

COMP5211: Machine Learning

Lecture 5

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Adagrad

$$\theta_{t+1} = \theta_t - \frac{\eta}{\sqrt{\varepsilon I + \text{diag}(G_t)}} \cdot g_t, \quad (1)$$

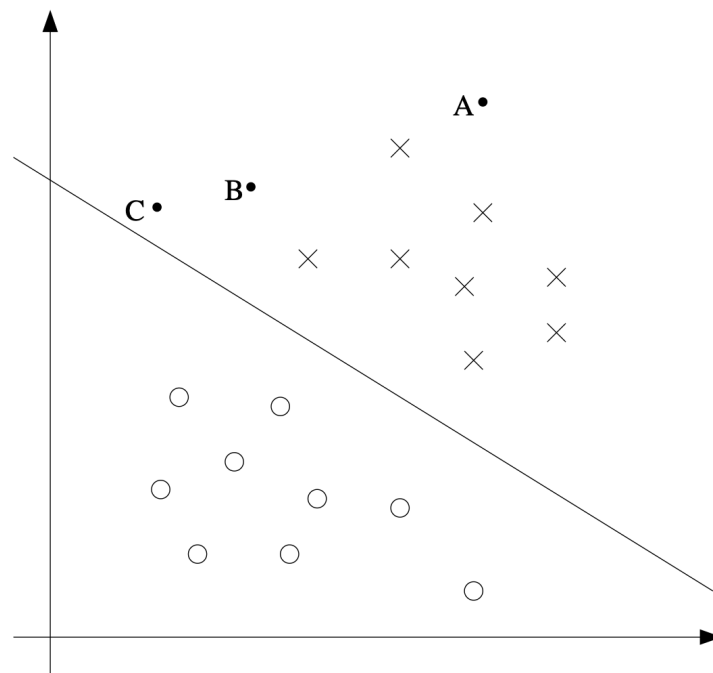
$$g_t = \frac{1}{n} \sum_{i=1}^n \nabla_{\theta} \mathcal{L}(x^{(i)}, y^{(i)}, \theta_t), \quad (2)$$

$$G_t = \sum_{\tau=1}^t g_{\tau} g_{\tau}^{\top}. \quad (3)$$

$$\begin{bmatrix} \theta_{t+1}^{(1)} \\ \theta_{t+1}^{(2)} \\ \vdots \\ \theta_{t+1}^{(m)} \end{bmatrix} = \begin{bmatrix} \theta_t^{(1)} \\ \theta_t^{(2)} \\ \vdots \\ \theta_t^{(m)} \end{bmatrix} - \begin{bmatrix} \frac{\eta}{\sqrt{\varepsilon + G_t^{(1,1)}}} & 0 & \cdots & 0 \\ 0 & \frac{\eta}{\sqrt{\varepsilon + G_t^{(2,2)}}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\eta}{\sqrt{\varepsilon + G_t^{(m,m)}}} \end{bmatrix} \cdot \begin{bmatrix} g_t^{(1)} \\ g_t^{(2)} \\ \vdots \\ g_t^{(m)} \end{bmatrix}. \quad (5)$$

Support Vector Machine Margin

- Intuition
 - Confidence of A,B,C
- Let our decision function as
 - $h_{w,b} = g(w^T x + b)$
 - $g(z) = 1$ if $z \geq 0$, $g(z) = -1$ otherwise
- Function margins: $\hat{y}^{(i)} = y^{(i)}(w^T x + b)$
- However, it will double just replace w with $2w$, b with $2b$



Support Vector Machine

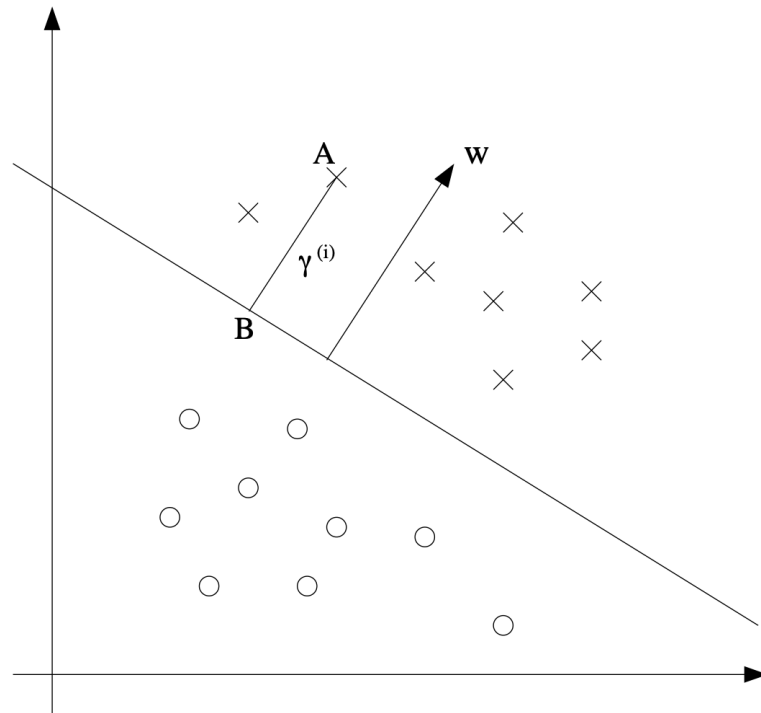
Margin

- Find point B:

- $w^T(x^{(i)} - \gamma^{(i)} \frac{w}{\|w\|}) + b = 0$

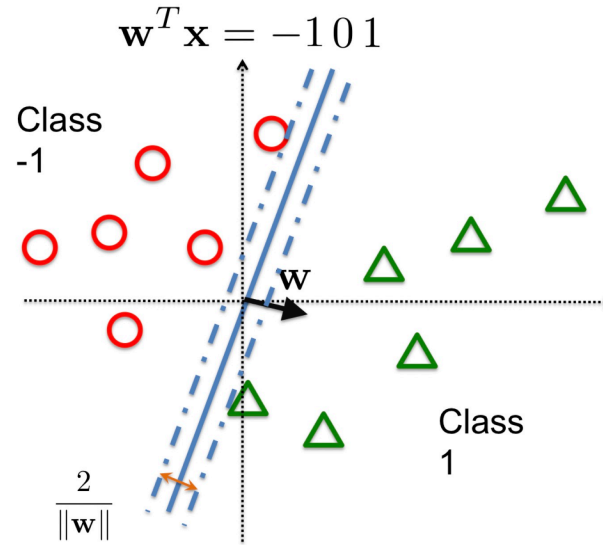
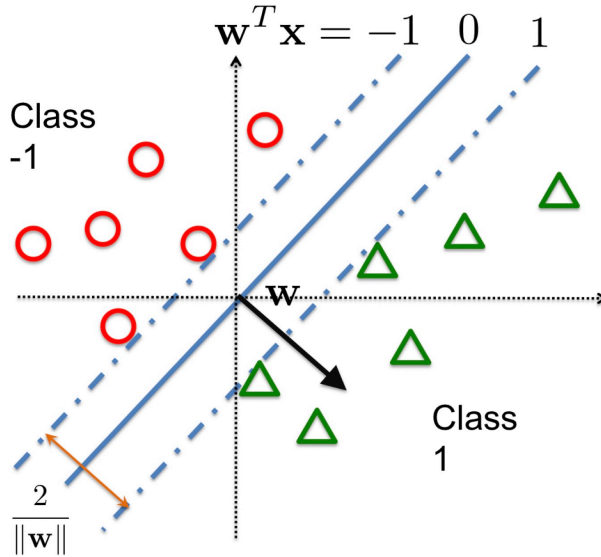
- Geometric margin: $\gamma^{(i)} = \frac{w^T x^{(i)} + b}{\|w\|}$

- $\gamma = \min_{i=1, \dots, N} \gamma^{(i)}$



Support Vector Machine

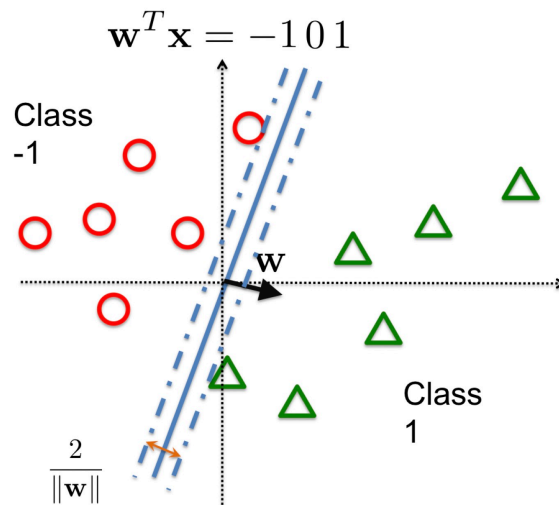
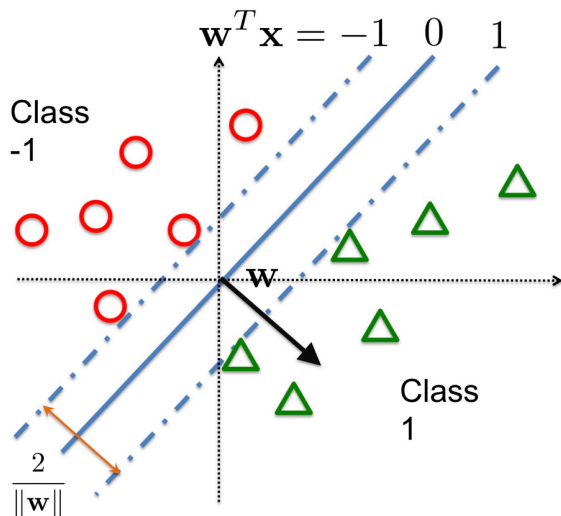
Max margin



Support Vector Machine

Linear SVM

- Goal: Find a hyperplane to separate these two classes of data: if $y_i = 1, w^T x_i \geq 1$; if $y_i = -1, w^T x_i \leq -1$



- Prefer a hyperplane with **maximum margin**

Support Vector Machine

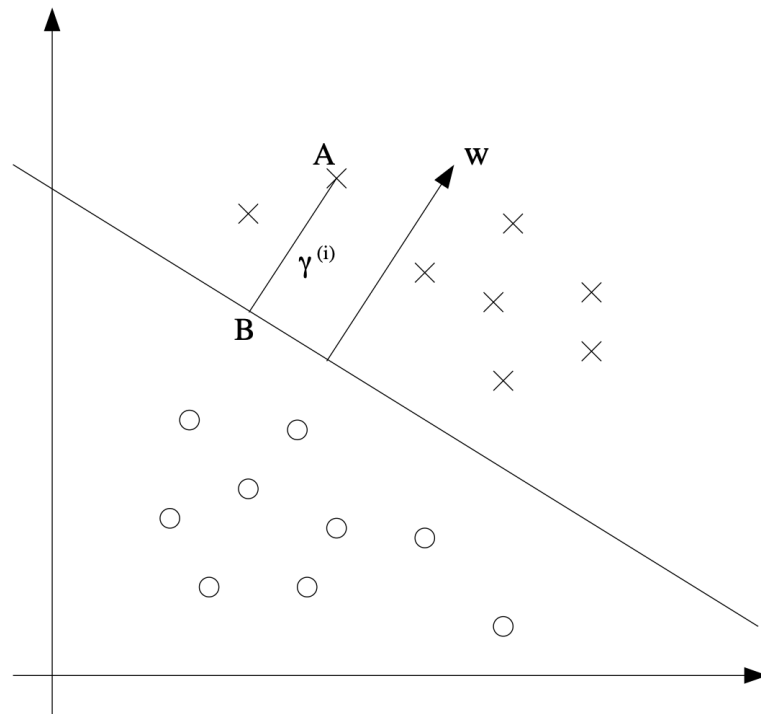
Margin

- $\gamma^{(i)} = \frac{w^T x^{(i)} + b}{\|w\|}$

- $\gamma = \min_{i=1, \dots, m} \gamma^{(i)}$

- $\max_{\gamma, w, b} \frac{\gamma}{\|w\|}$

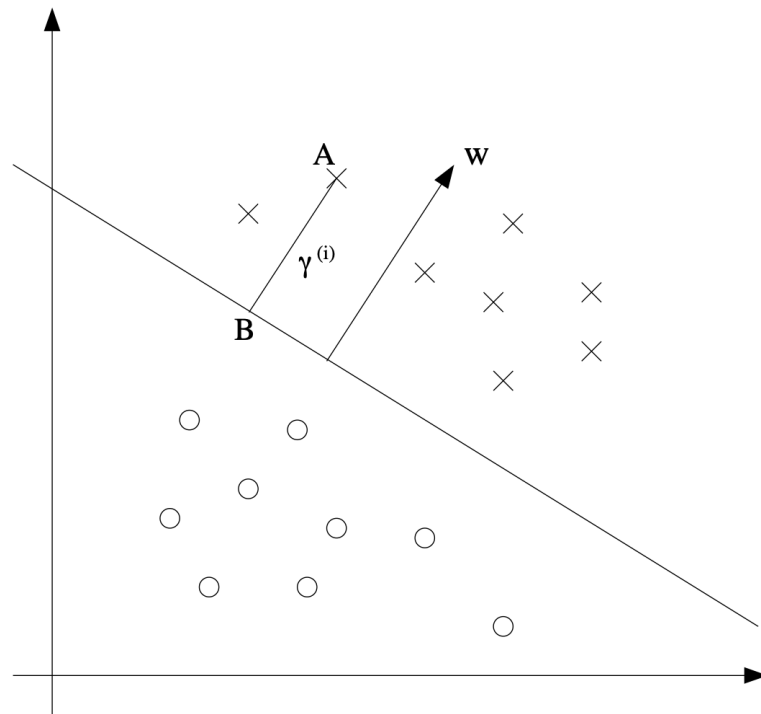
- s.t. $y_i(w^T x_i + b) \geq \gamma, i = 1, \dots, n$



Support Vector Machine

Margin

- $$\max_{\gamma, w, b} \frac{\gamma}{\|w\|}$$
- s.t. $y_i(w^T x_i + b) \geq \gamma, i = 1, \dots, n$
- γ will increase linearly with w, b
- Just set $\gamma = 1$



Support Vector Machine

Margin

$$\max_{\gamma, w, b} \frac{\gamma}{\|w\|}$$

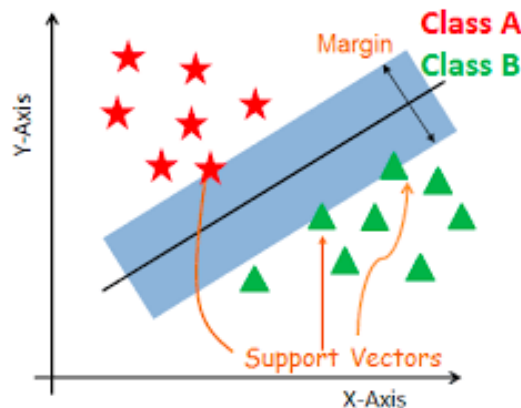
- s.t. $y_i(w^T x_i + b) \geq \gamma, i = 1, \dots, n$

- γ will increase linearly with w, b

- Just set $\gamma = 1$

$$\max_{w, b} \frac{1}{\|w\|}$$

- s.t. $y_i(w^T x_i + b) \geq 1, i = 1, \dots, n$



Support Vector Machine

Linear SVM (hard constraint)

- SVM primal problem (with hard constraints):

$$\begin{aligned} \min_w \quad & \frac{1}{2} w^T w \\ \text{s.t.} \quad & y_i(w^T x_i) \geq 1, i = 1, \dots, n \end{aligned}$$

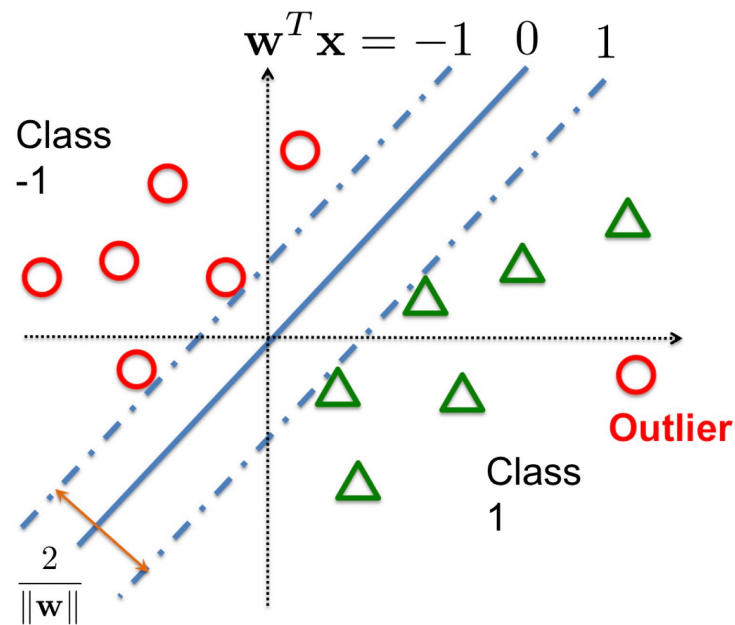
Support Vector Machine

Linear SVM (hard constraint)

- SVM primal problem (with hard constraints):

$$\min_w \quad \frac{1}{2} w^T w$$

- $\text{s.t. } y_i(w^T x_i) \geq 1, i = 1, \dots, n$
- What if there are outliers?



Support Vector Machine

Linear SVM

- Given training examples $(x_1, y_1), \dots, (x_n, y_n)$

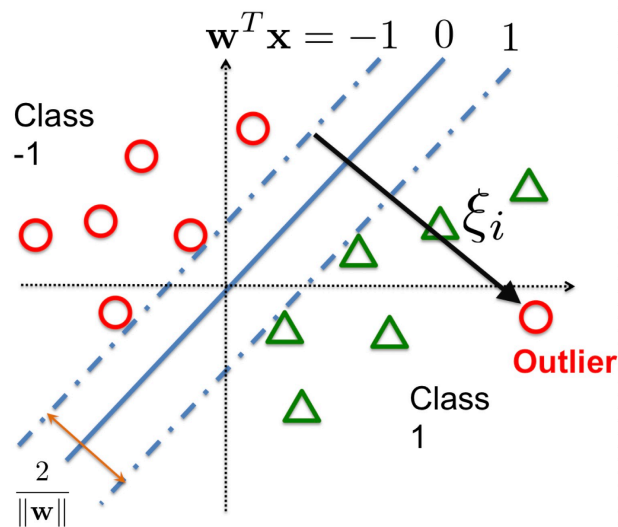
- Consider binary classification:

$$y_i \in \{+1, -1\}$$

- Linear Support Vector Machine (SVM):

$$\min_w \quad \frac{1}{2} w^T w + C \sum_{i=1}^n \xi_i$$

- s.t. $y_i(w^T x_i) \geq 1 - \xi_i, i = 1, \dots, n$
 $\xi_i \geq 0$



Support Vector Machine

Linear SVM

- SVM primal problem:

$$\min_w \quad \frac{1}{2}w^T w + C \sum_{i=1}^n \xi_i$$

- s.t. $y_i(w^T x_i) \geq 1 - \xi_i, i = 1, \dots, n$
 $\xi_i \geq 0$

- Equivalent to

- $$\arg \min_w \underbrace{C \sum_{i=1}^n \max(1 - y_i w^T x_i, 0)}_{\text{hinge loss}} + \underbrace{\frac{1}{2}w^T w}_{\text{L2 regularization}}$$

- None-differentiable when $y_i w^T x_i = 1$ for some i

Support Vector Machine

Linear SVM

- Given training examples $(x_1, y_1), \dots, (x_n, y_n)$

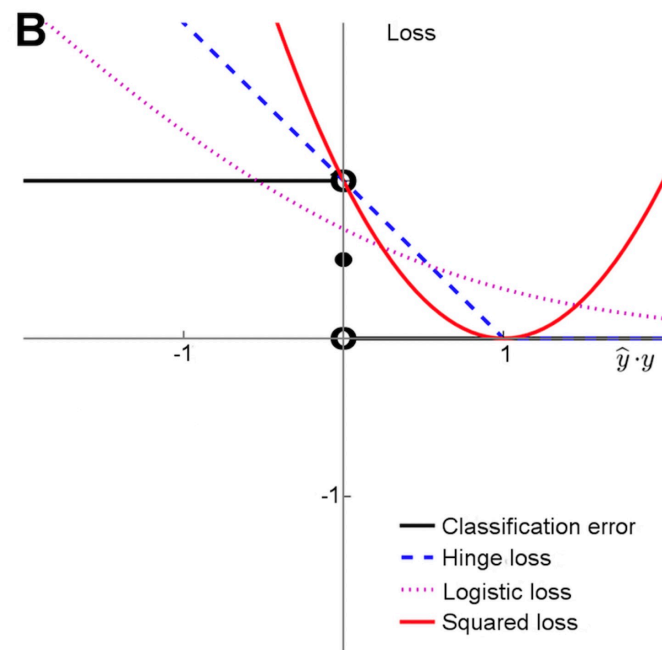
- Consider binary classification:

$$y_i \in \{+1, -1\}$$

- Linear Support Vector Machine (SVM):

$$\arg \min_w C \sum_{i=1}^n \max(1 - y_i w^T x_i, 0) + \frac{1}{2} w^T w$$

- (Hinge loss with l2 regularization)



Support Vector Machine

Stochastic subgradient method for SVM

- A sub gradient of $\ell_i(w) = \max(0, 1 - y_i w^T x_i)$:

$$\nabla_w \ell_i(w) = \begin{cases} -y_i x_i, & \text{if } 1 - y_i w^T x_i > 0 \\ 0, & \text{if } 1 - y_i w^T x_i < 0 \\ 0, & \text{if } 1 - y_i w^T x_i = 0 \end{cases}$$

- Stochastic subgradient descent for SVM:

For $t = 1, 2, \dots$

Randomly pick an index i

If $y_i \mathbf{w}^T \mathbf{x}_i < 1$, then

$$\mathbf{w} \leftarrow (1 - \eta_t) \mathbf{w} + \eta_t n C y_i \mathbf{x}_i$$

Else (if $y_i \mathbf{w}^T \mathbf{x}_i \geq 1$):

$$\mathbf{w} \leftarrow (1 - \eta_t) \mathbf{w}$$

Support Vector Machine

Linear SVM

- SVM primal problem:

$$\min_w \quad \frac{1}{2}w^Tw + C \sum_{i=1}^n \xi_i$$

- s.t. $y_i(w^Tx_i) \geq 1 - \xi_i, i = 1, \dots, n$
 $\xi_i \geq 0$

- Equivalent to

$$\arg \min_w \underbrace{C \sum_{i=1}^n \max(1 - y_i w^T x_i, 0)}_{\text{hinge loss}} + \underbrace{\frac{1}{2}w^Tw}_{\text{L2 regularization}}$$

- None-differentiable when $y_i w^T x_i = 1$ for some i
- Alternatively, we show how to derive the [dual form](#) of SVM

Support Vector Machine

Linear SVM dual

- SVM primal problem:

$$\min_w \quad \frac{1}{2}w^T w + C \sum_{i=1}^n \xi_i$$

- s.t. $y_i(w^T x_i) \geq 1 - \xi_i, i = 1, \dots, n$
 $\xi_i \geq 0$

- Equivalent to: (using Lagrange multiplier):

- $$\min_{w, \xi} \max_{\alpha \geq 0, \beta \geq 0} \frac{1}{2} \|w\|^2 + C \sum_i \xi_i - \sum_i \alpha_i (y_i w^T x_i - 1 + \xi_i) - \sum_i \beta_i \xi_i$$

Support Vector Machine

Linear SVM dual form

- SVM primal problem:

$$\min_w \quad \frac{1}{2}w^T w + C \sum_{i=1}^n \xi_i$$

- s.t. $y_i(w^T x_i) \geq 1 - \xi_i, i = 1, \dots, n$
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- Equivalent to: (using Lagrange multiplier):

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- Under certain condition (e.g., Slater's condition), exchanging \min , \max will not change the optimal solution:

$$\max_{\alpha \geq 0, \beta \geq 0} \min_{w, \xi} \frac{1}{2}\|w\|^2 + C \sum_i \xi_i - \sum_i \alpha_i (y_i w^T x_i - 1 + \xi_i) - \sum_i \beta_i \xi_i$$

Support Vector Machine

Linear SVM dual form

- Reorganize the equation:

- $$\max_{\alpha \geq 0, \beta \geq 0} \min_{w, \xi} \frac{1}{2} \|w\|^2 - \sum_i \alpha_i y_i w^T x_i + \sum_i \xi_i (C - \alpha_i - \beta_i) + \sum_i \alpha_i$$

- Now, for any given α, β , the minimizer of w will satisfy

- $$\frac{\partial L}{\partial w} = w - \sum_i \alpha_i y_i x_i = 0 \Rightarrow w^* = \sum_i y_i \alpha_i x_i$$

- Also, we have $C = \alpha_i + \beta_i$, otherwise ξ_i can make the objective function $-\infty$

- Substitue these two equations back we get

- $$\max_{\alpha \geq 0, \beta \geq 0, C = \alpha + \beta} -\frac{1}{2} \sum_{i,j} \alpha_i \alpha_j y_i y_j x_i^T x_j + \sum_i \alpha_i$$

Support Vector Machine

Linear SVM dual

- Therefore, we get the following dual problem

- $\max_{C \geq \alpha \geq 0} \left\{ -\frac{1}{2} \alpha^T Q \alpha + e^T \alpha \right\} := D(\alpha),$

- Where Q is an $n \times n$ matrix with $Q_{ij} = y_i y_j x_i^T x_j$

- Based on the derivations, we know

- Primal minimum = dual maximum (under Slater's condition)

- Let α^* be the dual solution and w^* be the primal solution, we have

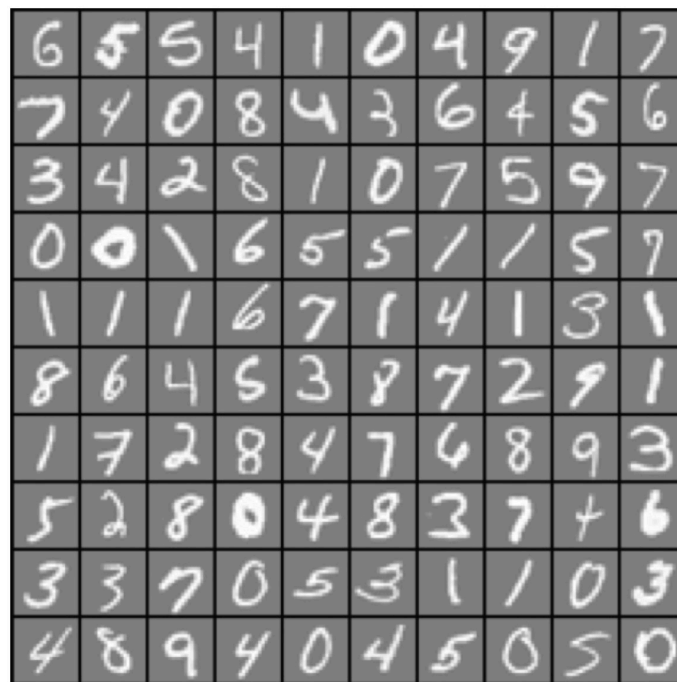
- $w^* = \sum_i y_i \alpha_i^* x_i$

- We can solve the dual problem instead of primal problem

Multi problem

Multi-class

- n data points, L labels, d features
- Input: training data $\{x_i, y_i\}_{i=1}^n$:
 - Each x_i is a d dimensional feature vector
 - Each $y_i \in \{1, \dots, L\}$ is the corresponding label
 - Each training data belongs to **one category**
- Goal: find a function to predict the correct label
 - $f(x) \approx y$



Multi problems

Reduction to binary classification

- Many algorithms for binary classification
- Idea: transform multi-class or multi-label problems to multiple binary classification problems
- Two approaches:
 - One versus All (OVA)
 - One versus One (OVO)

Multi problems

One Versus All (OVA)

- Multi-class/multi-label problems with L categories
- Build L different binary classifiers
- For the t -th classifier:
 - Positive samples: all the points in class t ($\{x_i : t \in y_i\}$)
 - Negative samples: all the points not in class t ($\{x_i : t \notin y_i\}$)
 - $f_t(x)$: the decision value for the t -th classifier
 - (Larger $f_t \Rightarrow$ higher probability that x in class t)
- Prediction:
 - $f(x) = \arg \max_t f_t(x)$
- Example: using SVM to train each binary classifier

Multi problems

One Versus One (OVO)

- Multi-class/multi-label problems with L categories
- Build $L(L - 1)$ different binary classifiers
- For the (s, t) -th classifier:
 - Positive samples: all the points in class s ($\{x_i : s \in y_i\}$)
 - Negative samples: all the points in class t ($\{x_i : t \notin y_i\}$)
 - $f_{s,t}(x)$: the decision value for the t -th classifier
 - (Larger $f_{s,t}(x) \Rightarrow$ label s has higher probability than label t)
 - $f_{t,s}(x) = -f_{s,t}(x)$
- Prediction:

$$f(x) = \arg \max_s \left(\sum_t f_{s,t}(x) \right)$$

- Example: using SVM to train each binary classifier

Multi problems

Another approach for multi-class classification

- OVA and OVO: decompose the problem by labels
 - But good binary classifiers may not imply good multi-class prediction
- Design a multi-class loss function and solve a single optimization problem

Multi problems

Another approach for multi-class classification

- OVA and OVO: decompose the problem by labels
 - But good binary classifiers may not imply good multi-class prediction
- Design a multi-class loss function and solve a single optimization problem
- Minimize the regularized training error:

$$\min_{w_1, \dots, w_L} \sum_{i=1}^n \text{loss}(x_i, y_i) + \lambda \sum_{j=1, \dots, L} w_j^T w_j$$

Multi problems

Main idea

- For simplicity, we assume a linear model
- Model parameters: w_1, \dots, w_L
- For each data point x , compute the decision value for each label:
 - $w_1^T x, w_2^T x, \dots, w_L^T x$
- Prediction:
 - $y = \arg \max_t w_t^T x$
- For training data x_i , y_i is the true label, so we want
 - $y_i \approx \arg \max_t w_t^T x_i, \quad \forall i$

Multi problems

Softmax

- The **predicted score** for each class:

- $w_1^T x_i, w_2^T x_i, \dots$

- Loss for the i-th data is defined by

- $-\log\left(\frac{e^{w_{y_i}^T x_i}}{\sum_j e^{w_j^T x_i}}\right)$ (probability of choosing the **correct label**)

- Solve a single optimization problem

- $\min_{w_1, \dots, w_L} \sum_{i=1}^n -\log\left(\frac{e^{w_{y_i}^T x_i}}{\sum_j e^{w_j^T x_i}}\right) + \lambda \sum_j w_j^T w_j$

Multi problems

Loss functions for multi-class classification

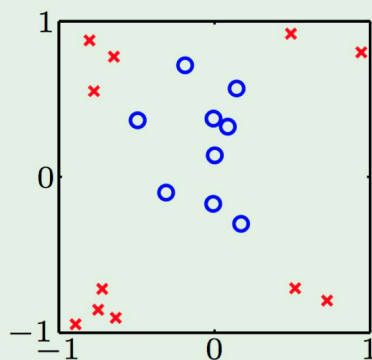
- Ranking based approaches: directly **minimizes the ranking loss**:
 - For multi-class classification, the score of y_i should be larger than other labels
- Soft-max loss:
 - Measure the **probability** of predicting correct class

Nonlinear transformation

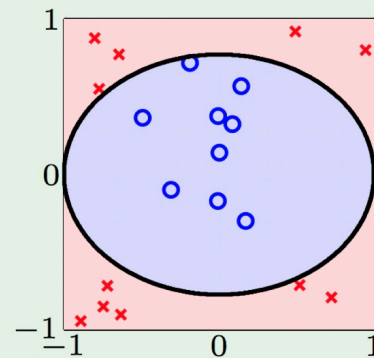
Linear hypotheses

- Up to now: linear hypotheses
 - Perception, Linear regression, Logistic regression, ...
- Many problems are not linearly separable

Data:



Hypothesis:

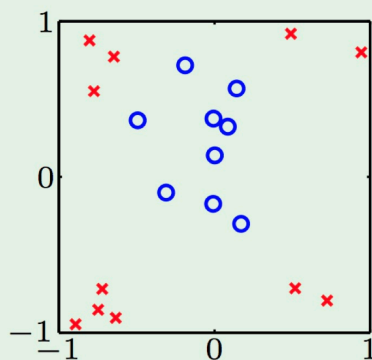


Nonlinear transformation

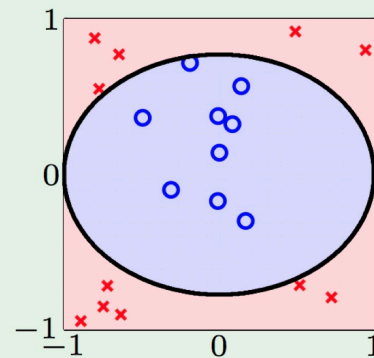
Circular Separable

- Data is not linear separable

Data:



Hypothesis:

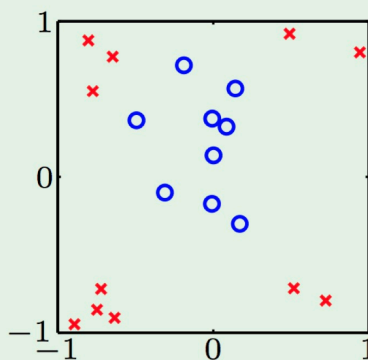


Nonlinear transformation

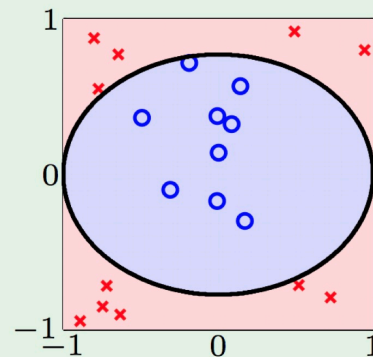
Circular Separable

- Data is not linear separable
- But circular separable by a circle of radius $\sqrt{0.6}$ centered at origin:
 - $h_{\text{SEP}}(x) = \text{sign}(-x_1^2 - x_2^2 + 0.6)$

Data:



Hypothesis:



Nonlinear transformation

Circular Separable and Linear Separable

- $h(x) = \text{sign}(0.6 \cdot 1 + (-1) \cdot x_1^2 + (-1) \cdot x_2^2)$

Nonlinear transformation

Circular Separable and Linear Separable

$$h(x) = \text{sign}(\underbrace{0.6}_{\tilde{w}_0} \cdot \underbrace{1}_{\tilde{z}_0} + \underbrace{(-1)}_{\tilde{w}_1} \cdot \underbrace{x_1^2}_{\tilde{z}_1} + \underbrace{(-1)}_{\tilde{w}_2} \cdot \underbrace{x_2^2}_{\tilde{z}_2})$$

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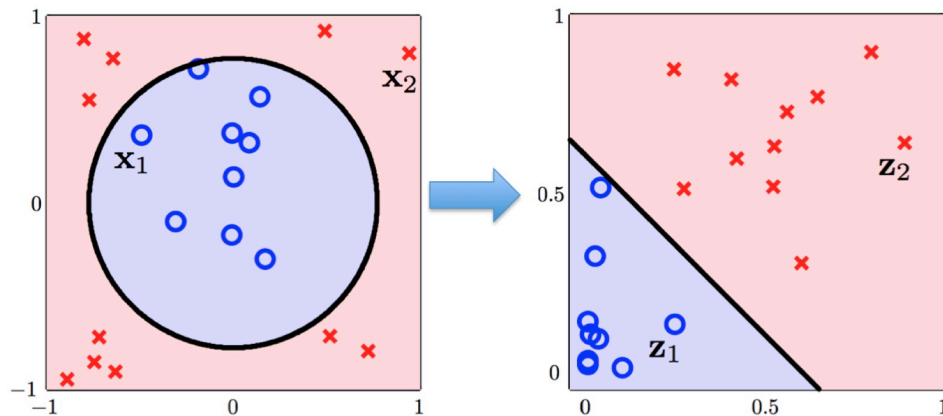
$$= \text{sign}(\tilde{w}^T z)$$

Nonlinear transformation

Circular Separable and Linear Separable

$$h(x) = \text{sign}(\underbrace{0.6}_{\tilde{w}_0} \cdot \underbrace{1}_{\tilde{z}_0} + \underbrace{(-1)}_{\tilde{w}_1} \cdot \underbrace{x_1^2}_{\tilde{z}_1} + \underbrace{(-1)}_{\tilde{w}_2} \cdot \underbrace{x_2^2}_{\tilde{z}_2})$$

- $= \text{sign}(\tilde{w}^T \tilde{z})$
- $\{(x_n, y_n)\}$ circular separable \Rightarrow
 $\{(z_n, y_n)\}$ linear separable

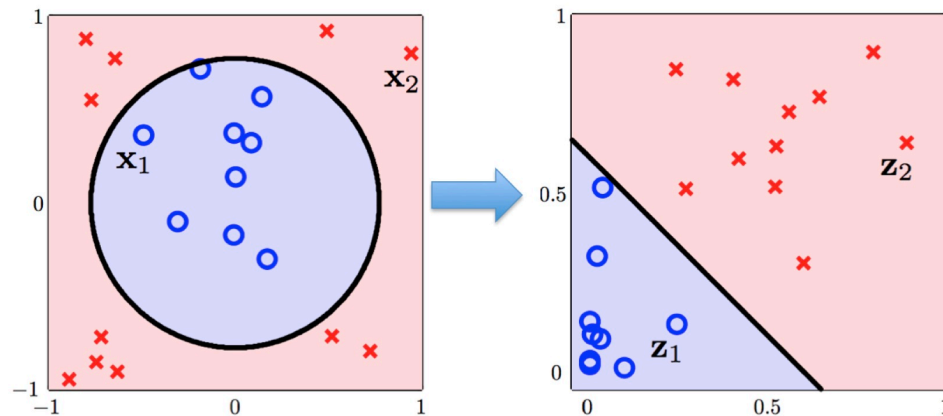


Nonlinear transformation

Circular Separable and Linear Separable

$$h(x) = \text{sign}(\underbrace{0.6}_{\tilde{w}_0} \cdot \underbrace{1}_{\tilde{z}_0} + \underbrace{(-1)}_{\tilde{w}_1} \cdot \underbrace{x_1^2}_{\tilde{z}_1} + \underbrace{(-1)}_{\tilde{w}_2} \cdot \underbrace{x_2^2}_{\tilde{z}_2})$$

- $= \text{sign}(\tilde{w}^T \tilde{z})$
- $\{(x_n, y_n)\}$ circular separable \Rightarrow
 $\{(z_n, y_n)\}$ linear separable
- $x \in \mathcal{X} \rightarrow x \in \mathcal{Z}$ (using a
nonlinear transformation ϕ)



Nonlinear Transformation

Definition

- Define nonlinear transformation
 - $\phi(\mathbf{x}) = (1, x_1^2, x_2^2) = (z_0, z_1, z_2) = \mathbf{z}$

Nonlinear Transformation

Definition

- Define nonlinear transformation
 - $\phi(\mathbf{x}) = (1, x_1^2, x_2^2) = (z_0, z_1, z_2) = \mathbf{z}$
- Linear hypotheses in \mathcal{Z} space:
 - $\text{sign}(\tilde{h}(\mathbf{z})) = \text{sign}(\tilde{h}(\phi(\mathbf{x}))) = \text{sign}(w^T \phi(\mathbf{x}))$

Nonlinear Transformation

Definition

- Define nonlinear transformation
 - $\phi(\mathbf{x}) = (1, x_1^2, x_2^2) = (z_0, z_1, z_2) = \mathbf{z}$
- Linear hypotheses in \mathcal{Z} -space:
 - $\text{sign}(\tilde{h}(\mathbf{z})) = \text{sign}(\tilde{h}(\phi(\mathbf{x}))) = \text{sign}(w^T \phi(\mathbf{x}))$
- Line in \mathcal{Z} -space \Leftrightarrow some quadratic curves in \mathcal{X} -space

Nonlinear Transformation

General Quadratic Hypothesis Set

- A “bigger ” \mathcal{F} space:
 - $\phi_2(\mathbf{x}) = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2)$

Nonlinear Transformation

General Quadratic Hypothesis Set

- A “bigger ” \mathcal{Z} -space:
 - $\phi_2(\mathbf{x}) = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2)$
- Linear in \mathcal{Z} -space \Leftrightarrow quadratic hypotheses in \mathcal{X} -space

Nonlinear Transformation

General Quadratic Hypothesis Set

- A “bigger ” \mathcal{Z} -space:
 - $\phi_2(\mathbf{x}) = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2)$
- Linear in \mathcal{Z} -space \Leftrightarrow quadratic hypotheses in \mathcal{X} -space
- The hypotheses space:
 - $\mathcal{H}_{\phi_2} = \{h(x) : h(x) = \tilde{w}^T \phi_2(x) \text{ for some } \tilde{w}\}$ (quadratic hypotheses)

Nonlinear Transformation

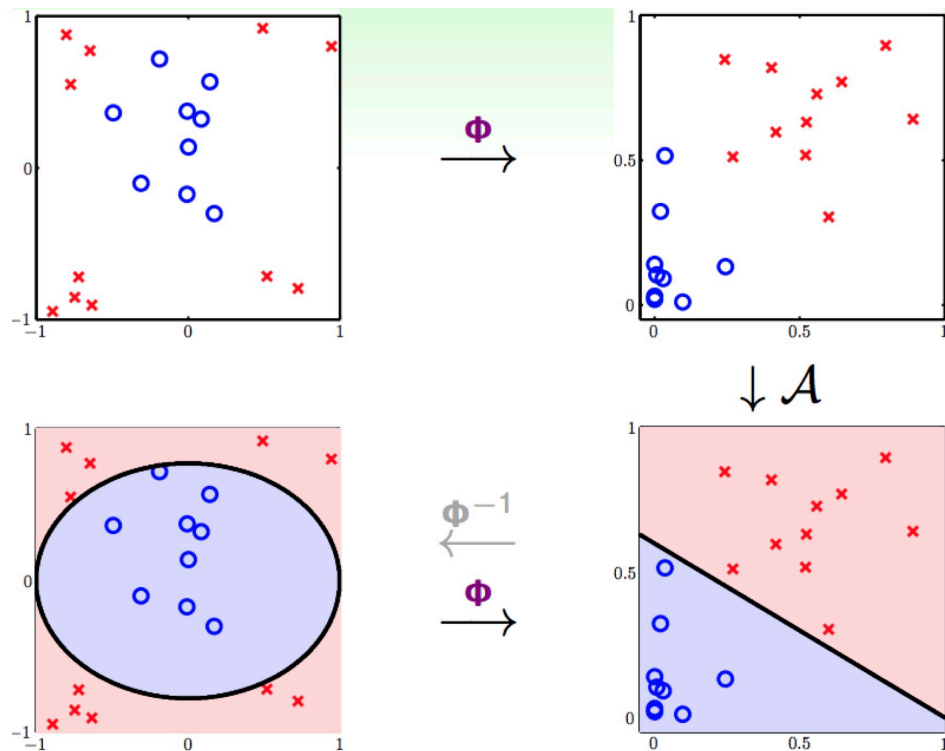
General Quadratic Hypothesis Set

- A “bigger ” \mathcal{Z} -space:
 - $\phi_2(\mathbf{x}) = (1, x_1, x_2, x_1^2, x_1x_2, x_2^2)$
- Linear in \mathcal{Z} -space \Leftrightarrow quadratic hypotheses in \mathcal{X} -space
- The hypotheses space:
 - $\mathcal{H}_{\phi_2} = \{h(x) : h(x) = \tilde{w}^T \phi_2(x) \text{ for some } \tilde{w}\}$ (quadratic hypotheses)
- Also include linear model as a degenerate case

Nonlinear transformation

Learning a good quadratic function

- Transform original data $\{x_n, y_n\}$ to $\{z_n = \phi(x_n), y_n\}$
- Solve a linear problem on $\{z_n, y_n\}$ using your favorite algorithm \mathcal{A} to get a good model \tilde{w}
- Return the model $h(x) = \text{sign}(\tilde{w}^T \phi(x))$



Nonlinear transformation

Polynomial mappings

- Can now freely do quadratic classification, quadratic regression

Nonlinear transformation

Polynomial mappings

- Can now freely do quadratic classification, quadratic regression
- Can easily extend to any degree of polynomial mappings

- E.g.,

$$\phi(x) = (x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2^2, x_1x_3^2, x_1x_2^2, x_2^2x_3, x_2^2x_3, x_1^3, x_2^3, x_3^3)$$

Support Vector Machine

SVM with nonlinear mapping

- SVM with nonlinear mapping $\varphi(\cdot)$:

$$\min_w \quad \frac{1}{2}w^T w + C \sum_{i=1}^n \xi_i$$

- s.t. $y_i(w^T \varphi(x_i)) \geq 1 - \xi_i, i = 1, \dots, n$

$$\xi_i \geq 0$$

- Hard to solve if $\varphi(\cdot)$ maps to very high or infinite dimensional space

Support Vector Machine

SVM with nonlinear mapping

- Similarly, we could derive

- $\max_{C \geq \alpha \geq 0} \left\{ -\frac{1}{2} \alpha^T Q \alpha + e^T \alpha \right\} := D(\alpha),$

- Where Q is an $n \times n$ matrix with $Q_{ij} = y_i y_j \varphi(x_i)^T \varphi(x_j)$

- Based on the derivations, we know

- Primal minimum = dual maximum (under Slater's condition)

- Let α^* be the dual solution and w^* be the primal solution, we have

- $w^* = \sum_i y_i \alpha_i^* \varphi(x_i)$

Nonlinear Transformation

The price we pay: computational complexity

- Q -th order polynomial transform:

$$\phi(x) = (1, x_1, x_2, \dots, x_d,$$

$$x_1^2, x_1x_2, \dots, x_d^2, \dots, x_d^2,$$

...

- $x_1^Q, x_1^{Q-1}x_2, \dots, x_d^Q)$

- $O(d^Q)$ dimensional vector \Rightarrow High computational cost

Support Vector Machine

Kernel trick

- Do **not** directly define $\varphi(\cdot)$
- Instead, define “kernel”
 - $K(x_i, x_j) = \varphi(x_i)^T \varphi(x_j)$

Support Vector Machine

Kernel trick

- Do **not** directly define $\varphi(\cdot)$
- Instead, define “kernel”
 - $K(x_i, x_j) = \varphi(x_i)^T \varphi(x_j)$
- Examples:
 - Gaussian kernel: $K(x_i, x_j) = e^{-\gamma \|x_i - x_j\|^2}$
 - Polynomial kernel: $K(x_i, x_j) = (\gamma x_i^T x_j + c)^d$
 - Other kernels for specific problems:
 - Graph kernels (Vishwanathan et al., “Graph Kernels”, JMLR, 2010)
 - Pyramid kernel for image matching (Grauman and Darrell, “The Pyramid Match Kernel: Discriminative Classification with Sets of Image Features”. In ICCV, 2005)
 - String kernel (Lodhi et al., “Text classification using string kernels”. JMLR, 2002)

Support Vector Machine

SVM with kernel

- Training: compute $\alpha = [\alpha_1, \dots, \alpha_n]$ by solving the quadratic optimization problem:

- $$\min_{0 \leq \alpha \leq C} \frac{1}{2} \alpha^T Q \alpha - e^T \alpha$$

- Where $Q_{ij} = y_i y_j K(x_i, x_j)$

Support Vector Machine

SVM with kernel

- Training: compute $\alpha = [\alpha_1, \dots, \alpha_n]$ by solving the quadratic optimization problem:

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- Where $Q_{ij} = y_i y_j K(x_i, x_j)$

- Prediction: for a test data x ,

- $w^T \varphi(x) = \sum_{i=1}^n y_i \alpha_i \varphi(x_i)^T \varphi(x) = \sum_{i=1}^n y_i \alpha_i K(x_i, x)$

Kernel trick

Kernel Ridge Regression

- Actually, this “kernel method” works for many different losses
- Example: ridge regression

- $$\min_w \frac{1}{2} \|w\|^2 + \frac{1}{2} \sum_{i=1}^n (w^T \varphi(x_i) - y_i)^2$$

- Dual problem:

- $$\min_{\alpha} \alpha^T Q \alpha + \|\alpha\|^2 - 2\alpha^T y$$

Kernel trick

Scalability

- Challenge for solving kernel SVMs (for dataset with n samples):
 - **Space:** $O(n^2)$ for storing the $n \times n$ kernel matrix (can be reduced in some cases);
 - **Time:** $O(n^3)$ for computing the exact solution

Kernel trick

Scalability

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 - **Time:** $O(n^3)$ for computing the exact solution
- Good packages available:
 - LIBSVM (can be called in scikit-learn)
 - LIBLINEAR (for linear SVM, can be called in scikit-learn)