ME564 L8

Last time $\dot{x} = Ax$

$x(t) = Te^{Dt}T^{-1}x(0)$

$A = TDT^{-1}$

in Matlab

```
>> [T,D] = eig(A);
>> d = diag(D);
>> etothed = exp(d);
          = exp(d*t);
>> etotheD = diag(etothed);
```

$\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$

$\rightarrow \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$

$\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \\ & & e^{\lambda_n t} \end{bmatrix}$

Next time $\dot{x} = f(x) \Rightarrow$ Linearize

Ex:

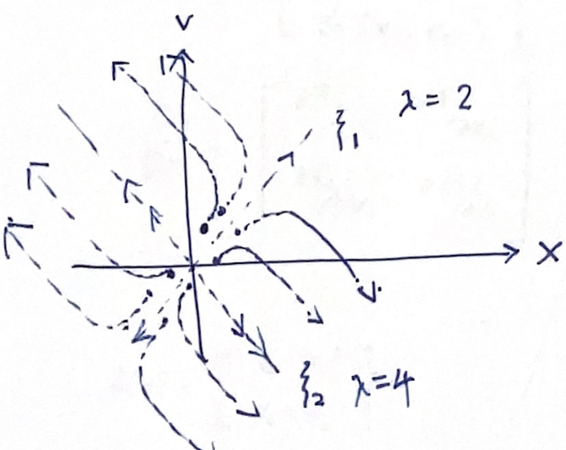
$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}}_{A}\begin{bmatrix} x \\ v \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & \\ & 4 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} .5e^{2t} & .5e^{2t} \\ .5e^{4t} & -.5e^{4t} \end{bmatrix}\begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

$$= 0.5\begin{bmatrix} e^{2t}+e^{4t} & e^{2t}-e^{4t} \\ e^{2t}-e^{4t} & e^{2t}+e^{4t} \end{bmatrix}\begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$



face portrait

$e^{At}\xi = e^{\lambda t}\xi$

$A\xi = \lambda\xi$

Ex:

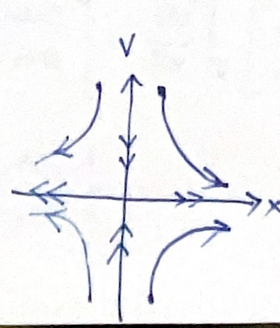
$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} \underbrace{1}_{\text{unstable}} & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \overbrace{e^{t}}^{\text{unstable}} & 0 \\ 0 & \underbrace{e^{-t}}_{\text{stable}} \end{bmatrix}\begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

Sun-Jupiter

Saddle point



saddle point $\rightarrow$ one stable, one unstable

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ 0 & -1 \end{bmatrix}\begin{bmatrix} .5 & .5 \\ .5 & -.5 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\ddot{x} + 4x = 0$$

$$(A - \lambda I) = \begin{bmatrix} -\lambda & 2 \\ -2 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - (-2 \cdot 2)$$

$$= \lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

↳ oscillate forever

$$e^{2it}$$

$$\longrightarrow T e^{Dt} T^{-1}$$

564 L9

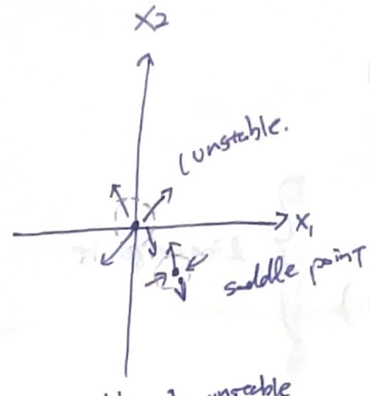
Linearize Nonlinear Dynamics

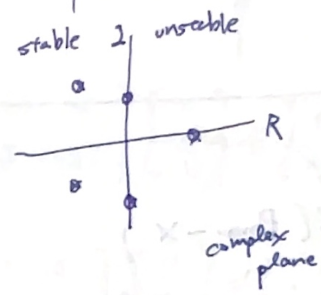
$\dot{X} = f(X)$

$\bar{X}$ is a fixed point

if $f(\bar{X}) = 0$

↳ is not moving





for longterm stability.

close enough to the fixed point, stable λ value will be proximating the origin λ. for λ values on the imaginary axis, little non-linearity could end up in either side

For X near $\bar{X}$, $\Delta X = X - \bar{X}$ is small

taylor expand @ $\bar{X}$

$$\dot{X} = f(X) = \underbrace{f(\bar{X})}_{\substack{=0 \\ \text{@ fixed point}}} + \frac{Df}{DX}\Big|_{\bar{X}} (X-\bar{X}) + \underbrace{\frac{1}{2!}\frac{D^2 f}{DX^2}\Big|_{\bar{X}} (X-\bar{X})^2 + \frac{1}{3!}\frac{D^3 f}{DX^3}(X-\bar{X})^3}_{\text{is small for } \Delta X \text{ small}}$$

$$\dot{X} - \dot{\bar{X}} = \frac{d}{dt}(\Delta X) = \underbrace{\frac{Df}{DX}\Big|_{\bar{X}}}_{\text{a matrix}} \cdot \Delta X$$

Ex:

$$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

$$\frac{Df}{DX} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}$$ Jacobian 1st derivative.

$$f(x) = \begin{bmatrix} x_1 - x_1^2 \\ x_1 + x_2 \end{bmatrix}$$

fixed points $x_1 = 0$, $x_2 = 0$; $x_1 = 1$, $x_2 = -1$

$$\frac{Df}{DX} = \begin{bmatrix} 1-2x_1 & 0 \\ 1 & 1 \end{bmatrix}$$

$\lambda = 1, 1$

Around $\bar{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \frac{Df}{DX}\Big|_{\bar{X}} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$

$\bar{X} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$ $\quad \lambda = \pm 1$

$$\ddot{\theta} = \frac{-g}{l} \sin(\theta)$$

$$\left.\begin{aligned} \dot{\theta} &= w \\ \dot{w} &= \frac{-g}{l} \sin(\theta) \end{aligned}\right\} \text{linearize it}$$

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ex2:

population

$$\dot{x} = f(x) = x(P_{max} - x)$$

max population

Logistic Equation

$$\left.\begin{aligned} \bar{x} &= 0 \\ \bar{x} &= P_{max} \end{aligned}\right\} \text{2 fix points}$$

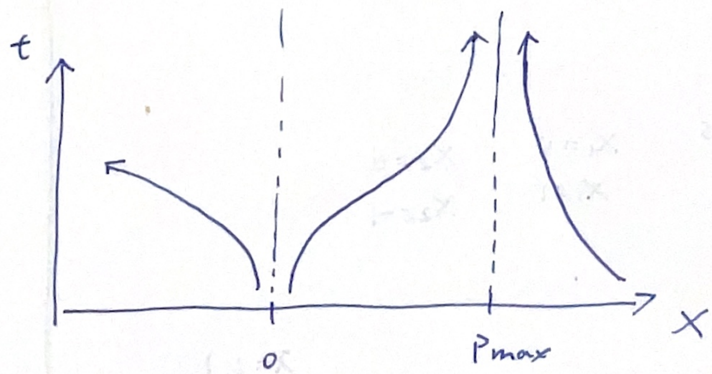
$$\frac{Df}{Dx} = P_{max} - 2x$$

$$\Delta\dot{x} = (P_{max} - 2\bar{x})\Delta x$$

$$\left(\frac{d}{dt}(\Delta x) = \frac{Df}{Dx}\Big|_{\bar{x}} \Delta x \right)$$

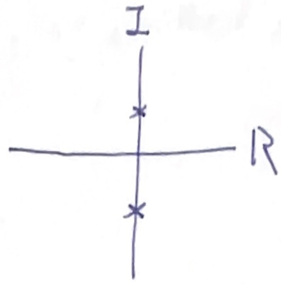
case 1 $\bar{x} = 0$ $\quad \frac{Df}{Dx} = P_{max}$

case 2 $\bar{x} = P_{max}$ $\quad \frac{Df}{Dx} = -P_{max}$



$$\frac{dx}{dt} = Ax$$

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} \qquad \text{eig}(A) = \pm 2i$$



ξ_1 for $\lambda_1 = 2i$: $[A - 2iI]\xi_1 = 0 \Rightarrow \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \xi_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

ξ_2 for $\lambda_2 = -2i$: $[A + 2iI]\xi_2 = 0 \Rightarrow \begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$x(t) = Te^{Dt}T^{-1}x(0) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{bmatrix} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} x(0)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{2it} & -ie^{2it} \\ e^{-2it} & ie^{-2it} \end{bmatrix} x(0)$$

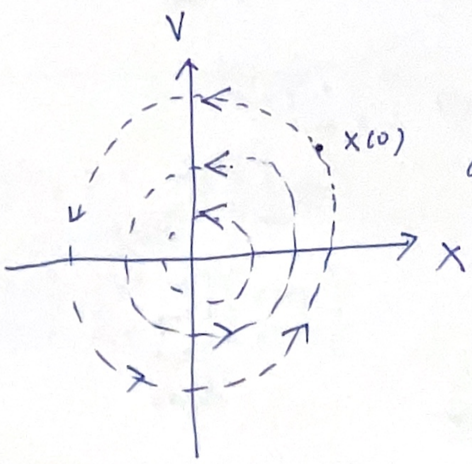
$$T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{2it} + e^{-2it} & i(-e^{2it} + e^{-2it}) \\ ie^{2it} - ie^{-2it} & e^{2it} + e^{-2it} \end{bmatrix} x(0)$$

$$\left(\begin{aligned} e^{i\theta} &= \cos(\theta) + i \sin(\theta) \\ e^{-i\theta} &= \cos(\theta) - i \sin(\theta) \end{aligned} \right)$$

$$= \frac{1}{2} \begin{bmatrix} 2\cos(2t) & i(-2i\sin(2t)) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix} x(0)$$

$$= \frac{1}{2} \begin{bmatrix} 2\cos(2t) & 2\sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix} \begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$



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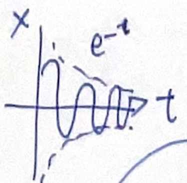
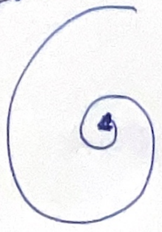
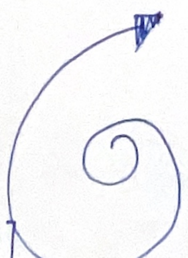




$\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix}$ damping



$\begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix}$ negative damping.