Kalman Filter

from Dr. Can

1. recursive algorithm

(an optimal recursive data processing algorithm)

an example when there are k measurements: intuitively, we take average

△ $$\hat{x}_k = \frac{1}{k}(z_1 + z_2 + \cdots + z_k)$$
$$= \frac{1}{k}(z_1 + z_2 + \cdots + z_{k-1}) + \frac{1}{k} z_k$$
$$= \frac{1}{k}\frac{k-1}{k-1}(z_1 + z_2 + \cdots + z_{k-1}) + \frac{1}{k} z_k$$
$$= \frac{k-1}{k}\hat{x}_{k-1} + \frac{1}{k} z_k$$
$$= \hat{x}_{k-1} - \frac{1}{k}\hat{x}_{k-1} + \frac{1}{k} z_k$$

$$\Rightarrow \hat{x}_k = \hat{x}_{k-1} + \frac{1}{k}(z_k - \hat{x}_{k-1})$$

$k\uparrow$, $\frac{1}{k} \to 0$ $\quad \hat{x}_k \to \hat{x}_{k-1}$ (measurement less important)

$k\downarrow$ $\quad \frac{1}{k}\uparrow$ $\quad z_k$ more important

△ let $\frac{1}{k} := K_k$ $\quad K_k$

$$\therefore \hat{x}_k = \hat{x}_{k-1} + K_k(z_k - \hat{x}_{k-1})$$

△ induce error into consideration

e_{est} $\quad e_{mea}$

$$K_k = \frac{e_{est_{k-1}}}{e_{est_{k-1}} + e_{mea_k}}$$

@k ① if $e_{est_{k-1}} \gg e_{mea_k}$:

$\to K_k \to 1$

$\to \hat{x}_k = \hat{x}_{k-1} + z_k - \hat{x}_{k-1}$

$= z_k$

② if $e_{est_{k-1}} \ll e_{mea_k}$:

$\to K_k \to 0$

$\to \hat{x}_k = \hat{x}_{k-1}$

△ KF algorithm

① calculate $K_k = \frac{e_{est_{k-1}}}{e_{est_{k-1}} + e_{mea_k}}$

② calculate $\hat{x}_k = \hat{x}_{k-1} + K_k(z_k - \hat{x}_{k-1})$

③ update $e_{est_k} = (1 - K_k) e_{est_{k-1}}$

2. Data fusion

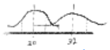
Covariance Matrix

State Space

Observation

△ Data Fusion

- e.g. $z_1 = 30\,g$ $\quad \sigma_1 = 2\,g$
  $z_2 = 32\,g$ $\quad \sigma_2 = 4\,g$



- what is $\hat{z}$?
  $\hat{z} = z_1 + k(z_2 - z_1)$
- then what is k?
  optimal k occurs @ $\sigma_{\hat{z}}$ has min.

$$\sigma_{\hat{z}}^2 = Var(z_1 + k(z_2 - z_1))$$
$$= Var(z_1 + kz_2 - kz_1)$$
$$= Var((1-k)z_1 + kz_2)$$
$$= Var((1-k)z_1) + Var(kz_2)$$
$$= (1-k)^2 Var(z_1) + k^2 Var(z_2)$$
$$= (1-k)^2\sigma_1^2 + k^2\sigma_2^2$$

$\Rightarrow$ min. value @ $\frac{d}{dk}\sigma_{\hat{z}}^2 = 0$

$$\Rightarrow \frac{d}{dk}\sigma_{\hat{z}}^2 = -2(1-k)\sigma_1^2 + 2k\sigma_2^2 = 0$$
$$= -\sigma_1^2 + k\sigma_1^2 + k\sigma_2^2 = 0$$
$$\Rightarrow k(\sigma_1^2 + \sigma_2^2) = \sigma_1^2$$
$$\Rightarrow k = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

△ Covariance Matrix

	x height	y weight	z age
P1	179	74	33
P2	187	80	31
P3	175	71	28
avg	180.3	75	30.7

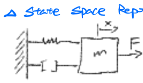
$\sigma_x^2 = \frac{1}{3}[(179-180.3)^2 + (187-180.3)^2 + (175-180.3)^2] = 24.89$

$\sigma_x\sigma_y = \frac{1}{3}[(179-180.3)(74-75) + (187-180.3)(80-75) + (175-180.3)(71-75)] = 18.7$

covariance matrix (control)

- $$P = \begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y & \sigma_x\sigma_z \\ \sigma_y\sigma_x & \sigma_y^2 & \sigma_y\sigma_z \\ \sigma_z\sigma_x & \sigma_z\sigma_y & \sigma_z^2 \end{bmatrix}$$
- $$a = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$
  $P = \frac{1}{3}a^T a$

△ State Space Representation



- $m\ddot{x} + B\dot{x} + kx = F = u$

dynamics

$x_1 = x$
$x_2 = \dot{x}$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{B}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{dynamics w/o input}}$$

$\dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{x} = \frac{1}{m}(F - B\dot{x} - kx)$

$= \frac{1}{m}(F - Bx_2 - kx_1)$

measurements

- $z_1 = x = x_1$
  $z_2 = \dot{x} = x_2$

- $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{B}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$\dot{x}_t = A \; x_t + B \; u_t$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$z_t = H \; x_t$

to discrete

$$x_k = Ax_{k-1} + Bu_k + w_{k-1}$$
$$z_k = Hx_k + v_k$$

$\therefore$ How to get $\hat{x}_k$

KF math

$$x_k = Ax_{k-1} + Bu_k + w_k$$
$$z_k = Hx_k + v_k$$

△ $P(w) \sim (0, Q)$

$Q = E[w w^T]$

$$\cdot\ E\left[\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\begin{bmatrix} w_1 & w_2 \end{bmatrix}\right] \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$
$$= \begin{bmatrix} E[w_1^2] & E[w_1w_2] \\ E[w_2w_1] & E[w_2^2] \end{bmatrix}$$

- $VAR = E[x^2] - \underbrace{E^2[x]}$ $= 0$ for w_1, w_2 $\quad \therefore E[a^2] = Var = \sigma^2$

$$\therefore \begin{bmatrix} E[w_1^2] & E[w_1w_2] \\ E[w_2w_1] & E[w_2^2] \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_{w_1}^2 & \sigma_{w_1}\sigma_{w_2} \\ \sigma_{w_2}\sigma_{w_1} & \sigma_{w_2}^2 \end{bmatrix}$$

△ $P(v) \sim (0, R)$

△ Priori

$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$ ① $\quad$ calculated

$z_k = Hx_k \to \hat{x}_{k_{mea}} = H^- z_k$ $\quad$ measured

△ Posteriori

$\hat{x}_k = \hat{x}_k^- + G(H^- z_k - \hat{x}_k^-)$

$G = K_k H$

$$\Rightarrow \hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$$

△ infer K_k

get K_k s.t. $\hat{x}_k \to x_k$

- let $e_k = x_k - \hat{x}_k$
  $P(e_k) \sim (0, P)$
  $$P = E[e e^T] = \begin{bmatrix} \sigma_{e_1}^2 & \sigma_{e_1}\sigma_{e_2} \\ \sigma_{e_2}\sigma_{e_1} & \sigma_{e_2}^2 \end{bmatrix}$$
- objective: minimize $tr(P) = \sigma_{e_1}^2 + \sigma_{e_2}^2$
- $P = E[e e^T]$
  $= E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T]$

$x_k - \hat{x}_k = x_k - (\hat{x}_k^- + K_k(z_k - H\hat{x}_k^-))$
$= x_k - \hat{x}_k^- - K_k z_k + K_k H\hat{x}_k^-$
$= x_k - \hat{x}_k^- - K_k H x_k - K_k v_k + K_k H\hat{x}_k^-$
$= (x_k - \hat{x}_k^-) - K_k H(x_k - x_k^-) - K_k v_k$
$= (I - K_k H)(x_k - \hat{x}_k^-) - K_k v_k$

$$= E[[(I - K_kH)e_k^- - K_kv_k][(I - K_kH)e_k^- - K_kv_k]^T]$$
$$= E[(I - K_kH)e_k^- e_k^{-T}(I - K_kH)^T$$
$$\cancel{- (I - K_kH)e_k^- v_k^T K_k^T} = 0$$
$$\cancel{- K_kv_k e_k^{-T}(I - K_kH)^T} = 0$$
$$+ K_kv_kv_k^TK_k^T]$$
$$= (I - K_kH)E(e_k^- e_k^{-T})(I - K_kH)^T + K_kE[v_kv_k^T]K_k^T$$
$$= (I - K_kH)P_k^-(I - K_kH)^T + K_kE[v_kv_k^T]K_k^T$$
$$= (P_k^- - K_kHP_k^-)(I - K_kH)^T + K_kRK_k^T$$

$$\Rightarrow P_k = P_k^- - K_kHP_k^- - P_k^-H^TK_k^T + K_kHP_k^-H^TK_k^T + K_kRK_k^T$$

$$\Rightarrow tr(P_k) = tr(P_k^-) - 2\,tr(K_kHP_k^-) + tr(K_kHP_k^-H^TK_k^T) + tr(K_kRK_k^T)$$

$$\frac{d\,tr(AB)}{dA} = B^T$$

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & \vdots \\ \vdots & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$\therefore tr(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}$

$$\therefore \frac{d\,tr(AB)}{dA} = \begin{bmatrix} \frac{\partial tr(AB)}{\partial a_{11}} & \frac{\partial tr(AB)}{\partial a_{12}} \\ \frac{\partial tr(AB)}{\partial a_{21}} & \frac{\partial tr(AB)}{\partial a_{22}} \end{bmatrix} = B^T$$

- $$\frac{d\,tr(P_k)}{dK_k} = 0 - 2(HP_k^-)^T + 2K_kHP_k^-H^T + 2K_kR = 0$$

$$\therefore K_k = P_k^-H^T(HP_k^-H^T + R)^{-1}$$

△ Priori/Posteriori: Error Covariance Matrix

- recall

$x_k = Ax_{k-1} + Bu_k + w_k$ $\quad w \sim P(0, Q)$

$z_k = Hx_k + v_k$ $\quad v \sim P(0, R)$

- Priori
  $\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$

  Posteriori
  $\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$

  Kalman gain

  $K_k = P_k^-H^T(HP_k^-H^T + R)^{-1}$

- $P_k^- := E[e_k^- e_k^{-T}]$

$\Rightarrow e_k^- = x_k - \hat{x}_k^-$
$= Ax_{k-1} + \cancel{Bu_k} + w_k - A\hat{x}_{k-1} - \cancel{Bu_k}$
$= A(x_{k-1} - \hat{x}_{k-1}) + w_k$
$= Ae_{k-1} + w_k$

$\therefore P_k^- = E[(Ae_{k-1} + w_k)(Ae_{k-1} + w_k)^T]$
$= E[Ae_{k-1}e_{k-1}^TA^T + \cancel{Ae_{k-1}w_k^T} + \cancel{w_ke_{k-1}^TA^T} + w_kw_k^T]$
$= AE[e_{k-1}e_{k-1}^T]A^T + E[w_{k-1}w_{k-1}^T]$
$= AP_{k-1}A^T + Q$

- $P_k = P_k^- - K_kHP_k^- - P_k^-H^TK_k^T + K_kHP_k^-H^TK_k^T + K_kRK_k^T$
  $= \cdots$ omitted $\cdots$
  $= (I - K_kH)P_k^-$

In sum:

$$\left.\begin{array}{ll} \text{priori} & \hat{x}_k^- = A\hat{x}_{k-1} + Bu_k \\ \text{cov.} & P_k^- = AP_{k-1}A^T + Q \end{array}\right\} \text{PREDICT}$$

posteriori	$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k)$
cov.	$P_k = (I - K_kH)P_k^-$ $((I-KH)P^-(I-KH)^T + KRK^T)$
Kalman gain	$K_k = P_k^-H^T(HP_k^-H^- + R)^{-1}$
