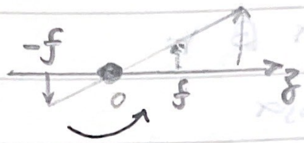
Pose Estimation via Vision

- pinhole model
- triangulation (2D-3D)
- ICP problem (3D-3D)
- optimization

pinhole model



$$\frac{z}{f} = \frac{X}{X'} = \frac{Y}{Y'}$$

$$\therefore \begin{cases} X' = f\dfrac{X}{z} \\ Y' = f\dfrac{Y}{z} \end{cases}$$

from $x'y'$ to u, v
enlarge α, β
translate c_x, c_y

$$\begin{cases} u = \alpha X' + c_x \\ v = \beta Y' + c_y \end{cases}$$

$$\Rightarrow \begin{cases} u = \alpha f \dfrac{X}{z} + c_x \\ v = \beta f \dfrac{Y}{z} + c_y \end{cases}$$

$$\Rightarrow \begin{cases} u = f_x \dfrac{X}{z} + c_x \\ v = f_y \dfrac{Y}{z} + c_y \end{cases}$$

let $s = z$

$$\Rightarrow s\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ z \end{bmatrix}$$

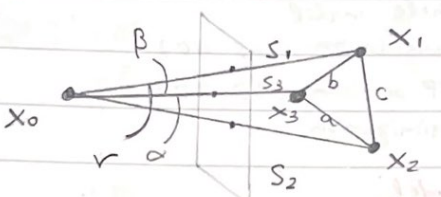
$$\begin{aligned} \Rightarrow s u &= K P_c \\ &= K T_{cw} P_w \\ &= K(R_{cw} P_w + t) \end{aligned}$$

No.

Date

Triangulation (2D-3D)

P3P



known X_1 X_2 X_3 α β γ

$$a^2 = S_2^2 + S_3^2 - 2S_2S_3\cos\alpha$$

$$b^2 = S_1^2 + S_3^2 - 2S_1S_3\cos\beta$$

$$c^2 = S_1^2 + S_2^2 - 2S_1S_2\cos\gamma$$

① $a^2 = S_2^2 + S_3^2 - 2S_2S_3\cos\alpha$

set $u = S_2/S_1$ $\quad v = S_3/S_1$

$$a^2 = S_1^2(u^2 + v^2 - 2uv\cos\alpha)$$

$$\therefore S_1^2 = \frac{a^2}{u^2+v^2-2uv\cos\alpha}$$

② $b^2 = S_1^2 + S_3^2 - 2S_1S_3\cos\beta$

set $u = S_2/S_1$ $\quad v = S_3/S_1$

$$b^2 = S_1^2(1 + v^2 - 2v\cos\beta)$$

$$\therefore S_1^2 = \frac{b^2}{1+v^2-2v\cos\beta}$$

③ $c^2 = S_1^2 + S_2^2 - 2S_1S_2\cos\gamma$

set $u = S_2/S_1$ $\quad v = S_3/S_1$

$$c^2 = S_1^2(1 + u^2 - 2u\cos\gamma)$$

$$\therefore S_1^2 = \frac{c^2}{1+u^2-2u\cos\gamma}$$

$$\therefore S_1^2 = \frac{a^2}{u^2+v^2-2uv\cos\alpha} = \frac{b^2}{1+v^2-2v\cos\beta} = \frac{c^2}{1+u^2-2u\cos\gamma}$$

parameterize u as v

$\Rightarrow A_4v^4 + A_3v^3 + A_2v^2 + A_1v + A_0 = 0$

get 4 (up to) v solutions

get v → solve u

→ solve S_1 S_2 S_3

now we have two pairs of 3D points

→ solve 3D-3D problem

EPnP

- we need reference points P to do odometry
- any reference points P can be expressed by control points $C_j \times 4$

$$\Rightarrow \underset{\text{known}}{P_i^w} = \sum_{j=1}^{4} \underset{\text{unknown}}{\alpha_{ij}} \underset{\text{known}}{C_j^w} \quad w/ \sum_{j=1}^{4}\alpha_{ij} = 1$$

side note 1:
- how to get C_j^w

side note 2:
- why 4 points, not 3, 2?

now try to get α (intermediate media)

$$P_i^c = R_{cw}P_i^w + t$$

$$= R_{cw}\left(\sum_{j=1}^{4}\alpha_{ij}C_j^w\right) + t$$

as $\sum_{j=1}^{4}\alpha_{ij} = 1$

$$t = \sum_{j=1}^{4}\alpha_{ij}t$$

$$\therefore P_i^c = R_{cw}\left(\sum_{j=1}^{4}\alpha_{ij}C_j^w\right) + \sum_{j=1}^{4}\alpha_{ij}t$$

$$= \sum_{j=1}^{4}\left[\alpha_{ij}\left(R_{cw}C_j^w + t\right)\right]$$

$$= \sum_{j=1}^{4}\alpha_{ij}C_j^c$$

Both {c} {w} share same α

$$\underset{\text{known}}{P_i^w} = \sum_{j=1}^{4}\underset{\text{unknown}}{\alpha_{ij}}\underset{\text{known}}{C_j^w}$$

get unknown α_{ij} → known

$$\underset{\text{unknown}}{P_i^c} = \sum_{j=1}^{4}\alpha_{ij}\left(R_{cw}C_j^w + t\right) = \sum_{j=1}^{4}\alpha_{ij}\underset{\text{known}}{C_j^c}$$

unknown

our objective is to get P_i^c

side note 3:
how to get α_{ij}?

$$P_i^c = \sum_{j=1}^{4} \alpha_{ij} (R_{cw} C_j^w + t) = \sum_{j=1}^{4} \alpha_{ij} C_j^c$$

$$\Rightarrow w_i \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} = \begin{bmatrix} f_u & 0 & u_c \\ 0 & f_v & v_c \\ 0 & 0 & 1 \end{bmatrix} \sum_{j=1}^{4} \alpha_{ij} \begin{bmatrix} x_j^c \\ y_j^c \\ z_j^c \end{bmatrix}$$

From Last Row:

$$w_i = \sum_{j=1}^{4} \alpha_{ij} z_j^c$$

we get $\begin{cases} w_i u_i = f_u \sum_{j=1}^{4} \alpha_{ij} x_j^c + u_c \sum_{j=1}^{4} \alpha_{ij} z_j^c \\ w_i v_i = f_v \sum_{j=1}^{4} \alpha_{ij} y_j^c + v_c \sum_{j=1}^{4} \alpha_{ij} z_j^c \end{cases}$

$$\Rightarrow \sum_{j=1}^{4} \left[\alpha_{ij} f_u x_j^c + \alpha_{ij} (u_c - u_i) z_j^c \right] = 0$$

$$\sum_{j=1}^{4} \left[\alpha_{ij} f_v y_j^c + \alpha_{ij} (v_c - v_i) z_j^c \right] = 0$$

we hv α, K matrix, u_i v_i

get $x_j^c\ y_j^c\ z_j^c \Rightarrow$ 4 control pts
3 dimension
12 unknown

$$Mx = 0$$

$$x \in \mathbb{R}^{12}$$

$$M \in \mathbb{R}^{2n \times 12}$$

no. of points $i = 0, 1 \ldots n$

solve x get c_j^c $(j = 1, 2, 3, 4)$

we then hv c_j^w $c_j^c \rightarrow$ 3D-3D (ICP problem)

we then also hv $P_i^c = \sum_{j=1}^{4} \alpha_{ij} C_j^c \rightarrow \boxed{P_i^w P_i^c \rightarrow \text{ICP}}$

side note 1:

how to get C_j^w?

① as long as c_j^w is invertible

② as per Lepetit et al.: get centre of weight as C_j^w $(j=1)$

$$C_{j=1}^w = \frac{1}{n} \sum_{i=1}^{n} P_i^w$$

let $A = \begin{bmatrix} P_1^{wT} & \cdots & C_1^{wT} \\ \vdots & & \vdots \\ P_n^{wT} & & C_1^{wT} \end{bmatrix}$

$A^T A \xrightarrow{\text{get}} \lambda_{c,i}$ value $i = 1, 2, 3$

$V_{c,i}$ vector $i = 1, 2, 3$

$$C_j^w = C_{j=1}^w + \lambda_{c,j-1}^{\frac{1}{2}} V_{c,j-1} \quad j = 2, 3, 4$$

side note 2:

why 4, not others?

$$P_i^w = \begin{bmatrix} x_i^w \\ y_i^w \\ z_i^w \end{bmatrix} = \begin{bmatrix} C_1^w & C_2^w & C_3^w \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\sum_{i=1}^{3} \alpha_{ij} = 1$$

4 equations w/ 3 unknown

overdefined.

$\therefore j = 4$

side note 3:

how to get α_{ij}?

4 equations 4 unknowns

$P_i^w = \sum_{j=1}^{4} \alpha_{ij} C_j^w$ w/ $\sum_{j=1}^{4} \alpha_{ij} = 1$

solve linear system

remark 3D-2D $4\alpha_s \rightarrow$ varies.

$4C_{js} \rightarrow$ const

ICP Problem

known data association

$Y = \{y_1, \ldots y_I\}$
$X = \{X_1, \ldots X_J\}$ } w $C = \{i, j\}$

$$\sum_{(i,j) \in C} \| y_i - R x_j - t \|^2 \rightarrow \min$$

matched points pairs:

X_n Y_n

$\bar{X}_n = R X_n + t$

$\sum \| Y_n - \bar{X}_n \|^2 p_n \rightarrow \min$

exist & let's derive <u>direct solution</u>

$$y_0 = \frac{\sum y_n p_n}{\sum p_n} \quad X_0 = \frac{\sum X_n p_n}{\sum p_n}$$

$$H = \sum (y_n - y_0)(X_n - X_0)^T p_n$$

$$\text{svd}(H) = U D V^T$$

$$R = V U^T$$

$$t = y_0 - R X_0$$

why is this a good solution? $\rightarrow$

No.

Date

$\bar{X}_n = RX_n + t$

$\bar{X}_n - y_0 = RX_n + t - y_0$

$\bar{X}_n - y_0 = R(X_n + \underbrace{R^T t - R^T y_0}_{})$ set as unknown varible X_0

$\bar{X}_n - y_0 = R(X_n - X_0)$

ICP (SVD) (cont'd)

$\sum \| y_n - \bar{X}_n \|^2 p_n \to \min$

$\Rightarrow \sum \| \underbrace{y_n - y_0}_{} - \underbrace{RX_n - t}_{\bar{X}_n} + \underbrace{y_0}_{} \|^2 p_n \to \min$

$\Rightarrow \sum \| y_n - y_0 - (\bar{X}_n - y_0) \|^2 p_n \to \min$

$\Rightarrow \sum \| y_n - y_0 - R(X_n - X_0) \|^2 p_n \to \min$

$\therefore R^* X_0^* = \underset{R, X_0}{\arg\min} \sum \| y_n - y_0 - R(X_n - X_0) \|^2 p_n$

$\Phi(X_0, R) = \sum [(y_n - y_0) - R(X_n - X_0)]^T [(y_n - y_0) - R(X_n - X_0)] p_n$

$\Phi(X_0, R) = \cancel{\sum (y_n - y_0)^T (y_n - y_0) p_n}$ no X_0, R

$+ \sum (X_n - X_0)^T (X_n - X_0) p_n$ no R

$- 2\sum (y_n - y_0)^T R (X_n - X_0) p_n$

① w.r.t X_0

$$\frac{\partial \Phi(X_0, R)}{\partial X_0} = -2\sum (X_n - X_0) p_n + 2\sum R^T (y_n - y_0) p_n$$

set $\frac{\partial \Phi(X_0, R)}{\partial X_0} = 0$

$\therefore 0 = -2\sum (X_n - X_0) p_n + 2\sum R^T (y_n - y_0) p_n$

$\Rightarrow \sum (X_n - X_0) p_n = 0$

$\sum X_n p_n - \sum X_0 p_n = 0$

$\Rightarrow X_0 = \dfrac{\sum X_n p_n}{\sum p_n}$

optimal value for X_0 is the weighted mean of points X_n

② w.r.t R

$R^* = \underset{R}{\arg\min} -2\sum (y_n - y_0)^T R (X_n - X_0) p_n$

$= \underset{R}{\arg\max}\, 2\sum (y_n - y_0)^T R (X_n - X_0) p_n$

let $b_n = y_n - y_0$

$a_n = X_n - X_0$

$R^* = \underset{R}{\arg\max} \sum b_n^T R a_n p_n$

$R^* = \underset{R}{\arg\max}\, tr(RH)$

$H = \sum (a_n b_n^T) p_n$

find R max $tr(RH)$

$SVD(H) = UDV^T$

$U^T U = I$

$V^T V = I$

$D = diag(d_i)$

$R = VU^T$

$H = UDV^T$

$tr(RH) = tr(VU^T UDV^T)$

$= tr(VDV^T)$

$= tr(VD^{\frac{1}{2}} D^{\frac{1}{2}} V^T)$ set $A = VD^{\frac{1}{2}}$

$= tr(AA^T)$

$\therefore tr(RH) = tr(AA^T)$

$tr(AA^T) \geq tr(R' AA^T)$

Schwarz inequality

$tr(RH) = tr(AA^T) \geq tr(R' AA^T)$

$\qquad\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad\quad tr(R' RH)$

any other rotation matrix

$\therefore$ choice $R = VU^T$

optimal to maxize the trace

$R'R$ is also another $SO(3)$

let $R'R = R''$

$tr(R''H)$ will alays $<$ $tr(RH)$

if $R = VU^T$

i.e, if RH can be written as AA^T

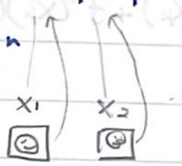
$tr(RH)$ will be max.

What's SVD

- for data reduction
- data-driven generalization of Fourier Transform (FFT)
- "tailored" to specific problem
- solve $Ax = b$
  for non-square A
- regression
- PCA
- correlation

$$\bar{X} = \begin{bmatrix} | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_m \\ | & | & | & & | \end{bmatrix}_{R^{n\times m}}$$

$x_k \in R^n$



$$\bar{X} = \begin{bmatrix} | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_m \\ | & | & | & & | \end{bmatrix}_{R^{n\times m}} = U\Sigma V^T,$$

eigen — importances — mixtures of U's to make X's

$$U\Sigma V^T = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \ \sigma_m \\ & 0 & \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}^T$$

eigen faces — order by importance — eigen time series, eigen mixtures

left singular vectors — singular values — right singular vectors

- u, v unitary — normal length, orthogonal

$UU^T = U^TU = I_{n\times n}$

$VV^T = V^TV = I_{m\times m}$

Σ diagonal $\sigma_1 \ge \sigma_2 \ge \cdots \sigma_m \ge 0$

$$\bar{X} = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_m \\ | & | & & | \end{bmatrix} = U\Sigma V^T = \overbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}}^{m} \overbrace{\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \ \sigma_m \\ & 0 & \end{bmatrix}}^{m\times m} \begin{bmatrix} -v_1^T- \\ -v_2^T- \\ \vdots \end{bmatrix}$$

guaranteed to exist, unique

$n \gg m$, usually

$= u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T \cdots + u_m\sigma_m v_m^T + 0$

$= \underbrace{\hat{U}}_{n\times m}\underbrace{\hat{\Sigma}}_{m\times m} V^T$ (economy SVD)

$[U, S, V]$ = svd(X, 'econ')

$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_m u_m v_m^T + 0$

$= \sigma_1 \, u_1 v_1^T + \sigma_2 \, u_2 v_2^T + \cdots + \sigma_m \, u_m v_m^T$ — rest negligible

truncate at rank r

$\hat{U}^T\hat{U} = I_{rr}$

$\hat{U}\hat{U}^T \ne I_{rr}$

$= \tilde{U}\tilde{\Sigma}\tilde{V}^T$ (Eckard-Young [1936] Theory)

$$\underset{\tilde{X} \text{ s.t. } \operatorname{rank}(\tilde{X}) = r}{\operatorname{argmin}} \|\bar{X} - \tilde{\bar{X}}\|_F = \tilde{U}\tilde{\Sigma}\tilde{V}^T$$

SVD + correlation

$$\bar{X}^T\bar{X} = {}_m\overset{n}{\begin{bmatrix} -x_1^T- \\ -x_2^T- \\ \vdots \end{bmatrix}} \; {}_n\underset{m}{\begin{bmatrix} | & | & \\ x_1 & x_2 & \cdots \\ | & | & \end{bmatrix}}$$

correlation matrix

$$= \begin{bmatrix} x_1^Tx_1 & x_1^Tx_2 & & x_1^Tx_m \\ x_2^Tx_1 & x_2^Tx_2 & \cdots & x_2^Tx_m \\ \vdots & \vdots & & \vdots \\ x_m^Tx_1 & x_m^Tx_2 & & x_m^Tx_m \end{bmatrix}$$

$x_i^Tx_j = \langle x_i, x_j \rangle$

$\bar{X} = \hat{U}\hat{\Sigma}V^T$

$\bar{X}^T = V\hat{\Sigma}\hat{U}^T$

$\bar{X}^T\bar{X} = V\hat{\Sigma}\hat{U}^T\hat{U}\hat{\Sigma}V^T$

$= V\hat{\Sigma}^2V^T$ (eigen decomposition)

$\bar{X}^T\bar{X}\underbrace{V}_{\text{eigenvectors}} = V\hat{\Sigma}^2 \rightarrow$ eigenvalues

$\bar{X}\bar{X}^T = \hat{U}\hat{\Sigma}V^TV\hat{\Sigma}\hat{U}^T = \hat{U}\hat{\Sigma}^2\hat{U}^T$

$\bar{X}\bar{X}^T\underbrace{\hat{U}}_{\text{eigenvectors}} = \hat{U}\hat{\Sigma}^2 \rightarrow$ eigenvales.

ICP

unknown data association

corresponding points $y_n, x_n \quad n = 1, \ldots, N$

weights $p_n \quad n = 1, \ldots, N$

find R, t s.t.

$\bar{x}_n = R x_n + t \quad n = 1, \ldots, N$

so that $\sum \|y_n - \bar{x}_n\|^2 p_n \rightarrow \min$

correspondencies unknown

guess $\longrightarrow$ point locations or point correspondences

$\downarrow$ iteration

ICP (Vanilla)

$\bar{x}_n = x_n$
$e = \infty$
while $e >$ thres
  c = correspondencies($\{y_n, \bar{x}_n\}$)
  (t, R) = compute $T(c)$
  $\bar{x}_n = R(x_n - x_0) + y_0$
  $e = E(t, R) = \Phi(R^Ty_0 - R^Tt, R)$
return $\{\bar{x}_n\}$

No.

Date

if easy → analytical form
→ through $\frac{dF}{dx} = 0$
→ else

Non-linear Least-Square Problem

$$\min_{X} F(x) = \frac{1}{2}\|f(x)\|_2^2 \qquad f(x): \mathbb{R}^n \to \mathbb{R}$$

objective: get X s.t. $F(x)$ has min

optimization algo

```
while true
    at i = k
    search ΔX_k
    get ||f(X_k + ΔX_k)||_2^2
    if ΔX_k < b
        return
    else
        continue
    X_{k+1} = X_k + ΔX_k
```

How to get ΔX_k?
A: different tricks
Here introduce

Gauss-Newton Method

$$f(x) = \frac{1}{2}\sum_{j=1}^{m} r_j(x)^2 = \frac{1}{2}\sum_{j=1}^{m}\|r(x)\|_2^2 \qquad \begin{cases} r_j(x) = \varphi(x; t_j) - y_j \\ j = 1, 2, 3 \dots m \\ r(x) = [r_1(x), r_2(x) \dots r_m(x)]^T \end{cases}$$

x is the objective parameter (e.g. pose) $X \in \mathbb{R}^n$

r is the residual result calculated from X (e.g. reprojection u, v) $m = 1, 2$

math preliminaries

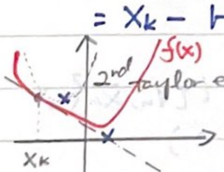
$$\nabla f(x_1, x_2) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) \qquad J(x) = \left[\frac{\partial r_j}{\partial x_i}\right]_{\substack{j=1\dots m \\ i=1\dots n}}$$

$$H(f(x_1, x_2)) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} \qquad = \begin{bmatrix} \nabla r_1(x)^T \\ \nabla r_2(x)^T \\ \vdots \\ \nabla r_m(x)^T \end{bmatrix}$$

recall Newton's Method

$$X_{k+1} = X_k + \Delta X_k$$

$$\Rightarrow X_{k+1} = X_k - [\nabla^2 f(X_k)]^{-1} \nabla f(X_k) = X_k - H^{-1} J_r(X_k)^T r(X_k)$$



2nd taylor expension $f(x) = f(X_k) + f'(X_k)(x - X_k) + \frac{1}{2} f''(X_k)(x - X_k)^2$

1st taylor expension $f(x) = f(X_k) + f'(X_k)(x - X_k)$

not a good value for ΔX_k, as it could over-shoot exagerattingly

much better. at least we hv a minimum value (quadratic)

1st order: $f(X_k) + f'(X_k)(x - X_k) = 0$

2nd order: $\underbrace{f(X_k) + f'(X_k)(x - X_k) + \frac{1}{2} f''(X_k)(x - X_k)^2}_{g(x)} = 0$

solve X value, and set it as next X_k.
Newton's method opt the second 2nd order.

$$\frac{d}{dx} g(x) = f'(X_k) + f''(X_k)(X - X_k)$$

$$X^* = \operatorname*{argmin}_{X} g(x)$$

$$\therefore 0 = \frac{d}{dx} g(x) = f'(X_k) + f''(X_k)\underbrace{(X - X_k)}_{= \Delta X}$$

$$\Delta X = \frac{-f'(X_k)}{f''(X_k)}$$

$$\therefore X_{k+1} = X_k - [\nabla^2 f(X_k)]^{-1} \nabla f(X_k) = X_k - H^{-1} J_r(X_k)^T r(X_k)$$

yet: still greedy
Hessian matrix not easy to solve

∴ **Gauss-Newton Method**

w/ $\nabla f = J_r^T r$

$$\nabla^2 f = J_r^T J_r + \sum_{i=1}^{m} r_i \nabla^2 r_i = J_r^T J_r + Q$$

$$\nabla^2 f \approx J_r^T J_r$$

substitute H as $J^T J$

$$\therefore X_{k+1} = X_k - \underbrace{[J_r(X_k)^T J_r(X_k)]^{-1} \cdot J_r(X_k)^T r(X_k)}_{\Delta X}$$

$$\therefore \boxed{\Delta X = -[J_r(X_k)^T J_r(X_k)]^{-1} J_r(X_k)^T r(X_k)}$$

$$f = \frac{1}{2}\sum_{j=1}^{m} r_j(x)^2 \qquad \frac{\partial f}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial r_j}{\partial x_i} r_j$$

1st order

$$\nabla f(x) = \nabla \frac{1}{2}\sum_{j=1}^{m}(r_j(x))^2 = \sum_{j=1}^{m} r_j(x) \nabla r_j(x) = J(x)^T r(x)$$

2nd order

$$\nabla^2 f(x) = \nabla \sum_{j=1}^{m} r_j(x) \nabla r_j(x) = \sum_{j=1}^{m} \nabla r_j(x) \nabla r_j(x)^T + \sum_{j=1}^{m} r_j(x) \nabla^2 r_j(x) = J(x)^T J(x) + \sum_{j=1}^{m} r_j(x) \nabla^2 r_j(x) = J(x)^T J(x) + Q(x)$$