

Control Bootcamp

P1

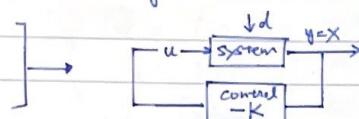
- passive control
- active control
 - open loop
 - close loop (w/ feedback)
 1. tackle w/ uncertainty
 2. tackle w/ instability
 3. tackle w/ disturbances
 4. efficiency

- $\dot{x} = Ax + Bu$
 $x(t) = e^{At}x(0)$

- $y = Cx$

close loop (w/ feedback)

1. tackle w/ uncertainty
2. tackle w/ instability
3. tackle w/ disturbances
4. efficiency



$$\Rightarrow u = -Kx$$

$$\begin{aligned} \dot{x} &= Ax - BKx \\ &= (A - BK)x \end{aligned}$$

determine K
→ make it stable

$$\left\{ \begin{array}{l} \text{from below} \\ = e^{-TDT^{-1}} \\ = T T^{-1} + T D T^{-1} + \frac{T D T^{-1} T D T^{-1}}{2!} + \dots \\ = T [I + D T^{-1} + \frac{D^2 T^{-2}}{2!} + \dots] T^{-1} \\ = T e^{D T^{-1}} \end{array} \right.$$

Linear System

$$\dot{x} = Ax \quad x \in \mathbb{R}^n \quad x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots$$

Eigenvalues & Eigenvectors : $AT = TD \Rightarrow [T, D] = \text{eig}(A)$

$$\left. \begin{array}{l} A \vec{\gamma} = \lambda \vec{\gamma} \\ T = [\vec{\gamma}_1, \vec{\gamma}_2, \dots, \vec{\gamma}_n] \\ D = \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \lambda_2 & & 0 \\ & & \ddots & \ddots \end{bmatrix} \end{array} \right\} \Rightarrow T^{-1}AT = D, \text{ let } \begin{array}{l} \vec{x} = T\vec{\gamma} \\ \vec{x} = T\vec{\gamma} = Ax \end{array}$$

$$\Rightarrow \dot{\vec{x}} = T^{-1}A T \vec{\gamma}$$

$$\Rightarrow \dot{\vec{\gamma}} = T^{-1}AT\vec{\gamma}$$

$$\Rightarrow \vec{\gamma} = T^{-1}AT\vec{\gamma}$$

$$\frac{d}{dt} \begin{bmatrix} \vec{\gamma}_1 \\ \vec{\gamma}_2 \\ \vdots \\ \vec{\gamma}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \lambda_2 & & 0 \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vec{\gamma}_1 \\ \vec{\gamma}_2 \\ \vdots \\ \vec{\gamma}_n \end{bmatrix}$$

in sum $\vec{\gamma}(t) = e^{Dt} \vec{\gamma}(0)$

$$= \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ 0 & e^{\lambda_2 t} & & 0 \\ & & \ddots & \ddots \\ & & 0 & e^{\lambda_n t} \end{bmatrix} \vec{\gamma}(0)$$

$$\begin{aligned} \therefore x(t) &= T e^{Dt} T^{-1} x(0) \\ &\underbrace{\qquad\qquad\qquad}_{\vec{\gamma}(0)} \\ &\underbrace{\qquad\qquad\qquad}_{\vec{\gamma}(t)} \\ &\underbrace{\qquad\qquad\qquad}_{x(t)} \end{aligned}$$

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P2

Stability Eigenvalues

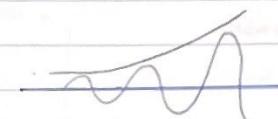
recall $\dot{x} = Ax \quad x \in \mathbb{R}^n$

$$\left. \begin{array}{l} [T, D] = \text{eig}(A); \\ D = \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \lambda_2 & \\ & & \ddots & \lambda_n \end{bmatrix} \\ e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ 0 & e^{\lambda_2 t} & \\ & & \ddots & 0 \end{bmatrix} \end{array} \right\} x(t) = T e^{Dt} T^{-1} x(0)$$

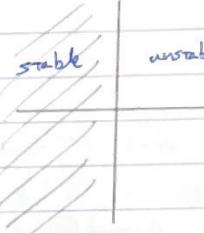
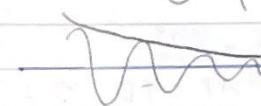
$\lambda = a + bi$

$$e^{\lambda t} = e^{at} [\cos(bt) + i \sin(bt)] \quad \lambda \in \mathbb{C}$$

if $a > 0$



if $a < 0$



(continuous case)

discrete system

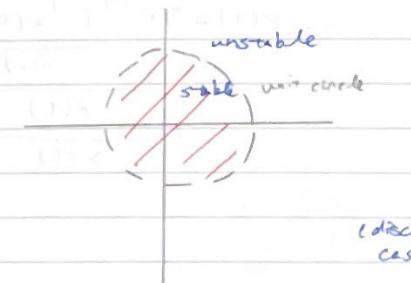
$$x_{k+1} = \tilde{A}x_k \quad x_k = x(k\Delta t)$$

$$\tilde{A} = e^{\tilde{A}\Delta t} \quad \text{continuous}$$

discrete

$$\begin{aligned} x_1 &= \tilde{A}x_0 & = \tilde{T}\tilde{D}\tilde{T}^{-1}x_0 \\ x_2 &= \tilde{A}x_1 = \tilde{A}^2x_0 & = \tilde{T}\tilde{D}^2\tilde{T}^{-1}x_0 \\ x_3 &= \tilde{A}^3x_0 & = \tilde{T}\tilde{D}^3\tilde{T}^{-1}x_0 \\ &\vdots & \vdots \\ x_n &= \tilde{A}^n x_0 & = \tilde{T}\tilde{D}^n\tilde{T}^{-1}x_0 \end{aligned}$$

recall \tilde{D} is a diagonal matrix
w/ $\lambda = a + bi = Re^{i\theta}$
 $\tilde{D}^n = R^n e^{i n \theta}$



(discrete case)

Linearizing Around a Fixed point

P3

From non-linear $\dot{x} = f(x) \Rightarrow$ to linear $\dot{x} = Ax \quad x \in \mathbb{R}^n$

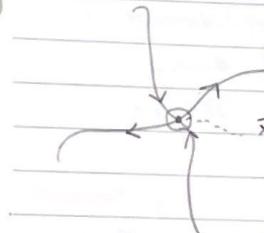
1. find fixed pts

$$\bar{x} \text{ s.t. } f(\bar{x}) = 0$$

2. linearize about \bar{x}

$$\frac{Df}{Dx}\Big|_{\bar{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_j} \end{bmatrix} \quad \text{e.g. } \dot{x}_1 = f_1(x_1, x_2) = x_1 x_2$$

$$\dot{x}_2 = f_2(x_1, x_2) = x_1^2 + x_2^2 \quad \frac{Df}{Dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_2 & x_1 \\ 2x_1 & 2x_2 \end{bmatrix}$$



$$\bar{x} = 0 \quad (\text{shift w/ } \bar{x}) - \text{assumed}$$

$$\dot{x} = f(x) = f(\bar{x}) + \frac{Df}{Dx}\Big|_{\bar{x}} \cdot (x - \bar{x}) + \frac{D^2f}{Dx^2}(x - \bar{x})^2 + \dots$$

$$\Delta \dot{x} = \frac{Df}{Dx}\Big|_{\bar{x}} \Delta x \Rightarrow \Delta \dot{x} = A \Delta x$$

$$\dot{x} = Ax$$

e.g. Pendulum

$$\ddot{\theta} = -\frac{g}{L} \sin(\theta) - \delta \dot{\theta} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin(x_1) - \delta x_2 \end{bmatrix}$$

$$\text{1. F.P. } \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$$\frac{Df}{Dx} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -\delta \end{bmatrix}$$

$$\text{case 1 } A = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix} \quad \lambda = \begin{bmatrix} -0.05 + 0.9987i \\ -0.05 - 0.9987i \end{bmatrix} \quad (\text{stable locally})$$

$$\text{case 2 } A = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \quad \lambda = \begin{bmatrix} -1.05/2 \\ 0.95/2 \end{bmatrix} \quad (\text{unstable saddle point})$$

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Controllability

affect stability

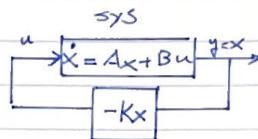
p4

△ recall: $\dot{x} = Ax \quad x \in \mathbb{R}^n$

△ now w/ control input:

$$\dot{x} = Ax + Bu$$

\mathbb{R}^n \mathbb{R}^m \mathbb{R}^n



"optimal"

for linear system.

when $u = -Kx$

$$\dot{x} = Ax - BKx$$

$$\dot{x} = (A - BK)x$$

by choosing K,
we can change
the "dynamics"

x can be "anywhere in \mathbb{R}^n "
 ↴ when or how to know?

△ $\Rightarrow \text{ctrb}(A, B)$ • $\dot{x} = Ax$

* $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \leftarrow \text{ctrb} : \text{as } x_1 \text{ is not coupled w/ } x_2 \text{ or } u,$
 making it impossible to twist anything on it

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \leftarrow \text{ctrb}$$

rather not obvious:

** $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \leftarrow \text{ctrb} : x_1 \text{ & } x_2 \text{ are coupled. twisting just one } u \text{ can do things on } x_1, x_2 \text{ simultaneously.}$

• $C = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$ i.f.f. $\text{rank}(C) = n$
 then sys is ctrb

: $C_ = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \rightarrow \text{ctrb } (\text{rank} = 1)$

*: $C_{**} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \text{ctrb } (\text{rank} = 2)$

• doing SVD on C gives you the controllability extent on each state
 $\begin{bmatrix} I &] \end{bmatrix}$ (most → least)
 (singular vectors)

△ Note that the controllability discussed here is "linear controllable".

Controllability / Reachability / Eigenvalue Placement

PS

△ recall: $\dot{x} = Ax + Bu$

$$C = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$$

$$\text{rank}(C) = n \Leftrightarrow \text{ctrb}$$

$$\Rightarrow \text{rank}(\text{ctrb}(A, B)) ;$$

△ Equivalences:

1. system is ctrb

2. Arbitrary eigenvalue (pole) placement

$$u = -kx \Rightarrow \dot{x} = (A - BK)x \quad \Rightarrow K = \text{place}(A, B, \text{eigs}) ;$$

arbitrary eigenvalue!

3. Reachability (full) in R^n can reach any vector in R^n given some u)

$$\text{Reachable set } R_t = \{ \vec{z} \in R^n \mid \text{there is an input } u(t) \text{ s.t. } x(t) = \vec{z} \}$$

$$R_t = R^n$$

Controllability & Discrete-Time Impulse Response

△ recall: $\dot{x} = Ax + Bu \quad x \in R^n$

$$C = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$$

△ $x_{k+1} = \tilde{A}x_k + \tilde{B}u_k$ ~ Impulse Response

$$\begin{aligned} \text{assume } x_0 &= 0 \\ \text{then } x_1 &= \tilde{B} = \tilde{A} \cdot 0 + \tilde{B} \cdot 1 & u_0 &= 1 \\ x_2 &= \tilde{A}\tilde{B} = \tilde{A} \cdot \tilde{B} + \tilde{B} \cdot 0 & u_1 &= 0 \\ x_3 &= \tilde{A}^2\tilde{B} = \tilde{A} \cdot \tilde{A}\tilde{B} + \tilde{B} \cdot 0 & u_2 &= 0 \\ &\vdots & u_3 &= 0 \end{aligned}$$

$$x_m = \tilde{A}^{m-1}\tilde{B} \cdot$$



if this can "hit" all axis in R^n $\left. \right\}$ just an intuition
 (I mean "hit" as in "affect")

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not yes or no.. not binary.
 "→ what extent"

P6

Degrees of Controllability & Gramians

- How controllable are different directions on \mathbb{R}^n

$$\Delta \quad x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

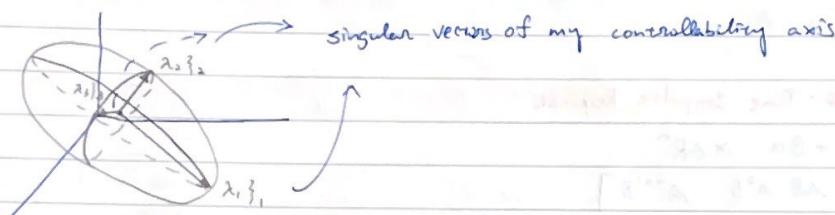
- controllability Gramian

$$W_t = \int_0^t e^{AT} B B^T e^{A^T \tau} d\tau \in \mathbb{R}^{n \times n}$$

$$W_t \{ \lambda \} \text{ larger eigenvalues, more controllable}$$

in discrete time

$$W_t \approx C C^T \leftrightarrow \text{svd of } C : [U, Z, V] = \text{svd}(C, \text{'econ'})$$



- stabilizability

lightly damped

stab. iff all unstable eigenvectors of A are in cont. space

(as if X is large, it is impossible to let each direction be controllable)

PBH Test (Popov - Belevitch - Hautus)

- (A, B) is contab i.f.s

$$\text{rank} [(A - \lambda I) B] = n \quad \forall \lambda \in \mathbb{C}$$

- rank(A - λI) = n except for eigenvalues of A)

∴ just need to perform PBH test @ λ

- B needs to have some component in each eigenvector direction

- if B is a random vector - i.e., $B = \text{randn}(n, 1)$

... (A, B) will be contab w/ high probability

(as it is hard to generate vector w.o. all components on eigenvector direction)

- rank [(A - λI) B]

Δ we multiply λ 2 columns → compute cont. rank

3x	3
----	---

4x	4
----	---

⋮	⋮
---	---

Δ on degenerate eigenvalues
 1-row values are close

Cayley - Hamilton Theorem

Every matrix A satisfies its own characteristic (eigen value) equation

$$\det(A - \lambda I) = 0$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

$$\Rightarrow A^n + a_{n-1}A^{n-1} + \dots + a_2A^2 + a_1A + a_0I = 0$$

(almost true for all A)

$$\Rightarrow A^n = -a_0I - a_1A - a_2A^2 - \dots - a_{n-1}A^{n-1}$$

$$\Rightarrow A^{2n} = \sum_{j=0}^{n-1} \alpha_j A^j \quad (\text{could be expressed as } n-1 \text{ or lower terms})$$

$$\Delta \dot{x} = Ax + Bu \quad x \in \mathbb{R}^n$$

$$e^{At} = I + A + \frac{A^2t}{2} + \dots$$

$$\Rightarrow = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2 + \dots + \alpha_{n-1}(t)A^{n-1}$$

no infinite

Reachability and controllability w/ Cayley-Hamilton

Reachability

If $\xi \in \mathbb{R}^n$ is reachable then we have $[AB-A]$ note that B applies to first n terms of ξ
then $\xi = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$ (for some $u(\tau)$)

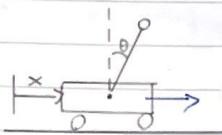
$$\begin{aligned} \xi &= \int_0^t (\phi_0(t-\tau)u(\tau)IB + \phi_1(t-\tau)u(\tau)AB + \dots + \phi_{n-1}(t-\tau)u(\tau)A^{n-1}B) d\tau \\ &= B \int_0^t \phi_0(t-\tau)u(\tau)d\tau + AB \int_0^t \phi_1(t-\tau)u(\tau)d\tau + \dots + A^{n-1}B \int_0^t \phi_{n-1}(t-\tau)u(\tau)d\tau \\ &= [B \quad AB \quad \dots \quad A^{n-1}B] \left[\begin{array}{c} \int_0^t \phi_0(t-\tau)u(\tau)d\tau \\ \int_0^t \phi_1(t-\tau)u(\tau)d\tau \\ \vdots \\ \int_0^t \phi_{n-1}(t-\tau)u(\tau)d\tau \end{array} \right] \end{aligned}$$

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P8

Inverted pendulum



u : force on the cart in \propto direction

$$\dot{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \quad \frac{d}{dt}x = f(x) \implies \dot{x} = Ax + Bu$$

fixed points: $\theta = 0, \pi$

$$\dot{\theta} = 0$$

$$\dot{x} = 0$$

x free

$$\dot{x} = (A - BK)x$$

refer to matlab

Pole placement

- △ $\gg K = \text{place}(A, B, \text{eigs})$
- $\gg \text{eig}(A - BK) = \text{eigs}$
- △ try to design K such that $[A - BK]$ matrix has stable poles

refer to matlab.

LQR

- △ $\gg K = \text{place}(A, B, \text{eigs})$
 - △ where are the best eigs?
- Linear Quadratic Regulator (LQR)

$$\Delta J = \int_0^\infty (x^T Q x + u^T R u) dt +$$

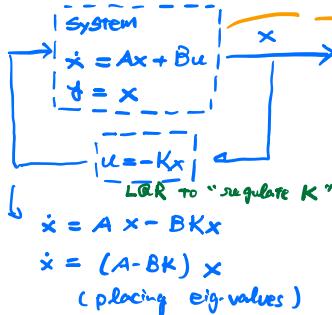
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \quad R = \begin{bmatrix} 0.001 \end{bmatrix}$$

$$\Delta \gg K = \text{lqr}(A, B, Q, R)$$

Motivation for Full-state estimation

Recall

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$$



I don't necessarily hv all states in real life

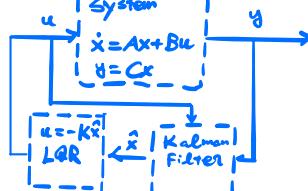
$$\dot{x} = Ax + Bu \quad (\text{controllability}) \quad \text{ctrb}(A, B)$$

$$y = Cx \quad (\text{observability}) \quad \text{obsv}(A, C)$$

Main Question here:

Can I estimate any state \hat{x} from measurement $y(t)$

hence:



Observability

- Duality exists between $\begin{bmatrix} AB \\ AC \end{bmatrix}$

- observability matrix

$$\Omega = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$$C^0 = [B \ AB \ A^2B \ \dots \ A^{n-1}B]$$

1. observable if

$$\gg \text{rank}(\text{obsv}(A, C)) = n$$

2. can estimate x from y

$$\gg [U, Z, V] = \text{svd}(\Omega)$$

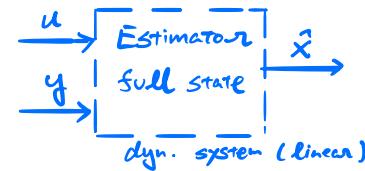
observability Gramian

V^T



In some direction, we hv higher signal to noise

Full State Estimation



$$\frac{d}{dt} \hat{x} = A \hat{x} + Bu + K_f(y - \hat{y})$$

$$\hat{y} = C \hat{x}$$

$$\begin{aligned} \frac{d}{dt} \hat{x} &= A \hat{x} + Bu + K_f y - K_f C \hat{x} \\ &= (A - K_f C) \hat{x} + [B \quad K_f] \begin{bmatrix} u \\ y \end{bmatrix} \end{aligned}$$

pick K_f
to place the eigen values to hv optimal choice

Kalman filter

- w_d - Gaussian
- V_d - Variance
- w_n - Gaussian
- V_n - Variance

- recall
 $\hat{x} = (A - K_f C) x$
 $\hat{e} = x - \hat{x}$

- cost function
 $J = E((x - \hat{x})^T(x - \hat{x}))$

$$\gg K_f = \text{arg} \min_{K_f} J(A, C, V_d, V_n)$$

Observability Example

- recall inverted pendulum

$$\begin{array}{l} \ddot{\theta} \rightarrow u \\ \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} \rightarrow x \\ x = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \\ \dot{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \\ \dot{x} = Ax + Bu \\ y = Cx \\ \gg \text{obsv}(A, C) \\ C = [1 \ 0 \ 0 \ 0] \end{array}$$

- subsystem: without $x \rightarrow \dot{x}$

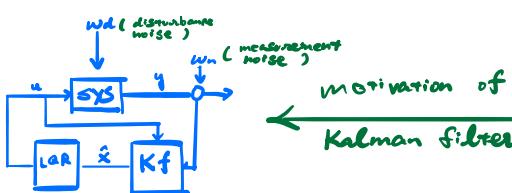
$$\begin{array}{l} x = \begin{bmatrix} \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \\ \text{with different measurement} \\ C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \text{meas. } \dot{x} \\ C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad \text{meas. } \theta \\ C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad \text{meas. } \dot{\theta} \\ \Rightarrow \text{det}(\text{gram}(\text{sys}, C, 0, 0)) \\ (\text{sys} = \text{ss}(A, B, C, D)) \end{array}$$

Kalman filter

real system:

$$\dot{x} = Ax + Bu + w_d$$

$$y = Cx + w_n$$



get the best K_f to place poles (eigs) based on w_d & w_n

Error $\epsilon = x - \hat{x}$

$$\frac{d}{dt} \epsilon = \frac{d}{dt} x - \frac{d}{dt} \hat{x}$$

$$= Ax + Bu - A\hat{x} + K_f C \hat{x} - K_f y - B u$$

$$= Ax - A\hat{x} + K_f C \hat{x} - K_f y$$

$$= A(x - \hat{x}) + K_f C(\hat{x} - x)$$

$$= A(x - \hat{x}) - K_f C(x - \hat{x})$$

$$= (A - K_f C)\epsilon$$

if observable,
then place eigs
by choosing K_f :

so that error
converge eventually