IEKF_for_ALAN_RPE_on_Manifold

given a dynamic & measurement model :

$$x_{k+1} = f(x_k, u_k, w_k)$$
$$y_k = h(x_k, v_k)$$

△ dynamic model remark: Note that all states are in the non-inertial frame

△ $x \in R$

$$x = \begin{bmatrix} p \\ q \\ b_g \\ b_a \\ v \end{bmatrix}, \quad x_{k+1} = f(x_k) = \begin{bmatrix} p_k + v_k \Delta t \\ q_k \otimes \exp(\omega \Delta t) \\ b_{g_k} \\ b_{a_k} \\ v_k + \Delta t\{R_k(a_k - b_{a_k}) + g(q_k)\} \end{bmatrix}, \quad F = \dots \text{ (omitted)}$$

$x \in \mathcal{M}$ $(SE(3))$

$$x = \begin{bmatrix} p \\ R \\ v \end{bmatrix}, \quad x_{k+1} = f(x_k) = \begin{bmatrix} p_k + v_k \Delta t \\ R(q)\,\Delta t \\ v_k \end{bmatrix}, \quad F = \begin{bmatrix} \frac{\partial p_k}{\partial p} & \frac{\partial p_k}{\partial R} & \frac{\partial p_k}{\partial v} \\ \frac{\partial R_k}{\partial p} & \frac{\partial R_k}{\partial R} & \frac{\partial R_k}{\partial v} \\ \frac{\partial v_k}{\partial p} & \frac{\partial v_k}{\partial R} & \frac{\partial v_k}{\partial v} \end{bmatrix} = \begin{bmatrix} I_3 & 0 & I_3 \Delta t \\ 0 & I_3 & 0 \\ 0 & -(R q)_\times & I_3 \end{bmatrix}$$

$\delta x = f'(x_k)$

$\dot{x}\Delta t$

$$J = \frac{\partial f(x)}{\partial x} \triangleq \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \in R^{n \times n} = \dots = \lim_{h \to 0} \frac{Jh}{h} = \frac{\partial f h}{\partial h} = J$$

① $\therefore \frac{\partial V_k}{\partial R} = \frac{R q t}{\partial R}$

$$= \lim_{\theta \to 0} \frac{(R \oplus \theta) \cdot q - R \cdot q}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(R \cdot \mathrm{Exp}(\theta)) \cdot q - R \cdot q}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(R \cdot (I + \theta_\times)) \cdot q - R \cdot q}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{R \theta_\times q}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{-R q_\times \theta}{\theta} \cdot t$$
$$= -R q_\times t \quad (J_R)$$

② $\therefore \frac{\partial V_k}{\partial R} = \frac{R q t}{\partial R}$

$$= \lim_{\theta \to 0} \frac{(\theta \oplus R) \cdot q - R q}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(\mathrm{Exp}(\theta) \cdot R) q - R q}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(I + \theta_\times) R q - R q}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{\theta_\times R q}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} -(R q)_\times$$

△ measurement model

△ $y_{k+1} = h(x_{k+1})$, let $(P, R) = \xi$

$$\Rightarrow \begin{bmatrix} u_i \\ v_i \\ \vdots \\ v \end{bmatrix} = \begin{bmatrix} f_x \frac{X'_i}{Z'_i} + c_x \\ f_y \frac{Y'_i}{Z'_i} + c_y \\ \vdots \\ d_{k+1} - d_k \end{bmatrix}$$

(derived from $su = KTP$, $d = T_{1:3,4}$, $\Rightarrow u = \frac{1}{s} KTP$)

$$\therefore H = \frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h_1}{\partial \xi} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial \xi} & \frac{\partial h_2}{\partial v} \\ \vdots & \vdots \\ \frac{\partial h_v}{\partial \xi} & \frac{\partial h_v}{\partial v} \end{bmatrix} = \begin{bmatrix} J_0 & 0 \\ J_1 & 0 \\ \vdots & \\ J_i & 0 \\ 0 & I_3 \end{bmatrix}$$

$[R^{2\times 6}]$

$[R^{2\times 9}]$

$$H = \begin{bmatrix} J_0 & 0' \\ J_1 & 0' \\ \vdots & \\ J_i & 0' \\ 0_{3\times 6} & I_3 \end{bmatrix}, \quad J_i = \begin{bmatrix} -\frac{f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} & \frac{f_x X' Y'}{Z'^2} & -f_x - \frac{f_x X'^2}{Z'^2} & \frac{f_x Y'}{Z'} \\ 0 & -\frac{f_y}{Z'} & \frac{f_y Y'}{Z'^2} & f_y + \frac{f_y Y'^2}{Z'^2} & -\frac{f_y X' Y'}{Z'^2} & -\frac{f_y X'}{Z'} \end{bmatrix} \in R^{2\times 6}$$

$0' \in R^{2\times 3}$

$$= \begin{bmatrix} 2 & \overbrace{\square}^{6} & 2 & \square & 2 \\ 2 & \square & & \square & 2 \\ & \vdots & & & \\ 3 & \square & & \square & 3 \\ & 6 & & 3 & \end{bmatrix}$$

Iterated Extended Kalman filter

△ prediction same as EKF

△ correction different :

- after receive $\hat{x}_{k-1}$
- $\hat{x}_k^0 = \hat{x}_{k-1}$ & start loop
- $H_k^i = \frac{\partial h(x)}{\partial x}\Big|_{x = \hat{x}_k^i}$
- $K_k^i = P_{k|k-1} H_k^{i\,T} (H_k^i P_{k|k-1} H_k^{i\,T} + R_k)^{-1}$
- $r_k^i = K_k^i (z_k - h(\hat{x}_k^i))$
- $\hat{x}_k^{i+1} = \hat{x}_k^i + r_k^i$
- break when $\|r_k^i\| < \epsilon$

in our case, we have additional parameters, thus residual slightly different

- embedded $\underbrace{Q, R}_{\text{determine } K \text{ LHS}}$ into one residual function

$$r = \begin{bmatrix} \alpha\, r_{meas} \\ \beta\, r_{prior} \end{bmatrix} = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \xi \ominus \hat{\xi}_k^i \end{bmatrix} \\ \vdots \\ H' \hat{x}_k^i - H' \hat{x}_k^0 \end{bmatrix}$$

- $\alpha = \frac{1}{\sqrt{\Lambda(R)}}$, $\beta = \frac{1}{\sqrt{\Lambda(Q)}}$

- $\frac{Dr}{Dx} = J$, $\Delta x = -[J J^T]^{-1} J f(x)$

Summary for ALAN-RPE

∴ Sec Prediction

$\hat{x}_{k+1}^- = \hat{x}_k \oplus \delta x$
$\quad = \hat{x}_k \oplus \mathrm{Exp}(\delta x)$ (e idex's method)

$\hat{P}_{k+1}^- = R_{k+1}(x_k, f(x_k))$

$P_{k+1}^- = F_k P_k F_k + Q$

$F_k = \frac{D f}{D x}\Big|_{x = \hat{x}_k}$

∴ for Correction (IEKF)

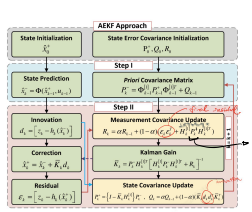
→ $\hat{x}_k^i := \hat{x}_k^-$

→ $r := \inf$

→ $i := 0$

→ while $r > \epsilon$

- $r = \begin{bmatrix} \alpha\, r_{meas} \\ \beta\, r_{prior} \end{bmatrix} = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \xi \ominus \hat{\xi}_k^i \end{bmatrix} \\ \vdots \\ H' \hat{x}_k^i - H' \hat{x}_k^0 \end{bmatrix}$
- $\alpha = \frac{1}{\sqrt{\Lambda(R)}}$, $\beta = \frac{1}{\sqrt{\Lambda(Q)}}$
- $\frac{Dr}{Dx} = J$, $\Delta x = -[J J^T]^{-1} J f(x)$
- $\hat{x}_k^{i+1} = \hat{x}_k^i \oplus \Delta x$



$P_k = \begin{bmatrix} [SE(3)] & \\ & [V] \end{bmatrix}$ $[R_{vc}]$ $[R_{vm}]$

Summary : after optimization : use that for H_k^{i+1}.

$\begin{bmatrix} [6\times 6] & \\ & [3\times 3] \end{bmatrix}$

△ Jacobian for a pinhole model during bundle adjustment

△ pinhole model :

$$s_i u_i = KTP_i \sim \text{3D points}$$

depth, Intrinsic, $SE(3)$

△ recall objective :

$$u_i = \frac{1}{s_i} KTP_i$$

△ let $P' \triangleq$ PCD in camera frame (intermediate variable)

$$P' = (TP)_{1:3} = [X', Y', Z']^T$$

$\therefore su = KTP$

$\Rightarrow su = KP'$

$$\Rightarrow \begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$

$$\therefore u = f_x \frac{X'}{Z'} + c_x$$
$$v = f_y \frac{Y'}{Z'} + c_y$$

△ $e = u - \frac{1}{s} KTP$

$$\frac{\partial e}{\partial \xi} = \lim_{\delta \xi \to 0} \frac{e(\delta \xi \oplus \xi) - e(\xi)}{\delta \xi} = \underbrace{\frac{\partial e}{\partial P'}}_{1} \underbrace{\frac{\partial P'}{\partial \delta \xi}}_{2}$$

△ $e = \begin{bmatrix} u - f_x \frac{X'}{Z'} + c_x \\ v - f_y \frac{Y'}{Z'} + c_y \end{bmatrix}$

recall Jacobian : $\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$

1 $\therefore \frac{\partial e}{\partial P'} = \begin{bmatrix} \frac{\partial e_1}{\partial X'} & \frac{\partial e_1}{\partial Y'} & \frac{\partial e_1}{\partial Z'} \\ \frac{\partial e_2}{\partial X'} & \frac{\partial e_2}{\partial Y'} & \frac{\partial e_2}{\partial Z'} \end{bmatrix}$

$$= \begin{bmatrix} -\frac{f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} \\ 0 & -\frac{f_y}{Z'} & \frac{f_y Y'}{Z'^2} \end{bmatrix} \quad \text{— ①}$$

let $P' := [P'; 1]$

2 $\frac{\partial P'}{\partial \delta \xi} = \frac{\partial TP}{\partial \delta \xi}$

Lie Group

$$T = \frac{D f(x)}{D x} = \frac{\partial y_n}{\partial x_n} = \frac{f(X \oplus (\tau)) \ominus f(X)}{\tau} \in R^{n \times n}$$
$$= \lim_{\tau \to 0} \frac{\log[f(X)^{-1} f(X \oplus \exp(\tau))]}{\tau}$$

remark: $\mathrm{Exp}(\tau)$ differentiate $\tau_{1:3}$ $\tau_{4:6}$ $\tau_{7:9}$ $\tau_{10:12}$ $\tau_{13:15}$ $\tau_{16:18}$ $\tau_{19:21}$ $\tau_{22:24}$ $\tau_{25:27}$ $\tau_{28:30}$ $\tau_{31:33}$ $\tau_{34:36}$ $\tau_{37:39}$ $\tau_{40:42}$ $\tau_{43:45}$ $\tau_{46:48}$ $\tau_{49:51}$ $\tau_{52:54}$ $\tau_{55:57}$ $\tau_{58:60}$ $\tau_{61:63}$ $\tau_{64:66}$ $\tau_{67:69}$ $\tau_{70:72}$ $\tau_{73:75}$ $\tau_{76:78}$ $\tau_{79:81}$ $\tau_{82:84}$ $\tau_{85:87}$ $\tau_{88:90}$ $\tau_{91:93}$ $\tau_{94:96}$ $\tau_{97:99}$ $\tau_{100:102}$ $\tau_{103:105}$ $\tau_{106:108}$ $\tau_{109:111}$ $\tau_{112:114}$ $\tau_{115:117}$ $\tau_{118:120}$