△ Kinematics

△ Generalized coordinates

$q = \begin{bmatrix} q_b \\ q_j \end{bmatrix}$ → base, → joints; $q_b = \begin{bmatrix} q_{bP} \\ q_{bR} \end{bmatrix} \in SE(3)$

△ Generalized velocities / accelerations

$u = \begin{bmatrix} {}_I v_B \\ {}_B \omega_{IB} \\ \dot\varphi_1 \\ \vdots \\ \dot\varphi_{n_j} \end{bmatrix}, \quad \dot u = \begin{bmatrix} {}_I a_B \\ {}_B \psi_{IB} \\ \ddot\varphi_1 \\ \vdots \\ \ddot\varphi_{n_j} \end{bmatrix} \in \mathbb{R}^{6+n_j}$

$u = E_{fb}\,\dot q, \quad E_{fb} = \begin{bmatrix} I_{3\times3} & & \\ & E_{R} & \\ & & I_{n_j\times n_j} \end{bmatrix}$

△ P / v of a pt on the robot

- P

${}_I r_{IQ}(q) = {}_I r_{IB}(q) + C_{IB}(q)\,{}_B r_{BQ}(q)$

- V

${}_I v_Q = {}_I v_B + \dot C_{IB}\,{}_B r_{BQ} + C_{IB}\,{}_B \dot r_{BQ}$

$= {}_I v_B + C_{IB}[{}_B\omega_{IB}]_\times {}_B r_{BQ} + C_{IB}\,{}_B\dot r_{BQ}$

$= {}_I v_B - C_{IB}[{}_B r_{BQ}]_\times {}_B\omega_{IB} + C_{IB}\,{}_B \dot r_{BQ}$

$= {}_I v_B - C_{IB}[{}_B r_{BQ}]_\times {}_B\omega_{IB} + C_{IB}\,{}_B J_{P_q}(q_j)\,\dot q_j$

${}_B \dot r_{BQ} = J_{q_j}\,\dot q_j$

$= \left[I_{3\times3} \;\; -C_{IB}[{}_B r_{BQ}]_\times \;\; C_{IB}\,{}_B J_{P_q}(q_j) \right] u$

$= {}_I J_Q(q)\,u$

△ Contact constraints

- C_i: contact point, cannot move

$$\begin{cases} {}_I r_{IC_i} = \text{const.} \\ {}_I \dot r_{IC_i} = {}_I J_{C_i}\,u = 0 \\ {}_I \ddot r_{IC_i} = {}_I J_{C_i}\,\dot u + {}_I \dot J_{C_i}\,u = 0 \end{cases}$$

$J_c = \begin{bmatrix} J_{C_1} \\ \vdots \\ J_{C_{n_c}} \end{bmatrix} \in \mathbb{R}^{3n_c \times n_n}$ — dim of u; $3\times$ no. of contact pt.

- Nullspace

$0 = \dot r_c = J_c \dot q \Rightarrow \dot q = J_c^{+} 0 + N_c \dot q_0 = N_c \dot q_0$

$0 = \ddot r_c = J_c \ddot q + \dot J_c \dot q \Rightarrow \ddot q = J_c^{+}(-\dot J_c \dot q) + N_c \ddot q_0$

- first priority

task 0: $\dot w_c = J_c \ddot q + \dot J_c \dot q = 0$

task 1: $\dot w_e = J_e \ddot q + \dot J_e \dot q = \dot w_e^*$

- system moved due to contact constraints

△ Contact Jacobian

- $J_c = \dfrac{\partial r}{\partial q}$ → constraints; → base & joints — relation between base motion / joint motion & constraints

$= [J_{c,b} \;\; J_{c,j}]$

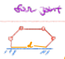
$= \left[\dfrac{\partial r_c}{\partial q_b} \;\; \dfrac{\partial r_c}{\partial q_j} \right]$

- $\text{rank}(J_{c,b}) \Rightarrow$ base is fully controllable

- $\underbrace{\text{rank}(J_c)}_{\text{total no. of independent constraints}} - \underbrace{\text{rank}(J_{c,b})}_{\text{no. of constraints contributes to motion of the base}}$

⇓

no. of kinematic constraints for joint actuators

e.g.  $\begin{cases} q_b = (\,) \\ q_j = (\,) \end{cases}$

$J_c = \mathbb{R}^{4\times7}$ — x, y constraints of 2 contact point

$\text{rank}(J_c) = 4$

$\text{rank}(J_{c,b}) = 3$

$4 - 3 = 1$ ("1" kinematic constraints of the actuators) "d"

△ generalized coordinates DON'T correspond to the degree of freedom

△ Dynamics

$M\ddot q + b + g = \tau + J^T F_c$ — control torque, external force

$M(q)\dot u + b(q,u) + g(q) = S^T\tau + J_c^T F_c$ — gravitational force, selection matrix / constraint

again

$q = \begin{bmatrix} q_b \\ q_j \end{bmatrix}$ — unactuated base, actuated joints; $u_j = S u = S\begin{bmatrix} u_b \\ u_j \end{bmatrix} = [0 \; I]\begin{bmatrix} u_b \\ u_j \end{bmatrix}$

that's why "selection matrix"

⇒ $M\ddot q_j + b + g = \tau + J^T F_c$

⇒ $M\ddot q + b + g = S^T\tau + J_c^T F_c$

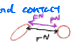
⇒ $M\dot u + b + g = S^T\tau + J_c^T F_c$ — equivalent in different structure; only joints to be actuated; $S = [0 \; I]$

$M\dot u + b + g + J_c^T F_c = S^T\tau$ — ↳ remember the direction of contact force

△ Soft contact

physical accuracy v. numerical stability

△ Hard contact



- closed contact

$r^N = 0$

$\dot r^N = 0$

- linear complementary constraint

$\ddot r^N \ge 0$

$F^N \ge 0$

$\ddot r^N F^N = 0$

△ Hard contact of an MBS

- $M\ddot q + b + g + J_c^T F_c = S^T\tau$ — (1)
- $\dot r_c = J_c \dot q = 0$

$\ddot r_c = J_c \ddot q + \dot J_c \dot q = 0$ — (2)

⇒ from (1)

$\ddot q = M^{-1}(S^T\tau - b - g - J_c^T F_c)$

w/ (2)

$\ddot r_c = J_c[M^{-1}(S^T\tau - b - g - J_c^T F_c)] + \dot J_c \dot q = 0$ — (3)

- get F_c — contact force from (3)

$F_c = (J_c M^{-1} J_c^T)^{-1}(J_c M^{-1}(S^T\tau - (b+g)) + \dot J_c \dot q)$ — given my constraints, actuations, get F_c

∴ w/ (1)

$M\ddot q + b + g + J_c^T F_c = S^T\tau$

⇒ $M\ddot q + b + g + J_c^T(\ldots) = S^T\tau$

⇒ $M\ddot q + b + g + J_c^T(\ldots) = S^T\tau$

define $N_c = I - M^{-1}J_c^T(J_c M^{-1} J_c^T)^{-1} J_c$

$J_c N_c = \cancel{J_c} - \cancel{J_c M^{-1} J_c^T (J_c M^{-1} J_c^T)^{-1} J_c} = 0$

⇒ $N_c^T M\ddot q + N_c^T(b+g) + \underbrace{N_c^T J_c^T (\ldots)}_{=0} = N_c^T S^T\tau$

⇒ $N_c^T M\ddot q + N_c^T(b+g) = N_c^T S^T\tau$

support consistent dynamics — independent of reaction force!

△ impulse transfer @ contact

$\int_{[t]} (M\dot u + b + g + J_c^T F_c - S^T\tau)\,dt$

$= M(u^+ - u^-) + J_c^T \mathcal{F}_c = 0$ — (1); $\mathcal{F}_c = F_c \cdot dt$

also

$\dot r_c^+ = J_c u^+ = 0$ — (2)

⇒ w/ (1)

$u^+ - u^- = -M^{-1}J_c^T \mathcal{F}_c$ — (3)

⇒ $J_c u^+ - J_c u^- = -J_c M^{-1} J_c^T \mathcal{F}_c$

⇒ $J_c u^- = J_c M^{-1} J_c^T \mathcal{F}_c$

⇒ $\dot r_c^- = \Lambda_c^{-1} \mathcal{F}_c$

⇒ $\mathcal{F}_c = \Lambda_c \dot r_c^- = \Lambda_c J_c u^-$ — end effector inertia

∴ $\Delta u = u^+ - u^-$ (from (3))

$= -M^{-1} J_c^T \mathcal{F}_c$

$= -M^{-1} J_c^T \Lambda_c J_c u^-$

$= -M^{-1}J_c^T(J_c M^{-1} J_c^T)^{-1} J_c u^-$ — change in generalized vel.

∴ $u^+ = u^- - M^{-1}J_c^T(J_c M^{-1}J_c^T)^{-1}J_c u^-$

$= [I - M^{-1}J_c^T(J_c M^{-1}J_c^T)^{-1}J_c]\,u^-$

$= N_c u^-$ — post-impact vel

N_c also maps $u^- \to u^+$

- $E_{loss} = \Delta E_{kin}$

$= -\frac{1}{2}\Delta u^T M \Delta u$

$= -\frac{1}{2}\Delta \dot r_c^T \Lambda_c \Delta \dot r_c$

$= -\frac{1}{2}\dot r_c^{-T} \Lambda_c \dot r_c^-$

△ Dynamics control

again

- $M(q)\dot u + b(q,u) + g(q) + J_c^T F_c = S^T\tau$
- $\dot r_c = J_c u$

$\ddot r_c = J_c \dot u + \dot J_c u = 0$

- $F_c = (J_c M^{-1} J_c^T)^{-1}(J_c M^{-1}(S^T\tau - b - g) + \dot J_c u)$
- $N_c = I - M^{-1}J_c^T(J_c M^{-1}J_c^T)^{-1}J_c$
- $N_c^T(M\ddot q + b + g) = N_c^T S^T\tau$

- inverse dynamics

$\tau^* = (N_c S^T)^+ N_c^T(M\ddot q + b + g)$

$\tau^* = (N_c^T S^T)^+ N_c^T(M\ddot q + b + g) + \mathcal{N}(N_c^T S^T)\tau_0$ — multiple solution; internal force direction

△ inducing QP

$\min_x \| A_i x - b_i \|_2$

w/ $x = \begin{bmatrix} \dot u \\ F_c \\ \tau \end{bmatrix}$

(1) $M(q)\dot u + b(q,u) + g(q) + J_c^T F_c = S^T\tau$

⇒ $A_1 = [\hat M \;\; \hat J_c^T \;\; -S^T]$

$b_1 = -\hat b - \hat g$

(2) $J\dot u + \dot J u = \dot w^*$

⇒ $A_2 = [\hat J_i \;\; 0 \;\; 0]$ — motion tasks

$b_2 = \dot w^* - \hat{\dot J}_i u$

(3) $F_i = F_i^*$

⇒ $A_3 = [0 \;\; I \;\; 0]$ — force tasks

$b = F_i^*$

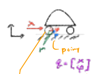
(4) $\min \|\tau\|_2$ — torque min

⇒ $A = [0 \;\; 0 \;\; I]$

$b = 0$

→ then solve QP!

△ case study



(1)

- contact constraints?

△ point on the wheel — point; $q = \begin{bmatrix} x \\ \varphi \end{bmatrix}$ (unactuated base + actuated wheel)

${}_I r_{IP} = \begin{bmatrix} x + r\sin\varphi \\ r + r\cos\varphi \\ 0 \end{bmatrix}$

△ Jacobian

$\dfrac{\partial r}{\partial q} = \begin{bmatrix} \frac{\partial r_1}{\partial x} & \frac{\partial r_1}{\partial \varphi} \\ \frac{\partial r_2}{\partial x} & \frac{\partial r_2}{\partial \varphi} \\ \frac{\partial r_3}{\partial x} & \frac{\partial r_3}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} 1 & r\cos\varphi \\ 0 & -r\sin\varphi \\ 0 & 0 \end{bmatrix}$

△ contact constraints

vel = 0 ← when P point now to the ground, establishing contact

${}_I \dot r_{IP}\big|_{\varphi=\pi} = J\big|_{\varphi=\pi}\,\dot q$ — $\dot r = 0$!

$= \begin{bmatrix} 1 & -r \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot x \\ \dot\varphi \end{bmatrix} = 0$

⇒ $\dot x - r\dot\varphi = 0$ (rolling constraint)

⇒ possible base motion

$\dot q = J_c^+ 0 + N_c \dot q_0$

$= N_c \dot q_0$ → $\mathcal{N}(J_c)$ ∴ $N J_c = 0$

$= \begin{bmatrix} r \\ 1 \end{bmatrix}\dot q_0$

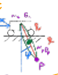
(2)

Floating base system with 12 actuated joint and 6 base coordinates (18DoF)

$J_c = [J_{c,b}\, J_{c,j}]$		1 contact			
total constraints (rank(J_c))	0	3	6	9	12
base constraints (rank$(J_{c,b})$)	0	3	5	6	6
internal constraints (rk(J_c) − rk$(J_{c,b})$)	0	0	1	3	6
uncontrollable DoF ($6 -$ rk$(J_{c,b})$)	6	3	1	0	0

(3) $N_c^T M\ddot q + N_c^T(b+g) = N_c^T S^T\tau$

△ EoM



△ Contact Jacobian

$J_c = [J_{c,b} \;\; J_{c,j}] = \begin{bmatrix} 1 & 2l\cos\varphi \\ 0 & 2l\sin\varphi \end{bmatrix}$

$q_b = \begin{bmatrix} q_b \\ q_j \end{bmatrix} = \begin{bmatrix} x \\ \varphi \end{bmatrix}$

${}_I r_P = \begin{bmatrix} x + 2l\sin\theta \\ -2l\cos\theta \end{bmatrix}$

$\dfrac{\partial r}{\partial q} = \begin{bmatrix} \frac{\partial r_x}{\partial x} & \frac{\partial r_x}{\partial \varphi} \\ \frac{\partial r_y}{\partial x} & \frac{\partial r_y}{\partial \varphi} \end{bmatrix}$

(a) both constraints active

- $\mathcal{N}(J_c) = 0$, no controllable subspace
- contact constraint cannot control the floating base; only change internal force

(b) vertical motion locked

$J_c = [J_{c,b} \;\; J_{c,j}] = [\,0 \;\; 2l\sin\varphi\,]$

$\mathcal{N}(J_c) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad N_c^T S^T = [1 \; 0]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$

- torque has no influence on motion
- only change constraint / reaction / internal force

(c) $J_c = [\,1 \;\; 2l\cos\varphi\,]$

$N_c = \begin{bmatrix} 2l\cos\varphi \\ -1 \end{bmatrix}$

$N_c^T S^T = [2l\cos\varphi \;\; -1]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1$

- can change the motion
- change the internal force, will change the motion

△ Control of legged robots

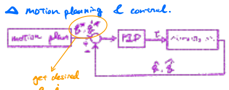
△ Static / dynamic stability

- static
  - ≥ 3 legs on ground
  - joints "freeze", will not fall
  - safe, slow, inefficient
- dynamic
  - < 3 legs on ground
  - will fall if no moving
  - hard, fast, efficient

△ control concepts

- kinematic control
  - high-gain joint position by traj. tracking
- impedance control w/ joint space inverse dynamics
  - low-gain joint control w/ model compensation
- support-consistent inverse dynamic control
  - projection of dynamics & desired accelerations into null space of contact constraints
- task-space inverse dynamics control
  - directly regulating in "taskspace" as sequential QP

△ motion planning & control

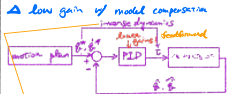


get desired $q, \dot q$

pure kinematic, w/ $q, \dot q$ then based on $q, \dot q$, do tracking

BUT: need accurate modelling of environs, to hv good motion planning otherwise, robot fail!

△ low gain w/ model compensation

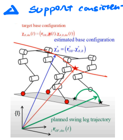


- recall $N_c^T M\ddot q + N_c^T(b+g) = N_c^T S^T\tau$

$\tau = (N_c^T S^T)^+ N_c^T(M\ddot q + b + g)$

support consistent dynamics, can better adapt to unknown terrain.

△ support consistent inverse dynamics



⇒ fully defines the motion of the system

- 12 actuated + 6 actuated = 18 DoF
- 3 swing leg + 6 base motion + 9 contact constraints = 18

⇒ $\ddot q_{des} = \begin{bmatrix} J_F \\ J_B \\ J_c \end{bmatrix}^{-1}\left(\begin{bmatrix} \ddot r_{OF,des} \\ \dot w_{B,des} \\ 0 \end{bmatrix} - \begin{bmatrix} \dot J_F \\ \dot J_B \\ \dot J_c \end{bmatrix}\dot q \right)$ — more task, then pseudo-inverse, can also hv priority

⇒ after having $\ddot q_{des}$

$\tau = (\hat N_c^T S^T)^+ \hat N_c^T(\hat M\ddot q_{des} + \hat b + \hat g)$

△ QP

$\min_x \| A_i x - b_i \|_2$

$x = \begin{bmatrix} \dot u \\ F_c \\ \tau \end{bmatrix}$

s.t.

- $M(q)\dot u + b(q,u) + g(q) + J_c^T F_c = S^T\tau$

$A = [\hat M \;\; \hat J_c^T \;\; -S^T] \quad b = -\hat b - \hat g$

- $J\dot u + \dot J u = \dot w^*$ (motion tasks)

$A = [\hat J_i \;\; 0 \;\; 0] \quad b = \dot w^* - \hat{\dot J}_i u$

- $F_i = F_i^*$ (force tasks)

$A = [0 \;\; I \;\; 0] \quad b = F_i^*$

- $\min \|\tau\|_2$ (torque min)

$A = [0 \;\; 0 \;\; I] \quad b = 0$

w/ priority

step 1: move base — try to move the base as good as possible. But not to violate the basics!

$\min_{\ddot q} \| \dot w_{B,des}(\tau) - J_B\ddot q - \dot J_B \dot q \|$

$M\ddot q + b + g + J_c^T F_c = S^T\tau$ — EoM

$J_c\ddot q + \dot J_c \dot q = 0$ — contact constraints

$F_{c,n,i} > F_{n,min}$ — minimal contact force

$\mu F_{c,n,i} > \|F_{c,t,i}\|_2$ — contact force in friction cone

step 2: move swing leg — try to swing it as good as possible, but not to violate step 1:

$\min_{\ddot q} \| \ddot r_{OF,des}(\tau) - J_F\ddot q - \dot J_F \dot q \|$

$C_1 = \ddot x_{B,des}(\tau) - J_B\ddot q - \dot J_B \dot q$ — step 1 not affected

$M\ddot q + b + g + J_c^T F_c = S^T\tau$ — EoM

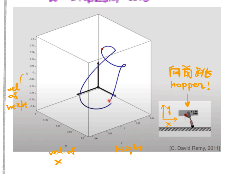
$F_{c,n,i} > F_{n,min}$ — minimal contact force

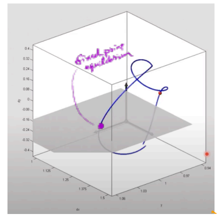
$\mu F_{c,n,i} > \|F_{c,t,i}\|_2$ — contact force in friction cone

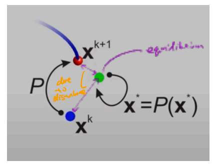
step 3: minimize $\|\tau\|_2$ + $\|F_{c,t,i}\|$

△ Stability of Legged Robot

- we want dynamic gait!
- limit cycle analysis: look at running on a step2step basis







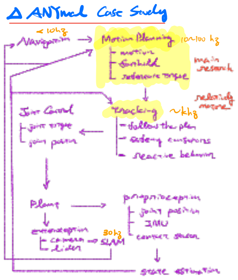
- Poincaré map $x_{k+1} = P(x_k)$
- fixed point $x^* = P(x^*)$
- linearization $\Delta x_{k+1} = \dfrac{\partial P}{\partial x}\Delta x_k$

$\text{Eig}\left(\dfrac{\partial P}{\partial x_k}\right) < 1$ ⇒ stable

△ Passive walker efficiency

$\dfrac{E_{used}}{\text{work}} = \dfrac{mgh}{mgd} = \dfrac{h}{d}$

△ ANYmal Case Study



△ Tracking of ANYmal

- inverse dynamics -based
- whole-body control
- get the "u"

- floating-base model

- $q = \begin{bmatrix} {}_I r_{IB} \\ q_{IB} \\ q_j \end{bmatrix}$ $u = \begin{bmatrix} {}_I v_B \\ {}_B \omega_{IB} \\ \dot{q}_j \end{bmatrix}$

  generalized coordinates   vel

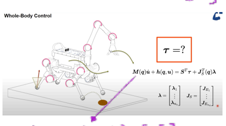
- $M(q)\dot{u} + h(q,u) = S^T \tau + J_S^T(q) \lambda$ external force / ground contacts

  $(b + g)$

  $\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{n_c} \end{bmatrix}$

  $J_S = \begin{bmatrix} J_{E_1} \\ \vdots \\ J_{E_{n_c}} \end{bmatrix}$

- ${}_I J_{E_k} = [I \quad -C_{IB} \cdot {}_B r_{BE_k}^{\times} \quad 0 \cdots {}_I J_{BE_k} \cdots 0]$



$$\begin{bmatrix} M_b \\ M_j \end{bmatrix} \dot{u} + \begin{bmatrix} h_b \\ h_j \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} + \begin{bmatrix} J_{S_b}^T \\ J_{E_j}^T \end{bmatrix} \frac{F_S}{\lambda}$$

get τ base on $\dot{u}$ & λ

$\Rightarrow \xi = \begin{bmatrix} \dot{u} \\ \lambda \end{bmatrix}$ $\Leftrightarrow$ break down into tasks

- EoM
- contact constraints
- force/torque limits

$\begin{cases} W(A\xi - b) = 0 \\ W'(D\xi - f) \le 0 \end{cases}$

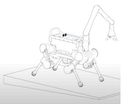
(solve the tasks parallelly/hierarchically

i.e., PROJECT THE CONSTRAINTS INTO THE NULL SPACE OF HIGHER PRIORITY TASKS!)

- examples

Priority	Task
1	Equations of Motion
2	No contact motion
3	Torque limits
4	Friction cone and λ modulation
5	Desired torso x, y position
6	Swing foot motion tracking
7	Limb kinematic limits
8	Main body yaw
9	Main body height
10	Main body roll and pitch
11	Contact force minimization

1	Equations of Motion
1	Force/torque limits
1	No motion at contact points
2	CoM motion tracking
2	Torso angular motion
2	Foot motion tracking
2	End-effector motion tracking
2	End-effector force tracking
2	Torso orientation adaptation
3	Contact force minimisation



- Then after the optimization:

  WE GET ξ, $\xi = [\begin{smallmatrix} \dot{u} \\ \lambda \end{smallmatrix}]$

  GET τ!

$\Rightarrow \tau^d = M_j(q)\dot{u}^* + h_j(q,u) - J_{S_j}(q)\lambda^*$

$\Rightarrow$ impedance

$\tau^{ref} = \tau^d + k_p \tilde{q} + k_D \dot{\tilde{q}}$

△ Planning of ANYmal

- motion
- foothold
- reference torque

} s.t. complex environments

- Traj. Opt / MPC
  - dynamics complexity? the "resolution" of the dynamics.
  - integrated/sequential optimization? w.r.t. timing/foothold locations

we are converging!
- complex/high-resolution dynamics
- integrated versions of optimization.

- foothold optimization

- first optimize the footholds i.e., what gait are we going to choose?

$\min_{\xi} \quad \frac{1}{2}\xi^T Q \xi + c^T \xi$

$\text{s.t.} \quad D\xi \le f$

$\xi = [p_{x_0}, p_{y_0}, \ldots p_{x_1}, p_{y_1}, \ldots]$

$\dot{\xi} = f(\xi)$?