Quadrotor Trajectory Planning

Minimum Snap/Jerk w/ Bezier Curve

Why use Bezier Curve to parameterize trajectory?
↳ parameterized w/ Bernstein Basis, controlled by control points

1. endpoint interpolation property
   will not pass thru middle points
2. Convex Hull properties
   if all control points stay within the corridor → safety ensurance
3. Hodograph
   derivatives

↳ These could ensure the trajectory stays within constraints

why minimum snap, should refer to differential flatness, a concept that is used in non-linear control.

Differential Flatness (see later section for quadrotor flatness mapping)

for dynamical system

$$\dot{x} = f(x) + g(x)u \qquad \begin{array}{l} f: \mathbb{R}^n \to \mathbb{R}^n \quad x \in \mathbb{R}^n \\ g: \mathbb{R}^n \to \mathbb{R}^n \quad u \in \mathbb{R}^m \\ \mathrm{rank}(g) = m \end{array}$$

we say that the system is differential flat if there exists

$z \in \mathbb{R}^m$ which can be determined by $x \in \mathbb{R}^n$ $u \in \mathbb{R}^m$

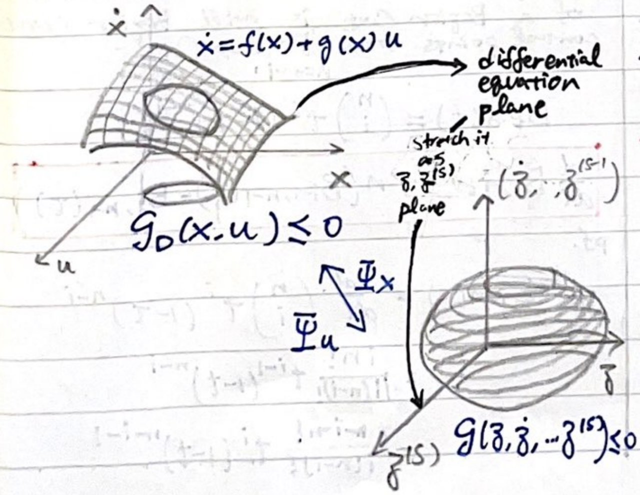
↓

differential outputs

$z \in \mathbb{R}^m$, could determine all state $x \in \mathbb{R}^n$ input $u \in \mathbb{R}^m$ through:

$$x = \Psi_x(z, \dot{z}, \cdots, z^{(s-1)}),$$
$$u = \Psi_u(z, \dot{z}, \cdots, z^{s})$$

Ψ_x, Ψ_u are transformation processes.



for quadrotor

$$z = (r, \psi) \in \mathbb{R}^3 \times SO(2)$$
$$\downarrow \Psi_x \times \Psi_u$$
$$x = \{r, v, R, \omega\} \in \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3) \times \mathbb{R}^3$$
$$u = \{f, T\} \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3$$

detailed in later section

∴ so we can then plan a trajectory without differential constraints

Minimum Snap/Jerk

represent the trajectory as piecewise polynomial

$$f_j(t) = \sum_{i=0}^{n} p_i t^i \qquad j = 0 \ldots m$$

- by harnessing flatness, we optimize the trajectory
- optimize it w/ snap$^{(4)}$ || jerk$^{(3)}$

$$f^{(4)}(t) = \sum_{i \geq 4} i(i-1)(i-2)(i-3)\, t^{i-4} p_i$$

$$J(t) = \int_{T_{j-1}}^{T_j} \left(f^{(4)}(t)\right)^2 dt$$

$$= p_i p_l \sum_{\substack{i \geq 4 \\ l \geq 4}} \left[\begin{array}{c} \vdots \\ \cdots \frac{i(i-1)(i-2)(i-3)\,l(l-1)(l-2)(l-3)}{i+l-7}\, t \\ \vdots \end{array} \right]_{T_{j-1}^{(i+l-7)}}^{T_j^{(i+l-7)}}$$

$$= p_i^T Q p_l$$

where
$$Q = \left[\begin{array}{c} \vdots \\ \cdots \frac{i(i-1)(i-2)(i-3)\,l(l-1)(l-2)(l-3)}{i+l-7}\, T^{i+l-7} \cdots \\ \vdots \end{array} \right]$$

$$\therefore \quad \min J(t) \quad \text{s.t.} \quad \begin{array}{l} Ap = b \ \text{(continuous constraints)} \\ Ap \leq b \ \text{(limitation constraints)} \end{array}$$

Minimum Snap/Jerk w/ Constraints.

Bernstein polynomials

from monomial basis → Bernstein basis

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$P_j(t) = c_j^0 b_n^0(t) + c_j^1 b_n^1(t) + \cdots + c_j^n b_n^n(t)$$

$$= \sum_{i=0}^{n} \underbrace{c_j^i}_{\text{control points}} \underbrace{b_n^i(t)}_{\text{bernstein basis}}$$

for n+1 points, polynomial has n order

No.

Date

Berstein Polynomials (cont'd)

From above, any polynomials:

monomial basis $\rightarrow$ Bernstein basis

$$P = MC$$

recall min. snap $\min J = \min P^T Q P$

here $= \min C^T M^T Q M C$

$$P_j(t) = C_j^0 b_n^0(t) + C_j^1 b_n^1(t) + \cdots + C_j^n b_n^n(t)$$
$$= \sum_{i=0}^{n} C_j^i b_n^i(t) \quad b_n^i(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

let $f_\mu(t)$ be the whole trajectory

$$f_\mu(t) = \begin{cases} s_1 \sum_{i=0}^{n} C_{\mu_1}^i b_n^i\left(\frac{t-T_0}{s_1}\right), & t \in [T_0, T_1] \\ s_2 \sum_{i=0}^{n} C_{\mu_2}^i b_n^i\left(\frac{t-T_1}{s_2}\right), & t \in [T_1, T_2] \\ \vdots \\ s_m \sum_{i=0}^{n} C_{\mu m}^i b_n^i\left(\frac{t-T_{m-1}}{s_m}\right), & t \in [T_{m-1}, T_m] \end{cases}$$

m^{th} peice trajectory

$$\tau = \frac{t - T_j}{s_{j+1}} \quad \text{normalization}$$

$$\tau = [0, 1]$$

$$J = \sum_{\mu \in \{x,y,z\}} \int_0^T \left(\frac{d^k f_\mu(t)}{dt^k}\right)^2 dt$$

on u axis, j^{th} trajectory

$s d\tau = dt$

$$J_{\mu_j} = \int_0^{s_j} \left(\frac{d^k f_{\mu_j}(t)}{dt^k}\right)^2 dt$$
$$= \int_0^1 \left(\frac{s_j d^k g_{\mu_j}(\tau)}{s_j^k d\tau}\right)^2 s_j d\tau$$
$$= \int_0^1 \frac{s_j^2}{s_j^{2k}} \cdot s_j \left(\frac{d^k g_{\mu_j}(\tau)}{d\tau}\right) d\tau$$
$$= \int_0^1 s_j^{3-2k} \frac{d^k g_{\mu_j}(\tau)}{d\tau} d\tau$$

e.g. for $t \in [0, T]$

$$J = \int_0^T \left(\frac{d^k (s \cdot \sum C_i b^i(\frac{t}{s}))}{dt^k}\right)^2 dt$$
$$= \int_0^1 \left(\frac{d^k (s \cdot \sum C_i b^i(\tau))}{s^k d\tau^k}\right)^2 s\, d\tau$$
$$= \int_0^1 s^{3-2k} \left(\frac{d^k \sum C_i b^i(\tau)}{d\tau^k}\right)^2 d\tau$$
$$= \int_0^1 s^{3-2k} \left(\frac{d^k \sum P_i \tau^i}{d\tau^k}\right)^2 d\tau$$

get Q from $P^T Q P$ or $C^T M^T Q M C$

$$Q = \begin{bmatrix} & \vdots & \\ \cdots & \frac{i(i-1)(i-2)(i-3)\,l(l-1)(l-2)(l-3)}{i+l-7} s^{-2k+3} & \cdots \\ & \vdots & \end{bmatrix}$$

get M from $C^T M^T Q M C$

pascal triangle

now we have Q & M

minimize objectives: $\min J = C^T M^T Q M C$

minimize conditions: $Ac = b$

$Ac \leq b$

$$c \in \mathbb{R}^{(n\_order+1) \times m}$$

Bezier Curve Hodograph

Hodograph implies that the derivatives of a Bezier Curve is still Bezier curve

control points from $P_{i,\ i=0\sim n+1} \rightarrow Q_{i,\ i=0\sim n}$

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$\boxed{\frac{d}{dt} B_{i,n}(t) = n\left(B_{i-1,n-1}(t) - B_{i,n-1}(t)\right)}$$

pf.

$$\frac{d}{dt} B_{i,n}(t) = \frac{d}{d\tau} \binom{n}{i} \tau^i (1-\tau)^{n-i}$$
$$= \frac{i\, n!}{i(n-i)!} t^{i-1} (1-t)^{n-i}$$
$$- \frac{(n-i)\, n!}{i(n-i)!} t^i (1-t)^{n-i-1}$$

$$\frac{d}{dt}B_{i,n}(t)=\frac{i\,n!}{i!(n-i)!}t^{i-1}(1-t)^{n-i}-\frac{(n-i)\,n!}{i!(n-i)!}t^{i}(1-t)^{n-i-1}$$

$$=\frac{n(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i}-\frac{n(n-1)!}{i!(n-k-1)!}t^{i}(1-t)^{n-i-1}$$

$$=n\left[\frac{(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i}-\frac{(n-1)!}{i!(n-k-1)!}t^{i}(1-t)^{n-i-1}\right]$$

$$=n\left(B_{i-1,n-1}(t)-B_{i,n-1}(t)\right) \#$$

$$F(t)=\sum_{i=0}^{n}B_{i,n}(t)P_i$$

$$\boxed{\frac{d}{dt}F(t)=\sum_{i=0}^{n-1}B_{i,n-1}(t)Q_i \quad Q_i=n(P_{i+1}-P_i)}$$

pf.

let $n=2$

$$F(t)=\sum_{i=0}^{2}B_{i,n}(t)P_i$$

$$=B_{0,2}(t)P_0+B_{1,2}(t)P_1+B_{2,2}(t)P_2$$

$$\frac{d}{dt}F(t)=2(\underbrace{\cancel{B_{-1,1}(t)}}_{=0}-B_{0,1}(t))P_0+2(B_{0,1}(t)-B_{1,1}(t))P_1+2(B_{1,1}(t)-\underbrace{\cancel{B_{2,1}(t)}}_{=0})P_2 \quad -①$$

when $n=1$, derivative of $F(t)$, $n=2$

$$F(t)=\sum_{i=0}^{1}B_{i,n}(t)Q_i=B_{0,1}(t)Q_0+B_{1,1}(t)Q_1 \quad -②$$

from ①

$$\frac{d}{dt}F(t)=(-2P_0+2P_1)B_{0,1}(t)+(-2P_1+2P_2)B_{1,1}(t)$$

$$=2(P_1-P_0)B_{0,1}(t)+2(P_2-P_1)B_{1,1}(t)$$

from②

$$=Q_0\,B_{0,1}(t)+Q_1\,B_{1,1}(t)$$

$$Q_0=2(P_1-P_0)$$
$$Q_1=2(P_2-P_1)$$

$$Q_i=n(P_{i+1}-P_i) \#$$

e.g.

$P_0\ P_1\ P_2\ P_3\ P_4$ $\quad n=4$ $\quad$ pts X5

$\frac{d}{dt}$ $n(P_1-P_0),\ n(P_2-P_1),\ n(P_3-P_2),\ n(P_4-P_3)$ pts X4

$\frac{d}{dt}$ $(n-1)\begin{pmatrix} n(P_2-P_1)-n(P_1-P_0) \\ n(P_3-P_2)-n(P_2-P_1) \\ n(P_4-P_3)-n(P_3-P_2)\end{pmatrix}$ pts X3

Equality Condition

starting condition
ending condition
continuous condition

$$Ac=b$$

$c\in R_{(n\_order+1)\times m}$

n_order: highest order of polynomial

m: no. of trajectories

e.g. $n\_order=7$
$m=5$

$c\in R_{(7+1)\times 5}=R_{40}$

$A\in R_{d\times(n\_order+1)\times m}=R_{d\times 40}$

$b\in R_d$ → $\times d$ constraints

starting condition, ending condition $j=0$, $j=f$

$a^0_{i,j}=c_0 \ (i,j)$ $\quad a^1_{i,j}=c_1$ $\quad a^2_{i,j}=c_2$

$P=a^0_{i,j}(s_j)'$

$V=\frac{d}{dt}a^0_{i,j}(s_j)^0=n(a^1_{i,j}-a^0_{i,j})$

$a=\frac{d}{dt^2}a^0_{i,j}(s_j)^{-1}=n(n-1)[(a^2_{i,j}-a^1_{i,j})\cdot(a^1_{i,j}-a^0_{i,j})]$ from above

↳ write everything into matrix

e.g. $n\_order=7$
$m=5$
$s_j=t$

$$Ac=b$$

$$\begin{bmatrix} 1t & 0 & 0 & \\ -7t^0, & 7t^0 & 0 & \\ 42t^{-1}, & -84t^{-1}, & 42t^{-1} & 0 \\ & 0 & & \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_8\end{bmatrix}=\begin{bmatrix} P \\ V \\ a \\ \vdots \\ 0\end{bmatrix}$$

continuous condition

$P_{n\_order+1,j}=P_{0,j+1}$ $\quad j=0,1,2,\dots m$

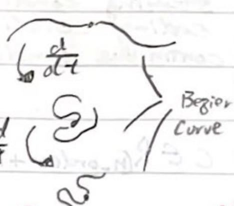
$V_{n\_order+1,j}=V_{0,j+1}$ $\quad j=0,1,2,\dots m$

$a_{n\_order+1,j}=a_{0,j+1}$ $\quad j=0,1,2,\dots m$

$$\therefore\begin{cases} P_{n\_order+1,j}-P_{0,j+1}=0 \\ V_{n\_order+1,j}-V_{0,j+1}=0 \\ a_{n\_order+1,j}-a_{0,j+1}=0 \\ j \\ \vdots \quad =0 \\ s_{:,:} \end{cases} \longrightarrow$$

Equality condition (cont'd)

$$\begin{cases} P_{n\_order+1,\,j} - P_{0,\,j+1} = 0 \\ V_{n\_order+1,\,j} - V_{0,\,j+1} = 0 \\ a_{n\_order+1,\,j} - a_{0,\,j+1} = 0 \end{cases}$$



$$\Rightarrow \begin{cases} a^{n\_order+1}_{\mu j} - a^{0}_{\mu j+1} = 0 \\ \frac{d}{dt} a^{n\_order+1}_{\mu j} - \frac{d}{dt} a^{0}_{\mu j+1} = 0 \\ \frac{d^2}{dt^2} a^{n\_order+1}_{\mu j} - \frac{d^2}{dt^2} a^{0}_{\mu j+1} = 0 \end{cases}$$

first pt of j+1 || last pt of j

$$\Rightarrow \begin{cases} a^{n\_order}_{\mu j}+1 - a^{0}_{\mu j+1} = 0 \\ n(a^{n\_order+1}_{\mu j} - a^{n\_order}_{\mu j}) - n(a^{1}_{\mu j+1}, a^{0}_{\mu j+1}) = 0 \\ n(n-1)\left[(a^{n\_order+1}_{\mu j} - a^{n\_order}_{\mu j}) - (a^{n\_order}_{\mu j} - a^{n\_order-1}_{\mu j})\right] \\ -n(n-1)\left[(a^{2}_{\mu j+1} - a^{1}_{\mu j+1}) - (a^{1}_{\mu j+1} - a^{0}_{\mu j+1})\right] = 0 \end{cases}$$

↳ write everything into matrix

e.g. $n\_order = 7$, $m = 5$ — 4 connection conditions

condition @ $j = 1, 2$ → 12 conditions

$$\begin{bmatrix} \cdots & & 1 & -1 & & \cdots \\ \cdots & -7 & 7 & 7 & -7 & \cdots \\ 42 & -84 & 42 & -42 & 84 & -42 \\ \cdots & & & & & \cdots \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_8 \\ \vdots \\ c_{40} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$R_{12\times 40}$ $\quad R_{40}$ $\quad R_{12}$

w, w/o t^{*n} does not matter as $Ac=0$

$\therefore \; Ac = b$

$A \in R_{d\times 40}$

$c \in R_{40}$

$b \in R_d$

d: no. of conditions
- starting
- ending
- continuous

Inequality condition (cont'd)

safety corridor

dynamic constraints

$$Ac \le b$$

- main reason for us to use bezier curve is that we can CONFINE all control points within our desired range
- all expression should be in the form of $Ac \le b$

$A \in R_{d\times[(n\_order+1)\times m]}$ — no. of trajectories

$C \in R_{(n\_order+1)\times m}$ — no. of ctrl pts per traj

$b \in R_d$ $\quad d \Rightarrow$ no. of conditions

safety corridor

let $a \le x \le b$

$c \le y \le d$

$e \le z \le f$

$$\Rightarrow \begin{cases} x \le b \\ -x \le -a \\ y \le d \\ -y \le -c \\ z \le f \\ -z \le -e \end{cases}$$

write everything in matrix $Ac \le b$

express everything in terms of control points

$$\Rightarrow t^{1}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ -1 & & & \\ & -1 & & \\ & & -1 & \ddots \end{bmatrix}\begin{bmatrix} C_0 \\ \vdots \\ C_{(n\_order+1)\times m} \end{bmatrix} \le b$$

ieq_P

$$\Rightarrow t^{0}\begin{bmatrix} -7 & 7 & & \\ & -7 & 7 & \\ & & & \ddots \\ 7 & -7 & & \\ & 7 & -7 & \ddots \end{bmatrix}\begin{bmatrix} C_0 \\ \vdots \\ C_{(n\_order+1)\times m} \end{bmatrix} \le b$$

ieq_V

$$\Rightarrow t^{-1}\begin{bmatrix} 42 & -84 & 42 & & \\ & 42 & -84 & 42 & \\ & & & & \ddots \\ -42 & 84 & -42 & & \\ & -42 & 84 & -42 & \ddots \end{bmatrix}\begin{bmatrix} C_0 \\ \vdots \\ C_{(n\_order+1)\times m} \end{bmatrix} \le b$$

ieq_V

e.g. $n\_order = 7$, $m = 5$

no. of P, V, a ctrl_pts $= 80$, $= 70$, $= 60$ $\quad \therefore A \in R_{210\times 40}$, $C \in R_{40}$, $b \in R_{210}$