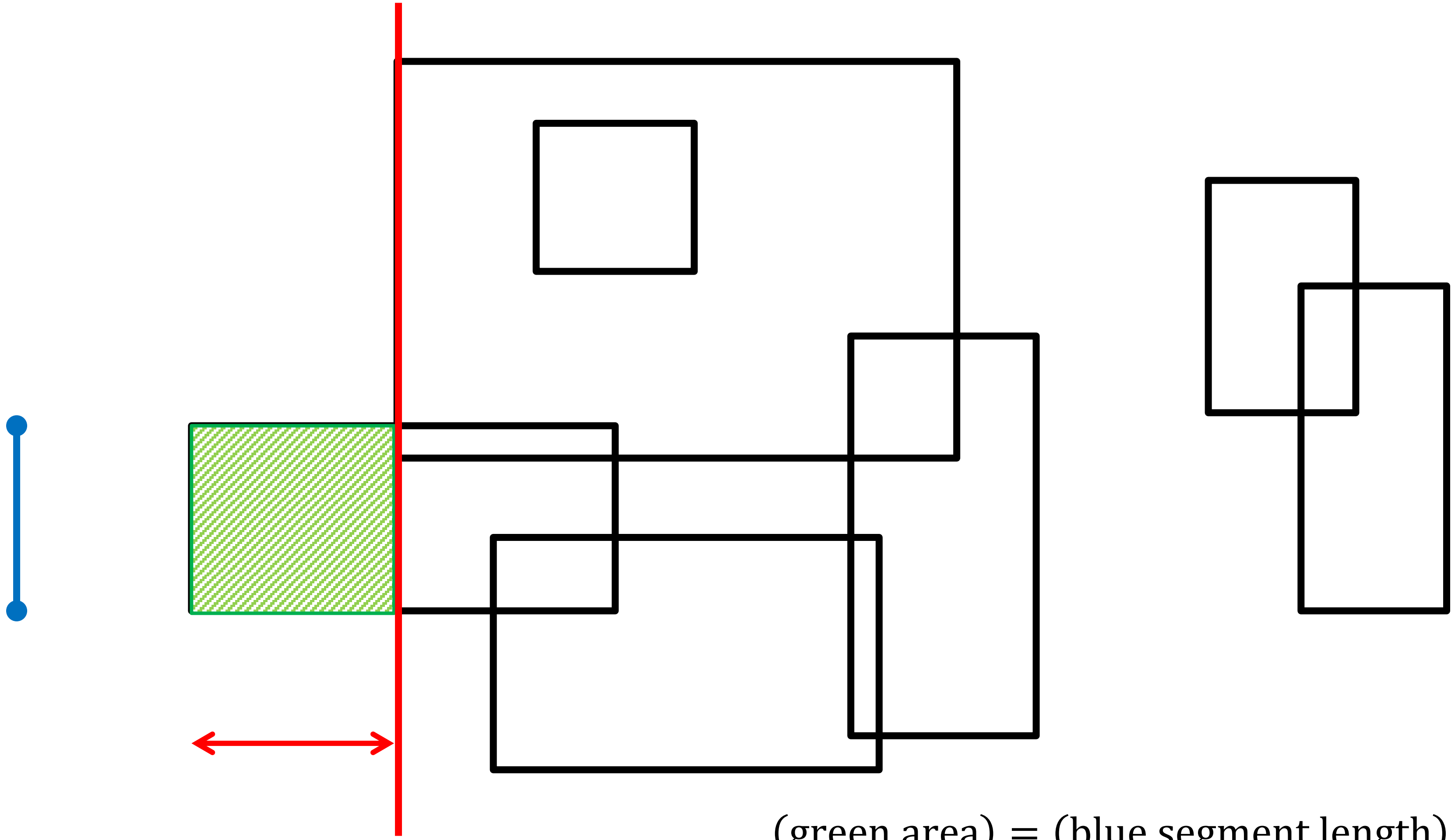


Example



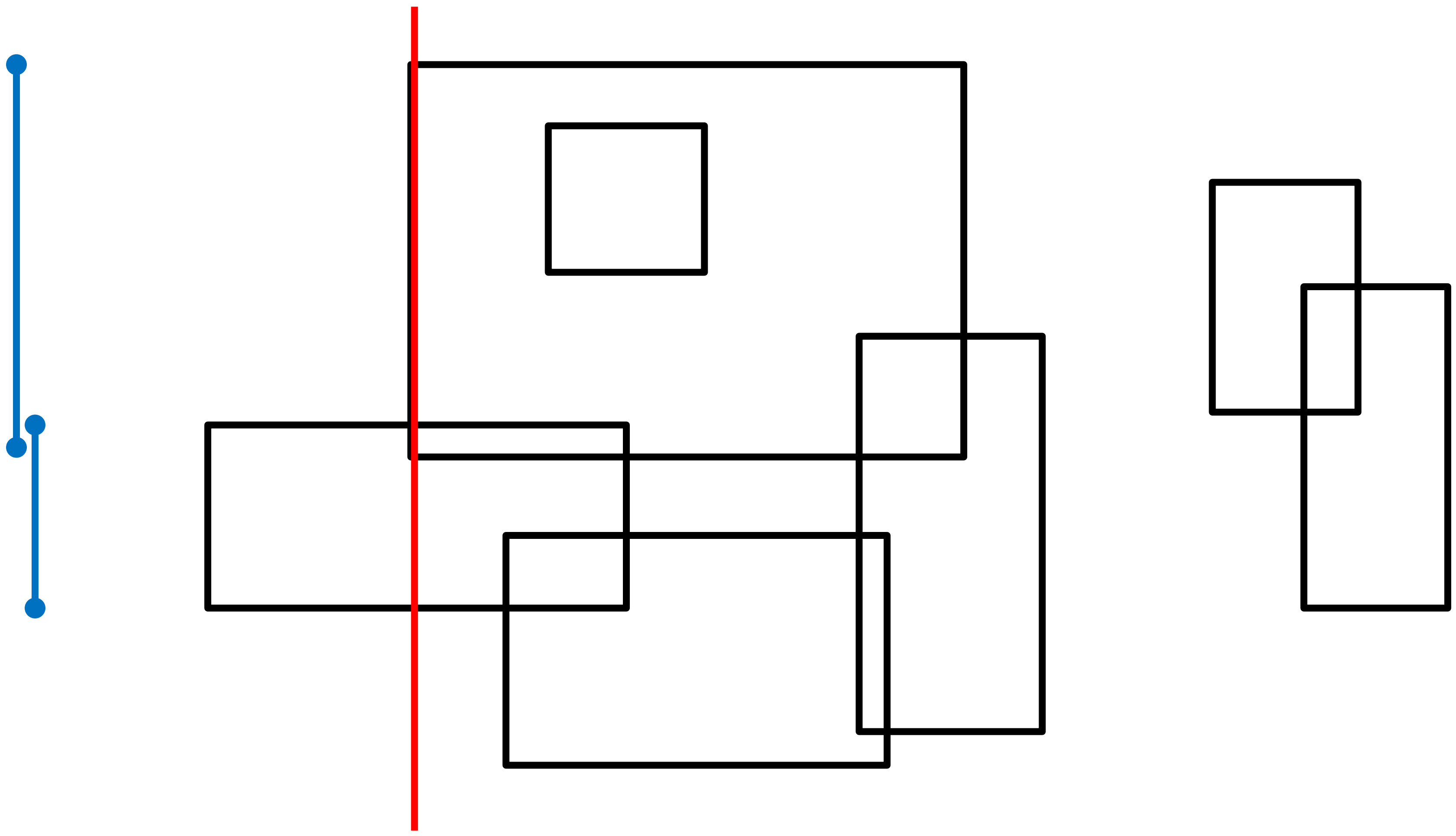
Blue interval is added
to the data structure

Example

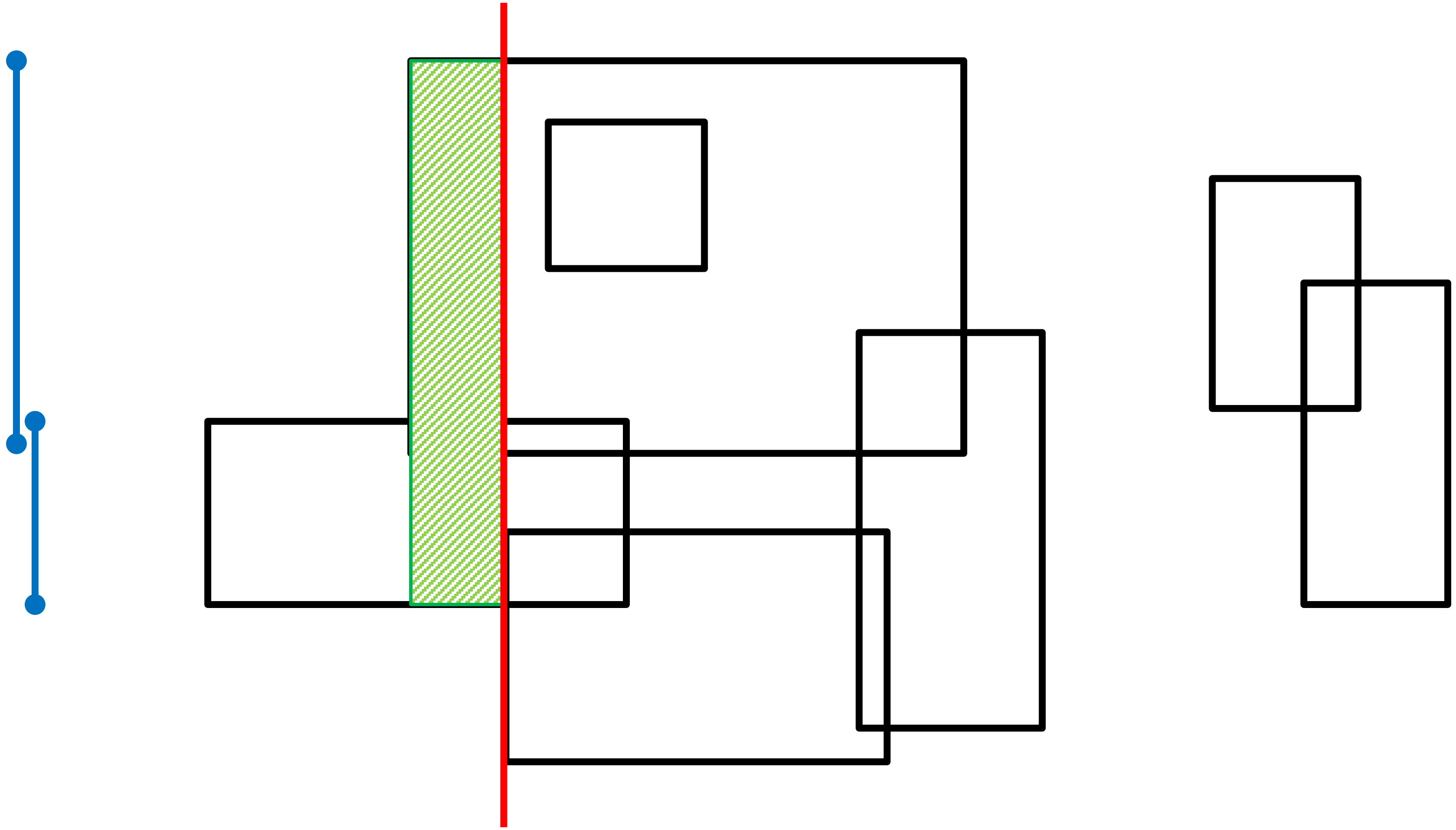


(green area) = (blue segment length)
× (distance that the sweep line traveled)

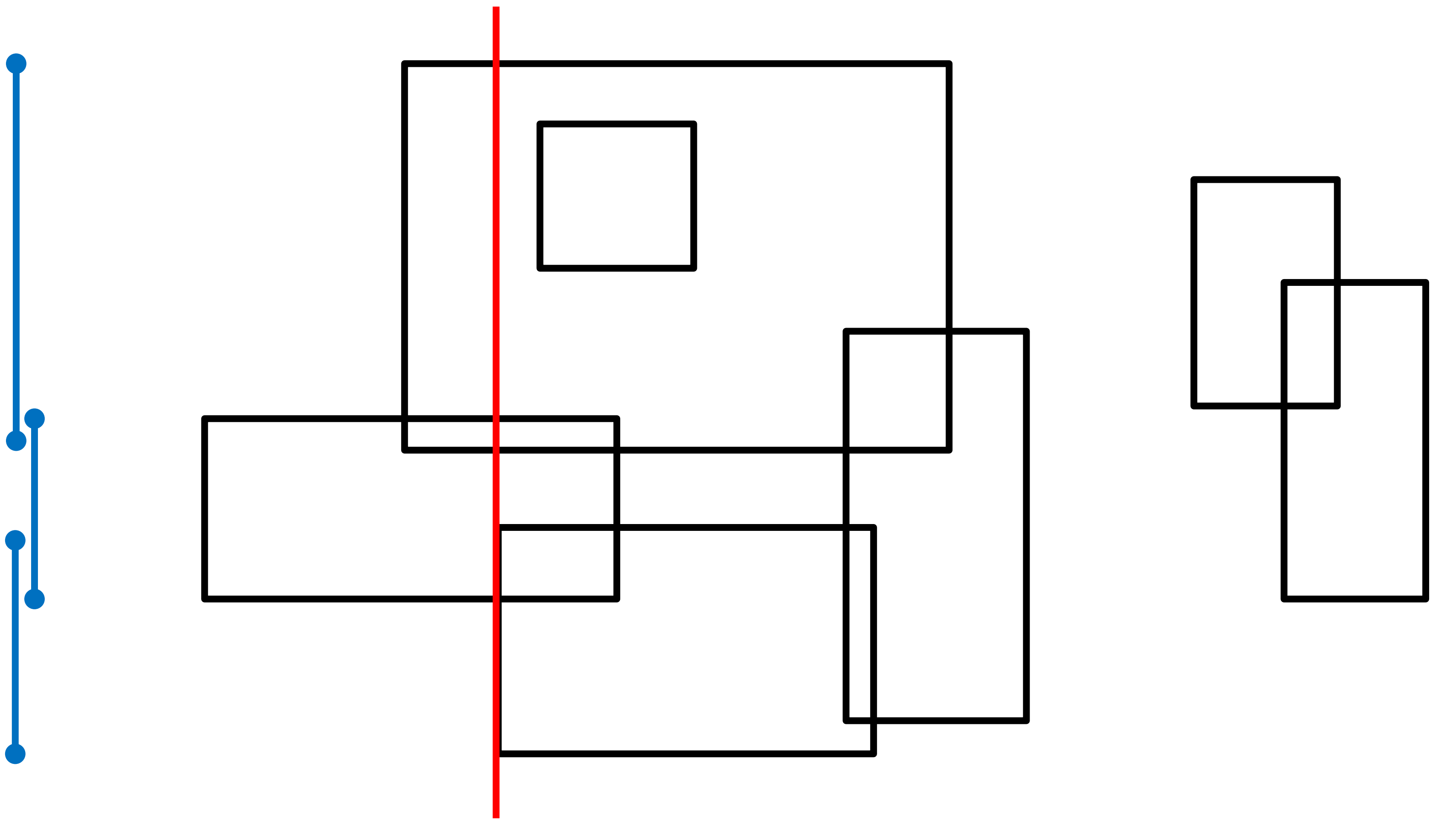
Example



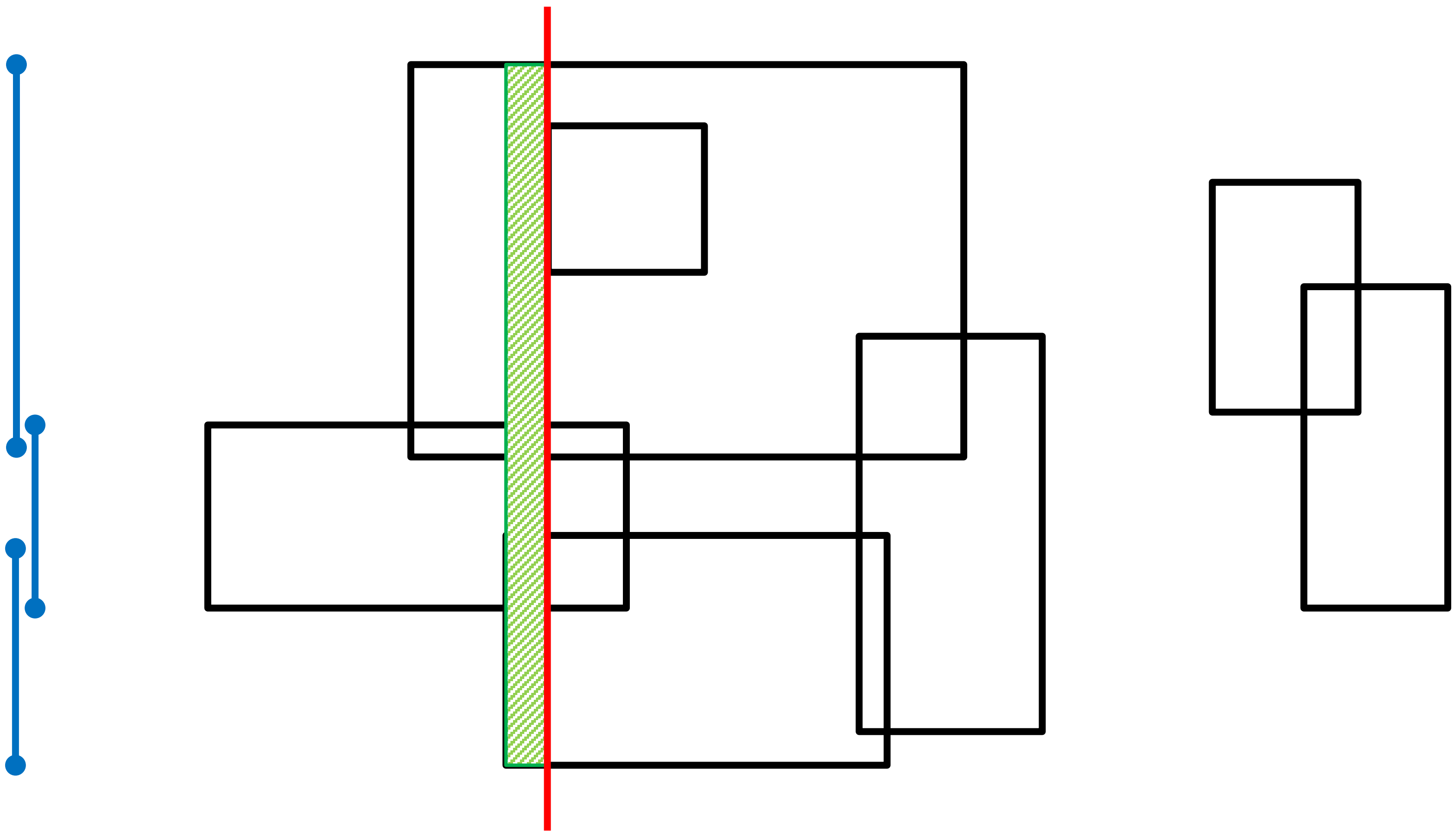
Example



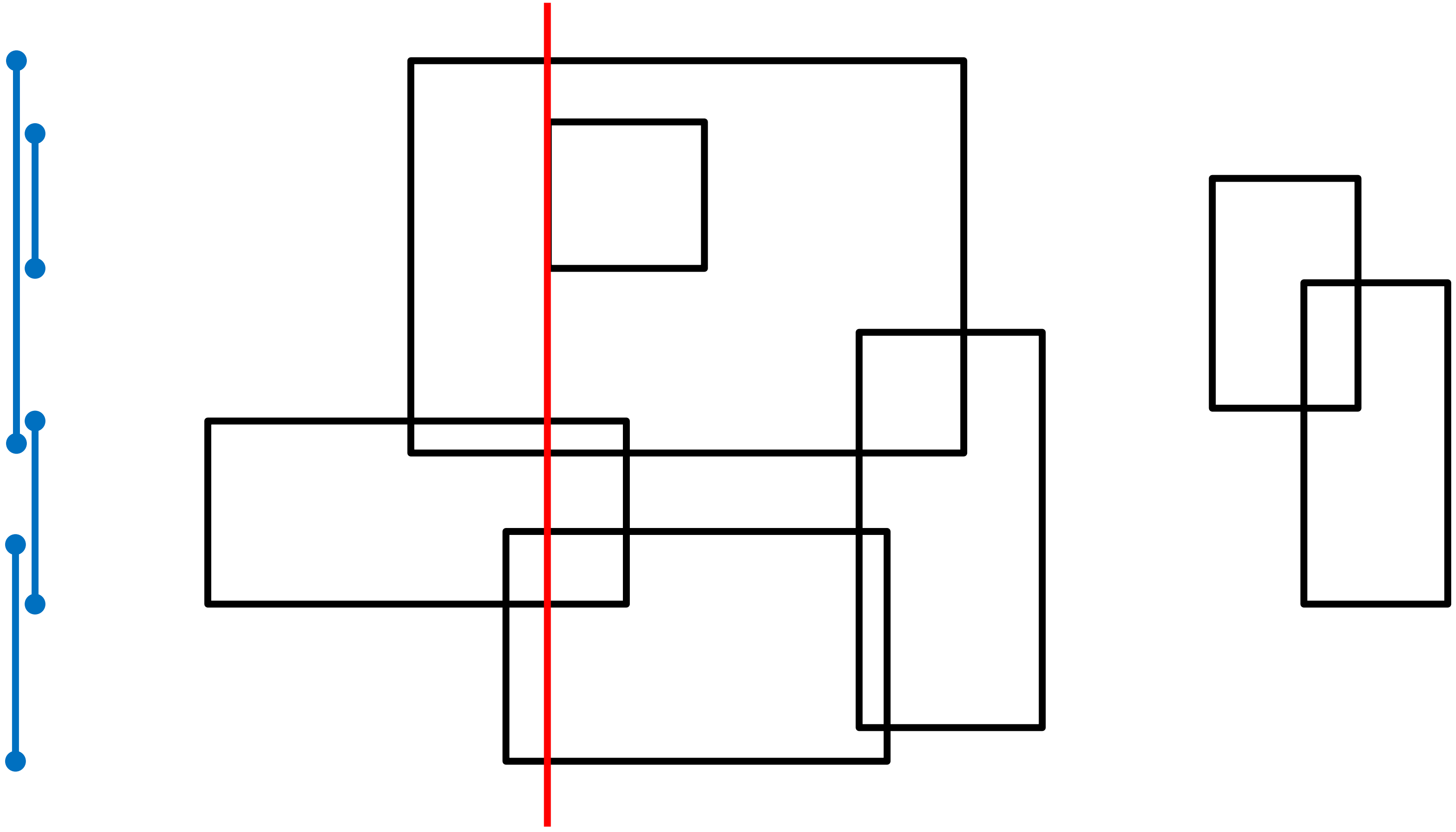
Example



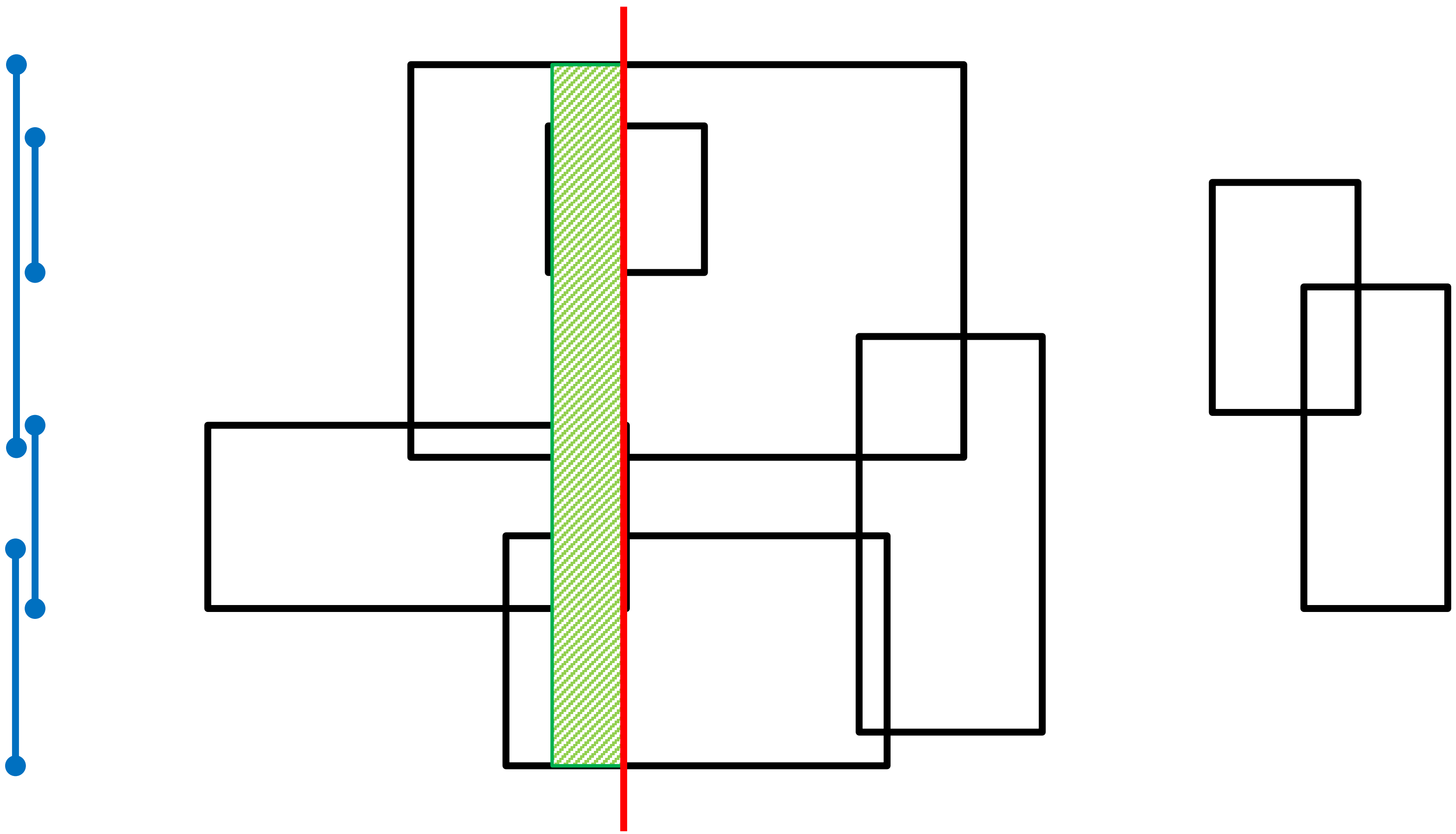
Example



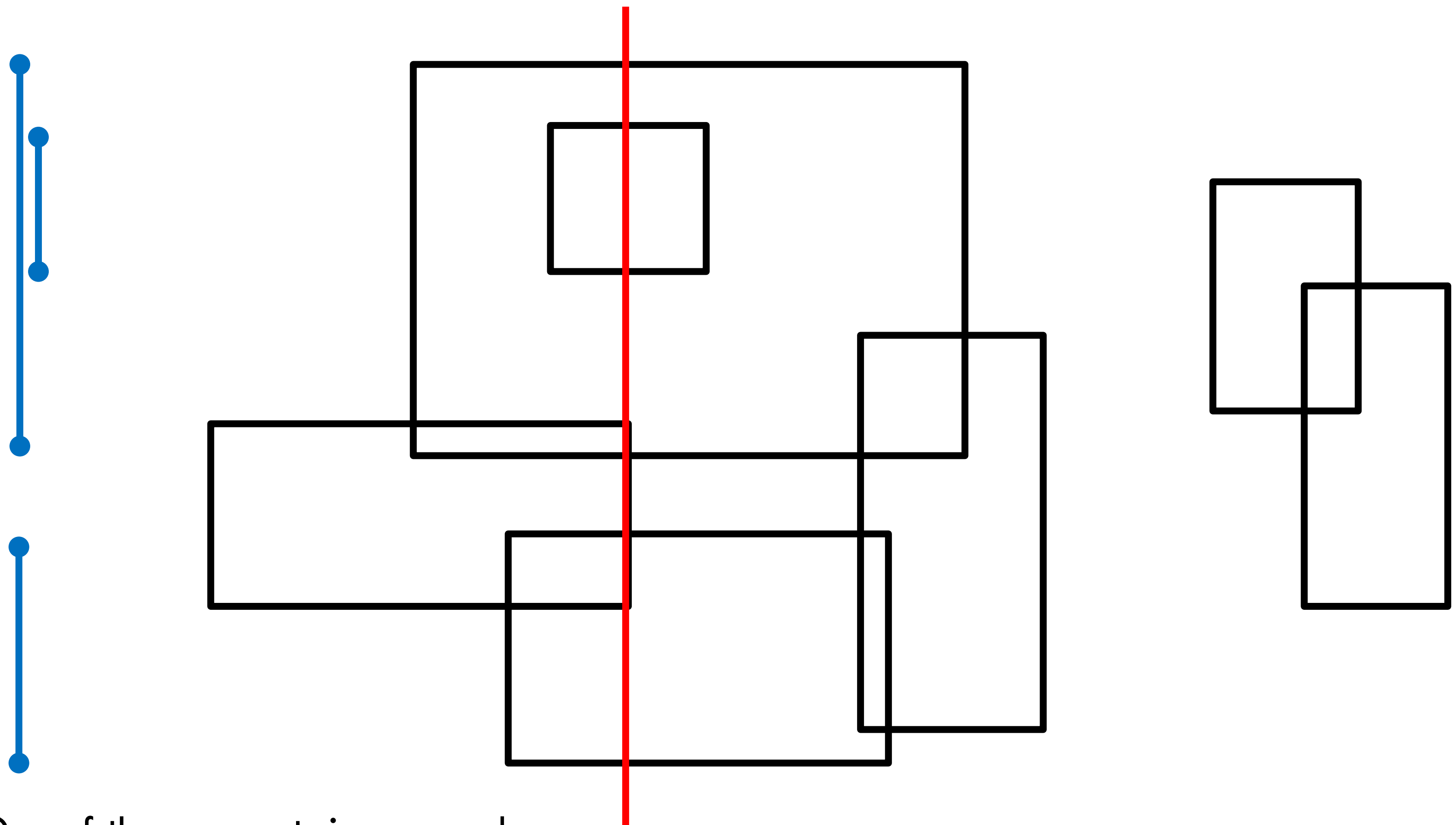
Example



Example

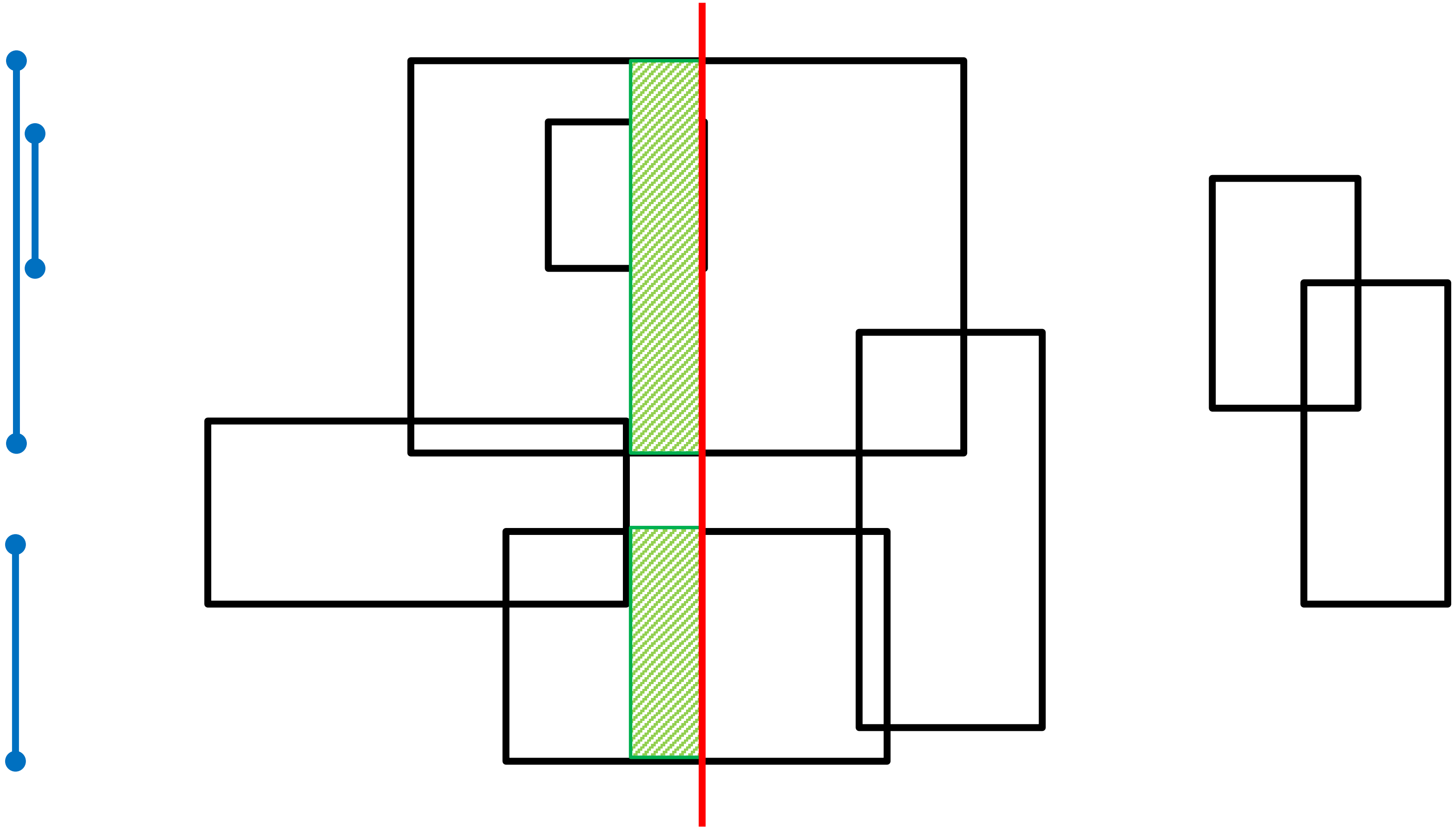


Example



One of the segments is removed

Example



Pseudopseudocode

- If the sweep line hits the left edge of a rectangle
  - Insert it to the data structure
- Right edge?
  - Remove it
- Move to the next event, and add the area(s) of the green rectangle(s)
  - Finding the length of the union of the blue segments is the hardest step
  - There is an easy $O(n)$ method for this step

Notes on Sweep Line Algorithms

- Sweep line algorithm is a generic concept
  - Come up with the right set of events and data structures for each problem
- Exercise problems
  - Finding the perimeter of the union of rectangles
  - Finding all k intersections of n line segments in $O\big((n+k)\log n\big)$ time

Intersecting Half-planes

- Representing a half-plane: $ax + by + c \le 0$
- The intersection of half-planes is a convex area
  - If the intersection is bounded, it gives a convex polygon
- Given n half-planes, how do we compute the intersection of them?
  - i.e. Find vertices of the convex area
- There is an easy $O(n^3)$ algorithm and a hard $O(n \log n)$ one
  - We will cover the easy one

Intersecting Half-planes

- For each half-plane $a_i x + b_i y + c_i \leq 0$, define a straight line $e_i: a_i x + b_i y + c_i = 0$
- For each pair of e_i and e_j:
  - Compute their intersection $p = (p_x, p_y)$
  - Check if $a_k p_x + b_k p_y + c_k \leq 0$ for all half-planes
    - If so, store p in some array P
    - Otherwise, discard p
- Find the convex hull of the points in P

Intersecting Half-planes

- The intersection of half-planes can be unbounded
  - But usually, we are given limits on the min/max values of the coordinates
  - Add four half-planes $x \geq -M$, $x \leq M$, $y \geq -M$, $y \leq M$ (for large M) to ensure that the intersection is bounded
- Time complexity: $O(n^3)$
  - Pretty slow, but easy to code

Note on Binary Search

- Usually, binary search is used to find an item of interest in a sorted array
- There is a nice application of binary search, often used in geometry problems
  - Example: finding the largest circle that fits into a given polygon
    - Don't try to find a closed form solution or anything like that!
    - Instead, binary search on the answer

Ternary Search

- Another useful method in many geometry problems
- Finds the minimum point of a “convex” function f
  - Not exactly convex, but let’s use this word anyway
- Initialize the search interval $[s, e]$
- Until $e - s$ becomes small:
  - $m_1 = s + (e - s)/3$, $m_2 = e - (e - s)/3$
  - If $f(m_1) \leq f(m_2)$, then set e to m_2
  - Otherwise, set s to m_1