IEKF_for_ALAN_RPE_on_Manifold

given a dynamic & measurement model :

$$x_{k+1} = f(x_k, u_k, w_k)$$
$$y_k = h(x_k, v_k)$$

△ dynamic model

remark: Note that all states are in the non-inertial frame

△ $x \in R$

$$x = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \quad x_{k+1} = f(x_k) = \begin{bmatrix} x_k + t u_k \\ y_k + t v_k \\ z_k + t w_k \\ \phi_k \\ \theta_k \\ \psi_k \\ u_k + tg(c\phi_k s\theta_k c\psi_k + s\phi_k s\psi_k) \\ v_k + tg(c\phi_k s\theta_k s\psi_k - s\phi_k c\psi_k) \\ w_k + tg(c\phi_k c\theta_k - 1) \end{bmatrix}, \quad F = \dots \text{ (omitted)}$$

$x \in \mathcal{M}$ (SE(3))

$$x = \begin{bmatrix} P \\ R \\ V \end{bmatrix}, \quad x_{k+1} = f(x_k) = \begin{bmatrix} P_k + V_k \Delta t \\ R \\ V + R(g)t \end{bmatrix}, \quad F = \begin{bmatrix} \frac{\partial P_k}{\partial P} & \frac{\partial P_k}{\partial R} & \frac{\partial P_k}{\partial V} \\ \frac{\partial R_k}{\partial P} & \frac{\partial R_k}{\partial R} & \frac{\partial R_k}{\partial V} \\ \frac{\partial V_k}{\partial P} & \frac{\partial V_k}{\partial R} & \frac{\partial V_k}{\partial V} \end{bmatrix} = \begin{bmatrix} I_3 & 0 & I_3\Delta t \\ 0 & I_3 & 0 \\ 0 & -R(g)_\times & I_3 \end{bmatrix}$$

$$\delta x = f'(x_k)$$

$$J = \frac{\partial f(x)}{\partial x} \triangleq \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \in R^{n\times m}$$
$$= \dots = \lim_{h\to 0} \frac{Jh}{h} = \frac{\partial Jh}{\partial h} = J$$

$$\therefore \frac{\partial V_k}{\partial R} = \frac{R g t}{\partial R}$$
$$= \lim_{\theta\to 0} \frac{(R \oplus \theta)\cdot g - R\cdot g}{\theta} \cdot t$$
$$= \lim_{\theta\to 0} \frac{(R\cdot \mathrm{Exp}(\theta))\cdot g - R\cdot g}{\theta} \cdot t$$
$$= \lim_{\theta\to 0} \frac{(R\cdot(I+\theta_\times))\cdot g - R\cdot g}{\theta} \cdot t$$
$$= \lim_{\theta\to 0} \frac{R\theta_\times \cdot g}{\theta} \cdot t$$
$$= \lim_{\theta\to 0} \frac{-R\, g_\times \theta}{\theta} \cdot t$$
$$= -R g_\times t$$

△ measurement model

△ $y_{k+1} = h(x^-_{k+1})$, let $(P, R) = \xi$

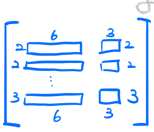
$$\Rightarrow \begin{bmatrix} u_i \\ v_i \\ \vdots \\ v \end{bmatrix} = \begin{bmatrix} f_x \frac{X'_i}{Z'_i} + c_x \\ f_y \frac{Y'_i}{Z'_i} + c_y \\ \vdots \\ d_{k+1} - d_k \end{bmatrix} \Big\} \; 0 \dots i$$

(derived from $su = KTP$, $d = T_{1:3,4}$ $\Rightarrow u = \frac{1}{s}KTP$)

$$\therefore H = \frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h_1}{\partial \xi} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial \xi} & \frac{\partial h_2}{\partial v} \\ \vdots & \vdots \\ \frac{\partial h_v}{\partial \xi} & \frac{\partial h_v}{\partial v} \end{bmatrix} = \begin{bmatrix} J_0 & 0 \\ J_1 & 0 \\ J_i & 0 \\ 0 & I_3 \end{bmatrix}$$

$$H = \begin{bmatrix} J_0 & 0' \\ J_1 & 0' \\ \vdots & \\ J_i & 0' \\ 0_{3\times 6} & I_3 \end{bmatrix}, \quad J_i = -\begin{bmatrix} \frac{-f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} & \frac{f_x X'Y'}{Z'^2} & -f_x - \frac{f_x X'^2}{Z'^2} & \frac{f_x Y'}{Z'} \\ 0 & \frac{-f_y}{Z'} & \frac{f_y Y'}{Z'^2} & f_y + \frac{f_y Y'^2}{Z'^2} & \frac{-f_y X'Y'}{Z'^2} & -\frac{f_y X'}{Z'} \end{bmatrix} \in R^{2\times 6}$$

$$0' \in R^{2\times 3}$$

= 

Iterated Extended Kalman Filter

△ prediction same as EKF

△ correction different :

- after receive z_{k+1}
- $\hat{x}^i_k = \hat{x}_{k|k-1}$ & start loop
- $H^i_k = \frac{\partial h(x)}{\partial x}\Big|_{x=\hat{x}^i_k}$
- $K^i_k = P_{k|k-1} H^{iT}_k (H^i_k P_{k|k-1} H^{iT}_k + R_k)^{-1}$
- $r^i_k = K^i_k (z_k - h(\hat{x}^i_k))$
- $\hat{x}^{i+1}_k = \hat{x}^i_k + r^i_k$
- break when $\|r^i_k\| < \epsilon$

in our case, we have additional param, thus residual slightly different

- embedded α, β into our residual function → determine K

$$r = \begin{bmatrix} \alpha\, r_{area} \\ \beta\, r_{align} \end{bmatrix} = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \underbrace{\begin{bmatrix} \begin{Bmatrix} u^i_k \\ \vdots \end{Bmatrix} - \frac{1}{s} K \hat{\chi}^i_k P \\ H' \hat{\chi}^i_k - H' \hat{x}^i_k \end{bmatrix}}_{LHS}$$

- $\alpha = \frac{1}{tr(R)}$, $\beta = \frac{1}{tr(Q)}$

- $\frac{Dr}{D\chi} = J$, $\Delta x = -[JJ^T]^{-1} J f(x)$

Summary for ALAN-RPE

∴ for Prediction

$\hat{\chi}^-_{k+1} = \chi_k \oplus \delta x$
$\quad = \chi_k \cdot \mathrm{Exp}(\delta x)$ (euler's method)
$\hat{\chi}^-_{k+1} = RK_4(x_k, f(x_k))$
$P^-_{k+1} = F_k P_k F_k + Q$
$F_k = \frac{Df}{D\chi}\Big|_{\chi=\chi_k}$

∴ for Correction (IEKF)

→ $\hat{\chi}^i_k := \hat{\chi}^-_k$
→ $r := \inf$
→ $i := 0$
→ while $r > \epsilon$

- $r = \begin{bmatrix} \alpha\, r_{area} \\ \beta\, r_{align} \end{bmatrix} = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \begin{bmatrix} \begin{Bmatrix} u^i_k \\ \vdots \end{Bmatrix} - \frac{1}{s} K \hat{\chi}^i_k P \\ H'\hat{\chi}^i_k - H'\hat{x}^i_k \end{bmatrix}$
- $\alpha = \frac{1}{tr(R)}$, $\beta = \frac{1}{tr(Q)}$
- $\frac{Dr}{D\chi} = J$, $\Delta x = -[JJ^T]^{-1} J f(x)$
- $\hat{\chi}^{i+1}_k = \hat{\chi}^i_k \oplus \Delta x$

△ Jacobian for a pinhole model during bundle adjustment

△ pinhole model :

$$s_i u_i = K T P_i$$

(s_i: depth; K: Intrinsic; T: SE(3); P_i: 3D points)

△ recall objective :

$$u_i - \frac{1}{s_i} K T P_i$$

intermediate variable

△ let $P' \triangleq$ PCD in camera frame

$$P' = (TP)_{1:3} = [X', Y', Z']^T$$

$\therefore su = KTP$

$\Rightarrow su = KP'$

$$\Rightarrow \begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$

$$\therefore u = f_x \frac{X'}{Z'} + c_x$$
$$v = f_y \frac{Y'}{Z'} + c_y$$

△ $e = u - \frac{1}{s} KTP$

$$\frac{\partial e}{\partial \delta\xi} = \lim_{\delta\xi \to 0} \frac{e(\delta\xi \oplus \xi) - e(\xi)}{\delta\xi} = \underbrace{\frac{\partial e}{\partial P'}}_{1} \underbrace{\frac{\partial P'}{\partial \delta\xi}}_{2}$$

△ $e = \begin{bmatrix} u - f_x \frac{X'}{Z'} + c_x \\ v - f_y \frac{Y'}{Z'} + c_y \end{bmatrix}$

recall Jacobian: $\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$

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$$\therefore \frac{\partial e}{\partial P'} = \begin{bmatrix} \frac{\partial e_1}{\partial X'} & \frac{\partial e_1}{\partial Y'} & \frac{\partial e_1}{\partial Z'} \\ \frac{\partial e_2}{\partial X'} & \frac{\partial e_2}{\partial Y'} & \frac{\partial e_2}{\partial Z'} \end{bmatrix} = \begin{bmatrix} -\frac{f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} \\ 0 & -\frac{f_y}{Z'} & \frac{f_y Y'}{Z'^2} \end{bmatrix} \quad \text{— ①}$$

let $P' := [P'; 1]$

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$$\frac{\partial P'}{\partial \delta\xi} = \frac{\partial TP}{\partial \delta\xi}$$

Lie Groups: $J = \frac{Df(\chi)}{D\chi} = \lim_{\tau\to 0} \frac{f(\chi \oplus \tau) \ominus f(\chi)}{\tau} \in R^{n\times m} = \lim_{\tau\to 0} \frac{\mathrm{Log}[f(\chi)^{-1} f(\chi \mathrm{Exp}(\tau))]}{\tau}$

e.g. remark: cannot differentiate w.r.t. R, as R is a matrix, hence, use lie algebra on the local tangent plane

$$= \lim_{\delta\xi\to 0} \frac{\exp(\delta\xi^\wedge)\exp(\xi^\wedge)P - \exp(\xi^\wedge)P}{\delta\xi}$$

$$\lim_{h\to 0} \frac{e^h e^x - e^x}{h} = \lim_{h\to 0} \frac{(1+h)e^x - e^x}{h} = \lim_{h\to 0} \frac{he^x}{h} = e^x$$

$$= \lim_{\delta\xi\to 0} \frac{(I+\delta\xi^\wedge)\exp(\xi^\wedge)P - \exp(\xi^\wedge)P}{\delta\xi}$$

$$= \lim_{\delta\xi\to 0} \frac{\delta\xi^\wedge \exp(\xi^\wedge)P}{\delta\xi}$$

$\exp(\xi^\wedge)P = T_P = RP + t$

$$= \lim_{\delta\xi\to 0} \frac{\begin{bmatrix} \delta\omega^\wedge & \delta\rho \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} RP+t \\ 1 \end{bmatrix}}{\delta\xi}$$

recall Jacobian: $\frac{\partial y}{\partial x} = [\dots]$

$$= \lim_{\delta\xi\to 0} \frac{\begin{bmatrix} \delta\omega^\wedge(RP+t) + \delta\rho \\ 0^T \end{bmatrix}}{\delta\xi}, \quad \delta\xi = \begin{bmatrix} \delta\rho \\ \delta\omega \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial\delta\rho}(\delta\omega^\wedge(RP+t)+\delta\rho) & \frac{\partial}{\partial\delta\omega}(\delta\omega^\wedge(RP+t)+\delta\rho) \\ \frac{\partial}{\partial\delta\rho} 0 & \frac{\partial}{\partial\delta\omega} 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & -(RP+t)^\wedge \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I & -P'^\wedge \\ 0 & 0 \end{bmatrix}$$

- Here $P' = [x', y', z']$

$$\therefore \frac{\partial P'}{\partial \delta\xi} = [I \;\; -P'^\wedge] = \begin{bmatrix} 1 & 0 & 0 & 0 & Z' & -Y' \\ 0 & 1 & 0 & -Z' & 0 & X' \\ 0 & 0 & 1 & Y' & -X' & 0 \end{bmatrix}$$

- as $\frac{\partial e}{\partial P'} = \begin{bmatrix} -\frac{f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} \\ 0 & -\frac{f_y}{Z'} & \frac{f_y Y'}{Z'^2} \end{bmatrix}$

& $\frac{\partial P'}{\partial \delta\xi} = \begin{bmatrix} 1 & 0 & 0 & 0 & Z' & -Y' \\ 0 & 1 & 0 & -Z' & 0 & X' \\ 0 & 0 & 1 & Y' & -X' & 0 \end{bmatrix}$

$$\therefore \frac{\partial e}{\partial \delta\xi} = \begin{bmatrix} -\frac{f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} \\ 0 & -\frac{f_y}{Z'} & \frac{f_y Y'}{Z'^2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & Z' & -Y' \\ 0 & 1 & 0 & -Z' & 0 & X' \\ 0 & 0 & 1 & Y' & -X' & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-f_x}{Z'} & 0 & \frac{f_x X'}{Z'^2} & \frac{f_x X'Y'}{Z'^2} & -f_x - \frac{f_x X'^2}{Z'^2} & \frac{f_x Y'}{Z'} \\ 0 & \frac{-f_y}{Z'} & \frac{f_y Y'}{Z'^2} & f_y + \frac{f_y Y'^2}{Z'^2} & \frac{-f_y X'Y'}{Z'^2} & -\frac{f_y X'}{Z'} \end{bmatrix}$$