- **optimization problem**

$$\underset{x}{\text{minimize}}\ f_0(x)$$
s.t. $f_i(x) \le 0$
$h_i(x) = 0$

- **feasible**: $x \in \operatorname{dom} f_0$
- **infeasible**: $p^* = \infty / -\infty$
- **optimal value**
  $p^* = \min\{f_0(x) \mid f_i(x) \le 0,\ h_i(x) = 0\}$
  x^* is optimal
- **locally optima**
  $f_0(x) = \inf\{f_0(z) \mid z \in D,\ \|z-x\|_2 \le R\}$
  $D = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=0} \operatorname{dom} h_i$
- **feasibility problem**
  find x s.t. $f_i(x) \le 0$, $h_i(x) = 0$ $\iff$ $\underset{x}{\text{minimize}}\ 0$ s.t. $f_i(x) \le 0$, $h_i(x) = 0$
- **convex optimization**
  $\underset{x}{\text{minimize}}\ f_0(x)$ → convex
  s.t. $f_i(x) \le 0$ → convex
  $a_i^T x = b_i$ → affine
- **quasiconvex optimization**
  $\underset{x}{\text{minimize}}\ f_0(x)$
  s.t. $f_i(x) \le 0$
  $a_i^T x = b_i$
- **local & global optima**
  $\underset{z}{\text{minimize}}\ f_0(z)$
  s.t. $f_i(z) \le 0$
  $h_i(z) = 0$
  $\|z-x\|_2 \le R$

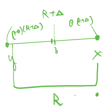
  x is locally optimal when
  $f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z-x\|_2 \le R\}$

fall back here

in convex
if x is local optimal
then x is global optimal
proof:
△ if $\exists y$ $f_0(y) < f_0(x)$, $\|y-x\|_2 > R$ } as if $\|y-x\|_2 < R$

△ we let
$$\theta = \frac{R}{2\|y-x\|_2} < \frac{1}{2}$$

△ we let
$z = (1-\theta)x + \theta y$
convex combination, in convex function
∴
$f_0(z) \le (1-\theta) f_0(x) + \theta f_0(y)$
$0 < \theta < \frac{1}{2}$

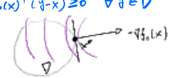


⇒ from here
$\|x-z\|_2 = \theta(R+\Delta)$
$\le \frac{1}{2}(R+\Delta)$

- we let $\|x-z\|_2 = \frac{1}{2}R$ to fullfil all the condition (we want z to be within $\|x-z\| < R$, s.t. x is local optimal)

△ w/ assumption $f_0(y) < f_0(x)$
we also hv
$\theta f_0(y) + (1-\theta) f_0(x) < f_0(x)$
$\Rightarrow f_0(z) = \theta f_0(y) + (1-\theta) f_0(x) < f_0(x)$
contradicts w/
$f_0(x) = \inf\{f_0(z) \mid z \in D, \|z-x\|_2 \le R\}$

- **optimality criterion**
  $\nabla f_0(x)^T (y-x) \ge 0 \quad \forall y \in D$



- **Geometric Programming**
  - monomial function
    $f: \mathbb{R}^n \to \mathbb{R}$ $\operatorname{dom} f = \mathbb{R}^n_{++}$
    $f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$
    $c > 0,\ a_i \in \mathbb{R}$
  - posynomial function
    $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$
    $c_k > 0,\ a_{ik} \in \mathbb{R}$
    sum of monomial function
  - transfer to convex problem
    $y_i = \log x_i$ $x_i = e^{y_i}$
    $f(x) = f(e^{y_1}, \dots, e^{y_n})$
    $= c(e^{y_1})^{a_1} \cdots (e^{y_n})^{a_n}$
    $= e^{a^T y + b}$ $(b = \log c)$
    monomial

original
$\underset{}{\text{minimize}}\ f_0(x)$
s.t. $f_i(x) \le 1$ (posy)
$h_i(x) = 1$ (mono)

convex equivalence
$\text{minimize} \log\left(\sum_{k=1}^{K} e^{a_{0k}^T y + b_{0k}}\right)$
s.t. $\log\left(\sum_{k=1}^{K} e^{a_{ik}^T y + b_{ik}}\right) \le 0$
$Gy + d = 0$

- **multicriterion optimization**

- **Duality**
  - The Lagrangian
    $\text{minimize}\ f_0(x)$
    s.t. $f_i(x) \le 0$ $i = 1, \dots, m$ → λ_i
    $h_i(x) = 0$ $i = 1, \dots, p$ → ν_i

    $\mathcal{L}(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$
  - The Lagrange Dual Function
    $g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu)$
    $\therefore \underbrace{g(\lambda, \nu)}_{\text{lower bound}} \le p^* = f_0(\tilde{x})$ → all feasible x ($\lambda \succeq 0$)
    $\therefore \underbrace{\lambda \succeq 0,\ (\lambda, \nu) \in \operatorname{dom} g}_{\text{dual feasible}}$
  - The Lagrange Dual Problem
    $\text{maximize}\ g(\lambda, \nu)$
    s.t. $\lambda \succeq 0$

- **Weak Duality**
  $\underset{\text{best bound}}{d^*} \le \underset{\text{optimal value}}{p^*}$
  Δ = duality gap
- **Strong Duality**
  - $d^* = p^*$
  - Slater's condition
    $\exists x \in \operatorname{relint} D$
    s.t. $f_i(x) < 0$, $Ax = b$

- **optimality conditions – complementary slackness**
  assume strong duality holds:
  $d^* = p^*$
  $g(\lambda^*, \nu^*) = f_0(x^*)$
  ⇒
  $f_0(x^*) = g(\lambda^*, \nu^*)$
  $= \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x)\right)$
  $\le f_0(x^*) + \sum_{i=1}^{m} \underbrace{\lambda_i^*}_{\ge 0} \underbrace{f_i(x^*)}_{\le 0} + \sum_{i=1}^{p} \nu_i^* \underbrace{h_i(x^*)}_{= 0}$
  $\le f_0(x^*)$

  $\begin{cases} \text{if } \lambda_i^* > 0 \Rightarrow f_i(x^*) = 0 \\ \text{if } f_i(x^*) < 0 \Rightarrow \lambda_i^* = 0 \end{cases}$
  complementary slackness!

  $\therefore \sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0$

- **KKT optimality conditions**
  - let $\underset{\text{primal}}{x^*}$ $\underset{\text{dual}}{(\lambda^*, \nu^*)}$ be optimal points
  - 0-duality gap

  $\therefore x^* = \underset{x}{\operatorname{argmin}}\ \mathcal{L}(x, \lambda^*, \nu^*)$

  - $\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0$

  ∴
  - $f_i(x^*) \le 0$
    $h_i(x^*) = 0$
    $\lambda_i^* \ge 0$
    $\lambda_i^* f_i(x^*) = 0$

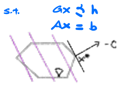
  → KKT optimality conditions!

- **Sufficiency & Necessity**
  if P then Q
  △ Q is necessary for P
  ① truth of Q is guaranteed by truth of P
  ② (falsity of Q guarantees falsity of P)
  ▲ P is sufficient for P
  ③ truth of P guarantees truth of Q
  ④ but falsity of P might still hv truth of Q

  如果"要退休"$_P$ 那"要40岁"$_Q$
  e.g.
  ① "人一定是40岁"$_Q$ 如果"这人退休"$_P$
  ② 如果"未40岁"$_Q$ 那"一定非退休"$_P$
  ③ 如果"要退休"$_P$ 那"一定40岁"$_Q$
  ④ 如果"不退休"$_P$ 那"可能40岁"$_Q$

  sum:
  if P then Q
  △ Q is P's necessary condition
  ▲ P is Q's sufficient condition

- **Convex problems**
  - Linear Programming
    $\underset{x}{\text{minimize}}\ c^T x + d$
    s.t. $Gx \preceq h$
    $Ax = b$



  - piecewise-linear minimization
    $f_0(x) = \max_{i=1,\dots,m} (a_i^T x + b_i)$

    $\underset{x}{\text{minimize}}\ f_0(x)$
    ⇕
    $\text{minimize}\ t$
    s.t. $a_i^T x + b_i \le t$ $i = 1, \dots, m$
  - Quadratic Program
    $\underset{x}{\text{minimize}}\ \frac{1}{2} x^T P x + q^T x + r$
    s.t. $Gx \preceq h$
    $Ax = b$

    $P \in S_+^n$
  - Quadratically Constrained Quadratic Program
    $\underset{x}{\text{minimize}}\ \frac{1}{2} x^T P_0 x + q_0^T x + r_0$
    s.t. $\frac{1}{2} x^T P_i x + q_i^T x + r_i \le 0$
    $Ax = b$
    $P_i \in S_+^n$ (cover a feasible region that's intersection of ellipsoids + affine set)

- **Second-order Cone Programming**
  $\underset{x}{\text{minimize}}\ f^T x$
  $\|A_i x + b_i\|_2 \le c_i^T x + d_i$
  $Fx = g$
  SOCP could be understood:
  $\begin{bmatrix} Ax+b \\ c^T x + d \end{bmatrix} \in \mathbb{R}^{k+1}$ lies in SOC
  $(Ax+b,\ c^T x + d) \succeq_Q 0$
  G.I. for cone Q
- e.g. Robust Linear Programming

- **Perturbation & Sensitivity Analysis**
  - recall
    $\underset{x}{\text{minimize}}\ f_0(x)$
    s.t. $f_i(x) \le 0$
    $h_i(x) = 0$

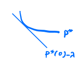
    perturbed by u_i, v_i

    $\underset{x}{\text{minimize}}\ f_0(x)$
    s.t. $f_i(x) \le u_i$
    $h_i(x) = v_i$
  - optimal value (becomes a function) w.r.t. (u, v)
    $p^*(u, v) = \inf\left\{f_0(x) \,\middle|\, \begin{matrix} \exists x \in D \\ f_i(x) \le u_i \\ h_i(x) = v_i \end{matrix}\right\}$
  - let (λ^*, ν^*) → dual optimal pt. of unperturbed problem
    $\Rightarrow p^*(0,0) = g(\lambda^*, \nu^*)$ (inf of Lagrange)
    $\le f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x)$
    $\le f_0(x) + \lambda^{*T} u + \nu^{*T} v$
    $\Rightarrow p^*(0,0) \le f_0(x) + \lambda^{*T} u + \nu^{*T} v$
    $\Rightarrow f_0(x) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$
    $\Rightarrow p^*(u,v) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$
    $(p^*(u,v) \in \{f_0(x) \mid x \in \operatorname{dom} f\})$

    $p^*(u,v) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$
    perturbation analysis

- **local sensitivity**
  assume
  - $p^*(u,v)$ differentiable @ $u=0$, $v=0$
  - strong duality holds

  $\Rightarrow p^*(u,v)$ @ $u=0, v=0$
  $\frac{\partial p^*(0,0)}{\partial u_i} = -\lambda_i^*$



  $\frac{\partial p^*(0,0)}{\partial v_i} = -\nu_i^*$

- w/ complementary slackness:
  $\lambda_i f_i(x) = 0$
  - $f_i(x) < 0$, $\lambda_i = 0$
    then: value @ $p^*(0,0)$ doesn't change w.r.t.
  - $f_i(x) = 0$, $\lambda_i^* > 0$
    then: value @ $p^*(0,0)$ changes w.r.t. u
    $\lambda_i^* \uparrow$ → perturb
    $\lambda_i^* \downarrow$ → perturb

  → tells us HOW ACTIVE IT IS

• Numerical Method of Differential Equations

• ODE 4 types

1 separable equations

$\frac{dy}{dx} = P(x)Q(x)$

e.g. $y' + 2xy = x$

$\Rightarrow \frac{dy}{dx} + 2xy = x$

$\Rightarrow \frac{dy}{dx} = x - 2xy$

$\Rightarrow \frac{dy}{dx} = x(1-2y)$

$\Rightarrow dy = x(1-2y)\,dx$

$\Rightarrow \frac{1}{1-2y} dy = x\,dx$

$\Rightarrow \int \frac{1}{1-2y} dy = \int x\,dx$

$\Rightarrow -\frac{1}{2}\ln(1-2y) = \frac{1}{2}x^2 + C$

$\Rightarrow e^{-\frac{1}{2}\ln(1-2y)} = e^{-\frac{1}{2}x^2+c}$

$\Rightarrow 1-2y = ce^{-x^2}$

$\Rightarrow y = -\frac{ce^{-x^2}-1}{2}$

2. homogenous method

$f(kx, ky) = f(x,y)$

e.g. $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

check $\frac{k^2x^2+k^2y^2}{kxky} = \frac{x^2+y^2}{xy}$

let $v = \frac{y}{x}$ $\quad \frac{1}{v} = \frac{x}{y}$

$y = vx$

$\frac{dy}{dx} = x\frac{dv}{dx} + v$

• $\frac{dy}{dx} = \frac{\frac{x^2}{x^2}+\frac{y^2}{x^2}}{\frac{xy}{x^2}} = \frac{1+(\frac{y}{x})^2}{\frac{y}{x}}$

$= \frac{1+v^2}{v}$

$\Rightarrow x\frac{dv}{dx} + v = \frac{1+v^2}{v}$

$\Rightarrow x\frac{dv}{dx} + v = \frac{1}{v} + v$

$\Rightarrow x\frac{dv}{dx} = \frac{1}{v}$

$\Rightarrow v\,dv = \frac{1}{x}dx$

$\Rightarrow \int v\,dv = \int \frac{1}{x}dx$

$\Rightarrow \frac{1}{2}v^2 = \ln(x) + C$

$\Rightarrow v^2 = 2\ln|x| + C$

$v = \pm\sqrt{2\ln|x|+C}$

$\frac{y}{x} = \pm\sqrt{2\ln|x|+C}$

$y = \pm x\sqrt{2\ln|x|+C}$

3. Integrating factor

$\frac{dy}{dx} + P(x)y = Q(x)$

e.g. $\frac{dy}{dx} + 1y = x$ $\quad P(x)=1$

$\Leftrightarrow \frac{dy}{dx} + P(x)y = 1$ ①

$\mu(x) = e^{\int P(x)dx}$

$\mu(x) = e^{\int 1\,dx}$

$= e^x$ ②

$\Rightarrow$ ① & ②

$\mu(x)\left[\frac{dy}{dx} + P(x)y = Q(x)\right]$

$\Rightarrow e^x\left[\frac{dy}{dx} + y = x\right]$

$\Rightarrow e^x\frac{dy}{dx} + ye^x = xe^x$

$\Rightarrow \frac{d}{dx}(e^x y) = xe^x$

$\Rightarrow \int \frac{d}{dx} e^x y = \int xe^x dx$ $\quad$ IBP

$\Rightarrow e^x y = xe^x - e^x + C$

$\Rightarrow y = x - 1 + \frac{c}{e^x}$

$= x - 1 + ce^{-x}$

• $\frac{dv}{dt} = g - \frac{c}{m}v$

$\frac{dv}{dt} + P(t)V = Q(t)$

$\Rightarrow \frac{dv}{dt} + \left(\frac{c}{m}\right)V = g$

$\Rightarrow \mu(t) = e^{\int \frac{c}{m}dt}$

$= e^{\frac{c}{m}t}$

$\Rightarrow e^{\frac{c}{m}t}\left(\frac{dv}{dt} + \left(\frac{c}{m}\right)v = g\right)$

$\Rightarrow e^{\frac{c}{m}t}\frac{dv}{dt} + \frac{c}{m}ve^{\frac{c}{m}t} = ge^{\frac{c}{m}t}$

$\Rightarrow \frac{d}{dt}e^{\frac{c}{m}t}v = ge^{\frac{c}{m}t}$

$\Rightarrow \int \frac{d}{dt}e^{\frac{c}{m}t}v = \int ge^{\frac{c}{m}t}dt$

$\Rightarrow e^{\frac{c}{m}t}v = \frac{m}{c}ge^{\frac{c}{m}t} + c'$

$\Rightarrow v = e^{-\frac{c}{m}t}\left[\frac{m}{c}ge^{\frac{c}{m}t} + c'\right]$

assume $v(0) = 0$

$v(0) = +\frac{m}{c}g + c' = 0$

$c' = -\frac{m}{c}g$

$\therefore v(t) = e^{-\frac{c}{m}t}\left[\frac{m}{c}ge^{\frac{c}{m}t} - \frac{m}{c}g\right]$

$= e^{-\frac{c}{m}t}\frac{m}{c}g\left[e^{\frac{c}{m}t} - 1\right]$

$= \frac{mg}{c}\left[1 - e^{-\frac{c}{m}t}\right]$

• linear ODE

- Laplace
- analytically solved
- $a_0(x)y + a_1(x)y' + \dots + a_n(x)y^{(n)} = b(x)$
- $Ly = f$

• Runge-Kutta Methods

△ Taylor series

$\frac{dy}{dx} = f(x,y)$

$\frac{d^{(n)}y}{dx^{(n)}} = f^{(n-1)}(x,y)$

$y_{i+1} = y_i + f(x_i,y_i)h$

Euler's Method

△ $y_{i+1} = y_i + f(x_i,y_i)h + \frac{f'(x_i,y_i)}{2!}h^2 + \dots + \frac{f^{(n-1)}(x_i,y_i)}{n!}h^n + O(h^{n+1})$

$\cong y_i + \left[f(x_i,y_i) + \frac{f'(x_i,y_i)}{2!}h + \dots + \frac{f^{(n-1)}(x_i,y_i)}{n!}h^{n-1}\right]h$

$= y_i + \phi(x_i,y_i,h)h$

△ $\phi(x_i,y_i,h) = f(x_i,y_i) + \frac{f'(x_i,y_i)}{2!}h + \dots + \frac{f^{(n-1)}(x_i,y_i)}{n!}h^{n-1}$

△ Runge-Kutta

acquire $\phi(x_i,y_i,h)$

w.o/ calculating higher-order term

$y_{i+1} = y_i + \phi(x_i,y_i,h)h$

$\Rightarrow \phi(x_i,y_i,h)$

$= a_1k_1 + a_2k_2 + \dots + a_{n-1}k_{n-1} + a_nk_n$

$k_1 = f(x_i,y_i)$

$k_2 = f(x_i + p_1h, y_i + q_{11}k_1h)$

$k_3 = f(x_i + p_2h, y_i + q_{21}k_1h + q_{22}k_2h)$

$\vdots$

$k_n = f(x_i + p_{n-1}h, y_i + q_{n-1,1}k_1h + q_{n-1,2}k_2h + q_{n-1,3}k_3h + \dots + q_{n-1,n-1}k_{n-1}h)$

$a, p, q \Rightarrow$ constants

$n =$ order $\Rightarrow$ RK n method!

e.g.

△ $y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$

$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$

$f(0) = 1$

$f(0.5) = ?$

△ RK4:

$y_{i+1} = y_i + \phi(x_i,y_i,h)h$

$\phi(x_i,y_i,h)$

$= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$

$k_1 = f(x_i,y_i)$

$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h)$

$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h)$

$k_4 = f(x_i + h, y_i + k_3h)$

△ ∴

$k_1 = f(0,1) = 8.5$

$k_2 = f(0+0.25, 1+0.5 \cdot 8.5 \cdot 0.5) = 4.21875$

$k_3 = f(0+0.25, 1+0.5 \cdot 4.21875 \cdot 0.5) = 4.21875$

$k_4 = f(0.5, 1+4.21875 \cdot 0.5) = 1.25$

$y(0.5) = 1 + \frac{0.5}{6}(8.5 + 2 \cdot 4.21875 + 2 \cdot 4.21875 + 1.25)$

$= 3.21875 + O(h^5)$

• system of equations

$$\begin{cases} \frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n) \\ \frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots, y_n) \\ \vdots \\ \frac{dy_n}{dx} = f_n(x, y_1, y_2, \dots, y_n) \end{cases}$$

e.g.

△ $$\begin{cases} \frac{dy_1}{dx} = -0.5y_1 \\ \frac{dy_2}{dx} = 4 - 0.3y_2 - 0.1y_1 \end{cases}$$

$y_1 = 4$ @ $x = 0$

$y_2 = 6$

△ $\dot{y} = f(x)$

$$= \begin{bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ -0.1 & -0.3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

△ RK4

$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h)$

$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h)$

$k_4 = f(x_i + h, y_i + k_3h)$

△ $k_{1,1} = f_1(0, 4) = -0.5 \cdot 4 = -2$

$k_{1,2} = f_2(0, 6) = 4 - 0.3 \cdot 6 - 0.1 \cdot 4 = 1.8$

△ $k_{2,1} = f_1(0 + \frac{1}{2} \cdot 0.5, 4 + \frac{1}{2} \cdot (-2) \cdot 0.5)$

$= -0.5 \cdot 3.5 = -1.75$

$k_{2,2} = f_2(0 + \frac{1}{2}0.5, 6 + \frac{1}{2} \cdot (1.8) \cdot 0.5)$

$= 4 - 0.3 \cdot 6.45 - 0.1 \cdot 3.5$

$= 1.715$

△ $k_{3,1} = f_1(0 + \frac{1}{2}0.5, 4 + \frac{1}{2}(-1.75) \cdot 0.5)$

$= -0.5 \cdot 3.5625$

$= -1.78125$

$k_{3,2} = f_1(0 + \frac{1}{2} \cdot 0.25, 6 + \frac{1}{2}(1.715) \cdot 0.5)$

$= 4 - 0.3 \cdot 6.42875 - 0.1 \cdot 3.5625$

$= 1.715125$

△ $k_{4,1} = f_1(0 + 0.5, 4 + (-1.78125) \cdot 0.5)$

$= -0.5 \cdot 3.109375$

$= -1.5546875$

$k_{4,2} = f_2(0 + 0.5, 6 + (1.715125 \cdot 0.5))$

$= 4 - 0.3 \cdot 6.8575625 - 0.1 \cdot 3.109375$

$= 1.63179375$

△ $y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$

$y = \begin{bmatrix} 4 \\ 6 \end{bmatrix} + \frac{0.5}{6}\left(\begin{bmatrix} -2 \\ 1.8 \end{bmatrix} + 2\begin{bmatrix} -1.75 \\ 1.715 \end{bmatrix} + 2\begin{bmatrix} -1.78125 \\ 1.715125 \end{bmatrix} + \begin{bmatrix} -1.5546875 \\ 1.63179375 \end{bmatrix}\right)$

$= \begin{bmatrix} 3.115234 \\ 6.875670 \end{bmatrix}$ #

• solving high-order ODE

$$\begin{cases} \frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x + y \\ y = 1 \quad \frac{dy}{dx} = 2 \quad \text{when } x = 0 \end{cases}$$

$$\begin{cases} \text{let } u = \frac{dy}{dx} \\ \frac{du}{dx} = \frac{d^2y}{dx^2} \end{cases}$$

$$\begin{cases} \Rightarrow \frac{du}{dx} + 2u = x + y \\ \Rightarrow \frac{du}{dx} = x + y - 2u \end{cases}$$

$$\begin{cases} \frac{dy}{dx} = u \\ \frac{du}{dx} = x + y - 2u \end{cases} \qquad \begin{matrix} y(0) = 1 \\ u(0) = 2 \end{matrix}$$

use this to propagate

$$\begin{cases} \frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = x + y \\ y = 1 \quad \frac{dy}{dx} = 2 \quad \frac{d^2y}{dx^2} = 3 \quad x = 0 \end{cases}$$

$$\begin{cases} \frac{dy}{dx} = u \\ \frac{d^2y}{dx^2} = v \end{cases} \qquad \frac{d^2y}{dx^2} = \frac{du}{dx} = v \qquad \frac{d^3y}{dx^3} = \frac{dv}{dx}$$

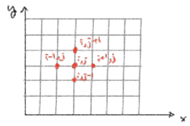
$$\begin{cases} \frac{dv}{dx} + 2v + 3u = x + y \\ \frac{dy}{dx} = u \\ \frac{du}{dx} = v \end{cases} \qquad \begin{matrix} y(0) = 1 \\ u(0) = 2 \\ v(0) = 3 \\ x = 0 \end{matrix}$$

use this to propagate

• PDE

• Laplace equation

$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$



$\frac{\partial^2 T}{\partial x^2} = \frac{\left(\frac{T_{i+1,j} - T_{i,j}}{\Delta x}\right) - \left(\frac{T_{i,j} - T_{i-1,j}}{\Delta x}\right)}{\Delta x}$

$= \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2}$

$\frac{\partial^2 T}{\partial y^2} = \frac{\left(\frac{T_{i,j+1} - T_{i,j}}{\Delta y}\right) - \left(\frac{T_{i,j} - T_{i,j-1}}{\Delta y}\right)}{\Delta y}$

$= \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2}$

$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

$\Rightarrow \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} = 0$

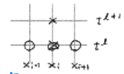
As $\Delta x = \Delta y$

$\Rightarrow T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$

$\Rightarrow T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1}, T_{i,j-1}}{4}$

• Parabolic Equation

$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$



$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$

$\frac{\partial^2 T}{\partial x^2} = \frac{\frac{T^l_{i+1} - T^l_i}{\Delta x} - \frac{T^l_i - T^l_{i-1}}{\Delta x}}{\Delta x}$

$\frac{\partial T}{\partial t} = \frac{T^{l+1}_i - T^l_i}{\Delta t}$

$\Rightarrow k\frac{T^l_{i+1} - 2T^l_i + T^l_{i-1}}{\Delta x^2} = \frac{T^{l+1}_i - T^l_i}{\Delta t}$

$\Rightarrow T^{l+1}_i - T^l_i = \frac{k\Delta t}{\Delta x^2}(T^l_{i+1} - 2T^l_i + T^l_{i-1})$

$\Rightarrow T^{l+1}_i = T^l_i + \lambda(T^l_{i+1} - 2T^l_i + T^l_{i-1})$

$\lambda = \frac{k\Delta t}{\Delta x^2}$

e.g.

$\Delta x = 2$ cm

$\Delta t = 0.1$ s

$k = 0.835$ cm²/s

$\lambda = 0.020875$

position

$$\begin{cases} T(0) = 100 & @ \forall t \\ T(10) = 50 & \\ T(x) = 0 & @ \; t = 0, 0 < x < 10 \end{cases}$$

$T^{l+1}_i = T^l_i + \lambda(T^l_{i+1} - 2T^l_i + T^l_{i-1})$

△ compute T @ $t = 0.1$ s, $x = 2$ cm

time

$T^1_1 = T^0_1 + 0.020875(T^0_2 - 2T^0_1 + T^0_0)$

position

$= 0 + 0.020875(0 - 2 \cdot 0 + 100)$

$= 2.0875$

△ compute T @ $t = 0.1$ s, $x = 4$ cm

$T^1_2 = T^0_2 + 0.020875(T^0_3 - 2T^0_2 + T^0_1)$

$= 0 + 0.020875(0 - 2 \cdot 0 + 0)$

$= 0$

△ compute T @ $t = 0.1$ s, $x = 6$ cm

$T^1_3 = T^0_3 + 0.020875(T^0_4 - 2T^0_3 + T^0_2)$

$= 0 + 0.020875(0 - 2 \cdot 0 + 0)$

$= 0$

△ compute T @ $t = 0.1$ s, $x = 8$ cm

$T^1_4 = T^0_4 + 0.020875(T^0_5 - 2T^0_4 + T^0_3)$

$= 0 + 0.020875(50 - 0 + 0)$

$= 1.0438$

△ Compute T @ $t = 0.2$ s, $x = 2$ cm

$T^2_1 = T^1_1 + \lambda(T^1_2 - 2T^1_1 + T^1_0)$

$= 2.0875 + 0.020875(0 - 2 \cdot 2.0875 + 100)$

$= 4.0878$

• convergence

$\lambda = \frac{k\Delta t}{\Delta x^2} \leq \frac{1}{2} \Rightarrow \Delta t \leq \frac{1}{2}\frac{\Delta x^2}{k}$

• $\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$

$$\begin{cases} \frac{\partial^2 T}{\partial x^2} = \frac{T^l_{i+1,j} - 2T^l_{i,j} + T^l_{i-1,j}}{\Delta x^2} \\ \frac{\partial^2 T}{\partial y^2} = \frac{T^l_{i,j+1} - 2T^l_{i,j} + T^l_{i,j-1}}{\Delta y^2} \\ \frac{\partial T}{\partial t} = \frac{T^{l+1}_{i,j} - T^l_{i,j}}{\Delta t} \end{cases}$$

$k\left(\frac{T^l_{i+1,j} - 2T^l_{i,j} + T^l_{i-1,j}}{\Delta x^2} + \frac{T^l_{i,j+1} - 2T^l_{i,j} + T^l_{i,j-1}}{\Delta y^2}\right)$

$= \frac{T^{l+1}_{i,j} - T^l_{i,j}}{\Delta t}$

$\Rightarrow T^{l+1}_{i,j} = T^l_{i,j} + \frac{k\Delta t}{\Delta x^2}(T^l_{i+1,j} - 2T^l_{i,j} + T^l_{i-1,j})$

$+ \frac{k\Delta t}{\Delta y^2}(T^l_{i,j+1} - 2T^l_{i,j} + T^l_{i,j-1})$

let $\Delta x = \Delta y$ $\quad \lambda = \frac{k\Delta t}{\Delta x^2} = \frac{k\Delta t}{\Delta y^2}$

$\Rightarrow T^{l+1}_{i,j} = T^l_{i,j} + \lambda(T^l_{i+1,j} - 2T^l_{i,j} + T^l_{i-1,j}$

$+ T^l_{i,j+1} - 2T^l_{i,j} + T^l_{i,j-1})$

• convergence

• 1-dimension:

$\lambda = \frac{k\Delta t}{\Delta x^2} \leq \frac{1}{2}$

• 2-dimension:

$\Delta t \leq \frac{1}{8}\frac{\Delta x^2 + \Delta y^2}{k}$

$\Downarrow$

if $\Delta x^2 = \Delta y^2$

$\Delta t \leq \frac{1}{4}\frac{\Delta x^2}{k}$