- Motivation
  - true-state
  - nominal-state
  - error-state

△ true-state = nominal-state ⊕ error state
△ nominal-state → takes no account in noise / model imperfection

# Definitions

| Magnitude | True | Nominal | Error | Composition | Measured | Noise |
|---|---|---|---|---|---|---|
| Full state ([1]) | $\mathbf{x}_t$ | $\mathbf{x}$ | $\delta\mathbf{x}$ | $\mathbf{x}_t = \mathbf{x} \oplus \delta\mathbf{x}$ | | |
| Position | $\mathbf{p}_t$ | $\mathbf{p}$ | $\delta\mathbf{p}$ | $\mathbf{p}_t = \mathbf{p} + \delta\mathbf{p}$ | | |
| Velocity | $\mathbf{v}_t$ | $\mathbf{v}$ | $\delta\mathbf{v}$ | $\mathbf{v}_t = \mathbf{v} + \delta\mathbf{v}$ | | |
| Quaternion ([2,3]) | $\mathbf{q}_t$ | $\mathbf{q}$ | $\delta\mathbf{q}$ | $\mathbf{q}_t = \mathbf{q} \otimes \delta\mathbf{q}$ | | |
| Rotation matrix ([2,3]) | $\mathbf{R}_t$ | $\mathbf{R}$ | $\delta\mathbf{R}$ | $\mathbf{R}_t = \mathbf{R}\,\delta\mathbf{R}$ | | |
| Angles vector ([4]) | | | $\delta\boldsymbol{\theta}$ | $\delta\mathbf{q} = e^{\delta\boldsymbol{\theta}/2}$<br>$\delta\mathbf{R} = e^{[\delta\boldsymbol{\theta}]_\times}$ | | |
| Accelerometer bias | $\mathbf{a}_{bt}$ | $\mathbf{a}_b$ | $\delta\mathbf{a}_b$ | $\mathbf{a}_{bt} = \mathbf{a}_b + \delta\mathbf{a}_b$ | | $\mathbf{a}_w$ |
| Gyrometer bias | $\boldsymbol{\omega}_{bt}$ | $\boldsymbol{\omega}_b$ | $\delta\boldsymbol{\omega}_b$ | $\boldsymbol{\omega}_{bt} = \boldsymbol{\omega}_b + \delta\boldsymbol{\omega}_b$ | | $\boldsymbol{\omega}_w$ |
| Gravity vector | $\mathbf{g}_t$ | $\mathbf{g}$ | $\delta\mathbf{g}$ | $\mathbf{g}_t = \mathbf{g} + \delta\mathbf{g}$ | | |
| Acceleration | $\mathbf{a}_t$ | | | | $\mathbf{a}_m$ | $\mathbf{a}_n$ |
| Angular rate | $\boldsymbol{\omega}_t$ | | | | $\boldsymbol{\omega}_m$ | $\boldsymbol{\omega}_n$ |

$R$: rotation matrix from body to inertial frame

NETWORKED CONTROL ROBOTICS LAB 4

- Dynamic

$p_t$, $v_t$, $q_t$, $a_{bt}$, $w_{bt}$, $g_t$

$$\dot{\mathbf{p}}_t = \mathbf{v}_t$$
$$\dot{\mathbf{v}}_t = \mathbf{a}_t$$
$$\dot{\mathbf{q}}_t = \frac{1}{2}\mathbf{q}_t \otimes \boldsymbol{\omega}_t$$
$$\dot{\mathbf{a}}_{bt} = \mathbf{a}_w$$
$$\dot{\boldsymbol{\omega}}_{bt} = \boldsymbol{\omega}_w$$
$$\dot{\mathbf{g}}_t = 0$$

- measurement

$$\mathbf{a}_m = \mathbf{R}_t^\top(\mathbf{a}_t - \mathbf{g}_t) + \mathbf{a}_{bt} + \mathbf{a}_n$$
$$\boldsymbol{\omega}_m = \boldsymbol{\omega}_t + \boldsymbol{\omega}_{bt} + \boldsymbol{\omega}_n$$

Sensor noise

Note: $a_m$, $\omega_m$ are measured in the body-fixed frame
$a_t$, $\omega_t$ are exoressed in the inertial frame

# True State Dynamics

$$\dot{\mathbf{p}}_t = \mathbf{v}_t \quad (235a)$$
$$\dot{\mathbf{v}}_t = \mathbf{R}_t(\mathbf{a}_m - \mathbf{a}_{bt} - \mathbf{a}_n) + \mathbf{g}_t \quad (235b)$$
$$\dot{\mathbf{q}}_t = \frac{1}{2}\mathbf{q}_t \otimes (\boldsymbol{\omega}_m - \boldsymbol{\omega}_{bt} - \boldsymbol{\omega}_n) \quad (235c)$$
$$\dot{\mathbf{a}}_{bt} = \mathbf{a}_w \quad (235d)$$
$$\dot{\boldsymbol{\omega}}_{bt} = \boldsymbol{\omega}_w \quad (235e)$$
$$\dot{\mathbf{g}}_t = 0 \quad (235f)$$

true state

noise term

$$\dot{x}_t = f_t(x_t, u, w)$$

input

$$\mathbf{a}_t = \mathbf{R}_t(\mathbf{a}_m - \mathbf{a}_{bt} - \mathbf{a}_n) + \mathbf{g}_t \quad (233)$$
$$\boldsymbol{\omega}_t = \boldsymbol{\omega}_m - \boldsymbol{\omega}_{bt} - \boldsymbol{\omega}_n. \quad (234)$$

# Nominal State Kinematics

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = \mathbf{R}(\mathbf{a}_m - \mathbf{a}_b) + \mathbf{g}$$
$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{q} \otimes (\boldsymbol{\omega}_m - \boldsymbol{\omega}_b) \quad (237)$$
$$\dot{\mathbf{a}}_b = 0$$
$$\dot{\boldsymbol{\omega}}_b = 0$$
$$\dot{\mathbf{g}} = 0.$$

$$x = \begin{bmatrix} p \\ v \\ q \\ a_b \\ w_b \\ g \end{bmatrix}$$

# Error State Kinematics

$$\boxed{\mathbf{p}_t = \mathbf{p} + \delta\mathbf{p}}$$

$\dot{\delta p}$ =(235)-(237)

$$\dot{\delta\mathbf{p}} = \delta\mathbf{v}$$
$$\dot{\delta\mathbf{v}} = -\mathbf{R}\left[\mathbf{a}_m - \mathbf{a}_b\right]_\times \delta\boldsymbol{\theta} - \mathbf{R}\delta\mathbf{a}_b + \delta\mathbf{g} - \mathbf{R}\mathbf{a}_n$$
$$\dot{\delta\boldsymbol{\theta}} = -\left[\boldsymbol{\omega}_m - \boldsymbol{\omega}_b\right]_\times \delta\boldsymbol{\theta} - \delta\boldsymbol{\omega}_b - \boldsymbol{\omega}_n$$
$$\dot{\delta\mathbf{a}}_b = \mathbf{a}_w$$
$$\dot{\delta\boldsymbol{\omega}}_b = \boldsymbol{\omega}_w$$
$$\dot{\delta\mathbf{g}} = 0.$$

$a_n$, $w_n$, $a_w$, $w_w$

Gaussian noises

# Discrete-time Nominal States

Taking the integration of (237) yields the discrete-time form as

$$\mathbf{p} \leftarrow \mathbf{p} + \mathbf{v}\,\Delta t + \frac{1}{2}(\mathbf{R}(\mathbf{a}_m - \mathbf{a}_b) + \mathbf{g})\,\Delta t^2 \quad (260a)$$
$$\mathbf{v} \leftarrow \mathbf{v} + (\mathbf{R}(\mathbf{a}_m - \mathbf{a}_b) + \mathbf{g})\,\Delta t \quad (260b)$$
$$\mathbf{q} \leftarrow \mathbf{q} \otimes \mathbf{q}\{(\boldsymbol{\omega}_m - \boldsymbol{\omega}_b)\,\Delta t\} \quad (260c)$$
$$\mathbf{a}_b \leftarrow \mathbf{a}_b \quad (260d)$$
$$\boldsymbol{\omega}_b \leftarrow \boldsymbol{\omega}_b \quad (260e)$$
$$\mathbf{g} \leftarrow \mathbf{g}\,, \quad (260f)$$

# Discrete-time Error States

Taking the integration of (238) yields the discrete-time form as

$$\delta\mathbf{p} \leftarrow \delta\mathbf{p} + \delta\mathbf{v}\,\Delta t \quad (261a)$$
$$\delta\mathbf{v} \leftarrow \delta\mathbf{v} + (-\mathbf{R}\left[\mathbf{a}_m - \mathbf{a}_b\right]_\times \delta\boldsymbol{\theta} - \mathbf{R}\delta\mathbf{a}_b + \delta\mathbf{g})\Delta t + \mathbf{v_i} \quad (261b)$$
$$\delta\boldsymbol{\theta} \leftarrow \mathbf{R}^\top\{(\boldsymbol{\omega}_m - \boldsymbol{\omega}_b)\Delta t\}\delta\boldsymbol{\theta} - \delta\boldsymbol{\omega}_b\Delta t + \boldsymbol{\theta}_\mathbf{i} \quad (261c)$$
$$\delta\mathbf{a}_b \leftarrow \delta\mathbf{a}_b + \mathbf{a_i} \quad (261d)$$
$$\delta\boldsymbol{\omega}_b \leftarrow \delta\boldsymbol{\omega}_b + \boldsymbol{\omega}_\mathbf{i} \quad (261e)$$
$$\delta\mathbf{g} \leftarrow \delta\mathbf{g}\,. \quad (261f)$$

$$\mathbf{V_i} = \sigma_{\tilde{\mathbf{a}}_n}^2 \Delta t^2 \mathbf{I} \qquad [m^2/s^2] \quad (262)$$
$$\boldsymbol{\Theta}_\mathbf{i} = \sigma_{\tilde{\boldsymbol{\omega}}_n}^2 \Delta t^2 \mathbf{I} \qquad [rad^2] \quad (263)$$
$$\mathbf{A_i} = \sigma_{\mathbf{a}_w}^2 \Delta t \mathbf{I} \qquad [m^2/s^4] \quad (264)$$
$$\boldsymbol{\Omega}_\mathbf{i} = \sigma_{\boldsymbol{\omega}_w}^2 \Delta t \mathbf{I} \qquad [rad^2/s^2] \quad (265)$$

Integration of covariance matrices

# ESKF v.s. EKF

| ESKF | EKF |
|---|---|
| $\hat{\delta\mathbf{x}} \leftarrow \mathbf{F_x}(\mathbf{x}, \mathbf{u}_m) \cdot \hat{\delta\mathbf{x}}$ (268)<br>$\mathbf{P} \leftarrow \mathbf{F_x}\,\mathbf{P}\,\mathbf{F_x}^\top + \mathbf{F_i}\,\mathbf{Q_i}\,\mathbf{F_i}^\top\,,$ (269) | $A = \frac{\partial f}{\partial x}\vert_{\hat{x}_{k-1},u_{k-1}}$ and $C = \frac{\partial g}{\partial x}\vert_{\hat{x}_k}$<br>$\hat{x}_k^- = f(\hat{x}_{k-1}, u_{k-1})$<br>$P_k^- = A_k P_{k-1} A_k^T + Q$ |
| $\mathbf{K} = \mathbf{P}\mathbf{H}^\top(\mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{V})^{-1}$ (274)<br>$\hat{\delta\mathbf{x}} \leftarrow \mathbf{K}(\mathbf{y} - h(\hat{\mathbf{x}}_t))$ (275)<br>$\mathbf{P} \leftarrow (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}$ (276) | $\mathcal{K}_k = P_k^- C^T (C P_k^- C^T + R)^{-1}$<br>$\hat{x}_k = \hat{x}_k^- + \mathcal{K}_k(y_k - \hat{y}_k)$<br>$P_k = (I - \mathcal{K}_k C) P_k^-$ |

# Injection of the Observed Error into the Nominal State

### 6.2 Injection of the observed error into the nominal state

After the ESKF update, the nominal state gets updated with the observed error state using the appropriate compositions (sums or quaternion products, see Table 3),

$$\mathbf{x} \leftarrow \mathbf{x} \oplus \hat{\delta\mathbf{x}}\,, \quad (282)$$

that is,

$$\mathbf{p} \leftarrow \mathbf{p} + \hat{\delta\mathbf{p}} \quad (283a)$$
$$\mathbf{v} \leftarrow \mathbf{v} + \hat{\delta\mathbf{v}} \quad (283b)$$
$$\mathbf{q} \leftarrow \mathbf{q} \otimes \mathbf{q}\{\hat{\delta\boldsymbol{\theta}}\} \quad (283c)$$
$$\mathbf{a}_b \leftarrow \mathbf{a}_b + \hat{\delta\mathbf{a}}_b \quad (283d)$$
$$\boldsymbol{\omega}_b \leftarrow \boldsymbol{\omega}_b + \hat{\delta\boldsymbol{\omega}}_b \quad (283e)$$
$$\mathbf{g} \leftarrow \mathbf{g} + \hat{\delta\mathbf{g}} \quad (283f)$$

# Error State Reset

Let us call the error reset function $g()$. It is written as follows,

$$\delta\mathbf{x} \leftarrow g(\delta\mathbf{x}) = \delta\mathbf{x} \ominus \hat{\delta\mathbf{x}}\,, \quad (284)$$

where $\ominus$ stands for the composition inverse of $\oplus$. The ESKF error reset operation is thus,

$$\hat{\delta\mathbf{x}} \leftarrow 0 \quad (285)$$
$$\mathbf{P} \leftarrow \mathbf{G}\,\mathbf{P}\,\mathbf{G}^\top\,. \quad (286)$$

where $\mathbf{G}$ is the Jacobian matrix defined by,

$$\mathbf{G} \triangleq \left.\frac{\partial g}{\partial \delta\mathbf{x}}\right|_{\hat{\delta\mathbf{x}}}\,. \quad (287)$$

Similarly to what happened with the update Jacobian above, this Jacobian is the identity on all diagonal blocks except in the orientation error. We give here the full expression and proceed in the following section with the derivation of the orientation error block, $\partial\delta\boldsymbol{\theta}^+/\partial\delta\boldsymbol{\theta} = \mathbf{I} - \left[\frac{1}{2}\hat{\delta\boldsymbol{\theta}}\right]_\times$,

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_6 & 0 & 0 \\ 0 & \mathbf{I} - \left[\frac{1}{2}\hat{\delta\boldsymbol{\theta}}\right]_\times & 0 \\ 0 & 0 & \mathbf{I}_9 \end{bmatrix}. \quad (288)$$