- optimization problem

minimize $f_0(x)$

s.t. $f_i(x) \le 0, \quad i = 1, \dots, m$

$h_i(x) = 0, \quad i = 1, \dots, p$

- dual problem

- $L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$
- $g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu)$

$= \inf_{x \in D} \left\{ f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right\}$

- maximize $g(\lambda, \nu)$

s.t. $\lambda \succeq 0$

- duality :
  - gives us lower bound
  - sometimes it's easier to solve (good properties for solver!)

- QCQP (quadratic constrained quadratic programming)

▲ minimize $x^T Q x$

s.t. $x^T A_k x = b_k \quad \forall k \in \{1, \dots, K\}$

non-convex QCQP

▲ $L(x, \lambda) = x^T Q x + \sum_{k=1}^{K} \lambda_k (b_k - x^T A_k x)$ (Lagrange multiplier)

$= \sum_k \lambda_k b_k + x^T (Q - \sum_{k=1}^{K} \lambda_k A_k) x$

$= b^T \lambda + x^T H(\lambda) x$

$[b_1 \dots b_K][\lambda_1 \dots \lambda_K]$ $\quad Q - \sum \lambda_k A_k$

▲ $g(\lambda) = \inf_x L(x, \lambda) = \begin{cases} b^T \lambda, & \text{if } H(\lambda) \succeq 0 \\ -\infty, & \text{o.w.} \end{cases}$ $\rightarrow \frac{\partial L}{\partial x} = 0$

▲ primal problem

minimize $x^T Q x$

s.t. $x^T A_k x = b_k \quad \forall k \in \{1, \dots, K\}$

$\Updownarrow$

$\theta_P(x) = \max_\lambda L(x, \lambda)$

$= \max_\lambda \begin{cases} x^T Q x & x \in D \\ \infty & \text{o.w.} \end{cases}$

$\left( \min_x \max_\lambda \; (b^T \lambda + x^T H(\lambda) x) \right)$ (1)

$H(\lambda)$: $0$

▲ dual problem

$\left( \begin{array}{l} \text{maximize } b^T \lambda \\ \text{s.t. } H(\lambda) \succeq 0 \\ \quad\; H(\lambda) x = 0 \end{array} \right)$ (2)

∴ from (1) & (2)

when $H(\lambda) x = 0$

$p^* = d^* = b^T \lambda$

strong duality holds

- Slater's condition: $\exists x \in \text{relint } D$ s.t. $f_i(x) < 0$, $Ax = b$

  especially: robotics application satisfy Slater's condition

This implies:

a. strong duality (if $H(\lambda) \succeq 0$)

b. $\hat{x}$ is optimal when $H\hat{x} = 0$ (global)

c. $\hat{x}$ is not optimal then what do we do?

▲ from above — certification problem

Find $H, \lambda$

s.t. $H = Q - \sum_{k=1}^{K} \lambda_k A_k$

$H \succeq 0$

$H\hat{x} = 0$

} strong duality holds!

↓

we have global optimal!

▲ SDP relaxation (which can allow us to solve it faster & solve it further!) dual of dual problem (omitted today)

minimize $x^T Q x$

s.t. $x A x = b$

- recall dual problem

maximize $b^T \lambda$

s.t. $H(\lambda) \succeq 0$

- Lagrangian:

$L'(\lambda, X) = b^T \lambda + \text{tr}\left( X H(\lambda) \right)$ $\quad X \succeq 0$ (matrix of Lagrangian multiplier)

note: $\langle X, Y \rangle = \text{tr}(X^T Y)$

- dual problem:

$L'(\lambda, X) = b^T \lambda + \text{tr}\left( X (Q - \sum_{k=1}^{K} \lambda_k A_k) \right)$

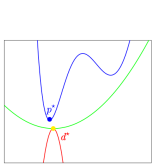
$= \text{tr}(QX) + [b_1 - \text{tr}(A_1 X) \quad b_K - \text{tr}(A_K X)] \lambda$

$g(X) = \sup_\lambda L'(\lambda, X) = \begin{cases} \text{tr}(QX) & \text{if } \text{tr}(A_k X) = b_k \\ \infty & \text{o.w.} \end{cases}$

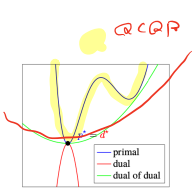
$\Rightarrow$

$\begin{array}{ll} \text{minimize}_X & \text{tr}(QX) \\ & \text{tr}(A_k X) = b_k \\ & X \succeq 0 \end{array}$ Semidefinite Programming!

$xx^T = X$




(a) Weak duality.




(b) Strong duality.