Exercise 2.1

1. **The equation of plane is 2x − y + 2z = 4, find (i) the intercepts (ii) the distance from the origin to the plane.**

Solution: Given plane is, $2x - y + 2z = 4$ ...... (i)

$$\Rightarrow \frac{2x}{4} - \frac{y}{4} + \frac{2z}{4} = 1$$

$$\Rightarrow \frac{x}{2} + \frac{y}{-4} + \frac{z}{2} = 1 \quad \text{.......(ii)}$$

Comparing (ii) with the equation $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ then the intercept made by (i) are

$a = 2,\quad b = -4\quad$ and $\quad c = 2.$

And, distance from origin to (i) is

$$d = \left| \pm \frac{2.0 - 0 + 2.0 - 4}{\sqrt{2^2 + (-1)^2 + 2^2}} \right| \quad \left[\because d = \left| \pm \frac{ax_1 + by_1 + cz_1 - d}{\sqrt{a^2 + b^2 + c^2}} \right|\right]$$

$$= \left| \pm \frac{-4}{\sqrt{9}} \right| = \frac{4}{3}$$

Thus distance of plane from origin is $\frac{4}{3}$.

2. **Find the intercepts made on the co-ordinate axes by the plane x + 2y − 2z = 9. Find also the direction cosines of the normal to this plane.**

Solution: Given plane is

$x + 2y - 2y = 9$ ...... (i)

$$\Rightarrow \frac{x}{9}+\frac{y}{9/2}+\frac{z}{-9/2}=1 \qquad \text{...... (ii)}$$

Comparing (ii) with the equation $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ then the intercepts made by (i) be

$a=9,\quad b=\frac{9}{2},\quad$ and $\quad c=\frac{-9}{2}$

And, comparing (i) with $ax+by+cz=d$ then we get,

$a=1,\quad b=2,\quad c=-2\quad$ and $d=9$

Then the direction cosines of the line normal to (i) are

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}} \qquad m=\frac{b}{\sqrt{a^2+b^2+c^2}} \qquad n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow l=\frac{1}{3} \qquad \Rightarrow m=\frac{2}{3} \qquad \Rightarrow n=\frac{-2}{3}$$

Thus, the direction cosines are $l=\frac{1}{3}$, $m=\frac{2}{3}$, $n=\frac{-2}{3}$.

3. Find the direction cosines of the line normal to the plane $6x-3y+2z=14$.

Solution: Given plane is, $\quad 6x-3y+2z=14 \qquad$ ...... (i)

Comparing (i) with $ax+by+cz=0$ then we get

$a=6,\quad b=-3,\quad c=2\quad$ and $\quad d=14.$

Then the direction cosines of the line normal to (i) are

$$\Rightarrow l=\frac{a}{\sqrt{a^2+b^2+c^2}} \qquad \Rightarrow m=\frac{b}{\sqrt{a^2+b^2+c^2}} \qquad \Rightarrow n=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow l=\frac{6}{\sqrt{36+9+4}} \qquad \Rightarrow m=\frac{-3}{\sqrt{36+9+4}} \qquad \Rightarrow n=\frac{2}{\sqrt{36+9+4}}$$

$$\Rightarrow l=\frac{6}{7} \qquad \Rightarrow m=\frac{-3}{7} \qquad \Rightarrow n=\frac{2}{7}$$

Thus, the direction cosines of the line normal to (i) are

$$l=\frac{6}{7},\quad m=-\frac{3}{7},\quad n=\frac{2}{7}.$$

4. The direction cosines of the line perpendicular to a plane from the origin are proportional to 1, 3, 1 and the length of the perpendicular is 2. Find the equation of the plane.

Solution: Given that the direction cosines of the line perpendicular to the plane are proportional to 1, 3, 1.

Then, the direction ratios of the plane are

$$a=\frac{1}{\sqrt{1^2+3^2+1^2}}=\frac{1}{\sqrt{11}},\ b=\frac{3}{\sqrt{1^2+3^2+1^2}}=\frac{3}{\sqrt{11}},\ c=\frac{1}{\sqrt{1^2+3^2+1^2}}$$

Also given that the length of perpendicular is 2. So, $P=2$.

Since we know that the equation of plane having direction ratios a, b, c and length of perpendicular p is

$$ax+by+cz=p \quad \Rightarrow \frac{x+3y+z}{\sqrt{11}}=2$$

$$\Rightarrow x+3y+z=\frac{2}{\sqrt{11}}.$$

5. Find the equation of the plane containing the lines through the origin with direction cosines proportional to 1, –2, 2 and 2, 3, –1.

Solution: We have any plane through the origin is

$$ax+by+cz=0 \qquad \text{...... (i)}$$

Since the plane (i) contains a line having direction cosines proportional to 1, -2, 2 and 2, 3, -1.

Clearly, the lines are perpendicular to (i)

So, using the condition of perpendicularity,

$$a-2b+2c=0 \qquad \text{...... (ii)}$$

and $\quad 2a+3b-c=0 \qquad$ ...... (iii)

Solving (ii) and (iii) we get,

$$\frac{a}{2-6}=\frac{b}{4+1}=\frac{c}{3+4}=k\ (\text{say})$$

$$\Rightarrow a=-4k,\ b=5k,\ c=7k$$

Then (i) becomes,

$$4x-5y-7z=0 \qquad [\because -k\neq 0]$$

This is equation of required plane.

Exercise 2.2

1. Find the equation of the plane through (1, 1, 1) and parallel to the plane $3x-4y+5z=0$.

Solution: Since the equation of plane that is parallel to $3x-4y+5z=0$ is

$$3x-4y+5z=k \qquad \text{...... (i)}$$

where k is some constant value.

Also, given that (i) passes through (1, 1, 1). So,

$$3-4+5=k \Rightarrow k=4$$

Thus (i) becomes, $3x-4y+5z=4$

This is required plane.

2. Find the angle between the following pair of planes: $x+3y+5z=0$ and $x-2y+z=20$.

Solution: Given planes are

$$x+3y+5z=0 \qquad \text{...... (i)}$$

and $\quad x-2y+z=20 \qquad$ ...... (ii)

Clearly, the direction ratios of (i) and (ii) are respectively,

1, 3, 5 and 1, -2, 1

So, the direction cosines of the line normal to (i) and (ii) are respectively,

$$\frac{1}{\sqrt{1^2+3^2+5^2}}, \frac{3}{\sqrt{1^2+3^2+5^2}}, \frac{5}{\sqrt{1^2+3^2+5^2}}$$

and $\frac{1}{\sqrt{1+4+1}}, \frac{-2}{\sqrt{1+4+1}}, \frac{1}{\sqrt{1+4+1}}$

i.e. $\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}$ and $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$

Let θ be the angle between (i) and (ii). Since, we have if θ be the angle between the planes having direction cosines of the lines normal to the plane are l_1, m_1, n_1 and l_2, m_2, n_2 then

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

So, the angle between (i) and (ii)

$$\cos\theta = \frac{1}{\sqrt{35}.\sqrt{6}}(1-6+5) = \frac{1}{\sqrt{35}.\sqrt{6}}.0 = 0 = \cos\frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

Thus angle between (i) and (ii) is $\theta = \frac{\pi}{2}$.

3. **Find the equation of the plane passing through the origin and containing the line joining the points (1, 1, 1) and (3, 4, -5).**

Solution: Since the equation of plane passing through the origin is

$$ax + by + cz = 0 \quad \text{...... (i)}$$

And, the direction ratios of the line joining (1, 1, 1) and (3, 4, -5) be

$a = 3 - 1 = 2$, $b = 4 - 1 = 3$, $c = -5 - 1 = -6$

So, the direction cosines of the line perpendicular to the line joining (1, 1, 1) and (3, 4, -5) are

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{2}{7}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{3}{7}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}} = -\frac{6}{7}$$

4. **Find the equation of the plane containing the point (1, -1, 2) and is perpendicular to the planes 2x + 3y – 2z = 5 and x + 2y – 3z = 8.**

Solution: Since the equation of plane containing the point (1, –1, 2) is

$$a(x-1) + b(y+1) + c(z-2) = 0 \quad \text{...... (i)}$$

Given that the plane (i) is perpendicular to the planes,

$$2x + 3y - 2z = 5 \quad \text{...... (ii)}$$

and $x + 2y - 3z = 8$ ...... (iii)

Then by condition of perpendicularity, we have

$$2a + 3b - 2c = 0 \quad \text{...... (iv)}$$

and $a + 2b - 3c = 0$ ...... (v)

Solving (iv) and (v) we get,

$$\frac{a}{-9+4} = \frac{b}{-2+6} = \frac{c}{4-3} = k \text{ (say)}$$

$$\Rightarrow a = -5k, b = 4k, c = k$$

Then (i) becomes,

$$5(x-1) - 4(y+1) - (z-2) = 0 \quad [\because -k \neq 0]$$

$$\Rightarrow 5x - 4y - z = 7$$

This is required plane.

5. **Find the equation of the plane through P(1, 4, -2) at right angle to OP.**

Solution: Since the direction ratios of the line of where P(1, 4, -2) is

1 – 0, 4 – 0, -2 – 0 i.e. 1, 4, -2.

So, the direction cosines of the line OP is

$$l = \frac{1}{\sqrt{1+16+4}} = \frac{1}{\sqrt{21}}, \quad m = \frac{4}{\sqrt{21}}, \quad n = \frac{-2}{\sqrt{21}}$$

Since we have the direction cosines of the line normal to the plane, are the direction ratios of the plane.

Now, equation of plane through (1, 4, -2) and having direction ratios $\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{-2}{\sqrt{21}}$ is,

$$\boxed{a(x-x_1) + b(y-y_1) + c(z-z_1) = 0}$$

i.e. $\frac{1}{\sqrt{21}}(x-1) + \frac{4}{\sqrt{21}}(y-4) - \frac{2}{\sqrt{21}}(z+2) = 0$

$$\Rightarrow (x-1) + 4(y-4) - 2(z+2) = 0$$

$$\Rightarrow x + 4y - 2z = 21$$

This is the equation of required plane.

Exercise 2.3

1. **Find the equation of plane through (4, 5, 1), (3, 9, 4), (–4, 4, 4).**

Solution: The equation of plane through (4, 5, 1), (3, 9, 4), (–4, 4, 4) be

$$\begin{vmatrix} x & y & z & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \\ -4 & 4 & 4 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x\begin{vmatrix} 5 & 1 & 1 \\ 9 & 4 & 1 \\ 4 & 4 & 1 \end{vmatrix} - y\begin{vmatrix} 4 & 1 & 1 \\ 3 & 4 & 1 \\ -4 & 4 & 1 \end{vmatrix} + z\begin{vmatrix} 4 & 5 & 1 \\ 3 & 9 & 1 \\ -4 & 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & 5 & 1 \\ 3 & 9 & 4 \\ -4 & 4 & 4 \end{vmatrix}$$

$= 0$

$$\Rightarrow x[5(4-4) - 1(9-4) + 1(36-16)] - y[4(4-4) - 1(3+4) + 1(12+16)] + z[4(9-4) - 5(3+4) + 1(12+36)] - 1[4(36-16) - 5(12+16) + 1(12+36)] = 0$$

$$\Rightarrow x(-5+20) - y(-7+28) + z(20-35+48) - 1(80-140+48) = 0$$

$$\Rightarrow 15x - 21y + 33z + 12 = 0$$

$$\Rightarrow 5x - 7y + 11z + 4 = 0 \qquad [\because 3 \neq 0]$$

This is equation of required plane.

2. Find the equation of the plane through the three points (1, 1, 1), (1, -1, 1), (-7, 3, -5) and show that it is perpendicular to xz-plane.

Solution: The equation of plane through (1, 1, 1), (1, -1, 1) and (-7, 3, -5) be

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ -7 & 3 & -5 & 1 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_3$ then

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ -7 & 3 & -5 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x[2(1+5)] - y\times 0 + z[-2(1+7)] - 1[-2(-5+7)] = 0$$

$$\Rightarrow 12x - 16z + 4 = 0$$

$$\Rightarrow 3x - 4z + 1 = 0 \qquad [\because 4 \neq 0]$$

This is the equation of required plane.

3. Show that the points (1, 3, -1), (1, 1, 0), (2, 5, 4) and (2, 7, 3) are coplanar.

Solution: Given points are (1, 3, - 1), (1, 1, 0), (2, 5, 4) and (2, 7, 3).

Since the points are coplanar if

$$\begin{vmatrix} 1 & 3 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 5 & 4 & 1 \\ 2 & 7 & 3 & 1 \end{vmatrix} = 0$$

Here,

$$\begin{vmatrix} 1 & 3 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 5 & 4 & 1 \\ 2 & 7 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 5 & 4 & 1 \\ 2 & 7 & 3 & 1 \end{vmatrix} \qquad [\because \text{Applying } R_1 \to R_1 - R_2]$$

$$= 0 - 2\begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 1 \\ 2 & 7 & 1 \end{vmatrix} - 0$$

$$= -2[1(4-3) - 0 + 1(6-8)] - 1[1(5-7) - 1(2-2) + 1(14-10)]$$

$$= -2(1-2) - 1(-2+4) = -2(-1) - 1(2)$$

$$= 2 - 2 = 0$$

This shows that the given points are coplanar.

4. Find the equation of the plane through the points (-1, 1, -1), (6, 2, 1) and normal to the plane 2x + y + z = 5.

Solution: Since the equation of plane through (-1, 1, -1) is

$$a(x+1) + b(y-1) + c(z+1) = 0 \qquad \text{...... (i)}$$

Also, (i) passes through (6, 2, 1). Then

$$a(6+1) + b(2-1) + c(1+1) = 0$$

$$\Rightarrow 7a + b + 2c = 0 \qquad \text{...... (ii)}$$

Again, given that, the plane (i) is normal to the plane 2x + y + z = 5. So,

$$2a + b + c = 0 \qquad \text{...... (iii)}$$

Solving (ii) and (iii) we get,

$$\frac{a}{1-2} = \frac{b}{4-7} = \frac{c}{7-2} = k \text{ (say)}$$

$$\Rightarrow a = -k, b = -3k, c = 5k$$

Then (i) becomes,

$$1(x+1) + 3(y-1) - 5(z+1) = 0 \qquad [\because -k \neq 0]$$

$$\Rightarrow x + 3y - 5z = 7$$

This is equation of required plane.

5. Obtain the equation of the plane, which passes through the point (-1, 3, 2) and is perpendicular to each of the two planes x + 2y + 2z = 5 and 3x + 3y + 2z = 8.

[2006 Spring Q. No. 1(a)]

Solution: The equation plane that passes through (-1, 3, 2) is,

$$a(x+1) + b(y-3) + c(z-2) = 0 \qquad \text{...... (i)}$$

Given that (i) is perpendicular to the plane x + 2y + 2z = 5. So,

$$a + 2b + 2 = 0 \qquad \text{...... (ii)}$$

Also, given that (i) is perpendicular to 3x + 3y + 2z = 8. So,

$$3a + 3b + 2c = 0 \qquad \text{...... (iii)}$$

Solving (ii) and (iii), we get

$$\frac{a}{4-6} = \frac{b}{6-2} = \frac{c}{3-6} = k \text{ (say)}$$

$$\Rightarrow a = -2k, b = 4k, c = -3k$$

Then (i) becomes,

$$2(x+1)-4(y-3)+3(z-2)=0 \quad [\because -k \neq 0]$$

$$\Rightarrow 2x-4y+3z+8=0$$

This is equation of required plane.

6. **Find the equation of the plane through the points (2, 2, 1) and (9, 3, 6) and is perpendicular to the plane $2x + 3y + 6z = 9$.**

Solution: Similar to Q. No. 4.

Exercise 2.4

1. **Find the equation of plane through the intersection of $x + 2y + 3z + 4 = 0$ and $4x + 3y + 2z + 1 = 0$ passing through (0, 0, 0).**

Solution: Since the equation of plane through the intersection of $x + 2y + 3y + 4 = 0$ and $4x + 3y + 2z + 1 = 0$ be,

$$(x+2y+3y+4)+\lambda(4x+3y+2z+1)=0 \quad \ldots\ldots (i)$$

Given that the plane (i) passes through (0, 0, 0). So,

$$4+\lambda=0 \Rightarrow \lambda=-4$$

Then (i) becomes,

$$(x+2y+3y+4)-4(4x+3y+2z+1)=0$$

$$\Rightarrow -15x-10y-5z=0$$

$$\Rightarrow 3x+2y+z=0 \quad [\because -5 \neq 0]$$

This is the equation of required plane.

2. **Find the equation of the plane through the intersection of the planes $x + 2y - 3z = 5$ and $5x + 7y + 3z = 10$ through (-1, 2, -3).**

[2006 Fall; 2010 Spring Q. No. 1(a)]

Solution: Similar to Q. No. 1.

3. **Find the equation of plane through the intersection of $2x + 3y + 10z = 8$, and $2x - 3y + 7z = 2$; and normal to the plane $3x - 2y + 4z = 5$.**

Solution: Since the equation of plane through the intersection of the planes $x + 2y - 3z = 5$ and $5x + 7y + 3z = 10$ be

$$(x+2y-3z-5)+\lambda(5x+7y+3z-10)=0$$

$$\Rightarrow (1+5\lambda)x+(2+7\lambda)y+(-3+3\lambda)z+(-5-10\lambda)=0 \quad \ldots\ldots (i)$$

Given that the plane (i) is normal to $3x - 2y + 4z = 5$. Then,

$$3(1+5\lambda)+(-2)(2+7\lambda)+4(-3+3\lambda)=0$$

$$\Rightarrow 3+15\lambda-4-14\lambda-12+12\lambda=0$$

$$\Rightarrow 13\lambda-13=0$$

$$\Rightarrow \lambda=1$$

Then (i) becomes,

$$(1+5)x+(2+7)y+(-3+3)z+(-5-10)=0$$

$$\Rightarrow 6x+3y-15=0$$

$$\Rightarrow 2x+3y-5=0$$

This is the equation of required plane.

4. **Find the equation of the plane, when the plane $x + 3y + 5z = 7$ is rotated through a right angle about its intersection with the plane $x - 2y - 6z = 8$.**

Solution: Since the equation of plane through the intersection of the planes $x + 3y + 5z = 7$ and $x - 2y - 6z = 8$ be

$$(x+3y+5z-7)+\lambda(x-2y-6z-8)=0$$

$$\Rightarrow (1+\lambda)x+(3-2\lambda)y+(5-6\lambda)z+(-7-8\lambda)=0 \quad \ldots\ldots (i)$$

Given that the plane $x + 3y + 5z = 7$ is at right angle to (i). So,

$$(1+\lambda).1+(3-2\lambda).3+(5-6\lambda).5=0$$

$$\Rightarrow 1+\lambda+9-6\lambda+25-30\lambda=0$$

$$\Rightarrow 35-35\lambda=0$$

$$\Rightarrow \lambda=1$$

Then (i) becomes,

$$2x+y-z-15=0$$

$$\Rightarrow 2x+y-z=15$$

This is the equation of required plane.

5. **Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$, $2x + y - z + 5 = 0$.**

Solution: Since the equation of plane through the intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$ be,

$$(x+2y+3z-4)+\lambda(2x+y-z+5)=0 \quad \ldots\ldots (i)$$

$$\Rightarrow (1+2\lambda)x+(2+\lambda)y+(3-\lambda)z+(4+5\lambda)=0 \quad \ldots\ldots (ii)$$

Given that (ii) is perpendicular to the plane $5x + 3y + 6z + 8 = 0$. So,

$$(1+2\lambda).5+(2+\lambda).3+(3-\lambda).6=0$$

$$\Rightarrow 5+10\lambda+6+3\lambda+18-6\lambda=0$$

$$\Rightarrow 7\lambda+29=0$$

$$\Rightarrow \lambda=-\frac{29}{7}$$

Then (i) becomes,

$$7(x+2y+3z-4)-29(2x+y-z+5)=0$$

$$\Rightarrow -51x-15y+32z-173=0$$

$$\Rightarrow 51x+15y-32z+173=0$$

This is the equation of required plane.

Exercise 2.5

1. **Show that the origin lies in the obtuse angle between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$. Find the bisector of the acute angle between the above two planes.**

Solution: Given planes are,

$$2x - y + 2z + 3 = 0 \quad \text{...... (i)}$$

$$3x - 2y + 6z + 8 = 0 \quad \text{...... (ii)}$$

Here,

$$a_1a_2 + b_1b_2 + c_1c_2 = 6 + 2 + 12 = 20 > 0$$

This shows that the angle between the normals to the given planes is acute. So, the origin lies in the obtuse angle.

And, the equation of bisector of the angle between the planes (i) and (ii) are,

$$\frac{2x - y + 2z + 3}{\sqrt{2^2 + 1^2 + 2^2}} = \pm\frac{3x - 2y + 6z + 8}{\sqrt{3^2 + (-2)^2 + 6^2}}$$

$$\Rightarrow \frac{2x - y + 2z + 3}{3} = \pm\frac{3x - 2y + 6z + 8}{\sqrt{9 + 4 + 36}}$$

$$\Rightarrow \frac{2x - y + 2z + 3}{3} = \pm\frac{3x - 2y + 6z + 8}{7}$$

Taking positive sign,

$$14x - 7y + 14z + 21 = 9x - 6y + 18z + 24$$

$$\Rightarrow 5x - y - 4z = 3 \quad \text{...... (iii)}$$

Taking negative sign

$$14x - 7y + 14z + 21 = -9x + 6y - 18z - 24$$

$$\Rightarrow 23x - 13y + 32z + 45 = 0 \quad \text{...... (iv)}$$

Let θ be the angle between (i) and (iii) then,

$$\cos\theta = \frac{10 + 1 - 8}{\sqrt{4 + 1 + 4}\sqrt{25 + 1 + 16}} = \frac{3}{3\sqrt{42}} = \frac{1}{\sqrt{42}}$$

i.e. $\frac{b}{h} = \frac{1}{\sqrt{42}}$

Then, $p = \sqrt{42 - 1} = \sqrt{41}$

So, $\tan\theta = \frac{\sqrt{41}}{1} > 1 = \tan 45° \Rightarrow \theta > 45°.$

Therefore, the bisector (iii) bisects the obtuse angle.

Next, let θ be the angle between (i) and (iv) then,

$$\cos\theta = \frac{46 + 13 + 64}{\sqrt{4 + 1 + 4}\sqrt{529 + 169 + 1024}} = \frac{123}{3\sqrt{1722}} = \frac{41}{\sqrt{1722}}$$

i.e. $\frac{b}{h} = \frac{41}{\sqrt{1722}}$

Then, $p = \sqrt{1722 - 41^2} = \sqrt{1722 - 1681} = \sqrt{41}$

So, $\tan\theta = \frac{\sqrt{41}}{41} = \frac{1}{\sqrt{41}} < 1 = \tan 45° \Rightarrow \theta < 45°$

This shows that (iv) is the bisector of the acute angle.

2. **Find the bisector of the angle between the planes $2x - 3y - 6z + 6 = 0$ and $12x - 4y + 3z = 3$ which contains the origin.**

Solution: Given planes are

$$2x - 3y - 6z + 6 = 0 \quad \text{...... (i)}$$

$$12x - 4y + 3z = 3 \quad \text{...... (ii)}$$

Here,

$$a_1a_2 + b_1b_2 + c_1c_2 = 24 + 12 - 18 = 18 > 0$$

This shows that the origin is contained in obtuse angle.

For the bisecting planes of (i) and (ii)

$$\frac{2x - 3y - 6z + 6}{\sqrt{4 + 9 + 36}} = \pm\frac{12x - 4y + 3z - 3}{\sqrt{144 + 16 + 19}}$$

$$\Rightarrow \frac{2x - 3y - 6z + 6}{7} = \pm\frac{12x - 4y + 3z - 3}{13}$$

Taking positive sign,

$$26x - 39y - 78z + 78 = 84x - 28y + 21z - 21$$

$$\Rightarrow 110x - 67y - 57z + 57 = 0$$

Let θ be the angle between (i) and (iii). Then

$$\cos\theta = \frac{116 - 33 - 594}{\sqrt{4 + 9 + 36}\sqrt{3364 + 121 + 9801}} = -\frac{511}{7\sqrt{13286}}$$

This gives the angle larger than 90°. That is impossible.

Next, let θ be the angle between (i) and (iv). Then,

$$\cos\theta = \frac{220 + 201 + 402}{\sqrt{4 + 9 + 36}\sqrt{12100 + 40401 + 161604}} = \frac{823}{7\sqrt{214105}}$$

i.e. $\frac{p}{h} = \frac{823}{7\sqrt{214105}}$

Then, $p^2 = 49\,(214105) - (823)^2 = 10491145 - 677329 = 9813816$

Therefore,

$$\tan\theta = \sqrt{\frac{9813816}{677329}} > 1 = \tan 45° \Rightarrow \theta = 45°$$

This shows that (iv) bisects the obtuse angle. Thus, the plane $110x - 67y - 57z + 57 = 0$ bisects the angle that contains origin.

3. Show that the origin lies in the obtuse angle between the planes $x + 2y + 2z - 9 = 0$ and $4x - 3y + 12z + 13 = 0$. Find the planes bisecting the angles between them and point out which bisects the acute angle.

Solution: Simular to Q. No. 1.

Exercise 2.6

(1) Show $2x^2 - y^2 + 2z^2 - yz + 5zx + xy = 0$ represents the pair of planes and find angle between them.

Solution: Given equation is,

$$2x^2 - y^2 + 2z^2 - yz + 5zx + xy = 0 \quad \ldots\ldots (i)$$

Comparing (i) with $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, we get

$$a = 2,\quad b = -1,\quad c = 2,\quad f = \frac{-1}{2},\quad g = \frac{5}{2},\quad h = \frac{1}{2}$$

Then, this equation (i) represents the pair of planes if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

i.e. $2\times(-1)\times2 + 2\times\frac{-1}{2}\times\frac{5}{2}\times\frac{1}{2} - 2\times\left(\frac{-1}{2}\right)^2 - (-1)\left(\frac{5}{2}\right)^2 - 2\left(\frac{1}{2}\right)^2 = 0$

$\Rightarrow -4 - \frac{5}{4} - \frac{1}{2} + \frac{25}{4} - \frac{1}{2} = 0$

$\Rightarrow \frac{-16 - 5 - 2 + 24 - 2}{4} = 0 \Rightarrow 0 = 0.$

This shows that the equation (i) represents a pair of planes.

Let, θ be angle between the pair of planes, then

$$\tan\theta = \frac{2\sqrt{f^2 + g^2 + h^2 - bc - ca - ab}}{a + b + c}$$

$$= \frac{2\sqrt{\frac{1}{4} + \frac{25}{4} + \frac{1}{4} - (-1)\times2 - 2\times2 - 2\times(-1)}}{2 - 1 + 2}$$

$$\Rightarrow \tan\theta = \frac{2\sqrt{\frac{27}{4} + 2 - 4 + 2}}{3} = \frac{2\times\frac{3\sqrt{3}}{2}}{3} = \sqrt{3} = \tan\frac{\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

Thus the angle between the pair of planes be $\frac{\pi}{3}$.

(2) Show $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ represents the pair of planes and find angle between them.

Solution: Given equation is,

$$2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0 \quad \ldots\ldots\ldots (i)$$

Comparing (i) with $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, we get

$$a = 2,\quad b = -6,\quad c = -12,\quad f = 9,\quad g = 1,\quad h = \frac{1}{2}$$

Then, this equation (i) represents the pair of planes if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

i.e. $2\times(-6)\times(-12) + 2\times9\times1\times\frac{1}{2} - 2(9)^2 - (-6)1^2 - (-12)\left(\frac{1}{2}\right)^2$

$= 144 + 9 - 162 + 6 + 3$

$= 162 - 162 = 0$

This shows that the equation (i) represents a pair of planes.

Let, θ be angle between the pair of planes, then

$$\tan\theta = \frac{2\sqrt{f^2 + g^2 + h^2 - bc - ca - ab}}{a + b + c}$$

$$= \frac{2\sqrt{9^2 + 1^2 + \left(\frac{1}{2}\right)^2 - (-6)\times(-12) - (-12)\times2 - 2\times(-6)}}{2 - 6 - 12}$$

$$= \frac{2\sqrt{81 + 1 + \frac{1}{4} - 72 + 24 + 12}}{-16} = \frac{2\sqrt{\frac{185}{4}}}{16} = \frac{\sqrt{185}}{-16}$$

Since, $\sec\theta = \sqrt{1 + \tan^2\theta}$

So, $\sec\theta = \sqrt{1 + \frac{185}{256}} = \sqrt{\frac{441}{256}} = \frac{21}{16}$

$\Rightarrow \cos\theta = \frac{16}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{16}{21}\right).$

Thus, the angle between the pair of planes be, $\cos^{-1}\left(\frac{16}{21}\right)$.

(3) Determine the value of k, where $2x^2 - y^2 - kz^2 + 3yz = 0$ represents a pair of planes.

Solution: Given equation is,

$$2x^2 - y^2 - kz^2 + 3yz = 0 \quad \ldots\ldots (i)$$

Comparing (i) with $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, we get

$$a = 2,\quad b = -1,\quad c = -k,\quad f = \frac{3}{2},\quad g = 0,\quad h = \frac{1}{2}$$

Then, this equation (i) represents the pair of planes if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

i.e. $2\times-1\times k + 2\times\frac{3}{2}\times 0\times\frac{1}{2} - 2\times\left(\frac{3}{2}\right)^2 - (-1)\,0^2 - (-k)\left(\frac{1}{2}\right)^2 = 0$

$\Rightarrow \quad 2k + 0 - \frac{9}{2} + \frac{k}{4} = 0$

$\Rightarrow \quad 2k + \frac{k}{4} = \frac{9}{2} \quad \Rightarrow \frac{9k}{4} = \frac{9}{2} \quad \Rightarrow k = 2.$

Thus the equation (i) will represents a pair of planes only if k = 2.

(4) Show $6x^2 - 4y^2 - 10z^2 + 3yz + 4zx - 11xy = 0$ represents the pair of perpendicular planes, find angle between them.

Solution: Given equation is,

$$6x^2 - 4y^2 - 10z^2 + 3yz + 4zx - 11xy = 0 \qquad \text{...... (i)}$$

Comparing (i) with $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, we get

$$a = 6,\ b = 4,\ c = -10,\ f = \frac{3}{2},\ g = 2,\ h = \frac{-11}{2}$$

Let, θ be angle between the pair of planes, then

$$\tan\theta = \frac{2\sqrt{f^2 + g^2 + h^2 - bc - ca - ab}}{a + b + c}$$

$$= \frac{2\sqrt{\left(\frac{3}{2}\right)^2 + 2^2 + \left(\frac{-11}{2}\right)^2 - 4\times(10) - (-10).6 - 6\times4}}{6 + 4 - 10}$$

$$= \frac{2\sqrt{\frac{9}{4} + 4 + \frac{121}{4} - 40 + 60 - 24}}{0} = \infty = \tan\frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

This shows the equation (i) represents a pair of perpendicular planes.

For the equation of the pair of planes, arrange (i) as,

$$6x^2 + (4z - 11y)x + (4y^2 + 3yz - 10z^2) = 0 \qquad \text{...... (ii)}$$

Comparing (ii) with quadratic equation $ax^2 + bx + c = 0$ then we get,

$$a = 6, \quad b = 4z - 11y, \quad c = 4y^2 + 3yz - 10z^2$$

Then, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

i.e. $x = \dfrac{-(4z - 11y) \pm \sqrt{(4z - 11y)^2 - 4\times6\,(4y^2 + 3yz - 10z^2)}}{2\times6}$

$\Rightarrow \quad x = \dfrac{-4z - 11y \pm \sqrt{16z^2 - 88yz + 121y^2 - 96y^2 - 72yz + 240z^2}}{12}$

$\Rightarrow \quad x = \dfrac{-4z - 11y \pm \sqrt{25y^2 - 160yz + 256z^2}}{12}$

$\Rightarrow \quad x = \dfrac{-4z - 11y \pm (\sqrt{5y - 16z})^2}{12}$

$\Rightarrow \quad x = \dfrac{-4z - 11y \pm 5y - 16z}{12}$

$\Rightarrow \quad x = \dfrac{-4z + 11y + 5y - 16z}{12}$

$\Rightarrow \quad x = \dfrac{-4z - 11y - 5y + 16z}{12}$

This gives, $12x = -20z + 16y$ and $x = \dfrac{6y + 12z}{12}$

$\Rightarrow \quad 12x - 16y + 20z = 0$ $\qquad \Rightarrow \quad 12x = 6y + 12z$

$\Rightarrow \quad 3x - 4y + 20z = 0$ $\qquad \Rightarrow \quad 12x - 6y - 12z = 0.$

$\Rightarrow \quad 2x - y - 2z = 0$

The pair of planes are $3x - 4y + 5z = 0$ and $2x - y - 2z = 0$.

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

SHORT QUESTIONS

2002: Find direction cosines of the normal to the plane $2x - y + 2z = 6$.

Solution: Given plane is

$$2x - y + 2z = 6 \qquad \text{........(1)}$$

So, the direction ratios of (1) are 2, -1, 2

Therefore, the direction cosines of (i) are

$$l = \frac{2}{\sqrt{4+1+4}} = \frac{2}{3},\ m = \frac{-1}{\sqrt{4+1+4}} = -\frac{1}{3},\ n = \frac{2}{\sqrt{4+1+4}} = \frac{2}{3}$$

Thus, the direction cosines of (i) are $\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$

2006 Spring: Find the angle between the planes $2x + 3y - z = 12$ and $x + 2y - z = 14$.

Solution: Given planes are

$$2x + 3y - z = 12 \text{ and } x + 2y - z = 14$$

Since we have if θ be the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Suppose that θ be the angle between given planes then

$$\cos\theta = \frac{2 + 6 - 1}{\sqrt{4 + 9 + 1}\sqrt{1 + 4 + 1}} = \frac{5}{\sqrt{14}\sqrt{6}} = \frac{5}{\sqrt{84}}$$

Thus angle between the planes is $\cos^{-1}\left(\frac{5}{\sqrt{84}}\right)$.