- Recall

$s = f(x;W) = Wx$ score function

$L_i = \sum_{j\neq y_i} \max(0, s_j - s_{y_i} + 1)$ SVM loss

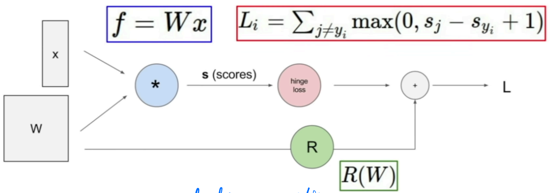
$L = \frac{1}{N}\sum_{i=1}^{N} L_i + \sum_k W_k^2$ data loss + regularization

want $\nabla_W L$

→ solve it with optimization { analytic gradient / numerical gradient

- analytic gradient (thru computational graphs)

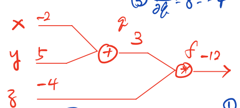
Computational graphs



- important to calculate gradient for neural network (complex function)
- e.g. find gradient

$f(x,y,z) = (x+y)z$

@ $x=-2,\ y=5,\ z=-4$



$q = x+y \qquad \frac{\partial q}{\partial x} = 1 \quad \frac{\partial q}{\partial y} = 1$

$f = q \times z \qquad \frac{\partial f}{\partial q} = z \quad \frac{\partial f}{\partial z} = q$

find $\frac{df}{dx}\ \frac{df}{dy}\ \frac{df}{dz}$

① $\frac{\partial f}{\partial f} = 1$

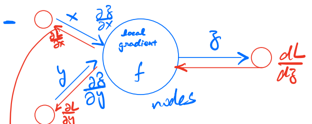
② $\frac{\partial f}{\partial z} = q = 3$

③ $\frac{\partial f}{\partial q} = z = -4$

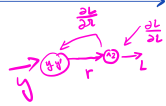
④ $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = -4\cdot 1 = -4$

⑤ $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = -4\cdot 1 = -4$

back-propagation $\qquad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial q}\frac{\partial q}{\partial x}$ chain rule



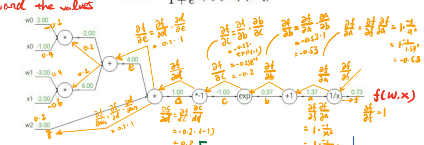
e.g.



derivative wrt L → $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial z}\frac{\partial z}{\partial x}$

Another example: $f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$

foward the values



backprop the gradient

backpropagation !

$f(x)=e^x \quad f'(x)=e^x$
$f_a(x)=ax \quad f_a'(x)=a$
$f(x)=\frac{1}{x} \quad f'(x)=-\frac{1}{x^2}$
$f_c(x)=c+x \quad f_c'(x)=1$

sigmoid gate

$\sigma(x) = \frac{1}{1+e^{-x}}$

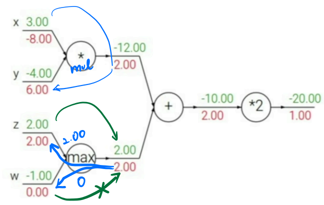
$\frac{\partial\sigma(x)}{\partial x} = \frac{e^{-x}}{(1+e^{-x})^2} = \left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)\left(\frac{1}{1+e^{-x}}\right)$

$= (1-\sigma(x))\,\sigma(x)$

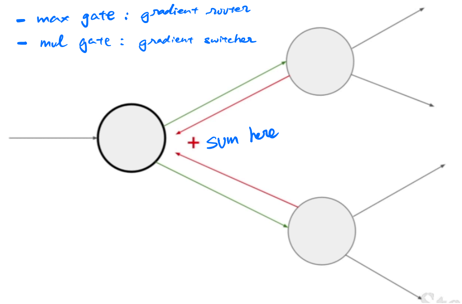
$(1-0.73)\ 0.73 = 0.2$

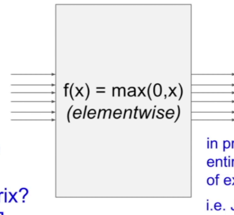
→ to combine into a more complex node

- other nodes (more complex ones)



- max gate : gradient router
- mul gate : gradient switcher





Q: what is the size of the Jacobian matrix? [4096 x 4096!]

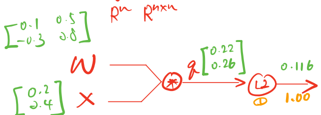
in practice we process an entire minibatch (e.g. 100) of examples at one time:

i.e. Jacobian would technically be a [409,600 x 409,600] matrix :\

yet it will be diagnoal (very sparse)

e.g. $f(x, W) = \|W\cdot x\|^2 = \sum_{i=1}^{n}(W\cdot x)_i^2$

$x \in \mathbb{R}^n,\ W \in \mathbb{R}^{n\times n}$



- $q = W\cdot(x) = \begin{bmatrix} W_{11}x_1 + \cdots + W_{1n}x_n \\ W_{21}x_1 + \cdots + W_{2n}x_n \\ \vdots \\ W_{n1}x_1 + \cdots + W_{nn}x_n \end{bmatrix}$

- $f(q) = \|q\|^2 = q_1^2 + \cdots + q_n^2$

- ① $\nabla_q f = 2q = \begin{bmatrix} 0.44 \\ 0.52 \end{bmatrix} \in \mathbb{R}^2$

② $\frac{\partial q_k}{\partial W_{ij}} = 1_{k=i}\,x_j$

$\therefore \frac{\partial f}{\partial W_{ij}} = \sum_k \frac{\partial f}{\partial q_k}\frac{\partial q_k}{\partial W_{ij}}$

$= \sum_k (2q_k)(1_{k=i}\,x_j)$

$= 2q_i x_j$

$\nabla_W f = 2q\cdot x^T$

$\frac{\partial q_k}{\partial x_i} = W_{k,i}$

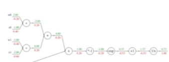
$\frac{\partial f}{\partial x_i} = \sum_k \frac{\partial f}{\partial q_k}\frac{\partial q_k}{\partial x_i}$

$= \sum_k 2q_k W_{k,i}$

$\nabla_x f = 2W^T q$

Note that: the gradient with respect to a variable should hv the same shape as the variable

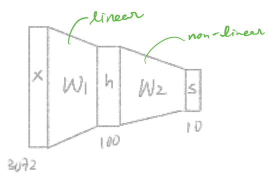
Modularized implementation: forward / backward API



Graph (or Net) object *(rough psuedo code)*

```
class ComputationalGraph(object):
    #...
    def forward(inputs):
        # 1. [pass inputs to input gates...]
        # 2. forward the computational graph:
        for gate in self.graph.nodes_topologically_sorted():
            gate.forward()
        return loss # the final gate in the graph outputs the loss
    def backward():
        for gate in reversed(self.graph.nodes_topologically_sorted()):
            gate.backward() # little piece of backprop (chain rule applied)
        return inputs_gradients
```

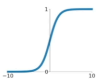
- Neural Network
  - Before | Linear score function : $f = Wx$
  - after | 2-Layer Neural Network : $f = W_2 \max(0, W_1 x)$



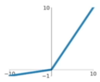
or even 3-layer ...

Activation functions

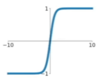
Sigmoid
$\sigma(x) = \frac{1}{1+e^{-x}}$



Leaky ReLU
$\max(0.1x, x)$

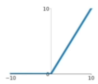


tanh
$\tanh(x)$

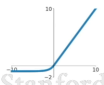


Maxout
$\max(w_1^T x + b_1, w_2^T x + b_2)$

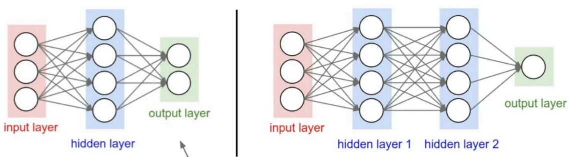
ReLU
$\max(0, x)$



ELU
$$\begin{cases} x & x \geq 0 \\ \alpha(e^x - 1) & x < 0 \end{cases}$$



Neural networks: Architectures



"2-layer Neural Net", or
"1-hidden-layer Neural Net"

"Fully-connected" layers

"3-layer Neural Net", or
"2-hidden-layer Neural Net"