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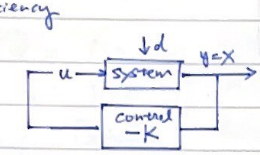
Control Bootcamp

- passive control
- active control
  - open loop
  - close loop (w/ feedback)
    1. tackle w/ uncertainty
    2. tackle w/ instability
    3. tackle w/ disturbances
    4. efficiency

- $\dot{x} = Ax + Bu$

  $x(t) = e^{At}x(0)$
- $y = Cx$



$\Rightarrow \quad u = -Kx$

$$\dot{x} = Ax - BKx = \underbrace{(A-BK)}_{\text{determine } K \text{ to make it stable}}x$$

Linear System

$\dot{x} = Ax \qquad x \in R^n \qquad x(t) = e^{At}x(0)$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

from below

$$= e^{TDT^{-1}t}$$
$$= TT^{-1} + TDT^{-1}t + \frac{TDT^{-1}TDT^{-1}}{2!}t^2 + \cdots$$
$$= T\left[I + Dt + \frac{D^2t^2}{2!} + \cdots\right]T^{-1}$$
$$= \boxed{Te^{Dt}T^{-1}}$$

Eigenvalues & Eigenvectors : $AT = TD$ >> [T, D] = eig(A);

$A\xi = \lambda\xi$

$T = [\xi_1, \xi_2, \cdots \xi_n]$

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$

$\Rightarrow T^{-1}AT = D$, let $\boxed{x = Tz}$, $\dot{x} = T\dot{z} = Ax$

$\Rightarrow \dot{x} = T\dot{z} = Ax$

$\Rightarrow T\dot{z} = ATz$

$\Rightarrow \dot{z} = T^{-1}ATz$

$\Rightarrow \boxed{\dot{z} = Dz}$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

in sum $z(t) = e^{Dt}z(0)$

$$= \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} z(0)$$

$$\therefore x(t) = \underbrace{T\underbrace{e^{Dt}\underbrace{T^{-1}x(0)}_{z(0)}}_{z(t)}}_{x(t)}$$

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~~Stability~~
Eigenvalues

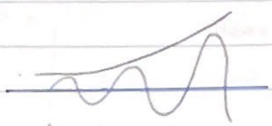
△ recall $\dot{x} = Ax \quad x \in \mathbb{R}^n$

$$\left.\begin{aligned} &[T, D] = \mathrm{eig}(A); \\ &D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \\ &e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{bmatrix} \end{aligned}\right\} \quad x(t) = T e^{Dt} T^{-1} x(0)$$

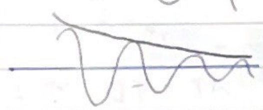
△ $\lambda = a + bi$

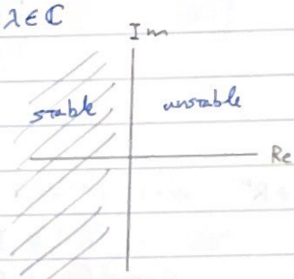
$$e^{\lambda t} = e^{at}\left[\cos(bt) + i\sin(bt)\right] \qquad \lambda \in \mathbb{C}$$

if $a > 0$



if $a < 0$





(continuous case)

△ discrete system

$x_{k+1} = \tilde{A} x_k \quad , \quad x_k = x(k\Delta t)$

$\underset{\text{discrete}}{\tilde{A}} = e^{A_{\text{continuous}} \Delta t}$

$$\tilde{A}\tilde{T} = \tilde{T}\tilde{D}$$

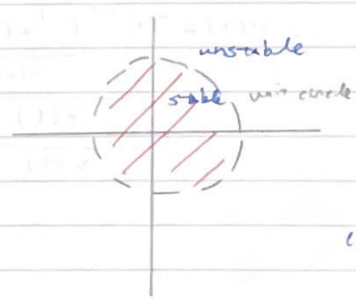
$$\tilde{A} = \tilde{T}\tilde{D}\tilde{T}^{-1}$$

$$\underbrace{\begin{aligned} x_1 &= \tilde{A} x_0 && = \tilde{T}\tilde{D}\tilde{T}^{-1} x_0 \\ x_2 &= \tilde{A} x_1 = \tilde{A}^2 x_0 && = \tilde{T}\tilde{D}^2\tilde{T}^{-1} x_0 \\ x_3 &= \tilde{A}^3 x_0 && = \tilde{T}\tilde{D}^3\tilde{T}^{-1} x_0 \\ &\vdots && \vdots \\ x_n &= \tilde{A}^n x_0 && = \tilde{T}\tilde{D}^n\tilde{T}^{-1} x_0 \end{aligned}}$$

recall $\tilde{D}$ is a diagonal matrix

w/ $\lambda = a + bi = Re^{i\theta}$

$\lambda^n = R^N e^{iN\theta}$



(discrete case)

Linearizing Around a Fixed Point

$\triangle$ from non-linear $\dot{x} = f(x) \Rightarrow$ to linear $\dot{x} = Ax \quad x \in \mathbb{R}^n$

1. find fixed pts

   $\bar{x}$ s.t. $f(\bar{x}) = 0$

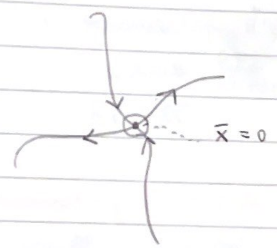
2. linearize about $\bar{x}$

$$\frac{Df}{Dx}\bigg|_{\bar{x}} = \left[\frac{\partial f_i}{\partial x_j}\right]_{ij}$$

e.g. $\dot{x}_1 = f_1(x_1, x_2) = x_1 x_2$

$\dot{x}_2 = f_2(x_1, x_2) = x_1^2 + x_2^2$

$$\frac{Df}{Dx} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} x_2 & x_1 \\ 2x_1 & 2x_2 \end{bmatrix}$$

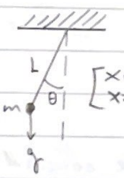


$\bar{x} = 0$ (shift w/ $\bar{x}$) assumed

$$\dot{x} = f(x)$$
$$= f(\bar{x}) + \frac{Df}{Dx}\bigg|_{\bar{x}} \cdot (x - \bar{x}) + \underbrace{\frac{D^2 f}{Dx^2} \cdot (x - \bar{x})^2 + \cdots}_{\text{too small, neglected}}$$

$$\Delta\dot{x} = \frac{Df}{Dx}\bigg|_{\bar{x}} \Delta x \Longrightarrow \Delta\dot{x} = A \Delta x$$
$$\dot{x} = Ax$$

e.g. Pendulum



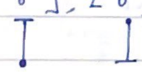
$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \delta\dot{\theta}$$

$\frac{g}{L} = 1$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin(x_1) - \delta x_2 \end{bmatrix}$$

1. F.P. $\bar{x} = \underset{\text{(case 1)}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}, \underset{\text{(case 2)}}{\begin{bmatrix} \pi \\ 0 \end{bmatrix}}$



$$\frac{Df}{Dx} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -\delta \end{bmatrix}$$

case 1 $\quad A = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix} \quad \lambda = \begin{matrix} -0.05 + 0.9987i \\ -0.05 + -0.9987i \end{matrix}$ (stable locally)

case 2 $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \quad \lambda = \begin{matrix} -1.0512 \\ 0.9512 \end{matrix}$ $\begin{pmatrix} \text{stable} \\ \text{unstable} \end{pmatrix}$ saddle point.

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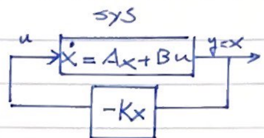
Controllability

affect stability

△ recall: $\dot{x} = Ax \qquad x \in R^n$

△ now, w/ control input:

$$\dot{x} = \underset{R^{n\times n}}{A}x + \underset{R^{n\times k}}{B}\,\underset{R^{k}}{u}$$



"optimal" for linear system.

when $u = -Kx$

$\dot{x} = Ax - BKx$

$\dot{x} = \underbrace{(A - BK)}_{\text{by choosing } K\text{, we can change the "dynamics"}}x$

△ intuition: if my system is controllable:

I can put the eigenvalues of $(A-BK)$ "anywhere I want", also indicating that x can be "anywhere in R^n"

→ when or how to know?

△ >> ctrb(A, B)

• $\dot{x} = Ax$

* $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$ ← uctrb: as x_1 is not coupled w/ x_2 or u, making it impossible to twist anything on it

$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$ ← ctrb

• rather not obvious:

** $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$ ← ctrb: x_1 & x_2 are coupled. twisting just one u, can do things on x_1, x_2 simultaneously.

• $\mathcal{C} = [B \quad AB \quad A^2B \ \dots \ A^{n-1}B]$ i.f.f $\text{rank}(\mathcal{C}) = n$ then sys is ctrb

*: $\mathcal{C}_{*} = \begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix}$ → uctrb (rank = 1)

**: $\mathcal{C}_{**} = \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}$ → ctrb (rank = 2)

• doing SVD on $\mathcal{C}$ gives you the controllability extent on each state (most → least) (singular vectors)

△ Note that the controllability discussed here is "linear controllable".

Controllability / Reachability / Eigenvalue Placement

△ recall: $\dot{x} = Ax + Bu$

$\mathcal{C} = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$

$\text{rank}(\mathcal{C}) = n \underset{\text{i.f.f.}}{\iff}$ ctrb

>> rank (ctrb (A, B)) ;

△ Equivalences:

1. system is ctrb
2. Arbitrary eigenvalue (pole) placement

   $u = -kx \implies \dot{x} = \underbrace{(A - BK)}_{\text{arbitrary eigenvalue!}} x$ >> K=place (A, B, eigs) ;
3. Reachability (full) in R^n can reach any vector in R^n given some u)

   Reachable set $R_t = \{ \xi \in R^n \mid \text{there is an input } u(t) \text{ s.t. } x(t) = \xi \}$

   $R_t = \mathbb{R}^n$

Controbility & Discrete-Time Impulse Response

△ recall: $\dot{x} = Ax + Bu \quad x \in R^n$

$C = [B \; AB \; A^2B \cdots A^{n-1}B]$

△ $x_{k+1} = \tilde{A} x_k + \tilde{B} u_k$ — Impulse Response

assume $x_0 = 0$ $\quad u_0 = 1$

then $x_1 = \tilde{B} = \tilde{A} \cdot 0 + \tilde{B} \cdot 1$ $\quad u_1 = 0$

$x_2 = \tilde{A}\tilde{B} = \tilde{A} \cdot \tilde{B} + \tilde{B} \cdot 0$ $\quad u_2 = 0$

$x_3 = \tilde{A}^2\tilde{B} = \tilde{A} \cdot \tilde{A}\tilde{B} + \tilde{B} \cdot 0$ $\quad u_3 = 0$

$\vdots$

$x_m = A^{m-1}B \cdot$ $\quad u_m = 0$

↓

if this can "hit" all axis in R^n } just an intuition
or "affect"

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not yes or no. not binary.
"to what extent"

Degrees of Controllability & Gramians

△ How controllable are different directions on R^n

△ $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$

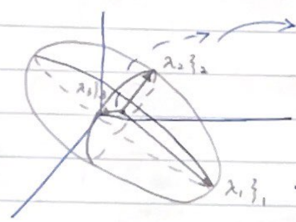
△ controllability Gramian

$$W_t = \int_0^t e^{A\tau} B B^T e^{A^T\tau}\, d\tau \quad \in R^{n \cdot n}$$

$W_t \xi = \lambda \xi$ larger eigenvalues, more controllable

in discrete time

$W_t \approx \mathcal{C}\mathcal{C}^T \iff$ svd of $\mathcal{C}$: $[U, \Sigma, V] = \text{svd}(\mathcal{C}, \text{'econ'})$



singular vectors of my controllability axis

△ stabilizability

stab. iff all unstable (lightly damped) eigenvectors of A are in ctrb. space

(as if x is large, it is impossible to let each direction be controllable.

PBH Test (Popov - Belevitch - Hautus)

△ (A, B) is ctrb i.f.f

$\text{rank}\, [\,(A-\lambda I)\ B\,] = n \quad \forall \lambda \in \mathbb{C}$

1. $\text{rank}(A-\lambda I) = n$ except for eigenvalues (of A)

   ∴ just need to perform PBH test @ λ

2. B needs to have some component in each eigenvector direction

3. if B is a random vector, i.e., $B = \text{randn}(n, 1)$

   ∴ (A, B) will be ctrb w/ high probability

   (as it is hard to generate vector w.o. all components

   on eigenvector direction

4. $\text{rank}\, [(A-\lambda I)\ B]$

   △ 2x multiple λ 2 columns → composite ctrb rank

   3x 3

   4x 4

   ⋮ ⋮

   △ or degenerate eigenvalues

   (two values are close

Caley - Hamilton Theorem

△ Every matrix A satisfies its own characteristics (eigen value) equation

$$\det(A-\lambda I) = 0$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

$$\Rightarrow A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + A\lambda + a_0 I = 0$$

(almost true for all A)

$$\Rightarrow A^n = -a_0 I - a_1 A - a_2 A^2 - \cdots - a_{n-1}A^{n-1}$$

$\Rightarrow A^{2n} = \sum_{j=0}^{n-1} \alpha_j A^j$ (could be expressed as $n-1$ or lower terms)

△ $\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n$

$$e^{At} = I + A + \frac{A^2t^2}{2}\cdots$$

$\Rightarrow \quad = \alpha_0(t) I + \alpha_1(t) A + \alpha_2(t) A^2 + \cdots + \alpha_{n-1}(t) A^{n-1}$ ← not infinite

Reachability and controllability w/ Cayley-Hamilton

△ Reachability

if $\xi \in \mathbb{R}^n$ is reachable

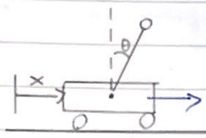
then $\xi = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$ (for some $u(\tau)$)

$$\xi = \int_0^t \left(\phi_0(t-\tau) u(\tau) IB + \phi_1(t-\tau) u(\tau) AB + \cdots + \phi_{n-1}(t-\tau) u(\tau) A^{n-1}B \right) d\tau$$

$$= B\int_0^t \phi_0(t-\tau) u(\tau) d\tau + AB\int_0^t \phi_1(t-\tau) u(\tau) d\tau + \cdots + A^{n-1}B\int_0^t \phi_{n-1}(t-\tau) u(\tau) d\tau$$

$$= \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \left.\begin{bmatrix} \int_0^t \phi_0(t-\tau) u(\tau) d\tau \\ \int_0^t \phi_1(t-\tau) u(\tau) d\tau \\ \vdots \\ \int_0^t \phi_{n-1}(t-\tau) u(\tau) d\tau \end{bmatrix}\right\} n \times q$$

Inverted Pendulum



u = force on the cart in x direction

$$x = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \qquad \frac{d}{dt}x = f(x) \xrightarrow{\left.\frac{Df}{Dx}\right|_{x_{fix}}} \dot{x} = Ax + Bu$$

fixed points : $\theta = 0, \pi$

$\dot{\theta} = 0$

$\dot{x} = 0$

x free $\qquad \dot{x} = (A - Bk)x$

refer to matlab

Pole placement

- △ >> K = place (A, B, eigs)
  >> eig (A − BK) = eigs
- △ try to design K such that [A−BK] matrix has stable poles
  # refer to matlab.

LQR

- △ >> K = place (A, B, eigs)
- △ where are the best eigs ?
  Linear Quadratic Regulator (LQR) ~ $O(n^3)$
- △ $J = \int_0^\infty (x^T Q x + u^T R u)\, dt$

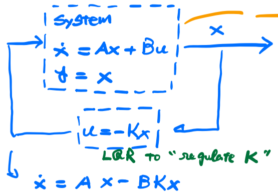
$$Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 10 & \\ 0 & & & 100 \end{bmatrix} \qquad R = \begin{bmatrix} 0.001 \end{bmatrix}$$

- △ >> K = lqr (A, B, Q, R)

Motivation for Full-State estimation

Recall

$\dot{x} = Ax + Bu$ $\quad x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$



LQR to "regulate K"

$\dot{x} = Ax - BKx$

$\dot{x} = (A - BK)x$

(placing eig-values)

I don't necessarily hv all states in real life

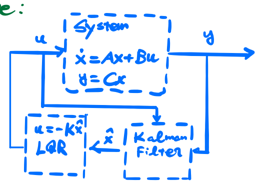
$\dot{x} = Ax + Bu$ (controllability) ctrb (A, B)

$y = Cx$ (observability) obsv (A, C)

Main Question here:

Can I estimate any state $\underline{x}$ from measurement $y(t)$

hence:



Observability

- Duality exists between AB, AC
- observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

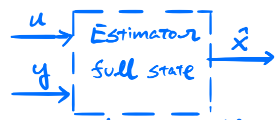
1. observable if
   >> rank (obsv (A, C)) = n
2. Can estimate x from y
3. >> [U, Σ, V] = svd ($\mathcal{O}$)

observability Gramian

V^T

In some direction, we hv higher signal to noise

Full State Estimation



dyn. system (linear)

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + \underbrace{K_f(y - \hat{y})}_{\text{update}}$$

filter

$$\hat{y} = C\hat{x}$$

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K_f y - K_f C\hat{x}$$

$$= (A - K_f C)\hat{x} + \begin{bmatrix} B & K_f \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix}$$

pick K_f to place the eigen values to hv optimal choice

Error $\varepsilon = x - \hat{x}$

$$\frac{d}{dt}\varepsilon = \frac{d}{dt}x - \frac{d}{dt}\hat{x}$$

$$= Ax + \cancel{Bu} - A\hat{x} + K_f C\hat{x} - K_f y - \cancel{Bu}$$

$$= Ax - A\hat{x} + K_f C\hat{x} - K_f \overset{Cx}{y}$$

$$= A(x - \hat{x}) + K_f C(\hat{x} - x)$$

$$= A(x - \hat{x}) - K_f C(x - \hat{x})$$

$$= \underline{(A - K_f C)}\,\varepsilon$$

if observable, then place eigs by choosing K_f: so that error converge eventually

Kalman filter

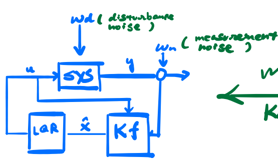
- w_d - Gaussian
  V_d Variance
  w_n - Gaussian
  V_n Variance
  &
- recall
  $\dot{\varepsilon} = (A - K_f C)\varepsilon$
  $\varepsilon = x - \hat{x}$
- cost function
  $J = \mathbb{E}\left((x - \hat{x})^T(x - \hat{x})\right)$
- >>> K_f = lqe (A, C, Vd, Vn)

Kalman filter

real system:

$\dot{x} = Ax + Bu + w_d$

$y = Cx + w_n$



motivation of Kalman filter

get the best K_f to place poles (eigs) based on w_d & w_n

Observability Example

- recall inverted pendulum

$x = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$ $\quad \dot{x} = Ax + Bu$, $y = Cx$, >> obsv (A, C), $C = [1\ 0\ 0\ 0]$

- subsystem : without x[0]

$x = \begin{bmatrix} \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$

with different measurement

C = [1 0 0] meas. $\dot{x}$
C = [0 1 0] meas. θ
C = [0 0 1] meas. $\dot{\theta}$

→ det (gram (sys, 'o'))
(sys = ss (A, B, C, D))

y	det Wo
$\dot{x}$	50
θ	.03
$\dot{\theta}$	.03

easier to observe w/ $\dot{x}$