ME564 L14

$\dot{x} = -2x + e^{t}$, $x(0) = 5$

$$
\begin{aligned}
x(t) &= e^{-2t}x(0) \\
&= e^{-2t}\cdot 5 + \int_0^t e^{-2(t-\tau)} e^{\tau}\, d\tau \\
&= 5e^{-2t} + e^{-2t}\int_0^t e^{2\tau} e^{\tau}\, d\tau \\
&= 5e^{-2t} + e^{-2t}\left[\frac{1}{3}e^{3\tau}\right]_0^t \\
&= 5e^{-2t} + e^{-2t}\left[\frac{1}{3}e^{3t} - \frac{1}{3}\right]
\end{aligned}
$$

Convolution Integral

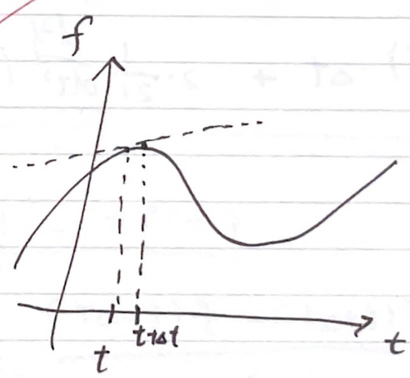
Computational Method (Numerical)

Numerical Differentiation:

Given a function (or data) $f(t)$

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

forward difference.



Units?

$$\frac{df}{dt} \approx \frac{f(t+\Delta t) - f(t)}{\Delta t} \quad \text{(finite difference)}$$

how good is this approximation?

check the error → use Taylor series

dominant error term $O(\Delta t)$

$$* \quad f(t+\Delta t) = f(t) + \frac{df}{dt}(t)\Delta t + \frac{1}{2}\frac{d^2f}{dt^2}(t)\cdot\Delta t^2 + \frac{1}{3!}\frac{d^3f}{dt^3}(t)\cdot\Delta t^3 + \cdots$$

$$** \quad f(t-\Delta t) = f(t) - \frac{df}{dt}(t)\Delta t + \frac{1}{2}\frac{d^2f}{dt^2}(t)\cdot\Delta t^2 - \frac{1}{3!}\frac{d^3f}{dt^3}(t)\cdot\Delta t^3 + \cdots$$

Forward Difference (FD) $\Rightarrow f' \approx \dfrac{f(t+\Delta t) - f(t)}{\Delta t} = \dfrac{1}{\Delta t}\left[\dfrac{df}{dt}(t)\Delta t + \dfrac{1}{2}\dfrac{d^2f}{dt^2}(t)\cdot\Delta t^2 + \dfrac{1}{3!}\dfrac{d^3f}{dt^3}(t)\Delta t^3 \ldots\right]$

$$= \underbrace{\frac{df(t)}{dt}}_{\text{hey this is what we want!}} + \underbrace{\frac{1}{2}\frac{d^2f(t)}{dt^2}\Delta t + \frac{1}{3!}\frac{d^3f(t)}{dt^3}\Delta t^2 \cdots}_{\text{error}}$$

(cont'd)

foword difference $\frac{df}{dt} = \frac{f(t+\Delta t)-f(t)}{\Delta t} + O(\Delta t)$

backward difference $\frac{df}{dt} = \frac{f(t)-f(t-\Delta t)}{\Delta t} + O(\Delta t)$

central difference $\frac{df}{dt} = \frac{f(t+\Delta t)-f(t-\Delta t)}{2\Delta t} + O(\Delta t^2)$

$f(t+\Delta t) - f(t-\Delta t)$

$= 2\frac{df(t)}{dt}\cdot\Delta t + \frac{2}{3!}\cdot\frac{d^3 f}{dt^3}\Delta t^3 + O(\Delta t^5)$

$\therefore \frac{f(t+\Delta t)-f(t-\Delta t)}{2\Delta t} = \frac{1}{2\Delta t}\left(2\frac{df(t)}{dt}\Delta t + \frac{2}{3!}\frac{d^3 f}{dt^3}\Delta t^3 + O(\Delta t^5)\right)$

$= \underbrace{\frac{df(t)}{dt}}_{\text{what we want}} + \underbrace{\frac{1}{3!}\frac{d^3 f}{dt^3}\Delta t^2 + \cdots}_{\text{error } O(\Delta t^2)}$

let's say we want to make our error 100X smaller,

if $O(\Delta t) \rightarrow \Delta t$ shrink to $\frac{1}{100}$

~~eff~~ $O(\Delta t^2) \rightarrow \Delta t$ shrink to $\frac{1}{10}$

$\therefore$ get less error with less reduction

No.

Date

e.g. measuring the velocity of a missile.

but when we dun hv $T+\Delta T$ data, we still use backword difference

and use higher order like $O(\Delta T^8)$

but what to do when $f(t)$ data is bad?

data error

e.g BD $\frac{df}{dt} = \frac{f(t)^{+e} - f(t-\Delta t)^{+e}}{\Delta t} + O(\Delta t)$

we shrink Δt yet enlarge "+e"

central difference sceheme $f''(t) = \frac{f(t+\Delta t) + f(t-\Delta t) - 2f(t)}{\Delta t^2}$

$= \frac{d^2 f(t)}{dt^2} + O(\Delta t^2)$

Question what do we do when we dun hv $f(t+\Delta T)$?