from Dr. Can

Kalman Filter

1. recursive algorithm

(an optimal recursive data processing algorithm)

an example when there are k measurements: intuitively, we take average

- $\hat{x}_k = \frac{1}{k}(z_1 + z_2 + \cdots + z_k)$

$$= \frac{1}{k}(z_1 + z_2 + \cdots + z_{k-1}) + \frac{1}{k} z_k$$
$$= \frac{1}{k}\frac{k-1}{k-1}(z_1 + z_2 + \cdots + z_{k-1}) + \frac{1}{k} z_k$$
$$= \frac{k-1}{k}\hat{x}_{k-1} + \frac{1}{k} z_k$$
$$= \hat{x}_{k-1} - \frac{1}{k}\hat{x}_{k-1} + \frac{1}{k} z_k$$

$\Rightarrow \hat{x}_k = \hat{x}_{k-1} + \frac{1}{k}(z_k - \hat{x}_{k-1})$

$k \uparrow$, $\frac{1}{k} \to 0$ $\quad \hat{x}_k \to \hat{x}_{k-1}$ (measurement less important)

$k \downarrow$ $\quad \frac{1}{k} \uparrow$ $\quad z_k$ more important

- let $\frac{1}{k} := K_k$ $\quad K_k$

$\therefore \hat{x}_k = \hat{x}_{k-1} + K_k(z_k - \hat{x}_{k-1})$

- induce error into consideration

e_{est} $\quad e_{mea}$

$$K_k = \frac{e_{est_{k-1}}}{e_{est_{k-1}} + e_{mea_k}}$$

@k ① if $e_{est_{k-1}} \gg e_{mea_k}$:

$\to K_k \to 1$

$\to \hat{x}_k = \hat{x}_{k-1} + z_k - \hat{x}_{k-1} = z_k$

② if $e_{est_{k-1}} \ll e_{mea_k}$:

$\to K_k \to 0$

$\to \hat{x}_k = \hat{x}_{k-1}$

- KF algorithm

① calculate $K_k = \frac{e_{est_{k-1}}}{e_{est_{k-1}} + e_{mea_k}}$

② calculate $\hat{x}_k = \hat{x}_{k-1} + K_k(z_k - \hat{x}_{k-1})$

③ update $e_{est_k} = (1 - K_k)\, e_{est_{k-1}}$

2. Data fusion

Covariance Matrix

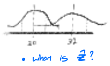
State Space

Observation

- Data Fusion

- eg. $z_1 = 30$ g $\quad \sigma_1 = 2$ g

$z_2 = 32$ g $\quad \sigma_2 = 4$ g



- what is $\hat{z}$?

$\hat{z} = z_1 + K(z_2 - z_1)$

- then what is K?

optimal K occurs @ $\sigma_{\hat{z}}$ has min.

$$\sigma_{\hat{z}}^2 = Var(z_1 + K(z_2 - z_1))$$
$$= Var(z_1 + Kz_2 - Kz_1)$$
$$= Var((1-K)z_1 + Kz_2)$$
$$= Var((1-K)z_1) + Var(Kz_2)$$
$$= (1-K)^2 Var(z_1) + K^2 Var(z_2)$$
$$= (1-K)^2 \sigma_1^2 + K^2 \sigma_2^2$$

$\Rightarrow$ min. value @ $\frac{d}{dK}\sigma_{\hat{z}}^2 = 0$

$\Rightarrow \frac{d}{dK}\sigma_{\hat{z}}^2 = -2(1-K)\sigma_1^2 + 2K\sigma_2^2 = 0$

$= -\sigma_1^2 + K\sigma_1^2 + K\sigma_2^2 = 0$

$\Rightarrow K(\sigma_1^2 + \sigma_2^2) = \sigma_1^2$

$\Rightarrow K = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$

- Covariance Matrix

	x height	y weight	z age
P1	179	74	33
P2	187	80	31
P3	175	71	28
avg	180.3	75	30.7

$\sigma_x^2 = \frac{1}{3}[(179-180.3)^2 + (187-180.3)^2 + (175-180.3)^2] = 24.89$

$\sigma_x\sigma_y = \frac{1}{3}[(179-180.3)(74-75) + (187-180.3)(80-75) + (175-180.3)(71-75)] = 18.7$

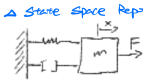
covariance matrix (contd)

- $P = \begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y & \sigma_x\sigma_z \\ \sigma_y\sigma_x & \sigma_y^2 & \sigma_y\sigma_z \\ \sigma_z\sigma_x & \sigma_z\sigma_y & \sigma_z^2 \end{bmatrix}$

- $a = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$

$P = \frac{1}{3}a^T a$

- State Space Representation



- $m\ddot{x} + B\dot{x} + kx = F = u$

dynamics

$x_1 = x$

$x_2 = \dot{x}$

$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{B}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{dynamics w/o input}}$

$\dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{x} = \frac{1}{m}(F - B\dot{x} - kx)$

$= \frac{1}{m}(F - Bx_2 - kx_1)$

measurements

- $z_1 = x = x_1$

$z_2 = \dot{x} = x_2$

- $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{B}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$

$\dot{X}_t = A\, X_t + B\, u_t$

$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$z_t = H\, x_t$

to discrete

$X_k = AX_{k-1} + Bu_k + W_{k-1}$

$z_k = HX_k + V_k$

$\therefore$ How to get $\hat{X}_k$

KF math

$X_k = AX_{k-1} + Bu_k + W_k$

$z_k = HX_k + V_k$

- $P(w) \sim (0, Q)$

$Q = E[w w^T]$

- $E\left[\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}[w_1\ w_2]\right]$ $\quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$= \begin{bmatrix} E[w_1^2] & E[w_1w_2] \\ E[w_2w_1] & E[w_2^2] \end{bmatrix}$

- $VAR = E[X^2] - \underbrace{E^2[X]}_{} = 0$ for w_1, w_2

$\therefore E[a^2] = Var = \sigma^2$

$\therefore \begin{bmatrix} E[w_1^2] & E[w_1w_2] \\ E[w_2w_1] & E[w_2^2] \end{bmatrix}$

$= \begin{bmatrix} \sigma_{w_1}^2 & \sigma_{w_1}\sigma_{w_2} \\ \sigma_{w_2}\sigma_{w_1} & \sigma_{w_2}^2 \end{bmatrix}$

- $P(V) \sim (0, R)$

- Priori

$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$ ① $\quad$ calculated

$z_k = Hx_k \to \hat{x}_{k_{mea}} = H^- z_k$ $\quad$ measured

- Posteriori

$\hat{x}_k = \hat{x}_k^- + G(H^- z_k - \hat{x}_k^-)$

$G = K_k H$

$\Rightarrow \hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$

- infer K_k

get K_k s.t. $\hat{x}_k \to x_k$

- let $e_k = x_k - \hat{x}_k$

$P(e_k) \sim (0, P)$

$P = E[e e^T] = \begin{bmatrix} \sigma_{e_1}^2 & \sigma_{e_1}\sigma_{e_2} \\ \sigma_{e_2}\sigma_{e_1} & \sigma_{e_2}^2 \end{bmatrix}$

- objective: minimize $tr(P) = \sigma_{e_1}^2 + \sigma_{e_2}^2$
- $P = E[e e^T]$

$= E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T]$

$x_k - \hat{x}_k = x_k - (\hat{x}_k^- + K_k(z_k - H\hat{x}_k^-))$

$= x_k - \hat{x}_k^- - K_k z_k + K_k H\hat{x}_k^-$

$= x_k - \hat{x}_k^- - K_k H x_k - K_k v_k + K_k H\hat{x}_k^-$

$= (x_k - \hat{x}_k^-) - K_k H(x_k - \hat{x}_k^-) - K_k v_k$

$= (I - K_k H)(x_k - \hat{x}_k^-) - K_k v_k$

$= E[[(I - K_k H)e_k^- - K_k v_k][(I - K_k H)e_k^- - K_k v_k]^T]$

$= E[(I - K_k H)e_k^- e_k^{-T}(I - K_k H)^T$

$\quad - \cancel{(I - K_k H)e_k^- v_k^T K_k^T} = 0$

$\quad - \cancel{K_k v_k e_k^{-T}(I - K_k H)^T} = 0$

$\quad + K_k v_k v_k^T K_k^T]$

$= (I - K_k H)E(e_k^- e_k^{-T})(I - K_k H)^T$

$\quad + K_k E[v_k v_k^T]K_k^T$

$= (I - K_k H)P_k^-(I - K_k H)^T + K_k E[v_k v_k^T]K_k^T$

$= (P_k^- - K_k H P_k^-)(I - K_k H)^T + K_k R K_k^T$

$\Rightarrow P_k = P_k^- - K_k H P_k^- - P_k^- H^T K_k^T + K_k H P_k^- H^T K_k^T + K_k R K_k^T$

$\Rightarrow tr(P_k) = tr(P_k^-) - 2tr(K_k H P_k^-) + tr(K_k H P_k^- H^T K_k^T) + tr(K_k R K_k^T)$

$\frac{d\,tr(AB)}{dA} = B^T$

$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & \cdots \\ \vdots & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

$\therefore tr(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}$

$\therefore \frac{d\,tr(AB)}{dA} = \begin{bmatrix} \frac{\partial tr(AB)}{\partial a_{11}} & \frac{\partial tr(AB)}{\partial a_{12}} \\ \frac{\partial tr(AB)}{\partial a_{21}} & \frac{\partial tr(AB)}{\partial a_{22}} \end{bmatrix} = B^T$

- $\frac{d\,tr(P_k)}{dK_k} = 0 - 2(HP_k^-)^T + 2K_k H P_k^- H^T + 2K_k R = 0$

$\therefore K_k = P_k^- H^T (H P_k^- H^T + R)^{-1}$

- Priori/Posteriori: Error Covariance Matrix

- recall

$X_k = AX_{k-1} + Bu_k + W_k$ $\quad w \sim P(0, Q)$

$z_k = HX_k + V_k$ $\quad v \sim P(0, R)$

- Priori

$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$

Posteriori

$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$

Kalman gain

$K_k = P_k^- H^T (H P_k^- H^T + R)^{-1}$

- $P_k^- := E[e_k^- e_k^{-T}]$

$\Rightarrow e_k^- = x_k - \hat{x}_k^-$

$= Ax_{k-1} + \cancel{Bu_k} + w_k - A\hat{x}_{k-1} - \cancel{Bu_k}$

$= A(x_{k-1} - \hat{x}_{k-1}) + w_k$

$= A e_{k-1} + w_k$

$\therefore P_k^- = E[(Ae_{k-1} + w_k)(Ae_{k-1} + w_k)^T]$

$= E[A e_{k-1} e_{k-1}^T A^T + \cancel{A e_{k-1} w_k^T} + \cancel{w_k e_{k-1}^T A^T} + w_k w_k^T]$

$= A\,E[e_{k-1} e_{k-1}^T]A^T + E[w_{k-1} w_{k-1}^T]$

$= A P_{k-1} A^T + Q$

- $P_k = P_k^- - K_k H P_k^- - P_k^- H^T K_k^T + K_k H P_k^- H^T K_k^T + K_k R K_k^T$

$= \cdots$ omitted $\cdots$

$= (I - K_k H)P_k^-$

In sum:

priori $\quad \hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$

cov. $\quad P_k^- = A P_{k-1} A^T + Q$ $\quad$ } PREDICT

posteriori $\quad \hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k)$

cov. $\quad P_k = (I - K_k H)P_k^-$ $((I-KH)P^-(I-KH)^T + KRK^T)$

Kalman gain $\quad K_k = P_k^- H^T (H P_k^- H^T + R)^{-1}$
