21/21

Good!

4.3 Prove $x^* = (1, \frac{1}{2}, -1)$ is optimal for

$$\text{minimize} \quad (1/2)x^TPx + q^Tx + r$$

$$\text{subject to} \quad -1 \le x_i \le 1, \quad i = 1, 2, 3$$

where $P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}$, $q = \begin{bmatrix} -22.0 \\ -14.5 \\ 13.0 \end{bmatrix}$, $r = 1$

sol$^n$

- recall for an objective function $f_0$, which is differentiable, $x, y \in \text{dom } f_0$, we have

$$f_0(y) \ge f_0(x) + \nabla f_0(x)^T(y-x) \quad \text{when } x \text{ is optimal}$$

$$\therefore \nabla f_0(x)^T(y-x) \ge 0$$

- we need to show that $\nabla f_0(x)^T(y-x) \ge 0$ @ $x = x^*$

$$\nabla f_0(x) = \frac{1}{2}(P+P^T)x + q^T$$

$$= \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ -1 \end{bmatrix} + \begin{bmatrix} -22.0 \\ -14.5 \\ 13.0 \end{bmatrix}$$

$$= \begin{bmatrix} 21 + -22 \\ 14.5 + -14.5 \\ -11 + 13.0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$\therefore [-1 \quad 0 \quad 2]\begin{bmatrix} y_1 - 1 \\ y_2 - \frac{1}{2} \\ y_3 + 1 \end{bmatrix} = \underbrace{-1(y_1 - 1)}_{a} + \underbrace{2(y_3 + 1)}_{b}$$

term a & term b $\in R^+$ when $-1 \le y_i \le 1$

$$\therefore \nabla f_0(x)^T(y-x) \ge 0$$

$\therefore x^* = (1, \frac{1}{2}, -1)$ is optimal ※

4.5 Show equivalent convex problem. show the following (a), (b) (c) convex problems are equivalent ( noted : $A \in R^{m\times n}$ $\begin{bmatrix} -a_1^T- \\ -a_2^T- \\ \vdots \\ -a_m^T- \end{bmatrix}$, $b \in R^m$, $M>0$ )

(a) minimize $\sum_{i=1}^{m} \phi(a_i^T x - b_i)$

$x \in R^n$,

$\phi : R \to R$

$$\phi(u) = \begin{cases} u^2 & |u| \le M \\ M(2|u| - M) & |u| > M \end{cases}$$

(b) minimize$_{x \in R^n, w \in R^m}$ $\sum_{i=1}^{m} \frac{(a_i^T x - b_i)^2}{(w_i + 1)} + M^2 1^T w$

subject to $w \succeq 0$

(c) minimize $\sum_{i=1}^{m} (u_i^2 + 2Mv_i)$

subject to $-u - v \preceq Ax - b \preceq u + v$

$0 \preceq u \preceq M1$

$v \succeq 0$

<u>sol<sup></sup></u>

$\Delta$ we could show that (a) is equivalent to (b) &

(a) is equivalent to (c)

(a) - (b)

$\Delta$ from (a), we could rewrite it as

minimize $\sum_{i=1}^{m} \phi(u_i)$, $\phi(u) = \begin{cases} u^2 & |u| \le M \\ M(2|u| - M) & |u| > M \end{cases}$

where $u_i = a_i^T x - b_i$

$\Delta$ from (b), we could rewrite it as

minimize$_{x \in R^n, w \in R^m}$ $\sum_{i=1}^{m} \left( \frac{u_i^2}{w_i + 1} + M^2 w_i \right)$, subject to $w_i \geq 0$

4.5 (cont'd)

soln (cont'd)

- we first determine $w$ with any fix $u$, i.e., for any value $u$, (from (b))

  what $w$ can minimize the objective function

$$\frac{\partial}{\partial w}\left(\frac{u^2}{(w+1)} + M^2 w\right)$$

$$= -\frac{u^2}{(w+1)^2} + M^2 \quad \text{——} \; ①$$

- from ①, gradient shows different cases:

  as $w \geq 0$:

  if $|u| > M$:

  $w \in [0, \frac{|u|}{M} - 1]$ decreases

  $w \in (\frac{|u|}{M} - 1, \infty)$ increases

  if $|u| \leq M$:

  gradient always positive, increases

  $w \in [0, \infty)$

- from above, we can know that (b) has

$$w = \begin{cases} \frac{|u|}{M} - 1, & |u| > M \\ 0, & |u| \leq M \end{cases}$$

to get optimal,
which can be substituted,

$$\underset{x \in R^n, w \in R^m}{\text{minimize}} \sum_{i=1}^{m} \phi(u_i), \quad \phi(u) = \begin{cases} \text{(i)}, & |u| \leq M \\ \text{(ii)}, & |u| > M \end{cases}$$

(i): $\frac{u^2}{w+1} + M^2 w = \frac{u^2}{0+1} + M^2 \cdot 0 = u^2$

(ii): $\frac{u^2}{w+1} + M^2 w = \frac{u^2}{\frac{|u|}{M} - 1 + 1} + M^2\left(\frac{|u|}{M} - 1\right)$

$$= \frac{u^2 M}{|u|} + M|u| - M^2$$

$$= |u| M + M|u| - M^2 = M(2|u| - M)$$

4.5 (cont'd)

$\text{sol}^n$ (cont'd)

△ therefore, (b) could eventually be rewrited as:

$$\text{minimize} \sum_{i=1}^{m} \phi(u_i), \quad \phi(u) = \begin{cases} u^2 & |u| \le M \\ M(2|u| - M) & |u| > M \end{cases}$$

which is equivalent to (a) —— A

(a) – (c)

△ from (c), as $u$, $v$ are non-negative,

$u + v = Ax - b$ should hold, or optimal value cannot be reached.

△ therefore,

$$u_i + v_i = |a_i^T x - b_i|$$

$$\Rightarrow v_i = |a_i^T x - b_i| - u_i$$

(c), could be written as

$$\underset{x,u}{\text{minimize}} \sum_{i=1}^{m} \left(u_i^2 + 2M\left(|a_i^T x - b_i| - u_i\right)\right)$$

subject to $0 \le u_i \le M$

$u_i \le |a_i^T x - b_i|$, $\boxed{\text{let } |a_i^T x - b_i| = s}$

△ $u^2 + 2M(s - u)$

$= u^2 - 2Mu + 2Ms \implies u = M$ has minimum value

but $u \le |s|$

△ ∴ if $s \le M$, i.e., $|a^T x - b| \le M$

$u = s$, i.e., $u = |a^T x - b|$ has minimum value.

$u^2 + 2M(|a^T x - b| - u) = s^2 = |a^T x - b|^2$

∴ if $s > M$, i.e., $|a^T x - b| > M$

$u = M$ has minimum value,

$$\begin{aligned} u^2 + 2M(|a^T x - b| - u) &= M^2 + 2M(s - M) \\ &= M^2 + 2Ms - 2M^2 \\ &= M(2s - M) \\ &= M(2|a^T x - b| - M) \end{aligned}$$

4.5 (cont'd)

sol$^n$ (cont'd)

△ therefore, (c) could eventually be rewrited as :

minimize $\sum_{i=1}^{m} \phi(a_i^T x - b_i)$

$$\phi(a^T x - b) = \begin{cases} |a^T x - b|^2 & |a^T x - b| \le M \\ M(2|a^T x - b| - M) & |a^T x - b| > M \end{cases} \quad \text{— (B)}$$

△ Therefore, from (A) & (B), we could show that

(a) – (b) – (c) are equivalent #

4.11 formulate the following as LP, $A \in R^{m \times n}$, $b \in R^m$

(a) Minimize $\|Ax - b\|_\infty$

sol$^n$

△ $\|Ax-b\|_\infty$ is the $l$-infinity norm

△ could be rewritten as

$$\begin{aligned} \underset{x,t}{\text{minimize}} \quad & t \\ \text{subject to} \quad & \|Ax-b\|_\infty \le t \end{aligned}$$ , through epigraph problem form

$$\Rightarrow \begin{aligned} \underset{x,t}{\text{minimize}} \quad & t \\ \text{subject to} \quad & \max_{1 \le i \le m} |a_i^T x - b_i| \le t \end{aligned}$$ (definition of $l$-norm)

$$\Rightarrow \begin{aligned} \underset{x,t}{\text{minimize}} \quad & t \\ \text{subject to} \quad & |a_1^T x - b_1| \le t \\ & |a_2^T x - b_2| \le t \\ & \vdots \\ & |a_m^T x - b_m| \le t \end{aligned}$$

$$\Rightarrow \begin{aligned} \underset{x,t}{\text{minimize}} \quad & t \\ \text{subject to} \quad & -t \le a_1^T x - b_1 \le t \\ & -t \le a_2^T x - b_2 \le t \\ & \vdots \\ & -t \le a_m^T x - b_m \le t \end{aligned}$$

$$\Rightarrow \begin{aligned} \underset{x,t}{\text{minimize}} \quad & t \\ \text{subject to} \quad & -t\mathbf{1} \preceq Ax - b \preceq t\mathbf{1} \end{aligned}$$

△ objective & constraint functions are affine, hence LP #

4.11 (cont'd)

(b) Minimize $\|Ax-b\|_1$

Sol$^n$

- $\|Ax-b\|_1$ is $\ell$-1 norm
- could be re-written as

$$\begin{aligned} &\text{minimize} && \mathbf{1}^T t \\ &\text{subject to} && -t \preceq Ax-b \preceq t \end{aligned}$$ , through epigraph problem form

$$\Rightarrow \begin{aligned} &\text{minimize} && \sum_{i=1}^{m} t_i \\ &\text{subject to} && -t_i \le a_i^T x - b_i \le t_i \quad \forall i. \end{aligned}$$

- objective & constraints are affine, hence LP ∀

(c) Minimize $\|Ax-b\|_1$

subject to $\|x\|_\infty \le 1$

Sol$^n$

- $\|Ax-b\|_1$ ($\ell$-1 norm), $\|x\|_\infty$ ($\ell$-$\infty$ norm)
- could be written as

$$\begin{aligned} &\underset{x,t}{\text{minimize}} && \mathbf{1}^T t \\ &\text{subject to} && -t \preceq Ax-b \preceq t \\ &&& \max_{1\le i\le n} |x_i| \le 1 \end{aligned}$$ through epigraph problem form

$$\Rightarrow \begin{aligned} &\underset{x,t}{\text{minimize}} && \mathbf{1}^T t \\ &\text{subject to} && -t \preceq Ax-b \preceq t \\ &&& x_1 \le 1 \\ &&& x_2 \le 1 \\ &&& \vdots \\ &&& x_n \le 1 \end{aligned}$$

$$\Rightarrow \begin{aligned} &\underset{x,t}{\text{minimize}} && \mathbf{1}^T t \\ &\text{subject to} && -t \preceq Ax-b \preceq t \\ &&& -1 \preceq x \preceq 1 \end{aligned}$$

- objective & constraints are affine, hence LP ∀

4.11 (cont'd)

(d) Minimize $\|x_i\|_1$

subject to $\|Ax-b\|_\infty \le 1$

Sol$^n$

$\triangle$ $\|x_i\|_1$ is $l$-1 norm

$\|Ax-b\|_\infty$ is $l$-$\infty$ norm

$\triangle$ could be written as

$$\underset{x,t}{\text{minimize}} \quad 1^T t$$

$$\text{subject to} \quad -t \preceq x \preceq t$$

$$\max_{1\le i\le m} |a_i^T x - b_i| \le 1$$

through epigraph problem form

$$\Rightarrow \quad \underset{x,t}{\text{minimize}} \quad 1^T t$$

$$\text{subject to} \quad -t \preceq x \preceq t$$

$$-1 \le a_1^T x - b_1 \le 1$$

$$\vdots$$

$$-1 \le a_m^T x - b_m \le 1$$

$$\Rightarrow \quad \underset{x,t}{\text{minimize}} \quad 1^T t$$

$$\text{subject to} \quad -t \preceq x \preceq t$$

$$-1 \preceq Ax-b \preceq 1$$

$\triangle$ objective & constraint functions are affine, hence LP ※

(e) Minimize $\|Ax-b\|_1 + \|x\|_\infty$

Sol$^n$

$\triangle$ $\|Ax-b\|_1$ $l$-1 norm

$\|x\|_\infty$ $l$-$\infty$ norm

$\triangle$ could be written as

$$\underset{t,u,x}{\text{minimize}} \quad 1^T t + u$$

$$\text{subject to} \quad -t \preceq Ax-b \preceq t$$

$$-u1 \preceq x \preceq u1$$

(derivation from (a), (b), (c), (d))

$\triangle$ objective & constraint functions are affine, hence LP ※

PE

4.19

consider the problem

$$\text{minimize} \quad \frac{\|Ax-b\|_1}{c^Tx+d} \qquad A \in R^{m\times n}$$

$$\text{subject to} \quad \|x\|_\infty \leq 1 \qquad b \in R^m, \; c \in R^n, \; d \in R^n$$

$d>0$

$d > \|c\|_1$ , as $\|x\|_\infty \leq 1$ , $c^Tx+d > 0$ for all $x$ feasible

sum of magnitude $c_1, c_2, \dots, c^n < d$

(a) show that it is quasiconvex optimization problem

soln

△ recall : $f: R^n \to R$ is quasiconvex

if dom $f$

&

$$S_\alpha = \{x \in \text{dom } f \mid f(x) \leq \alpha\} \qquad \alpha \in R$$

is convex

△ assume any $\alpha \in R$

$$\frac{\|Ax-b\|_1}{\underbrace{c^Tx+d}_{>0}} \leq 0$$

$$\therefore \|Ax-b\|_1 \leq \alpha(c^Tx+d)$$

$$\Rightarrow \|Ax-b\|_1 - \alpha(c^Tx+d) \leq 0$$

convex function
α sublevel set
is also convex

where $S_\alpha = \{x \in \text{dom } f \mid \underbrace{\|Ax-b\|_1}_{\text{convex}} - \underbrace{\alpha(c^Tx+d)}_{\text{convex}} \leq 0\}$ (nonnegative weighted sum)

which is a convex set

△ $\therefore$ it is a quasiconvex problem

4.19 (cont'd)

(b) Show that it is equivalent to convex optimization problem

$$\begin{aligned} \underset{y,t}{\text{minimize}} \quad & \|Ay - bt\|_1 \\ \text{subject to} \quad & \|y\|_\infty \le t \\ & c^T y + dt = 1 \end{aligned}$$

Sol$^n$

△ from (a) we know that $c^T x + d > 0$ as $d > \|c\|_1$, $\|x\|_\infty \le 1$

$\therefore$
$$\begin{aligned} \text{minimize} \quad & \frac{\|Ax - b\|_1}{c^T x + d} \\ \text{subject to} \quad & \|x\|_\infty \le 1 \end{aligned}$$

$$\Rightarrow \begin{aligned} \text{minimize} \quad & \left\| \frac{Ax}{c^T x + d} - \frac{b}{c^T x + d} \right\|_1 \\ \text{subject to} \quad & \|x\|_\infty \le 1 \end{aligned} \quad \text{——— ①}$$

△ we need to prove that ① is equivalent to (b)

can do it through Introducing equality constraints

$\Rightarrow$ we thus let $y = \dfrac{x}{c^T x + d}$ $\quad t = \dfrac{1}{c^T x + d}$

$\therefore$ $xt = y$

△ inequality constraint $\|x\|_\infty \le 1$

$$\Rightarrow \|x\|_\infty t \le t \quad (t > 0)$$

$$\Rightarrow \|xt\|_\infty \le t$$

$$\Rightarrow \|y\|_\infty \le t \quad \longrightarrow ②$$

△ equality constraint $c^T y + dt$

$$= c^T \frac{x}{c^T x + d} + d \cdot \frac{1}{c^T x + d}$$

$$= \frac{c^T x + d}{c^T x + d}$$

$$= 1 \quad \longrightarrow ③$$

△ from ② & ③ the two is equivalent

\#

4.23 formulate

$$\text{minimize} \quad \|Ax-b\|_4 = \left(\sum_{i=1}^{m} (a_i^T x - b_i)^4\right)^{1/4}$$

as a QCQP $A \in R^{m \times n}$, $b \in R^m$

soln

- for a QCQP, objective function & inequality constraint function are quadratic, while equality constraint function is affine
- through equivalent problems operation, the above could be re-written as:

$$\text{minimize} \quad \|Ax-b\|_4$$

$$\Rightarrow \text{minimize} \quad \sum_{i=1}^{m} (a_i^T x - b_i)^4$$

$$\Rightarrow \text{minimize} \quad \sum_{i=1}^{m} t_i^2$$

$$\left.\begin{array}{l} u_i^2 \le t_i \\ a_i^T x - b_i = u_i \end{array}\right\} i = 1, \dots, m$$

- a QCQP ※

4.28 a robust quadratic programming problem is defined as

$$\begin{array}{ll} \text{minimize} & \sup_{P \in \mathcal{E}} \left(\frac{1}{2} x^T P x + q^T x + r\right) \\ \text{subject to} & Ax \preceq b \end{array}$$

where $\mathcal{E}$ is the set of possible matrices $P$,

for the following express robust QP as convex problem.

(a) finite set of matrices: $\mathcal{E} = \{P_1, \dots P_K\}$, where $P_i \in S_+^n$, $i=1, \dots, K$

soln

- as $P_1, \dots P_K \in S_+^n$
- problem could be expressed as

$$\begin{array}{lll} \underset{x,t}{\text{minimize}} & t & \\ \text{subject to} & \frac{1}{2} x^T P_i x + q^T x + r \le t & i=1, \dots, K \\ & Ax \preceq b & \end{array}$$

- above shows a QCQP problem ※

4.28 (cont'd)

(b) A set of

$$\mathcal{E} = \{ P \in S^n \mid -rI \preceq P - P_0 \preceq rI \}$$

where $r \in R$ & $P_0 \in S^n_+$

<u>sol^n</u>

$\triangle$ $P = P_0 + P - P_0$

$= P_0 + r_i I_i$ for some $r_i I_i$

$\therefore \sup_{P \in \mathcal{E}} \left( \frac{1}{2} x^T P x + q^T x + r \right)$

occurs when, for any random $x$, $P = P_0 + rI \in S^n_+$

$\triangle$ can hence be written as

$$\text{minimize} \quad \frac{1}{2} x^T (P_0 + rI) x + q^T x + r$$
$$\text{subject to} \quad Ax \preceq b$$

$\triangle$ above shows a QP problem #

(c) An ellipsoid of matrices:

$$\mathcal{E} = \left\{ P_0 + \sum_{i=1}^{K} P_i u_i \;\middle|\; \|u\|_2 \le 1 \right\}$$

$P_i \in S^n_+, \; i = 1, \dots, K$

<u>sol^n</u>

$\triangle$ minimize $\sup_{P \in \mathcal{E}} \left( \frac{1}{2} x^T P x + q^T x + r \right)$

subject to $Ax \preceq b$

$\Rightarrow$ minimize $\frac{1}{2} x^T \overset{ⓐ}{\boxed{\sup_{\|u\|_2 \le 1} \left( P_0 + \sum_{i=1}^{K} P_i u_i \right) x}} + q^T x + r$

subject to $Ax \preceq b$

$\Rightarrow$ minimize $\frac{1}{2} x^T P_0 x + \overset{ⓑ}{\boxed{\frac{1}{2} \left( \sum_{i=1}^{K} (x^T P_i x)^2 \right)^{\frac{1}{2}}}} + q^T x + r.$

from ⓐ to ⓑ

$\sup_{\|u\|_2 \le 1} \left\{ \sum_{i=1}^{K} u_i (x^T P x) \right\}$

$\Rightarrow \left\| \sum_{i=1}^{K} u_i (x^T P x) \right\|_2$

$\Rightarrow$ minimize$_{x, y}$ $\frac{1}{2} x^T P_0 x + \|y\|_2 + q^T x + r$

Subject to $\frac{1}{2} x^T P_i x \le y_i, \; i = 1, \dots, K$

$Ax \preceq b$

4.28 (cont'd)

(c) (cont'd)

Soln (cont'd)

△ we then try to express it as a SOCP

$$\begin{array}{ll} \underset{x,\, y}{\text{minimize}} & \frac{1}{2}x^T P_0 x + \|y\|_2 + q^T x + r \\ \text{subject to} & \frac{1}{2}x^T P_i x \le y_i \ , \quad i=1,\dots,K \\ & Ax \preceq b \end{array}$$

$$\Rightarrow \begin{array}{ll} \underset{s,t,x,y}{\text{minimize}} & s + t + q^T x + r \\ \text{subject to} & \frac{1}{2}x^T P_0 x \le s \\ & \|y\|_2 \le t \\ & \frac{1}{2}x^T P_i x \le y_i \quad i=1,\dots,K \\ & Ax \preceq b \end{array}$$

$\Rightarrow$ for $\frac{1}{2}x^T P_0 x \le s$

$$\Rightarrow 0 \ge \frac{1}{2}x^T P_0 x - s$$

$$\Rightarrow 0 \ge 2x^T P_0 x - 4s$$

$$\Rightarrow 0 \ge 2x^T P_0 x + (1-s)^2 - (1+s)^2$$

$$\Rightarrow (1+s)^2 \ge 2x^T P_0 x + (1-s)^2$$

$$1+s \ge \left\| \begin{bmatrix} \frac{1}{\sqrt{2}} P_0^{\frac{1}{2}} x \\ 1-s \end{bmatrix} \right\|_2$$

for $\frac{1}{2}x^T P_i x \le y_i$

$$\Rightarrow 0 \ge \frac{1}{2}x^T P_i x - y_i$$

$$\Rightarrow 0 \ge 2x^T P_i x - 4y_i$$

$$\Rightarrow 0 \ge 2x^T P_i x + (1-y_i)^2 - (1+y_i)^2$$

$$\Rightarrow (1+y_i)^2 \ge 2x^T P_i x + (1-y_i)^2$$

$$1+y_i \ge \left\| \begin{bmatrix} \frac{1}{\sqrt{2}} P_i^{\frac{1}{2}} x \\ 1-y_i \end{bmatrix} \right\|_2$$

$$\Rightarrow \begin{array}{ll} \underset{s,t,x,y}{\text{minimize}} & s + t + q^T x + r \\ \text{subject to} & \left\| \begin{bmatrix} \frac{1}{\sqrt{2}} P_0^{\frac{1}{2}} x \\ 1-s \end{bmatrix} \right\|_2 \le s + 1 \\ & \left\| \begin{bmatrix} \frac{1}{\sqrt{2}} P_i^{\frac{1}{2}} x \\ 1-y_i \end{bmatrix} \right\|_2 \le y_i + 1 \ , \quad i=1,\dots,K \\ & \|y\|_2 \le t \\ & Ax \preceq b \end{array}$$

which is in SOCP form

△ above shows a SOCP problem ※

4.30

Express the desribed problem as a geometric program

soln

△ energy cost due to heat loss $\alpha_1 \frac{Tr}{w}$

pipe cost $\alpha_2 r$

insulation cost $\alpha_3 rw$

fixed velo. of heat flow $\alpha_4 Tr^2$

△ $\underset{T,r,w}{\text{maximize}}$ $\alpha_4 T r^2$

subject to $\alpha_1 \frac{Tr}{w} + \alpha_2 r + \alpha_3 rw \le C_{max}$

$T_{min} \le T \le T_{max}$

$r_{min} \le r \le r_{max}$

$w_{min} \le w \le w_{max}$

$w \le 0.1 r$

△ convert to a GP:

$\underset{T,r,w}{\text{minimize}}$ $\alpha_4^{-1} T^{-1} r^{-2}$ $\qquad x = \begin{bmatrix} T \\ r \\ w \end{bmatrix}$

subject to $\frac{\alpha_1}{C_{max}} T r w^{-1} + \frac{\alpha_2}{C_{max}} r + \frac{\alpha_3}{C_{max}} rw \le 1$

$\frac{1}{T_{min}} T^{-1} \le 1$

$\frac{1}{T_{max}} T \le 1$

$\frac{1}{r_{min}} r^{-1} \le 1$

$\frac{1}{r_{max}} r \le 1$

$\frac{1}{w_{min}} w^{-1} \le 1$

$\frac{1}{w_{max}} w \le 1$

$10 w r^{-1} \le 1$

※

4.33

express the following as convex optimization problems.

(a) minimize $\max\{P(x), q(X)\}$, where $p \cdot q$ are posynomials

sol$^n$

$\Delta$ minimize $\max\{P(x), q(x)\}$ $p \cdot q$ are posynomials, which is a GP

$\Delta$ could be rewritten as

$$\begin{aligned} \underset{x,t}{\text{minimize}} \quad & t \\ \text{s.t.} \quad & P(x) \leq t \\ & q(x) \leq t \end{aligned}$$

$$\Rightarrow \begin{aligned} \underset{x,t}{\text{minimize}} \quad & t \\ \text{s.t.} \quad & t^{-1}P(x) \leq 1 \\ & t^{-1}q(x) \leq 1 \end{aligned}$$

$\Rightarrow$ let $x_i = e^{y_i}$, $t = e^{u}$

to Convex form

$$\Rightarrow \begin{aligned} \underset{y,u}{\text{minimize}} \quad & \log(e^{u}) \\ \text{s.t.} \quad & \log\left((e^{u})^{-1}P(e^{y_1}, \dots e^{y_n})\right) \leq \log 1 \\ & \log\left((e^{u})^{-1}q(e^{y_1}, \dots e^{y_n})\right) \leq \log 1 \end{aligned}$$

$$\Rightarrow \begin{aligned} \underset{y,u}{\text{minimize}} \quad & u \\ \text{s.t.} \quad & -u + \overbrace{\log P(e^{y_1}, \dots e^{y_n})}^{①} \leq 0 \\ & -u + \underbrace{\log q(e^{y_1}, \dots e^{y_n})}_{②} \leq 0 \end{aligned}$$

where term ① & ② could be expressed affine

$\Delta$ therefore, above shows that GP could be expressed as convex optimization problem #

(b)

Minimize $\exp(p(x)) + \exp(q(x))$ , $p, q$ are posynomials.

Sol$^n$

$\triangle$ is equivalent to

$$\begin{aligned} \underset{t_1, t_2, x}{\text{minimize}} \quad & \exp(t_1) + \exp(t_2) \\ \text{s.t.} \quad & p(x) \le t_1 \\ & q(x) \le t_2 \end{aligned}$$

$$\Rightarrow \begin{aligned} \underset{t_1, t_2, x}{\text{minimize}} \quad & \exp(t_1) + \exp(t_2) \\ \text{s.t.} \quad & t_1^{-1} p(x) \le 1 \\ & t_2^{-1} q(x) \le 1 \end{aligned}$$

$\Rightarrow$ let $x_i = e^{y_i}$

which is equivalent to take logarithm on constraints function.

$$\Rightarrow \begin{aligned} \underset{t_1, t_2, x}{\text{minimize}} \quad & \exp(t_1) + \exp(t_2) \\ \text{s.t.} \quad & \underbrace{\log(t_1^{-1})}_{a} + \underbrace{\log(P(e^{y_1}, \dots e^{y_n}))}_{1} \le 0 \\ & \underbrace{\log(t_2^{-1})}_{b} + \underbrace{\log(q(e^{y_1}, \dots e^{y_n}))}_{2} \le 0 \end{aligned}$$

where term 1 & term 2 are affine

term a & term b are convex

$\triangle$ hence, it could be converted as a convex optimization problem #

(c) Minimize $\dfrac{p(x)}{r(x) - q(x)}$

s.t. $r(x) > q(x)$

Sol$^n$

$\triangle$ as it could be re-written as

$$\begin{aligned} \text{minimize} \quad & \underbrace{p(x)}_{\text{posynomial}} \underbrace{[r(x) - q(x)]^{-1}}_{\text{posynomial}} \\ \text{s.t.} \quad & \underbrace{r(x)^{-1} q(x)}_{\text{posynomial}} < 1 \end{aligned}$$

$\triangle$ it is a GP

$\triangle$ GP could be expressed as convex optimization problem :

4.33 (cont'd)

(c) (cont'd)

soln (cont'd)

△ minimize $p(x)\,[r(x)-q(x)]^{-1}$

s.t. $r(x)^{-1}q(x) < 1$

$\Rightarrow$ let $x_i = e^{y_i}$

$[r(x)-q(x)]^{-1} = s(x)$

$\Rightarrow$ minimize $p(x)\,s(x)$

s.t. $r(x)^{-1}q(x) < 1$

$\Rightarrow$ $\underset{y}{\text{minimize}}$ $p(e^{y_1}, e^{y_2} \ldots e^{y_n})\; s(e^{y_1}, \ldots e^{y_n})$

s.t. $r(e^{y_1}, \ldots, e^{y_n})^{-1} q(e^{y_1}, \ldots e^{y_n}) < 1$

$\Rightarrow$ take logarithm ( convert to a equivalent optimization problem )

$\underset{y}{\text{minimize}}$ $\left(\log(p(e^{y_1}, e^{y_2} \ldots e^{y_n})) + \log(s(e^{y_1}, \ldots e^{y_n}))\right)^{ⓐ} < 0$

s.t. $\left(\log(r(e^{y_1}, \ldots e^{y_n})^{-1}) + \log(q(e^{y_1} \ldots e^{y_n})\right)^{ⓑ} < 0$

where term ⓐ & term ⓑ are affine .

△ above shows that it could be expressed as convex optimization problem #

4.38

(a) show that generalized eigenvalues of $(A, B)$ are real . given by $\lambda_i = a_i / b_i$, $i = 1, \ldots, n$

soln

△ $\det(\lambda B - A)$

$= \det\left( R^T [\lambda \operatorname{diag}(b) - \operatorname{diag}(a)] R \right)$

△ to find eigenvales , we set $\det(\lambda B - A) = 0$

△ $\det(R^T [\lambda \operatorname{diag}(b) - \operatorname{diag}(a)] R) = 0$

$\Rightarrow (\det R)^2 (\lambda b_1 - a_1)(\lambda b_2 - a_2) \cdots (\lambda b_n - a_n) = 0$

$a_i, b_i > 0$ hence $\lambda_i \in R$, $\lambda_i = \dfrac{a_i}{b_i}$ #

4.38

(b) Express the solution of the SDP

$$\begin{array}{ll} \text{minimize} & ct \\ \text{subject to} & tB \preceq A \end{array}$$

with variable $t \in \mathbb{R}$, in terms of $a$ & $b$

<u>sol^n</u>

△ from above, the solution is subject to $tB \preceq A$, where $A, B \in S^n$

△ from the problem,

$$tB \preceq A$$

$$\Rightarrow \quad t\, R^{T^{-1}} \operatorname{diag}(b)\, R \preceq R^{T^{-1}} \operatorname{diag}(a)\, R$$

$$\Rightarrow \quad t \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{bmatrix} \preceq \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$$

$$\therefore \quad tb \preceq a$$

△ solution $\Rightarrow$

$$\begin{array}{ll} \underset{t}{\operatorname{argmin}} & ct \\ \text{subject to} & tb \preceq a \end{array}$$

which is a linear programming problem #

4.40 Express the following as SDPs

(a) LP

soln

$\Delta$ recall semidefinite programming form:

minimize $c^T x$

subject to $x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0$

$Ax = b$

$G, F_1, F_2 \ldots F_n \in S^K$

$A \in \mathbb{R}^{p \times n}$

$\Delta$ recall LP form

minimize $c^T x + d$

subject to $Gx \preceq h$

$Ax = b$

$\Rightarrow$ minimize $c^T x + d$

subject to $Gx - h \preceq 0$

$Ax = b$

$\Rightarrow$ minimize $c^T x + d$

subject to $\mathrm{diag}(Gx - h) \preceq 0$

$Ax = b$

which is in SDP form $\divideontimes$

4.40 (cont'd)

(b) QP, QCQP, SOCP

soln

△ QP

△ recall SDP:

minimize $c^T x$

subject to $x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0$

$Ax = b$

△ recall QP:

minimize $\frac{1}{2} x^T P x + q^T x + r$

subject to $Gx \preceq h$

$Ax = b$

$\Rightarrow$ minimize$_{t, x}$ $t + q^T x + r$

subject to $\frac{1}{2} x^T P x \le t$

$Gx \preceq h$

$Ax = b$

Schur complement

$C - B^T A^{-1} B \succeq 0 \iff \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0$

$\therefore t - \frac{1}{2} x^T P x \ge 0$

$\Rightarrow tI - \frac{1}{\sqrt{2}} x^T (P^{\frac{1}{2}}) I^{-1} (P^{\frac{1}{2}})^T x \frac{1}{\sqrt{2}}$

$\iff \begin{bmatrix} I & \frac{1}{\sqrt{2}} (P^{\frac{1}{2}})^T x \\ \frac{1}{\sqrt{2}} x^T P^{\frac{1}{2}} & tI \end{bmatrix} \succeq 0$

$\Rightarrow$ minimize$_{t, x}$ $t + q^T x + r$

subject to $\begin{bmatrix} I & \frac{1}{\sqrt{2}} (P^{\frac{1}{2}})^T x \\ \frac{1}{\sqrt{2}} x^T P^{\frac{1}{2}} & tI \end{bmatrix} \succeq 0$

$\mathrm{diag}(Gx - h) \preceq 0$

$Ax = b$

which is in SDP form ※

△ QCQP

△ recall SDP:

minimize $c^T x$

subject to $x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0$

$Ax = b$

△ recall QCQP:

minimize $\frac{1}{2} x^T P_0 x + q_0^T x + r_0$

subject to $\frac{1}{2} x^T P_i x + q_i^T x + r_i \le 0, \quad i = 1, \ldots, m$

$Ax = b$

4.40 (cont'd)

(b) (cont'd)

soln (cont'd)

∴ could be re-formulated as :

$$\begin{aligned} \underset{t,x}{\text{minimize}} \quad & t + q_0^T x + r_0 \\ \text{subject to} \quad & \tfrac{1}{2} x^T P_0 x \le t \\ & \tfrac{1}{2} x^T P_i x + q_i^T x + r_i \le 0 \qquad i = 1, \dots, m \\ & Ax = b \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \underset{u,t,x}{\text{minimize}} \quad & t + q_0^T x + r_0 \\ \text{subject to} \quad & \tfrac{1}{2} x^T P_0 x \le t \\ & \tfrac{1}{2} x^T P_i x \le u_i \qquad i = 1, \dots, m \\ & u_i + q_i^T x + r_i \le 0 \qquad i = 1, \dots, m \\ & Ax = b \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \underset{u,t,x}{\text{minimize}} \quad & t + q_0^T x + r_0 \\ \text{subject to} \quad & \begin{bmatrix} I & \frac{1}{\sqrt{2}} (P^{\frac{1}{2}})^T x \\ \frac{1}{\sqrt{2}} x^T P^{\frac{1}{2}} & t\, I \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} I & \frac{1}{\sqrt{2}} (P_i^{\frac{1}{2}})^T x \\ \frac{1}{\sqrt{2}} x^T P_i^{\frac{1}{2}} & u_i I \end{bmatrix} \succeq 0 \qquad i = 1, \dots, m \\ & u_i + q_i^T x + r_i \le 0 \qquad i = 1, \dots, m \\ & Ax = b \end{aligned}$$

which is in SDP form #

4.40 (cont'd)

(b) (cont'd)

solⁿ (cont'd)

△ SOCP

△ recall SDP:

minimize $c^T x$

subject to $x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0$

$Ax = b$

△ recall SOCP:

minimize $f^T x$

subject to $\|A_i x + b_i\|_2 \le c_i^T x + d_i$ , $i = 1, \dots, m$

$Fx = g$

△ could be re-formulated as :

since

$$(c_i^T x + d_i)^2 \ge (A_i x + b_i)^T (A_i x + b_i)$$

$$\Rightarrow \underbrace{c_i^T x + d_i}_{C} \ge \underbrace{(A_i x + b_i)^T}_{B^T} \underbrace{\frac{I}{(c_i^T x + d_i)}^{-1}}_{A^{-1}} \underbrace{(A_i x + b_i)}_{B}$$

$$\Rightarrow \begin{bmatrix} (c_i^T x + d_i) I & (A_i x + b_i) \\ (A_i x + b_i)^T & (c_i^T x + d_i) \end{bmatrix} \succeq 0$$

$\therefore$ minimize $f^T x$

subject to $\begin{bmatrix} (c_i^T x + d_i) I & (A_i x + b_i) \\ (A_i x + b_i)^T & (c_i^T x + d_i) \end{bmatrix}$ , $i = 1, \dots, m$

$Fx = g$

which is in SDP form ※

4.40 (cont'd)

(c) matrix fractional optimization problem

$$\text{minimize} \quad \underbrace{(Ax+b)^T F(x)^{-1} (Ax+b)}_{= g}$$

$A \in R^{m \times n}$
$b \in R^m$
$F_i \in S^m$

$$F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n$$

$$\text{dom } g = \{ x \mid F(x) \succ 0 \}$$

soln

△ similiar to previous, first recall SDP form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{aligned}$$

△ hence the problem could be rewritten as :

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & (Ax+b)^T F(x)^{-1} (Ax+b) \le t \end{aligned}$$

$\to t - (Ax+b)^T F(x)^{-1} (Ax+b) \ge 0$

$\Rightarrow$ recall Schruz complement $C - B^T A^{-1} B \succeq 0 \iff \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0$

$$\Rightarrow \quad \begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & \begin{bmatrix} F(x) & Ax+b \\ (Ax+b)^T & tI \end{bmatrix} \succeq 0 \end{aligned}$$

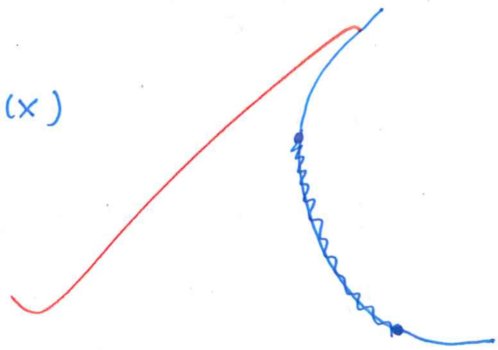
which is in SDP form #

4.52 show that all Pareto optimal values, set of Pareto optimal values satisfies

$$P \subseteq \mathcal{O} \cap \mathrm{bd}\,\mathcal{O}$$

soln

- recall definition of Pareto optimal values:
  a feasible point $x$ is Pareto optimal if
  $f_0(x)$ is a minimal element of the set of $\mathcal{O}$ (achievable values)
  $\therefore$ $x$ is Pareto optimal
  $y$, $f_0(y) \preceq_K f_0(x)$,
  $f_0(y) = f_0(x)$
- $(f_0(x) - K) \cap \mathcal{O} = \{f_0(x)\}$
- to prove $P \subseteq \mathcal{O} \cap \mathrm{bd}\,\mathcal{O}$
  let $f_0(x) \in \mathrm{int}\,\mathcal{O}$.
  then $\exists\, f_0(x')$, that is $f_0(x') \preceq f_0(x)$
  which contradicts to the definition.
- Hence, $P$ must be on $\mathrm{bd}\,\mathcal{O}$



$\therefore$ $P \subseteq \mathcal{O} \cap \mathrm{bd}\,\mathcal{O}$

4.53 suppose $f_0(x)$ for vector optimization problem is convex. show

$$A = \mathcal{O} + K = \{t \in \mathbb{R}^q \mid f_0(x) \preceq_K t \text{ for some feasible } x\}$$

is convex. show minimal elements of $A$ is equivalent to minimal elements of $\mathcal{O}$.

soln

(a) - show $A$ is convex

- let $f_0(x_1) \preceq_K t_1$
  $f_0(x_2) \preceq_K t_2$
  we show $\forall\, \theta \in [0,1]$
  $\theta t_1 + (1-\theta) t_2$ also satisfies
  $f_0(\theta x_1 + (1-\theta) x_2) \preceq_K \theta t_1 + (1-\theta) t_2$

- as $f_0$ is $K$-convex
  $f_0(\theta x_1 + (1-\theta) x_2) \preceq_K \theta f(x_1) + (1-\theta) f(x_2)$
  and as $f(x_1) \preceq t_1$
  $f(x_2) \preceq t_2$
  $\therefore$ $f_0(\theta x_1 + (1-\theta) x_2)$
  $\preceq_K \theta f(x_1) + (1-\theta) f(x_2)$
  $\preceq_K \theta t_1 + (1-\theta) t_2$ $\therefore$ hence conv

4.53 (cont'd)

(b)

soln

$\triangle$ $A = \mathcal{O} + K$

let $p \in A$ minimal

$\Downarrow$

if $q \preceq_K p$

$q = p$

(def. of minimal.

also let $p = \hat{p} + z$ , $\hat{p} \in \mathcal{O}$, $z \in K$

$\triangle$ we let $z \neq 0$

$\therefore$ $p \succneq_K 0$

$\therefore$ if any $q \preceq_K p$

$q$ might not be $p$

$\therefore$ against def. of minimal element, where

minimal element :

if $q \preceq_K p$

$q = p$

$\therefore$ $z$ must be zero

$\therefore$ $p = \hat{p}$

where $p \in A$

$\hat{p} \in \mathcal{O}$

$\therefore$ minimal elements of $A$ are the same as minimal elements of $\mathcal{O}$ $\maltese$