- Newton's 2nd Law
- Hamilton's Principle

integral by part

$\int u\,dv$
$= uv - \int v\,du$

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\,dt = 0$$

e.g.

$$T = \frac{1}{2}\int_0^l m\dot{w}^2\,d\xi$$

$$V = \frac{1}{2}\int_0^l N\left(\frac{\partial w}{\partial \xi}\right)^2 d\xi + \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial \xi^2}\right)^2 d\xi$$

$E \Rightarrow$ modulus
$I \Rightarrow bh^3/12$

$$\int_{t_1}^{t_2} \delta L\,dt = 0$$

$$L \equiv T - V$$

$$\delta\int_{t_1}^{t_2}\left[\int_0^l \frac{1}{2}m\dot{w}^2\,d\xi - \int_0^l \frac{1}{2}N\left(\frac{\partial w}{\partial \xi}\right)^2 d\xi - \int_0^l \frac{1}{2}EI\left(\frac{\partial^2 w}{\partial \xi^2}\right)^2 d\xi\right] dt$$

ⓐ $\delta \frac{1}{2}m\dot{w}^2 = \cancel{\frac{1}{2}}\cancel{2}\cdot m\dot{w}\,\delta\dot{w}$

ⓑ $\delta \frac{1}{2}N\left(\frac{\partial w}{\partial \xi}\right)^2 = \cancel{\frac{1}{2}}\cancel{2}\cdot N\frac{\partial w}{\partial \xi}\,\delta\left(\frac{\partial w}{\partial \xi}\right)$

ⓒ $\delta \frac{1}{2}EI\left(\frac{\partial^2 w}{\partial \xi^2}\right)^2 = \cancel{\frac{1}{2}}\cancel{2}\,EI\frac{\partial^2 w}{\partial \xi^2}\,\delta\left(\frac{\partial^2 w}{\partial \xi^2}\right)$

$\therefore$

ⓐ' $\int_{t_1}^{t_2}\int_0^l \delta\frac{1}{2}m\dot{w}^2\,d\xi\,dt$

$= \int_{t_1}^{t_2}\int_0^l m\dot{w}\,\delta\dot{w}\,d\xi\,dt$

$= \int_0^l m\dot{w}\,\delta w\Big|_{t_1}^{t_2} d\xi - \int_{t_1}^{t_2}\int_0^l m\ddot{w}\,\delta w\,d\xi\,dt$

ⓑ' $\int_{t_1}^{t_2}\int_0^l \delta\frac{1}{2}N\left(\frac{\partial w}{\partial \xi}\right)^2 d\xi\,dt$

$= \int_{t_1}^{t_2}\int_0^l N\frac{\partial w}{\partial \xi}\,\delta\left(\frac{\partial w}{\partial \xi}\right) d\xi\,dt$

$= \int_{t_1}^{t_2} N\frac{\partial w}{\partial \xi}\,\delta w\Big|_0^l dt - \int_0^l\int_{t_1}^{t_2} N\frac{\partial^2 w}{\partial \xi^2}\,\delta w\,dt\,d\xi$

ⓒ' $\int_{t_1}^{t_2}\int_0^l \delta\frac{1}{2}EI\left(\frac{\partial^2 w}{\partial \xi^2}\right)^2 d\xi\,dt$

$= \int_{t_1}^{t_2}\int_0^l EI\frac{\partial^2 w}{\partial \xi^2}\,\delta\left(\frac{\partial^2 w}{\partial \xi^2}\right) d\xi\,dt$

$\rightarrow \int_{t_1}^{t_2} EI\frac{\partial^2 w}{\partial \xi^2}\,\delta\left(\frac{\partial w}{\partial \xi}\right)\Big|_0^l dt - \int_0^l\int_{t_1}^{t_2} EI\frac{\partial^3 w}{\partial \xi^3}\,\delta\frac{\partial w}{\partial \xi}\,dt\,d\xi$

$= \int_{t_1}^{t_2} EI\frac{\partial^2 w}{\partial \xi^2}\,\delta\left(\frac{\partial w}{\partial \xi}\right)\Big|_0^l dt - \int_{t_1}^{t_2} EI\frac{\partial^3 w}{\partial \xi^3}\,\delta w\Big|_0^l dt + \int_0^l\int_{t_1}^{t_2} EI\frac{\partial^4 w}{\partial \xi^4}\,\delta w\,dt\,d\xi$

$= \int_{t_1}^{t_2}\Big[\,EI\frac{\partial^2 w}{\partial \xi^2}\,\delta\left(\frac{\partial w}{\partial \xi}\right)\Big|_0^l - EI\frac{\partial^3 w}{\partial \xi^3}\,\delta w\Big|_0^l + \int_0^l EI\frac{\partial^4 w}{\partial \xi^4}\,\delta w\,d\xi\,\Big]\,dt$

ⓐ' − ⓑ' − ⓒ'

assume $t_1, t_2 \rightarrow \delta w = 0$

$= \cancel{\int_0^l m\dot{w}\,\delta w\Big|_{t_1}^{t_2} d\xi}$

$\left(-\int_{t_1}^{t_2}\int_0^l m\ddot{w}\,\delta w\,d\xi\,dt\right)$

$-\int_{t_1}^{t_2} N\frac{\partial w}{\partial \xi}\,\delta w\Big|_0^l dt$

$\left(+\int_0^l\int_{t_1}^{t_2} N\frac{\partial^2 w}{\partial \xi^2}\,\delta w\,dt\,d\xi\right)$

$-\int_{t_1}^{t_2} EI\frac{\partial^2 w}{\partial \xi^2}\,\delta\left(\frac{\partial w}{\partial \xi}\right)\Big|_0^l dt$

$+\int_{t_1}^{t_2} EI\frac{\partial^3 w}{\partial \xi^3}\,\delta w\Big|_0^l dt$

$\left(-\int_{t_1}^{t_2}\int_0^l EI\frac{\partial^4 w}{\partial \xi^4}\,\delta w\,d\xi\,dt\right)$

Boundary Conditions

$$= -\int_{t_1}^{t_2}\int_0^l \overbrace{m\ddot{w}\,\delta w - N\frac{\partial^2 w}{\partial \xi^2}\,\delta w + EI\frac{\partial^4 w}{\partial \xi^4}\,\delta w}^{EOM}\,d\xi\,dt$$

$$+\int_{t_1}^{t_2} \underbrace{EI\frac{\partial^3 w}{\partial \xi^3}\,\delta w\Big|_0^l - EI\frac{\partial^2 w}{\partial \xi^2}\,\delta\left(\frac{\partial w}{\partial \xi}\right)\Big|_0^l - N\frac{\partial w}{\partial \xi}\,\delta w\Big|_0^l}_{B}\,dt$$

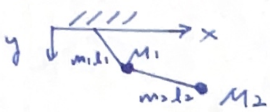
$$B = \left(EI\frac{\partial^3 w}{\partial \xi^3} - N\frac{\partial w}{\partial \xi}\right)\delta w - EI\frac{\partial^2 w}{\partial \xi^2}\,\delta\left(\frac{\partial w}{\partial \xi}\right)$$
$$= 0$$

Lagrange's Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = Q_\theta$$

$$L \equiv T - V$$

e.g. double pendulum



- $r_{M_1} = \begin{bmatrix} x_{M_1} \\ y_{M_1} \end{bmatrix} = \begin{bmatrix} l_1 \sin\theta_1 \\ l_1 \cos\theta_1 \end{bmatrix}$

$r_{M_2} = \begin{bmatrix} x_{M_2} \\ y_{M_2} \end{bmatrix} = \begin{bmatrix} l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ l_1\cos\theta_1 + l_2\cos\theta_2 \end{bmatrix}$

$\Rightarrow \dot r_{M_1} = \begin{bmatrix} l_1 \dot\theta_1 \cos\theta_1 \\ -l_1 \dot\theta_1 \sin\theta_1 \end{bmatrix}$

$\dot r_{M_2} = \begin{bmatrix} l_1 \dot\theta_1 \cos\theta_1 + l_2 \dot\theta_2 \cos\theta_2 \\ -l_1 \dot\theta_1 \sin\theta_1 - l_2 \dot\theta_2 \sin\theta_2 \end{bmatrix}$

- on l_1 & on l_2

$\dot r_{\xi_1} = \begin{bmatrix} \xi_1 \dot\theta_1 \cos\theta_1 \\ -\xi_1 \dot\theta_1 \sin\theta_1 \end{bmatrix}$

$\dot r_{\xi_2} = \begin{bmatrix} l_1 \dot\theta_1 \cos\theta_1 + \xi_2 \dot\theta_2 \cos\theta_2 \\ -l_1 \dot\theta_1 \sin\theta_1 - \xi_2 \dot\theta_2 \sin\theta_2 \end{bmatrix}$

- $T = \overset{a}{\frac{1}{2} M_1 \dot r_{M_1}^2} + \overset{b}{\frac{1}{2} M_2 \dot r_{M_2}^2}$

$\overset{c}{+ \int_0^{l_1} \frac{1}{2} m_1(\xi_1) \dot r_{\xi_1}^2 \, d\xi_1}$

$\overset{d}{+ \int_0^{l_2} \frac{1}{2} m_2(\xi_2) \dot r_{\xi_2}^2 \, d\xi_2}$

- $V = -M_1 g\, l \cos\theta_1$

$- \int_0^l m(\xi)\, g\, \xi \cos\theta_1 \, d\xi$

$- M_2\, g\, (l\cos\theta_1 + l\cos\theta_2)$

$- \int_0^l m(\xi)\, g\, (l\cos\theta_1 + \xi\cos\theta_2)\, d\xi$

- $L = T - V$

$q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ $\dot q = \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0$$

$M_{11}\ddot q_1 + K_{11} q_1 = 0$

let $q_1(t) = \bar q_1 e^{i w_1 t}$

$\Rightarrow M_{11}(i w_1)^2 \cancel{\bar q_1 e^{i w_1 t}} + K_{11} \cancel{\bar q_1 e^{i w t}} = 0$

$\Rightarrow [-M_{11} w_1^2 + K_{11}]\, \bar q_1 = 0$

$w_1^2 = \frac{K_{11}}{M_{11}}$

$\Rightarrow K_{11} = w_1^2 M_{11}$

$\Rightarrow M_{11}[\ddot q_1 + w_1^2 q_1] = 0$

$\Rightarrow M_{22}[\ddot q_2 + w_2^2 q_2] = 0$

Rayleigh – Ritz Modal Method

- to approximate solution of w
- expressed in eigen modes

"w/ continuum sys", $f(\xi, \tau)$, can use modes & turn then back to ODE

$w(\xi, t) = e^{iwt}\, \psi(\xi)$

(here we assume this, such that we can see what kind of ψ we need)

what is $\psi(\xi)$? what kinda shape function are we looking at?

Q: why is it that:

$w(\xi,t) = e^{iwt}\psi(\xi)$

or

$w(\xi,t) = \sum_n q_n(t)\psi_n(\xi)$

$q_n = \bar q_n e^{i w_n t}$

$m\ddot w - N\frac{\partial^2 w}{\partial \xi^2} = 0$

$w(\xi, t) = e^{iwt}\psi(\xi)$

$\Rightarrow \left[-m^2 w^2 \psi - N\frac{\partial^2\psi}{\partial\xi^2} = 0\right] e^{iwt}$

assume $\psi = Ae^{\lambda\xi}$

$\Rightarrow -m^2 w^2 \cancel{Ae^{\lambda\xi}} - N\lambda^2 \cancel{Ae^{\lambda\xi}} = 0$

$A = 0$ is trivial

$\Rightarrow -mw^2 = N\lambda^2$

$\Rightarrow \lambda = \pm i\sqrt{mw^2/N}$

$\therefore w(\xi, t) = e^{iwt}\left(A_1 e^{i\sqrt{mw^2/N}\,\xi} + A_2 e^{-i\sqrt{mw^2/N}\,\xi}\right)$

$= e^{iwt}\left(B_1 \sin\sqrt{\frac{mw^2}{N}}\,\xi + B_2\cos\sqrt{\frac{mw^2}{N}}\,\xi\right)$

w/ $w(\xi=0, t) \longrightarrow B_2 = 0$

w/ $w(\xi=l, t) \longrightarrow \sin\sqrt{\frac{mw^2}{N}}\, l = 0$

$\longrightarrow \frac{mw^2}{N} l = n\pi$

$\longrightarrow \psi = \frac{n\pi\xi}{l}$ $(n = 0, 1, 2 \ldots)$

back to string/beam

$T = \frac{1}{2}\int_0^l m\dot w^2\, d\xi$

$V = \frac{1}{2}\int_0^l N\left(\frac{\partial w}{\partial \xi}\right)^2 d\xi$

$w(\xi, t) = \sum_n q_n(t)\psi_n(\xi)$

$\Rightarrow \dot w^2 = \left[\sum_n \dot q_n \psi_n\right]\left[\sum_r \dot q_r \psi_r\right]$

$\dot w^2 = \sum_n\sum_r \dot q_n \dot q_r \psi_n \psi_r$

$\Rightarrow T = \frac{1}{2}\sum_n\sum_r \dot q_n \dot q_r \int_0^l m\psi_n\psi_r\, d\xi$

$= \frac{1}{2}\sum_n\sum_r \dot q_n \dot q_r M_{nr}$

$\Rightarrow V = \frac{1}{2}\int_0^l N\left(\frac{\partial w}{\partial w}\right)^2 d\xi$

$\left(\frac{\partial w}{\partial \xi}\right)^2 = \sum_n\sum_r q_n q_r \frac{\partial\psi_n}{\partial\xi}\frac{\partial\psi_n}{\partial\xi}$

$V = \frac{1}{2}\sum_n\sum_r q_n q_r \int_0^l N\frac{\partial\psi_n}{\partial\xi}\frac{\partial\psi_r}{\partial\xi}\, d\xi$

$= \frac{1}{2}\sum_n\sum_r q_n q_r K_{nr}$

* the reason we are using Rayleigh – Ritz system is becos, we are dealing w/ continuum system, where we hv infinite dimension of freedom. so, we are using $w(\xi, t) = \sum_n^\infty q_n(t)\psi_n(\xi)$ to approximate it, reducing it to finite-dimension.

* Here, all the systems are linear, i.e., superposition when we hv non-orthogonal systems, we hv $\psi_i\psi_j \neq 0$ on the off-diagonal terms, but system is still "linear", cos linearity is defined w.r.t. $q \to$ in coordinate!

$\Rightarrow T = \frac{1}{2} \begin{bmatrix} \text{—} \dot q_n \text{—} \end{bmatrix} [M_{nr}] \begin{bmatrix} \dot q_r \end{bmatrix}$

$V = \frac{1}{2} \begin{bmatrix} \text{—} q_n \text{—} \end{bmatrix} [K_{nr}] \begin{bmatrix} q_r \end{bmatrix}$

diagonal when orthogonal

$\Rightarrow T = \frac{1}{2}\sum_n \dot q_n^2 M_{nn}$

$V = \frac{1}{2}\sum_n q_n^2 K_{nn}$ $\Big\} \longrightarrow \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$

$L \equiv T - V$

for mode 1 $\Big\{$ $\cdot\ \frac{\partial T}{\partial \dot q_1} = \frac{1}{2}\cdot 2\cdot \dot q_1 M_{11}$ $\cdot\ \frac{\partial V}{\partial q_1} = \frac{1}{2}\cdot 2\, q_1 K_{11} = K_{11} q_1$

$\cdot\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_1}\right) = M_{11}\ddot q_1$

revisit of eigenvalue problem:

- after Lagrange w/ Rayleigh-Ritz
- $M\ddot{q} + Kq = 0$

  w/ $q = \bar{q}e^{\lambda t}$

$\Rightarrow (\lambda^2 M + K)\bar{q}e^{\lambda t} = 0$

$\Rightarrow (\lambda^2 M + K)\bar{q} = 0$

recall

$Ax = 0$

$Null(A) = \{x \mid Ax = 0\}$

- $\exists A^{-1}$, $x = 0$ only soln
- $!\exists A^{-1}$, $\exists x != 0$, non-trivial $Null(A)$

  ↳ i.e., $det(A) = 0 \Leftrightarrow Null(A) \neq \{0\}$

$\therefore$ we want $\bar{q} \neq 0$

$det(\lambda^2 M + K) = 0$

$\because M \succ 0 \quad K \succ 0$

$\lambda = \pm i\omega$

- again $M\ddot{q} + Kq = 0$

  w/ $q = \bar{q}e^{i\omega t}$

$\Rightarrow (-M\omega^2 + K)\bar{q} = 0$

$\Rightarrow K\bar{q} = \omega^2 M\bar{q}$

$Av = \lambda v$

$\therefore$ eigenvalue problem ※

Impulse response

- $M_{nn}[\ddot{q}_n + 2\rho_n\omega_n\dot{q}_n + \omega_n^2 q_n] = \delta(t)$

  recall $\int_{-\epsilon}^{\epsilon}\delta(t)\,dt = 1$ ← impulse

$M_{nn}\left[\dot{q}_n\big|_{t=+\epsilon} - \dot{q}_n\big|_{t=-\epsilon}\right] + 0 + 0 = 1$

$\dot{q}_n(t = 0^+) = 1/M_{nn}$

$\Rightarrow \underset{w/\,\delta(t)}{q_n(t)} \equiv I_n = \frac{1}{M_{nn}\omega_n[1-\rho_n^2]^{1/2}} e^{-\rho_n\omega_n t}\sin(\omega_n[1-\rho_n^2]t)$

Convolution Integral Response

$q_n(t) = \int_0^t I_n(t-\tau)\,Q_n(\tau)\,d\tau$

if $Q_n(\tau) = \delta(\tau)$

then $q_n(t) = I_n(t)$ → impulse response

Fourier Transform of Convolution Integral Response

$q^* = I_n^* Q_n^*$

where $q_n^* \equiv \int_{-\infty}^{\infty} q_n(t)\,e^{-i\omega t}\,dt$

$Q_n^* \equiv \int_{-\infty}^{\infty} Q_n(t)\,e^{-i\omega t}\,dt$

$I_n^* \equiv \int_{-\infty}^{\infty} I_n(t)\,e^{-i\omega t}\,dt$

$I_n^* \equiv H_n(\omega)$ → Transfer Function

* w/ system being linear, Q_T → sum of infinitesimal impulses

$Q(T) = \int_0^t Q(\tau)\,\delta(t-\tau)\,d\tau$

$\Rightarrow q(t) = \int_0^t Q(\tau)\,I_n(t-\tau)\,d\tau$

Linear System

- superposition

  $y_1 = f(x_1)$

  $y_2 = f(x_2)$

  $\Rightarrow \alpha y_1 + \beta y_2 = f(\alpha x_1 + \beta x_2)$

- recall

  $M\ddot{q} + Kq = Q$

$\Rightarrow M_{nn}[\ddot{q}_n + \omega_n^2 q_n] = Q_n$

where $K = \omega_n^2 M_{nn}$

- Add damping → this emperically determined

  $M_{nn}[\ddot{q}_n + 2\rho_n\omega_n\dot{q}_n + \omega_n^2 q_n] = Q_n$

let $q_n = \bar{q}_n e^{\beta t}$, $Q_n = 0$ $\qquad \beta = \alpha + \gamma i$

w/ damping, we now hv time decaying term. which is α

$\Rightarrow M_{nn}[\beta^2 + 2\rho_n\omega_n\beta + \omega_n^2]\bar{q}_n = 0$

$\beta = \frac{-2\rho_n\omega_n \pm [(2\rho_n\omega_n)^2 - 4\omega_n^2]^{1/2}}{2}$

$= -\rho_n\omega_n \pm i\omega_n[1-\rho_n^2]^{1/2}$

$\Rightarrow$ so, $q_n = Ae^{\beta_1 t} + Be^{\beta_2 t}$

or $q_n = e^{-\rho_n\omega_n t}[C\sin(\omega_n[1-\rho_n^2]^{1/2}t) + D\cos(\omega_n[1-\rho_n^2]^{1/2}t)]$

for I.C. $q_n(t=0) = 0$, $D = 0$

$\dot{q}_n(t=0) = \dot{q}_{n_0}$

$\Rightarrow \dot{q}_n = -\rho_n\omega_n e^{-\rho_n\omega_n t}C\sin(\omega_n[1-\rho_n^2]^{1/2}t) + e^{-\rho_n\omega_n t}[C\cos(\omega_n[1-\rho_n^2]^{1/2}t)\,\omega_n[1-\rho_n^2]^{1/2}]$

$\therefore \dot{q}_n(t=0) = \omega_n[1-\rho_n^2]^{1/2}C = \dot{q}_{n_0}$

$\therefore C = \frac{\dot{q}_{n_0}}{\omega_n[1-\rho_n^2]^{1/2}}$

$\Rightarrow q_n(t) = \frac{\dot{q}_{n_0}}{\omega_n[1-\rho_n^2]^{1/2}} e^{-\rho_n\omega_n t}\sin(\omega_n[1-\rho_n^2]^{1/2}t)$

transient response, $q_n \to 0$ as $t \to \infty$

$M_{nn}[\ddot{q}_n + 2\rho_n\omega_n\dot{q}_n + \omega_n^2 q_n] = Q_n$ → special case: $\begin{cases} Q_n = \bar{Q}_n e^{i\omega t} \\ q_n = \bar{q}_n e^{i\omega t} \end{cases}$

$\Rightarrow M_{nn}\bar{q}_n[-\omega^2 + 2\rho_n\omega_n(i\omega) + \omega_n^2] = \bar{Q}_n$

$\Rightarrow \bar{q}_n/\bar{Q}_n = \frac{1}{M_{nn}[-\omega^2 + 2\rho_n\omega_n i\omega + \omega_n^2]} \equiv H_n(\omega)$ (TF) → this can also be derived by getting the F.T. of impulse response

$\Rightarrow$ @ resonance $\quad \omega \to \omega_n$, $H_n \to H_{n,max}$

$H_n(\omega = \omega_n) = \frac{1}{M_{nn}[2\rho_n\omega_n^2 i]}$

Random Dynamics

recall :

$$q_n(t) = \int_0^t I_n(t-\tau)\, Q_n(\tau)\, d\tau$$

if $Q_n \to$ random

we can still use $q_n(t)$

but, we hv better method:

mean: $\overline{q(t)} \equiv \frac{1}{2T}\int_{-T}^{T} q(t)\, dt \quad T\to\infty$

square mean: $\overline{q^2} \equiv \frac{1}{2T}\int_{-T}^{T} q^2(t)\, dt \quad T\to\infty$

$\Downarrow$

$$M_{nn}\left[\ddot{q}_n + 2\rho_n\omega_n \dot{q}_n + \omega_n^2 q_n\right] = Q_n$$

$$\Rightarrow \frac{M_{nn}}{2T}\int_{-T}^{T}\ddot{q}_n\, dt = \frac{1}{2T}\int_{-T}^{T} Q_n(\tau)\, dt$$

$$\Rightarrow \frac{M_{nn}}{2T}\dot{q}_n\Big|_{-T}^{T} = \frac{1}{2T}\int_{-T}^{T} Q(\tau)\, dt$$

$$\Rightarrow M_{nn}\omega_n^2 \frac{1}{2T}\int_{-T}^{T} q(\tau)\, dt = \frac{1}{2T}\int_{-T}^{T} Q(\tau)\, dt$$

$$\Rightarrow M_{nn}\omega_n^2\, \overline{q_n} = \overline{Q_n}$$

$$\Rightarrow \frac{\overline{q_n}}{\overline{Q_n}} = \frac{1}{M_{nn}\omega_n^2} = H_n(\omega = 0)$$

$$M_{nn}\ddot{q}_n(t) = M_{nn}\omega_n^2 q_n(t)$$

$$\frac{M_{nn}}{2T}\int_{-T}^{T}\ddot{q}_n(t)\, dt = \frac{M_{nn}\omega_n^2}{2T}\int_{-T}^{T} q_n(t)\, dt$$

Correlation Func. of Random Dynamics

$$\left.\begin{aligned} \phi_q(\tau) &\equiv \frac{1}{2T}\int_{-T}^{T} q(t)\, q(t+\tau)\, dt \\ \phi_Q(\tau) &\equiv \frac{1}{2T}\int_{-T}^{T} Q(t)\, Q(t+\tau)\, dt \end{aligned}\right\} (a)$$

- when $\tau\uparrow$ $\phi \to 0$, correlation is low↓, random function
- when $\tau\uparrow$ $\phi \neq 0$, correlation is high, not random

- recall

$$\left.\begin{aligned} q(t) &= \int_0^t I(t-\tau_1)\, Q(\tau_1)\, d\tau_1 \\ q(t+\tau) &= \int_0^t I(t+\tau-\tau_2)\, Q(\tau_2)\, d\tau_2 \end{aligned}\right\} (b)$$

put (b) in (a)

$$\phi_q(\tau) \equiv \frac{1}{2T}\int_{-T}^{T}\left[\int_{-\infty}^{\infty} I(t-\tau_1)Q(\tau_1)\, d\tau_1\right]\left[\int_{-\infty}^{\infty} I(t+\tau-\tau_2)\, Q(\tau_2)\, d\tau_2\right] dt \quad \text{— (c)}$$

- F.T. on (c)

power spectrum density

$$\Phi_q(\omega) = H_{qQ}(\omega)\, H_{qQ}(-\omega)\, \Phi_Q(\omega)$$

P.S.D $\leftrightarrow$ FT of correlation f.

$$\begin{cases} \Phi_q(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\, \phi_q(t)\, dt \\ \phi_q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, \Phi_q(\omega)\, d\omega \end{cases}$$

$$\therefore \overline{q^2} = \phi_q(\tau=0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_q(\omega)\, d\omega$$

→ 1. give me PSD of input (from experiment)
2. times T.F. $H_{qQ}(\omega)H_{qQ}(-\omega)$
3. get $\Phi_q(\omega)$
4. integrate $\Phi_q(\omega) \to$ get $\phi_q(t)$

$$\Rightarrow \overline{q^2} = \phi_q(\tau=0) = \frac{1}{\pi}\int_0^{\infty}\Phi_q(\omega)\, d\omega$$

better coefficient

---

filter for measuring PSD

- filter $q(t) \to$ ⌈filter⌋ $\to q(t)$

only pretain a certain freq. band

ω_c – center freq.

- $\Delta\omega \to$ bandwidth

$$\overline{q^2} = \Phi_q(\omega=\omega_c)\,\Delta\omega$$

$$\Phi_q(\omega=\omega_c) = \frac{\overline{q^2}}{\Delta\omega}$$

- also for Φ_Q

PSD :
- basically gives you the distribution of power across frequencies
- determine which frequencies are dominant (contain more power) (W/Hz)
- FT of time series → what Hz are present
  PSD → How much does Hz contribute
- " PSD is FT of correlation
  correlation function basically tells me how related 2 signal at different time are.
  ∴ when some freqs are dominant, correlation ↑ ($\phi\uparrow$)
  when some freqs are random, correlation ↓ ($\phi\downarrow$)
  doing FT of correlation → see strong dominant freq^2 → "power"

$$\text{w/ } \Phi_q(\omega) = H_{qQ}(\omega)\, H_{qQ}(-\omega)\, \Phi_Q(\omega)$$

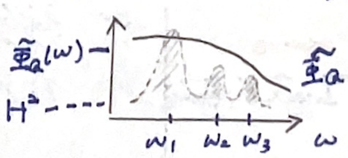
$$\& \quad \phi_q(t=0) = \int_0^{\infty}\tilde{\Phi}_q(\omega)\, d\omega$$

$$\Rightarrow \phi_q(t=0) = \int \tilde{\Phi}_Q(\omega)\, H(\omega)H(-\omega)\, d\omega$$

$$\text{w/ } H(\omega) = \frac{1}{M_n}\frac{1}{[-\omega^2 + 2\rho_n\omega_n i\omega + \omega_n^2]}$$

then

$$\phi_q(t=0) = \int \tilde{\Phi}_Q(\omega)\frac{1}{M_n^2[(-\omega^2+\omega_n^2) + (2\rho_n\omega_n\omega)^2]}\, d\omega$$



$$\therefore \phi_q(t=0) \cong \tilde{\Phi}_Q(\omega_1)\int_{\omega_1-\epsilon}^{\omega_1+\epsilon} H^2\, d\omega + \tilde{\Phi}_Q(\omega_2)\int_{\omega_2-\epsilon}^{\omega_2+\epsilon} H^2\, d\omega + \cdots$$

$$\text{w/ } \int_{-\infty}^{\infty} H^2\, d\omega = \frac{\pi}{4}\frac{1}{M_n^2}\frac{1}{\omega_n^3\rho_n}$$

$$\therefore \phi_q(\tau=0) \cong \sum_n\left[\tilde{\Phi}_Q(\omega_n)\frac{1}{M_n^2\omega_n^3\rho_n}\right]$$

w/ this :
- faster calculation
- compared to sim/conv.

PSD!!!

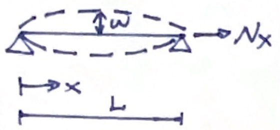
$$\tilde{\Phi}_q = \frac{1}{\pi}\Phi_q = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\phi_q(\tau)\, d\tau$$

$$\Rightarrow \phi_q(t) = \frac{\pi}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\,\tilde{\Phi}_q(\omega)\, d\omega$$

$$\Rightarrow \phi_q(\tau) = \int_0^{\infty} e^{i\omega t}\,\tilde{\Phi}_q(\omega)\, d\omega$$

$$\therefore \overline{q^2} = \phi_q(t=0) = \int_0^{\infty}\tilde{\Phi}_q(\omega)\, d\omega$$

Nonlinear Systems



- $N_x = T + \frac{Ehb}{L} \frac{1}{2} \int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx$

  $T > 0$ tension
  $T < 0$ compression

- $\frac{Ehb}{L} \frac{1}{2} \int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx$ strain

  arc length

  $\Rightarrow \int_0^L \frac{\sqrt{1+\left(\frac{\partial w}{\partial x}\right)^2} dx}{L} - L$

  $\Rightarrow \sqrt{1+\left(\frac{\partial w}{\partial x}\right)^2} \cong 1 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$

- nonlinear term induced
- usually we use 1st mode analysis

  $\Rightarrow w(x,t) = q(t) \sin \frac{\pi x}{L}$

Duffing's Equation

- $\ddot{q} + \alpha q + q^3 = \frac{Q}{M} = F$

  $\alpha > 0$ unbuckled
  $\alpha < 0$ buckled

  $\alpha = \omega_n^2 [1 - P/P_{buckle}]$ — as $P > P_{buckle}$, $\alpha < 0 \rightarrow$ buckled.

- case I: $F = 0$

  $\alpha q_s + q_s^3 = 0$

  $\Rightarrow q_s(\alpha + q_s^2) = 0$

  buckled up & down

  ↳ for $\alpha > 0$, $q_s = 0$
  ↳ for $\alpha < 0$, $q_s = 0$ or $q_s^2 = -\alpha > 0$

Harmonic Balance

- let $q = A \cos \omega t$

  $\Rightarrow -\omega^2 A \cos \omega t + \alpha A \cos \omega t + (A \cos \omega t)^3 = 0$

  $\Rightarrow -\omega^2 A \cos \omega t + \alpha A \cos \omega t + A^3\left(\frac{3}{4} \cos \omega t + \cancel{\frac{1}{4} \cos 3\omega t}\right) = 0$

  $\Rightarrow \left(-\omega^2 A + \alpha A + \frac{3}{4} A^3\right) \cos \omega t = 0$ — we only balance lowest oscillation

  $\Rightarrow \omega^2 = \alpha + \frac{3}{4} A^2$ — this can be neglected as $A\downarrow$; if $A >> \epsilon$, then system enters nonlinear

---

Van der Pol

- $\ddot{q} + \epsilon\left[1 - (q/q')^2\right] \dot{q} + \omega_0^2 q = 0$

  $q = A \cos \omega t$

Harmonic Balance

$\Rightarrow -\omega^2 A \cos \omega t - \epsilon\left[1 - \frac{A^2 \cos^2 \omega t}{q_{ref}^2}\right] A\omega \sin \omega t + \omega_0^2 A \cos \omega t = 0$

$\Rightarrow -\omega^2 A + \omega_0^2 A = 0$

$\Rightarrow \omega = \omega_0$

$\Rightarrow 1 - \frac{A^2}{2 q_{ref}^2} = 0$

Perturbation

- $q = q_0 + \epsilon q_1$
  - ϵ^0: $\ddot{q}_0 + \omega_0^2 q_0 = 0$ $q_0 = A \cos \omega_0 t$
  - ϵ^1: $\ddot{q}_1 - \left[1 - (q_0/q_{ref})^2\right] \dot{q}_0 + \omega_0^2 q_1 = 0$

    $q_1 = B \cos \omega t$

    let $1 - (q_0/q_{ref})^2 = 0$

    $\therefore (A/q_{ref})^2 \approx 2$

Perturbation Analysis

$\ddot{q} + \alpha q + \epsilon q^3 = F$

- case 1 $F = 0$

  $\Rightarrow q = q_0 + \epsilon q_1 + \cdots$

$q = q_0 + \epsilon q_1$

$\Rightarrow \ddot{q} + \alpha q + \epsilon q^3 = 0$

$\Rightarrow \ddot{q}_0 + \epsilon \ddot{q}_1 + \alpha q_0 + \alpha \epsilon q_1 + \epsilon(q_0 + \epsilon q_1)^3 = 0$

$\Rightarrow \ddot{q}_0 + \alpha q_0 + \epsilon \ddot{q}_1 + \epsilon \alpha q_1 + \epsilon q_0^3 + HOT = 0$

$\Rightarrow \ddot{q}_0 + \alpha q_0 + \epsilon(\ddot{q}_1 + \alpha q_1 + q_0^3) = 0$

- ϵ^0: $\ddot{q}_0 + \alpha q_0 = 0$

  ϵ^1: $\ddot{q}_1 + \alpha q_1 = -q_0^3$ → will end up in resonance

$\therefore \omega^2 = \alpha + \epsilon \beta_1$

$\Rightarrow \ddot{q} + (\omega^2 - \epsilon \beta_1) q + \epsilon q^3 = 0$

$\Rightarrow \epsilon^0$: $\ddot{q}_0 + \omega^2 q_0 = 0$

ϵ^1: $\ddot{q}_1 + \omega^2 q_1 - \beta_1 q_0 + q_0^3 = 0$

$\Rightarrow$ let $-\beta_1 q_0 + q_0^3 = 0$

or $-\beta_1 A \cos \omega t + A^3 \cos^3 \omega t = 0$

or $(-\beta_1 A + A^3 \frac{3}{4}) \cos \omega t = 0$

$\Rightarrow \beta_1 = \frac{3}{4} A^3$

w/ $\epsilon = 1$

$\Rightarrow \omega^2 = \alpha + \frac{3}{4} A^3$

Duffing's Equation

$\ddot{q} + \alpha q + q^3 = 0$

$\Rightarrow \alpha q_s + q_s^3 = 0$

$\Rightarrow q_s = 0$

$q_s^2 = -\alpha$

- $q = q_s + \hat{q}(t)$

$\Rightarrow \ddot{\hat{q}} + \alpha(q_s + \hat{q}) + (q_s + \hat{q})^3 = 0$

$\Rightarrow \ddot{\hat{q}} + \alpha(q_s + \hat{q}) + (q_s^3 + 3q_s^2\hat{q} + 3q_s\hat{q}^2 + \hat{q}^3) = 0$

if $\hat{q} << q_s$

$\Rightarrow \ddot{\hat{q}} + \cancel{\alpha q_s} + \alpha\hat{q} + \cancel{q_s^3} + 3q_s^2\hat{q} \cong 0$

$\therefore \ddot{\hat{q}} + (\alpha + 3q_s^2)\hat{q} = 0$

- case 1; $q_s = 0$

$\ddot{\hat{q}} + \alpha\hat{q} = 0$

$\rightarrow$ if $\alpha > 0$

$\hat{q} = A\sin\sqrt{\alpha} + B\cos\sqrt{\alpha}$

neutrally stable

$\rightarrow$ if $\alpha < 0$

$\hat{q} = Ce^{-\sqrt{|\alpha|}t} + De^{\sqrt{|\alpha|}t}$

unstable

- case 2 : $q_s^2 = -\alpha$

$\ddot{\hat{q}} - 2\alpha\hat{q} = 0$

$\rightarrow$ if $\alpha < 0$

$\hat{q} = E\sin\sqrt{|-\alpha|} + F\cos\sqrt{|-\alpha|}$

neutrally stable

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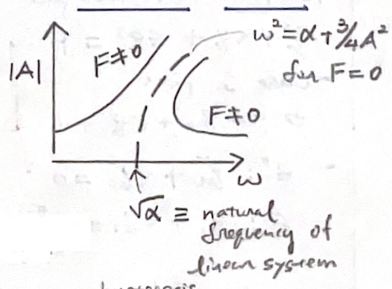
w/ Force

$\ddot{q} + \alpha q + q^3 = F(\tau)$

w/ $F = \bar{F}\cos\omega t$

$q = A\cos\omega t$



case 1.

- $\alpha > 0$ & $q = q_s + \hat{q}(t)$

$q_s(\alpha + q_s^2) = 0$

$q_s = 0$

$\therefore \ddot{q} + \alpha q + q^3 = F$

$\Rightarrow \ddot{\hat{q}} + \alpha\hat{q} + \hat{q}^3 = F$

w/ $\hat{q} = A\cos\omega\tau$

$F = \bar{F}\cos\omega t$

$\Rightarrow -\omega^2 A\cos\omega t + \alpha A\cos\omega\tau + \frac{A^3}{4}(3\cos\omega t + \cos 3\omega\tau) = \bar{F}\cos\omega\tau$

$\Rightarrow (-\omega^2 + \alpha)A + \frac{3}{4}A^3 = \bar{F}$ *

case 2

- $\alpha < 0$ & $q = q_s + \hat{q}(\tau)$

$q_s(\alpha + q_s^2) = 0$

$q_s = 0,\ \pm\sqrt{-\alpha}$

$\therefore \ddot{q} + \alpha q + q^3 = F$

$\Rightarrow (q_s + \hat{q})'' + \alpha(q_s + \hat{q}) + (q_s + \hat{q})^3 = F$

$\Rightarrow \hat{q}'' + \cancel{\alpha q_s} + \underline{\alpha\hat{q}} + \cancel{q_s^3} + \underline{3q_s^2\hat{q}} + 3q_s\hat{q}^2 + \hat{q}^3 = F$

$\Rightarrow \hat{q}'' + (\alpha + 3q_s^2)\hat{q} + 3q_s\hat{q}^2 + \hat{q}^3 = F$

$\Rightarrow \hat{q}'' + (-2\alpha)\hat{q} + 3q_s\hat{q}^2 + \hat{q}^3 = F$

w/ $\hat{q} = A\cos\omega t$

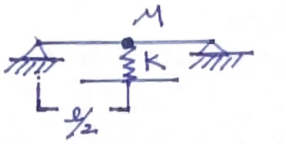
$F = \bar{F}\cos\omega t$

$-\omega^2 A\cos\omega t - 2\alpha A\cos\omega t + \cancel{3q_s\frac{A^2}{2}(1+\cos 2\omega t)}$

$+ \frac{A^3}{4}(3\cos\omega t + \cancel{\cos 3\omega t}) = \bar{F}\cos\omega t$

$-\omega^2 A - 2\alpha A + \frac{3}{4}A^3 = \bar{F}$ *

Component Mode Analysis



$$\triangle \begin{cases} T = \frac{1}{2}\sum_n M_{nn}\dot{q}_n^2 + \frac{1}{2}M\dot{\zeta}^2 \\ V = \frac{1}{2}\sum_n K_n q_n^2 + \frac{1}{2}K\zeta^2 \\ f(t) = \zeta - \sum_n q_n \psi_n(\xi=\xi_s) \end{cases}$$

$$\psi_n(\xi) \equiv \sin\frac{n\pi\xi}{l}$$

$$\triangle \quad L = T - V + \lambda f$$

$$= \frac{1}{2}\sum_n M_{nn}\dot{q}_n^2 + \frac{1}{2}M\dot{\zeta}^2 - \frac{1}{2}\sum_n K_n q_n^2 - \frac{1}{2}K\zeta^2 + \lambda\left(\zeta - \sum_n q_n\psi_n(\xi=\xi_s)\right) = 0$$

$\triangle$ EoM

$$q' = [q_1, q_2 \dots q_n, \zeta, \lambda]$$

$$\Rightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}'}\right] - \frac{\partial L}{\partial q'} = 0$$

$$\Rightarrow \begin{cases} M_{nn}\ddot{q}_n + K_n q_n - \lambda\psi_n(\xi=\xi_s) = 0 \\ M\ddot{\zeta} + K\zeta + \lambda = 0 \\ \zeta - \sum_n q_n\psi_n(\xi=\xi_s) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} M_{nn}[\ddot{q}_n + \omega_{nb}^2 q_n] - \lambda\psi_n(\xi=\xi_s) = 0 \\ M[\ddot{\zeta} + \omega_{ns}^2\zeta] + \lambda = 0 \\ \zeta - \sum_n q_n\psi_n(\xi=\xi_s) = 0 \end{cases}$$

$\triangle$ assume

$$\lambda = \bar{\lambda}e^{i\omega t}$$
$$\zeta = \bar{\zeta}e^{i\omega t}$$
$$q_n = \bar{q}_n e^{i\omega t}$$

$$\Rightarrow \begin{cases} M_{nn}[-\omega^2 + \omega_{nb}^2]\bar{q}_n - \bar{\lambda}\psi_n(\xi=\xi_s) = 0 \\ M[-\omega^2 + \omega_{ns}^2]\bar{\zeta} + \bar{\lambda} = 0 \\ \bar{\zeta} - \sum_n \bar{q}_n\psi_n(\xi=\xi_s) = 0 \end{cases}$$

$$\Rightarrow \left.\begin{aligned} \bar{q}_n &= \frac{\bar{\lambda}\psi_n(\xi=\xi_s)}{M_{nn}[-\omega^2+\omega_{nb}^2]} \\ \bar{\zeta} &= \frac{-\bar{\lambda}}{M[-\omega^2+\omega_{ns}^2]} \end{aligned}\right]$$

$$\left[\frac{-\bar{\lambda}}{M[-\omega^2+\omega_{ns}^2]} - \sum_n \frac{\bar{\lambda}\psi_n^2(\xi=\xi_s)}{M_{nn}[-\omega^2+\omega_{nb}^2]}\right] = 0$$

$$\Rightarrow \bar{\lambda}\left\{\overbrace{\underbrace{\frac{1}{M[-\omega^2+\omega_{ns}^2]}}_{\text{spring}}}^{D_s} + \overbrace{\underbrace{\sum_n \frac{\psi_n^2(\xi=\xi_s)}{M_{nn}[-\omega^2+\omega_{nb}^2]}}_{\text{beam}}}^{D_b}\right\} = 0$$

- natural frequency of the component system.
- when $n = 2, 4, 6$, $\psi_n(\xi) = 0$ only spring's natural frequency left

$\Rightarrow$

Plate Theory

10/21

$$D \equiv \frac{Eh^3}{12(1-\nu^2)}$$

poisson's ratio

$$\triangle \quad T = \frac{1}{2}\iint m\left(\frac{\partial w}{\partial t}\right)^2 dx\,dy$$

$$V = \frac{1}{2}\iint D\left[w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx}w_{yy} + 2(1-\nu)w_{xy}^2\right]dx\,dy$$

$\triangle$ Hamilton's Principle:

$$D\nabla^4 w + m\frac{\partial^2 w}{\partial t^2} = 0$$

& B.C.

$\triangle$ Lagrange's Equation

$$w(x,y,t) = \sum_n q_n(t)\psi_n(x,y)$$

Acoustics wave equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{a_\infty^2}\frac{\partial^2\phi}{\partial t^2} = 0$$

ϕ: velocity potential

- from graph $n > 2$, beam will be dominating, as $D_s \to 0$
- reason we are plotting $+D_b$, $-D_s$ is that we want to know when $D_b + D_s = 0$
  e.g. @ ω_1 we hv $D_b = k$, $-D_s = k$, $k < 0$
  $\therefore D_b + D_s = k - k = 0$
  @ ω_2 we hv $D_b = l$, $-D_s = l$, $l > 0$
  $\therefore D_b + D_s = l - l = 0$