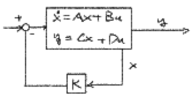
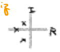
△ LQR



① in Linear system
we let $u = -Kx$
s.t.
$\dot{x} = Ax + Bu$
$\Rightarrow \dot{x} = Ax - BKx$
$= (A - BK)x$

set K so that $A-BK$ is Hurwitz



② LQR, we optimize K

△ LQR formulation

$$J = \int_0^\infty (x^TQx + u^TRu)\,dt$$

$\Rightarrow u = -Kx$

$\Rightarrow \dot{x} = Ax + Bu$
$y = Cx + Du$

$Q \succ 0$, $R \succeq 0$

△ Ricatti equation

minimize $J = \int_0^\infty (x^TQx + u^TRu)\,dt$
s.t. $\dot{x} = Ax + Bu$
`>> [k,P] = lqr(A,B,Q,R)`

sol^n

1. Brute-force
2. Learning Algorithms (gradient descent)
3. Analytic approach
   - introduce $P = P^T$

$$J = x_0^TPx_0 - x_0P^Tx_0 + \int_0^\infty (x^TQx + u^TRu)\,dt$$

$$\Rightarrow J = x_0^TPx_0 + \int_0^\infty \left[\frac{d}{dt}(x^TPx) + x^TQx + u^TRu\right]dt$$

$[x^TPx]_0^\infty = 0 - x_0P^Tx_0$

$\frac{d}{dt}(x^TPx) = \dot{x}^TPx + x^TP\dot{x}$
$= (Ax+Bu)^TPx + x^TP(Ax+Bu)$

$$\Rightarrow J = x_0^TPx_0 + \int_0^\infty [(Ax+Bu)^TPx + x^TP(Ax+Bu) + x^TQx + u^TRu]\,dt$$

$$= x_0^TPx_0 + \int_0^\infty [x^T(A^TP + PA + Q)x + \underbrace{u^TRu + x^TPBu + u^TB^TPx}]\,dt$$

$u^TRu + x^TPBu + u^TB^TPx$
$\Leftrightarrow (u + R^{-1}B^TPx)^TR(u + R^{-1}B^TPx) - x^T(PBR^{-1}B^TP)x$

$$= x_0^TPx_0 + \int_0^\infty [\,x^T(A^TP + PA + Q - PBR^{-1}B^TP)x + (u + R^{-1}B^TPx)^TR(u + R^{-1}B^TPx)\,]\,d\tau$$

$\to$ get an "u" s.t. the $\int$ is minimized

$\Rightarrow u = -R^{-1}B^TPx$
$u = -K\,x$ $\quad \therefore K = R^{-1}B^T\underset{\smile}{P}$

what is this

let
$A^TP + PA + Q - PBR^{-1}B^TP = 0$
(Algebraic Riccati Equation - ARE)

- find P that solves ARE
- Use P to get K

(stable one)

△ Tracking Trajectories

$x \to x_d$

let $e = x - x_d$
$v = u - u_d$
$f(x,u) = f(x) + g(x)u$
$\dot{x} = f(x,u)$

$\Rightarrow \dot{e} = \dot{x} - \dot{x}_d$
$= f(x) + g(x)u - (f(x_d) + g(x_d)u_d)$
$= f(e + x_d) - f(x_d)$
$+ g(e + x_d)(v + u_d) - g(x_d)u_d$
$= F(e, v, x_d(t), u_d(t))$

$\Rightarrow$ linearize around $e = 0$
$v = Ke = K(x - x_d)$

$\Rightarrow u = K(x - x_d) + u_d$

△ TL;DR of OCP

- general OCP

$$\underset{u(\cdot)}{\text{minimize}} \int_0^\infty \ell(x(t), u(t))\,dt$$

s.t. $\quad x(0) = x_0$
$\quad \dot{x}(t) = f(x(t), u(t))$
$\quad x(t) \in \mathcal{X}$
$\quad u(t) \in \mathcal{U}$
$\quad \} \forall t \in [0, \infty)$

- LQR

$$\underset{u(\cdot)}{\text{minimize}} \int_0^\infty x(t)^TQx(t) + u(t)^TRu(t)\,dt$$

s.t. $\quad x(0) = x_0$
$\quad \dot{x}(t) = Ax(t) + Bu(t)$
$\quad x(t) \in \mathcal{X}$
$\quad u(t) \in \mathcal{U}$
$\quad \} \forall t \in [0, \infty)$

- MPC

$$\underset{u(\cdot)}{\text{minimize}} \int_t^{t+N} \ell(x(t), u(\tau))\,d\tau$$

s.t. $\quad x(t) = x_0$
$\quad \dot{x}(\tau) = f(x(\tau), u(\tau))$
$\quad x(\tau) \in \mathcal{X}$
$\quad u(\tau) \in \mathcal{U}$
$\quad \} \forall \tau \in [t, t+N]$

△ Standard formulation

$$\ell(x,u) = \|x_u - x^r\|_Q^2 + \|u - u^r\|_R^2$$

$$J_N(x,u) = \sum_{k=0}^{N-1} \ell(x_u(k), u(k))$$

$\Rightarrow \underset{u}{\text{minimize}}\ J_N(x_0, u)$

s.T. $\quad x_u(k+1) = f(x_u(k), u(k))$
$\quad x_u(0) = x_0$
$\quad u(k) \in \mathcal{U} \quad \forall k \in [0, N-1]$
$\quad x(k) \in \mathcal{X} \quad \forall k \in [0, N]$

△ mobile robot as an example

$x = [x, y, \theta]^T$

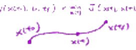
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \frac{r}{2} \begin{bmatrix} (\dot{\phi}_R + \dot{\phi}_L)\cos\theta \\ (\dot{\phi}_R + \dot{\phi}_L)\sin\theta \\ (\dot{\phi}_R - \dot{\phi}_L)/D \end{bmatrix}$$

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \frac{r}{2}(\dot{\phi}_R + \dot{\phi}_L) \\ \frac{r}{2D}(\dot{\phi}_R - \dot{\phi}_L) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

△ Optimal Nonlinear Control

- $\frac{d}{dx}x = f(x(\tau), u(\tau), \tau)\,d\tau$
- $J(x(t), u(t), t_0, t_f)$
  $= Q(x(t_f), t_f) + \int_{t_0}^{t_f} \ell(x(\tau), u(\tau))\,d\tau$
- $V(x(t_0), t_0, t_f) = \min_{u(t)} J(x(t), u(t), t_0, t_f)$



$V(x(t_0), t_0, t_f)$
$= V(x(t_0), t_0, \tau) + V(x(\tau), \tau, t_f)$

- $-\frac{\partial V}{\partial t} = \min_{u(t)}\left\{\left(\frac{\partial V}{\partial x}\right)^T f(x(t), u(t)) + \ell(x(t), u(t))\right\}$

Hamilton Jacobi Bellman (HJB) eqn