- **optimization problem**

  minimize $f_0(x)$

  s.t. $f_i(x) \le 0, \quad i=1,\dots,m$

  $h_i(x) = 0, \quad i=1,\dots,p$

- **duality**:
  - gives us lower bound
  - sometimes it's easier to solve (good properties for solver!)

- **dual problem**
  - $L(x,\lambda,\nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$
  - $g(\lambda,\nu) = \inf_{x\in D} L(x,\lambda,\nu) = \inf_{x\in D}\left\{ f_0(x) + \sum_{i=1}^{m}\lambda_i f_i(x) + \sum_{i=1}^{p}\nu_i h_i(x) \right\}$
  - maximize $g(\lambda,\nu)$

    s.t. $\lambda \succeq 0$

- **QCQP** (quadratic constrained quadratic programming)

  minimize$_x$ x^TQx

  s.t. $x^TA_kx = b_k \quad \forall k \in \{1,\dots,k\}$

  non-convex QCQP

  Lagrange multiplier

  $$L(x,\lambda) = x^TQx + \sum_{k=1}^{K} \lambda_k (b_k - x^TA_kx)$$
  $$= \sum_k \lambda_k b_k + x^T\left(Q - \sum_{k=1}^{K}\lambda_k A_k\right)x$$
  $$= \underbrace{b^T\lambda}_{\begin{bmatrix} b_1 \\ \vdots \\ b_k\end{bmatrix}^T \begin{bmatrix}\lambda_1 \\ \vdots \\ \lambda_k\end{bmatrix}} + x^T \underbrace{H(\lambda)}_{Q-\sum\lambda_k A_k} x$$

  $$g(\lambda) = \inf_x L(x,\lambda) = \begin{cases} b^T\lambda, & \text{if } H(\lambda)\succeq 0 \\ -\infty, & \text{o.w.}\end{cases} \rightarrow \frac{\partial L}{\partial x} = 0$$

- **primal problem**

  minimize$_x$ x^TQx

  s.t. $x^TA_kx = b_k \quad \forall k \in \{1,\dots,K\}$

  $\Updownarrow$

  $$\theta_P(w) = \max_\lambda \mathcal{L}(x,\lambda) = \max_\lambda \begin{cases} x^TQx & x \in D \\ \infty & \text{o.w.}\end{cases}$$

  $$\left(\text{minimize}_x \ \max_\lambda \ (b^T\lambda + x^T\underbrace{H(\lambda)x}_{0}) \qquad (1) \right)$$

- **dual problem**

  maximize $b^T\lambda$

  s.t. $H(\lambda) \succeq 0$

  $H(\lambda)x = 0$ (2)

$\therefore$ from (1) & (2)

when $H(\lambda)x = 0$

$p^* = d^* = b^T\lambda$

strong duality holds

Slater's condition

$\exists x \in \text{relint } D$

s.t.

$f_i(x) < 0$

$Ax = b$

empirically: robotics application satisfy Slater's condition

This implies:

a. strong duality (if $H(\lambda)\succeq 0$)

b. $\hat{x}$ is optimal when $H\hat{x} = 0$ (global)

c. $\hat{x}$ is not optimal then what do we do?

- from above — certification problem

  find H, λ

  s.t. $H = Q - \sum_{i=1}^{K}\lambda_k A_k$

  $H \succeq 0$

  $H\hat{x} = 0$

  strong duality holds!

  $\downarrow$

  we have global optimal!

- **SDP relaxation** (which can allow us to solve it faster & solve it further!)

  dual of dual problem

  (omitted today)

  minimize$_x$ x^TQx

  s.t. $xAx = b$

  - recall dual problem

    maximize $b^T\lambda$

    s.t. $H(\lambda) \succeq 0$

  - Langragian:

    $$L'(\lambda, X) = b^T\lambda + \mathrm{tr}(XH(\lambda))$$

    $X \succeq 0$ — matrix of langragian multiplier

    note: $\langle X, Y\rangle = \mathrm{tr}(X^TY)$

  - dual problem:

    - $L'(\lambda,X) = b^T\lambda + \mathrm{tr}\left(X\left(Q - \sum_{k=1}^{K}\lambda_k A_k\right)\right)$

      $= \mathrm{tr}(QX) + [b_1 - \mathrm{tr}(A_1X) \quad b_k - \mathrm{tr}(A_kX)]\lambda$

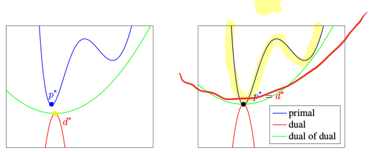
    - $g(X) = \sup_\lambda L'(\lambda,X) = \begin{cases} \mathrm{tr}(QX) & \text{if } \mathrm{tr}(A_kX) = b_k \\ \infty & \text{o.w.}\end{cases}$

    $\Rightarrow$
    $$\boxed{\begin{aligned} &\text{minimize}_X \ \mathrm{tr}(QX) \\ &\mathrm{tr}(A_kX = b_k) \\ &X \succeq 0 \end{aligned}}$$

    Semidefinite Programming!

    $xx^T = X$

QCQP



(a) Weak duality. (b) Strong duality.