— proof the false statement

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Midterm Exam for ELEC 5470

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2.5

(3') **Problem 1.** Is the set $\{a \in \mathbf{R}^k \mid p(0) = 1,\ |p(t)| \le 1 \text{ for every } t \in [\alpha, \beta]\}$, where

$$p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1},$$

convex? Please explain.

△ $P(t) = \begin{bmatrix} 1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} [t^0, t^1, \dots, t^{k-1}]$ a_1

and $|1 + a_2 t + \cdots + a_k t^{k-1}| \le 1$

△ let x, y be two points in the set

$\Rightarrow x, y \in \{a \in \mathbb{R}^k \dots\}$

$\Rightarrow |1 + x_2 t + \cdots + x_k t^{k-1}| \le 1$

$\Rightarrow |1 + y_2 t + \cdots + y_k t^{k-1}| \le 1$

△ let $z = \theta x + (1-\theta) y$ where $\theta \in [0,1]$

we need to show $|1 + z_2 t + \cdots + z_k t^{k-1}| \le 1$ to prove the set convex.

△ $\left| 1 + (\theta x_2 + (1-\theta) y_2) t + \cdots + (\theta x_k + (1-\theta) y_k) t^{k-1} \right| = \left| 1 + \theta (x_2 t + \cdots + x_k t^{k-1}) + (1-\theta)(y_2 t + \cdots + y_k t^{k-1}) \right|$

should be ≤ 1, as $\theta \in [0,1]$

△ thus, set $\{a \in \mathbb{R}^k \mid p(0) = 1, |p(t)| \le 1, \forall t \in [\alpha, \beta]\}$ is convex #

Skipped steps

▲ $p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1}$

$\Rightarrow -1 \le a_1 + a_2 t + \cdots + a_k t^{k-1} \le 1$ where $t \in [\alpha, \beta]$

▲ from above, it can be told that it is a slab with infinite combination of t

▲ $\{a \in \mathbb{R}^k \mid -1 \le a_1 + a_2 t + \cdots + a_k t^{k-1} \le 1\}$ — ①

▲ $\{a \in \mathbb{R}^k \mid a_1 = 1\}$ from $P(0) = 1$ — ②

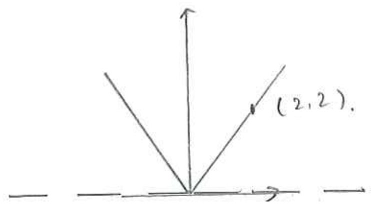
▲ the set is an intersection of ① & ②

Also, $x_1 = 1$, $y_1 = 1$

$\Rightarrow z_1 = 1$

▲ intersection of convex set is convex.

...



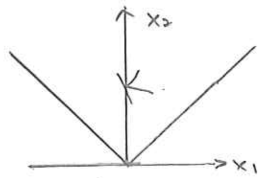
(3') **Problem 2.** Describe the dual cone for $K = \{(x_1, x_2) | \, |x_1| \leq x_2\}$. Please show the proof.

3

△ recall def. of dual cone.

$K^* = \{ y \mid x^T y \geq \text{ for all } x \in K \}$

△ as described in the given problem $K = \{ (x_1, x_2) \mid |x_1| \leq x_2 \}$



Show it is a self-dual cone:

$\begin{cases} K \subset K^* \\ K^* \subset K \end{cases}$

which is a cone $\subseteq \mathbb{R}^2$,

with two rays at right angle.

△ therefore, for all $y^T x \geq 0$,

all y is confined within the cone to satisfy $\{ y \mid x^T y \geq 0 \}$

△ therefore, $K^* = \{ y \mid |y_1| \leq y_2 \}$

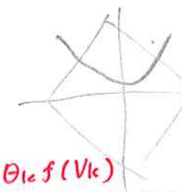
which is self-dual

- counter example + Jensen's inequality
- $Z \in P$, and $f(z) > f(V_i)$
- as $Z \in P$ and is convex hull of $\text{conv}\{V_1, \ldots V_k\}$
  - z can be expressed as $z = \theta_1 V_1 + \cdots + \theta_k V_k \quad \theta \succeq 0 \quad 1^T\theta = 1$.
  - recall Jensen's inequality: $f(\theta_1 V_1 + \cdots + \theta_k V_k) \leq \theta_1 f(V_1) + \theta_2 f(V_2) + \cdots + \theta_k f(V_k)$

$z = f(\theta_1 V_1 + \cdots + \theta_k V_k) \leq \theta_1 f(V_1) + \cdots + \theta_k f(V_k) < \theta_1 f(z) + \cdots + \theta_k f(z) = f(z)$

$\downarrow$ contradiction

hence: statement must be true.



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(5') **Problem 3.** Show that the maximum of a convex function f over the polyhedron $\mathcal{P} = \textbf{conv}\{v_1, ..., v_k\}$ is achieved at one of its vertices, i.e.,

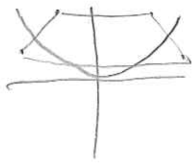
$$\sup_{x \in \mathcal{P}} f(x) = \max_{i=1,\ldots,k} f(v_i).$$

$\triangle$ polyhedron $P = \text{conv}\{V_1, \ldots V_k\}$

could be first expressed as a series of halfspaces.

$P = \text{conv}\{V_1, \ldots, V_k\}$

$= \{x \mid \theta_1 V_1 + \cdots + \theta_k V_k, \ \theta \succeq 0, \ 1^T\theta = 1\}$

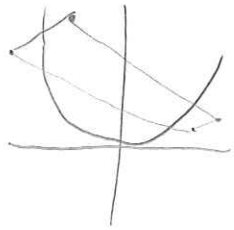


$\Rightarrow x = \theta_1 V + \theta_2 V_2 + \cdots + \theta_k V_k$

$= (1 - \theta_2 - \theta_3 - \cdots - \theta_k) V_1 + V_2\theta_2 + \cdots + V_k\theta_k$

$= V_1 + (V_2 - V_1)\theta_2 + (V_3 - V_1)\theta_3 + \cdots + (V_k - V_1)\theta_k$

let $B = [V_2 - V_1 \ \cdots \ V_k - V_1] \quad y = [\theta_2, \ldots, \theta_k]$



$x = V_1 + By$

let $A = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$ where

$\cancel{x}B = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} B = \begin{bmatrix} I \\ 0 \end{bmatrix}$

$\therefore Ax = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} (V_1 + By)$

$\begin{cases} A_1 x = A_1 V_0 + A_1 By \\ A_2 x = A_2 V_0 + A_2 By \end{cases}$

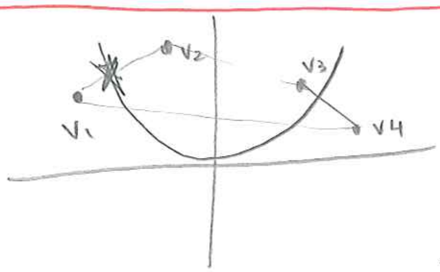
$\begin{cases} A_1 x = A_1 V_0 + y \\ A_2 x = A_2 V_0 \end{cases}$

$A_1 x \succeq A_1 V_0$

$A_2 x = A_2 V_0$

No, I'm sure it's Correct.

$\triangle$ Yet, the problem statement might be false, counter example below



the maximum of a convex function over $x \in P$ is at ✱, does not neccessary lies on the one of the vertices ✱

$$\left(\frac{\sqrt{\alpha_k}\, y_k}{x_k} \cdot \sqrt{\alpha_k}\right)^2 = \frac{\alpha_k^2 y_k^2}{x_k^2}.$$

$$\left(\frac{\sqrt{\alpha_k}\, y_k}{x_k}\right)^2 \quad (\sqrt{\alpha_k})^2$$

$$\left(\frac{\sqrt{\alpha_k}\, y_k}{x_k} \cdot \frac{\sqrt{\alpha_k}\, y_k}{x_k}\right) x_k x_k$$

5

(7') **Problem 4.** Show that the function

$$f(x) = \prod_{k=1}^{n} x_k^{\alpha_k}, \qquad \mathbf{dom} f = \mathbf{R}^n_{++},$$

is concave, where α_k's are non-negative numbers with $\sum_{k=1}^{n} \alpha_k \leq 1$.

$\triangle$ $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots x_n^{\alpha_n}$

$\alpha_1 y_1$

$\alpha_1 \geq 0,\ \alpha_2 \geq 0 \cdots \alpha_n \geq 0$

$\alpha_1 + \alpha_2 + \cdots + \alpha_k \leq 1$

$\frac{\alpha_1 y_1^2}{x_1^2} \cdot \alpha_1$

$$\triangle\ \nabla f(x) = \begin{bmatrix} \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots x_n^{\alpha_n} \\ \alpha_2 x_1 \quad x_2^{\alpha_2 - 1} \cdots x_n^{\alpha_n} \\ \vdots \\ \alpha_n x_1 x_2 \cdots x_n^{\alpha_n - 1} \end{bmatrix} \Rightarrow \left[\alpha_k \frac{\prod_{k=1}^{n} x_k^{\alpha_k}}{x_k}\right]_k$$

$\sqrt{\frac{\alpha_1 y_1^2}{x_1^2} + \frac{\alpha_2 y_2^2}{x_2^2}} \sqrt{\alpha_1 + \alpha_2}$

$$\triangle\ \nabla^2 f(x) = \begin{bmatrix} \frac{\alpha_1(\alpha_1 - 1)}{x_1^2} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} & \frac{\alpha_1 \alpha_2}{x_1 x_2} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} & \cdots \\ \frac{\alpha_2 \alpha_1}{x_2 x_1} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} & \frac{\alpha_2(\alpha_2 - 1)}{x_2^2} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} & \vdots \\ \vdots & \vdots & \ddots \\ \frac{\alpha_n \alpha_1}{x_n x_1} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} & \cdots & \frac{\alpha_n(\alpha_n - 1)}{x_n^2} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \end{bmatrix}$$

$$\left(\frac{\alpha_k y_k}{x_k}\right)^2 - \frac{\alpha_k y_k^2}{x_k^2}$$

$$(a^T a)(b^T b) \geq (a^T b)^2$$

$a = \frac{\sqrt{\alpha_k}\, y_k}{x_k} \quad b = \sqrt{\alpha_k}$

$$(a^T a)(b^T b) = \left(\frac{\sqrt{\alpha_k}\, y_k}{x_k}\right)^2 \cdot \sqrt{\alpha_k}$$

$$= \frac{\alpha_k^2 y_k^2}{x_k^2}.$$

$\triangle$ to show function is concave,

for any vector $y \in \mathbb{R}^n$, $y^T \nabla^2 f(x) y \leq 0$

$\triangle$ for $\nabla^2 f(x)$:

$$\boxed{\begin{array}{ll} i = j & \frac{\alpha_i \alpha_i}{x_i x_i} \prod_{k=1}^{n} x_k^{\alpha_k} - \frac{\alpha_i}{x_i x_i} \prod_{k=1}^{n} x_k^{\alpha_k} \\ i \neq j & \frac{\alpha_i \alpha_j}{x_i x_j} \prod_{k=1}^{n} x_k^{\alpha_k} \end{array}}$$

$1\times2 \quad 2\times2 \quad 2\times1$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$[y_1\ y_2] \begin{bmatrix} A & C \\ B & D \end{bmatrix}$$

$$[y_1 A + y_2 B \quad y_1 C + y_2 D] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$y_1^2 A + y_2^2 B + y_1^2 C + y_2^2 D$

Correct until here

$\triangle$ $y^T \nabla^2 f(x) y$:

$$\left(\frac{\alpha_k y_k}{x_k}\right)^2 - \frac{\alpha_k y_k^2}{x_k^2}$$

we want

$$= \left(\prod_{k=1}^{n} x_k^{\alpha_k}\right) \left[\left(\sum_{k=1}^{n} \frac{\alpha_k y_k}{x_k}\right)^2 - \left(\sum_{k=1}^{n} \frac{\alpha_k^2 y_k^2}{x_k^2} \frac{1}{\alpha_k}\right)\right]$$

while $\sum_{k=1}^{n} \frac{1}{\alpha_k} \geq 1$

therefor

wrong

$\triangle$ using Cauchy - Schwarz inequality

$(a^T a)(b^T b) \geq (a^T b)^2$, $\therefore y^T \nabla^2 f(x) y \leq 0$, $\therefore \nabla^2 f(x) \preceq 0$ $\therefore f$ conca

$a = \frac{\alpha_k}{x_k}$ $b = y_k$, $\sum \frac{1}{\alpha_k} \geq 1$

$$\left(\sum_{i=1}^{n} \frac{\alpha_k y_k}{x_k} \right)^2 - \left(\sum_{i=1}^{n} \frac{\alpha_k y_k^2}{x_k} \right).$$

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(7') **Problem 5.** Suppose we are given matrices $\bar{A} \in \mathbf{R}^{m \times n}$ and $V \in \mathbf{R}^{m \times n}$, where every entry V_{ij} of the matrix V is non-negative. Define the following set:

$$\mathcal{U} = \{A \in \mathbf{R}^{m \times n} \mid \bar{A}_{ij} - V_{ij} \leq A_{ij} \leq \bar{A}_{ij} + V_{ij},\ \forall i = 1, ..., m,\ j = 1, ..., n\}.$$

Consider the following *robust LP* with decision variable $x \in \mathbf{R}^n$:

$$\begin{array}{ll} \text{minimize} & c^{\mathsf{T}} x \\ \text{subject to} & Ax \preceq b \text{ for all } A \in \mathcal{U}. \end{array}$$

Express this problem as an LP. The LP you construct should be efficient, i.e., it should not have dimensions that grow exponentially with m or n.

$\triangleright$ $\bar{A} \in R^{m \times n}$ $\quad V \in R^{m \times n}$

$\mathcal{U} = \{ A \in R^{m \times n} \mid \bar{A}_{ij} - V_{ij} \leq A_{ij} \leq \bar{A}_{ij} + V_{ij}\ \ \forall i = 1 \ldots m,\ j = 1 \ldots n \}$

$\triangleright$ could express $A_{ij} = \bar{A}_{ij} + \Delta A_{ij}$

minimize $C^T x$
subject to $Ax \preceq b$.

$\Rightarrow$ ~~minimize $C^T x$ subject to $(\bar{A}+V)$~~

$\triangle$ minimize $C^T x$
subject to $Ax \preceq b$, $\forall A \in \mathcal{U}$

scalar $\quad$ vector

$\Rightarrow$ minimize $C^T x$
subject to $(\bar{A}_{ij} + \Delta A_{ij})\, x \preceq b$ $\quad$ don't understand

as V_{ij} entries ≥ 0 $\quad Vx \succeq 0$

$\therefore$ hence if $(\bar{A} + V)x \preceq b$

so does $(\bar{A} - V)x \preceq b$.

minimize $C^T x$
subject to $\bar{A}x + Vx \preceq b$
$\bar{A}x - Vx \preceq b$

minimize $C^T x$
subject to $\bar{A}x + V|x| \preceq$

minimize $C^T x$
subject to
$\bar{A}x + Vy \preceq$
$-y \preceq x \preceq y$

$\triangle \Rightarrow$ hence the robust LP could be reformulated as

minimize $C^T x$
subject to $(\bar{A} + V)x \preceq b$