Atlas : mechanics + optimization
Anymal : RL-based
manipulation : Imitation-learning

Δ underactuated

$\begin{cases} \dot{x} = f(x,u) \\ y = g(x) \end{cases}$ $\begin{cases} \dot{x} = Ax+Bu \\ y = Cx+Du \end{cases}$

Δ second-order nonlinear system

$\ddot{q} = f(q,\dot{q},u)$

or

$\dot{x} = f(x,u) = \begin{bmatrix} \dot{q} \\ f(q,\dot{q},u) \end{bmatrix}$

Δ "control affine" nonlinear systems

$\ddot{q} = f_1(q,\dot{q}) + f_2(q,\dot{q})u$ (1)

based on this, define underactuated

def

(1) is fully actuated in $q, \dot{q}$ i.f.f. $f_2(q,\dot{q})$ is full row rank (matrix)

i.e., $\dim(q) = m$, $\dim(u) = n$

$f_2(q,\dot{q}) \in \mathbb{R}^{m \times n}$, $\mathrm{rank}[f_2(q,\dot{q})] = m$

controllable : can choose some inputs over the time to achieve some goal

actuatable : can choose some inputs to cause acceleration

(1) is underactuated iff in $q, \dot{q}$ $\mathrm{rank}[f_2(q,\dot{q})] < m$

$\forall q, \dot{q}$, $\mathrm{rank}[f_2(q,\dot{q})] < m$

⇔ "system" is underactuated

Δ Feedback equivalence (fully-actuated)

given $\ddot{q} = f_1(q,\dot{q}) + f_2(q,\dot{q})u$

$\ddot{q}_d$

⇒ then $u = f_2^{-1}(q,\dot{q})[\ddot{q}_d - f_1(q,\dot{q})]$

⇒ $\ddot{q} = \ddot{q}_d$

give me an "acceleration", I give u an "u"

⇒ feedback equivalent to $\ddot{q} = u$ (double integrator which we hv optimal ctrl fun)

Δ feedback equivalence be broken: input saturations, state constraints, model uncertainty

Δ Manipulation Eqs

$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$ (+τ)

mass matrix, coriolis terms, gravity, torque input

$M \succ 0$

⇒ $\ddot{q} = M^{-1}(q)[-C(q)\dot{q} + \tau_g(q) + Bu]$

$\ddot{q} = f_1(q,\dot{q}) + f_2(q,\dot{q})u$ is in this form!

Q! whether $f_2(q,\dot{q})$ is full rank relies on "B" as $\underline{M^{-1}(q)}\,\underline{B}$ (full rank, not sure!)

Δ Dynamic Programming

$ml\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$

Δ Control as an optimization

- given trajectory $x(\cdot), u(\cdot)$ — $\forall t$ $x(t)$ $t \in [0, T_{max}]$
- Assign a score (scalar, reward, cost)
  e.g. time, avg. distance
- subject to constraints

e.g. minimum-time for double integrator

$\ddot{q} = u$

goal : drive to $q = \dot{q} = 0$ in minimum-time from any initial condition

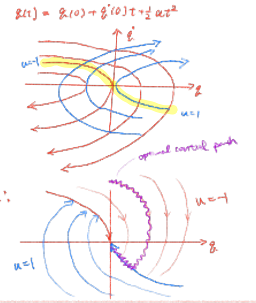
Δ intuition: "bang-bang" policy (accelerate as much as possible) (unsmooth)

Δ intuition #2:

$\ddot{q} = u$ $u = -1$ $u = 1$

$\dot{q}(t) = \dot{q}(0) - t$

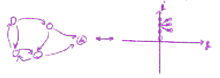
$q(t) = q(0) + \dot{q}(0)t + \frac{1}{2}at^2$



Q: How to generalize ?

Δ Dynamic Programming

minimum-time ≅ shortest path problems



- weighted shortest path
- DP
  discrete states $s_i \in S$
  discrete actions $a_i \in A$
  discrete time $s[n+1] = f(s[n], a[n])$
  "edge cost" $g(s,a)$
  total cost $\sum g(s,a)$
  Key idea : Additive cost
  e.g. time to goal $g(s,a) = \begin{cases} 1 & \forall s \neq s_{goal} \\ 0 & \text{o.w.} \end{cases}$
  $\int g(s,a)\,dt$

Δ DP

- DP is a recursive algorithm
- solve backwards from the goal
- cost-to-go

$J^*(s_i) = \min_{a[\cdot]} \sum_{n=0}^{\infty} g(s[n], a[n])$ (watch for convergence)

$s[0] = s_i$ → hard to search

⇔ $J^*(s_i) = \min_a [g(s,a) + J^*(f(s,a))]$ (single)

condition to certify optimality

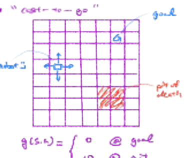
- Algorithm of DP (= value iteration)

$\hat{J}^*$ ← estimate of optimal cost-to-go

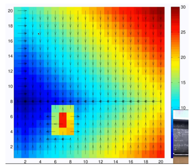
$\forall i$ $\hat{J}^*(s_i) \leftarrow \min_a [g(s,a) + \hat{J}^*(f(s,a))]$

$\hat{J}^* \longrightarrow J^* + c$

- "cost-to-go"



$g(s,a) = \begin{cases} 0 & @ \text{ goal} \\ 10 & @ \text{ pit} \\ 1 & @ \text{ o.w.} \end{cases}$



- Caveats
  - accuracy (discretization errors)
  - scalability (dimensions ~5) (curse of the dimensionality)
  - model is known
  - cost function
  - assume "full state" feedback

Δ $\lim_{\Delta x, \Delta u, \Delta t \to 0}$ DP

- my minimum-time double integrator
  $\ddot{q} = u$, $|u| \leq 1$
  optimal policy is "bang-bang"
- we want "discrete" → "continuous"

Δ optimal cost-to-go

- discrete
  $s[n+1] = f(s[n], a[n])$
- cost
  $\sum \ell(s[n], a[n])$
  $\forall s$ $J^*(s) = \min_a [\ell(s,a) + J^*(f(s,a))]$ (cost-to-go function)
  ∴ $\pi^*(s) = \arg\min_a [\ell(s,a) + J^*(f(s,a))]$ (optimal controller/policy)

Δ discrete v continuous

	discrete	continuous
states	s	x
actions	a	u
time	n	t

Δ informal derivation

$x[n+1] \approx x[n] + h f_c(x[n], u[n])$

euler-integration $t_{n+1} - t_n$

$\ell_d(x,u) = h\ell_c(x,u)$

∴ $J^*(x) = \min_u [h\ell_c(x,u) + J^*(x + hf_c(x,u))]$

$J^*(x) + \frac{\partial J^*}{\partial x} h f_c(x,u)$

⇒ $0 = \min_u [\ell_c(x,u) + \frac{\partial J^*}{\partial x} f_c(x,u)]$

Hamilton-Jacobi-Bellman Eqn.

HJB

e.g. infinite-time Grid world

$J^*(x) = c$

$\frac{dJ}{dt} = \frac{\partial J}{\partial x} f(x,u)$

for u^* cost-to-go

$0 = \ell(x,u^*) + \frac{\partial J^*}{\partial x}\frac{\partial x}{\partial t}$ ⇒ $\frac{dJ}{dt} = -\ell(x,u^*)$ (running cost)

⇒ if you are following optimal policy your cost ↓

HJB sufficiency theorem

e.g. double integrator / Quadratic cost

$\ddot{q} = u$ $\ell(x,u) = q^2 + \dot{q}^2 + u^2$ (no constraints)

claim:

$\pi^*(x) = -q - \sqrt{3}\dot{q}$

$J^*(x) = \sqrt{3}q^2 + 2q\dot{q} + \sqrt{3}\dot{q}^2$ $\left(x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}\right)$

proof:

(1) $0 = \min_u [q^2 + \dot{q}^2 + u^2 + \frac{\partial J}{\partial x} f(x,u)]$

$\left(\frac{\partial J}{\partial x} f(x,u) = [\frac{\partial J}{\partial q} \frac{\partial J}{\partial \dot{q}}]\begin{bmatrix} \dot{q} \\ u \end{bmatrix}\right)$

$= \min_u [q^2 + \dot{q}^2 + u^2 + \frac{\partial J}{\partial q}\dot{q} + \frac{\partial J}{\partial \dot{q}}u]$

$= \min_u [0]$

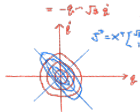
(2) $u = \arg\min_u [u^2 + \frac{\partial J}{\partial \dot{q}}u]$

⇒ $\frac{\partial}{\partial u}(\cdot) = 2u + \frac{\partial J}{\partial \dot{q}} = 0$

⇒ $u = -\frac{1}{2}\frac{\partial J}{\partial \dot{q}}$

$= -\frac{1}{2}(2q + 2\sqrt{3}\dot{q})$

$= -q - \sqrt{3}\dot{q}$



$J^* = x^T \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix} x$

Δ LQR

- $\dot{x} = Ax + Bu$
  $\ell(x,u) = x^TQx + u^TRu$
  $Q \succeq 0$ $R \succ 0$
- inf horizon
  $J^*(x) = x^TSx$ (cost-to-go) (can capture the dynamics)
  $S \succeq 0$

$\frac{\partial J}{\partial x} = 2x^TS$

$0 = \min_u [x^TQx + u^TRu + \frac{\partial J}{\partial x}(Ax+Bu)]$

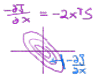
⇒ $\frac{\partial [\,]}{\partial u} = 0$, $u = -R^{-1}B^TSx$
$= -Kx$

Riccati Equation

$0 = Q - SBR^{-1}B^TS + A^TS + SA$

- insights :

$-\frac{\partial J}{\partial x} = -2x^TS$ (where we want our cost-to-go func. to decrease)



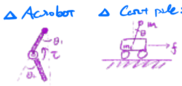
$u = -R^{-1}B^TSx$
(gradient)
(underactuated B matrix to adjust Sx)
(weighting on B^TSx)

Δ DP so far

- Tabular (discrete → graph search)
- Linear dynamics + convex objective (LQR)

↳ here, use acrobot, cartpole, quadrotor to demonstrate DP

Δ Acrobot Δ Cart pole:



Δ Equations of motion :

$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$

- Acrobot	- Cart pole
$q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$	$q = \begin{bmatrix} x_{cart} \\ \theta_{pole} \end{bmatrix}$
$u = [\tau]$	$u = [f_{cart}]$
$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\lvert u \rvert \leq u_{max}$	$\lvert u \rvert \leq u_{max}$
	$\lvert x \rvert \leq x_{max}$

↳ target: swing up + balance

soln 1. tabular value iteration ?
yes, not elegant

2. LQR ?
Beautiful solution for balancing (need more for swing-up)

Δ LQR for nonlinear systems

$\dot{x} = f(x,u)$ ⇔ $\dot{x} = Ax + Bu$

⇒ Chose nominal x_0, u_0 (want a fixed pt)

⇒ $\dot{x} \approx f(x_0,u_0) + \underbrace{\frac{\partial f}{\partial x}\Big|_{x=x_0, u=u_0}}_{A}(x-x_0) + \underbrace{\frac{\partial f}{\partial u}\Big|_{x=x_0, u=u_0}}_{B}(u-u_0)$

let
$\bar{x} = x - x_0$
$\dot{\bar{x}} = \dot{x} - \dot{x}_0$
$\bar{u} = u - u_0$

⇒ $\dot{\bar{x}} = A\bar{x} + B\bar{u}$

e.g. simple pendulum

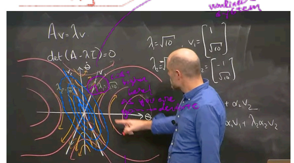
- $ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = \tau$
  $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ $u = [\tau]$
- linearize about $\theta = \pi$, $\dot{\theta} = 0$, $u = 0$
- $\dot{x} = \begin{bmatrix} \dot{\theta} \\ \frac{1}{ml^2}[\tau - b\dot{\theta} - mgl\sin\theta] \end{bmatrix}$
- check fixed pt ✓
  $\dot{x}_0 = \begin{bmatrix} 0 \\ \frac{1}{ml^2}[0 - b\cdot 0 - mgl\sin\pi] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- linearization
  $\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta & -\frac{b}{ml^2} \end{bmatrix}\Bigg|_{\theta=\pi} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix}$
  $\frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$

let $m = l = 1$ $g = 10$

$A = \begin{bmatrix} 0 & 1 \\ 10 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

orange: approximation of a nonlinear system



red: nonlinear system

- LQR Design

$\begin{cases} \dot{\bar{x}} = A\bar{x} + B\bar{u} \\ J = \int_0^\infty \bar{x}^TQ\bar{x} + \bar{u}^TR\bar{u}\,dt \\ [K, S] = lqr(A, B, Q, R) \\ u = -Kx = -R^{-1}B^TSx \\ J^* = \bar{x}^TS\bar{x} \end{cases}$

⇒ Stable

$\dot{x} = \underbrace{(A - BK)}_{\text{Hurwitz}}x$

Δ Is (A,B) controllable?

underactuated but controllable

Δ controllable : given $x(0)$ can I find $u(t)$ $\forall t \in [t_0, t_f]$ s.t. $x(t_f) = 0$

Is (A,B) stablizable

Δ stablizable : given $x(0)$ can I find $u(t)$ $\forall t \in [t_0, \infty)$ s.t. $\lim_{t \to \infty} x(t) = 0$

(no control input)

Δ Nonlinear Dynamics

(kinetic energy) $T = \frac{1}{2}ml^2\dot{\theta}^2$ $v_{mass} = l\dot{\theta}$

(potential energy) $U = -mgl\cos\theta$

Lagrangian mechanics

$L = K - U$

$K = \frac{1}{2}mv^2$
$= \frac{1}{2}m\dot{s}^2$
$= \frac{1}{2}ml^2\dot{\theta}^2$ $\theta = \frac{s}{l}$

$U = mgy$
$= -mgl\cos\theta$

∴ $L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

Euler-lagrange equation

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$

$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

$\frac{\partial L}{\partial \theta} = ml^2\ddot{\theta} + mgl\sin\theta = Q$ (generalized force)

$Q = \underbrace{-b\dot{\theta}}_{\text{friction}} + \underbrace{u}_{\text{torque}}$

∴ $ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$

⇕

$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$

⇕

$\ddot{\theta} = f(\theta, \dot{\theta}, u)$

⇕

$\theta(t) = ?$

Δ Nonlinear Dynamics Questions

- What is $\lim_{t \to \infty} \theta(t)$?
- Will my robot fall down ?

$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$ (N·m)

$kg\frac{m^2}{s^2}\frac{1}{s}$ $kg\,m^2\frac{1}{s^2}$ $kg\frac{m^2}{s^2}$

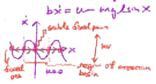
case : $b\dot{\theta} \gg ml^2$

$b\sqrt{\frac{l}{g}} \gg ml^2$ (heavily damped region)

⇒ linear approximation!

⇒ $b\dot{\theta} = u - mgl\sin\theta$

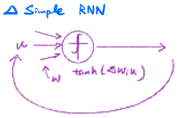
$b\dot{x} = u - mgl\sin x$ $x \in \mathbb{R}$



Δ Stability

- local stability (do not leave a ball)
  - in the sense of Lyapunov (i.s.L.)
    $\begin{cases} \epsilon, \delta, \text{ small positive constants} \\ \forall \epsilon, \exists \delta \text{ s.t. } \|x(0) - x^*\| < \delta \\ \Rightarrow \forall t \|x(t) - x^*\| < \epsilon \end{cases}$
  - locally attractive — get to the region
    $\{\lim_{t\to\infty} x(t) \to x^*$
  - asymptotically stable — i.s.L + attractive
    $\{$ i.s.L + attractive
  - exponentially stable — get to the region w/ rate
    $\begin{cases} \forall t \\ \|x(t) - x^*\| < Ce^{-\alpha t} \\ C, \alpha > 0 \end{cases}$

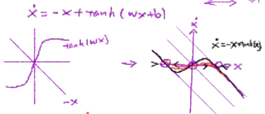
Δ Simple RNN

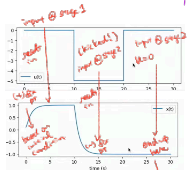


Autapse

±b offset

$\dot{x} = -x + \tanh(wx + b)$





Δ 2nd-order system

$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$

$\dot{x} = f(x,u)$

$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \frac{1}{ml^2}[u - b\dot{\theta} - mgl\sin\theta] \end{bmatrix}$

★ Essence of controls for underactuated robotics :

we hv all these dynamics

→ plug w/ u, s.t. phase portraits behave as we want!!!

△ Lyapunov Analysis

recall DP:
- Tabular — easy to explode
- LQR — only for linear case
- Approximate DP (NN) — works quite well (but solve a different DP, e.g. discounting)

→ all are trying to get "cost-to-go" function $J^*(x)$

△ now Lyapunov ⟷ optimal value
(easy to find) (hard to find)
good enough — very good

might replace the original optimal val.

△ Example : stability analysis of simple pendulum

$ml\ddot{\theta} + mgl\sin\theta = -b\dot{\theta}$

△ $\theta(t)$ ⟶ hard to solve
cannot be analysis
△ Lyapunov instead !!!

△ $E = K + U$
$= \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$

△ $\frac{d}{dt}E(x)$, $x = \begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix}$

$= \frac{\partial E}{\partial x}\frac{\partial x}{\partial t}$

$= \frac{\partial E}{\partial \theta}\dot{\theta} + \frac{\partial E}{\partial \dot{\theta}}\ddot{\theta}$

$= mgl\sin\theta\,\dot{\theta} + ml^2\dot{\theta}\ddot{\theta}$

$= \dot{\theta}(mgl\sin\theta + ml^2\ddot{\theta})$

$= -b\dot{\theta}^2 \le 0$ if $b > 0$

△ General Energy function

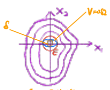
△ given $\dot{x} = f(x)$ no u
→ want to prove stability of $x^* = 0$
→ construct a differentiable function $V(x)$, s.t.

$$\begin{cases} V(0)=0, & V(x)>0,\ x\ne 0 \quad \text{PD} \\ \dot{V}(0)=0, & \dot{V}(x)\le 0,\ x\ne 0 \quad \text{NSD} \end{cases}$$

sufficient condition

→ then, x^* is stable i.s.L.

△ i.s.L def
$\forall \epsilon > 0,\ \exists \delta > 0$
s.t. $\|x(0) - x^*\| < \delta$
$\forall t\ \|x(t) - x^*\| < \epsilon$



I start @ V=0.82, my V(t), t>0, V<0.82, [will not leave the subset V≤0.82], ∃ε that contains V≤0.82

e.g. pendulum
$V(x) = E + mgl$

△ Asymptotically Stable

$$\begin{cases} V(0)=0, & V(x)>0,\ x\ne 0 \quad \text{PD} \\ \dot{V}(0)=0, & \dot{V}(x)<0,\ x\ne 0 \quad \text{NSD} \end{cases}$$

"<" not "≤"
otherwise $\dot{V}(x) = 0$,
⟶ limit cycles.

△ Global Stability

Global Asymptotically stability (GAS)

$$\begin{cases} V(0)=0, & V(x)>0,\ x\ne 0 \quad \text{PD} \\ \dot{V}(0)=0, & \dot{V}(x)\le 0,\ x\ne 0 \quad \text{NSD} \end{cases}$$
+
$\lim_{|x|\to\infty} V(x) = \infty$ "radially" unbounded

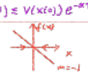
△ Regional Stability

$$\begin{cases} V(0)=0, & V(x)>0,\ x\ne 0 \quad \text{PD} \\ \dot{V}(0)=0, & \dot{V}(x)\le 0,\ x\ne 0 \quad \text{NSD} \end{cases}$$
$\forall x \in D \subset R^n$

△ Exponential Stability

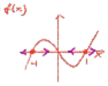
$$\begin{cases} V(0)=0, & V(x)>0,\ x\ne 0 \quad \text{PD} \\ \dot{V}(0)=0, & \dot{V}(x)\le -\alpha V(x),\ x\ne 0 \quad \text{NSD} \end{cases}$$
⇕
$V(x(t)) \le V(x(0))e^{-\alpha t}$

e.g. $\dot{x} = -x$



$V(x) = x^2$
$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x}$
$= 2x(-x)$
$= -2x^2 < 0$ { i.S.L ✓, asymptotic ✓
$\le 2V(x)$ { exponential ✓
$\lim_{|x|\to\infty} V(x) = \infty$ { global stable

e.g. $\dot{x} = -x + x^3 = f(x)$



{ $x^* = 0$ a f.p.
{ $x \in (-1, 1)$ R.O.A.

$V(x) = x^2$
$\dot{V}(x) = 2x(-x + x^3)$
$= 2x^2(x^2 - 1)$
$= \begin{cases} 0 & x = 0, \pm 1 \\ > 0 & |x| > 1 \\ < 0 & |x| < 1,\ x \ne 0 \end{cases}$

• sublevel set of V ⟶ invariant set
$V(x) \le \rho$ (inside the R.O.A. of x^*)

△ General form of R.O.A.

if $V(x) > 0$, $\dot{V}(x) < 0$
$\forall x \in \{x \mid V(x) < \rho,\ \rho > 0\}$ — invariant set
then $V(x(0)) < \rho$
⇒ $\lim_{t\to\infty} V \to 0$, $x \to 0$
and $\{x \mid V(x) < \rho\}$ is inside R.O.A.

△ LaSalle's Theorem

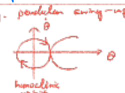
△ Lyapunov ⟶ Cost-to-go funcs.

HJB:
$0 = \min_u [\ell(x,u) + \frac{\partial J^*}{\partial x} f(x,u)]$
$u^* = \pi^*(x)$
⇒ $0 = \ell(x, u^*) + \frac{\partial J^*}{\partial x} f(x, u^*)$
$= \ell(x, u^*) + \frac{\partial J^*}{\partial t}$
⇒ $\dot{J}^*(x) = -\ell(x, u^*)$
"cost-to-go" function's derivative has to be decreasing !!!
⇕ relaxation
$\dot{V}(x) < 0$ ⟶ way more easy !!!

HJB discrete
$J^*(x) = \min_u [\ell(x,u) + J^*(f(x,u))]$
$\Delta J^*(x) = J^*(f(x,u^*)) - J^*(x)$
⇕ $= -\ell(x, u^*)$
$\Delta V(x) \le 0$

△ Lyapunov-based controller

e.g. pendulum swing-up



homoclinic orbit

• $E^d = mgl$
$V(x) = \frac{1}{2}(E(x) - E^d(x))^2$
$= \frac{1}{2}\tilde{E}^2(x)$

• $ml^2\ddot{\theta} + mgl\sin\theta = u$
$\dot{E} = u\dot{\theta}$
$\dot{\tilde{E}} = \dot{E} - \dot{E}^d = u\dot{\theta}$

• let $u = -k\dot{\theta}\tilde{E}$, $k > 0$
making
$\dot{\tilde{E}} = -k\dot{\theta}^2\tilde{E}$

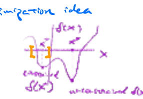
△ Promise : find V(x) should be easier

- Global Lyapunov analysis for the pendulum
- input: pendulum dynamics
  parametric family of trig/poly for Lyapunov — polynomial over $1, \cos\theta, \sin\theta, \dot{\theta}$
  output: coefficient of the polynomials
  + certificate that Lyapunov conditions satisfied $\forall x$?

→ Lyapunov analysis w/ convex optimization

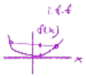
△ some basic optimization idea

$\min_x f(x)$
s.t. $g_i(x) \le 0$



constrained $f(x)$ — unconstrained $f(x^*)$

• convex optimization
i.f.f $f(x)$ is convex function
$\{g_i(x)\}$ is a convex set



LP → QP → SOCP ⟶ SDP
easy to solve ⟶ hard to solve

△ SDP

linear objective, linear constraints
PSD matrix constraints.

△ Q! How do we compute Lyapunov func?

1. Parameterize Lyapunov candidate
let's say
$\dot{x} = f(x)$ (linear NN)
$V(x) = \sum_i \alpha_i \phi_i(x) = \alpha^T\phi(x)$ — nonlinear basis

2. Use convex optimization search over α to satisfy Lyapunov condition.
find α: $V(0) = 0$, $\forall i\ V(x_i) > \epsilon x_i^T x_i$
⇒ $V(x_i) > \epsilon x^T x$ — linear constraint in α
$\dot{V}(0) = 0$, $\forall i\ \dot{V}(x_i) = \frac{\partial V}{\partial x}\Big|_{x=x_i} f(x_i)$
$= \alpha^T \frac{\partial \phi}{\partial x} f(x_i)$
$< -\epsilon x^T x$ — linear constraint
How do we make sure for all x?
⇕
$\min_\alpha |\dot{V}(x) + 1|_1$

△ Robust stability analysis

$\dot{x} = Ax$, $A \in [A^-, A^+]$ uncertain
⇒ $A = \sum \beta_i A_i$ $\beta_i \ge 0$, $\sum\beta_i = 1$
⇕
find $P > 0$ s.t. $\forall i$: $PA_i + A_i^T P < 0$
⇒ implies $P(\sum \beta_i A_i) + (\sum \beta_i A_i)^T P < 0$

△ $\dot{x} = Ax$ is it stable?

$V(x) = x^T P x$, $P \succ 0$
$\dot{V}(x) = 2x^T P A x$
$= x^T P A x + x^T A^T P x$ — Lyapunov for linear system
$\dot{V}(x) < 0$ if $(PA + A^T P) \prec 0$
find α s.t. $P = [\ldots]$ — pick $Q \succ 0$ & $PA + A^T P = -Q$ (Lyapunov eq.)
⇕ $P \succ 0$, $PA + A^T P \preceq 0$
find α, β: $P \succ 0$ $P_2 \succ 0$
s.t. $PA + A^T P = -P_2$

$V(x) = \sum \alpha_i \phi(x)$, $\alpha_i \ge 0$
☆ $V(x) = \phi^T(x) G \phi(x)$, $G \succ 0$
$\dot{V}(x) = \frac{\partial V}{\partial x} f(x)$
⟶ construct $-\phi^T(x) G_2 \phi(x)$
⟶ set $\dot{V}(x) = -\phi^T G_2 \phi$

△ DP sum

$\frac{d}{dt}x = f(x(t), u(t), t)\,dt$

• $J(x(t), u(t), t_0, t_f)$
$= Q(x(t_f), t_f) + \int_{t_0}^{t_f} \ell(x(\tau), u(\tau))\,d\tau$

• $V(x(t_0), t_0, t_f)$ (cost-to-go)
$= \min_{u(t)} J(x(t), u(t), t_0, t_f)$

• $-\frac{\partial V}{\partial t} = \min_{u(t)} \left(\left(\frac{\partial V}{\partial x}\right)^T f(x(t), u(t)) + \ell(x(t), u(t)) \right)$
⟺ $\min_{u(t)} \left(\left(\frac{\partial J^*}{\partial x}\right)^T f(x(t), u(t)) + \ell(x(t), u(t)) \right)$
Hamilton-Jacobi-Bellman (HJB)

• $V(x(t_0), t_0, t_f)$
$= V(x(t_0), t_0, t) + V(x(t), t, t_f)$

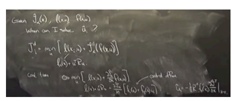
Bellman Optimality

⟺ $J^*(s_i) = \min_a [\ell(s,a) + J^*(f(s,a))]$

proof — total derivative
$\frac{d}{dt}V(x(t), t, t_f) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right)^T \frac{\partial x}{\partial t}$
$= \frac{d}{dt}\min_{u(t)} J(x(t), u(t), t, t_f)$
$= \frac{d}{dt}\min_{u(t)} Q(x(t_f), t_f) + \int_t^{t_f} \ell(x(\tau), u(\tau))\,d\tau$
$= \min_{u(t)} \frac{d}{dt}\left[Q(x(t_f), t_f) + \int_t^{t_f} \ell(x(\tau), u(\tau))\,d\tau\right]$
$= \min_{u(t)}\ 0 + \frac{d}{dt}\left(\left[\mathcal{L}(x(\tau), u(\tau))\right]_t^{t_f}\right)$
$= \min_{u(t)} \frac{d}{dt}\left(\mathcal{L}(x(t_f), u(t_f)) - \mathcal{L}(x(t), u(t))\right)$
$= \min_{u(t)} -\ell(x(t), u(t))$
⇕
$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right)^T \frac{\partial x}{\partial t} = \min_{u(t)} -\ell(x(t), u(t))$
⇕
$-\frac{\partial V}{\partial t} = \min_{u(t)} \left(\left(\frac{\partial V}{\partial x}\right)^T \frac{\partial x}{\partial t} + \ell(x, u) \right)$
$= \min_{u(t)} \left(\frac{\partial V^T}{\partial x} f(x,u) + \ell(x,u) \right)$

☆ Get V ⟶ Get u
- two point bounded value problem
- shooting method

△ Discrete-Time HJB
• $x_{n+1} = F(x_n, u_n)$
• $J(x_0, \{u_k\}_{k=0}^{n}, n)$
$= Q(x_n, t_n) + \sum_{k=0}^{n} \ell(x_k, u_k)$
• $V(x_0, n)$
$= \min_{\{u_k\}_{k=0}^{n}} J(x_0, \{u_k\}_{k=0}^{n}, n)$
• $V(x_0, n)$
$= V(x_0, k) + V(x_k, n-k)\ \forall k \in (0, n)$
• $V(x_k, n)$
$= \min_{u_k}\left[\ell(x_k, u_k) + V(x_{k+1}, n-1)\right]$, $x_{k+1} = F(x_k, u_k)$
⟺ vs $J(s_i) = \min_a [\ell(s,a) + J(f(s,a))]$
Discrete HJB
⇕
$V(x_k, n)$
$= \min_{u_k}\left(\ell(x_k, u_k) + V(F(x_k, u_k), n-1)\right)$
⇕
$V(x) = \min_u (\ell(x,u) + V(F(x,u)))$
↓
$\pi(x) = \arg\min_u (\ell(x,u) + V(F(x,u)))$



- use optimal u : recall CT LQR: $u^* = -K^* x = -R^{-1}B^T S^* x$
- instead of sampling u_i — DT LQR: $u^* = -K^* x = -(R + B^T S^* B)^{-1} B^T S^* A x$
only sample $x_i \in X_s \subset X$
⇒ $J_i^d = x_i^T Q x_i + \bar{u}_i^T R \bar{u}_i + \hat{J}_S(Ax_i + B\bar{u}_i)$
w/ optimal u w/ current S

△ DP (cont'd)

key idea
1) LQR (linear optimal control)
2) local linearization
3) $Q = \begin{bmatrix} 10 & & & \\ & 10 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ $R = [1]$

@ balancing

Q: How do we swing up?
"value iteration"
⇒ Approximate DP

△ Value iteration on a mesh for Pendulum Swing-up

△ mesh q 51 bins — odd for "0"
$\dot{q}$ 51 bins
u 9 bins

△ Acrobot
51^4 bins ↗ explode !!!
9 bins

△ NN! Approximating Value Functions

$\hat{J}_\alpha(x)$ — parameters + bias

- value iteration update
• recall:
$\forall s_i \in S$
$\hat{J}(s_i) = \min_a [\ell(s_i, a) + \hat{J}(f(s_i, a))]$
$s[n+1] = f(s[n], a[n])$
$\sum_{n=0}^{\infty} \ell(s[n], a[n])$

• Draw samples
$x_i \in X_s \subset X$
$u_j \in U_s \subset U$
$\forall ij$ $x'_{ij} = f(x_i, u_j)$ next state samples
$\ell_{ij} = \ell(x_i, u_j)$
desired J @ i: $J_i^d = \min_j [\ell_{ij} + \hat{J}_\alpha(x'_{ij})]$

⇒ $\min_\alpha$ loss $\sum_i [\hat{J}_\alpha(x_i) - J_i^d]^2$
⇕
$\min_\alpha$ loss $\sum_i \left[\hat{J}_\alpha(x_i) - \min_j [\ell_{ij} + \hat{J}_\beta(x'_{ij})]\right]^2$
"target network"

"here we do the training based on the samples. To hv a better sampling, RL kicks in, & guide the state space, to a subset, that is closer to the optima."

Q: when does it work?
does it converge
↓ △ linear approximator
△ works @ special case : linear NN
• $\hat{J}_\alpha(x) = \alpha^T\phi(x)$ — nonlinear basis "input features" & fixed
$= \sum_j \alpha_j \phi_j(x)$
⇒ $\min_\alpha \sum_i [\alpha^T\phi(x) - J_i^d]^2$
linear least squares
has closed-form sol^n
⇒ SGD is guaranteed to converge to global minima

△ Back to LQR ; connection to approximator ; w/ discrete LQR

$J^*(x) = x^T S^* x$ — NN ⟶ linear approximator
loss $\sum_i [x_i^T \hat{S} x_i - J_i^d]^2$
$\hat{S} \leftarrow \hat{S} - \eta \frac{\partial loss}{\partial S}$

Based on this, loss explodes ↑

why? LQR tries to solve inf horizon, cost cannot go to inf.
How does normal LQR avoid inf?
$x \to 0$, $u \to 0$ (exact 0)
How about here, when we do sampling? — probably seen in RL
- discounting $\sum_{n=0}^{\infty} \gamma^n \ell(x[n], u[n])$ $0 \le \gamma \le 1$
optimal : LQR($\sqrt{\gamma}A$, B, Q, $\frac{1}{\gamma}$R)
pretend that is more stable
$x[n+1] = Ax[n] + Bu[n]$
$|\text{eig}(A)| \le 1$.

$|NFA| \leq |A|$