ME564 L7

① Eigenvalues, eigenvectors & diagonalize $\dot{x} = Ax$

② geometry of e-vals, e-vecs

③ evals evecs in general

④ examples

⑤ solution to $\dot{x} = Ax$

Recall to diagonalize ODE $\dot{x} = Ax$

we need a coord x-form $x = Tz$ s.t. $\dot{z} = Dz$.

$\boxed{AT = TD}$ eigenvalue equation

cols are e-vecs (A)

diag of D are e-val (A)

"Eigen" = characteristic or latent

$A\xi = \lambda \xi$

single column vector

for special vec ξ

for special val λ

Ex:

$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$

try $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad Ax = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ different direction

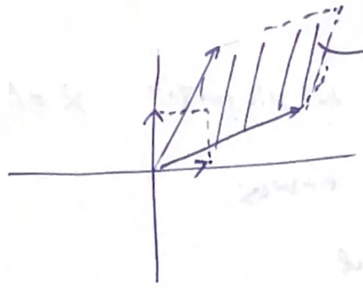
try $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad Ax = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ different direction

try $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad Ax = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ same direction $\rightarrow$ e-v, w/ $\lambda = 2$, $\xi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

try $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad Ax = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$ same direction $\rightarrow$ e-v, w $\lambda = 4$, $\xi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$Ax = \lambda x = \lambda I x$$

$$[A - \lambda I]\, x = 0$$



$\det(B)$

case 1 : $x = 0$ (boring)

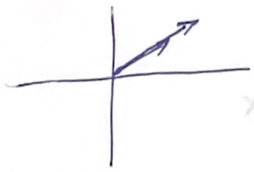
case 2 : $\det[A - \lambda I] = 0$

is singular

- cannot be invertible
- column co-dependent

characteristic equation

$\det(B)$



$Bx \Rightarrow$ the resulted vectors collapse on the same direction, dimension lost.

Step 1

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$[A - \lambda I]$$

$$= \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix}$$

$$(3-\lambda)^2 - (-1 \times -1)$$

$$= \lambda^2 - 6\lambda + 8$$

$$= (\lambda - 4)(\lambda - 2)$$

$\lambda = 2, 4$

$\lambda_1 = 2$

$\lambda_2 = 4$

Step 2

find x given λ

- $\lambda_1 = 2$

$$A - 2I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left.\begin{aligned} x_1 - x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}\right\} \xrightarrow{\text{true if}} x_1 = x_2 \Rightarrow \xi_1 = \begin{bmatrix} t \\ t \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- $\lambda_2 = 4$

$$A - 4I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left.\begin{aligned} -x_1 - x_2 &= 0 \\ -x_1 - x_2 &= 0 \end{aligned}\right\} \xrightarrow[\text{iff}]{\text{true if}} x_1 = -x_2 \Rightarrow \xi_2 = \begin{bmatrix} t \\ -t \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A\xi = \lambda \xi$$

$$(A - \lambda I)\,\xi = 0$$

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>>[T, D] = eig(A);
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$$AT = TD$$

$$A = TDT^{-1}$$

$$A^2 = TD^2T^{-1}$$

$$A^3 = TD^3T^{-1}$$

$$\vdots$$

$$\dot{x} = Ax \Rightarrow x(t) = \underbrace{e^{At}}\, x(0)$$

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots$$

$$= TT^{-1} + TDT^{-1}t + \frac{1}{2!}TD^2T^{-1}t^2 + \frac{1}{3!}TD^3T^{-1}t^3 + \dots$$

$$= T\left[I + Dt + \frac{1}{2!}D^2t^2 + \frac{1}{3!}D^3t^3 + \dots\right]T^{-1}$$

$$= T\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & \vdots & e^{\lambda_n t} \end{bmatrix}T^{-1}$$

$$\Rightarrow \boxed{e^{At} = Te^{Dt}T^{-1}} \ \text{*}$$

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>> [T, D] = eig(A);
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$$T = \begin{bmatrix} | & | & | & & | \\ \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_n \\ | & | & | & & | \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{bmatrix}$$

$$x = Tz$$

$$\dot{x} = Ax$$

$$x(t) = \underbrace{T\underbrace{e^{Dt}\underbrace{T^{-1}x(0)}_{z(0)}}_{z(t) = e^{Dt}z(0)}}_{x(t) = Tz(t)}$$