ME564 L4.

second ODE

① Harmonic Oscillator
- Taylor Series
- guess

② Damped Harmonic Oscillator

③ Higher order Linear ODEs

Guess.

x (sol)

$m\ddot{x} = -kx \Rightarrow \ddot{x} = -x$

$m=1$
$k=1$

Guess : $x(t) = \cos(t)\, x(0)$ — I.C. $\dot{x}(0)=0$

$\dot{x}(t) = -\sin(t)\, x(0)$ ✓

$\ddot{x}(t) = -\cos(t)\, x(0)$ ✓

natural frequency
$\omega_n = \sqrt{\frac{k}{m}}$

for general m, k $\Rightarrow x(t) = \cos\left(\sqrt{\frac{k}{m}}\, t\right) x(0)$

$\cos \lambda t$

Taylor Series

$x(t) = \underline{C_0} + \underline{C_1 t} + \underline{C_2 t^2} + \underline{C_3 t^3} + C_4 t^4 + \dots + O(t^5)$ (assume it is continuous)

$\dot{x}(t) = C_1 + 2C_2 t + 3C_3 t^2 + 4C_4 t^3 + \dots$

$\ddot{x}(t) = \underline{2C_2} + \underline{3\cdot 2\cdot C_3 t} + \underline{4\cdot 3\cdot C_4 t^2} + \underline{5\cdot 4\cdot C_5 t^3} + \dots$

$\ddot{x} = -x$

$C_2 = \frac{-1}{2} C_0 = -\frac{1}{2} x(0)$

$C_3 = \frac{-1}{3\cdot 2} C_1 = \frac{-1}{3!} \dot{x}(0)$

$C_4 = \frac{-1}{4\cdot 3} C_2 = \frac{-1}{4\cdot 3}\,\frac{-1}{2}\, C_0 = \frac{1}{4!} C_0 = \frac{1}{4!} x(0)$

$\Leftarrow$ I.C.

$C_0 = x(0)$ initial position

$C_1 = \dot{x}(0)$ initial velocity.

$\longrightarrow$ odd $\rightarrow x(0)$
even $\rightarrow \dot{x}(0)$

(2) $\ddot{X} = -X$

$X(t) = C_0 + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4 + \ldots + O(t^5)$

$\dot{X}(t) = C_1 + 2C_2 t + 3C_3 t^2 + 4C_4 t^3 + \ldots$

$\ddot{X}(t) = 2C_2 + 3\cdot 2\cdot C_3 t + 4\cdot 3 C_4 t^2 + 5\cdot 4 C_5 t^3 + \ldots$

$\ddot{X} = -X$

$\cancel{C_0 = -2C_2}$

$\cancel{C_2 = -1}$

$2C_2 = -C_0$

$C_2 = -\frac{1}{2} C_0 \quad \text{— (1)} \rightarrow \quad C_2 = -\frac{1}{2} X(0)$

$3\cdot 2\cdot C_3 = -C_1$

$C_3 = -\frac{1}{3\cdot 2} C_1 \quad \text{— (2)} \rightarrow \quad C_3 = -\frac{1}{3!} \dot{X}(0)$

$4\cdot 3\cdot C_4 = -C_2$

$C_4 = \frac{-1}{4\cdot 3} C_2 \quad \text{— (3)} \rightarrow \quad C_4 = \frac{1}{4!} X(0)$

$(1) \rightarrow (3)$

$C_4 = \frac{-1}{4\cdot 3} \cdot -\frac{1}{2} C_0$

$= \frac{1}{4\cdot 3\cdot 2} C_0$

$= \frac{1}{4!} C_0$

I.C. (initial condition)

$C_0 = X(0)$ initial position

$C_1 = \dot{X}(0)$ initial velocity

$5\cdot 4\cdot C_5 = -C_3$

$C_5 = \frac{-1}{5\cdot 4} C_3$

$= \frac{-1}{5\cdot 4} \cdot -\frac{1}{3!} \dot{X}(0)$

$= \frac{1}{5!} \dot{X}(0)$

Assume $\dot{X}(0) = 0$ or sth else

$$X(t) = \underbrace{\left[1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 - \frac{1}{6!} t^6 \ldots\right]}_{\cos(t)} X(0) + \underbrace{\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \ldots\right]}_{\sin(t)} \dot{X}(0)$$

$\Rightarrow X(t) = \cos(t) X(0)$ [when $\sin(t) = 0$]

e.g.

$\ddot{X} + 2\dot{X} + X = 0$

step 1: assume Taylor

step 2: express x, $\dot{x}$, $\ddot{x}$ in terms of Taylor series

step 3: match the coefficients

(3) Guess again

$x(t) = e^{\lambda t}$

$\dot{x}(t) = \lambda e^{\lambda t} = \lambda x(t)$

$\ddot{x}(t) = \lambda^2 e^{\lambda t}$

$\ddot{x} = -x \Rightarrow \lambda^2 e^{\lambda t} = -e^{\lambda t}$

$\lambda^2 = -1 \quad \lambda = \pm i$

$$\boxed{\begin{gathered} ☆ \quad x(t) = k_1 e^{it} + k_2 e^{-it} \quad ☆ \\ k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \end{gathered}}$$

$x(0) = k_1 + k_2$

$\dot{x}(0) = ik_1 e^{it} - ik_2 e^{-it} \xrightarrow[t=0]{} i(k_1 - k_2)$ set $\dot{x}(0) = 0$

$\therefore k_1 = k_2$

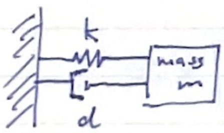
$\dot{x}(0) = 0 \rightarrow k_1 = k_2$

$x(0) = 2k_1 \rightarrow \quad k_1 = k_2 = \frac{x(0)}{2}$

$x(t) = \frac{x(0)}{2}\left[\cos(t) + \cancel{i\sin(t)}\right] + \frac{x(0)}{2}\left[\cos(t) - \cancel{i\sin(t)}\right]$

$= x(0)\cos(t)$

e.g.



$m\ddot{x} + d\dot{x} + kx = 0 \qquad w = \sqrt{\frac{k}{m}}$

$\ddot{x} + \xi\dot{x} + w^2 x = 0$

$\ddot{x} + \xi\dot{x} + w^2 x = 0$

linear ODE

let $x(t) = e^{\lambda t}$

$\dot{x}(t) = \lambda e^{\lambda t}$

$\ddot{x}(t) = \lambda^2 e^{\lambda t}$

$\lambda^2 e^{\lambda t} + \xi \lambda e^{\lambda t} + w^2 e^{\lambda t} = 0$

$\Rightarrow \underbrace{\left[\lambda^2 + \xi\lambda + w^2\right]}_{= 0} \underbrace{e^{\lambda t}}_{\neq 0} = 0$

$\lambda^2 + \xi\lambda + w^2 = 0 \qquad \frac{b' \pm \sqrt{b^2 - 4ac}}{-2a}$

$\lambda_{1,2} = \frac{-\xi \pm \sqrt{\xi^2 - 4w^2}}{2}$

$\rightarrow x(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$

now plug in I.C.

solve in Matlab.

$\ddot{x} + \xi\dot{x} + w^2 x = 0$

$\dot{x} = v$

$\dot{v} = -\xi v - w^2 x$

$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w^2 & -\xi \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$