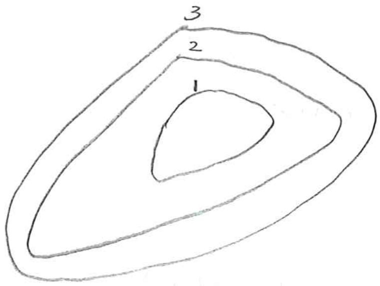
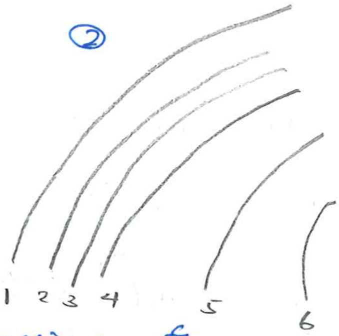
20.5 / 20.5

3.2 Infer function f with curve of level shown below:

①



②



infer whether the above functions f are convex, concave, quasiconvex, and quasiconcave.

sol^n

- definition of convex:

  $f: \mathbb{R}^n \to \mathbb{R}$ is convex when $\operatorname{dom} f$ is a convex set

  $x \in \operatorname{dom} f$

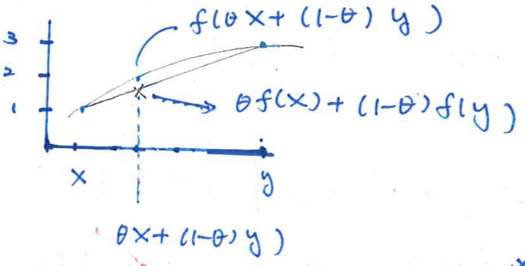
  $y \in \operatorname{dom} f$

  $\theta \in [0, 1]$

  $f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$

- function ① is not convex,

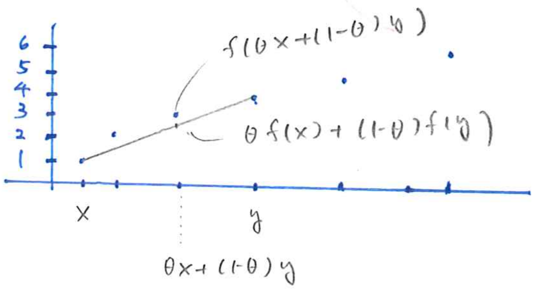
  as $f(\theta x + (1-\theta) y)$ could $> \theta f(x) + (1-\theta) f(y)$ at some points, which is as shown below:



※

- function ② is also not convex,

  as $f(\theta x + (1-\theta) y)$ could $> \theta f(x) + (1-\theta) f(y)$ at some points which could be shown graphically as below:



※

3.2 (cont'd)

sol^n (cont'd)

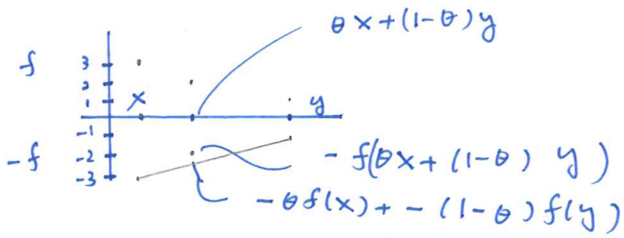
△ definition of concave

$f: R^n \to R$ is concave if $-f$ is convex

- function ① is not concave as

$-f(\theta x + (1-\theta) y)$ could $> -\theta f(x) + -(1-\theta) f(y)$ at some points

which is as shown below:



#

- function ② is concave as

$-f(\theta x + (1-\theta) y) \leq -\theta f(x) + -(1-\theta) f(y)$ holds at all points from the given level curve.

furthermore if f is concave, then α- superlevel set of f

α-superlevel set $\{x \in \text{dom} f \mid f(x) \geq \alpha\}$ is a convex set.

which holds for function ② #

△ definition of quasiconvex

$f: R^n \to R$ is quasiconvex when · dom f is convex &

all sublevel sets of

$$S_\alpha = \{x \in \text{dom} f \mid f(x) \leq \alpha\} \quad \forall \alpha \in R \quad \text{is convex.}$$

- function ① is quasiconvex, as all α-sublevel sets are convex,

i.e., $\alpha=1, \alpha=2, \alpha=3$ from the given level curve.

#

- function ② is not quasiconvex, as no α-sublel sets are convex,

i.e., $\alpha=1, \alpha=2 \cdots\cdots \alpha=6$ from the given level curve.

#

3.2 (cont'd)

sol^n (cont'd)

△ definition of quasiconcave

$f: R^n \to R$ is quasiconcave when $-f$ is quasiconvex,

i.e., all superlevel sets of

$$S_\alpha = \{x \in \text{dom} f \mid f(x) \geq \alpha\} \quad \forall \alpha \in R \quad \text{is convex}$$

- function ① is not quasiconcave, as no α-superlevel sets are convex
i.e., $\alpha = 1, \alpha = 2, \alpha = 3$ from the given curve. ※

- function ② is quasiconcave, as all α-superlevel sets are convex,
i.e., $\alpha = 1, \alpha = 2 \ldots \alpha = 6$ from the given curve.
Additionally, function ② is concave, hence function ② is quasi-concave. ※

3.5 $f: R \to R$ is convex, $R_+ \subseteq \text{dom} f$

show whether

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt \qquad \text{dom} F = R_{++}$$

is convex.

$\int_0^x f(t)dt$ · $\int_{g(a)}^{g(b)} f(u)du$

$t = g(x) = sx$ $\quad u = g(x)$

$\int_0^1 \quad ds \quad \int_a^b f(g(x))g'(x)$

sol^n

△ applying integration by change of variables,

$$F(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad \text{let } t = xs$$

$dt = x.$

$\int f(g(x))g'(x)dy = \int f(u)du$

$u = g(x)$

$\int_a^b f(g(x))g'(x)dy = \int_{g(a)}^{g(b)} f(u$

$$\Rightarrow F(x) = \frac{1}{x}\int_0^1 f(sx)\,x\,ds$$

$$= \int_0^1 f(sx)\,ds$$

△ from the given hint, "for each s, $f(sx)$ is convex in x, so $\int_0^1 f(sx)\,ds$ is convex".

△ Hence $F(x) = \frac{1}{x}\int_0^x f(t)dt$ $\quad$ $f: R \to R$ is convex,
$R_+ \subseteq \text{dom} f$
$\text{dom} F = R_{++}$

is convex ※

3.11

$R^n \to R$ is a differentiable convex function

show its gradient ∇f is monotone. show converse true or not.

Sol^n

△ recall first-order conditions

$$\begin{cases} f(y) \geq f(x) + \nabla f(x)^T (y-x) & \forall\, x, y \in \operatorname{dom} f \\ f(x) \geq f(y) + \nabla f(y)^T (x-y) & \forall\, x, y \in \operatorname{dom} f \end{cases} \quad \begin{matrix} — ① \\ — ② \end{matrix}$$

△ from ① + ②

$$\Rightarrow \cancel{f(y)} + \cancel{f(x)} \geq \cancel{f(x)} + \cancel{f(y)} + \nabla f(x)^T (y-x) + \nabla f(y)^T (x-y)$$

$$\Rightarrow \quad 0 \geq \nabla f(x)^T (y-x) + \nabla f(y)^T (x-y)$$

$$\Rightarrow \left(\nabla f(x)^T - \nabla f(y)^T \right)(x-y) \geq 0$$

$$\Updownarrow$$

$$(\psi(x) - \psi(y))^T (x-y) \geq 0 \qquad \psi : R^n \to R^n, \quad x, y \in \operatorname{dom} \psi$$

$\therefore$ the gradient ∇f is monotone ※

△ converse : every monotone mapping is a gradient of a convex function.

the other direction is not true.

Not all fields are gradients ※

3.16 determine the following functions, whether they are

convex
concave
quasiconvex
quasi concave

sol^n

△ def. of convex

$f: R^n \to R$ is convex when dom f is a convex set

$$f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y)$$

$\forall x, y \in \text{dom } f.$
$\theta \in [0, 1]$

△ def. of concave

$f: R^n \to R$ is concave if $-f$ is convex

△ def. of quasiconvex

$f: R^n \to R$ is quasiconvex when dom$f$ is convex &
all sublevel sets of

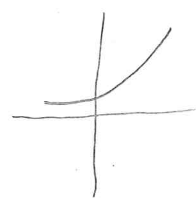
$$S_\alpha = \{x \in \text{dom} f \mid f(x) \le \alpha\} \quad \forall \alpha \in R$$ is convex.

△ def. of quasiconcave

$f: R^n \to R$ is quasiconcave when $-f$ is quasi convex.
i.e., all super level sets of

$$S_\alpha = \{x \in \text{dom} f \mid f(x) \ge \alpha\} \quad \forall \alpha \in R$$ is convex

(a) $f(x) = e^x - 1$ on R



△ $\forall x, y \in$ dom f

any x, y satisfy $f(\theta x + (1-\theta) y) \le \theta f(x) + (1-\theta) f(y)$

$\therefore$ convex ※

△ $-f$ is not convex

$\therefore$ not concave

3.16 (cont'd)
soln (con'td)
(a) (cont'd)

Δ f is convex

$\therefore$ f is quasiconvex ※

Δ superlevel sets of f is convex

$\therefore$ f is quasiconcave ※

(b) $f(x_1, x_2) = x_1 x_2$ on R^2_{++}

Δ recall second order condition:

f is convex i.f.f. dom f is convex &
$\nabla^2 f(x) \succeq 0$

$\Rightarrow \nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

which is not positive semidefinite

$\therefore$ f is not convex ※

Δ recall second order condition:

f is concave i.f.f dom$f$ is convex &
$\nabla^2 f(x) \preceq 0$

$\Rightarrow \nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

which is not negative semidefinite,

$\therefore$ f is not concave ※

Δ not all sublevel sets of f is convex,

e.g. $S_\alpha = \{x \in \text{dom} f \mid f(x) \le 1\}$

$x_1 = 2$ $\quad x_1' = 0.5$
$x_2 = 0.5$ $\quad x_2' = 2$

let $(x_1'', x_2'') = \theta(x_1, x_2) + (1-\theta)(x_1', x_2')$, $\theta \in [0, 1]$

$x_1'', x_2'' \not\le 1$. $\therefore$ S_α not convex

$\therefore$ f is not quasiconvex ※

3.16 (cont'd)

Sol^n (cont'd)

(b) (cont'd)

△ all superlevel sets of f is convex,

recall 2.11 hyperbolic set is convex

∴ f is quasiconcave ※

(c) $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on R^2_{++}

recall second order condition

$$\nabla^2 f(x_1, x_2) = \frac{1}{x_1 x_2} \begin{bmatrix} \frac{2}{x_1^2} & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & \frac{2}{x_2^2} \end{bmatrix} \succeq 0$$

△ f is convex ※

△ f is not concave ※

△ f is quasiconvex, from f is convex and (b)'s quasiconcavity ($S_\alpha' = \{ x \in \text{dom} f \mid \frac{1}{x_1 x_2} \leq \alpha \}$

$\Rightarrow S_\alpha = \{ x \in \text{dom} f \mid x_1 x_2 \geq \frac{1}{\alpha} \}$)

from 2.11.

∴ f is quasiconvex.

△ ∴ f is not quasiconcave

(d) $f(x_1, x_2) = \frac{x_1}{x_2}$ on R^2_{++}

recall second order condition

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix} \Rightarrow \text{not positive/negative semidefinit}$$

△ f is not convex

△ f is not concave

3.16 (cont'd)

sol^n (cont'd)

(d) (cont'd)

△ the sublevel sets of f

$$S_\alpha = \left\{ x \in \mathrm{dom}\, f \;\middle|\; \frac{x_1}{x_2} \le \alpha \right\}$$

for x_1, x_2 on $\mathbb{R}^2_{++}$, S_α is halfspace with $\frac{x_1}{x_2} \le \alpha \Rightarrow x_1 \le \alpha x_2$

$\therefore$ f is quasiconvex ✻

△ similarly, the superlevel sets of

$$S_\alpha = \left\{ x \in \mathrm{dom}\, f \;\middle|\; \frac{x_1}{x_2} \ge \alpha \right\}$$

for x_1, x_2 on $\mathbb{R}^2_{++}$, S_α is halfspace with $\frac{x_1}{x_2} \ge \alpha \Rightarrow x_1 \ge \alpha x_2$

$\therefore$ f is quasiconcave ✻

(e) $f(x_1, x_2) = \frac{x_1^2}{x_2}$ on $\mathbb{R} \times \mathbb{R}_{++}$

recall second-order condition

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \frac{2}{x_2} & \frac{-2x_1}{x_2^2} \\ \frac{-2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2x_1}{x_2} \end{bmatrix} \underbrace{\frac{2}{x_2}}_{>0} \begin{bmatrix} 1 & -2\frac{x_1}{x_2} \end{bmatrix} \succeq 0$$

△ f is convex ✻

△ f is not concave ✻

△ f is quasiconvex, as f is convex. ✻

△ f is not quasiconcave

as not all superlevel sets of f

$$S_\alpha = \left\{ x \in \mathrm{dom}\, f \;\middle|\; \frac{x_1^2}{x_2} \ge \alpha \right\}$$

are convex.

$\frac{x_1^2}{x_2} \ge \alpha$

$\frac{1}{2} \quad \frac{-1}{2} \quad \ge 0.5$

for instance $\begin{cases} (x_1, x_2) = (1, 0.5) \\ (x_1', x_2') = (-1, 0.5) \end{cases}$ $\alpha = 1 \Rightarrow \begin{cases} \frac{1^2}{0.5} \ge 1 \\ \frac{(-1)^2}{0.5} \ge 1 \end{cases}$

$(x_1'', x_2'') = \theta (x_1, x_2) + (1-\theta)(x_1', x_2') \quad \theta \in [0, 1]$

3.1b (cont'd)

Sol^n (cont'd)

(e) (cont'd)

for $\theta = 0.5$

$$\frac{x_1''^{2}}{x_2''} \ngeq 1$$

hence, superlevel set is not convex.

f is not quasiconcave ※

(f) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ $\alpha \in [0, 1]$ on R^2_{++}

recall second-order condition.

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & (1-\alpha)(-\alpha) x_1^{\alpha} x_2^{-\alpha-1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{x_1} \\ \frac{-1}{x_2} \end{bmatrix} \underbrace{\left(- \underbrace{\alpha}_{\geq 0} \underbrace{(1-\alpha)}_{\geq 0} \underbrace{x_1^{\alpha} x_2^{1-\alpha}}_{\geq 0} \right)}_{\leq 0} \begin{bmatrix} \frac{1}{x_1} & \frac{-1}{x_2} \end{bmatrix} \preceq 0$$

△ f is not convex ※

△ f is concave ※

△ f is not quasiconvex ※
as not all sub-level sets of f is convex.

△ f is quaiconcave, as f is concave ※

≤ 1

$4^{0.5}$ $4^{0.5}$

$16^{0.5}$ $16^{0.5}$ ≤ 16

3.17

show $f(x) = \left(\sum_{i=1}^{n} x_i^p \right)^{1/p}$ with $\mathrm{dom}\, f = \mathbb{R}_{++}^n$, $p<1$, $p \neq 0$ is concave

<u>sol^n</u>

- as hinted, follow the procedure of proofing log-sum-exp & geometric mean.

- recall second-order condition, f is concave when hessian of f $\nabla^2 f(x) \preceq 0$.

- $$\nabla f(x) = \frac{\partial}{\partial x} \left(\sum_{i=1}^{n} x_i^p \right)^{1/p}$$

$$= \frac{\partial}{\partial x} \left(x_1^p + x_2^p + x_3^p + \cdots + x_n^p \right)^{1/p}$$

$$= \frac{1}{\cancel{p}} \left(x_1^p + x_2^p + \cdots + x_n^p \right)^{\frac{1-p}{p}} \cdot \cancel{p} x_1^{p-1}$$
$$+ \frac{1}{\cancel{p}} \left(x_1^p + x_2^p + \cdots + x_n^p \right)^{\frac{1-p}{p}} \cdot \cancel{p} x_2^{p-1}$$
$$\vdots$$
$$+ \frac{1}{\cancel{p}} \left(x_1^p + x_2^p + \cdots + x_n^p \right)^{\frac{1-p}{p}} \cdot \cancel{p} x_n^{p-1}$$

$$= \begin{bmatrix} \left(\sum_{i=1}^{n} x_i^p \right)^{\left(\frac{1-p}{p}\right)} (x_1)^{p-1} \\ \left(\sum_{i=1}^{n} x_i^p \right)^{\left(\frac{1-p}{p}\right)} (x_2)^{p-1} \\ \vdots \\ \left(\sum_{i=1}^{n} x_i^p \right)^{\left(\frac{1-p}{p}\right)} (x_n)^{p-1} \end{bmatrix} = \begin{bmatrix} \left(\frac{\left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}}}{x_1} \right)^{1-p} \\ \left(\frac{\left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}}}{x_2} \right)^{1-p} \\ \vdots \\ \left(\frac{\left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}}}{x_n} \right)^{1-p} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{f(x)}{x_1} \right)^{1-p} \\ \left(\frac{f(x)}{x_2} \right)^{1-p} \\ \vdots \\ \left(\frac{f(x)}{x_n} \right)^{1-p} \end{bmatrix}$$

3.17 (cont'd)

soln (cont'd)

$$\triangle \quad \nabla^2 f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} f(x) & \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} f(x) & \cdots & \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_n} f(x) \\ \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} f(x) & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2} f(x) & \ddots & \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_n} f(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_1} f(x) & \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_2} f(x) & \cdots & \frac{\partial}{\partial x_n}\frac{\partial}{\partial x_n} f(x) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x_1}(x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1-p}{p}} x_1^{p-1} & \cdots & \\ \frac{\partial}{\partial x_2}(x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1-p}{p}} x_1^{p-1} & \ddots & \vdots \\ \vdots & & \\ \frac{\partial}{\partial x_n}(x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1-p}{p}} x_1^{p-1} & \cdots & \frac{\partial}{\partial x_n}(x_1^p + x_2^p + \cdots x_n^p)^{\frac{1-p}{p}} x_n^{p-1} \end{bmatrix}$$

$i=j$ & $i\neq j$ are two different cases, element-wise

take $i=j=1$ as instance

$$\frac{\partial}{\partial x_1}(x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1-p}{p}} (x_1)^{p-1}$$

$$= \frac{1-p}{\cancel{p}}(x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1-2p}{p}} (x_1)^{p-1} \cdot \cancel{p} \cdot x_1^{p-1}$$

$$+ (x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1-p}{p}} (p-1)\, x_1^{p-2}$$

$$= (1-p)(x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1}{p}} \cdot (x_1^p + x_2^p + \cdots + x_n^p)^{-2} (x_1)^{2(p-1)}$$
$$+ \left[(x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1}{p}}\right]^{1-p} (p-1)\, x_1^{(p-2)}$$

$$= (1-p)\, f(x) \cdot f(x)^{-2p} (x_1)^{2(p-1)} + [f(x)]^{1-p} (p-1)\, x_1^{(p-2)}$$

$$= \frac{1-p}{f(x)} \frac{f(x)^{2-2p}}{x_1^{2-2p}} + (-1) \cdot \frac{1-p}{x_1} \frac{f(x)^{1-p}}{x_1^{1-p}}$$

$$= \frac{1-p}{f(x)}\left[\left(\frac{f(x)}{x_1}\right)^2\right]^{1-p} - \frac{1-p}{x_1}\left(\frac{f(x)}{x_1}\right)^{1-p}$$

$$\Rightarrow \left[\nabla^2 f(x)\right]_{ij,\, i=j} = \frac{1-p}{f(x)}\left[\left(\frac{f(x)}{x_i}\right)^2\right]^{1-p} - \frac{1-p}{x_i}\left(\frac{f(x)}{x_i}\right)^{1-p}$$

3.17 (cont'd)

sol^n (cont'd)

take $i=1$, $j=2$ as instance

$$\frac{\partial}{\partial x_1}(x_1^p+x_2^p+\cdots+x_n^p)^{\frac{1-p}{p}}x_2^{p-1}$$

$$= \frac{1-p}{\cancel{p}}(x_1^p+x_2^p+\cdots+x_n^p)^{\frac{1-2p}{p}}\cdot \cancel{p}x_1^{p-1}\cdot x_2^{p-1}$$

$$= (1-p)\,[f(x)]^{1-2p}\cdot(x_1^{1-p})^{-1}\cdot(x_2^{1-p})^{-1}$$

$$= \frac{1-p}{f(x)}\,\frac{f(x)^{2-2p}}{(x_1x_2)^{1-p}} = \frac{1-p}{f(x)}\left[\frac{f(x)^2}{x_1x_2}\right]^{1-p}$$

$$\Rightarrow [\nabla^2 f(x)]_{ij,\ i\neq j} = \left(\frac{1-p}{f(x)}\right)\left[\frac{f(x)^2}{x_ix_j}\right]^{1-p}$$

	a	b	c
a	a^2	ab	ac
b	ab	b^2	bc
c	ac	bc	c^2

$= (a+b+c)^2$

$\triangle$ for $\nabla^2 f(x) \preceq 0$

any vector y $\quad y^T\nabla^2 f(x)\,y \leq 0$

from hessian matrix

$$[\nabla^2 f(x)]_{ij,\ i=j} = \boxed{\left(\frac{1-p}{f(x)}\right)\left[\left(\frac{f(x)}{x_i}\right)^2\right]^{1-p}} - \underbrace{\left(\frac{1-p}{x_i}\right)\left(\frac{f(x)}{x_i}\right)^{1-p}}$$

$\hookrightarrow$ let terms related to this = sum 2

$$[\nabla^2 f(x)]_{ij,\ i\neq j} = \boxed{\left(\frac{1-p}{f(x)}\right)\left[\frac{f(x)^2}{x_ix_j}\right]^{1-p}}$$

$\hookrightarrow$ let terms related to this = sum$_1$

let $y^T\nabla^2 f(x)\,y = \text{sum}_1 + \text{sum}_2$

3.17 (cont'd)

soln (cont'd)

$$\text{sum}_1 = \frac{1-p}{f(x)} \cdot \left[y_1 \left(\frac{f(x)^2}{x_1 x_1}\right)^{1-p} y_1 + y_1 \left(\frac{f(x)^2}{x_1 x_2}\right)^{1-p} y_2 + \cdots + y_2 \left(\frac{f(x)^2}{x_2 x_1}\right)^{1-p} y_1 + y_2 \left(\frac{f(x)^2}{x_2 x_2}\right)^{1-p} y_2 + \cdots + y_n \left(\frac{f(x)^2}{x_n x_n}\right)^{1-p} y_n \right]$$

$$= \frac{1-p}{f(x)} \cdot \left[\frac{f(x)^{1-p}}{x_1^{1-p}} y_1 + \frac{f(x)^{1-p}}{x_2^{1-p}} y_2 + \cdots + \frac{f(x)^{1-p}}{x_n^{1-p}} y_n \right]^2$$

$$= \frac{1-p}{f(x)} \cdot \left[\sum_{i=1}^{n} \left(\frac{f(x)}{x_i}\right)^{1-p} y_i \right]^2$$

$$\text{sum}_2 = -y_1 \left(\frac{1-p}{x_1}\right)\left(\frac{f(x)}{x_1}\right)^{1-p} y_1 - y_2 \left(\frac{1-p}{x_2}\right)\left(\frac{f(x)}{x_2}\right)^{1-p} y_2 - \cdots - y_n \left(\frac{1-p}{x_n}\right)\left(\frac{f(x)}{x_n}\right)^{1-p} y_n$$

$$= -\left(\frac{1-p}{f(x)}\right) \cdot \left[y_1 \left(\frac{f(x)}{x_1}\right)^{2-p} y_1 + y_2 \left(\frac{f(x)}{x_2}\right)^{2-p} y_2 + \cdots + y_n \left(\frac{f(x)}{x_n}\right)^{2-p} y_n \right]$$

$$= -\left(\frac{1-p}{f(x)}\right) \cdot \sum_{i=1}^{n} \left(\frac{f(x)}{x_i}\right)^{2-p} y_i^2$$

2 - p

$$\therefore \quad y^T \nabla^2 f(x) y = \left(\frac{1-p}{f(x)}\right) \left[\underbrace{\left[\sum_{i=1}^{n} \left(\frac{f(x)}{x_i}\right)^{1-p} y_i \right]^2}_{①} - \underbrace{\sum_{i=1}^{n} \left(\frac{f(x)}{x_i}\right)^{2-p} y_i^2}_{②} \right]$$

Δ to prove $y^T \nabla^2 f(x) y \leq 0$, we apply Cauchy-Schwarz inequality

$(a^T a)(b^T b) \geq (a^T b)^2$, where $[a]_i = \left(\frac{f(x)}{x_i}\right)^{1-p/2} y_i$, $[b]_i = \left(\frac{f(x)}{x_i}\right)$

$$\Rightarrow \underbrace{\left[\sum_{i=1}^{n} \left(\frac{f(x)}{x_i}\right)^{1-p} y_i \right]^2}_{①} \leq \underbrace{\left[\sum_{i=1}^{n} \left(\frac{f(x)}{x_i}\right)^{2-p} y_i^2 \right]}_{②}$$

as

$$\left[\sum_{i=1}^{n} \left(\frac{f(x)}{x_i}\right)^{1-p/2} y_i \left(\frac{f(x)}{x_i}\right)^{-p/2} \right]^2 \leq \left[\sum_{i=1}^{n} \left[\left(\frac{f(x)}{x_i}\right)^{1-p/2} y_i \right] \right]^2 \left[\sum \left(\frac{f(x)}{x_i}\right)^{-p/2} \right]^2 \frac{1}{\left[\left(\frac{f(x)}{x_i}\right)^{\cdots}\right.}$$

3.17 (cont'd)

sol^n (cont'd)

where

$$\sum_{i=1}^{n}\left[\left(\frac{f(x)}{x_i}\right)^{-p/2}\right]^2 = 1$$

recall

$$f(x) = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} = (x_1^p + x_2^p + \cdots + x_n^p)^{1/p}$$

$$\sum_{i=1}^{n}\left[\left(\frac{f(x)}{x_i}\right)^{-p/2}\right]^2$$

$$= \sum_{i=1}^{n}\left(\frac{(x_1^p + x_2^p + x_n^p)^{\frac{1}{p}}}{x_i}\right)^{-p}$$

$$= \sum_{i=1}^{n}\left(\frac{x_i}{(x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1}{p}}}\right)^{p}$$

$$= \sum_{i=1}^{n}\frac{x_i^p}{x_1^p + x_2^p + \cdots + x_n^p}$$

$$= \frac{x_1^p + x_2^p + \cdots + x_n^p}{x_1^p + x_2^p + \cdots + x_n^p}$$

therefore, statement ① ≤ ② holds

&

① − ② ≤ 0 holds

&

therefore,

$y^T \nabla^2 f(x) y^T \leq 0$, $\nabla^2 f(x) \preceq 0$

∴ $f(x)$ w/ dom $f = \mathbb{R}^n_{++}$ is concave ※

3.18

(a) show $f(X) = tr(X^{-1})$ is convex on $\mathrm{dom} f = S^n_{++}$

<u>solⁿ</u>

△ a function is convex i.f.f. it is convex when restricted to any line that intersects its domain,

i.e.

$g(t) = f(X + tV)$ is convex in t

where $\mathrm{dom}\ g = \{ t \mid X + tV \in \mathrm{dom} f \}$

△ using this fact,

$\mathrm{dom} f = S^n_{++}$

define $g(t) = f(Z + tV)$, $Z \in S^n_{++}$, $V \in S_n$

$$
\begin{aligned}
g(t) &= tr\left((Z+tV)^{-1}\right) \\
&= tr\left(\left[Z(I + tZ^{-\frac{1}{2}} V Z^{-\frac{1}{2}}\right]^{-1}\right) \\
&= tr\left(Z^{-1}(I + tQ\Lambda Q^T)^{-1}\right) \\
&= tr\left(Z^{-1}(QQ^T + tQ\Lambda Q^T)^{-1}\right) \\
&= tr\left(Z^{-1} Q (I + t\Lambda)^{-1} Q^T\right) \\
&= tr\left(Q^T Z^{-1} Q (I + t\Lambda)^{-1}\right) \\
&= \sum_{i=1}^{n} \left[(Q^T Z^{-1} Q)\right]_{ii} (1 + t\lambda_i)^{-1}
\end{aligned}
$$

let $Z^{-\frac{1}{2}} V Z^{-\frac{1}{2}}$ undergo eigenvalue decomposition

$Z^{-\frac{1}{2}} V Z^{-\frac{1}{2}} = Q\Lambda Q^T$

orthogonal matrix

eigen values on the diagonal

△ with $X + tV \in \mathrm{dom} f$

$(1 + t\lambda_i)^{-1}$ is a convex function.

△ Nonnegative weighted sums is one of the operations that preserve convexity.

△ ∴ $f(X)$ is convex ※

3.18

(b) show $f(X) = (\det X)^{\frac{1}{n}}$ is concave on $\mathrm{dom}\, f = S^n_{++}$

soln

△ similar to (b)

we let $g(t) = f(Z + tV)$, $Z \in S^n_{++}$ & $V \in S^n$

$$g(t) = (\det(Z + tV))^{\frac{1}{n}}$$

$$= \left[\det Z(I + tZ^{-\frac{1}{2}} V Z^{-\frac{1}{2}})\right]^{\frac{1}{n}}$$

$$= \left[\det Z \det(I + tZ^{-\frac{1}{2}} V Z^{-\frac{1}{2}})\right]^{\frac{1}{n}}$$

$$= \left[\det Z \det(I + tQ\Lambda Q^T)\right]^{\frac{1}{n}}$$

$$= (\det Z)^{\frac{1}{n}} \cdot \left[\det(I + t\underbrace{Q}\Lambda\underbrace{Q^T})\right]^{\frac{1}{n}}$$

$\det(Q) = \det(Q^T) = 1$

$\det(A) = \lambda_1 * \lambda_2 * \cdots * \lambda_n$

$$= (\det Z)^{\frac{1}{n}} \cdot \left(\prod_{i=1}^{n} 1 + t\lambda_i\right)^{\frac{1}{n}}$$

△ $\prod_{i=1}^{n} (1 + t\lambda_i)^{\frac{1}{n}}$ ⟹ is concave, as it is a geometric mean. (on $\mathrm{dom} f = R^n_{++}$)

$(\det Z)^{\frac{1}{n}}$ ⟹ non negative

∴ $f(x)$ is concave ☆

3.19

(a) show $f(x) = \sum_{i=1}^{r} \alpha_i x_{[i]}$ is convex $f(x)$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 0$

$x_{[i]}$ is the i^{th} largest component of x

hint: $f(x) = \sum_{i=1}^{k} x_{[i]}$ is convex on R^n

<u>soln</u>

△ given hint $f(x)' = \sum_{i=1}^{k} x_{[i]}$ is convex on R^n,
try to express $f(x) = \sum_{i=1}^{r} \alpha_i x_{[i]}$ as $f(x)'$

△ $f(x) = \alpha_1 x_{[1]} + \alpha_2 x_{[2]} + \alpha_3 x_{[3]} + \cdots + \alpha_r x_{[r]}$

$$= \alpha_r (x_{[1]} + x_{[2]} + \cdots + x_{[r-1]} + x_{[r]})$$
$$+ (\alpha_{r-1} - \alpha_r)(x_{[1]} + x_{[2]} + \cdots + x_{[r-1]})$$
$$+ \cdots$$
$$+ (\alpha_2 - \alpha_3)(x_{[1]} + x_{[2]})$$
$$+ (\alpha_1 - \alpha_2)(x_{[1]})$$

$$= \alpha_r \sum_{i=1}^{r} x_{[i]} + (\alpha_{r-1} - \alpha_r) \sum_{i=1}^{r-1} x_{[i]} + \cdots + (\alpha_1 - \alpha_2) \sum_{i=1}^{1} x_{[i]}$$

△ as $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 0$

$\alpha_r, \alpha_{r-1} - \alpha_r, \alpha_{r-2} - \alpha_{r-1}, \ldots, \alpha_1 - \alpha_2 \in R^+$

△ ∴ it is a nonnegative sum of convex functions, which is one of the operation preserve convexity.

△ $f(x)$ is hence convex ✱

3.19

(b) $T(x, w) = x_1 + x_2 \cos w + x_3 \cos 2w + \cdots + x_n \cos(n-1)w$

show $f(x) = -\int_0^{2\pi} \log T(x, w)\, dw$

is convex on $\{x \in R^n \mid T(x, w) > 0,\ 0 \le w \le 2\pi\}$

<u>sol^n</u>

△ $\log T(x, w) = -\log(x_1 + x_2 \cos w + x_3 \cos 2w + \cdots + x_n \cos(n-1)w)$

△ which is convex for a fixed w.

△ hence,

$$f(x) = \int_0^{2\pi} g(x, w)\, dw$$

which is a non-negative sum of convex optimization which is an operation that preserve convexity

△ $\therefore$ $f(x)$ is convex #

3.23

(a) show that for $p > 1$,

$$f(x,t) = \frac{|x_1|^p + \cdots |x_n|^p}{t^{p-1}}$$ is convex on

$\{(x,t) \mid t > 0\}$

Sol^n

△ which could be expressed as a perspective function of
$\|x\|_p^p = |x_1|^p + \cdots + |x_n|^p$

△ perspective function operation preserves convexity

△ ∴ $f(x,t)$ is convex ✗

(b) Show that $$f(x) = \frac{\|Ax+b\|_2^2}{c^Tx+d}$$

is convex on $\{x \mid c^Tx+d > 0\}$, where $A \in R^{m\times n}$, $b \in R^m$, $c \in R^n$ & $d \in R$

Sol^n

△ as mentioned in the problem, $\{x \mid c^Tx+d > 0\}$,
it could be interpretated as $f(y,t) = \frac{y^Ty}{t}$,

where $y = Ax+b$, $t = c^Tx+d$

△ $f(y,t) = \frac{y^Ty}{t}$ is convex, as perspective function operation preserves convexity

△ affine mapping also preserves convexity

△ ∴ $f(x)$ is convex ✗

3.31

$$g(x) = \inf_{\alpha>0} \frac{f(\alpha x)}{\alpha} \qquad f \text{ is convex}$$

soln

(a) show g is homogeneous

Δ for a function to be homogeneous, $f(\gamma x, \gamma y) = \gamma^n f(x, y)$

$\therefore \; g(tx) = t\,g(x)$

new α

Δ $g(tx) = \inf_{\alpha>0} \frac{f(\alpha t x)}{\alpha} = t \inf_{\alpha>0} \frac{f(\alpha t x)}{\alpha t} = t\,g(x)$

$\Delta \therefore$ $g(x)$ is homogeneous

(b) show g is the largest homogeneous underestimator of f.

if $h(x) \le f(x)$ for x

then $h(x) \le g(x)$

soln

$$\underbrace{h(x) = \frac{h(\alpha x)}{\alpha}}_{\text{homogeneous}} \le \frac{f(\alpha x)}{\alpha}, \text{ as } h \text{ is the underestimator,}$$

and $\alpha > 0$

$\inf_{\alpha>0} \frac{f(\alpha x)}{\alpha} \ge \frac{h(\alpha x)}{\alpha}$ holds

$\therefore \; g(x) \ge \frac{h(\alpha x)}{\alpha} = h(x)$

$g(x) \ge h(x)$ ※

3.31 (cont'd)

Sol^n (cont'd)

(c) show that g is convex

Sol^n

recall

△ $g(x) = \inf_{\alpha > 0} \frac{f(\alpha x)}{\alpha}$

let $\alpha = \frac{1}{t}$

$= \inf_{t>0} \frac{f(\frac{x}{t})}{\frac{1}{t}}$

$= \inf_{t>0} t f(\frac{x}{t})$

$= \inf_{t>0} h(x,t)$

which is a perspective function operation

which preserves convexity

△ $\therefore g(x)$ is convex ※

3.36

(a) derive the conjugate function of $f(x) = \max_{i=1,\dots,n} x_i$ on R^n

Sol^n

△ recall conjugate function :

$f: R^n \to R$

$f^*: R^n \to R$, $f^*(y) = \sup_{x \in \text{dom} f} (y^T x - f(x))$

△ $f^*(y) = \sup_{x \in R^n} (y^T x - x_m) = \sup_{x \in R^n} \left(\sum_{i \neq m}^{n} y_i x_i + y_m x_m - x_m \right)$

(let x_m be the value of $f(x)$)

3.36 (cont'd)

(a) (cont'd)

Sol^n (cont'd)

$\because$ from $\sup_{x \in R^n} \left(\sum_{i \neq m}^{n} y_i x_i + y_m x_m - x_m \right)$ — (#)

only when $\sum_{i=1}^{n} y_i = 1$ and $y_1, y_2 \ldots y_n \geq 0$

$\sup_{x \in R^n} \left(\sum_{i \neq m}^{n} y_i x_i + y_m x_m - x_m \right)$ is bounded.

e.g.

case ① $y_i < 0$, let random $x_i \to -\infty$

$$\sup_{x \in R^n} \left(\underbrace{\sum_{i \neq m}^{n} y_i x_i}_{ⓐ} + \underbrace{y_m x_m}_{ⓑ} - \underbrace{x_m}_{ⓒ} \right)$$

term ⓐ $\to \infty$

$\therefore$ (#) $\to \infty$

case ② $y_i \geq 0$ (all i's), $\sum_{i=1}^{n} y_i < 1$, let $x_1, x_2, \cdots x_n \to -\infty$

$$\sup_{x \in R^n} \left(\underbrace{\sum_{i \neq m}^{n} y_i x_i}_{ⓐ} + \underbrace{y_m x_m}_{b} - \underbrace{x_m}_{c} \right)$$

term ⓒ $\to \infty$

$\therefore$ (#) $\to \infty$

case ③ $y_i \geq 0$ (all i's) $\sum_{i=1}^{n} y_i > 1$, let random $x_i \to \infty$

$$\sup_{x \in R^n} \left(\underbrace{\sum_{i \neq m}^{n} y_i x_i}_{ⓐ} + \underbrace{y_m x_m}_{ⓑ} - \underbrace{x_m}_{ⓒ} \right)$$

term ⓐ $\to \infty$

$\therefore$ (#) $\to \infty$

3.36 (cont'd)

(a) (cont'd)

soln (cont'd)

case ④ $y_i \geq 0$ (all is) $\sum_{i=1}^{n} y_i = 0$, let random $x_i \to \infty$

$$\sup_{x \in R^n} \Big(\underbrace{\sum_{i \neq m}^{n} y_i x_i}_{a} + \underbrace{y_m x_m}_{b} - \underbrace{x_m}_{c} \Big)$$

$$a + b \leq c$$

find $\sup_{x \in R^n} (a + b - c)$ at

$$a + b = c$$

$\therefore \sup_{x \in R^n} \left(\sum_{i \neq m}^{n} y_i x_i + y_m x_m - x_m \right)$ for case

④ $= 0$

$\triangle$
$$f^*(y) = \begin{cases} 0, & y_1, y_2 \ldots y_n \geq 0, \ \sum_{i=1}^{n} y_i = 1 \\ \infty, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0, & y \succeq 0, \ \sum_{i=1}^{n} y_i = 1 \\ \infty, & \text{otherwise} \end{cases}$$

(d) derive conjugate function $f(x) = x^P$ on R_{++}, $P > 1$, case 1

$P < 1$, case 2

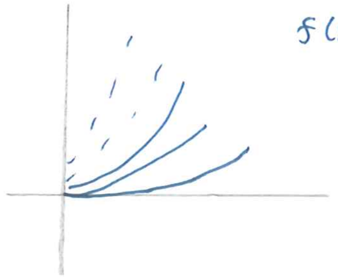
soln

case 1, $P > 1$:

$\triangle$

recall

$$f^*(y) = \sup_{x \in \mathrm{dom} f} (y^T x - f(x))$$



$f(x) = x^p$ on R_{++}

3.36 (cont'd)

(d) (cont'd)

Sol^n (cont'd)

$$\sup_{x \in dom f} \left(y^T x - f(x) \right)$$

$$\Rightarrow \sup_{x \in dom f} \left(y^T x - x^p \right)$$

$\Rightarrow$ has sup @ $\frac{d}{dx}(y^T x - x^p) = 0$

$$\Rightarrow y - p x^{p-1} = 0$$

$$\Rightarrow y = p x^{p-1}$$

$$\Rightarrow x^{p-1} = \frac{y}{p}$$

$$\Rightarrow x = \left(\frac{y}{p}\right)^{\frac{1}{p-1}}$$

$$\Rightarrow @ \; x = \left(\frac{y}{p}\right)^{\frac{1}{p-1}}$$

p = 2

y = -1

$$\Rightarrow y \cdot \left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}}$$

$-1 \cdot \left(\frac{-1}{2}\right)^1 - \left(\frac{-1}{2}\right)$

$\triangle \therefore \sup_{x \in dom f} \left(y^T x - f(x) \right) = y \left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}}$

yet, the above only holds when $y \geq 0$

for $y < 0$, $y^T x$ & x^p intersect @ 0

$$\therefore \sup_{x \in dom f} \left(y^T x - f(x) \right) = 0$$

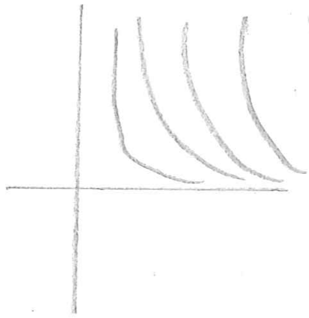
$$\therefore \sup_{x \in dom f} \left(y^T x - f(x) \right) = \begin{cases} y \left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

$\divideontimes$

3.36 (cont'd)

(d) (cont'd)

Sol^n (cont'd)



$f(x) = x^p$ on R_{++}

case 2, $P < 1$

- for $y > 0$

$\sup_{x \in \mathrm{dom} f} (y^T x - x^p)$ is unbounded $\Rightarrow \infty$

- for $y = 0$

no solution

- for $y < 0$

similar operation to case I

$$\sup_{x \in \mathrm{dom} f} (y^T x - x^p) = y \left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}}$$

※

3.42

$$W(x) = \sup \left\{ T \;\middle|\; |x_1 f_1(t) + \cdots + x_n f_n(t) - f_0(t)| \le \epsilon \quad t \in [0, T] \right\}$$

show $w(x)$ is quasiconcave.

Sol^n

- for a function to be quasiconcave, the superlevel set of - i.e., $\{x \mid f(x) \ge \alpha\}$ is convex for all α.

- $|x_1 f_1(t) + \cdots + x_n f_n(t) - f_0(t)| \le \epsilon$

$\Rightarrow -\epsilon \le |x_1 f_1(t) + \cdots + x_n f_n(t) - f_0(t)| \le \epsilon$

3.42 (cont'd)

sol^n (cont'd)

① for different x,
② get T, that satisfy $|\cdots| \le \epsilon$
③ get $\sup\{T\}$.

△ from 0 to T,

for each t

$$-\epsilon \le x_1 f_1(t) + \cdots + x_n f_n(t) - f_0(t) \le \epsilon$$

X is a convex set subject to halfspaces.

△ $$W(x) = \sup\{ T \mid |x_1 f_1(t) + \cdots + x_n f_n(t) - f_0(t)| \le \epsilon,\ t \in [0, T] \}$$

superlevel set of $W(x) = \{ x \mid W(x) \ge \alpha \}$,

which is equivalent to

from 0 to α for t,

$$-\epsilon \le x_1 f_1(t) + \cdots + x_n f_n(t) - f_0(t) \le \epsilon$$

which is convex, (subject to halfspaces).

△ $W(x)$ is hence quasiconvex ✡

3.47

show f is log-concave i.f.f $\dfrac{f(y)}{f(x)} \le \exp\left(\dfrac{\nabla f(x)^T (y-x)}{f(x)} \right)$

($f: R^n \to R$, $\operatorname{dom} f$ is convex, $f(x) > 0$, $x \in \operatorname{dom} f$,

$x, y \in \operatorname{dom} f$)

sol^n

△ take log of the inequality

$$\log\left(\frac{f(y)}{f(x)}\right) \le \log\left(\exp\left(\frac{\nabla f(x)^T (y-x)}{f(x)} \right) \right)$$

$$\Rightarrow \log f(y) - \log f(x) \le \frac{\nabla f(x)^T (y-x)}{f(x)}$$

3.47 (cont'd)

soln (cont'd)

$\Rightarrow$ which is in a first-order condition form of convex function

$$\Rightarrow \log f(y) \le \log f(x) + \frac{\nabla f(x)^T}{f(x)} (y-x)$$

$$\Rightarrow -\log f(y) \ge -\log f(x) + \nabla(-\log f(x))(y-x)$$

$$\Rightarrow g(y) \ge g(x) + \nabla g(x)(y-x)$$

$\Rightarrow$ where $g(x) = -\log f(x)$ ※

3.49

(a) show $f(x) = e^x/(1+e^x)$, $\operatorname{dom} f = R$ is log-concave.

soln

Δ for a function to be log-concave:

a function $f: R^n \to R$, $f(x) > 0$, $x \in \operatorname{dom} f$

and $\log f$ is concave

$$\Delta \quad \log(f(x)) = \log(e^x/(1+e^x))$$

$$= \log e^x - \log(1+e^x)$$

$$= \underbrace{x}_{ⓐ} - \underbrace{\log(1+e^x)}_{ⓑ}$$

Δ a is concave

$-\log(1+e^x)$ is concave

Δ nonnegative weighted sum operation

Δ $\therefore$ $f(x)$ is log-concave ※

3.49 (cont'd)

(b) show $f(x) = \dfrac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$, $\mathrm{dom}\, f = \mathbb{R}^n_{++}$

is log-concave

Sol^n

△ we need to show $\log f(x)$ is concave

△ $f(x) = \dfrac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots \frac{1}{x_n}}$

△ $\log f(x) = \log \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right)^{-1}$

$= -\log \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right)$

△ similar to 3.17, recall second-order condition, try to show $\nabla^2 \log f(x) \preceq 0$, we let $g(x) = \log f(x)$

△ $\nabla g(x) = \dfrac{\partial}{\partial x} -\log \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right)$

$$= \begin{bmatrix} -\dfrac{(-1)\, x_1^{-2}}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \\ -\dfrac{(-1)\, x_2^{-2}}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \\ \vdots \\ -\dfrac{(-1)\, x_n^{-2}}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \end{bmatrix}$$

3.49 (cont'd)

(b) (cont'd)

sol^n (cont'd)

△ $$\nabla^2 g(x) = \begin{bmatrix} \frac{\partial}{\partial x_1}\left(\frac{x_1^{-2}}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}\right) & \frac{\partial}{\partial x_1}\left(\frac{x_2^{-2}}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}\right) & \cdots \\ \frac{\partial}{\partial x_2}\left(\frac{x_1^{-2}}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}\right) & \ddots & \vdots \\ \frac{\partial}{\partial x_n}\left(\frac{x_1^{-2}}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}\right) & \cdots & \frac{\partial}{\partial x_n}\left(\frac{x_n^{-2}}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2x_1^{-3}}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} + -1\cdot\frac{-1\cdot x_1^{-2}\cdot x_1^{-2}}{\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)^2} & & \\ -1\frac{x_1^{-2}(-1)x_2^{-2}}{\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)^2} & \ddots & \\ & & \frac{-2x_n^{-3}}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} + \frac{x_n^{-4}}{\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)^2} \end{bmatrix}$$

△ from above

$$\left[\nabla^2 g(x)\right]_{ij,\ i=j} = \frac{-2x_i^{-3}}{\sum_{i=1}^{n}\frac{1}{x_i}} + \frac{x_i^{-4}}{\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^2}$$

$$\left[\nabla^2 g(x)\right]_{ij,\ i\neq j} = \frac{x_i^{-2}x_j^{-2}}{\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^2}$$

△ to show $\nabla^2 g(x) \preceq 0$, we need to show

for any vector y

$$y^T \nabla^2 g(x)\, y \leq 0$$

	a	b	c
a			
b			
c			

3.49 (cont'd)

(b) (cont'd)

sol^n (cont'd)

- again

sum 1   sum 2

$$[\nabla^2 g(x)]_{ij},\ i=j = \boxed{\frac{x_i^{-4}}{\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2}} - \boxed{\frac{2x_i^{-3}}{\sum_{i=1}^{n} \frac{1}{x_i}}}$$

$$[\nabla^2 g(x)]_{ij},\ i\neq j = \boxed{\frac{x_i^{-2}x_j^{-2}}{\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2}}$$

$$\Rightarrow\ y^T \nabla^2 g(x)\, y = \text{sum 1} + \text{sum 2}$$

$$= \frac{1}{\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2}\Big[\, y_1^T x_1^{-4} y_1 + y_2^T x_2^{-4} y_2 + \cdots + y^T x_n^{-4} y$$

$$+ y_1 x_1^{-2} x_2^{-2} y_2 + \cdots + y_{n-1} x_{n-1}^{-2} x_n^{-2} y_n$$

$$+ y_2 x_2^{-2} x_1^{-2} y_1 + \cdots + y_n x_n^{-2} x_{n-1}^{-2} y_{n-1}$$

$$+$$

$$-2\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)\left(y_1 x_1^{-3} y_1 + y_2 x_2^{-3} y_2 + \cdots + y_n x_n^{-3} y_n\right)\Big]$$

$$= \frac{1}{\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2}\left[\left(\sum_{i=1}^{n} \frac{y_i}{x_i^2}\right)^2 + (-2)\cdot\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)\left(\sum_{i=1}^{n} \frac{y_i}{x_i^3}\right)\right]$$

$$\Rightarrow\ \underbrace{\frac{1}{\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^2}}_{\geq 0}\left[\left(\sum_{i=1}^{n} \frac{y_i}{x_i^2}\right)^2 - 2\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)\left(\sum_{i=1}^{n} \frac{y_i}{x_i^3}\right)\right] \leq 0$$

$$\Rightarrow\ \left(\sum_{i=1}^{n} \frac{y_i}{x_i^2}\right)^2 - 2\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)\left(\sum_{i=1}^{n} \frac{y_i}{x_i^3}\right) \leq 0$$

3.49 (cont'd)

(b) (cont'd)

sol^n (cont'd)

Δ recall Cauchy-Schwarz inequality.

$(a^T a)(b^T b) \geq (a^T b)^2$

here $a_i = x_i^{-\frac{1}{2}}$ $b_i = y_i^{\frac{1}{2}} x_i^{\frac{3}{2}}$

Δ hence $y^T \nabla^2 g(x) y \leq 0$,

hence $g(x)$ is concave,

hence $\log f(x)$ is concave,

hence $f(x)$ is log-concave ✱

3.57

✱ show that function $f(X) = X^{-1}$ is matrix convex on S_{++}^n

sol^n

Δ to show convexity, we show for any vector y

$g(X) = y^T X^{-1} y$ is convex

Δ as per example 3.4, we could establish convexity of g through epigraph

$$\text{epi}\, g = \{(X, t) \mid X \succ 0,\ y^T X^{-1} y \leq t,\ y \in R^n\}$$

$$= \left\{ (X, t) \,\middle|\, \begin{bmatrix} X & y \\ y^T & t \end{bmatrix} \succeq 0,\ X \succ 0 \right\}$$

which is from Schur complement condition for PSD

3.57 (cont'd)

(b) (cont'd)

soln (cont'd)

△ it is an linear matrix inequality.

∴ epi g is convex

∴ $g(X)$ is convex,

∴ $f(X)$ is convex ※

3.59

show $f: R^n \to R^m$, dom f convex

is K-convex ($K \subseteq R^m$, proper convex cone)
($\to \succeq_K$)

i.f.f.

$$\sum_{i,j=1}^{n} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} y_i y_j \succeq_K 0 \qquad x \in \text{dom} f, \quad y \in R^n$$

R^m

soln

△ according to the dual characterization of K-convexity, f is K-convex i.f.f. for every $w \succeq_{K^*} 0$, $w^T f$ is convex.

△ we need to show $\sum_{i,j=1}^{n} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} y_i y_j \succeq_K 0$

generalized inequality induced by cone K

so that $f(x)$ ~~is~~ K-convex.

△ $w^T f$ is convex when $\nabla^2 (w^T f) \succeq 0$

△ $\nabla^2 w^T f \succeq 0$ when any $y \in R^n$, $y^T \nabla^2 w^T f \, y \geq 0$

3-59 (cont'd)

Sol^n (cont'd)

$\triangle$ $\nabla^2 (w^T f(x))$

$$= \nabla^2 \left([w_1 \ w_2 \ \cdots \ w_m] \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix} \right)$$

$$= \nabla^2 (w_1 f_1(x) + w_2 f_2(x) + \cdots + w_m f_m(x))$$

$$= \sum_{k=1}^{m} w_k \nabla^2 f_k(x) \quad \in R^{n\times n}$$

$\triangle$ $y^T \nabla^2 (w^T f(x)) y$

$$= \sum_{k=1}^{m} \left(\sum_{\substack{i=1\\j=1}}^{n} y_i w_k \nabla^2 f_k(x) y_j \right)$$

$$= \sum_{k=1}^{m} w_k \sum_{\substack{i=1\\j=1}}^{n} \nabla^2 f_k(x) y_i y_j$$

where we need

$$\underbrace{\sum_{k=1}^{m} w_k}_{\geq 0} \underbrace{\sum_{\substack{i=1\\j=1}}^{n} \nabla^2 f_k(x) y_i y_j}_{} \geq 0$$

$\downarrow$ for $k = 1, 2, \ldots, m$

$$= \sum_{\substack{i=1\\j=1}}^{n} \frac{\partial^2 (f(x))}{\partial x_i \partial x_j} y_i y_j \succeq 0$$

$\forall$