

# Quadrotor Trajectory Planning

## Minimum Snap/Jerk w/ Bezier Curve

Why use Bezier Curve to parameterize trajectory?  $\rightarrow$  parameterized by Bernstein Basis, controlled by control points

1. endpoint interpolation property will not pass thru middle points
2. Convex Hull properties is all control points stay within the corridor  $\rightarrow$  safety assurance
3. Hodograph derivatives

$\rightarrow$  These could ensure the trajectory stays within constraints

Why minimum snap, should refer to differential flatness, a concept that is used in non-linear control.

Differential Flatness (see later section for quadrotor flatness mapping)

for dynamical system

$$\dot{x} = f(x) + g(x)u$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^m$$

$$g: \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^m$$

$$\text{rank}(g) = m$$

we say that the system is differential flat if there exists

$$z \in \mathbb{R}^m \text{ which can be determined by } x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

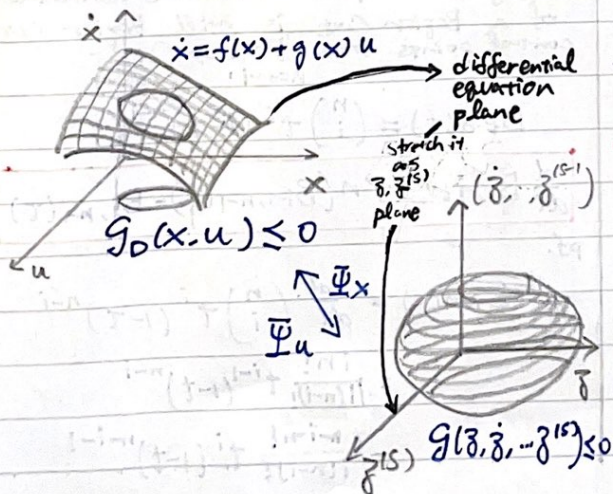
differential outputs

$z \in \mathbb{R}^m$ , could determine all state  $x \in \mathbb{R}^n$  input  $u \in \mathbb{R}^m$  through:

$$x = \Psi_x(z, \dot{z}, \dots, z^{(s-1)})$$

$$u = \Psi_u(z, \dot{z}, \dots, z^{(s)})$$

$\Psi_x, \Psi_u$  are transformation processes.



for quadrotor

$$z = (r, \psi) \in \mathbb{R}^3 \times \text{SO}(2)$$

$$\downarrow \Psi_x \Psi_u$$

$$x = \{r, v, R, \omega\} \in \mathbb{R}^3 \times \mathbb{R}^3 \times \text{SO}(3) \times \mathbb{R}^3$$

$$u = \{f, \tau\} \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3$$

detailed in later section

$\therefore$  So we can then plan a trajectory without differential constraints

Minimum Snap/Jerk

represent the trajectory as piecewise polynomial

$$f_j(t) = \sum_{i=0}^n P_i t^i \quad j = 0 \dots m$$

- by harnessing flatness, we optimize the trajectory

- optimize it w/ snap (4) || jerk (3)

$$f^{(4)}(t) = \sum_{i=4}^n i(i-1)(i-2)(i-3) t^{i-4} P_i$$

$$J(t) = \int_{T_{j-1}}^{T_j} (f^{(4)}(t))^2 dt$$

$$= P_i P_e \sum_{i=4}^n \sum_{e=4}^n \frac{i(i-1)(i-2)(i-3)e(e-1)(e-2)(e-3)}{i+e-7} t^{i+e-7}$$

$$= P_i^T Q P_e$$

$$\text{where } Q = \left[ \dots \frac{i(i-1)(i-2)(i-3)e(e-1)(e-2)(e-3)}{i+e-7} \dots \right]$$

$$\therefore \min J(t) \text{ s.t. } \begin{cases} AP = b \text{ (continuous constraints)} \\ AP \leq b \text{ (discontinuity constraints)} \end{cases}$$

Minimum Snap/Jerk w/ Constraints.

Bernstein polynomials

from monomial basis  $\rightarrow$  Bernstein basis

$$b_{i,n}(\tau) = \binom{n}{i} \tau^i (1-\tau)^{n-i}$$

$$P_j(\tau) = C_j^0 b_n^0(\tau) + C_j^1 b_n^1(\tau) + \dots$$

$$\dots + C_j^n b_n^n(\tau)$$

$$= \sum_{i=0}^n C_j^i b_n^i(\tau) \quad \text{bernstein basis}$$

$\rightarrow$  control points

for  $n+1$  points, polynomial has  $n$  order



### Bernstein Polynomials (control)

From above, any polynomials:

monomial  $\rightarrow$  Bernstein basis

$$P = MC$$

recall min. snap  $\min J = \min P^T Q P$

here  $= \min C^T M^T Q M C$

$$P_j(t) = C_j^0 b_n^0(t) + C_j^1 b_n^1(t) + \dots + C_j^n b_n^n(t) \\ = \sum_{i=0}^n C_j^i b_n^i(t) \quad b_n^i(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

let  $f_u(t)$  be the whole trajectory

$$f_u(t) = \begin{cases} s_1 \sum_{i=0}^n C_{u1}^i b_n^i\left(\frac{t-T_0}{s_1}\right), & t \in [T_0, T_1] \\ s_2 \sum_{i=0}^n C_{u2}^i b_n^i\left(\frac{t-T_1}{s_2}\right), & t \in [T_1, T_2] \\ \vdots \\ s_m \sum_{i=0}^n C_{um}^i b_n^i\left(\frac{t-T_{m-1}}{s_m}\right), & t \in [T_{m-1}, T_m] \end{cases}$$

with piece trajectory

$$\tau = \frac{t - T_j}{s_{j+1}} \quad \text{normalization}$$

$$\tau \in [0, 1]$$

$$J = \sum_{u \in \mathcal{U}} \int_0^T \left( \frac{d^k f_u(t)}{dt^k} \right)^2 dt$$

on u axis, jth trajectory

$$s dt = d\tau$$

$$J_{u_j} = \int_0^{s_j} \left( \frac{d^k f_{u_j}(t)}{dt^k} \right)^2 dt \\ = \int_0^1 \left( \frac{s_j d^k g_{u_j}(\tau)}{s_j^k d\tau} \right)^2 s_j d\tau \\ = \int_0^1 \frac{s_j^{2-k}}{s_j^{2k}} \cdot s_j \left( \frac{d^k g_{u_j}(\tau)}{d\tau} \right)^2 d\tau \\ = \int_0^1 s_j^{3-2k} \frac{d^k g_{u_j}(\tau)}{d\tau} d\tau$$

e.g. for  $t \in [0, T]$

$$J = \int_0^T \left( \frac{d^k (s \cdot \sum C_i b^i(\frac{t}{s}))}{dt^k} \right)^2 dt \\ = \int_0^1 \left( \frac{d^k (s \cdot \sum C_i b^i(\tau))}{s^k d\tau^k} \right)^2 s d\tau \\ = \int_0^1 s^{3-2k} \left( \frac{d^k \sum C_i b^i(\tau)}{d\tau^k} \right)^2 d\tau \\ = \int_0^1 s^{3-2k} \left( \frac{d^k \sum P_i \tau^i}{d\tau^k} \right)^2 d\tau$$

get  $Q$  from  $P^T Q P$  or  $C^T M^T Q M C$

$$Q = \begin{bmatrix} \dots & \frac{1!(1-1)(1-2)(1-3)\dots(1-k+1)(k-2)(k-3)\dots(1-2k+3)}{i+k-1} s^{-2k+3} & \dots \end{bmatrix}$$

get  $M$  from  $C^T M^T Q M C$

pascal triangle

now we have  $Q$  &  $M$

minimize objectives:  $\min J = C^T M^T Q M C$

minimize conditions:  $Ac = b$   
 $Ac \leq b$

$$C \in \mathbb{R}^{(n_{order}+1) \times m}$$

### Bézier Curve Homography

Homography implies that the derivatives of a Bézier Curve is still Bézier curve  
control points from  $P_i \rightarrow Q_i$   
 $i=0 \dots n-1$   $i=0 \dots n$

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$\frac{d}{dt} B_{i,n}(t) = n(B_{i-1,n-1}(t) - B_{i,n-1}(t))$$

pf.

$$\frac{d}{dt} B_{i,n}(t) = \frac{d}{dt} \binom{n}{i} t^i (1-t)^{n-i} \\ = \frac{i n!}{i!(n-i)!} t^{i-1} (1-t)^{n-i} \\ - \frac{(n-i)n!}{i!(n-i)!} t^i (1-t)^{n-i-1}$$



$$\begin{aligned}
 B_{i,n}(t) &= \frac{n!}{i!(n-i)!} t^{i-1} (1-t)^{n-i} - \frac{(n-i)!}{i!(n-i-1)!} t^i (1-t)^{n-i-1} \\
 &= \frac{n(n-1)!}{i!(n-i)!} t^{i-1} (1-t)^{n-i} - \frac{n(n-1)!}{i!(n-i-1)!} t^i (1-t)^{n-i-1} \\
 &= n \left[ \frac{(n-1)!}{i!(n-i)!} t^{i-1} (1-t)^{n-i} - \frac{(n-1)!}{i!(n-i-1)!} t^i (1-t)^{n-i-1} \right] \\
 &= n(B_{i-1,n-1}(t) - B_{i,n-1}(t))
 \end{aligned}$$

$$F(t) = \sum_{i=0}^n B_{i,n}(t) P_i$$

$$\frac{d}{dt} F(t) = \sum_{i=0}^{n-1} B_{i,n-1}(t) Q_i \quad Q_i = n(P_{i+1} - P_i)$$

PS.

let  $n=2$

$$F(t) = \sum_{i=0}^2 B_{i,2}(t) P_i$$

$$\begin{aligned}
 &= B_{0,2}(t) P_0 \\
 &+ B_{1,2}(t) P_1 \\
 &+ B_{2,2}(t) P_2
 \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dt} F(t) &= 2(B_{0,1}(t) - B_{0,2}(t)) P_0 \\
 &+ 2(B_{0,1}(t) - B_{1,1}(t)) P_1 \\
 &+ 2(B_{1,1}(t) - B_{2,1}(t)) P_2 = 0
 \end{aligned}$$

when  $n=1$ , derivative of  $F(t)$ ,  $n=2$

$$\begin{aligned}
 F(t) &= \sum_{i=0}^1 B_{i,1}(t) Q_i \\
 &= B_{0,1}(t) Q_0 \\
 &+ B_{1,1}(t) Q_1 = 0
 \end{aligned}$$

from ①

$$\frac{d}{dt} F(t) = (-2P_0 + 2P_1) B_{0,1}(t) + (-2P_1 + 2P_2) B_{1,1}(t)$$

$$= 2(P_1 - P_0) B_{0,1}(t) + 2(P_2 - P_1) B_{1,1}(t)$$

$$\text{from ②} \quad = Q_0 B_{0,1}(t) + Q_1 B_{1,1}(t)$$

$$Q_0 = 2(P_1 - P_0)$$

$$Q_1 = 2(P_2 - P_1)$$

$$Q_i = n(P_{i+1} - P_i)$$

e.g.  $P_0 \ P_1 \ P_2 \ P_3 \ P_4 \quad n=4 \quad \text{pts} \times 5$

$$\frac{d}{dt} \begin{pmatrix} n(P_1 - P_0) & n(P_2 - P_1) & n(P_3 - P_2) & n(P_4 - P_3) \end{pmatrix} \text{pts} \times 4$$

$$\frac{d}{dt} \begin{pmatrix} (n-1) \begin{pmatrix} n(P_2 - P_1) - n(P_1 - P_0) \\ n(P_3 - P_2) - n(P_2 - P_1) \\ n(P_4 - P_3) - n(P_3 - P_2) \end{pmatrix} \end{pmatrix} \text{pts} \times 3$$

Equality Condition

starting condition

ending condition

continuous condition

$$Ac = b$$

$$C \in R_{(n\text{-order}+1) \times m}$$

$n$ -order:  
highest order of  
polynomial  
 $m$ :  
no. of parameters

e.g.  $n\text{-order} = 7$   
 $m = 5$

$$C \in R_{(7+1) \times 5} = R_{40}$$

$$A \in R_{dx(n\text{-order}+1) \times m} = R_{dx40}$$

$$b \in R_d \rightarrow dx \text{ constraints}$$

starting condition, ending condition  $j=0$   
 $j=f$

$$a_{0,j} = c_0 \quad (u_j) \quad a_{f,j} = c_1 \quad a_{f,j} = c_2$$

$$P = a_{f,j} (S_j)^f$$

$$V = \frac{d}{dt} a_{f,j} (S_j)^f = n(a'_{f,j} - a_{f,j})$$

$$a = \frac{d}{dt} a_{f,j} (S_j)^f = n(n-1)(a''_{f,j} - a'_{f,j}) \cdot (a'_{f,j} - a_{f,j})$$

Write everything into matrix from above

e.g.  $n\text{-order} = 7$   
 $m = 5$   
 $S_j = t$

$$Ac = b$$

$$\begin{bmatrix} 1t & 0 & 0 \\ -7t^0 & 7t^0 & 0 \\ 42t^1 & -84t^1 & 42t^1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix} = \begin{bmatrix} P \\ V \\ a \\ \vdots \\ 0 \end{bmatrix}$$

continuous condition

$$P_{n\text{-order}+1,j} = P_{0,j+1} \quad j = 0, 1, 2, \dots, m$$

$$V_{n\text{-order}+1,j} = V_{0,j+1} \quad j = 0, 1, 2, \dots, m$$

$$a_{n\text{-order}+1,j} = a_{0,j+1} \quad j = 0, 1, 2, \dots, m$$

$$\therefore \begin{cases} P_{n\text{-order}+1,j} - P_{0,j+1} = 0 \\ V_{n\text{-order}+1,j} - V_{0,j+1} = 0 \\ a_{n\text{-order}+1,j} - a_{0,j+1} = 0 \end{cases}$$

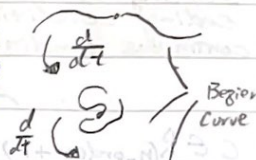
$$\vdots$$

$$S_{j+1} = 0$$



## Equality condition (cont'd)

$$\begin{cases} P_{n\text{-order}+1,j} - P_{0,j+1} = 0 \\ V_{n\text{-order}+1,j} - V_{0,j+1} = 0 \\ A_{n\text{-order}+1,j} - A_{0,j+1} = 0 \end{cases}$$



$$\Rightarrow \begin{cases} a_{n\text{-order}+1,j} - a_{0,j+1} = 0 \\ \frac{d}{dt} a_{n\text{-order}+1,j} - \frac{d}{dt} a_{0,j+1} = 0 \\ \frac{d^2}{dt^2} a_{n\text{-order}+1,j} - \frac{d^2}{dt^2} a_{0,j+1} = 0 \end{cases} \begin{cases} \text{first pt of } j+1 \\ \parallel \\ \text{last pt of } j \end{cases}$$

$$\Rightarrow \begin{cases} a_{n\text{-order}+1,j} - a_{0,j+1} = 0 \\ n(a_{n\text{-order}+1,j} - a_{0,j+1}) - n(a_{0,j+1} - a_{0,j+1}) = 0 \\ n(n-1)[a_{n\text{-order}+1,j} - a_{0,j+1}] - n(n-1)(a_{0,j+1} - a_{0,j+1}) = 0 \end{cases}$$

↳ write everything into matrix

e.g.  $n\text{-order} = 7$   
 $m = 5$   
condition @  $j = 1, 2$

4 connection conditions  
↓  
12 conditions

$$\begin{bmatrix} \dots & 1 & -1 & \dots \\ \dots & -7 & 7 & -7 & \dots \\ 42 & -84 & 42 & -42 & 84 & -42 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} C_0 \\ \vdots \\ C_8 \\ \vdots \\ C_{40} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$R_{12 \times 40}$

$R_{40 \times 1}$

$R_{12}$

w, w/o  $t^n$   
does not matter  
as  $AC = 0$

$$Ac = b$$

$$A \in R_{d \times 40}$$

$$c \in R_{40}$$

$$b \in R_d$$

d: no. of conditions  
starting  
ending  
continuous

## Inequality condition (cont'd)

safety corridor  
dynamic constraints

$$Ac \leq b$$

- main reason for us to use bezier curve is that we can **CONFINE** all control points within our desired range  
- all expression should be in the form of  $Ac \leq b$

$$A \in R_{d \times (n\text{-order}+1) \times m}$$

$$C \in R_{(n\text{-order}+1) \times m}$$

$$b \in R_d$$

$d \Rightarrow$  no. of conditions

safety corridor  
let  $a \leq x \leq b$   
 $c \leq y \leq d$   
 $e \leq z \leq f$

$$\Rightarrow \begin{cases} x \leq b \\ -x \leq -a \\ y \leq d \\ -y \leq -c \\ z \leq f \\ -z \leq -e \end{cases}$$

write everything in matrix  
 $Ac \leq b$   
express everything in terms of control points

$$\Rightarrow t^1 \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ -1 & \dots & -1 \end{bmatrix} \begin{bmatrix} C_0 \\ \vdots \\ C_{(n\text{-order}+1) \times m} \end{bmatrix} \leq b$$

ieq-p

$$\Rightarrow t^0 \begin{bmatrix} -7 & 7 & \dots & 7 & -7 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 7 & -7 & \dots & 7 & -7 \end{bmatrix} \begin{bmatrix} C_0 \\ \vdots \\ C_{(n\text{-order}+1) \times m} \end{bmatrix} \leq b$$

ieq-v

$$\Rightarrow t^1 \begin{bmatrix} 42 & -84 & 42 & \dots & 42 & -84 & 42 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -42 & 84 & -42 & \dots & -42 & 84 & -42 \end{bmatrix} \begin{bmatrix} C_0 \\ \vdots \\ C_{(n\text{-order}+1) \times m} \end{bmatrix} \leq b$$

ieq-v

e.g.  $n\text{-order} = 7$   
 $m = 5$

no. of p ctrl-ptcs = 80  
= 70

$$A \in R_{210 \times 40}$$

$$C \in R_{40}$$

$$b \in R_{210}$$