# Pose Estimation via Vision
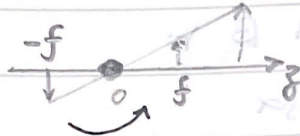
- pinhole model
- triangulation (2D-3D)
- ICP problem (3D-3D)
- optimization

## pinhole model



$$\frac{3}{f} = \frac{X}{X'} = \frac{Y}{Y'}$$

$$\therefore \begin{cases} X' = f \dfrac{X}{3} \\ Y' = f \dfrac{Y}{3} \end{cases}$$

from $x'y'$ to $u, v$
      enlarge $\alpha, \beta$
      translate $c_x, c_y$

$$\begin{cases} u = \alpha X' + c_x \\ v = \beta Y' + c_y \end{cases}$$

$$\Rightarrow \begin{cases} u = \alpha f \dfrac{X}{3} + c_x \\ v = \beta f \dfrac{Y}{3} + c_y \end{cases}$$

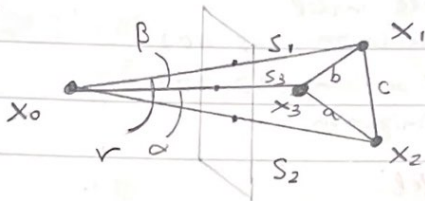$$\Rightarrow \begin{cases} u = f_x \dfrac{X}{3} + c_x \\ v = f_y \dfrac{Y}{3} + c_y \end{cases}$$

let $S = 3$

$$\Rightarrow S \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ 3 \end{bmatrix}$$

$$\Rightarrow S u = K P_c$$
$$= K T_{cw} P_w$$
$$= K (R_{cw} P_w + t)$$

## Triangulation (2D-3D)

### P3P



known $X_1$ $X_2$ $X_3$ $\alpha$ $\beta$ $\gamma$

$a^2 = S_2^2 + S_3^2 - 2 S_2 S_3 \cos\alpha$

$b^2 = S_1^2 + S_3^2 - 2 S_1 S_3 \cos\beta$

$c^2 = S_1^2 + S_2^2 - 2 S_1 S_2 \cos\gamma$

① $a^2 = S_2^2 + S_3^2 - 2 S_2 S_3 \cos\alpha$

set $u = \frac{S_2}{S_1}$, $v = \frac{S_3}{S_1}$

$a^2 = S_1^2 (u^2 + v^2 - 2uv\cos\alpha)$

$\therefore S_1^2 = \dfrac{a^2}{u^2 + v^2 - 2uv\cos\alpha}$

② $b^2 = S_1^2 + S_3^2 - 2 S_1 S_3 \cos\beta$

set $u = \frac{S_2}{S_1}$, $v = \frac{S_3}{S_1}$

$b^2 = S_1^2 (1 + v^2 - 2v\cos\beta)$

$\therefore S_1^2 = \dfrac{b^2}{1 + v^2 - 2v\cos\beta}$

③ $c^2 = S_1^2 + S_2^2 - 2 S_1 S_2 \cos\gamma$

set $u = \frac{S_2}{S_1}$, $v = \frac{S_3}{S_1}$

$c^2 = S_1^2 (1 + u^2 - 2u\cos\gamma)$

$\therefore S_1^2 = \dfrac{c^2}{1 + u^2 - 2u\cos\gamma}$

$\therefore S_1^2 = \dfrac{a^2}{u^2 + v^2 - 2uv\cos\alpha} = \dfrac{b^2}{1 + v^2 - 2v\cos\beta} = \dfrac{c^2}{1 + u^2 - 2u\cos\gamma}$

parameterize $u$ as $v$

$\Rightarrow A_4 v^4 + A_3 v^3 + A_2 v^2 + A_1 v + A_0 = 0$

get 4 (up to) $v$ solutions

get $v \to$ solve $u$

$\to$ solve $S_1$ $S_2$ $S_3$

now we have two pairs of 3D points
$\to$ solve 3D-3D problem

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## EPnP

- we need reference points $P$ to do odometry
- any reference points $P$ can be expressed by control points $C_j$ ×4

$\Rightarrow P_i^w = \sum_{j=1}^{4} \alpha_{ij} C_j^w$ w/ $\sum_{j=1}^{4}\alpha_{ij} = 1$

known ↓ known

unknown

side note 1:
- how to get $C_j^w$

side note 2:
- why 4 points, not 3, 2 ?

now try to get $\alpha$ (intermediate media)

$P_i^c = R_{cw} P_i^w + t$

$= R_{cw} \left( \sum_{j=1}^{4} \alpha_{ij} C_j^w \right) + t$

as $\sum_{j=1}^{4}\alpha_{ij} = 1$

$t = \sum_{j=1}^{4} \alpha_{ij} t$

$\therefore P_i^c = R_{cw} \left( \sum_{j=1}^{4} \alpha_{ij} C_j^w \right) + \sum_{j=1}^{4}\alpha_{ij} t$

$= \sum_{j=1}^{4} \left[ \alpha_{ij} \left( R_{cw} C_j^w + t \right) \right]$

$= \sum_{j=1}^{4} \alpha_{ij} C_j^c$

Both $\{C_j^c\}$ $\{C_j^w\}$ share same $\alpha$

$P_i^w = \sum_{j=1}^{4} \alpha_{ij} C_j^w$

known ↓ known

unknown

get unknown $\alpha_{ij} \to$ known

$P_i^c = \sum_{j=1}^{4} \alpha_{ij} (R_{cw} C_j^w + t) = \sum_{j=1}^{4} \alpha_{ij} C_j^c$

unknown ↓ known

our objective is to get $P_i^c$

side note 3:
how to get $\alpha_{ij}$?

$$P_i^c = \sum_{j=1}^{4} \alpha_{ij}(R_{cw}C_j^w + t) = \sum_{j=1}^{4} \alpha_{ij} C_j^c$$

$$\Rightarrow w_i \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & u_c \\ 0 & f_v & v_c \\ 0 & 0 & 1 \end{bmatrix} \sum_{j=1}^{4} \alpha_{ij} \begin{bmatrix} x_j^c \\ y_j^c \\ z_j^c \end{bmatrix}$$

From Last Row:

$$w_i = \sum_{j=1}^{4} \alpha_{ij} z_j^c$$

we get $\begin{cases} w_i u_i = f_u \sum_{j=1}^{4} \alpha_{ij} x_j^c + u_c \sum_{j=1}^{4} \alpha_{ij} z_j^c \\ w_i v_i = f_v \sum_{j=1}^{4} \alpha_{ij} y_j^c + v_c \sum_{j=1}^{4} \alpha_{ij} z_j^c \end{cases}$

$$\Rightarrow \sum_{j=1}^{4} \left[ \alpha_{ij} f_u x_j^c + \alpha_{ij}(u_c - u_i) z_j^c \right] = 0$$

$$\sum_{j=1}^{4} \left[ \alpha_{ij} f_v y_j^c + \alpha_{ij}(v_c - v_i) z_j^c \right] = 0$$

we hv $\alpha$, K matrix, $u_i$ $v_i$

get $x_j^c$ $y_j^c$ $z_j^c$ $\Rightarrow$ 4 control pts
3 dimension
12 unknown

$$Mx = 0$$
$$x \in \mathbb{R}^{12}$$
$$M \in \mathbb{R}^{2n \times 12}$$
no. of points $i = 0, 1 \dots n$

solve $x$ get $C_j^c$ ($j = 1, 2, 3, 4$)
we then hv $a_j^w$ $C_j^c$ $\to$ 3D-3D (ICP problem)
we then also hv $P_i^c = \sum_{j=1}^{4} \alpha_{ij} C_j^c \to$ <mark>$P_i^w P_i^c \to ICP$</mark>

side note 1:
how to get $C_j^w$?
① as long as $C_j^w$ is invertible
② as per Lepetit et al. : get centre of
weight as $C_j^w$ ($j = 1$)

$$C_{j=1}^{w} = \frac{1}{n} \sum_{i=1}^{n} P_i^w$$

let $A = \begin{bmatrix} P_1^{wT} & \cdots & C_1^{wT} \\ \vdots & & \vdots \\ P_n^{wT} & & C_1^{wT} \end{bmatrix}$

$A^T A \xrightarrow{get} \lambda_{c,i}$ value $i = 1, 2, 3$
$V_{c,i}$ vector $i = 1, 2, 3$

$$C_j^w = C_{j=1}^{w} + \lambda_{c,j-1}^{\frac{1}{2}} V_{c,j-1} \quad j = 2, 3, 4$$

side note 2:
why 4, not others?

$$P_i^w = \begin{bmatrix} x_i^w \\ y_i^w \\ z_i^w \end{bmatrix} = \begin{bmatrix} C_1^w & C_2^w & C_3^w \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\sum_{i=1}^{3} \alpha_{ij} = 1$$

4 equations w/ 3 unknown
overdefined.

$\therefore$ $j = 4$

side note 3:
how to get $\alpha_{ij}$?

4 equations    4 unknowns
$$P_i^w = \sum_{j=1}^{4} \alpha_{ij} C_j^w \quad w/ \sum_{j=1}^{4} \alpha_{ij} = 1$$
solve linear system

remark 3D-2D   $4\alpha_s \to$ varies.
$4C_j^c \to$ const

ICP Problem
known data association
$$Y = \{y_1, \dots, y_I\}$$
$$X = \{x_1, \dots, x_J\}$$ $w c = \{i, j\}$

$$\sum_{(i,j) \in c} \| y_i - Rx_j - t \|^2 \to min$$

matched points pairs:
$x_n$ $y_n$
$$\overline{x}_n = Rx_n + t$$
$$\sum \| y_n - \overline{x}_n \|^2 P_n \to min$$

exist & let's derive direct solution

$$y_0 = \frac{\sum y_n P_n}{\sum P_n} \quad x_0 = \frac{\sum x_n P_n}{\sum P_n}$$

$$H = \sum (y_n - y_0)(x_n - x_0)^T P_n$$
$$svd(H) = U D V^T$$
$$R = V U^T$$
$$t = y_0 - Rx_0$$

why is this a good solution? $\longrightarrow$

$$\bar{X}_n = RX_n + t$$
$$\bar{X}_n - y_0 = RX_n + t - y_0$$
$$\bar{X}_n - y_0 = R(X_n + R^T t - R^T y_0)$$
$$\bar{X}_n - y_0 = R(X_n - X_0) \quad \text{set as unknown variable } X_0$$

## ICP (SVD) (cont'd)

$$\sum \|y_n - \bar{X}_n\|^2 P_n \to \min$$
$$\Rightarrow \sum \|y_n - y_0 - \underbrace{RX_n - t + y_0}_{\bar{X}_n}\|^2 P_n \to \min$$
$$\Rightarrow \sum \|y_n - y_0 - (\bar{X}_n - y_0)\|^2 P_n \to \min$$
$$\Rightarrow \sum \|y_n - y_0 - R(X_n - X_0)\|^2 P_n \to \min$$

$$\therefore R^* X_0^* = \operatorname*{argmin}_{R, X_0} \sum \|y_n - y_0 - R(X_n - X_0)\|^2 P_n$$

$$\Phi(X_0, R) = \sum [(y_n - y_0) - R(X_n - X_0)]^T [(y_n - y_0) - R(X_n - X_0)] P_n$$

$$\Phi(X_0, R) = \sum (y_n - y_0)^T (y_n - y_0) P_n \quad \text{no } X_0, R$$
$$+ \sum (X_n - X_0)^T (X_n - X_0) P_n \quad \text{no } R$$
$$- 2\sum (y_n - y_0)^T R (X_n - X_0) P_n$$

① w.r.t $X_0$

$$\frac{\partial \Phi(X_0, R)}{\partial X_0} = -2\sum (X_n - X_0) P_n + 2\sum R^T (y_n - y_0) P_n$$

set $\dfrac{\partial \Phi(X_0, R)}{\partial X_0} = 0$

$$\therefore \quad 0 = -2\sum (X_n - X_0) P_n + 2\sum R^T (y_n - y_0) P_n$$
$$\Rightarrow \sum (X_n - X_0) P_n = 0$$
$$\sum X_n P_n - \sum X_0 P_n = 0$$
$$\Rightarrow X_0 = \frac{\sum X_n P_n}{\sum P_n}$$

Optimal value for $X_0$ is the weighted mean of points $X_n$

② w.r.t $R$

$$R^* = \operatorname*{argmin}_R -2\sum (y_n - y_0)^T R (X_n - X_0) P_n$$
$$= \operatorname*{argmax}_R 2\sum (y_n - y_0)^T R (X_n - X_0) P_n$$

let $b_n = y_n - y_0$
$a_n = X_n - X_0$

$$R^* = \operatorname*{argmax}_R \sum b_n^T R a_n P_n$$
$$R^* = \operatorname*{argmax}_R \operatorname{tr}(RH)$$

$$H = \sum (a_n b_n^T) P_n$$

find $R$ max $\operatorname{tr}(RH)$

$$SVD(H) = UDV^T$$
$$U^T U = I$$
$$V^T V = I$$
$$D = \operatorname{diag}(d_i)$$

$$R = VU^T$$
$$H = UDV^T$$
$$\operatorname{tr}(RH) = \operatorname{tr}(VU^T UDV^T)$$
$$= \operatorname{tr}(VDV^T)$$
$$= \operatorname{tr}(VD^{\frac{1}{2}} D^{\frac{1}{2}} V^T) \quad \text{let}$$
$$= \operatorname{tr}(AA^T) \quad A = VD^{\frac{1}{2}}$$

$$\therefore \operatorname{tr}(RH) = \operatorname{tr}(AA^T)$$
$$\underline{\operatorname{tr}(AA^T) \geq \operatorname{tr}(R'AA^T)}$$
Schwarz inequality

$$\operatorname{tr}(RH) = \operatorname{tr}(AA^T) \geq \operatorname{tr}(R'AA^T)$$
$$\parallel$$
$$\operatorname{tr}(R'RH)$$

any other rotation matrix

$\therefore$ choice $R = VU^T$

optimal to maximize the trace

$R'R$ is also another $SO(3)$.
let $R'R = R''$

$\operatorname{tr}(R''H)$ will always $< \operatorname{tr}(RH)$
if $R = VU^T$

i.e, if $RH$ can be written as $AA^T$
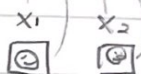$\operatorname{tr}(RH)$ will be max.

# What's SVD

- for data deduction
- data-driven generalization of Fourier Transform (FFT)
- "tailored" to specific problem
- solve $Ax = b$
  for non-square $A$
- regression
- PCA
- correlation

$$\bar{X} = \left[ \begin{array}{ccccc} | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_m \\ | & | & | & & | \end{array} \right]_{R^{n \times m}}$$

$$x_k \in R^n$$



$$\bar{X} = \left[ \begin{array}{cccc} | & | & | & | \\ x_1 & x_2 & x_3 \cdots x_m \\ | & | & | & | \end{array} \right]_{R^{n \times m}} = U \Sigma V^T$$

eigen    importances    mixtures of U's to make X's

$$U \Sigma V^T = \left[ \begin{array}{ccc} | & | & | \\ u_1 & u_2 \cdots u_n \\ | & | & | \end{array} \right] \left[ \begin{array}{ccc} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_m \\ & 0 & \end{array} \right] \left[ v_1 v_2 \cdots v_m \right]^T$$

eigen faces    order by importance    eigen-time series, eigen mixtures

left singular vectors    singular values    right singular vectors

- $u, v$ unitary    normal length orthogonal

$$UU^T = U^T U = I_{n \times n}$$
$$VV^T = V^T V = I_{m \times m}$$
$$\Sigma \text{ diagonal} \quad \sigma_1 \geq \sigma_2 \geq \cdots \sigma_n \geq 0$$

$$\bar{X} = \left[ \begin{array}{ccc} | & | & | \\ x_1 & x_2 \cdots x_n \\ | & | & | \end{array} \right] = U \Sigma V^T = \left[ \begin{array}{ccc} | & | & | \\ u_1 & u_2 \cdots u_n \\ | & | & | \end{array} \right]^{m} \left[ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_m \\ 0 \end{array} \right]^{m \times m} \left[ \begin{array}{c} -v_1^T- \\ -v_2^T- \end{array} \right]$$

guaranteed to exist unique

$n \gg m$, usually

$$= u_1 \sigma_1 v_1^T + u_2 \sigma_2 v_2^T + \cdots + u_m \sigma_m v_m^T + 0$$

$$= \hat{U} \hat{\Sigma} V^T \text{ (economy SVD)}$$

$$[U, S, V]$$
$$= svd(X, 'econ')$$

$$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_m u_m v_m^T + 0$$

$$= \sigma_1 \left[ \begin{array}{c} u_1 v_1^T \end{array} \right] + \sigma_2 \left[ \begin{array}{c} u_2 v_2^T \end{array} \right] + \cdots + \sigma_m \left[ \begin{array}{c} u_m v_m^T \end{array} \right]$$

rest negligible

$\hat{U}^T \hat{U} = I_{rr}$
$\hat{U} \hat{U}^T \neq I_{rr}$

truncate at rank r

$$= \tilde{U} \tilde{\Sigma} \tilde{V}^T \text{ (Eckard-Young [1936] Theory)}$$

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$$\underset{\tilde{X} \text{ s.t. } rank(\tilde{X}) = r}{\arg\min} \| \bar{X} - \tilde{\bar{X}} \|_F = \tilde{U} \tilde{\Sigma} \tilde{V}^T$$

## SVD + correlation

$$\bar{X}^T \bar{X} = m \underset{\substack{\longleftarrow x_1^T \longrightarrow \\ \longleftarrow x_2^T \longrightarrow \\ \vdots}}{} \quad n \left| \begin{array}{c} | & | \\ x_1 & x_2 \cdots \\ | & | \end{array} \right. \quad \text{correlation matrix}$$

$$= \left[ \begin{array}{cccc} x_1^T x_1 & x_1^T x_2 & & x_1^T x_m \\ x_2^T x_1 & x_2^T x_2 & \cdots & x_2^T x_m \\ \vdots & & & \vdots \\ x_m^T x_1 & x_m^T x_2 & & x_m^T x_m \end{array} \right]$$

$$x_i^T x_j = \langle x_i, x_j \rangle$$

$\bar{X} = \hat{U} \hat{\Sigma} V^T$
$\bar{X}^T = V \hat{\Sigma} \hat{U}^T$

$$\bar{X}^T \bar{X} = V \hat{\Sigma} \hat{U}^T \hat{U} \hat{\Sigma} V^T$$
$$= V \hat{\Sigma}^2 V^T \text{ (eigen decomposition)}$$

$$\bar{X}^T \bar{X} V = V \hat{\Sigma}^2 \longrightarrow \text{eigenvalues}$$
eigenvectors

$$\bar{X} \bar{X}^T = \hat{U} \hat{\Sigma} V^T V \hat{\Sigma} \hat{U}^T = \hat{U} \hat{\Sigma}^2 \hat{U}^T$$

$$\bar{X} \bar{X}^T \hat{U} = \hat{U} \hat{\Sigma}^2 \longrightarrow \text{eigenvalues}$$
eigenvectors

## ICP

unknown data association

corresponding points $y_n, x_n$   $n = 1, \ldots, N$
weights $p_n$   $n = 1, \ldots, N$
find $R, t$   s.t.

$$\bar{x}_n = R x_n + t \quad n = 1, \ldots, N$$
so that
$$\sum \| y_n - \bar{x}_n \|^2 p_n \longrightarrow \min$$

Correspondences unknown

guess $\longrightarrow$ point locations
or
iteration $\longrightarrow$ point correspondences

### ICP (Vanilla)

$\bar{x}_n = x_n$
$e = \infty$
while $e > thres$
   $c = $ correspondences ($\{y_n, \bar{x}_n\}$)
   $(t, R) = $ compute $T(c)$
   $\bar{x}_n = R(x_n - x_0) + y_0$
   $e = E(t, R) = \Sigma (R^T y_0 - R^T t, R)$
return $\{\bar{x}_n\}$

if easy $\rightarrow$ analytical form
$\rightarrow$ through $\frac{dF}{dx} = 0$
$\rightarrow$ else

## Non-linear Least-Square Problem

$\min_{x} F(x) = \frac{1}{2} \|f(x)\|_2^2 \qquad f(x): R^n \rightarrow R$

objective: get $X$ s.t. $F(x)$ has min

optimization algo
while true
   at $i = k$
   search $\Delta X_k$
   get $\|f(X_k + \Delta X_k)\|_2^2$
   if $\Delta X_k < b$
     return
   else
     continue
   $X_{k+1} = X_k + \Delta X_k$

How to get $\Delta X_k$?
A: different tricks
Here introduce
### Gauss-Newton Method

$f(x) = \frac{1}{2} \sum_{j=1}^{m} r_j(x)^2$
$= \frac{1}{2} \sum_{j=1}^{m} \|r(x)\|_2^2$

$\begin{cases} r_j(x) = \emptyset(x; \tau_j) - y_j \\ j = 1, 2, 3 \dots m \\ r(x) = [r_1(x), r_2(x) \\ \qquad \dots r_m(x)]^T \end{cases}$

$X$ is the objective parameter
    (e.g. pose) $X \in R^n$
$r$ is the residual result calculated
  from $X$ (e.g. reprojection $u, v$)
    $m = 1, 2$

math preliminaries
$\nabla f(x_1, x_2) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right) \quad J(x) = \left[ \frac{\partial r_j}{\partial x_i} \right]_{\substack{j=1\dots m \\ i=1\dots n}}$
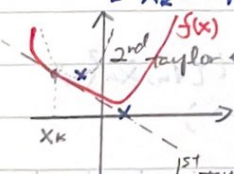
$H(f(x_1, x_2)) = \begin{pmatrix} \frac{\partial f}{\partial x_1^2} & \frac{\partial f}{\partial x_1 \partial x_2} \\ \frac{\partial f}{\partial x_2 \partial x_1} & \frac{\partial f}{\partial x_2^2} \end{pmatrix} = \begin{bmatrix} \nabla r_1(x)^T \\ \nabla r_2(x)^T \\ \vdots \\ \nabla r_m(x)^T \end{bmatrix}$

recall Newton's Method
$X_{k+1} = X_k + \Delta X_k$
$\Rightarrow X_{k+1} = X_k - [\nabla^2 f(X_k)]^{-1} \nabla f(X_k)$
$\qquad = X_k - H^{-1} J_r(X_k)^T r(X_k)$



$2^{nd}$ taylor expansion $f(x) =$
$\qquad f(x_k) + f'(x_k)(x - x_k)$
$\qquad + \frac{1}{2} f''(x_k)(x - x_k)^2$
$f(x) =$
$1^{st}$ taylor expansion $f(x_k) + f'(x_k)(x - x_k)$
not a good value for $\circ X_k$.
as it could over-shoot exagerratingly

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$1^{st}$ order: $f(X_k) + f'(X_k)(x - X_k) = 0$

$2^{nd}$ order: $\underbrace{f(X_k) + f'(X_k)(x - X_k) + \frac{1}{2} f''(X_k)(x - X_k)^2}_{g(x)} = 0$

solve $X$ value,
and set it as next $X_k$.
Newton's method opt the second $2^{nd}$ order.

$\frac{d}{dx} g(x) = f'(x_k) + f''(x_k)(x - x_k)$

$x^* = \underset{x}{\arg\min}\, g(x)$

$\therefore \quad 0 = \frac{d}{dx} g(x) = f'(x_k) + f''(x_k)(x - x_k)$

$\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad \Delta x$

$\Delta x = \dfrac{-f'(x_k)}{f''(x_k)}$

$\therefore \quad X_{k+1} = X_k - [\nabla^2 f(X_k)]^{-1} \nabla f(X_k)$
$\qquad\qquad = X_k - H^{-1} J_r(X_k)^T r(X_k)$

yet: still greedy
  Hessian matrix not easy to solve

$\therefore$ Gauss-Newton Method
w/ $\nabla f = J_r^T r$
$\nabla^2 f = J_r^T J_r + \sum_{i=1}^{m} r_i \nabla^2 r_i$
$\qquad = J_r^T J_r + Q$
$\nabla^2 f \approx J_r^T J_r$

substitute $H$ as $J^T J$

$\therefore X_{k+1} = X_k - \underbrace{[J_r(X_k)^T J_r(X_k)]^{-1} \cdot J_r(X_k)^T r(X_k)}_{\Delta X}$

$\boxed{\Delta x = - [J_r(X_k)^T J_r(X_k)]^{-1} J_r(X_k)^T r(X_k)}$

$f = \frac{1}{2} \sum_{j=1}^{m} r_j(x)^2$
$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial r_j}{\partial x_i} r_j$

$1^{st}$ order
$\nabla f(x) = \nabla \frac{1}{2} \sum_{j=1}^{m} (r_j(x))^2$
$\qquad = \sum_{j=1}^{m} r_j(x) \nabla r_j(x)$
$\qquad = J(x)^T r(x)$

$2^{nd}$ order
$\nabla^2 f(x) = \nabla \sum_{j=1}^{m} r_j(x) \nabla r_j(x)$
$\qquad = \sum_{j=1}^{m} \nabla r_j(x) \nabla r_j(x) + \sum_{j=1}^{m} r_j(x) \nabla^2 r_j(x)$
$\qquad = J(x)^T J(x) + \sum_{j=1}^{m} r_j(x) \nabla^2 r_j(x)$
$\qquad = J(x)^T J(x) + Q(x)$

much better. at least we hv a minimum value
(quadratic)