IEKF_for_ALAN_RPE_on_Manifold

given a dynamic & measurement model:

$$x_{k+1} = f(x_k, u_k, w_k)$$
$$y_k = h(x_k, v_k)$$

△ dynamic model

remark: Note that all states are in the non-inertial frame

△ $x \in R$

$$x = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \quad x_{k+1} = f(x_k) = \begin{bmatrix} x_k + t u_k \\ y_k + t v_k \\ z_k + t w_k \\ \phi_k \\ \theta_k \\ \psi_k \\ u_k + tg(c\phi_k s\theta_k c\psi_k + s\phi_k s\psi_k) \\ v_k + tg(c\phi_k s\theta_k s\psi_k - s\phi_k c\psi_k) \\ w_k + tg(c\phi_k c\theta_k - 1) \end{bmatrix}, \quad F = \dots \text{ (omitted)}$$

$x \in \mathcal{M}$ (SE(3))

$$x = \begin{bmatrix} P \\ R \\ V \end{bmatrix}, \quad x_{k+1} = f(x_k) = \begin{bmatrix} P_k + \boxed{V_k \Delta t} \\ R \\ V + \boxed{R(g)t} \end{bmatrix}_{\dot{x}\Delta t}, \quad F = \begin{bmatrix} \frac{\partial P_k}{\partial P} & \frac{\partial P_k}{\partial R} & \frac{\partial P_k}{\partial V} \\ \frac{\partial R_k}{\partial P} & \frac{\partial R_k}{\partial R} & \frac{\partial R_k}{\partial V} \\ \frac{\partial V_k}{\partial P} & \frac{\partial V_k}{\partial R} & \frac{\partial V_k}{\partial V} \end{bmatrix}$$

$$\delta x = f'(x_k)$$

$$= \begin{bmatrix} I_3 & 0 & I_3 \Delta t \\ 0 & I_3 & 0 \\ 0 & -(Rg)_\times & I_3 \end{bmatrix}$$

$$J = \frac{\partial f(x)}{\partial x} \triangleq \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \in R^{n \times m}$$
$$= \dots = \lim_{h \to 0} \frac{Jh}{h} = \frac{\partial Jh}{\partial h} = J$$

① $\therefore \frac{\partial V_k}{\partial R} = \frac{Rgt}{\partial R}$

$$= \lim_{\theta \to 0} \frac{(R \oplus \theta) \cdot g - R \cdot g}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(R \cdot Exp(\theta)) \cdot g - R \cdot g}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(R \cdot (I + \theta_\times)) \cdot g - R \cdot g}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{R\theta_\times \cdot g}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{-R\, g_\times \theta}{\theta} \cdot t$$
$$= -Rg_\times t \quad (J_R)$$

② $\therefore \frac{\partial V_k}{\partial R} = \frac{Rgt}{\partial R}$

$$= \lim_{\theta \to 0} \frac{(\theta \oplus R) \cdot g - Rg}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(Exp(\theta) \cdot R) g - Rg}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{(I + \theta_\times) Rg - Rg}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} \frac{\theta_\times Rg}{\theta} \cdot t$$
$$= \lim_{\theta \to 0} -(Rg)_\times$$

△ measurement model

△ $y_{k+1} = h(x_{k+1}^-)$, let $(P, R) = \xi$

$$\Rightarrow \begin{bmatrix} u_i \\ v_i \\ \vdots \\ v \end{bmatrix} = \begin{bmatrix} f_x \frac{X'_i}{Z'_i} + c_x \\ f_y \frac{Y'_i}{Z'_i} + c_y \\ \vdots \\ d_{k+1} - d_k \end{bmatrix} \Big\} \text{ O.S.}$$

(derived from $su = KTP$ $\Rightarrow u = \frac{1}{s} KTP$, $d = T_{1:3,4}$)

$$\therefore H = \frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h_1}{\partial \xi} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial \xi} & \frac{\partial h_2}{\partial v} \\ \vdots & \vdots \\ \frac{\partial h_v}{\partial \xi} & \frac{\partial h_v}{\partial v} \end{bmatrix} = \begin{bmatrix} J_0 & 0 \\ J_1 & 0 \\ \vdots & \\ J_i & 0 \\ 0 & I_3 \end{bmatrix}$$

$[R^{2\times 6}]$
$[R^{2\times 9}]$

$$H = \begin{bmatrix} J_0 & O' \\ J_1 & O' \\ \vdots & \\ J_i & O' \\ O_{3\times 6} & I_3 \end{bmatrix}, \quad J_i = -\begin{bmatrix} \frac{-f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} & \frac{f_x X' Y'}{Z'^2} & -f_x - \frac{f_x X'^2}{Z'^2} & \frac{f_x Y'}{Z'} \\ 0 & \frac{-f_y}{Z'} & \frac{f_y Y'}{Z'^2} & f_y + \frac{f_y Y'^2}{Z'^2} & \frac{-f_y X' Y'}{Z'^2} & -\frac{f_y X'}{Z'} \end{bmatrix} \in R^{2\times 6}$$

$$O' \in R^{2\times 3}$$

$$= \begin{bmatrix} {}_2[\;6\;] & [\;3\;]_2 \\ {}_2[\;\;] & [\;\;]_2 \\ \vdots & \\ {}_3[\;6\;] & [\;3\;]_3 \end{bmatrix}$$

Iterated Extended Kalman Filter

△ prediction same as EKF

△ correction different:

- after receive $\hat{x}_{k|k-1}$
- $\hat{x}_k^0 = \hat{x}_{k|k-1}$ & start loop
- $H_k^i = \frac{\partial h(x)}{\partial x}\Big|_{x = \hat{x}_k^i}$
- $K_k^i = P_{k|k-1} H_k^{iT} (H_k^i P_{k|k-1} H_k^{iT} + R_k)^{-1}$
- $r_k^i = K_k^i (z_k - h(\hat{x}_k^i))$
- $\hat{x}_k^{i+1} = \hat{x}_k^i + r_k^i$
- break when $\|r_k^i\|_2 < \epsilon$

in our case, we have additional param, thus residual slightly different

- embedded $\underbrace{Q, R}$ into one residual function
  ↳ determine K
  LHS

$$r = \begin{bmatrix} \alpha r_{mea} \\ \beta r_{dyn} \end{bmatrix} = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \begin{bmatrix} \left.\begin{bmatrix} u \\ v \end{bmatrix}_i \right\} - \frac{1}{s} K \hat{\chi}_k^i P \\ \vdots \\ H' \hat{\chi}_k^0 - H' \hat{x}_k^i \end{bmatrix}$$

- $\alpha = \frac{1}{tr(R)}$, $\beta = \frac{1}{tr(Q)}$
- $\frac{Dr}{D\chi} = J$, $\Delta x = -[JJ^T]^{-1} J f(x)$

Summary for ALAN-RPE

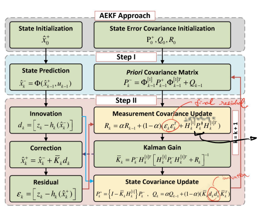
$\therefore$ for Prediction

$$\hat{\chi}_{k+1}^- = \chi_k \oplus \delta x$$
$$= \chi_k \circ Exp(\delta x) \quad \text{(euler's method)}$$
$$\hat{\chi}_{k+1}^- = RK_4(x_k, f'(x_k))$$
$$P_{k+1}^- = F_k P_k F_k + Q$$
$$F_k = \frac{Df}{D\chi}\Big|_{\chi = \chi_k}$$

$\therefore$ for Correction (IEKF)

→ $\hat{\chi}_k^0 := \hat{\chi}_k^-$
→ $r := \inf$
→ $i := 0$
→ while $r > \epsilon$

- $r = \begin{bmatrix} \alpha r_{mea} \\ \beta r_{dyn} \end{bmatrix} = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \begin{bmatrix} \left.\begin{bmatrix} u \\ v \end{bmatrix}_i \right\} - \frac{1}{s} K \hat{\chi}_k^i P \\ \vdots \\ H' \hat{\chi}_k^0 - H' \hat{x}_k^i \end{bmatrix}$
- $\alpha = \frac{1}{tr(R)}$, $\beta = \frac{1}{tr(Q)}$
- $\frac{Dr}{D\chi} = J$, $\Delta x = -[JJ^T]^{-1} J f(x)$
- $\hat{\chi}_k^{i+1} = \hat{\chi}_k^i \oplus \Delta x$



set Q first/

→ refer to stochastic modelling for GNSS paper

$$P_k = \begin{bmatrix} [\,SE(3)\,] & \\ & [V] \end{bmatrix} \qquad [R_{6\times 6}]\ [R_{3\times 3}]$$

$$\begin{bmatrix} [\,6\times 6\,] & \\ & [3\times 3] \end{bmatrix}$$

Summary: after optimization: use that for $H_k^{[1]}$.

△ Jacobian for a pinhole model during bundle adjustment

△ pinhole model:

$$s_i u_i = KTP_i$$

(s_i ~ depth, K ~ Intrinsic, T ~ SE(3), P_i ~ 3D points)

△ recall objective:

$$u_i - \frac{1}{s_i} KTP_i$$

intermediate variable

△ let $P' \triangleq$ PCD in camera frame

$$P' = (TP)_{1:3} = [X', Y', Z']^T$$

$$\therefore su = KTP$$
$$\Rightarrow sv = KP'$$
$$\Rightarrow \begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$
$$\therefore u = f_x \frac{X'}{Z'} + c_x$$
$$v = f_y \frac{Y'}{Z'} + c_y$$

△ $e = u - \frac{1}{s} KTP$

$$\frac{\partial e}{\partial \delta\xi} = \lim_{\delta\xi \to 0} \frac{e(\delta\xi \oplus \xi) - e(\xi)}{\delta\xi} = \underbrace{\frac{\partial e}{\partial P'}}_{1} \underbrace{\frac{\partial P'}{\partial \delta\xi}}_{2}$$

△ $e = \begin{bmatrix} u - f_x \frac{X'}{Z'} + c_x \\ v - f_y \frac{Y'}{Z'} + c_y \end{bmatrix}$

recall Jacobian: $\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$

1 $\therefore \frac{\partial e}{\partial P'} = \begin{bmatrix} \frac{\partial e_1}{\partial X'} & \frac{\partial e_1}{\partial Y'} & \frac{\partial e_1}{\partial Z'} \\ \frac{\partial e_2}{\partial X'} & \frac{\partial e_2}{\partial Y'} & \frac{\partial e_2}{\partial Z'} \end{bmatrix}$

$$= \begin{bmatrix} -\frac{f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} \\ 0 & -\frac{f_y}{Z'} & \frac{f_y Y'}{Z'^2} \end{bmatrix} \quad \text{— ①}$$

let $P' := [P'; 1]$

2 $\frac{\partial P'}{\partial \delta\xi} = \frac{\partial TP}{\partial \delta\xi}$

Lie Groups
$J = \frac{Df(\chi)}{D\chi} = \lim_{\tau \to 0} \frac{f(\chi \oplus \tau) \ominus f(\chi)}{\tau} \in R^{n\times m}$
$= \lim_{\tau \to 0} \frac{Log[f(\chi)^{-1} f(\chi Exp(\tau))]}{\tau}$
e.g. remark: cannot differentiate T w.r.t. R, as R is a matrix. hence, use lie algebra on the local tangent plane

$$= \lim_{\delta\xi \to 0} \frac{exp(\delta\xi^\wedge) exp(\xi^\wedge) P - exp(\xi^\wedge) P}{\delta\xi}$$
$$= \lim_{\delta\xi \to 0} \frac{(I + \delta\xi^\wedge) exp(\xi^\wedge) P - exp(\xi^\wedge) P}{\delta\xi}$$
$$= \lim_{\delta\xi \to 0} \frac{\delta\xi^\wedge exp(\xi^\wedge) P}{\delta\xi}$$

$\lim_{h \to 0} \frac{e^h e^x - e^x}{h} = \lim_{h \to 0} \frac{(1+h) e^x - e^x}{h} = \lim_{h \to 0} \frac{h e^x}{h} = e^x$

$exp(\xi^\wedge) P = Tp = Rp + t$

$$= \lim_{\delta\xi \to 0} \frac{\begin{bmatrix} \delta\omega^\wedge & \delta\rho \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} Rp + t \\ 1 \end{bmatrix}}{\delta\xi}$$

recall Jacobian: $\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$

$$= \lim_{\delta\xi \to 0} \frac{\begin{bmatrix} \delta\omega^\wedge (Rp + t) + \delta\rho \\ 0^T \end{bmatrix}}{\delta\xi} \quad \delta\xi \to \begin{bmatrix} \delta\rho \\ \delta\omega \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial \delta\rho}(\delta\omega^\wedge(Rp+t) + \delta\rho) & \frac{\partial}{\partial \delta\omega}(\delta\omega^\wedge(Rp+t) + \delta\rho) \\ \frac{\partial}{\partial \delta\rho} 0 & \frac{\partial}{\partial \delta\omega} 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & -(Rp + \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & -P'^\wedge \\ 0 & 0 \end{bmatrix}$$

- Here $P' = [x', y', z']$

$$\therefore \frac{\partial P'}{\partial \delta\xi} = [I \;\; -P'^\wedge]$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & Z' & -Y' \\ 0 & 1 & 0 & -Z' & 0 & X' \\ 0 & 0 & 1 & Y' & -X' & 0 \end{bmatrix}$$

- as $\frac{\partial e}{\partial P'} = \begin{bmatrix} -\frac{f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} \\ 0 & -\frac{f_y}{Z'} & \frac{f_y Y'}{Z'^2} \end{bmatrix}$

& $\frac{\partial P'}{\partial \delta\xi} = \begin{bmatrix} 1 & 0 & 0 & 0 & Z' & -Y' \\ 0 & 1 & 0 & -Z' & 0 & X' \\ 0 & 0 & 1 & Y' & -X' & 0 \end{bmatrix}$

$$\therefore \frac{\partial e}{\partial \delta\xi} = \begin{bmatrix} -\frac{f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} \\ 0 & -\frac{f_y}{Z'} & \frac{f_y Y'}{Z'^2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & Z' & -Y' \\ 0 & 1 & 0 & -Z' & 0 & X' \\ 0 & 0 & 1 & Y' & -X' & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} & \frac{f_x X' Y'}{Z'^2} & -f_x - \frac{f_x X'^2}{Z'^2} & \frac{f_x Y'}{Z'} \\ 0 & \frac{-f_y}{Z'} & \frac{f_y Y'}{Z'^2} & f_y + \frac{f_y Y'^2}{Z'^2} & \frac{-f_y X' Y'}{Z'^2} & -\frac{f_y X'}{Z'} \end{bmatrix}$$