Atlas : mechanics + optimization
Anymal : RL-based
manipulation : imitation-learning

△ underactuated

$\begin{cases} \dot{x} = f(x,u) \\ y = g(x) \end{cases}$ $\quad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

△ second-order nonlinear system

$\ddot{q} = f(q, \dot{q}, u)$

or

$\dot{x} = \tilde{f}(x,u) = \begin{bmatrix} \dot{q} \\ f(q,\dot{q},u) \end{bmatrix}$

△ "control affine" nonlinear systems

$\ddot{q} = f_1(q,\dot{q}) + f_2(q,\dot{q})u \quad (1)$

based on this, define underactuated

def

(1) is fully actuated in $q, \dot{q}$ i.f.f.

$f_2(q,\dot{q})$ is full row rank (matrix)

i.e., $\dim(q) = m$, $\dim(u) = n$

$f_2(q,\dot{q}) \in \mathbb{R}^{m \times n}$, $\operatorname{rank}[f_2(q,\dot{q})] = m$

controllable : can choose some inputs over time to achieve some goal

actuable : can choose some inputs to cause acceleration

(1) is underactuated iff in $q, \dot{q}$ $\operatorname{rank}[f_2(q,\dot{q})] < m$

$\forall q, \dot{q}$, $\operatorname{rank}[f_2(q,\dot{q})] < m$

$\Rightarrow$ "system" is underactuated

△ Feedback equivalence (fully-actuated)

given

$\ddot{q} = f_1(q,\dot{q}) + f_2(q,\dot{q})u$

& $\ddot{q}^d$

$\Rightarrow$ then $u = f_2^{-1}(q,\dot{q})[\ddot{q}^d - f_1(q,\dot{q})]$

$\Rightarrow \ddot{q} = \ddot{q}^d$

give me an "acceleration",

I give u an "u"

$\Rightarrow$ feedback equivalent to $\ddot{q} = u$

(double integrator

which we hv optimal sol^n for)

△ feedback equivalence be broken;

input saturations

state constraints

model uncertainty

△ Manipulator Eqs

$\underbrace{M(q)\ddot{q}}_{\text{mass matrix}} + \underbrace{C(q,\dot{q})\dot{q}}_{\text{coriolis terms}} = \underbrace{\tau_g(q)}_{\text{gravity}} + \underbrace{Bu}_{\text{torque input}} \; (+?)$

actuator selector Matrix

$M \succ 0$

$\Rightarrow \ddot{q} = M^{-1}(q)[-C(q,\dot{q})\dot{q} + \tau_g(q) + Bu]$

$\ddot{q} = f_1(q,\dot{q}) + f_2(q,\dot{q})u$

is in this form!

α! whether $f_2(q,\dot{q})$ is full rank relies on "B"

as $\underbrace{M^{-1}(q)}_{\text{full rank}}\underbrace{B}_{\text{not sure!}}$

△ Dynamic Programming

$ml\ddot{\theta} + b\dot{\theta} + mgl s\theta = u$

△ Control as an optimization

• given trajectory $x(\cdot), u(\cdot)$

$\forall t \;\; x(t) \;\; t \in [0, T_{max}]$

• Assign a score (scalar reward cost)

e.g. time, avg distance

• subject to constraints

e.g. minimum time for double integrator

$\ddot{q} = u$

goal : drive to $q = \dot{q} = 0$ in minimum-time from any initial condition

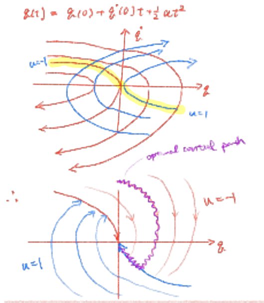
△ intuition: "bang-bang" policy (accelerate as much as possible) (unsmooth)

△ intuition #2:

$\ddot{q} = u \quad u = -1 \qquad u = 1$

$\dot{q}(t) = \dot{q}(0) - t$

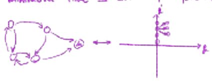
$q(t) = q(0) + \dot{q}(0)t + \frac{1}{2}at^2$



Q: How to generalize ?

△ Dynamic Programming

minimum-time $\cong$ shortest path problems



• weighted shortest path

• DP

discrete states $s_i \in S$

discrete actions $a_i \in A$

discrete time $s[n+1] = f(s[n], a[n])$

"edge cost" $g(s,a)$

total cost $\sum g(s,a)$

Key idea : Additive cost

e.g. time to goal $g(s,a) = \begin{cases} 1 & \forall s \neq s_{goal} \\ 0 & \text{o.w.} \end{cases}$

$\int g(s,a)\,dt$

△ DP

• DP is a recursive algorithm

• solve backwards from the goal

• cost-to-go

$J^*(s_i) = \min_{a[\cdot]} \sum_{n=0}^{\infty} g(s[n], a[n])$ (watch for convergence)

$s[0] = s_i$

hard to search

$\Leftrightarrow J^*(s_i) = \min_a [g(s,a) + J^*(f(s,a))]$

single

condition to certify optimality

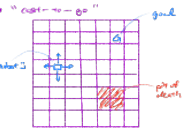
• Algorithm of DP (= value iteration)

$\hat{J}^* \leftarrow$ estimate of optimal cost-to-go

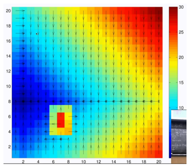
$\forall i \quad \hat{J}^*(s_i) \leftarrow \min_a [g(s,a) + \hat{J}^*(f(s,a))]$

$\hat{J}^* \longrightarrow J^* + c$

• "cost-to-go"



$g(s,a) = \begin{cases} 0 & @ \text{ goal} \\ 10 & @ \text{ pit} \\ 1 & @ \text{ o.w.} \end{cases}$



• Caveats

- accuracy (discretization errors)

- scalability (dimensions ~5) curse of the dimensionality

- model is known

- cost function

- assume "full state" feedback

△ $\lim_{\substack{\Delta x \to 0 \\ \Delta u \to 0 \\ \Delta t \to 0}}$ DP

• eg minimum-time double integrator

$\ddot{q} = u, \; |u| \le 1$

optimal policy is "bang-bang"

• we want "discrete" → "continuous"

△ optimal cost-to-go

• discrete

$s[n+1] = f(s[n], a[n])$

• cost

$\sum \ell(s[n], a[n])$

$\forall s \;\; J^*(s) = \min_a [\ell(s,a) + J^*(f(s,a))]$ (cost-to-go function)

$\therefore \pi^*(s) = \operatorname{argmin}_a [\ell(s,a) + J^*(f(s,a))]$ (optimal controller/policy)

△ discrete v continuous

	discrete	continuous
states	s	x
actions	a	u
time	n	t

△ informal derivation

$x[n+1] \approx x[n] + h f_c(x[n], u[n])$

euler-integration $\quad t_{n+1} - t_n$

$\ell_d(x,u) = h\ell_c(x,u)$

$\therefore J^*(x) = \min_u [h\ell_c(x,u) + J^*(x + hf_c(x,u))]$

$J^*(x) + \frac{\partial J^*}{\partial x}\Big|_{x=x} hf_c(x,u)$

$\Rightarrow 0 = \min_u \left[\ell_c(x,u) + \frac{\partial J^*}{\partial x} f_c(x,u)\right]$

Hamilton-Jacobi-Bellman Eqn. HJB

e.g. infinite-time Grid world

$J^*(x) = c$

$\frac{dJ}{dt} = \frac{\partial J}{\partial x} f(x,u)$

for u^* — cost-to-go

$0 = \ell(x,u^*) + \frac{\partial J^*}{\partial x}\frac{\partial x}{\partial t} \Rightarrow \frac{\partial J}{\partial t} = -\underbrace{\ell(x,u^*)}_{\text{running cost}}$

$\Rightarrow$ if you are following optimal policy your cost ↓

HJB sufficiency theorem

e.g. double integrator / Quadratic cost

$\ddot{q} = u \quad \ell(x,u) = q^2 + \dot{q}^2 + u^2$ (no constraints)

claim:

$\pi^*(x) = -q - \sqrt{3}\dot{q}$

$J^*(x) = \sqrt{3}q^2 + 2q\dot{q} + \sqrt{3}\dot{q}^2 \quad \left(x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}\right)$

proof:

(1) $0 = \min_u \left[q^2 + \dot{q}^2 + u^2 + \frac{\partial J}{\partial x} f(x,u)\right]$

$\left(\frac{\partial J}{\partial x} f(x,u) = \left[\frac{\partial J}{\partial q}\frac{\partial J}{\partial \dot{q}}\right]\begin{bmatrix} \dot{q} \\ u \end{bmatrix}\right)$

$= \min_u \left[q^2 + \dot{q}^2 + u^2 + \frac{\partial J}{\partial q}\dot{q} + \frac{\partial J}{\partial \dot{q}}u\right]$

$= \min_u [0]$

(2) $u = \operatorname{argmin}_u \left[u^2 + \frac{\partial J}{\partial \dot{q}}u\right]$

$\Rightarrow \frac{\partial}{\partial u}(\cdot) = 2u + \frac{\partial J}{\partial \dot{q}} = 0$

$\Rightarrow u = -\frac{1}{2}\frac{\partial J}{\partial \dot{q}}$

$= -\frac{1}{2}(2q + 2\sqrt{3}\dot{q})$

$= -q - \sqrt{3}\dot{q}$ ✱

$J^* = x^T \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix} x$

△ LQR

• $\dot{x} = Ax + Bu$

$\ell(x,u) = x^TQx + u^TRu$

$Q \succeq 0 \quad R \succ 0$

• inf horizon

$J^*(x) = x^TSx$ (cost-to-go) can capture the dynamics

$S \succeq 0$

$\frac{\partial J}{\partial x} = 2x^TS$

$0 = \min_u \left[x^TQx + u^TRu + \frac{\partial J}{\partial x}(Ax + Bu)\right]$

$\Rightarrow \frac{\partial [\cdot]}{\partial u} = 0, \quad u = -R^{-1}B^TSx$

$= -Kx$

Riccati Equation

$0 = Q - SBR^{-1}B^TS + A^TS + SA$

• insights :

$-\frac{\partial J}{\partial x} = -2x^TS$ (where we want our cost-to-go func. to decrease)

$-\frac{\partial J}{\partial x}$

$u = -R^{-1}B^TSx$

(gradient)

(underactuated B matrix to adjust Sx)

(weighting on B^TSx)

△ DP so far

- Tabular (discrete → graph search)

- Linear dynamics + convex objective (LQR)

↳ here, use acrobot, cartpole, quadrotors to demonstrate DP

△ Acrobot △ Cart pole:

△ Equations of motion :

$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$

- Acrobot	- Cart pole
$q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$	$q = \begin{bmatrix} x_{cart} \\ \theta_{pole} \end{bmatrix}$
$u = [\tau]$	$u = [f_{cart}]$
$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\lvert u \rvert \le u_{max}$	$\lvert u \rvert \le u_{max}$
	$\lvert x \rvert \le x_{max}$

↳ target: swing up + balance

sol^n 1. Tabular value iteration ?

yes, not elegant

2. LQR ?

Beautiful solution for balancing (need more for swing-up)

△ LQR for nonlinear systems

$\dot{x} = f(x,u) \Longleftrightarrow \dot{x} = Ax + Bu$

$\Rightarrow$ Chose nominal x_0, u_0 (usually a fixed pt)

$\Rightarrow \dot{x} \approx f(x_0,u_0) + \underbrace{\frac{\partial f}{\partial x}\Big|_{\substack{x=x_0 \\ u=u_0}}}_{A}(x - x_0) + \underbrace{\frac{\partial f}{\partial u}\Big|_{\substack{x=x_0 \\ u=u_0}}}_{B}(u - u_0)$

let

$\bar{x} = x - x_0$

$\dot{\bar{x}} = \dot{x} - \dot{x}_0$

$\bar{u} = u - u_0$

$\Rightarrow \dot{\bar{x}} = A\bar{x} + B\bar{u}$

e.g. simple pendulum

• $ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = \tau$

$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \quad u = [\tau]$

• linearize about $\theta = \pi, \dot{\theta} = 0, u = 0$

• $\dot{x} = \begin{bmatrix} \dot{\theta} \\ \frac{1}{ml^2}[\tau - b\dot{\theta} - mgl\sin\theta] \end{bmatrix}$

• check fixed pt ✓

$\dot{x}_0 = \begin{bmatrix} 0 \\ \frac{1}{ml^2}[0 - b\cdot 0 - mgl\sin\pi] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

• linearization

$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta & -\frac{b}{ml^2} \end{bmatrix}\Bigg|_{\theta=\pi} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix}$

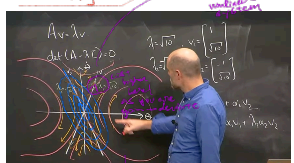
$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$

let $m = l = 1 \quad g = 10$

$A = \begin{bmatrix} 0 & 1 \\ 10 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

orange: approximate of a nonlinear system



red: nonlinear system

• LQR Design

$\begin{cases} \dot{\bar{x}} = A\bar{x} + B\bar{u} \\ J = \int_0^\infty \bar{x}^TQ\bar{x} + \bar{u}^TR\bar{u}\,dt \\ [K,S] = \operatorname{lqr}(A,B,Q,R) \\ u = -Kx = -R^{-1}B^TSx \\ J^* = \bar{x}^TS\bar{x} \end{cases}$

$\Rightarrow$ Stable

$\dot{x} = \underbrace{(A - BK)}_{\text{Hurwitz}}x$

△ Is (A,B) controllable?

underactuated but controllable

△ controllable : given $x(0)$

can I find $u(t) \;\forall t \in [t_0, t_f]$

s.t. $x(t_f) = 0$

Is (A,B) stablizable

△ stablizable : given $x(0)$

can I find $u(t) \;\forall t \in [t_0, \infty)$

s.t. $\lim_{t\to\infty} x(t) = 0$

(no control input)

△ Nonlinear Dynamics

$T = \frac{1}{2}ml^2\dot{\theta}^2$ (kinetic energy) $\quad v_{tan} = l\dot{\theta}$

(potential energy)

$U = -mgl\cos\theta$

Lagrangian mechanics

$L = K - U$

$K = \frac{1}{2}mv^2$

$= \frac{1}{2}ml\dot{\theta}^2$

$= \frac{1}{2}ml^2\dot{\theta}^2$

$U = mgy$

$= -mgl\cos\theta$

$\therefore L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

Euler-Lagrange equation

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$

$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

$\frac{\partial L}{\partial \theta} = ml^2\ddot{\theta} + mgl\sin\theta = Q$

generalized force

$Q = -\underbrace{b\dot{\theta}}_{\text{friction}} + \underbrace{u}_{\text{torque}}$

$\therefore ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$

⇕

$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$

⇕

$\ddot{\theta} = f(\theta, \dot{\theta}, u)$

⇕

$\theta(t) = ?$

△ Nonlinear Dynamics Questions

• What is $\lim_{t\to\infty} \theta(t)$?

• Will my robot fall down ?

$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u \quad (N \cdot m)$

$kg\frac{m^2}{s^2}\frac{1}{s} \quad kg\,m^2\frac{1}{s^2} \qquad kg\frac{m^2}{s^2}$

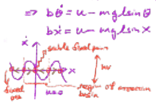
case : $b\dot{\theta} \gg ml^2$

$b\sqrt{\frac{l}{g}} \gg ml^2$ (heavily damped region)

$\Rightarrow$ linear approximation!

$\Rightarrow b\dot{\theta} = u - mgl\sin\theta$

$b\dot{x} = u - mgl\sin x \quad x \in \mathbb{R}$



△ Stability

• local stability (do not leave a ball)

- in the sense of Lyapunov (i.s.L.)

$\begin{cases} \epsilon, \delta, \text{ small positive constants} \\ \forall \epsilon, \exists \delta, \text{ s.t. } \|x(0) - x^*\| < \delta \\ \quad \Rightarrow \forall t \; \|x(t) - x^*\| < \epsilon \end{cases}$

- locally attractive — get to the region

$\left\{\lim_{t\to\infty} x(t) \longrightarrow x^*\right.$

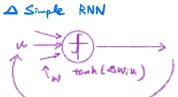
- asymptotically stable — i.s.L + attractive

{ isL + attractive

- exponentially stable — get to the region w/ rate

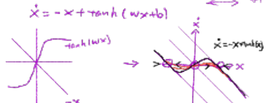
$\begin{cases} \forall t \\ \|x(t) - x^*\| < Ce^{-\alpha t} \\ C, \alpha > 0 \end{cases}$

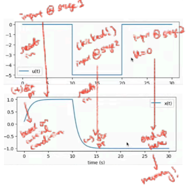
△ Simple RNN



$\dot{x} = -x + \tanh(wx + b)$

±b offset





△ 2^nd-order system

$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$

$\dot{x} = f(x,u)$

$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \frac{1}{ml^2}[u - b\dot{\theta} - mgl\sin\theta] \end{bmatrix}$

✱ Essence of controls for underactuated robots :

we hv all these dynamics

→ plug w/ u, s.t. phase portraits behave as we want!!!

△ Lyapunov Analysis

recall DP:
- Tabular — easy to explode
- LQR — only for linear case
- Approximate DP (NN) — works quite well (but solve a different DP, e.g. discounting)

→ all are trying to get "cost-to-go" function $J^*(x)$

△ now Lyapunov ⟷ optimal value
(easy to find) (hard to find)
good enough — very good

might replace the original optimal ul.

△ Example: stability analysis of simple pendulum

$ml\ddot\theta + mgl\sin\theta = -b\dot\theta$

△ $\theta(t)$ ⟶ hard to solve, cannot be analysis

△ Lyapunov instead !!!

△ $E = K + U$
$= \frac{1}{2}ml^2\dot\theta^2 - mgl\cos\theta$

△ $\frac{d}{dt}E(x)$, $x = \begin{bmatrix}\theta\\ \dot\theta\end{bmatrix}$

$= \frac{\partial E}{\partial x}\frac{\partial x}{\partial t}$

$= \frac{\partial E}{\partial \theta}\dot\theta + \frac{\partial E}{\partial \dot\theta}\ddot\theta$

$= mgl\sin\theta\,\dot\theta + ml^2\dot\theta\ddot\theta$

$= \dot\theta(mgl\sin\theta + ml^2\ddot\theta)$

$= -b\dot\theta^2 \le 0$ if $b>0$

△ General Energy function

△ given $\dot x = f(x)$ no u.

→ want to prove stability of $x^* = 0$

→ construct a differentiable function $V(x)$, s.t.

$\begin{cases} V(0)=0, & V(x)>0,\ x\neq 0 & \text{PD} \\ \dot V(0)=0, & \dot V(x)\le 0,\ x\neq 0 & \text{NSD}\end{cases}$

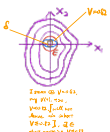
sufficient condition

→ then, x^* is stable i.s.L.

△ i.s.L def

$\forall \epsilon>0,\ \exists\delta>0$

s.t. $\|x(0)-x^*\|<\delta$
$\forall t\ \|x(t)-x^*\|<\epsilon$



e.g. pendulum
$V(x) = E + mgl$

△ Asymptotically Stable

$\begin{cases} V(0)=0, & V(x)>0,\ x\neq 0 & \text{PD} \\ \dot V(0)=0, & \dot V(x)<0,\ x\neq 0 & \text{NSD}\end{cases}$

"<" not "≤"
otherwise $\dot V(x)=0$, ⟶ limit cycles.

△ Global Stability
Global Asymptotically stability (GAS)

$\begin{cases} V(0)=0, & V(x)>0,\ x\neq 0 & \text{PD} \\ \dot V(0)=0, & \dot V(x)\le 0,\ x\neq 0 & \text{NSD}\end{cases}$
+
$\lim_{|x|\to\infty} V(x) = \infty$ "radially unbounded"

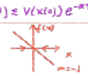
△ Regional Stability

$\begin{cases} V(0)=0, & V(x)>0,\ x\neq 0 & \text{PD} \\ \dot V(0)=0, & \dot V(x)\le 0,\ x\neq 0 & \text{NSD}\end{cases}$
$\forall x\in D \subset \mathbb{R}^n$

△ Exponential Stability

$\begin{cases} V(0)=0, & V(x)>0,\ x\neq 0 & \text{PD} \\ \dot V(0)=0, & \dot V(x)\le -\alpha V(x),\ x\neq 0 & \text{NSD}\end{cases}$
⇕
$V(x(t)) \le V(x(0))e^{-\alpha t}$

e.g. $\dot x = -x$

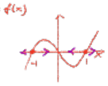


$V(x) = x^2$
$\dot V(x) = \frac{\partial V}{\partial x}\dot x$
$= 2x(-x)$
$= -2x^2 < 0$
$\le 2V(x)$

{ i.s.L ✓, asymptotic ✓ }
{ exponential ✓ }

$\lim_{|x|\to\infty} V(x) = \infty$ { global stable }

e.g. $\dot x = -x + x^3 = f(x)$



$x^*=0$ is f.p.
$x\in(-1,1)$ R.O.A.

$V(x) = x^2$
$\dot V(x) = 2x(-x+x^3)$
$= 2x^2(x^2-1)$
$= \begin{cases} 0 & x=0,\ \pm 1 \\ >0 & |x|>1 \\ <0 & |x|<1,\ x\neq 0\end{cases}$

• sublevel set of V ⟶ invariant set
$V(x)\le \rho$ (inside the R.O.A. of x^*)

△ General form of R.O.A.

if $V(x)>0$, $\dot V(x)<0$
$\forall x\in\{x \mid V(x)<\rho,\ \rho>0\}$ — invariant set
then $V(x(0))<\rho$
⇒ $\lim_{t\to\infty} V\to 0$, $x\to 0$
and $\{x \mid V(x)<\rho\}$ is inside R.O.A.

△ LaSalle's Theorem

△ Lyapunov ⟶ Cost-to-go funcs.

HJB:
$0 = \min_u\left[\ell(x,u) + \frac{\partial J^*}{\partial x}f(x,u)\right]$
$u^* = \pi^*(x)$

⇒ $0 = \ell(x,u^*) + \frac{\partial J^*}{\partial x}f(x,u^*)$
$= \ell(x,u^*) + \frac{\partial J^*}{\partial t}$

⇒ $\dot J^*(x) = -\ell(x,u^*)$

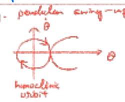
"cost-to-go" function's derivative has to be decreasing!!!

⇕

$\dot V(x) < 0$ ⟶ way more easy!!!

△ Lyapunov-based controller

e.g. pendulum swing-up



homoclinic orbit

• $E^d = mgl$
$V(x) = \frac{1}{2}(E(x) - E^d(x))^2$
$= \frac{1}{2}\tilde E^2(x)$

• $ml^2\ddot\theta + mgl\sin\theta = u$
$\dot E = u\dot\theta$
$\dot{\tilde E} = \dot E - \dot E^d = u\dot\theta$

• let $u = -k\dot\theta\tilde E$, $k>0$
making
$\dot{\tilde E} = -k\dot\theta^2\tilde E$

△ DP sum $\int_t^{t+dt} x = f(x(t),u(t))dt$

• $J(x(t), u(t), t_0, t_f)$
$= Q(x(t_f), t_f) + \int_{t_0}^{t_f}\ell(x(t), u(t))\,dt$

• $V(x(t_0), t_0, t_f)$ (cost-to-go)
$= \min_{u(t)} J(x(t), u(t), t_0, t_f)$

• $-\frac{\partial V}{\partial t} = \min_{u(t)}\left(\left(\frac{\partial V}{\partial x}\right)^T f(x(t),u(t)) + \ell(x(t),u(t))\right)$

⟺ $\min_{u(t)}\left(\left(\frac{\partial J^*}{\partial x}\right)^T f(x(t),u(t)) + \ell(x(t),u(t))\right)$

Hamilton-Jacobi-Bellman (HJB)

• $V(x(t_0), t_0, t_f)$
$= V(x(t_0), t_0, t) + V(x(t), t, t_f)$

Bellman Optimality

proof
$\frac{d}{dt}V(x(t), t, t_f) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right)^T\frac{\partial x}{\partial t}$ — total derivative

$= \frac{d}{dt}\min_{u(t)} J(x(t), u(t), t, t_f)$

$= \frac{d}{dt}\min_{u(t)} Q(x(t_f), t_f) + \int_t^{t_f}\ell(x(t),u(t))\,dt$

$= \min_{u(t)}\frac{d}{dt}\left[Q(x(t_f),t_f) + \int_t^{t_f}\ell(x(t),u(t))\,dt\right]$

$= \min_{u(t)}\ 0 + \frac{d}{dt}\left(\left[\mathcal{L}(x(t),u(t))\right]_t^{t_f}\right)$

$= \min_{u(t)}\frac{d}{dt}\left(\mathcal{L}(x(t_f),u(t_f)) - \mathcal{L}(x(t),u(t))\right)$

$= \min_{u(t)} -\ell(x(t),u(t))$

⇕
$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right)^T\frac{\partial x}{\partial t} = \min_{u(t)} -\ell(x(t),u(t))$

⇕
$-\frac{\partial V}{\partial t} = \min_{u(t)}\left(\left(\frac{\partial V}{\partial x}\right)^T\frac{\partial x}{\partial t} + \ell(x,u)\right)$
$= \min_{u(t)}\left(\frac{\partial V}{\partial x}^T f(x,u) + \ell(x,u)\right)$

★ Get V ⟶ Get u
- two point boundary value problem
- shooting method

△ Discrete-Time HJB

• $x_{k+1} = F(x_k, u_k)$
• $J(x_0, \{u_k\}_{k=0}^{n}, n)$
$= Q(x_n, t_n) + \sum_{k=0}^{n}\ell(x_k, u_k)$

• $V(x_0, n)$
$= \min_{\{u_k\}_{k=0}^{n}} J(x_0, \{u_k\}_{k=0}^{n}, n)$

• $V(x_0, n)$
$= V(x_0, k) + V(x_k, n-k)\ \forall k\in(0,n)$

• $V(x_k, n)$
$= \min_{u_k}\left[\ell(x_k,u_k) + V(x_{k+1}, n-1)\right]$ — $x_{k+1} = F(x_k,u_k)$

⟺ $\forall\ J(x) = \min_u[\ell(x,u) + J(F(x,u))]$

Discrete HJB

⇕
$V(x_k, n)$
$= \min_{u_k}\left(\ell(x_k,u_k) + V(F(x_k,u_k), n-1)\right)$

⇕
$V(x) = \min_u(\ell(x,u) + V(F(x,u)))$
↓
$\pi(x) = \arg\min_u(\ell(x,u) + V(F(x,u)))$

△ DP (cont'd)

key idea
1) LQR (linear optimal control)
2) local linearization
3) $Q = \begin{bmatrix}10 & & & \\ & 10 & & \\ & & 1 & \\ & & & 1\end{bmatrix}$ $R = [1]$

@ balancing

Q: How do we swing up?
"value iteration"

⇒ Approximate DP

△ Value iteration on a mesh for Pendulum Swing-up

△ mesh q: 51 bins — odd for "0"
$\dot q$: 51 bins
u: 9 bins

△ Acrobot
51^4 bins
9 bins — explode!!!

△ NN! Approximating Value Functions

$\hat J_\alpha(x)$
— parameters + bias

- value iteration update
• recall:
$\forall s_i\in S$
$\hat J(s_i) = \min_a\left[\ell(s_i,a) + \hat J(f(s_i,a))\right]$
$s[n+1] = f(s[n], a[n])$
$\sum_{n=0}^{\infty}\ell(s[n], a[n])$

• Draw samples
$x_i\in X_s\subset X$
$u_j\in U_s\subset U$
$\forall ij$ $x_{ij} = f(x_i, u_j)$ — next state samples
$\ell_{ij} = \ell(x_i, u_j)$

desired $J^d_i = \min_j\left[\ell_{ij} + \hat J_\alpha(x_{ij})\right]$

⇒ $\min_\alpha$ loss $\sum_i\left[\hat J_\alpha(x_i) - J_i^d\right]^2$
⇕
$\min_\alpha$ loss $\sum_i\left[\hat J_\alpha(x_i) - \min_j\left[\ell_{ij} + \hat J_\beta(x_{ij})\right]\right]^2$
"target network"

"here we do the training based on the samples. To have a better sampling, RL kicks in, & guide the state space, to a subset, that is closer to the optimum."

Q: when does it work? does it converge

△ linear approximator
△ works @ special case: linear NN

• $\hat J_\alpha(x) = \alpha^T\phi(x)$ — nonlinear basis "input features" & fixed
$= \sum_j \alpha_j\phi_j(x)$

⇒ $\min_\alpha \sum_i\left[\alpha^T\phi(x) - J_i^d\right]^2$

linear least squares has closed-form soln
⇒ SGD is guaranteed to converge to global minima

△ Back to LQR; connection to approximator; w/ discrete LQR

$J^*(x) = x^T S^* x$ — S^*: NN ⟶ linear approximator

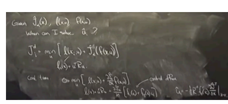
loss $\sum_i\left[x_i^T\hat S x_i - J_i^d\right]^2$

$\hat S \leftarrow \hat S - \eta\frac{\partial\,\text{loss}}{\partial S}$

Based on this, loss explodes↑

why? LQR tries to solve inf horizon, cost cannot go to inf.
How does normal LQR avoid inf?
$x\to 0$, $u\to 0$ (exact 0)
How about here, when we do sampling? — popularly seen in RL

- discounting. $\sum_{n=0}^{\infty}\gamma^n\ell(x[n],u[n])$
$0\le\gamma\le 1$
optimal: LQR($\sqrt{\gamma}A$, B, Q, $\frac{1}{\gamma}R$)
pretend that is more stable
$x[n+1] = Ax[n] + Bu[n]$
$|\mathrm{eig}(A)|\le 1$.



- use optimal u: recall
CT LQR: $u^* = -K^*x = -R^{-1}B^TS^*x$
DT LQR: $u^* = -K^*x = -(R + B^TS^*B)^{-1}B^TS^*Ax$
- instead of sampling u_i, only sample $x_i\in X_s\subset X$

⇒ $J_i^d = x_i^TQx_i + \underline{\hat u_i^T}R\underline{\hat u_i} + \hat J_\alpha(Ax_i + Bu_i)$
w/ optimal u w/ current $\hat S$

$|NFA| \leq |A|$