

MES64 L22

vector calculus

Div, Grad, Curl

 $\nabla \cdot \vec{f}$  Div $\nabla f$  Grad $\nabla \times \vec{f}$  Curl $\nabla$  Del, Nabla & vectors

of Partial Derivatives

$$\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

 $\nabla$  is a linear operator

$$\nabla(f+g) = \nabla f + \nabla g \quad f(x,y,z)$$

$$\nabla(\alpha f) = \alpha \nabla f \quad g(x,y,z)$$

 $\alpha$  is a number

$$[A] [v_1 + v_2]$$

matrix mult.p.

is linear operator.

Gradient

 $\nabla f(x,y,z)$  is called gradient.

how fast my function is changing.

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

input

scalar function

output

vector field

Ex. 1.

Mass in Gravity Field

$$\text{Newton's Law: } \vec{F} = -\frac{mMG}{r^3} \vec{r}$$

$$= -\nabla V \text{ (potential field)}$$

Gravitational Potential.

$$V = -\frac{mMG}{r} \text{ (scalar)}$$

$$= -\frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

linear about earth radius.

$$\frac{1}{r_0 + h} \text{ (linearized)}$$

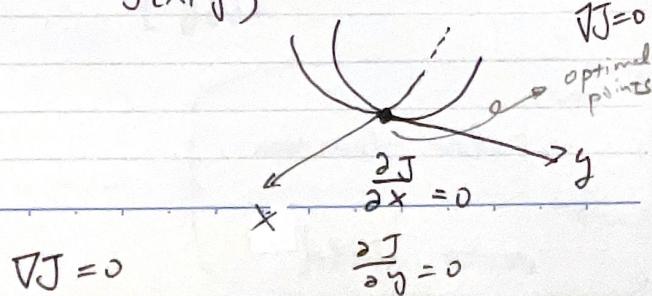
$$\frac{\partial V}{\partial x} = \frac{mMGx}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial V}{\partial y} = \frac{mM Gy}{(x^2 + y^2 + z^2)^{3/2}}$$

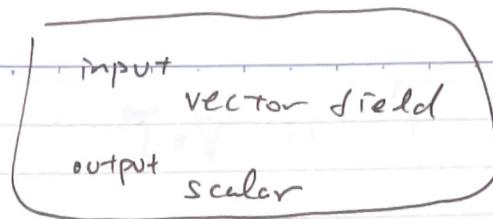
$$\frac{\partial V}{\partial z} = \frac{mM Gz}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\vec{F} = -\nabla V = \begin{bmatrix} -m \frac{M G x}{r^3} \\ -m \frac{M G y}{r^3} \\ -m \frac{M G z}{r^3} \end{bmatrix}$$

$$J(x, y)$$



Divergence.



Since  $\nabla$  is a vector, I can take inner product w/  $\vec{f}$

$$\vec{f}(x, y, z) = \underbrace{\begin{bmatrix} f_1(x, y, z) \\ f_2(x, y, z) \\ f_3(x, y, z) \end{bmatrix}}_{\text{vector field}} \quad (\text{is still a function})$$

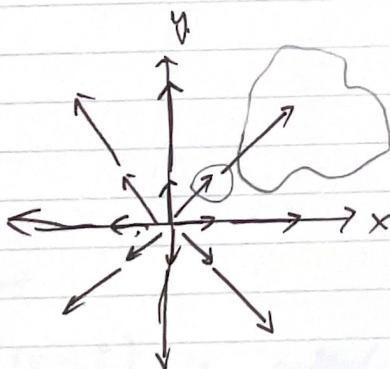
$\nabla \cdot \vec{f}$  is the divergence of  $\vec{f}$

$$\nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

How much will this vector field diverge.

Ex 1.  $\vec{f} = x\vec{i} + y\vec{j}$

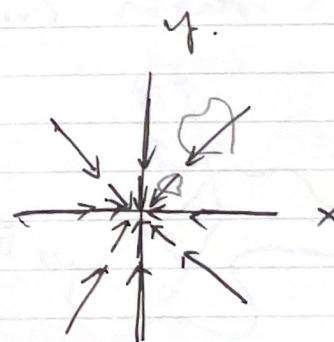
$$\nabla \cdot \vec{f} = 1 + 1 = 2$$



flow is diverging

Ex 2.  $\vec{f} = -x\vec{i} - y\vec{j}$

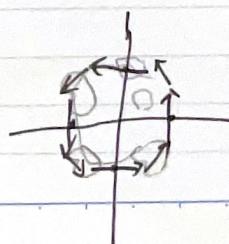
$$\nabla \cdot \vec{f} = -2$$



flow is converging

Ex 3.  $\vec{f} = -y\vec{i} + x\vec{j}$

$$\nabla \cdot \vec{f} = 0 + 0 = 0$$



flow is rotation.

Curl of  $\vec{f}$  is  $\nabla \times \vec{f}$

$$\nabla_x = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}$$

$$\nabla_x \vec{f} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$\vec{f} = x y \vec{i} - \sin(\theta) \vec{j} + \vec{k}$$

$$\nabla \times \vec{f} = \vec{i} (\cos(\theta)) - \vec{j} (0) + \vec{k} (-x)$$

measures rotation (vorticity)

