17/17

2.5

Find the distance between two parallel hyperplanes

$\{x \in R^n \mid a^T x = b_1\}$,

$\{x \in R^n \mid a^T x = b_2\}$

sol^n

- for a hyperplane $a^T x = b$, the normal vector is a.
  let L be a line passing through hyperplane 1 ($a^T x = b_1$) at x_1 :

  $L = x_1 + at \quad \forall t \in R$

- as hyperplane 1 & 2 are parallel
  and $L = x_1 + at$ is a line starting from x_1 with normal vector a
  hyperplane 2 has intersecting point with L

  let $x_2 = x_1 + at$

  $\Rightarrow a^T x_2 = a^T(x_1 + at) = b_2$

  $\Rightarrow a^T(x_1 + at) = b_2$

  $\Rightarrow a^T x_1 + a^T a t = b_2$

  $\Rightarrow b_1 + a^T a t = b_2$

  $\Rightarrow t = \dfrac{|b_2 - b_1|}{a^T a}$

- distance $= \|x_2 - x_1\|_2 = \|at\|_2 = \dfrac{|b_2 - b_1|}{\|a\|}$ #

2.11 Show hyperbolic set $\{x \in R_+^2 \mid x_1 x_2 \geq 1\}$ is convex.

<u>sol^n</u> $\{x \in R_+^2 \mid \prod_{i=1}^{n} x_i \geq 1\}$ is convex.

△ definition of convex set :

A set C is convex if

$$\theta x_1 + (1-\theta) x_2 \in C, \ \forall\ x_1, x_2 \in C$$
$$\theta \in [0,1]$$

△ to proof $\{x \in R_+^2 \mid x_1 x_2 \geq 1\}$ is convex

any two points within the set should satify the above condition.

let a, b be 2 points in the hyperbolic set

let $C = \theta a + (1-\theta) b$, $\theta \in [0,1]$

$C_1 \cdot C_2$ should $C_1 \cdot C_2 \geq 1$

△ as given by the hint, when $a, b \geq 0$ & $\theta \in [0,1]$

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b$$

$$\Rightarrow a^\theta b^{1-\theta} \leq C$$

$$\Rightarrow (a_1 \cdot a_2)^\theta \cdot (b_1 \cdot b_2)^{1-\theta} \leq C_1 \cdot C_2$$

$$\Rightarrow 1 \leq (a_1 a_2)^\theta \cdot (b_1 \cdot b_2)^{1-\theta} \leq C_1 C_2$$

$$\therefore C \in \{x \in R_+^2 \mid x_1 x_2 \geq 1\}$$

$\therefore$ set $\{x \in R_+^2 \mid x_1 x_2 \geq 1\}$ is convex ⋕

2.11 (cont'd)

- for more generalized case $\{x \in R_+^n \mid \prod_{i=1}^n x_i \geq 1\}$ is convex

similarly, from the hint

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b \quad ,$$

for $a, b \in$ hyperbolic Set

$$\prod_{i=1}^n a_i \geq 1 \quad , \quad \prod_{i=1}^n b_i \geq 1$$

$$\therefore \quad 1 \leq \left(\prod_{i=1}^n a_i\right)^\theta \left(\prod_{i=1}^n b_i\right)^{(1-\theta)} \leq \prod_{i=1}^n (\theta a_i + (1-\theta) b_i)$$

where $\theta a + (1-\theta) b = c$

$$\prod_{i=1}^n c_i \geq 1$$

$\therefore$ set $\{x \in R_+^n \mid \prod_{i=1}^n x_i \geq 1\}$ is convex $\divideontimes$

2.12 (a) (b) (c) (d) (f) (g) are convex, below shows why:

soln

(a) set $\{x \in R^n \mid \alpha \leq a^T x \leq \beta\}$

- above describes an intersection of two halfspaces (polyhedron)

$\alpha \leq a^T x$, $a^T x \leq \beta$

as halfspaces are convex, so is the set

(b) set $\{x \in R^n \mid \alpha_i \leq x_i \leq \beta_i, \ i = 1 \dots n\}$

- above describes a special case of halfspaces of intersection (polyhedron)

as halfspaces are convex, so is the set.

(c) set $\{x \in R^n \mid a_1^T x \leq b_1, \ a_2^T x \leq b_2\}$

- above describes an intersection of two half spaces. (polyhedron)

as halfspaces are convex, so is the set.

(d) set $\{x \mid \|x - x_0\| \leq \|x - y\|_2 \text{ for all } y \in S\}$ where $S \subseteq R^n$

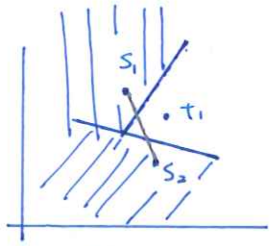
- above describes for different point in S, there exists different half-planes that form different halfspaces. Therefore, the set is convex.

2.12 (cont'd)

(e)

$\triangle$ set $\{x \mid dist(x, S) \le dist(x, T)\}$ is not neccessarily convex, as some counter example could be found:

let $S, T \subseteq R^2$



the striped area being the set,

yet, $\theta s_1 + (1-\theta) s_2 \notin C$, i.e.,

the line segment points between s_1, s_2

does not all lie in set $\{x \mid dist(x,S) \le dist(x,T)\}$.

Hence, the set is not convex ※

(f) set $\{x \mid x + S_2 \subseteq S_1\}$ where $S_1, S_2 \subseteq R^n$, S_1 is convex.

$\triangle$ let $x_1 + s_2 = s_1 \in S_1$ — (1)

$x_2 + s_2 = s_1' \in S_1$ — (2)

to proof set $\{x \mid x + S_2 \subseteq S_1\}$

we need

$\theta x_1 + (1-\theta) x_2 + s_2 \in S_1$ — (3)

$\triangle$ from (1) & (2)

we have $\theta s_1 + (1-\theta) s_1' \in S_1$ (as it is convex)

$\theta s_1 + (1-\theta) s_1'$

$= \theta (x_1 + s_2) + (1-\theta)(x_2 + s_2)$

$= \theta x_1 + \cancel{\theta s_2} + (1-\theta) x_2 + \cancel{(1-\theta) s_2}$

$= \theta x_1 + (1-\theta) x_2 + s_2$ — (4)

from (3) & (4) we then infer

set $\{x \mid x + S_2 \subseteq S_1\}$ is convex

※

2.12 (cont'd)

(g)

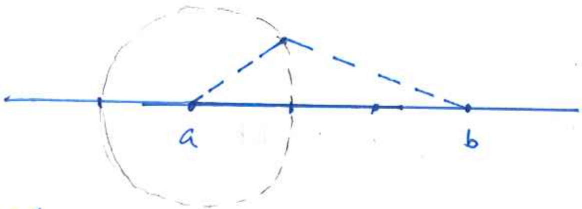
set $\{x \mid \|x-a\|_2 \le \theta \|x-b\|_2\}$ $a \neq b$ & $0 \le \theta \le 1$.

- if $\theta = 0$

  set is a single point — (1)

- if $0 < \theta < 1$

  set is a ball, below shows $\theta = 0.5$ on R^2 — (2)



- if $\theta = 1$

  set is a halfspace — (3)

- from (1), (2), (3) are all convex.

  hence set $\{x \mid \|x-a\|_2 \le \theta \|x-b\|_2\}$ is convex.

2.14 $S \subseteq R^n$, $\|\cdot\|$ is a norm on R^n $\qquad \inf_{x \in R^{+}} (x) = 0$

sol^n

(a) $S_a = \{x \mid dist(x,S) \le a\}$, $\underset{S \text{ is convex}}{a \ge 0}$, $dist(x,S) = \overset{\text{infimum}}{\inf_{y \in S}} \|x-y\|$

- again, S_a is convex when $s_1, s_2 \in S_a$ satisfy

  $$\theta s_1 + (1-\theta) s_2 \in S_a \quad \forall\ \theta \in [0,1]$$

- let $s' = \theta s_1 + (1-\theta) s_2$

  check whether $dist(s', S) \le a$

  $\Rightarrow dist(\theta s_1 + (1-\theta) s_2, S) = \inf_{y \in S} \|\theta s_1 + (1-\theta) s_2 - y\|$

  $\Rightarrow \inf_{y \in S} \|\theta s_1 + (1-\theta) s_2 - y\| = \inf_{y_1, y_2 \in S} \|(\theta s_1 + (1-\theta) s_2 - (\theta y_1 + (1-\theta) y_2)\|$

  (S is convex so that $\exists$ $(\theta y_1 + (1-\theta) y_2 = y)$)

  $= \inf_{y_1, y_2 \in S} \|\theta (s_1 - y_1) + (1-\theta)(s_2 - y_2)\|$

  $\inf_{y_1, y_2 \in S} \|\theta (s_1 - y_1) + (1-\theta)(s_2 - y_2)\| \overset{\text{triangle inequality}}{\le} \inf_{y_1, y_2 \in S} (\theta \|s_1 - y_1\| + (1-\theta) \|s_2 - y_2\|)$

2.14 (cont'd)

(a) (cont'd)

we got

$$\inf_{y_1, y_2 \in S} \| \theta(s_1 - y_1) + (1-\theta)(s_2 - y_2) \| \leq \inf_{y_1 y_2 \in S} \left(\theta \| s_1 - y_1 \| + (1-\theta) \| s_2 - y_2 \| \right)$$

$$= \theta \underbrace{\inf_{y_1 \in S} \| s_1 - y_1 \|}_{\leq a} + (1-\theta) \underbrace{\inf_{y_2 \in S} \| s_2 - y_2 \|}_{\leq a}$$

$$\therefore \quad \leq a$$

Therefore, $dist(s', S) \leq a$ holds,

$S_a = \{ x \mid dist(x, S) \leq a \}$ where $a \geq 0$, S is convex

S_a is convex ※

B(x,a)

(b) $S_{-a} = \{ x \mid B(x, a) \subseteq S \}$ where $a \geq 0$, S is convex

△ again, S_{-a} is convex when $s_1, s_2 \in S_{-a}$ satisfy $\theta s_1 + (1-\theta) s_2 \in S_{-a}$

△ let some $a = a \geq 0$

$s_1 + a \in S$

$s_2 + a \in S$

we need

$\theta s_1 + (1-\theta) s_2 + a \in S$ as well

△ $\theta(s_1 + a) + (1-\theta)(s_2 + a) \in S$ as S is convex

$\Rightarrow \theta s_1 + \cancel{\theta a} + (1-\theta) s_2 + \cancel{(1-\theta) a}$

$\Rightarrow \theta s_1 + (1-\theta) s_2 + a \in S$

△ Hence $S_{-a} = \{ x \mid B(x, a) \subseteq S \}$ is convex

where $a \geq 0$

S is convex

2.15

- $P = \{ p \mid 1^T p = 1,\ p \succeq 0 \} \quad p \in R^n$

the above shows a set of R^n vectors that are pdfs.

- for a vector p, $p_i = \text{prob}(X = a_i)$, $\forall\ i = 1, \dots, n$
- below determine sets subject to below conditions' convexity:

sol^n

(a) $\alpha \le E f(x) \le \beta$

$\Rightarrow E f(x) = \sum_{i=1}^{n} p_i f(a_i)$

$\Rightarrow \quad \alpha \le \sum_{i=1}^{n} p_i f(a_i) \le \beta$

i.e., $\alpha \le \sum_{i=1}^{n} p_i f(a_i)$

$\sum_{i=1}^{n} p_i f(a_i) \le \beta$

which is the intersection of two halfspaces

hence convex ※

(b) $\text{prob}(X > \alpha) \le \beta$

$\Rightarrow \text{prob}(X > \alpha)$

$\Rightarrow \quad \sum p_i$ where i satisfy $a_i > \alpha$

$\Rightarrow \quad \sum p_i \le \beta$

(where i satisfy $a_i > \alpha$)

which is a half space

hence convex ※

(c) $E|X^3| \le \alpha E|X|$

from (a)

$E f(x) = \sum_{i=1}^{n} p_i f(a_i)$

$\Rightarrow E|X^3| - \alpha E|X| \le 0$

$\Rightarrow \quad E|X^3| - \alpha E|X|$

$= \sum_{i=1}^{n} p_i |a_i^3| - \alpha \sum_{i=1}^{n} p_i |a_i|$

$= \sum_{i=1}^{n} p_i \left(|a_i|^3 - \alpha |a_i| \right)$

$\Rightarrow \sum_{i=1}^{n} p_i \left(|a_i|^3 - \alpha |a_i| \right) \le 0$

which is a halfspace

hence convex ※

2.15 (cont'd)

(d)

$EX^2 \leq \alpha$

$\Rightarrow EX^2 = \sum_{i=1}^{n} p_i a_i^2$

$\Rightarrow \sum_{i=1}^{n} p_i a_i^2 \leq \alpha$

is a halfspace

hence convex $_*$

(e)

$EX^2 \geq \alpha$

$\Rightarrow EX^2 = \sum_{i=1}^{n} p_i a_i^2$

$\Rightarrow \sum_{i=1}^{n} p_i a_i^2 \geq \alpha$

is a halfspace

hence convex $_*$

(f, g)

$\mathrm{var}(x) = E(x - Ex)^2$

$\Rightarrow E(x - Ex)^2 = EX^2 - [Ex]^2$

$= \sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i\right)^2$

let $b =$ vector of $a_i^2 = [a_1^2, a_2^2, a_3^2 \cdots a_n^2]$

$a =$ vector of $a_i = [a_1, a_2, a_3 \dots a_n]$

$A = aa^T$

$\Rightarrow E(x - Ex)^2 = b^T p - p^T A p$

(f) $E(x - Ex)^2 \leq \alpha$

$\Rightarrow b^T p - p^T A p \leq \alpha$

$\Rightarrow \underbrace{p^T A p - b^T p} \geq -\alpha$

$g(p)$ is convex function as A is positive semi-definite

set subject to this constraint is not convex, as shown below

for p_1 p_2 satisfy $\begin{cases} p_1^T A p_1 - b^T p_1 \geq -\alpha \\ p_2^T A p_2 - b^T p_2 \geq -\alpha \end{cases}$

& for $\theta \in [0, 1]$, $p_3 = \theta p_1 + (1-\theta) p_2$

$p_3^T A p_3 - b^T p_3 \ngeq -\alpha$

hence not convex $_*$

2.15 (cont'd)

(g) conversely, from (f, g), (f)

$$E(x - Ex)^2 \geq \alpha$$

$$\Rightarrow b^T p - p^T A p \geq \alpha$$

$$\Rightarrow p^T A p - b^T p \leq -\alpha$$

is convex ※

(h, i)

$$\text{quartile}(x) = \inf\{\beta \mid \text{prob}(x \leq \beta) \geq 0.25\}$$

$$= \inf\{\beta \mid \sum_{i:a_i \leq \beta} p_i \geq 0.25\}$$

which β could let $p_1 + p_2 + p_3 + \cdots + p_i$ (here i let $a_i \leq \beta$) be at least 0.25

(h) $\text{quartile}(X) \geq \alpha$

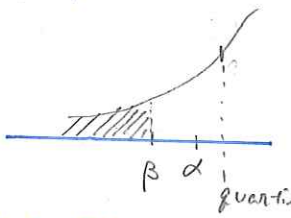
$\Rightarrow \text{prob}(X \leq \beta) < 0.25 \quad \forall \ \beta < \alpha$

as shown in figure ⓐ, ⓑ,

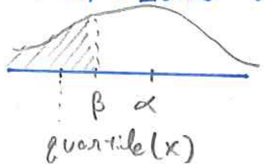
in ⓑ when $\text{prob}(X \leq \beta) \geq 0.25 \ \forall \ \beta < \alpha$ ⓑ

$\text{quartile}(X) \not\geq \alpha$

ⓐ $\text{prob}(X \leq \beta) < 0.25$ when $\beta < \alpha$



$\text{prob}(X \leq \beta) \geq 0.25$ when $\beta < \alpha$



$\Rightarrow \text{prob}(X \leq \beta) < 0.25 \quad \forall \ \beta < \alpha$

$$= \sum_{i:a_i < \alpha} p_i < 0.25$$

which is a halfspace, hence convex ※

(i) conversely, $\text{quartile}(x) \leq \alpha$

$\Rightarrow \text{prob}(X \leq \beta) \geq 0.25 \quad \forall \ \beta < \alpha$

as shown in figure ⓐ ⓑ,

in ⓐ when $\text{prob}(X \leq \beta) < 0.25 \ \forall \ \beta < \alpha$

$\text{quartile}(x) \not\leq \alpha$

$\Rightarrow \text{prob}(X \leq \beta) \geq 0.25 \quad \forall \ \beta < \alpha$

$$= \sum_{i:a_i < \alpha} p_i \geq 0.25$$

which is a halfspace, hence convex ※

2.16

S_1, S_2 are convex sets in $R^{m \times n}$

partial sum of S_1, S_2

$$S = \{ (x, y_1 + y_2) \mid x \in R^m, y_1, y_2 \in R^n, (x, y_1) \in S_1, (x, y_2) \in S_2 \}$$, is S convex?

sol^n

- let there be 2 points in set S

$P_1 (x, y_1 + y_2)$, $P_2 (x', y_1' + y_2')$ 2 points from partial sum.

- for S to be convex

$P_3 = \theta P_1 + (1-\theta) P_2 \in S \quad \forall \theta \in [0, 1]$

- $\theta (x, y_1 + y_2) + (1-\theta)(x', y_1' + y_2')$

$$= \left(\theta x + (1-\theta) x', \ \theta(y_1 + y_2) + (1-\theta)(y_1' + y_2') \right)$$

$$= (\underbrace{\theta x + (1-\theta)x', \ \theta y_1 + (1-\theta) y_1'} + \underbrace{\theta y_2 + (1-\theta) y_2'})$$

which is also a partial sum from S_1, S_2

$(\theta x + (1-\theta) x', \ \theta y_1 + (1-\theta) y_1') \in S_1$

$(\theta x + (1-\theta) x', \ \theta y_2 + (1-\theta) y_2') \in S_2$

hence, S is convex ※

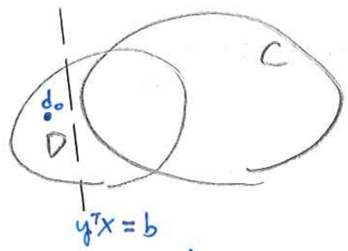
2.26

Show for set $C =$ set D, supporting function of set C and set D should be equal

$$S_C(y) = \sup \{ y^T x \mid x \in C \}$$

soln

△ if $C \neq D$, it indicates :



△ $d_0 \in D$ $\quad d_0 \notin C$

there exist $y^T x = b$ that could separate d_0 from C

△ according to seperating hypeplane theorem

$d_0 \cap C = \varnothing$

then $\exists\, y \neq 0$ & $b \in \mathbb{R}$ s.t.

$$y^T d_0 < b$$
$$y^T x > b, \ \forall x \in C$$

△ therefore, there exists

$$\sup \{ y^T x \mid x \in C \} < y^T d_0 < b < \sup \{ y^T x \mid x \in D \}$$

△ from above

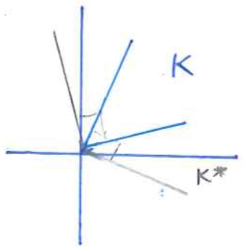
$$\sup \{ y^T x \mid x \in C \} \neq \sup \{ y^T x \mid x \in D \}$$

#

2.32

Find dual cone of $\{Ax \mid x \succeq 0\}$, where $A \in R^{m \times n}$

soln



△ set $\{Ax \mid x \succeq 0\}$

let K be the cone of one set $\{Ax \mid x \succeq 0\}$

△ K^* is the dual cone

where

$$\{y \mid A^T y \succeq 0\}$$

2.33

monotone nonnegative cone :

$$K_{m+} = \{x \in R^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$$

which is a set of nonnegative vectors with their components sorted in non-increasing order.

soln

(a) is K_{m+} a proper cone

△ a proper cone satisfies the following :

- K is convex
- K is closed
- K is solid, i.e., $int(K) \neq \emptyset$
- K is pointed . no line

△ K is convex

K_{m+} is defined with $x_1 \geq 0$, $x_2 \geq 0$, ... $x_n \geq 0$, a series of halfplane and halfspaces. Hence a polyhedron. Hence convex.

△ K is closed ($\lim_{k \to \infty} x_k = x$. $x_k \in C$ $\forall k = 1, 2, \ldots$)
as mentioned above K_{m+} is defined with inequalities, it is closed.

△ K is solid
$int(K_{m+}) \neq \emptyset$ as there exists points satisfy strict inequality

△ K is pointed ($x \in K, -x \in K \Rightarrow x = 0$) (but do hv line segments!)
which in this case obey the above. contains no line

2.33 (cont'd)

(a) (cont'd)

K_{m+} is a proper cone #

(b) Find dual cone K^*_{m+}

▵ for dual cone

$$K^* = \{ y \mid x^T y \geq 0, \ \forall \ x \in K \}$$

▵ from hint

$$\sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_n y_n$$

$$= (x_1 - x_2) y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)(y_1 + y_2 + y_3) + \cdots + (x_{n-1} - x_n)(y_1 + \cdots + y_{n-1}) + x_n (y_1 + \cdots + y_n)$$

▵ for $K_{m+} = \{ x \in R^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \}$

$x_1 - x_2 \geq 0$
$x_2 - x_3 \geq 0$
$\vdots$
$x_{n-1} - x_n \geq 0$
$x_n \geq 0$

▵ so we need to let

$y_1 \geq 0$
$y_1 + y_2 \geq 0$
$y_1 + y_2 + y_3 \geq 0$
$\vdots$

to have dual cone K^*_{m+}

▵ Hence

$$K^*_{m+} = \{ y \mid \sum_{i=1}^{k} y_i \geq 0, \ k = 1, 2, 3, \ldots, n \}$$

#