On the Stability of Constrained MPC Without Terminal Constraint

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Abstract—The usual way to guarantee stability of model predictive control (MPC) strategies is based on a terminal cost function and a terminal constraint region. This note analyzes the stability of MPC when the terminal constraint is removed. This is particularly interesting when the system is unconstrained on the state. In this case, the computational burden of the optimization problem does not have to be increased by introducing terminal state constraints due to stabilizing reasons. A region in which the terminal constraint can be removed from the optimization problem is characterized depending on some of the design parameters of MPC. This region is a domain of attraction of the MPC without terminal constraint. Based on this result, it is proved that weighting the terminal cost, this domain of attraction of the MPC controller without terminal constraint is enlarged reaching (practically) the same domain of attraction of the MPC with terminal constraint; moreover, a practical procedure to calculate the stabilizing weighting factor for a given initial state is shown. Finally, these results are extended to the case of suboptimal solutions and an asymptotically stabilizing suboptimal controller without terminal constraint is presented.

Index Terms—Asymptotic stability, predictive control, suboptimal control.

I. INTRODUCTION AND PROBLEM STATEMENT

Consider a system described by a nonlinear invariant discrete time model

$$x^+ = f(x, u) \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector and x^+ is the successor state. The system is subject to constraints on both states and control actions, and they are given by

$$x \in X, \ u \in U \tag{2}$$

where X is a closed set and U a compact set, both of them containing the origin. In what follows, x_k and u_k will denote the state and the control action applied to the system at sampling time k. A sequence of control actions to be applied to the system at current state x is denoted as

$$\mathbf{u}(x) = \{u(0; x), u(1; x), \ldots, u(N-1; x)\}$$

where its dependence with x may be omitted. The predicted state of the system at time j, when the initial state is x (at time 0) and the control sequence $\mathbf{u}$ is applied, will be denoted as $x(j) = \phi(j; x, \mathbf{u})$.

It has been proved that this class of systems can be stabilized by a MPC control law $K_N(x)$; this control law is obtained by solving a constrained optimization problem at each sampling time and applying it to the system in a receding horizon way. The finite horizon nominal MPC optimization problem with terminal cost and terminal constraint

is the standard way of formulating the MPC controller [1], and it will be denoted as general MPC. This optimization problem, denoted as $P_N(x, \Omega)$, is given by

$$\min_{\mathbf{u}} V_N(x, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i), u(i)) + F(x(N))$$
$$s.t. \ x(i) \in X, u(i) \in U, \qquad i = 0, \ldots, N-1 \quad x(N) \in \Omega$$

where $x(i) = \phi(i; x, \mathbf{u})$, $\ell(x, u)$ is the stage cost, which is assumed to be a positive definite function on x (i.e., there exists a $\mathcal{K}$ function[1] $\gamma_1(\cdot)$ such that $\ell(x, u) \geq \gamma_1(\|x\|)$ [2]); $F(x)$ is the terminal cost and $\Omega \subseteq X$ is the terminal region. In what follows, $\mathbf{u}^*(x)$ denotes the optimal solution to $P_N(x, \Omega)$ and $x^*(i) = \phi(i; x, \mathbf{u}^*(x)), i = 0, \ldots, N$, denotes the optimal predicted trajectory. The set of states where the optimization problem $P_N(x, \Omega)$ is feasible (and hence $K_N(x)$ is defined) is denoted by $X_N(\Omega)$.

The terminal cost and the terminal constraint are usually chosen satisfying the following assumption.

Assumption 1: Let $F(x)$ be a control Lyapunov function (CLF) and let Ω be a set given by $\Omega = \{x \in \mathbb{R}^n : F(x) \leq \alpha\}$, with $\alpha > 0$, such that $\Omega \subseteq X$ and for all $x \in \Omega$:

$$\alpha_1(\|x\|) \leq F(x) \leq \alpha_2(\|x\|) \tag{3}$$
$$\min_{u \in U}\{F(f(x, u)) - F(x) + \ell(x, u)\} \leq 0 \tag{4}$$

where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ are $\mathcal{K}$-functions.

In [3], it is proved that if the terminal cost and terminal set satisfy assumption 1, the optimal cost of $P_N(x, \Omega)$ is a Lyapunov function and the model predictive control (MPC) control law stabilizes asymptotically the system in $X_N(\Omega)$. If the terminal constraint is removed from the optimization problem, then the optimal cost may not be a Lyapunov function for the system and, moreover, the feasibility may be lost. However, there are some predictive controllers with guaranteed stability which do not consider an explicit terminal constraint, as in [4]–[7]. Notice that the removal of the terminal constraint may be interesting, for instance, if the system is not constrained on the states. In this case, the terminal constraint is the only one that depends on the predicted state of the system. So, the removal of this constraint makes the problem much easier to solve and the computational burden is reduced, but at the expense of a reduction of the domain of attraction.

In [5], stability is guaranteed by considering a quadratic terminal cost function $F(x) = a \cdot x^T \cdot P \cdot x$ and it is proved that, for any stabilizable initial state, there is a triple $(a, P, \text{and } N)$ such that the system is stabilized. Based on these results, stability of MPC with a CLF as terminal cost for a class of unconstrained nonlinear systems is analyzed in [6]. In [7], using a slightly modified Lypaunov function as terminal cost it is proved that the MPC without terminal constraint stabilizes the system asymptotically for any initial state where the terminal constraint is not active; that is, in $\hat{X}_N = \{x : F(x^*(N)) \leq \alpha\}$.

This note presents some novel results on this topic. Generalizing previous results presented in [5] to the general MPC, a region where the terminal constraint is satisfied in the optimization problem is characterized. This region is a domain of attraction of the MPC without terminal constraint. This characterization allows us to prove that this region can be enlarged by weighting the terminal cost. Furthermore, it is proved that a larger weighting factor implies a bigger domain of attraction. Thus, the proposed MPC, by means of weighting the terminal cost, can stabilize the system at any initial state such that $x_0 \in X_N(\tilde{\Omega})$, where

[1]A function $\gamma(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{K}$ function if it is continuous, strictly increasing and $\gamma(0) = 0$

$\tilde{\Omega}$ denotes the interior of Ω. Notice that this region is almost the same that the domain of attraction of the MPC with terminal constraint. The note also shows how to choose this weighting factor for a given initial state. The local optimality property can be maintained by means of a proposed practical procedure to adapt the weighting factor. Another contribution of this note is a practical algorithm that allows us to stabilize asymptotically the system under suboptimal solutions of the optimization problem.

II. CHARACTERIZATION OF A DOMAIN OF ATTRACTION

In this section, a region where the terminal constraint is not active is obtained. This region is a domain of attraction of MPC without terminal constraint, which is derived from the optimization problem without terminal constraint: $P_N(x, X)$. In what follows, $V_N^*(x)$ denotes the optimal cost of $P_N(x, X)$, $\mathbf{u}^*(x)$ denotes the optimal solution of $P_N(x, X)$ and $x^{*}(i) = \phi(i; x, \mathbf{u}^*(x))$, $i = 0, \ldots, N$, denotes the optimal predicted trajectory.

The characterization of a domain of attraction, as well as further results, is based on the following lemma which generalizes other similar ones [5], [7] to the MPC with a CLF as terminal cost.

Lemma 1: Consider the optimization problem $P_N(x, X)$ such that $F(x)$ and Ω satisfy Assumption 1. Let $\mathbf{u}^*(x)$ be the optimal sequence of inputs for any $x \in X_N(X)$. If $x^*(N) \notin \Omega$, then $x^*(j) \notin \Omega$, for any $j = 0, \ldots, N - 1$.

Proof: Assume that $x^*(N) \notin \Omega$ and that an $i \in [0, N-1]$ exists such that $x^*(i) \in \Omega$. Consider that $u^*(x)$ is the optimal solution of $P_N(x, X)$ and $\hat{\mathbf{u}}$ is the solution of $P_{N-i}(x^*(i), X)$. Then, in virtue of the optimality principle, we have that $\hat{\mathbf{u}} = \{u^*(i; x), \ldots, u^*(N - 1; x)\}$ and consequently the optimal predicted trajectory is the same, that is

$$\phi(j; x^*(i), \hat{\mathbf{u}}) = \phi(i + j; x, \mathbf{u}^*(x)) = x^*(i + j)$$

for all $j = 0, \ldots, N - i$.

Since for all $x \in \Omega$, $V_N^*(x) \leq F(x)$ [1], we have that

$$F(x^*(i)) \geq V_{N-i}^*(x^*(i)) \geq F(x^*(N)) > \alpha.$$

Hence, $x^*(i) \notin \Omega$, which contradicts the assumption, proving the lemma.

Since $\ell(x, u)$ is positive definite and the origin is in the interior of Ω, the following assumption can be established.

Assumption 2: Let d be a positive constant such that $\ell(x, u) > d$, $\forall x \notin \Omega$ and $\forall u \in U$.

This definition for d and Lemma 1 lead us to the following theorem, where a domain of attraction of the MPC without terminal constraint is characterized. The main advantage of this region is its explicit dependence on some of the design parameters of the MPC as the prediction horizon, the stage cost and the terminal region. This feature allows us to analyze the effect of these parameters on the size of the region.

Theorem 1: Consider $F(x)$ and $\Omega = \{x \in \mathbb{R}^n : F(x) \leq \alpha\}$ such that satisfy Assumption 1; let d be a constant such that Assumption 2 holds; then the MPC controller with $N \geq 1$ derived from $P_N(x, X)$ stabilizes asymptotically the system (1) subject to (2) for any initial state in

$$\Gamma_N = \{x \in \mathbb{R}^n : V_N^*(x) \leq \ell(x, K_N(x)) + (N-1) \cdot d + \alpha\}.$$

Proof: First, it is proved by contradiction that for any $x \in \Gamma_N$, the optimal solution satisfies the terminal constraint. From Lemma 1, it can be inferred that if the optimal sequence is such that the terminal region is not reached, then all the trajectory of the system is out of Ω and hence

$$V_N^*(x) > \ell(x, K_N(x)) + (N-1) \cdot d + \alpha$$

for all $N \geq 1$ and, hence, $x \notin \Gamma_N$. Therefore, for all $x \in \Gamma_N$, we have that

$$V_N^*(x) \leq \ell(x, K_N(x)) + (N-1) \cdot d + \alpha$$

and consequently the optimal solution of the MPC satisfies the terminal constraint.

Second, it is proved that Γ_N is a positively invariant set for the closed loop system. Consider that $x \in \Gamma_N$, then $x^*(N) \in \Omega$. The terminal constraint satisfaction, as well as the assumption considered for the terminal cost and the terminal region, make that the monotonicity property of the optimal cost [1] holds, that is, $V_N^*(x) \leq V_{N-1}^*(x)$ for all $x \in X_{N-1}(\Omega)$. By virtue of this property, we have that

$$\begin{aligned} V_N^*(x^*(1)) &\leq V_{N-1}^*(x^*(1)) = V_N^*(x) - \ell(x, K_N(x)) \\ &\leq (N-1) \cdot d + \alpha \end{aligned} \tag{5}$$

and, consequently, $x^*(1) \in \Gamma_N$.

From standard arguments in MPC (see, for instance, [1]), we have that $V_N^*(x) \geq \ell(x, K_N(x)) \geq \ell(x, 0) \geq \gamma_1(\|x\|)$ and for all $x \in \Omega$, $V_N^*(x) \leq F(x) \leq \alpha_2(\|x\|)$. From (5), we derive that $V_N^*(x^*(1)) \leq V_N^*(x) - \ell(x, K_N(x))$. Therefore, the optimal cost is a strictly decreasing Lyapunov function, which proves the asymptotic stability of the closed loop system [2].

It is easy to show that the set Γ_N contains the terminal region. Using similar arguments, it can be shown that the set

$$\Upsilon_N = \{x \in \mathbb{R}^n : V_N^*(x) \leq N \cdot d + \alpha\}$$

is also a domain of attraction of the system controlled by the proposed controller and it is contained in Γ_N. Notice that to use this result an explicit expression of the optimal cost is not required, but only the solution of the optimization problem at a given state.

III. ENLARGING THE DOMAIN OF ATTRACTION BY WEIGHTING THE TERMINAL COST

It is well known that increasing the prediction horizon, the domain of attraction of the MPC (with or without terminal constraint [7]) is enlarged. It is easy to prove that Γ_N is also enlarged in this case. However, a drawback of this procedure is that the computational burden is increased.

In this section, it is proved that the domain of attraction of the MPC without terminal constraint is enlarged by weighting the terminal cost and a practical procedure to obtain the stabilizing weighting factor for a given state is presented. Finally it is shown a decaying sequence of weighting factors to enhance the optimality of the obtained MPC controller.

Consider $F(x)$ and $\Omega = \{x \in \mathbb{R}^n : F(x) \leq \alpha\}$ which verify Assumption 1 and consider a weighted terminal cost given by $F_\lambda(x) = \lambda \cdot F(x)$, where $\lambda \geq 1$. It is easy to show that $F_\lambda(x)$ also satisfies Assumption 1 in $\Omega = \{x \in \mathbb{R}^n : F_\lambda(x) \leq \lambda \cdot \alpha\}$.

In what follows, $V_{N,\lambda}(x, \mathbf{u})$ denotes the functional cost considering $F_\lambda(x)$ as terminal cost function. Analogously, to emphasize its dependence with λ, the region Υ_N is denoted as $\Upsilon_N(\lambda)$. This set is given by

$$\Upsilon_N(\lambda) = \{x \in \mathbb{R}^n : V_{N,\lambda}^*(x) \leq N \cdot d + \lambda \cdot \alpha\}.$$

Notice that this set is a domain of attraction of the MPC without terminal constraint. In the following theorem, it is proved that the set $\Upsilon_N(\lambda)$ is enlarged if λ is increased.

Theorem 2: Consider $F(x)$ and $\Omega = \{x \in \mathbb{R}^n : F(x) \leq \alpha\}$ such that satisfy Assumption 1. Consider the MPC controller with $N \geq 1$ derived from $P_N(x, X)$ using a weighted terminal cost $F_\lambda(x)$, with $\lambda \geq 1$. Then for all $\lambda_0 \leq \lambda_1$, $\Upsilon_N(\lambda_0) \subseteq \Upsilon_N(\lambda_1)$.

Proof: For all $x \in \Upsilon_N(\lambda_0)$, we have that

$$
\begin{aligned}
V_{N,\lambda_0}^*(x) &= \sum_{i=0}^{N-1} \ell(x^*(i), u^*(i)) + \lambda_0 \cdot F(x^*(N)) \\
&= \sum_{i=0}^{N-1} \ell(x^*(i), u^*(i)) + \lambda_1 \cdot F(x^*(N)) \\
&\quad - (\lambda_1 - \lambda_0) \cdot F(x^*(N)) \\
&\geq V_{N,\lambda_1}^*(x) - (\lambda_1 - \lambda_0) \cdot F(x^*(N)).
\end{aligned}
$$

Considering that $x \in \Upsilon_N(\lambda_0)$, then $F(x^*(N)) \leq \alpha$ and hence $V_{N,\lambda_0}^*(x) \geq V_{N,\lambda_1}^*(x) - (\lambda_1 - \lambda_0) \cdot \alpha$. Thus, given any $x \in \Upsilon_N(\lambda_0)$, i.e., $V_{N,\lambda_0}^*(x) \leq N \cdot d + \lambda_0 \cdot \alpha$, we have that

$$
V_{N,\lambda_1}^*(x) \leq V_{N,\lambda_0}^*(x) + (\lambda_1 - \lambda_0) \cdot \alpha \leq N \cdot d + \lambda_1 \cdot \alpha
$$

and then $x \in \Upsilon_N(\lambda_1)$, which completes the proof. ∎

Analogously, the enlargement of $\Gamma_N(\lambda)$ can be proved. This property allows us to state the following theorem.

Theorem 3: Let $\tilde{\Omega}$ denote the interior of Ω. For any $x_0 \in X_N(\tilde{\Omega})$, there exists a finite constant λ such that $x_0 \in \Upsilon_N(\lambda)$, and hence, the system can be stabilized using the MPC without terminal constraint.

Proof: Assume that the initial state $x_0 \in X_N(\tilde{\Omega})$. Thus, there exists a sequence of feasible control inputs $\mathbf{u}$ such that the terminal state is in the interior of Ω, that is, $F(x(N)) < \alpha$, where $x(N) = \phi(N; x_0, \mathbf{u})$. Then, there exists a constant $\rho \in [0, 1)$ such that $x_0 \in X_N(\Omega_\rho)$, where $\Omega_\rho = \{x : F(x) \leq \rho \alpha\}$. Let $\mathbf{u}$ be a (suboptimal) solution of $P_N(x, \Omega_\rho)$, then the associated cost $V_{N,1}(x_0, \mathbf{u})$ is given by

$$
V_N(x_0, \mathbf{u}) = \sum_{i=0}^{N-1} \ell(x(i), u(i)) + F(x(N)) = L_N + F(x(N))
$$

where $x(i) = \phi(i; x_0, \mathbf{u})$ and the sum of the stage cost along the trajectory is denoted as L_N. If $V_N(x_0, \mathbf{u}) \leq N \cdot d + \alpha$, then the theorem is proved with $\lambda = 1$. In other case, if we consider

$$
\lambda = \frac{L_N - N \cdot d}{(1 - \rho) \cdot \alpha} \tag{6}
$$

then we have that

$$
V_{N,\lambda}^*(x_0) \leq V_{N,\lambda}(x_0, \mathbf{u}) \leq L_N + \rho \cdot \lambda \cdot \alpha = N \cdot d + \lambda \cdot \alpha.
$$

Therefore, the initial state $x_0 \in \Upsilon_N(\lambda) \subseteq \Gamma_N(\lambda)$, and hence it is stabilized by the MPC controller. ∎

Remark 1 (Weighting Factor Calculation): Theorem 3 allows us to obtain a practical method to calculate a (probably conservative) stabilizing weighting factor. Given an initial state, this can be derived from (6). Furthermore, defining D as a constant such that for all $x \in X$ and $u \in U$, $\ell(x, u) \leq D$, it is easy to see that $L_N \leq N \cdot D$, and hence the weighting factor

$$
\lambda = \frac{N \cdot (D - d)}{(1 - \rho) \cdot \alpha} \tag{7}
$$

is a (conservative) stabilizing weighting factor for all initial state contained $X_N(\Omega_\rho)$. Conversely, a given weighting factor λ is stabilizing for any initial state contained in $X_N(\Omega_\rho)$ where

$$
\rho = \max\left(1 - \frac{N(D - d)}{\lambda\alpha}, 0\right).
$$

From this statement one can derive that for any $x \in X_N(\Omega_\rho)$, $V_{N,\lambda}^*(x) \leq Nd + \lambda \cdot \alpha$ and hence $X_N(\Omega_\rho) \subseteq \Upsilon_N(\lambda)$.

Remark 2 (Local Optimality): If the terminal cost is the unconstrained optimal cost, then the obtained MPC controller is optimal in the region $\hat{X}_N$ [7]. Therefore, the usage of a weighted terminal cost enlarges the domain of attraction at expense of a loss of the optimality (i.e., a worse closed loop performance); moreover the greater λ, the worse is $F_\lambda(x)$ an as approximation to the optimal cost in Ω and, thus, the greater is the difference with the optimal controller.

In order to reduce this effect and maintain the optimality property locally, the weighting factor can be reduced at each sample time as it is proposed as follows:

$$
\lambda_{k+1} = \begin{cases} \lambda_k - \frac{\ell(x_k, K_N(x_k))}{\alpha}, & \text{if } V_{N,1}(x_k, \mathbf{u}^*(k)) > N \cdot d + \alpha \\ 1, & \text{if } V_{N,1}(x_k, \mathbf{u}^*(k)) \leq N \cdot d + \alpha \end{cases}
$$

where $\mathbf{u}^*(k)$ is the optimizer of the optimization problem at sample time k.

In effect, consider that $V_{N,\lambda_k}^*(x_k) \leq N \cdot d + \lambda_k \cdot \alpha$ and assume that $V_{N,1}(x_k, \mathbf{u}^*(k)) > N \cdot d + \alpha$, then

$$
\begin{aligned}
V_{N,\lambda_{k+1}}^*(x_{k+1}) &\leq V_{N,\lambda_k}^*(x_{k+1}) \\
&\leq V_{N,\lambda_k}^*(x_k) - \ell(x_k, K_N(x_k)) \\
&\leq N \cdot d + \lambda_k \cdot \alpha - \ell(x_k, K_N(x_k)) \\
&= N \cdot d + \lambda_{k+1} \cdot \alpha.
\end{aligned}
$$

If $V_{N,1}(x_k, \mathbf{u}^*(k)) \leq N \cdot d + \alpha$ then

$$
V_{N,1}^*(x_{k+1}) \leq V_{N,1}^*(x_k) \leq V_{N,1}(x_k, \mathbf{u}^*(k)) \leq N \cdot d + \alpha.
$$

Therefore, the chosen λ_{k+1} ensures that for any x_k, $x_{k+1} \in \Upsilon_N(\lambda_{k+1})$.

IV. SUBOPTIMAL CONTROLLER

The previously presented results are based on Theorem 1, which requires the optimality of the solution. However, as it is pointed out in [8], the computation of the optimal solution may be difficult or too demanding due to the nonlinear nature of the optimization problem; hence stability under suboptimality of the obtained solutions should be addressed. In this section the presented results are extended to the case of suboptimal solutions of the optimization problem; asymptotic stability under suboptimality is proved and a constructive way of implementing the controller is presented. This is a novel result on this topic that has not been addressed in [5] and [7]. In [6], the optimality is relaxed assuming that it is possible to obtain a feasible better solution satisfying the terminal constraint.

First, an extension of Lemma 1 to the case of feasible solutions is presented. Based on this, a suboptimal controller is proposed in such a way that asymptotic stability is proved and no additional assumptions have to be made.

Lemma 2: Let $P_N(x, X)$ be an optimization problem such that $F(x)$ and $\Omega = \{x \in \mathbb{R}^n : F(x) \leq \alpha\}$ satisfy Assumption 1. Let $\mathbf{u} = \mathbf{u}(x)$ be a feasible solution to $P_N(x, X)$ such that $V_N(x, \mathbf{u}) \leq N \cdot d + \alpha$. Then the terminal region is reached along the system evolution, that is, there exists an $i \in [0, N]$ such that $x(i) = \phi(i; x, \mathbf{u}) \in \Omega$

Proof: It is proved by contradiction. Consider that $x(i) \notin \Omega$ for all $i \in [0, N]$. Then, we have that $\ell(x(i), u(i)) > d$ and, hence, $V_N(x, \mathbf{u}) > N \cdot d + \alpha$, which contradicts the fact $V_N(x, \mathbf{u}) \leq N \cdot d + \alpha$. ∎

From this result, the following lemma is derived.

Lemma 3: Let $P_N(x, X)$ be an optimization problem such that $F(x)$ and $\Omega = \{x \in \mathbb{R}^n : F(x) \leq \alpha\}$ satisfy Assumption 1. Let $\mathbf{u}$ be a feasible solution to $P_N(x, X)$ such that $V_N(x, \mathbf{u}) \leq N \cdot d + \alpha$ and let x^+ be the successor state, that is $x^+ = f(x, u(0; x))$. Then there exists a feasible solution $\hat{\mathbf{u}}$ to the optimization problem $P_N(x^+, X)$ such that

- $V_N(x^+, \hat{\mathbf{u}}) < V_N(x, \mathbf{u})$, for all $x \neq 0$;
- If $x^+ \in \Omega$ then $V_N(x^+, \hat{\mathbf{u}}) \leq F(x^+)$.

Proof: Let $i \in [0, N]$ be the maximum i such that $x(i) = \phi(i; x, \mathbf{u}) \in \Omega$, which exists in virtue of Lemma 2. Hence, we have that $x(i) \in \Omega$ and, if $i < N$, $x(j) \notin \Omega$ for all $j \in [i + 1, N]$.

Calculate a sequence of control inputs $\hat{\mathbf{u}}$ given by

$$\hat{\mathbf{u}} = \{u(1; x), \ldots, u(i - 1; x), h(\hat{x}(i - 1)), \ldots, h(\hat{x}(N - 1))\} \quad (8)$$

where $\hat{x}(j) = \phi(j; x^+, \hat{\mathbf{u}})$ and

$$h(x) = \arg \min_{u \in U} \{F(f(x, u)) - F(x) + \ell(x, u)\}.$$

Note that the computational burden of this solution is quite reduced compared with the cost required for the computation of $P_N(x, X)$. Moreover, we only need a feasible solution such that $F(f(x, u)) - F(x) + \ell(x, u) \leq 0$.

It is easy to see that $\hat{x}(j - 1) = x(j)$, for all $j \in [1, i]$; then $\hat{x}(i - 1) \in \Omega$. Thus, in virtue of the invariance of Ω in (4), $\hat{x}(j) \in \Omega \subseteq X$ for all $j \in [i, N]$. Consequently, $\hat{\mathbf{u}}$ is a feasible solution to $P_N(x^+, X)$.

By definition we have that $\hat{u}(j - 1; x^+) = u(j; x)$ for all $j \in [1, i - 1]$ and hence $\hat{x}(j - 1) = x(j)$ for all $j \in [1, i]$. Denoting $\Delta V_N = V_N(x, \mathbf{u}) - V_N(x^+, \hat{\mathbf{u}})$, we have that if $i < N$ then

$$\Delta V_N = \left\{ \ell(x, u(0; x)) + \sum_{j=i}^{N-1} \ell(x(j), u(j; x)) + F(x(N)) \right\}$$
$$- \left\{ \sum_{j=i-1}^{N-1} \ell(\hat{x}(j), \hat{u}(j; x^+)) + F(\hat{x}(N)) \right\}.$$

Since $x(j) \notin \Omega$ for $j \in [i + 1, N]$, we have that $\ell(x(j), u(j; x)) > d$ and $F(x(N)) > \alpha$. However, $\hat{x}(j) \in \Omega$ and $\hat{u}(j; x^+) = h(\hat{x}(j))$ for $j \in [i - 1, N]$ and, thus

$$\sum_{j=i-1}^{N-1} \ell(\hat{x}(j), h(\hat{x}(j))) + F(\hat{x}(N)) \leq F(\hat{x}(i - 1)) \leq \alpha.$$

Hence

$$V_N(x, \mathbf{u}) - V_N(x^+, \hat{\mathbf{u}}) > \left\{ (N - i - 1) \cdot d + \alpha \right\} - \left\{ \alpha \right\} \geq 0.$$

If $i = N$, then

$$\Delta V_N = \left\{ \ell(x, u(0; x)) + F(x(N)) \right\}$$
$$- \left\{ \ell(\hat{x}(N - 1), h(\hat{x}(N - 1))) + F(\hat{x}(N)) \right\}.$$

Since $\hat{x}(N - 1) = x(N) \in \Omega$, in virtue of Assumption 1, we have that

$$\Delta V_N = V_N(x, \mathbf{u}) - V_N(x^+, \hat{\mathbf{u}}) \geq \ell(x, u(0; x)) > 0$$

for all $x \neq 0$. Consequently, the first statement is proved.

If $x^+ \in \Omega$, then two candidate feasible solutions are considered: $\hat{\mathbf{u}}$ derived from (8) and $\hat{\mathbf{u}}_h$ given by

$$\hat{\mathbf{u}}_h = \{h(\hat{x}(0)), \ldots, h(\hat{x}(N - 1))\}. \quad (9)$$

It is clear that $V_N(x, \mathbf{u}) > V_N(x^+, \hat{\mathbf{u}})$. Moreover, in virtue of (4), $V_N(x^+, \hat{\mathbf{u}}_h) \leq F(x^+)$.

Then considering the candidate with the minimum cost, both conditions are simultaneously satisfied. Thus, the second statement is proved. ∎

Remark 3: From this lemma, a practical implementation of the suboptimal controller is derived. Assume that at $k = 0$, a feasible solution $\mathbf{u}(x_0)$ to $P_N(x_0, X)$ such that $V_N(x_0, \mathbf{u}(x_0)) \leq N \cdot d + \alpha$ is available. For $k > 1$, the following hold.

1) Compute the maximum i such that $\phi(i; x_{k-1}, \mathbf{u}(x_{k-1})) \in \Omega$.
2) Calculate

$$\hat{\mathbf{u}}(x_k) = \{ u(1; x_{k-1}), \ldots, u(i - 1; x_{k-1})$$
$$h(\hat{x}(i - 1)), \ldots, h(\hat{x}(N - 1))\}.$$

3) If $x_k \in \Omega$
 a) Calculate

$$\hat{\mathbf{u}}_h(x_k) = \{h(x_k), \ldots, h(\hat{x}(N - 1))\}$$

with $\hat{x}(i) = \phi(i; x_k, \hat{\mathbf{u}}_h(x_k))$.
 b) If $V_N(x_k, \hat{\mathbf{u}}_h(x_k)) < V_N(x_k, \hat{\mathbf{u}}(x_k))$, make $\hat{\mathbf{u}}(x_k) = \hat{\mathbf{u}}_h(x_k)$.
4) Try to obtain a feasible solution $\mathbf{u}(x_k)$ to $P_N(x_k, X)$ such that

$$V_N(x_k, \mathbf{u}(x_k)) < V_N(x_k, \hat{\mathbf{u}}(x_k)).$$

If this is not achieved, make $\mathbf{u}(x_k) = \hat{\mathbf{u}}(x_k)$.
5) Apply $u(0, x_k)$, make $k = k + 1$ and go to step 1)

In the following theorem, it is proved that the proposed suboptimal receding horizon controller guarantees asymptotic stability of the closed-loop system.

Theorem 4: Let $F(x)$ and $\Omega = \{x \in \mathbb{R}^n : F(x) \leq \alpha\}$ satisfy Assumption 1 and let d be a positive constant such that Assumption 2 holds; consider the proposed MPC controller derived from the suboptimal solution of $P_N(x, X)$. Then this controller stabilizes the system asymptotically for all feasible initial state such that $V_N(x_0, \mathbf{u}(x_0)) \leq N \cdot d + \alpha$.

Proof: Before the beginning of the proof some definitions are given [2]: a function $\gamma(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{K}$ function if it is continuous, strictly increasing and $\gamma(0) = 0$. The function $\gamma^{-1}(\cdot)$ denotes a function such that $\gamma^{-1}(\gamma(a)) = a$ for all $a \geq 0$. If $\gamma(\cdot)$ is a $\mathcal{K}$ function, then $\gamma^{-1}(\cdot)$ is also a $\mathcal{K}$ function.

First we prove that the suboptimal controller is well defined, that is, for any x_k and a feasible solution $\mathbf{u}(x_k)$ to $P_N(x_k, X)$ such that $V_N(x_k, \mathbf{u}(x_k)) \leq N \cdot d + \alpha$, there exists a feasible solution $\mathbf{u}(x_{k+1})$ associated to its successor state x_{k+1} such that $V_N(x_{k+1}, \mathbf{u}(x_{k+1})) \leq N \cdot d + \alpha$. This fact is derived from lemma 3: since $V_N(x_0, \mathbf{u}(x_0)) \leq N \cdot d + \alpha$, then by induction we have that $V_N(x_k, \mathbf{u}(x_k)) \leq N \cdot d + \alpha$ for all $k \geq 0$.

Now, we prove that the closed loop system is stable at the origin, that is, for any given $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that for all x_0 satisfying $\|x_0\| \leq \delta$, then $\|x_k\| \leq \epsilon$ for all $k \geq 0$.

For all $x \in \Omega$ we have that $\ell(x, 0) \leq V_N(x, \mathbf{u}(x)) \leq F(x)$ in virtue of lemma 3. Then, the suboptimal cost is a positive definite function locally bounded above by a (control) Lyapunov function. Consequently, there exists a couple of $\mathcal{K}$ functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ such that

$$\gamma_1(\|x\|) \leq \ell(x, 0) \leq V_N(x, \mathbf{u}(x)) \leq F(x) \leq \gamma_2(\|x\|).$$

Notice that the set $\Psi = \{x \in \mathbb{R}^n : V_N(x, \mathbf{u}(x)) \leq \gamma_1(\gamma_2^{-1}(\alpha))\}$ is contained in Ω, since for all $x \in \Psi$ we have that $\gamma_1(\|x\|) \leq V_N(x, \mathbf{u}(x)) \leq \gamma_1(\gamma_2^{-1}(\alpha))$ which yields $\|x\| \leq \gamma_2^{-1}(\alpha)$ and hence $F(x) \leq \gamma_2(\|x\|) \leq \alpha$.

Consider a given constant $\epsilon > 0$ such that $\epsilon \leq \gamma_2^{-1}(\alpha)$. Let us take $\delta = \gamma_2^{-1}(\gamma_1(\epsilon))$ then for all $\|x_0\| \leq \delta$, we have that

$$V_N(x_0, \mathbf{u}(x_0)) \leq F(x_0) \leq \gamma_2(\|x_0\|) \leq \gamma_1(\epsilon) \leq \gamma_1(\gamma_2^{-1}(\alpha))$$

and hence $x_0 \in \Psi \subseteq \Omega$. Therefore, from lemma 3, we have that $V_N(x_k, \mathbf{u}(x_k)) \leq V_N(x_0, \mathbf{u}(x_0)) \leq \gamma_1(\gamma_2^{-1}(\alpha))$ and the system evolution remains in Ω. Then

$$\gamma_1(\|x_k\|) \leq V_N(x_k, \mathbf{u}(x_k)) \leq V_N(x_0, \mathbf{u}(x_0)) \leq \gamma_1(\epsilon)$$

which leads to $\|x_k\| \leq \epsilon$ for all $k \geq 0$. Therefore, the origin is a stable steady state of the closed loop system.

Finally, we prove that the closed loop system asymptotically converges to the origin. Since $V_N(x_0, \mathbf{u}(x_0)) \leq N \cdot d + \alpha$, from lemma 3 we derive that the suboptimal cost $V_N(x_k, \mathbf{u}(x_k))$ is strictly decreasing along the system evolution. Since the suboptimal cost is a positive definite function, asymptotic convergence to the origin is inferred [2], which completes the proof. ∎

Notice that the potential domain of attraction of the suboptimal controller is the same that the one of the optimal controller and this can be enlarged by weighting the terminal cost.

Remark 4: In [8], the stability of predictive controllers under suboptimality is analyzed and it is proved that any feasible solution which provides a decreasing cost stabilizes the system. However, in [8], an additional condition for stability is required: for all x in a neighborhood of the origin, the suboptimal solution is such that $\|\mathbf{u}\| \leq \sigma(\|x\|)$, where $\sigma()$ is a $\mathcal{K}$ function. We show in this work that the condition $V_N(x, \mathbf{u}(x)) \leq F(x)$ for all $x \in \Omega$ is a practical and sufficient condition for stability.

V. CONCLUSION

In this note, a domain of attraction of the MPC without terminal constraint is characterized and asymptotic stability of the optimal and suboptimal controller is proved. Based on this, it is proved that by weighting the terminal cost, the domain of attraction is enlarged. Since this region depends explicitly on the ingredients of the MPC, practical methods for designing a stabilizing MPC controller without terminal constraint are presented. The new results presented are relevant from a practical point of view, since they allows us to obtain almost the same domain of attraction that the one of the general MPC. Moreover, a suboptimal procedure is proposed to guarantee asymptotic stability in case of suboptimality of the controller without terminal constraint.

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Robust Stability and Stabilization of Uncertain Discrete-Time Markovian Jump Linear Systems

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Abstract—This note deals with robust stability and control of uncertain discrete-time linear systems with Markovian jumping parameters. Systems with polytopic-type parameter uncertainty in either the state-space model matrices, or in the transition probability matrix of the Markov process, are considered. This note develops methods of robust stability analysis and robust stabilization in the mean square sense which are dependent on the system uncertainty. The design of both mode-dependent and mode-independent control laws is addressed. The proposed methods are given in terms of linear matrix inequalities. Numerical examples are provided to demonstrate the effectiveness of the derived results.

Index Terms—Discrete-time systems, Markovian jump linear systems, robust stability, robust stabilization.

I. INTRODUCTION

The study of linear systems with Markovian jumping parameters has been attracting an increasing attention over the past decade. This class of systems, referred to as Markov jump linear (MJL) systems, is very appropriate to model plants whose structure is subject to random abrupt changes due to, for instance, random component failures, abrupt environment disturbance, changes of the operating point of a linearized model of a nonlinear system, etc. A number of control problems related to discrete-time MJL systems has been analyzed by several authors; see, e.g., [1]–[6], [8]–[14], [16]–[21], and the references therein. In particular, with regard to uncertain discrete-time MJL systems, robust stability and control of systems with norm-bounded uncertainty has been studied in [3], [6], [12], and [20], whereas polytopic uncertain systems have been treated in, e.g., [4]. A common feature of the