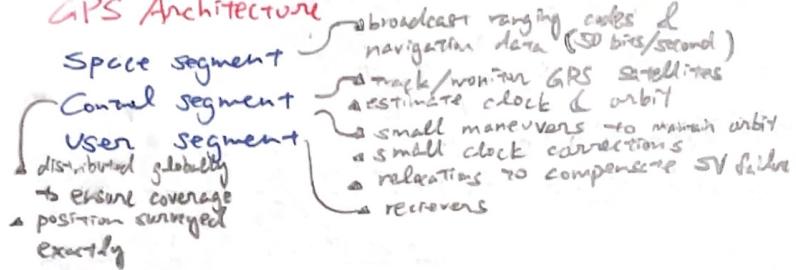


Ch 1 Basics

GPS Architecture



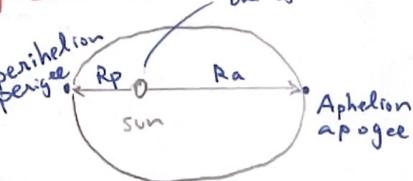
Satellite position representation
Kepler's three law

The law of orbit

$$R_a = a(1+e)$$

$$R_p = a(1-e)$$

$e = \text{eccentricity of the ellipse}$



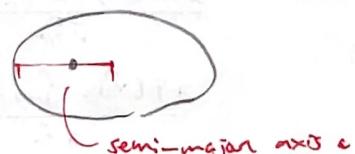
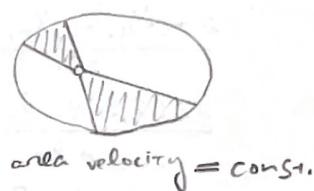
The law of areas

$$\frac{A_0}{T_{1-t_0}} = \frac{A_1}{T_2-t_1}$$

The law of period

$$\frac{R_0^3}{T_0^2} = \frac{R_1^3}{T_1^2}$$

semi-major axis
orbital period

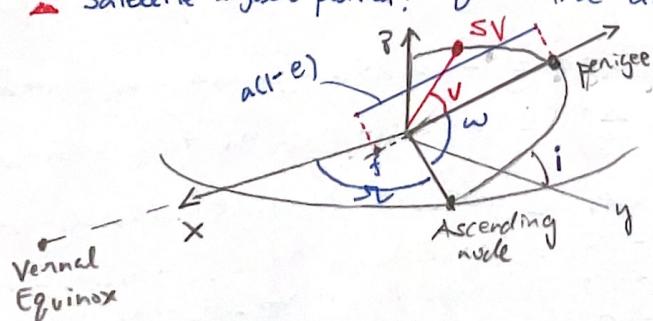


Keplerian Elements

orbit shape: a semimajor axis
 e eccentricity

orbit orientation: i inclination
relative to earth
 Ω right ascension of ascending node
 ω angle of perigee

Satellite angular position: v true anomaly



Coordinate Transformation

Keplerian parameters

Peri-focal coordinates

Earth centered inertial

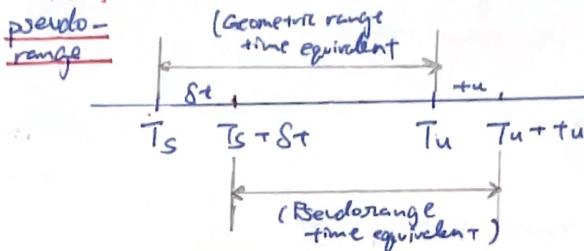
Earth centered earth-fixed (World Geodetic System WGS-84)

Factors that affect Satellite orbit/clock

- perturbation forces
 - non sphericity of earth's gravitational potential
 - 3rd body effects
 - solar radiation pressure
- relativity effects
 - less gravity \rightarrow time faster
 - faster speed \rightarrow time slower

Ch 2 Positioning

TOA & GNSS measurement



$$P = c[(T_u + \delta u) - (T_s + \delta t)] + \sum \delta$$

$$= c(T_u - T_s) + c(\delta u - \delta t) + \sum \delta$$

Geometric Range Clock offset from system time
delay & noise

- $T_u + \delta u$, $T_s + \delta t$ are obtained from

- Ranging codes &
- non-synchronized clock

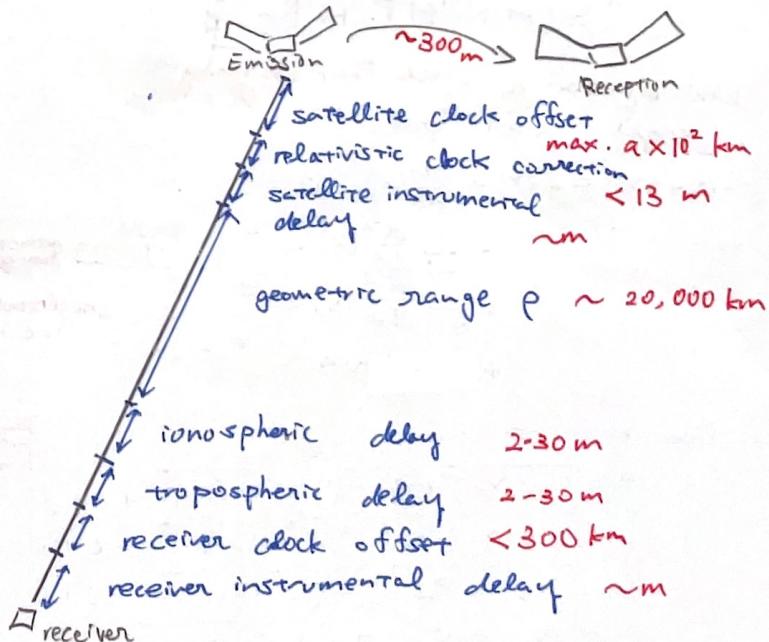
Carrier phase

$$\rho = \lambda \cdot \Delta \varphi = \lambda (\varphi_r(T_2) - \varphi_s(T_1))$$

phase of signal reception phase of signal transmission

- much more precise
- yet w/ ambiguity on integer of existing wavelengths

pseudo-range error source



Least Squares

$x \in \mathbb{R}^n$ (tracking state)
 $y \in \mathbb{R}^m$ (observation state)

$$\begin{cases} y = Hx + n \\ \hat{x} = \underset{x}{\operatorname{argmax}} P(y|x) \end{cases}$$

$$n \sim N(0, \Sigma)$$

$$\therefore \hat{x} = \underset{x}{\operatorname{argmax}} \frac{1}{(2\pi\Sigma)^{N/2}} e^{-\frac{1}{2\Sigma} \|y - Hx\|^2}$$

$$= \underset{x}{\operatorname{argmin}} \|y - Hx\|^2$$

$$\therefore \text{we set } \frac{d}{dx} \|y - H\hat{x}\|^2 = 2H^T H \hat{x} - 2H^T y$$

$$\hat{x} = (H^T H)^{-1} H^T y$$

△ weighted version

$$\hat{x} = \underset{x}{\operatorname{argmax}} \frac{1}{(2\pi)^{\frac{N}{2}} |R_n|^{1/2}} e^{-\frac{1}{2} (y - Hx)^T R_n^{-1} (y - Hx)}$$

$$= \underset{x}{\operatorname{argmin}} (y - Hx)^T R_n^{-1} (y - Hx)$$

$$\therefore \hat{x} = (H^T R_n^{-1} H)^{-1} H^T R_n^{-1} y$$

linearization (Single Point Positioning)

△ recall Taylor Series

$$f(x+\Delta x) = f(x) + \frac{df}{dx}(x) \cdot \Delta x$$

$$+ \frac{1}{2!} \frac{d^2 f}{dx^2}(x) \Delta x^2$$

$$+ \frac{1}{3!} \frac{d^3 f}{dx^3}(x) \Delta x^3$$

$$+ \mathcal{O}(\Delta x^4)$$

recall pseudo-range
 $r = c(t_u - t_j)$ geometry
 $+ c(t_u - \delta_t)$

 $+ c(t_u - \delta_t)$ (neglected here)

 $= r + c(t_u - \delta_t)$ reception offset

 δ_t satellite offset

known here broadcasted

△ linearization to solve

$$\begin{aligned} P_1 &= \sqrt{(x_1 - \hat{x}_u)^2 + (y_1 - \hat{y}_u)^2 + (\hat{z}_1 - \hat{z}_u)^2} + c\hat{t}_u \\ P_2 &= \sqrt{(x_2 - \hat{x}_u)^2 + (y_2 - \hat{y}_u)^2 + (\hat{z}_2 - \hat{z}_u)^2} + c\hat{t}_u \\ P_3 &= \sqrt{(x_3 - \hat{x}_u)^2 + (y_3 - \hat{y}_u)^2 + (\hat{z}_3 - \hat{z}_u)^2} + c\hat{t}_u \\ P_4 &= \sqrt{(x_4 - \hat{x}_u)^2 + (y_4 - \hat{y}_u)^2 + (\hat{z}_4 - \hat{z}_u)^2} + c\hat{t}_u \end{aligned}$$

known known

4 unknowns

$$\Rightarrow P_j = \sqrt{(x_j - \hat{x}_u)^2 + (y_j - \hat{y}_u)^2 + (\hat{z}_j - \hat{z}_u)^2} + c\hat{t}_u$$

$$= f(x_u, y_u, \hat{z}_u, \hat{t}_u)$$

assume a known, approximate position location $(\hat{x}_u, \hat{y}_u, \hat{z}_u)$ & \hat{t}_u (time bias)

$$\hat{P}_j = \sqrt{(x_j - \hat{x}_u)^2 + (y_j - \hat{y}_u)^2 + (\hat{z}_j - \hat{z}_u)^2} + c\hat{t}_u$$

$$= f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u) \quad \text{--- } \oplus$$

$$\therefore x_u = \hat{x}_u + \Delta x_u \quad \hat{z}_u = \hat{z}_u + \Delta \hat{z}_u$$

$$y_u = \hat{y}_u + \Delta y_u \quad \hat{t}_u = \hat{t}_u + \Delta \hat{t}_u$$

$$\therefore f(x_u, y_u, \hat{z}_u, \hat{t}_u) = f(\hat{x}_u + \Delta x_u, \hat{y}_u + \Delta y_u, \hat{z}_u + \Delta \hat{z}_u, \hat{t}_u + \Delta \hat{t}_u)$$

∴ TS of $f(x_u, y_u, \hat{z}_u, \hat{t}_u)$:

$$\begin{aligned} &f(x_u, y_u, \hat{z}_u, \hat{t}_u) \\ &= f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u) + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial x_u} \Delta x_u \\ &\quad + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial y_u} \Delta y_u \\ &\quad + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{z}_u} \Delta \hat{z}_u \\ &\quad + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{t}_u} \Delta \hat{t}_u \\ &\quad + \mathcal{O}(\Delta^2) \quad \text{--- } \oplus \end{aligned}$$

(from above)

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial x_u} = - \frac{x_j - \hat{x}_u}{\hat{r}_j}$$

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial y_u} = - \frac{y_j - \hat{y}_u}{\hat{r}_j}$$

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{z}_u} = - \frac{z_j - \hat{z}_u}{\hat{r}_j}$$

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{t}_u} = c$$

$$\hat{r}_j = \sqrt{(x_j - \hat{x}_u)^2 + (y_j - \hat{y}_u)^2 + (z_j - \hat{z}_u)^2}$$

△ put \oplus & \oplus into \oplus

$$\text{we get } P_j = \hat{P}_j - \frac{x_j - \hat{x}_u}{\hat{r}_j} \Delta x_u - \frac{y_j - \hat{y}_u}{\hat{r}_j} \Delta y_u - \frac{z_j - \hat{z}_u}{\hat{r}_j} \Delta \hat{z}_u - c \Delta \hat{t}_u$$

$$\Rightarrow P_j - \hat{P}_j = -\Delta x_u - \Delta y_u - \Delta \hat{z}_u + c \Delta \hat{t}_u$$

$$\Rightarrow \Delta P_j = \Delta x_u \Delta x_u + \Delta y_u \Delta y_u + \Delta \hat{z}_u \Delta \hat{z}_u - c \Delta \hat{t}_u$$

$$\Rightarrow \Delta P_1 = \Delta x_1 \Delta x_u + \Delta y_1 \Delta y_u + \Delta \hat{z}_1 \Delta \hat{z}_u - c \Delta \hat{t}_u$$

$$\Delta P_2 = \Delta x_2 \Delta x_u + \Delta y_2 \Delta y_u + \Delta \hat{z}_2 \Delta \hat{z}_u - c \Delta \hat{t}_u$$

$$\Delta P_3 = \Delta x_3 \Delta x_u + \Delta y_3 \Delta y_u + \Delta \hat{z}_3 \Delta \hat{z}_u - c \Delta \hat{t}_u$$

$$\Delta P_4 = \Delta x_4 \Delta x_u + \Delta y_4 \Delta y_u + \Delta \hat{z}_4 \Delta \hat{z}_u - c \Delta \hat{t}_u$$

$$\Rightarrow \Delta P = \begin{bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{bmatrix} \quad H = \begin{bmatrix} \Delta x_1 & \Delta y_1 & \Delta \hat{z}_1 & 1 \\ \Delta x_2 & \Delta y_2 & \Delta \hat{z}_2 & 1 \\ \Delta x_3 & \Delta y_3 & \Delta \hat{z}_3 & 1 \\ \Delta x_4 & \Delta y_4 & \Delta \hat{z}_4 & 1 \end{bmatrix} \quad \Delta X = \begin{bmatrix} \Delta x_u \\ \Delta y_u \\ \Delta \hat{z}_u \\ \Delta \hat{t}_u \end{bmatrix}$$

∴ $\Delta P = H \Delta X \rightarrow$ could be > 4 over-determined

get ΔX (can use least-square)

$$\text{get } (x_u, y_u, \hat{z}_u, \hat{t}_u) = (\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u) + \Delta X$$

Dilution of Precision (DOP)



$$\begin{aligned} \text{- PDOP/GDOP} &= \sqrt{D_{11} + D_{22} + D_{33} + D_{44}} \\ \text{- HDOP} &= \sqrt{D_{11} + D_{22}} \\ \text{- VDOP} &= \sqrt{D_{33}} \\ \text{- TDOP} &= \sqrt{D_{44}} / c \end{aligned}$$

$$H\Delta x = \Delta p$$

$$H = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 \\ a_{21} & a_{22} & a_{23} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & 1 \end{bmatrix}$$

$$\Delta x = (H^T H)^{-1} H^T \Delta p$$

user equivalent range error

$$\text{cov}(\Delta x) = (H^T H)^{-1} \sigma_{\text{VERE}}^2$$

$$(H^T H)^{-1} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix}$$

- better geometry, less DOP
- VDOP > HDOP, no variation vertically
- precise → concentrated data
accurate → ground-truth-closed data

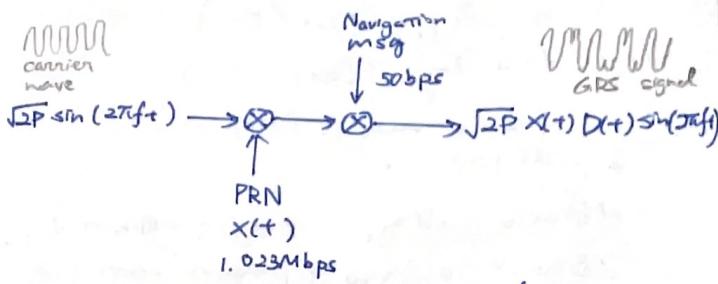
SV clock correction

- » satellite clock offset
 - monitored by ground station network
 - rebroadcasted to receiver
- » clock correction model
 - thru polynomial
 - polynomial coefficients a_0, a_1, a_2
 - prediction model (hours → days)
 - reference by Toc → operation center

Ch3 GPS/GNSS Signal

Signal Structure

- signal = carrier wave + ephemeris
satellite clock/position
navigation msg
PRN code
for Time of Arrival



- carrier wave @ 1575.42 MHz, L1, is modulated by
 - coarse/acquisition (C/A) code civilians
 - precision/secure (P/Y) code military

PRN (Pseudo Random Noise) Code

- △ for measurement of time of arrival (TOA)
- △ generation pattern: Gold Code

- △ 1 second : 50 bit
 - 1000 ms : 50 bit
 - 20 ms : 1 bit
 - 20 ms : 20 repetition of C/A
 - 1 ms : 1 repetition of C/A
 - 1 repetition of C/A : $1023 \frac{1}{6}$

- △ C/A : 1.023 M cps
- P/Y : 10.23 M cps

Signal @ L-band lower frequency gets easier to be affected by ionosphere

Auto-correlation

- ① Satellite transmits code-phase with timestamp $T_s(n)$
- ② Receiver receives code-phase with timestamp $T_r(n)$
n here stands for epoch n
- ③ on timeline of "Receiver", a replica code-phase is generated at $T_s(n)$ on timeline "receiver"
- ★ ④ conduct auto-correlation by shifting the replica code-phase to match received code-phase on timeline receiver
- ⑤ the shifted Δt is the inferred propagation time
- ⑥ pseudorange = $c \cdot \Delta t$
- ⑦ this pseudorange = $c[(T_u + t_u) - (T_s + \delta t)] + \Delta S$

Carrier Phase: resolve fractional difference/
integer ambiguity (N)
could get accuracy of 1-2cm

Power of GPS

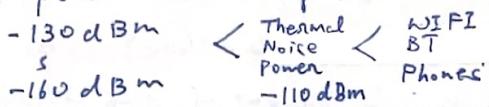
- Bel $B = \log_{10} (P_{out}/P_{in})$
- Decibel dB = $\log_{10} (P_{out}/P_{in})$
- set reference value (Pin)
1 milliwatt :

$$\text{dBm} = 10 \log_{10} (\text{power in milliwatt}) / 1 \text{ milliwatt}$$

$$\text{dBW} = 10 \log_{10} (\text{power in watt}) / 1 \text{ watt}$$

△ Path Loss

- transmit power is 27W
- received power reduced by 10^{16}



△ Antenna Gain: ratio of $\frac{\text{entire sphere area}}{\text{spot area}}$

Ch4 Differential Positioning & Precise Positioning

Errors

$$\delta t_p = \delta t_{\text{atm}} + \delta t_{\text{noise inter.}} + \delta t_{\text{multi-path}} + \delta t_{\text{hw}}$$

- atmosphere
- receiver noise + interference
- multipath
- receiver hardware offset

△ Satellite Clock Error

- offset of SV time to GPS time
- curve-fit to model the clock correction (polynomial coefficients in navmsg)
- not location-dependent for users

△ Ephemeris Error

- due to perturbation forces
- curve-fit to model the error
- location-dependent for users (increases with separation)

base station receiver

△ Ionosphere Delay

- 電離層
- affects electromagnetic wave propagation
- dispersive medium 15XX MHz
- $f > f_p \approx 10^6 \text{ Hz}$ can travel thru
- phase velocity well exceed group velocity
- amount of retardation of V_{group}
= advance of carrier phase w.r.t. free-space propagation
- GPS:

PRN & navmsg delayed sham carrier phase

→ ionospheric divergence

- removed by dual receiver
- depend on frequency f , elevation angle φ' (@ iono. intercept), total electron content TEC
- location-dependent for users (increases w/ separation)

base station receiver

△ Troposphere Delay

- 對流層

- nondispersive, but with refraction
- dry (hydroscopic) net (nonhydroscopic)
 - 90% to delay
 - predict accurately
 - extends to 40km height
 - utilize troposphere model to predict

- lesser to delay
- difficult to predict
- extends to 10km height

basically depends on weather

- Location-dependent for users

△ Receiver Noise

- thermal noise jitter
- interference/jamming (malicious signal)

△ Multipath



- depends on environment
- receiver location
- satellite elevation angle
- receiver signal processing
- antenna gain pattern
- signal characteristic
- improvements :
 - detect received signal pattern thru → signal/noise density ratio to see → reflection/refraction
 - signal processing : reduce early/late correlator
 - antenna : increase directivity

△ Hardware Bias

- Satellite Bias
 - signals and carrier frequency not totally sync'd.
- Receiver/Equipment Bias
 - signals are delayed in hardware

I. From above, usually (single point), dominant pseudorange error:
"Ionospheric Delay"

→ could be removed w/ dual receiver

2. spatially correlated :

- Ephemeris Errors
- Ionospheric Error
- Tropospheric Error
- multipath

Real-Time Kinematic (RTK)

- base receiver + rover receiver (22)
- base receiver process GPS signals
- rover receiver process GPS signals
- base station broadcast its measurement to rover receiver
- rover receiver could then resolve some errors like ionospheric errors
- harnesses both pseudo-range (code) & carrier phase
 - * thru base correction & RTK algo
 - m → cm
 - * single-band baseline < 10km
multi-band baseline < 60km
 - * base unit could be provided thru "local service provider"
- m → cm
- △ typical algo.
 1. KF
 2. unknowns: x_{rover} , v_{rover} , ambig.
 3. measurements: double differenced carrier phase, double differenced pseudo-range
 4. Integer ambig. solution:
 - a. EKF update
 - b. solve ambig. improve accuracy convergence time

Precise Point Positioning (PPP)

- single receiver
- relies on precise satellite clock orbit products connection T from global stations.

- cm-level.

- PPP performance \rightarrow RTK

Ch 5 Augmentation Systems

differential positioning (relative positioning)

- 2 receivers are close
- track difference between the 2
 - can be tracked more accurately
 - iono + tropo + clock + ephemeris errors are cancelled.
- $\Delta X = (P_1, t_1) - (P_2, t_2)$
 - ↳ know ΔX , know reference get rover.
- base station, with high-performance GPS receiver, could calculate error/correction & send correction \rightarrow rover.
- DGPS = reference station + datalink + user receiver

- Local Area Differential GPS
Wide Area Differential GPS

Ground Based Augmentation Systems (GBAS)

= pseudorange correctables
- use / don't use

- Aviation
- DGPS

Satellite Based Augmentation Systems (SBAS)

- Aviation
- DGPS
- iono., clock, ephemeris.
- GEO
- monitored / not monitored
- use / don't use

- send corrections
- send integrity msgs.