- **optimization problem**

  minimize $f_0(x)$

  s.t. $f_i(x) \le 0,\quad i=1,\dots,m$

  $h_i(x) = 0,\quad i=1,\dots,p$

- **duality**:
  - gives us lower bound
  - sometimes it's easier to solve (good properties for solver!)

- **dual problem**
  - $L(x,\lambda,\nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$
  - $g(\lambda,\nu) = \inf_{x\in D} L(x,\lambda,\nu) = \inf_{x\in D}\left\{ f_0(x) + \sum_{i=1}^{m}\lambda_i f_i(x) + \sum_{i=1}^{p}\nu_i h_i(x) \right\}$
  - maximize $g(\lambda,\nu)$

    s.t. $\lambda \succeq 0$

- **QCQP** (quadratic constrained quadratic programming)

▲ minimize$_x$ $x^T Q x$

s.t. $x^T A_k x = b_k \quad \forall k \in \{1,\dots,k\}$

non-convex QCQP

Lagrange multiplier

▲ $L(x,\lambda) = x^TQx + \sum_{k=1}^{k} \lambda_k (b_k - x^T A_k x)$

$= \sum_k \lambda_k b_k + x^T\left(Q - \sum_{k=1}^{k}\lambda_k A_k\right)x$

$= \underbrace{b^T\lambda}_{\begin{bmatrix} b_1 \\ b_k \end{bmatrix}^T \begin{bmatrix} \lambda_1 \\ \lambda_k \end{bmatrix}} + x^T \underbrace{H(\lambda)}_{Q-\sum \lambda_k A_k} x$

▲ $g(\lambda) = \inf_x L(x,\lambda) = \begin{cases} b^T\lambda, & \text{if } H(\lambda)\succeq 0 \\ -\infty, & \text{o.w.} \end{cases} \rightarrow \frac{\partial L}{\partial x} = 0$

▲ primal problem

minimize$_x$ x^TQx

s.t. $x^TA_kx = b_k \quad \forall k \in \{1,\dots,k\}$

$\Updownarrow$

$\theta_P(w) = \max_\lambda \mathcal{L}(x,\lambda)$

$= \max_\lambda \begin{cases} x^TQx & x \in D \\ \infty & \text{o.w} \end{cases}$

minimize$_x$ $\max_\lambda \left(b^T\lambda + x^T \underbrace{H(\lambda)x}_{0}\right)$ (1)

▲ dual problem

maximize $b^T\lambda$

s.t. $H(\lambda) \succeq 0$

$H(\lambda)x = 0$ (2)

∴ from (1) & (2)

when $H(\lambda)x = 0$

$p^* = d^* = b^T\lambda$

strong duality holds

Slater's condition

$\exists x \in \text{relint } D$

s.t.

$f_i(x) < 0$

$Ax = b$

empirically: robotics application satisfy Slater's condition

This implies:

a. strong duality (if $H(\lambda)\succeq 0$)

b. $\hat{x}$ is optimal (global) when $H\hat{x} = 0$

c. $\hat{x}$ is not optimal then what do we do?

▲ from above

certification problem

find H, λ

s.t. $H = Q - \sum_{k=1}^{k} \lambda_k A_k$

$H \succeq 0$

$H\hat{x} = 0$

strong duality holds!

$\downarrow$

we have global optimal!

▲ SDP relaxation (which can allow us to solve it faster & solve it further!)

dual of dual problem

(omitted today)

minimize$_x$ x^TQx

s.t. $xAx = b$

- recall dual problem

  maximize $b^T\lambda$

  s.t. $H(\lambda)\succeq 0$

- Langragian:

  $L'(\lambda, X) = b^T\lambda + \mathrm{tr}(X H(\lambda))$

  $X \succeq 0$ matrix of langragian multiplier

  note: $\langle X, Y\rangle = \mathrm{tr}(X^TY)$

- dual problem:

  - $L'(\lambda, X) = b^T\lambda + \mathrm{tr}\left(X\left(Q - \sum_{k=1}^{k}\lambda_k A_k\right)\right)$

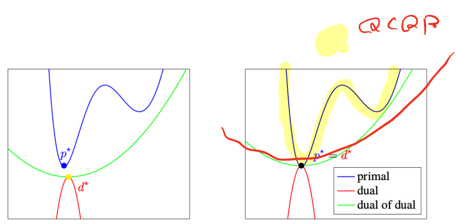
    $= \mathrm{tr}(QX) + [b_1 - \mathrm{tr}(A_1X) \quad b_k - \mathrm{tr}(A_kX)]\lambda$

  - $g(X) = \sup_\lambda L'(\lambda, X) = \begin{cases} \mathrm{tr}(QX) & \text{if } \mathrm{tr}(A_kX) = b_k \\ \infty & \text{o.w.} \end{cases}$

$\Rightarrow$ $\boxed{\begin{array}{ll} \text{minimize}_X & \mathrm{tr}(QX) \\ & \mathrm{tr}(A_kX = b_k) \\ & X \succeq 0 \end{array}}$

Semidefinite Programming!

$xx^T = X$



(a) Weak duality. (b) Strong duality.