- **optimization problem**

$$\underset{x}{\text{minimize}}\ f_0(x) \quad \text{s.t.}\ f_i(x) \le 0,\ h_i(x) = 0$$

- feasible: $x \in \text{dom } f_0$
- infeasible: $p^* = \infty / -\infty$
- optimal value
  $p^* = \min\{f_0(x) \mid f_i(x) \le 0,\ h_i(x) = 0\}$
  x^* is optimal
- locally optima
  $f_0(x) = \inf\{f_0(z) \mid z \in D,\ \|z-x\|_2 \le R\}$
  $D = \bigcap_{i=0}^{m} \text{dom} f_i \cap \bigcap_{i=0} \text{dom} h_i$
- feasibility problem

$$\text{find } x \quad \text{s.t. } f_i(x) \le 0,\ h_i(x) = 0 \iff \underset{x}{\text{minimize}}\ 0 \quad \text{s.t. } f_i(x)\le 0,\ h_i(x) = 0$$

- convex optimization

$$\underset{x}{\text{minimize}}\ f_0(x) \to \text{convex} \quad \text{s.t. } f_i(x) \le 0 \ \text{(convex)},\ a_i^T x = b_i \ \text{(affine)}$$

- quasiconvex optimization

$$\underset{x}{\text{minimize}}\ f_0(x) \rightsquigarrow \text{quasiconvex} \quad \text{s.t. } f_i(x) \le 0,\ a_i^T x = b_i$$

$S_\alpha = \{x \in D \mid f(x) \le \alpha\}$, $\alpha = R$, S_α = convex, $f(x) \in$ quasiconvex

- local & global optima

$$\underset{z}{\text{minimize}}\ f_0(z) \quad \text{s.t. } f_i(z) \le 0,\ h_i(z) = 0,\ \|z-x\|_2 \le R$$

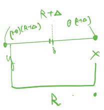
x is locally optimal when
$f_0(x) = \inf\{f_0(z) \mid z \text{ feasible},\ \|z-x\|_2 \le R\}$ — fall back here

in convex
if x is local optimal
then x is global optimal

proof:
△ if $\exists y$: $f_0(y) < f_0(x)$, $\|y-x\|_2 > R$ } as if $\|y-x\|_2 < R$

△ we let $\theta = \dfrac{R}{2\|y-x\|_2} < \dfrac{1}{2}$

△ we let
$z = (1-\theta)x + \theta y$
convex combination, in convex function
∴ $f_0(z) \le (1-\theta) f_0(x) + \theta f_0(y)$
$0 < \theta < \frac{1}{2}$



⇒ from here
$\|x-z\|_2 = \theta(R+\triangle)$
$\le \frac{1}{2}(R+\triangle)$
- we let $\|x-z\|_2 = \frac{1}{2}R$ to fulfil all the condition (we want z to be within $\|x-z\| < R$, s.t. x is local optimal)

△ w/ assumption $f_0(y) < f_0(x)$
we also hv
$\theta f_0(y) + (1-\theta) f_0(x) < f_0(x)$
⇒ $f_0(z) = \theta f_0(y) + (1-\theta) f_0(x) < f_0(x)$
contradicts w/
$f_0(x) = \inf\{f_0(z) \mid z \in D,\ \|z-x\|_2 \le R\}$

- optimality criterion
$\nabla f_0(x)^T (y-x) \ge 0 \quad \forall y \in D$

- **Geometric Programming**
- monomial function
  $f: R^n \to R$, $\text{dom} f = R^n_{++}$
  $f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$
  $c > 0,\ a_i \in R$
- posynomial function
  $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$
  $c_k > 0,\ a_{ik} \in R$
  sum of monomial function
- transfer to convex problem
  $y_i = \log x_i$, $x_i = e^{y_i}$
  $f(x) = f(e^{y_1}, \dots, e^{y_n})$
  $= c(e^{y_1})^{a_1} \cdots (e^{y_n})^{a_n}$
  $= e^{a^T y + b}$ ($b = \log c$)
  monomial

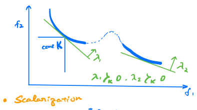
original:
$\text{minimize } f_0(x)$
$\text{s.t. } f_i(x) \le 1$, $h_i(x) = 1$
(mono, posy)

convex equivalence:
$\text{minimize } \log\left(\sum_{k=1}^{K} e^{a_{0k}^T y + b_{0k}}\right)$
$\text{s.t. } \log\left(\sum_{k=1}^{K} e^{a_{ik}^T y + b_{ik}}\right) \le 0$
$Gy + d = 0$

- **multicriterion optimization**

$$\text{minimize } f_0(x) = [F_1(x), \dots, F_q(x)] \quad \text{s.t. } f_i(x) \le 0,\ Ax = b$$

- Pareto optimal
$(f_0(x) - K) \cap = f_0(x)$



- Scalarization

$$\text{minimize } \lambda^T f_0(x) \quad \text{s.t. } f_i(x) \le 0,\ h_i(x) = 0$$

⇒ $\forall x^{po}$, $\exists \lambda \succ_{K^*} 0$
s.t. $\tilde{x}$ is an optimal point of the scalarized problem

- **Duality**
- The Lagrangian

$$\text{minimize } f_0(x) \quad \text{s.t. } f_i(x) \le 0,\ i = 1,\dots,m;\ h_i(x) = 0,\ i = 1,\dots,p$$

(λ_i, ν_i)

$\mathcal{L}(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$

- The Lagrange Dual Function
$g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu)$
∴ $g(\lambda, \nu) \le p^* = f_0(\tilde{x})$ (all feasible x) — lower bound
∴ $\lambda \succeq 0$, $(\lambda, \nu) \in \text{dom } g$ — dual feasible
- The Lagrange Dual Problem

$$\text{maximize } g(\lambda, \nu) \quad \text{s.t. } \lambda \succeq 0$$

- Weak Duality
$d^* \le p^*$
(best bound, optimal value)
△ = duality gap
- Strong Duality
- $d^* = p^*$
- Slater's condition
$\exists x \in \text{relint } D$
s.t. $f_i(x) < 0$, $Ax = b$

- optimality conditions — complementary slackness
assume strong duality holds:
$d^* = p^*$
$g(\lambda^*, \nu^*) = f_0(x^*)$
⇒
$f_0(x^*) = g(\lambda^*, \nu^*)$
$= \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x)\right)$
$\le f_0(x^*) + \underbrace{\sum_{i=1}^{m} \lambda_i^* f_i(x^*)}_{\ge 0,\ \le 0} + \underbrace{\sum_{i=1}^{p} \nu_i^* h_i(x^*)}_{= 0}$
$\le f_0(x^*)$

$\begin{cases} \text{if } \lambda_i^* > 0 \Rightarrow f_i(x^*) = 0 \\ \text{if } f_i(x^*) < 0 \Rightarrow \lambda_i^* = 0 \end{cases}$
complementary slackness!

∴ $\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0$

- KKT optimality conditions
- let x^* (primal), (λ^*, ν^*) (dual) be optimal points
- 0-duality gap
∴ $x^* = \underset{x}{\operatorname{argmin}}\ \mathcal{L}(x, \lambda^*, \nu^*)$

- $\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0$
∴
- $f_i(x^*) \le 0$
$h_i(x^*) = 0$
$\lambda_i^* \ge 0$
$\lambda_i^* f_i(x^*) = 0$

→ KKT optimality conditions!

- **Sufficiency & Necessity**

if P then Q

△ Q is necessary for P
① truth of Q is guaranteed by truth of P
② (falsity of Q guarantees falsity of P)

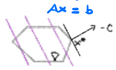
▲ P is sufficient for Q
③ truth of P guarantees truth of Q
④ but falsity of P might still hv truth of Q

如果"要选总统"(P) 那"要40岁"(Q)
e.g.
① "人一定是40岁"(Q) 如果"这人选总统"(P)
② 如果"未40岁"(Q) 那"一定非总统"(P)
③ 如果"要选总统"(P) 那"一定40岁"(Q)
④ 如果"不选总统"(P) 那"可能40岁"(Q)

sum:
if P then Q
△ Q is P's necessary condition
△ P is Q's sufficient condition

- **Convex problems**
- Linear Programming

$$\underset{x}{\text{minimize}}\ c^T x + d \quad \text{s.t. } Gx \preceq h,\ Ax = b$$



- piecewise-linear minimization
$f_0(x) = \max_{i=1,\dots,m} (a_i^T x + b_i)$

$$\underset{x}{\text{minimize}}\ f_0(x)$$

⇕

$$\text{minimize } t \quad \text{s.t. } a_i^T x + b_i \le t \quad i = 1,\dots,m$$

- Quadratic Program

$$\underset{x}{\text{minimize}}\ \tfrac{1}{2} x^T P x + q^T x + r \quad \text{s.t. } Gx \preceq h,\ Ax = b$$

$P \in S^n_+$

- Quadratically Constrained Quadratic Program

$$\underset{x}{\text{minimize}}\ \tfrac{1}{2} x^T P_0 x + q_0^T x + r_0 \quad \text{s.t. } \tfrac{1}{2} x^T P_i x + q_i^T x + r_i \le 0,\ Ax = b$$

$P_i \in S^n_+$ (cover a feasible region that's intersection of ellipsoids + affine set)

- Second-order Cone Programming

$$\underset{x}{\text{minimize}}\ f^T x$$

$\|A_i x + b_i\|_2 \le c_i^T x + d_i$
$Fx = g$
SOCP could be understood:
$\begin{bmatrix} A_x + b \\ c^T x + d \end{bmatrix} \in R^{k+1}$ lies in SOC
$(Ax + b,\ c^T x + d) \succeq_Q 0$
G.I. for cone Q

- e.g. Robust Linear Programming

- **Perturbation & Sensitivity Analysis**
- recall

$$\underset{x}{\text{minimize}}\ f_0(x) \quad \text{s.t. } f_i(x) \le 0,\ h_i(x) = 0$$

perturbed by u_i, v_i

$$\underset{x}{\text{minimize}}\ f_0(x) \quad \text{s.t. } f_i(x) \le u_i,\ h_i(x) = v_i$$

- optimal value (becomes a function) w.r.t. (u, v)
$p^*(u, v) = \inf\left\{f_0(x) \,\middle|\, \exists x \in D,\ f_i(x) \le u_i,\ h_i(x) = v_i\right\}$
- let (λ^*, ν^*) → dual optimal pt. of unperturbed problem
⇒ $p^*(0,0) = g(\lambda^*, \nu^*)$
$\le f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x)$
$\le f_0(x) + \lambda^{*T} u + \nu^{*T} v$
⇒ $p^*(0,0) \le f_0(x) + \lambda^{*T} u + \nu^{*T} v$
⇒ $f_0(x) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$
⇒ $p^*(u, v) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$
$(p^*(u,v) \in \{f_0(x) \mid x \in \text{dom} f\})$

$p^*(u, v) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$
perturbation analysis

- local sensitivity
assume
- $p^*(u, v)$ differentiable @ $u = 0$, $v = 0$
- strong duality holds

⇒ $p^*(u, v)$ @ $u = 0, v = 0$
$\dfrac{\partial p^*(0,0)}{\partial u_i} = -\lambda_i^*$

$\dfrac{\partial p^*(0,0)}{\partial v_i} = -\nu_i^*$

- w/ complementary slackness:
$\lambda_i f_i(x) = 0$
- $f_i(x) < 0$, $\lambda_i = 0$
then: value @ $p^*(0,0)$ doesn't change w.r.t. u
- $f_i(x) = 0$, $\lambda_i^* > 0$
then: value @ $p^*(0,0)$ changes w.r.t. u
$\lambda_i^* \uparrow$ → perturb ↑
$\lambda_i^* \downarrow$ → perturb ↓

→ tells us HOW ACTIVE IT IS

• Numerical Method of Differential Equations

• ODE 4 types

1 separable equations

$$\frac{dy}{dx}=P(x)Q(x)$$

e.g. $y'+2xy=x$

$\Rightarrow \frac{dy}{dx}+2xy=x$

$\Rightarrow \frac{dy}{dx}=x-2xy$

$\Rightarrow \frac{dy}{dx}=x(1-2y)$

$\Rightarrow dy=x(1-2y)dx$

$\Rightarrow \frac{1}{1-2y}dy=x\,dx$

$\Rightarrow \int\frac{1}{1-2y}dy=\int x\,dx$

$\Rightarrow -\frac{1}{2}\ln(1-2y)=\frac{1}{2}x^2+C$

$\Rightarrow e^{-\frac{1}{2}\ln(1-2y)}=e^{\frac{1}{2}x^2+C}$

$\Rightarrow 1-2y=Ce^{-x^2}$

$\Rightarrow y=-\frac{Ce^{-x^2}-1}{2}$

2. homogenous method

$$f(kx,ky)=f(x,y)$$

e.g. $\frac{dy}{dx}=\frac{x^2+y^2}{xy}$

check $\frac{k^2x^2+k^2y^2}{kxky}=\frac{x^2+y^2}{xy}$

let $v=\frac{y}{x}$ $\quad \frac{1}{v}=\frac{x}{y}$

$y=vx$

$\frac{dy}{dx}=x\frac{dv}{dx}+v$

• $\frac{dy}{dx}=\frac{\frac{x^2}{x^2}+\frac{y^2}{x^2}}{\frac{xy}{x^2}}=\frac{1+(\frac{y}{x})^2}{\frac{y}{x}}$

$=\frac{1+v^2}{v}$

$\Rightarrow x\frac{dv}{dx}+v=\frac{1+v^2}{v}$

$\Rightarrow x\frac{dv}{dx}+v=\frac{1}{v}+v$

$\Rightarrow x\frac{dv}{dx}=\frac{1}{v}$

$\Rightarrow v\,dv=\frac{1}{x}dx$

$\Rightarrow \int v\,dv=\int\frac{1}{x}dx$

$\Rightarrow \frac{1}{2}v^2=\ln(x)+C$

$\Rightarrow v^2=2\ln|x|+C$

$v=\pm\sqrt{2\ln|x|+C}$

$\frac{y}{x}=\pm\sqrt{2\ln|x|+C}$

$y=\pm x\sqrt{2\ln|x|+C}$

3. Integrating factor

$$\frac{dy}{dx}+P(x)y=Q(x)$$

e.g. $\frac{dy}{dx}+1y=x$ $\quad P(x)=1$

$\Leftrightarrow \frac{dy}{dx}+P(x)y=1$ ①

$\mu(x)=e^{\int P(x)dx}$

$\mu(x)=e^{\int 1dx}$

$=e^x$ ②

$\Rightarrow$ ① & ②

$\mu(x)\left[\frac{dy}{dx}+P(x)y=Q(x)\right]$

$\Rightarrow e^x\left[\frac{dy}{dx}+y=x\right]$

$\Rightarrow e^x\frac{dy}{dx}+ye^x=xe^x$

$\Rightarrow \frac{d}{dx}(e^xy)=xe^x$

$\Rightarrow \int\frac{d}{dx}e^xy=\int xe^x dx$ (IBP)

$\Rightarrow e^xy=xe^x-e^x+C$

$\Rightarrow y=x-1+\frac{C}{e^x}$

$=x-1+Ce^{-x}$

• $\frac{dv}{dt}=g-\frac{c}{m}v$

$$\frac{dv}{dt}+P(t)V=Q(t)$$

$\Rightarrow \frac{dv}{dt}+\left(\frac{c}{m}\right)V=g$

$\Rightarrow \mu(t)=e^{\int\frac{c}{m}dt}$

$=e^{\frac{c}{m}t}$

$\Rightarrow e^{\frac{c}{m}t}\left(\frac{dv}{dt}+\left(\frac{c}{m}\right)v=g\right)$

$\Rightarrow e^{\frac{c}{m}t}\frac{dv}{dt}+\frac{c}{m}ve^{\frac{c}{m}t}=ge^{\frac{c}{m}t}$

$\Rightarrow \frac{d}{dt}e^{\frac{c}{m}t}v=ge^{\frac{c}{m}t}$

$\Rightarrow \int\frac{d}{dt}e^{\frac{c}{m}t}v=\int ge^{\frac{c}{m}t}dt$

$\Rightarrow e^{\frac{c}{m}t}v=\frac{m}{c}ge^{\frac{c}{m}t}+c'$

$\Rightarrow v=e^{-\frac{c}{m}t}\left[\frac{m}{c}ge^{\frac{c}{m}t}+c'\right]$

assume $V(0)=0$

$v(0)=+\frac{m}{c}g+c'=0$

$c'=-\frac{m}{c}g$

$\therefore v(t)=e^{-\frac{c}{m}t}\left[\frac{m}{c}ge^{\frac{c}{m}t}-\frac{m}{c}g\right]$

$=e^{-\frac{c}{m}t}\frac{m}{c}g\left[e^{\frac{c}{m}t}-1\right]$

$=\frac{mg}{c}\left[1-e^{-\frac{c}{m}t}\right]$

• linear ODE

- Laplace
- analytically solved
- $a_0(x)y+a_1(x)y'+\dots+a_n(x)y^{(n)}=b(x)$
- $Ly=f$

• Runge-Kutta Methods

Δ Taylor series

$\frac{dy}{dx}=f(x,y)$

$\frac{d^{(n)}y}{dx^{(n)}}=f^{(n-1)}(x,y)$

$y_{i+1}=y_i+f(x_i,y_i)h$ Euler's Method

Δ $y_{i+1}=y_i+f(x_i,y_i)h+\frac{f'(x_i,y_i)}{2!}h^2+\dots+\frac{f^{(n-1)}(x_i,y_i)}{n!}h^n+O(h^{n+1})$

$\simeq y_i+\left[f(x_i,y_i)+\frac{f'(x_i,y_i)}{2!}h+\dots+\frac{f^{(n-1)}(x_i,y_i)}{n!}h^{n-1}\right]h$

$=y_i+\phi(x_i,y_i,h)h$

Δ $\phi(x_i,y_i,h)$

$=f(x_i,y_i)+\frac{f'(x_i,y_i)}{2!}h+\dots+\frac{f^{(n-1)}(x_i,y_i)}{n!}h^{n-1}$

Δ Runge-Kutta

acquire $\phi(x_i,y_i,h)$

w.o/ calculating higher-order term

$y_{i+1}=y_i+\phi(x_i,y_i,h)h$

$\Rightarrow \phi(x_i,y_i,h)$

$=a_1k_1+a_2k_2+\dots+a_{n-1}k_{n-1}+a_nk_n$

$k_1=f(x_i,y_i)$

$k_2=f(x_i+p_1h,\ y_i+q_{11}k_1h)$

$k_3=f(x_i+p_2h,\ y_i+q_{21}k_1h+q_{22}k_2h)$

$\vdots$

$k_n=f(x_i+p_{n-1}h,\ y_i+q_{n-1,1}k_1h+q_{n-1,2}k_2h+\dots+q_{n-1,n-1}k_{n-1}h)$

$a,p,q\Rightarrow$ constants

$n=$ order $\Rightarrow$ RK n method!

e.g.

Δ $y=-0.5x^4+4x^3-10x^2+8.5x+1$

$\frac{dy}{dx}=-2x^3+12x^2-20x+8.5$

$f(0)=1$

$f(0.5)=?$

Δ RK4:

$y_{i+1}=y_i+\phi(x_i,y_i,h)h$

$\phi(x_i,y_i,h)$

$=\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$

$k_1=f(x_i,y_i)$

$k_2=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_1h)$

$k_3=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_2h)$

$k_4=f(x_i+h,\ y_i+k_3h)$

Δ ∴

$k_1=f(0,1)=8.5$

$k_2=f(0+0.25,\ 1+0.5\cdot8.5\cdot0.5)=4.21875$

$k_3=f(0+0.25,\ 1+0.5\cdot4.21875\cdot0.5)=4.21875$

$k_4=f(0.5,\ 1+4.21875\cdot0.5)=1.25$

$y(0.5)=1+\frac{0.5}{6}(8.5+2\cdot4.21875+2\cdot4.21875+1.25)$

$=3.21875+O(h^5)$

• system of equations

$$\begin{cases}\frac{dy_1}{dx}=f_1(x,y_1,y_2,\dots,y_n)\\ \frac{dy_2}{dx}=f_2(x,y_1,y_2,\dots,y_n)\\ \vdots\\ \frac{dy_n}{dx}=f_n(x,y_1,y_2,\dots,y_n)\end{cases}$$

e.g.

Δ $\begin{cases}\frac{dy_1}{dx}=-0.5y_1\\ \frac{dy_2}{dx}=4-0.3y_2-0.1y_1\end{cases}$

$y_1=4$ @ $x=0$
$y_2=6$

Δ $\dot{y}=f(x)$

$$=\begin{bmatrix}\frac{dy_1}{dx}\\ \frac{dy_2}{dx}\end{bmatrix}=\begin{bmatrix}-0.5 & 0\\ -0.1 & -0.3\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}+\begin{bmatrix}0\\ 4\end{bmatrix}$$

Δ RK4

$y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$

$k_1=f(x_i,y_i)$

$k_2=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_1h)$

$k_3=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_2h)$

$k_4=f(x_i+h,\ y_i+k_3h)$

Δ $k_{1,1}=f_1(0,4)=-0.5\cdot4=-2$

$k_{1,2}=f_2(0,6)=4-0.3\cdot6-0.1\cdot4=1.8$

Δ $k_{2,1}=f_1(0+\frac{1}{2}\cdot0.5,\ 4+\frac{1}{2}\cdot(-2)\cdot0.5)$

$=-0.5\cdot3.5=-1.75$

$k_{2,2}=f_2(0+\frac{1}{2}0.5,\ 6+\frac{1}{2}\cdot(1.8)\cdot0.5)$

$=4-0.3\cdot6.45-0.1\cdot3.5$

$=1.715$

Δ $k_{3,1}=f_1(0+\frac{1}{2}0.5,\ 4+\frac{1}{2}(-1.75)\cdot0.5)$

$=-0.5\cdot3.5625$

$=-1.78125$

$k_{3,2}=f_1(0+\frac{1}{2}\cdot0.25,\ 6+\frac{1}{2}(1.715)\cdot0.5)$

$=4-0.3\cdot6.42875-0.1\cdot3.5625$

$=1.715125$

Δ $k_{4,1}=f_1(0+0.5,\ 4+(-1.78125)\cdot0.5)$

$=-0.5\cdot3.109375$

$=-1.5546875$

$k_{4,2}=f_2(0+0.5,\ 6+(1.715125\cdot0.5))$

$=4-0.3\cdot6.8575625-0.1\cdot3.109375$

$=1.63179375$

Δ $y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$

$y=\begin{bmatrix}4\\6\end{bmatrix}+\frac{0.5}{6}\left(\begin{bmatrix}-2\\1.8\end{bmatrix}+2\begin{bmatrix}-1.75\\1.715\end{bmatrix}+2\begin{bmatrix}-1.78125\\1.715125\end{bmatrix}+\begin{bmatrix}-1.5546875\\1.63179375\end{bmatrix}\right)$

$=\begin{bmatrix}3.115234\\6.857670\end{bmatrix}$

• solving high-order ODE

$$\begin{cases}\frac{d^2y}{dx^2}+2\frac{dy}{dx}=x+y\\ y=1\quad \frac{dy}{dx}=2\quad \text{when } x=0\end{cases}$$

$\begin{cases}\text{let}\quad u=\frac{dy}{dx}\\ \frac{du}{dx}=\frac{d^2y}{dx^2}\end{cases}$

$\begin{cases}\Rightarrow \frac{du}{dx}+2u=x+y\\ \Rightarrow \frac{du}{dx}=x+y-2u\end{cases}$

$\begin{cases}\frac{dy}{dx}=u\\ \frac{du}{dx}=x+y-2u\end{cases}$ $\quad y(0)=1$, $u(0)=2$

use this to propagate

$\begin{cases}\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}+3\frac{dy}{dx}=x+y\\ y=1\quad \frac{dy}{dx}=2\quad \frac{d^2y}{dx^2}=3\quad x=0\end{cases}$

$\begin{cases}\frac{dy}{dx}=u\\ \frac{d^2y}{dx^2}=v\end{cases}$ $\quad \frac{d^2y}{dx^2}=\frac{du}{dx}=v$, $\quad \frac{d^3y}{dx^3}=\frac{dv}{dx}$

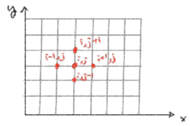
$\begin{cases}\frac{dv}{dx}+2v+3u=x+y\\ \frac{dy}{dx}=u\\ \frac{du}{dx}=v\end{cases}$ $\quad y(0)=1$, $u(0)=2$, $v(0)=3$, $x=0$

use this to propagate

• PDE

• Laplace equation

$$\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}=0$$



$$\frac{\partial^2T}{\partial x^2}=\frac{\left(\frac{T_{i+1,j}-T_{i,j}}{\Delta x}\right)-\left(\frac{T_{i,j}-T_{i-1,j}}{\Delta x}\right)}{\Delta x}$$

$$=\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}$$

$$\frac{\partial^2T}{\partial y^2}=\frac{\left(\frac{T_{i,j+1}-T_{i,j}}{\Delta y}\right)-\left(\frac{T_{i,j}-T_{i,j-1}}{\Delta y}\right)}{\Delta y}$$

$$=\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}$$

$$\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}=0$$

$\Rightarrow \frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}=0$

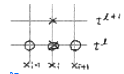
As $\Delta x=\Delta y$

$\Rightarrow T_{i+1,j}+T_{i-1,j}+T_{i,j+1}+T_{i,j-1}-4T_{i,j}=0$

$\Rightarrow T_{i,j}=\frac{T_{i+1,j}+T_{i-1,j}+T_{i,j+1}+T_{i,j-1}}{4}$

• Parabolic Equation

$$k\frac{\partial^2T}{\partial x^2}=\frac{\partial T}{\partial t}$$



$$k\frac{\partial^2T}{\partial x^2}=\frac{\partial T}{\partial t}$$

$$\frac{\partial^2T}{\partial x^2}=\frac{\frac{T_{i+1}^l-T_i^l}{\Delta x}-\frac{T_i^l-T_{i-1}^l}{\Delta x}}{\Delta x}$$

$$\frac{\partial T}{\partial t}=\frac{T_i^{l+1}-T_i^l}{\Delta t}$$

$\Rightarrow k\frac{T_{i+1}^l-2T_i^l+T_{i-1}^l}{\Delta x^2}=\frac{T_i^{l+1}-T_i^l}{\Delta t}$

$\Rightarrow T_i^{l+1}-T_i^l=\frac{k\Delta t}{\Delta x^2}(T_{i+1}^l-2T_i^l+T_{i-1}^l)$

$\Rightarrow T_i^{l+1}=T_i^l+\lambda(T_{i+1}^l-2T_i^l+T_{i-1}^l)$

$\lambda=\frac{k\Delta t}{\Delta x^2}$

e.g.

$\Delta x=2$ cm

$\Delta t=0.1$ s

$k=0.835$ cm²/s

$\lambda=0.020875$

$\begin{cases}T(0)=100 & @\ \forall t\\ T(10)=50 & \\ T(x)=0 & @\ t=0,\ 0<x<10\end{cases}$

position

$T_i^{l+1}=T_i^l+\lambda(T_{i+1}^l-2T_i^l+T_{i-1}^l)$

Δ compute T @ $t=0.1$ s, $x=2$ cm

time

$T_1^1=T_1^0+0.020875(T_2^0-2T_1^0+T_0^0)$

position

$=0+0.020875(0-2\cdot0+100)$

$=2.0875$

Δ compute T @ $t=0.1$ s, $x=4$ cm

$T_2^1=T_2^0+0.020875(T_3^0-2T_2^0+T_1^0)$

$=0+0.020875(0-2\cdot0+0)$

$=0$

Δ compute T @ $t=0.1$s, $x=6$cm

$T_3^1=T_3^0+0.020875(T_4^0-2T_3^0+T_2^0)$

$=0+0.020875(0-2\cdot0+0)$

$=0$

Δ compute T @ $t=0.1$s, $x=8$ cm

$T_4^1=T_4^0+0.020875(T_5^0-2T_4^0+T_3^0)$

$=0+0.020875(50-0+0)$

$=1.0438$

Δ Compute T @ $t=0.2$s, $x=2$cm

$T_1^2=T_1^1+\lambda(T_2^1-2T_1^1+T_0^1)$

$=2.0875+0.020875(0-2\cdot2.0875+100)$

$=4.0878$

• convergence

$\lambda=\frac{k\Delta t}{\Delta x^2}\leq\frac{1}{2}\Rightarrow\Delta t\leq\frac{1}{2}\frac{\Delta x^2}{k}$

• $\frac{\partial T}{\partial t}=k\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}\right)$

$$\begin{cases}\frac{\partial^2T}{\partial x^2}=\frac{T_{i+1,j}^l-2T_{i,j}^l+T_{i-1,j}^l}{\Delta x^2}\\ \frac{\partial^2T}{\partial y^2}=\frac{T_{i,j+1}^l-2T_{i,j}^l+T_{i,j-1}^l}{\Delta y^2}\\ \frac{\partial T}{\partial t}=\frac{T_{i,j}^{l+1}-T_{i,j}^l}{\Delta t}\end{cases}$$

$k\left(\frac{T_{i+1,j}^l-2T_{i,j}^l+T_{i-1,j}^l}{\Delta x^2}+\frac{T_{i,j+1}^l-2T_{i,j}^l+T_{i,j-1}^l}{\Delta y^2}\right)$

$=\frac{T_{i,j}^{l+1}-T_{i,j}^l}{\Delta t}$

$\Rightarrow T_{i,j}^{l+1}=T_{i,j}^l+\frac{k\Delta t}{\Delta x^2}(T_{i+1,j}^l-2T_{i,j}^l+T_{i-1,j}^l)$
$+\frac{k\Delta t}{\Delta y^2}(T_{i,j+1}^l-2T_{i,j}^l+T_{i,j-1}^l)$

let $\Delta x=\Delta y$ $\quad \lambda=\frac{k\Delta t}{\Delta x^2}=\frac{k\Delta t}{\Delta y^2}$

$\Rightarrow T_{i,j}^{l+1}=T_{i,j}^l+\lambda(T_{i+1,j}^l-2T_{i,j}^l+T_{i-1,j}^l$
$+T_{i,j+1}^l-2T_{i,j}^l+T_{i,j-1}^l)$

• convergence

• 1-dimension:

$$\lambda=\frac{k\Delta t}{\Delta x^2}\leq\frac{1}{2}$$

• 2-dimension:

$$\Delta t\leq\frac{1}{8}\frac{\Delta x^2+\Delta y^2}{k}$$

⇓

if $\Delta x^2=\Delta y^2$

$$\Delta t\leq\frac{1}{4}\frac{\Delta x^2}{k}$$

- Linear Programming
- Example problem

a_1 origin 1, costs c_{ij}, terminal 1 b_1, terminal 2 b_2, terminal 3 b_3

a_2 origin 2

minimize $[c_{11}\ c_{12}\ c_{13}]\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\end{bmatrix}+[c_{21}\ c_{22}\ c_{23}]\begin{bmatrix}x_{21}\\x_{22}\\x_{23}\end{bmatrix}$

s.t. $\begin{cases}x_{11}+x_{12}+x_{13}=a_1\\x_{21}+x_{22}+x_{23}=a_2\end{cases}$ $\begin{cases}x_{11}+x_{21}=b_1\\x_{12}+x_{22}=b_2\\x_{13}+x_{23}=b_3\end{cases}$ $x_{ij}\ge0$

⇒ minimize$_x$ $[c_{11}\,c_{12}\,c_{13}\,c_{21}\,c_{22}\,c_{23}]\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\\x_{21}\\x_{22}\\x_{23}\end{bmatrix}$

s.t. $\begin{bmatrix}1&1&1&0&0&0\\0&0&0&1&1&1\\1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\end{bmatrix}\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\\x_{21}\\x_{22}\\x_{23}\end{bmatrix}=\begin{bmatrix}a_1\\a_2\\b_1\\b_2\\b_3\end{bmatrix}$, $x\ge0$

⇒ minimize$_x$ c^Tx s.t. $Ax=b$, $x\ge0$

- Transform to standard form

minimize$_x$ c^Tx s.t. $Ax\le b$, $x\ge0$ ⇕ minimize$_{x,y}$ c^Tx s.t. $Ax+y=b$, $x\ge0$, $y\ge0$

minimize$_x$ c^Tx s.t. $Ax\ge b$, $x\ge0$ ⇕ minimize$_{x,y}$ c^Tx s.t. $Ax-y=b$, $x\ge0$, $y\ge0$

minimize$_{x,y}$ c^Tx+y s.t. $Ax+y=b$, $x\ge0$ (no $y\ge0$)

⇓ let $y=u-v$

minimize$_{x,u,v}$ c^Tx+u-v s.t. $Ax+u-v=b$, $x\ge0$ $u\ge0$ $v\ge0$

minimize$_{x,y,z}$ c^Tx+y+z s.t. $Ax+y+z=b$, $y+z=d$, $x\ge0$ $y\ge0$ (no $z\ge0$)

⇕ let $z=b-Ax-y$

minimize$_{x,y}$ $c^Tx+y+b-Ax-y$ s.t. $y+b-Ax-y=d$, $x\ge0$ $y\ge0$

⇕

minimize$_{x,y}$ $c^Tx+b-Ax$ s.t. $b-Ax=d$, $x\ge0$ $y\ge0$

- Solution - Simplex method

高中線性歸化

- Solution - Basic Feasible solution (BFS)

minimize$_x$ c^Tx s.t. $Ax=b$, $x\ge0$

let $\tilde{x}$ be
1. $\tilde{x}\in D$, i.e., $A\tilde{x}=b$, $\tilde{x}\ge0$
2. cols of A corresponding to non-zero of $\tilde{x}$ are linearly independent

Theorem: $\tilde{x}$ is an extreme (rank(A) = m) i.f.f it is BFS

1. Basic/non-Basic variables
- for any BFS $\tilde{x}$, we rewrite:

$\tilde{x}'=\begin{pmatrix}x_B\\x_N\end{pmatrix}$ B: non-zero

$A=(B, N)$ N: zero

$c=\begin{pmatrix}c_B\\c_N\end{pmatrix}$

- $Ax=b$ ⇕ $Bx_B+Nx_N=b$ ⇕ $Bx_B=b$; rank(B) $=m$, $x_B=B^{-1}b$, $x_N=0^{n-m}$

- $Bx_B+Nx_N=b$

$\begin{cases}x_B=B^{-1}(b-Nx_N)\ge0^{m}\\x_N\ge0^{n-m}\end{cases}$

2. minimize$_x$ c^Tx s.t. $Ax=b$, $x\ge0^n$ ⟺ minimize$_x$ $c_B^Tx_B+c_N^Tx_N$ s.t. $Bx_B+Nx_N=b$, $x_B\ge0^{m}$, $x_N\ge0^{n-m}$

⟺ minimize$_x$ $c_B^T(B^{-1}(b-Nx_N))+c_N^Tx_N$ s.t. $B^{-1}(b-Nx_N)\ge0^{m}$, $x_N\ge0^{n-m}$

- Simplex method: the Tableau

minimize x_1-x_2 s.t. $x_1-x_2\le2$, $x_1+x_2\le6$, $x_1\ge0$, $x_2\ge0$

$x_1=0$, $x_2=6$, $x_3=8$, $x_4=0$

⇕

minimize $P=x_1-x_2$ s.t. $x_1-x_2+s_1=2$, $x_1+x_2+s_2=6$, $x_1\ge0$, $x_2\ge0$, $s_1\ge0$, $s_2\ge0$

$-x_1+x_2=P$

$-P-x_1+x_2=0$

⇒ $P+x_1-x_2=0$

$$\begin{array}{cccccc}x_1&x_2&s_1&s_2&P&\\\end{array}\left[\begin{array}{ccccc|c}1&-1&1&0&0&2\\1&1&0&1&0&6\\1&-1&0&0&1&0\end{array}\right]\begin{array}{c}-2\\6\\\end{array}$$

⇓

$$\begin{bmatrix}2&0&1&1&0&8\\1&1&0&1&0&6\\1&-1&0&0&1&0\end{bmatrix}$$

⇓

$$\begin{bmatrix}2&0&1&1&0&8\\1&1&0&1&0&6\\2&0&0&1&1&6\end{bmatrix}$$

maximize $P=5x+4y$ s.t. $3x+5y\le78$, $4x+y\le36$, $x,y\ge0$

⇓

maximize$_{x,y,s_1,s_2}$ $P=5x+4y$ s.t. $3x+5y+s_1=78$, $4x+y+s_2=36$, $x,y,s_1,s_2\ge0$

$$\begin{array}{cccccc}x&y&s_1&s_2&P&\\\end{array}\begin{bmatrix}3&5&1&0&0&78\\4&1&0&1&0&36\\-5&-4&0&0&1&0\end{bmatrix}\begin{array}{c}26\\9\\\end{array}$$

⇓

$$\begin{bmatrix}3&5&1&0&0&78\\1&1/4&0&1/4&0&9\\-5&-4&0&0&1&0\end{bmatrix}$$

⇓

$$\begin{bmatrix}0&17/4&1&-3/4&0&51\\1&1/4&0&1/4&0&9\\0&-11/4&0&5/4&1&45\end{bmatrix}\begin{array}{c}12\\36\\\end{array}$$

⇓

$$\begin{bmatrix}0&1&4/17&-3/17&0&12\\1&1/4&0&1/4&0&9\\0&-11/4&0&5/4&1&45\end{bmatrix}$$

⇓

$$\begin{bmatrix}0&1&4/17&-3/17&0&12\\1&0&-1/17&5/17&0&6\\0&0&11/17&13/17&1&78\end{bmatrix}$$

$x=6$

$y=12$

$P=78$

- Integer/Mixed-Integer Linear Programming

ILP/MILP

minimize$_x$ c^Tx

s.t. $Ax=b$, $x\ge0$, $x\in\{x\mid x_i\in\mathbb{Z}, i\in\mathcal{I}\}$

- Solution - Branch & Bound
- Branch: partition D onto subsets (check objective value in subsets)
- Bound: acquire a bound w.o./ integer constraints

e.g.

maximize$_x$ $9x_1+5x_2+6x_3$

s.t. $6x_1+3x_2+5x_3\le10$, $-x_1+x_3\le0$, $x_i\in\{0,1\}$ $i=1,2,3$

⇕

maximize$_x$ $9x_1+5x_2+6x_3$

s.t. $6x_1+3x_2+5x_3\le10$, $-x_1+x_3\le0$, $x_i\le1$ $i=1,2,3$, $x_i\ge0$ $i=1,2,3$, $x_i\in\mathbb{Z}$ $i=1,2,3$

△ Bound:
- w.o./ $\mathbb{Z}$ constraints: $x_1=1$, $x_2=1$, $x_3=0.2$, $P=15.2$ (UB)
- Rounded down solution: $x_1=1$, $x_2=1$, $x_3=0$, $P=14$ (LB)

△ Branch

UB 15.2 (1, 1, 0.2)

LB 14 (1, 1, 0)

$x_3=0$: feasible; $x_3=1$: infeasible

UB 14 (1,1,0)

LB 14 (1,1,0)

no need to check hv

UB=LB

- Solutions to Optimization Problems.
- recall Duality

minimize$_x$ $f_0(x)$ s.t. $f_i(x)\le0$, $h_i(x)=0$

△ Lagrangian:

$\mathcal{L}(x,\lambda,\nu)=f_0(x)+\sum_{i=1}^m\lambda_if_i(x)+\sum_{i=1}^p\nu_ih_i(x)$

△ Lagrangian Dual function:

let $g(\lambda,\nu)=\inf_{x\in D}\{\mathcal{L}(x,\lambda,\nu)\}$

maximize$_{\lambda,\nu}$ $g(\lambda,\nu)$ s.t. $\lambda\ge0$

△ let x^*,λ^*,ν^*, $d^*=g(\lambda^*,\nu^*)$ be optimal

$p^*\ge d^*$

△ Strong duality

Slater's condition

$\exists x\in$ relint D s.t. $f_i(x)<0$, $h_i(x)=0$

△ complementary slackness

$f_0(x^*)=g(\lambda^*,\nu^*)$

$=\inf_x f_0(x)+\sum_{i=1}^m\lambda_i^*f_i(x)+\sum_{i=1}^p\nu_i^*h_i(x)$

$\le f_0(x^*)+\sum_{i=1}^m\lambda_i^*f_i(x^*)+\sum_{i=1}^p\nu_i^*h_i(x^*)$

$\le f_0(x^*)$

$\lambda_i^*f_i(x^*)=0$

⇒ $\begin{cases}f_i(x^*)<0\to\lambda_i^*=0\\\lambda_i^*>0\to f_i(x^*)=0\end{cases}$

△ KKT optimality conditions

$\begin{cases}f_i(x^*)\le0\\h_i(x^*)=0\\\lambda^*\ge0\\\lambda_i^*f_i(x^*)=0\\\nabla f_0(x^*)+\sum_{i=1}^m\lambda_i^*\nabla f_i(x^*)+\sum_{i=1}^p\nu_i^*\nabla h_i(x^*)=0\end{cases}$

this will be used later

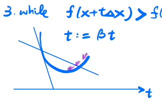
- Descent Method
1. given $x^{(0)}\in$ dom f
2. repeat
   1) descent direction Δx
   2) stepsize $t\ge0$
   3) $x^{k+1}=x^k+t^k\Delta x^k$
3. Stop $\|\Delta f(x)\|_2\le\eta$

- Line Search
- Exact Line Search

$t=\arg\min_{s\ge0} f(x+s\Delta x)$

- Back-tracking Line Search
1. given Δx^k, $f(x^k)$, $\alpha\in(0, 0.5)$, $\beta\in(0,1)$
2. set $t:=1$
3. while $f(x+t\Delta x)>f(x)+\alpha t\nabla f(x)^T\Delta x$

$t:=\beta t$



- Primal - dual Interior-point method

minimize $tf_0(x)+\phi(x)$ s.t. $Ax=b$

KKT:

$\begin{cases}Ax^*(t)=b\\t\nabla f_0(x^*(t))+\nabla\phi(x^*(t))+A^T\hat{\nu}=0\end{cases}$

⇒ $\begin{cases}Ax^*(t)=b\\t\nabla f_0(x^*(t))+\sum_{i=1}^m\frac{1}{-f_i(x^*(t))}\nabla f_i(x^*(t))+A^T\hat{\nu}=0\end{cases}$

⇒ $\begin{cases}Ax=b\\\nabla f_0(x)+\sum_{i=1}^m-\frac{1}{f_i(x)t}\nabla f_i(x)+A^T\nu=0\end{cases}$ $\frac{\hat{\nu}}{t}=\nu$

⇒ $\begin{cases}\nabla f_0(x)+\sum_{i=1}^m\lambda_i\nabla f_i(x)+A^T\nu=0\\-\lambda_if_i(x)=1/t\\Ax=b\end{cases}$

⇒ $r_t(x,\lambda,\nu)=\begin{bmatrix}\nabla f_0(x)+Df(x)^T\lambda+A^T\nu\\-\text{diag}(\lambda)f(x)-1/t\,\mathbf{1}\\Ax-b\end{bmatrix}$ r_{dual}, r_{cent}, r_{primal}

⇒ $r_t(x+\Delta x,\lambda+\Delta\lambda,\nu+\Delta\nu)=\begin{bmatrix}\nabla^2f_0(x)+\sum_{i=1}^m\lambda_i\nabla^2f_i(x)&Df(x)^T&A^T\\-\text{diag}(\lambda)Df(x)&-\text{diag}(f(x))&0\\A&0&0\end{bmatrix}\begin{bmatrix}\Delta x\\\Delta\lambda\\\Delta\nu\end{bmatrix}=\begin{bmatrix}-r_{dual}\\-r_{cent}\\-r_{pri}\end{bmatrix}$

- Unconstrained problem
- Gradient descent
1. given $x_0\in$ dom f
2. repeat
   - $\Delta x^k=-\nabla f(x)$
   - t: from line search
   - $x^{k+1}=x^k+t\Delta x^k$
3. until $\|\nabla f(x)\|\le\eta$

- Newton's method
1. given: $x^0\in$ dom f
2. repeat:

$\Delta x^k=-\nabla^2f(x^k)^{-1}\nabla f(x^k)$

$\lambda^{k^2}=\nabla f(x^k)\nabla^2f(x^k)\nabla f(x^k)$

t: from line search

$x^{k+1}=x^k+t\Delta x$

3. until $\lambda^{k^2}/2\le\epsilon$

on Newton's Method

$f(x)$

⇒ $\hat{f}(x+v)=f(x)+\nabla f(x)^Tv+\frac{1}{2}v^T\nabla^2f(x)v$

⇒ $\nabla\hat{f}(x+v)=0$

⇒ $\nabla f(x)+\nabla^2f(x)v=0$

⇒ $v=-\nabla^2f(x)^{-1}\nabla f(x)$

- Equality-constrained Problem

minimize $f(x)$ s.t. $Ax=b$

recall

Slater's condition

$\exists x\in$ relint D s.t. $Ax=b$

KKT optimality condition

$f_i(x^*)\le0$

$h_i(x^*)=0$

$\lambda^*\ge0$

$\lambda_i^*f_i(x^*)=0$

$\nabla f_0(x^*)+\sum_{i=1}^m\lambda_i^*\nabla f_i(x^*)+\sum_{i=1}^p\nu_i^*\nabla h_i(x^*)=0$

△ Here @ optimal $\tilde{x},\tilde{\nu}$

$\begin{cases}A\tilde{x}=b\\\nabla f(\tilde{x})+A^T\tilde{\nu}=0\end{cases}$

- Newton's Method
1. @ x

addition to x: $x+v$

∴ $x+v\in$ dom f

$A(x+v)=b$

∴ $Av=0$ —①

2. $\hat{f}(x+v)$

$=f(x)+\nabla f(x)v+\frac{1}{2}v^T\nabla^2f(x)v$

∴ minimize $f(x)+\nabla f(x)v+\frac{1}{2}v^T\nabla^2f(x)v$ s.t. $Av=0$ —②

③ $\tilde{v}$ is optimal of ②

∴ $\exists w$ s.t. Lagrange multiplier of ②

$\nabla f(x)+\nabla^2f(x)\tilde{v}+A^Tw=0$

and recall $A\tilde{v}=0$

from KKT $\mathcal{L}(v,w)=\hat{f}(v)+w(Av)$

∴ $A\tilde{v}=0$

$\nabla f(x)+\nabla^2f(x)\tilde{v}+A^Tw=0$

⇒ $\begin{bmatrix}\nabla^2f(x)&A^T\\A&0\end{bmatrix}\begin{bmatrix}\tilde{v}\\w\end{bmatrix}=\begin{bmatrix}-\nabla f(x)\\0\end{bmatrix}$

∴ $\tilde{v}=\Delta x_{nt}$.

- Inequality-Constrained Problem
- Barrier Method

minimize $f_0(x)$ s.t. $f_i(x)\le0$, $Ax=b$

⇕

minimize $tf_0(x)+\phi(x)$ s.t. $Ax=b$

$\phi(x)=-\sum_{i=1}^m\log(-f_i(x))$

1. given $x^0\in$ dom f

$t^0:=t^0>0$

$\mu>1$

$\epsilon>0$

2. repeat
   a. solve

   minimize$_x$ $tf_0(x)+\phi(x)$ s.t. $Ax=b$ (get x^*, w/ t^k)

   b. $x:=x^*(t)$
   c. until $m/t<\epsilon$
   d. $t:=\mu t$