No.

Date

ME564 L17

Numerical → we value : stability
accuracy

$\dot{x} = f(x)$, x_0 → Chaos : no closed-form solution

Forward Euler $\quad x_{k+1} = x_k + \Delta t\, f(x_k)$
$\qquad\qquad\qquad\quad = [I + \Delta t A]\, x_k$

Backward Euler $\quad x_{k+1} = x_k + \Delta t\, f(x_{k+1})$
$\qquad\qquad\qquad\quad = [I - \Delta t A]^{-1}\, x_k$

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Accuracy & Stability

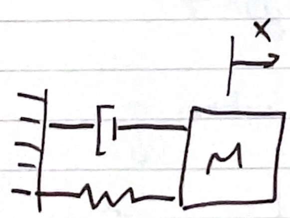
recall

Forward Euler : $\quad x_{k+1} = M x_k$

$x_0 \to x_1 \to x_2 \to x_3 \to \dots \to x_N$

$x_0 \to M x_0 \to M^2 x_0 \to M^3 x_0 \to \cdots \to M^N x_0$

↳ what kinda matrix gives stability



$$\ddot{x} + 2\zeta\omega_0 \dot{x} + \omega_0^2 x = 0$$

$\zeta = \dfrac{c}{2\sqrt{km}}$ damping ratio

$\omega_0 = \sqrt{k/m}$ natural frequency

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ v \end{bmatrix}$$

→ use Forward Euler

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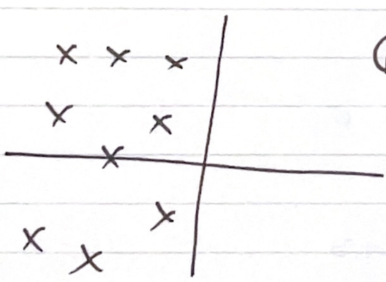
	FE	BE	ODE
Local Error	$O(\Delta t^2)$	$O(\Delta t^2)$	$O(\Delta t^5)$
Global Error	$O(\Delta t)$	$O(\Delta t)$	$O(\Delta t^4)$

(Not exactly ODE)

Continuous ODEs

$$\dot{x} = Ax$$

$$\boxed{\text{stable when: } \mathrm{Re}(\mathrm{eig}(A)) < 0}$$



discrete iteration

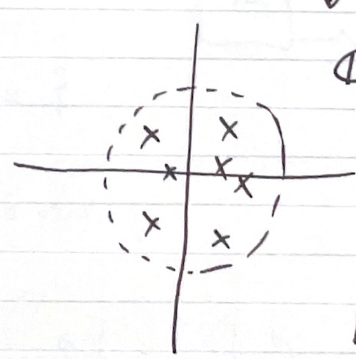
$$x_{k+1} = M x_k$$

$$\boxed{\text{stable when: All } |\mathrm{eig}(M)| < 1}$$

$$M = TDT^{-1}$$

$$M^k = TD^kT^{-1}$$

stable when



$$\lim_{k\to\infty} M^k = 0$$

$$M^k = TD^kT^{-1}$$

$$\begin{bmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_N^k \end{bmatrix}$$

Try $\lambda = 1.1 \to \infty$

Try $\lambda = 0.9 \to 0$

$\lambda = Re^{i\theta}$

$\lambda^N = R^N e^{iN\theta}$

Ex: $\dot{y} = \lambda y \quad y(0) = y_0$ Stable if $Re(\lambda) < 0$

Forward Euler: $y_{k+1} = (1 + \Delta t \lambda)\, y_k$: $y_0 \to y_1 \to y_2 \to \cdots \to y_N$

$y_N = (1+\Delta t\lambda)^N y_0$

Back. Euler: $y_{k+1} = (1 - \Delta t \lambda)^{-1} y_k$

$y_N = (1-\Delta t\lambda)^{-N} y_0$

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when is F.E stable iff

$$|1 + \Delta t \lambda| < 1$$

unstable if

$$|1 + \Delta t \lambda| > 1$$

assume $Re(\lambda) < 0 \to$ stable

despite being stable,

numerically, system could be unstable

when $|1+\Delta t\lambda| > 1$

e.g. $\Delta t \to$ too large.

B.E stable iff

$$|(1 - \Delta t \lambda^{-1})| < 1$$

unstable if

$$|(1 - \Delta t \lambda^{-1})| > 1$$

F.E. $\qquad \Delta t\lambda = z \qquad$ B.E.