

ME564 L23

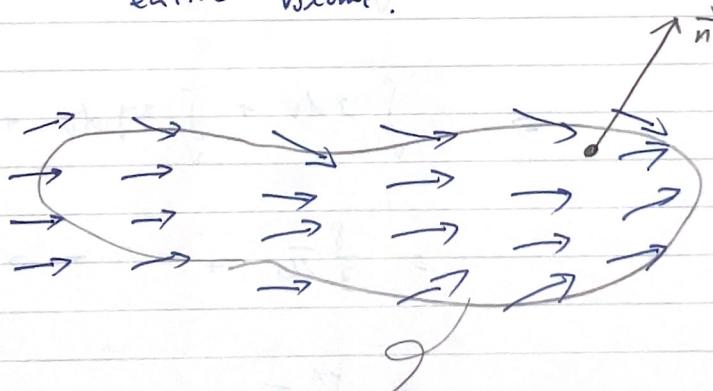
Gauss's Divergence Theorem

Overview of Topics

- ① Double & Triple Integrals
- ② Gauss's Thm $\nabla \cdot$
 - * Example
 - * Continuing Eqn.

Gauss's Divergence Theorem

→ In words : The flux of a vector field out of a close surface
 = the integral of the $\nabla \cdot$ of the vector field over the
 (divergence) entire volume.

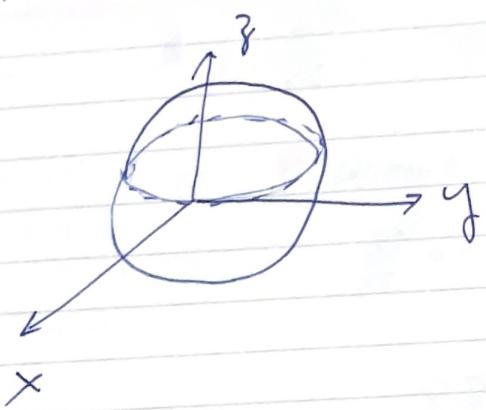


∂V (boundary of my volume).

in math

$$\iint_{\partial V} \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

$\underbrace{}$
 flux \int
 of mass through ∂V



$$\vec{F} = 2x\vec{i} + y^2\vec{j} + z^2\vec{k}$$

$$S = \{x^2 + y^2 + z^2 = 1\}$$

compute flux through S

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_V \nabla \cdot \vec{F} \, dV$$

$$\Rightarrow \nabla \cdot \vec{F} = 2 + 2y + 2z$$

$$\Rightarrow \iiint_V (2 + 2y + 2z) \, dV$$

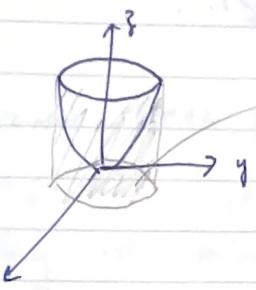
$$\Rightarrow \int_V 2 \, dV + \int_V 2y \, dV + \int_V 2z \, dV.$$

$$= \frac{8}{3}\pi + 0 + 0$$

$$= \frac{8}{3}\pi$$

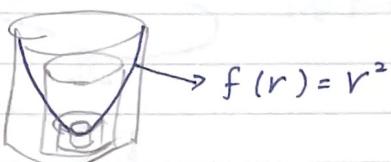
ex

$$f(x, y) = x^2 + y^2 \\ = r^2$$

find volume under $f(x, y)$

for $0 \leq x^2 + y^2 \leq 3$

$0 \leq r^2 \leq 3$



$$V = \iiint dV = \int_{r=0}^{\sqrt{3}} \int_{\theta=0}^{2\pi} \int_{z=0}^{f(r)} dz \, d\theta \, dr.$$

$$= \int_{r=0}^{\sqrt{3}} \int_{\theta=0}^{2\pi} r^2 \, d\theta \, dr$$

$$= \int_0^{\sqrt{3}} 2\pi r^2 \, dr$$

$$= \left[2\pi \frac{r^3}{3} \right]_0^{\sqrt{3}}$$

$$= 2\pi \sqrt{3}$$

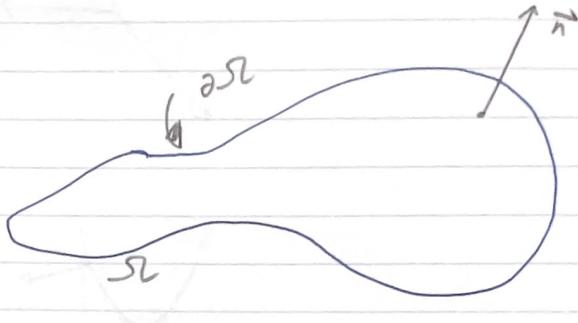
$$\vec{F} = (3x - 2xy) \vec{i} - y \vec{j} + 2yz \vec{k}$$

$$\iint_S \vec{F} \cdot \vec{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dV = \iiint_V 2 \, dV = 2 \cdot 4\pi \sqrt{3}$$

$$\nabla \cdot \vec{F} = 3 - 2y - 1 + 2y = 2$$

Conservation of Mass & Continuity Eqn.

→ consider a volume Ω containing some mass (fluid)
we also hv a velocity field \vec{V}



let ρ be the density.

Any change in mass in Ω must

correspond to mass being carried in/out by \vec{V}

$$\frac{d}{dt} \iiint_{\Omega} \rho d\Omega = - \iint_{\partial\Omega} \rho \vec{V} \cdot \vec{n} dS$$

rate of change Δ in mass.

$$= - \iiint_{\Omega} \rho \nabla \cdot \vec{V} d\Omega$$

$$\iiint_{\Omega} \left[\frac{d}{dt} \rho + \nabla \cdot (\rho \vec{V}) \right] d\Omega = 0$$

Integral form of mass conservation!

for all Ω !! \Rightarrow

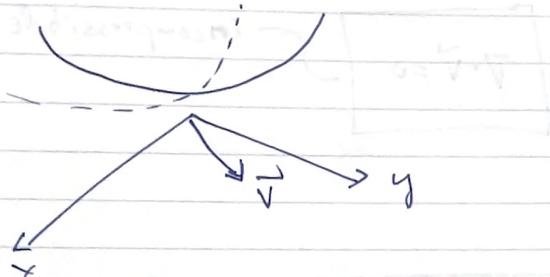
$$\frac{d}{dt} \rho + \nabla \cdot (\rho \vec{V}) = 0 \text{ everywhere!}$$

linear PDE

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 Δ directional derivative

The gradient ∇f of a function f can be used to compute the directional derivative of f in some vector direction \vec{v}



$$\nabla_v f = (\nabla f) \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

$$\text{if } \vec{v} = \vec{i} : \frac{\partial f}{\partial x}$$

$$\text{if } \vec{v} = \vec{j} : \frac{\partial f}{\partial y}$$

$$\text{if } \vec{v} = \vec{k} : \frac{\partial f}{\partial z}$$

$$\frac{\partial^2}{\partial t^2} \rho + \nabla \cdot (\rho \vec{v}) = 0$$

$$\nabla \cdot (\rho \vec{v}) = \nabla \cdot \begin{bmatrix} \rho v_1 \\ \rho v_2 \\ \rho v_3 \end{bmatrix} = (\rho_x v_1 + \rho_{xx} v_1) +$$

$$+ (\rho_y v_2 + \rho_{xy} v_2) +$$

$$+ (\rho_z v_3 + \rho_{xz} v_3)$$

$$= \rho \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) + (\rho_x v_1 + \rho_y v_2 + \rho_z v_3)$$

$$= \rho \nabla \cdot \vec{v} + \underbrace{(\nabla \rho) \cdot \vec{v}}_{\|\vec{v}\| \cdot D_{\vec{v}} \rho}$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \vec{v} + (\nabla \rho) \cdot \vec{v} = 0$$

if ρ constant everywhere

$$\frac{\partial \rho}{\partial t} = 0 \quad \& \quad \nabla \rho = \vec{0} \quad \dots \quad D_{\vec{v}} \rho = 0 \text{ for all } \vec{v}$$

$$\rho \nabla \cdot \vec{v} = 0 \Rightarrow \boxed{\nabla \cdot \vec{v} = 0} \quad \text{incompressible}$$

A

Vector field \vec{v}

① \vec{v} is often the solⁿ of a PDE. (Navier - Stokes)

$$\textcircled{2} \quad \dot{x} = \vec{v}(x, t)$$

$$x(t) = x_0 + \int_0^t \vec{v}(x(\tau), \tau) d\tau$$

