Robot Dynamics
Fundamentals

vector

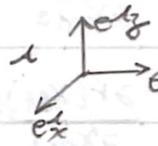
r / $\vec{r}$

$B \to P$ $\quad r_{BP}$

in $\mathcal{A}$ frame $\quad {}_{\mathcal{A}}r_{BP}$

orthonormal basis of R^3

$$(e_x^{\mathcal{A}}, e_y^{\mathcal{A}}, e_z^{\mathcal{A}}) := \text{orthonormal basis of } R^3$$



parameterization of ${}_{\mathcal{A}}r_{AB} = x\, e_x^{\mathcal{A}} + y e_y^{\mathcal{A}} + z e_z^{\mathcal{A}}$

Cartesian coordinates

$$\chi_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\triangle\ {}_{\mathcal{A}}r_{AB} = x e_x^{\mathcal{A}} + y e_y^{\mathcal{A}} + z e_z^{\mathcal{A}}$$

Cylindral coourdinates

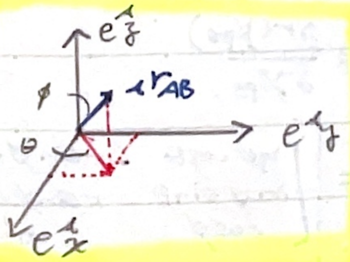
$$\chi_{Pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}$$

$$\triangle\ {}_{\mathcal{A}}r_{AB} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix}$$

Spherical Coordinates

$$\chi_{Ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$$

$$\triangle\ {}_{\mathcal{A}}r_{AB} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\phi\cos\theta \\ r\sin\phi\sin\theta \\ r\cos\phi \end{pmatrix}$$



ex 1-1.

$${}_{\mathcal{A}}r_{AP} = {}_{\mathcal{A}}r_{AB} + {}_{\mathcal{A}}r_{BP}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

get $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$: $\begin{cases} \chi_{Pc} = \\ \chi_{Pz} = \\ \chi_{Ps} = \end{cases}$

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$: $\begin{cases} \chi_{Pc} = \\ \chi_{Pz} = \\ \chi_{Ps} = \end{cases}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$: $\begin{cases} \chi_{Pc} = \\ \chi_{Pz} = \\ \chi_{Ps} = \end{cases}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad \chi_{Pc} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix} = \begin{pmatrix} r\sin\phi\cos\theta \\ r\sin\phi\sin\theta \\ r\cos\phi \end{pmatrix}$

$$\therefore \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\cos 0^\circ \\ 1\sin 0^\circ \\ z \end{pmatrix} = \begin{pmatrix} 1\sin 90^\circ\cos 0^\circ \\ 1\sin 90^\circ\sin 0^\circ \\ 1\cos 90^\circ \end{pmatrix}$$

$$\chi_{Pz} = \begin{pmatrix} 1 \\ 0^\circ \\ 0 \end{pmatrix}$$

$$\chi_{Ps} = \begin{pmatrix} 1 \\ 0^\circ \\ 90^\circ \end{pmatrix}$$

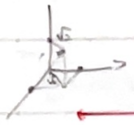
$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad \chi_{Pc} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix} = \begin{pmatrix} r\sin\phi\cos\theta \\ r\sin\phi\sin\theta \\ r\cos\phi \end{pmatrix}$

$$\therefore \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cos 90^\circ \\ 1\sin 90^\circ \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\sin 45^\circ\cos 90^\circ \\ \sqrt{2}\sin 45^\circ\sin 90^\circ \\ \sqrt{2}\cos 45^\circ \end{pmatrix}$$

$$\chi_{Pz} = \begin{pmatrix} 1 \\ 90^\circ \\ 1 \end{pmatrix}$$

$$\chi_{Ps} = \begin{pmatrix} \sqrt{2} \\ 90^\circ \\ 45^\circ \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \chi_{Pc} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\cos 45^\circ \\ \sqrt{2}\sin 45^\circ \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}\sin(\operatorname{atan}\sqrt{2})\cos 45^\circ \\ \sqrt{3}\sin(\operatorname{atan}\sqrt{3})\sin 45^\circ \\ \sqrt{3}\cos(\operatorname{atan}\sqrt{2}) \end{pmatrix}$



$$\chi_{Pz} = \begin{pmatrix} \sqrt{2} \\ 45^\circ \\ 1 \end{pmatrix}$$

$$\chi_{Ps} = \begin{pmatrix} \sqrt{3} \\ 45^\circ \\ \operatorname{atan}\sqrt{2} \end{pmatrix}$$

vector derivatives (linear velocity)

$r = r(\chi)$ $\quad$ vector being represented by specific parameterization

$$\dot{r} = \dot{r}(\chi)\dot{\chi}$$

$$\Rightarrow \dot{r} = \underbrace{\frac{\partial r}{\partial \chi}}_{E_P(\chi)} \dot{\chi}$$

$$\dot{r} = E_P(\chi)\dot{\chi} \qquad \dot{r} \in R^3,\ E_P(\chi) \in R^{3\times 3}$$

$$E_P(\chi)^{-1}\dot{r} = \dot{\chi}$$

$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ Cartesian coordinates

$$\chi_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore E_{Pc}(\chi_{Pc}) = E_{Pc}^{-1}(\chi_{Pc}) = I$$

Cylindral coordinates

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix}$$

matrix calculus

vector-by-vector

$$\frac{\partial y}{\partial x} \quad (x, y \in R^n)$$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \ddots & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$$

No.

Date

column picture !

$r(x_{pz}) = [\rho\cos\theta \ \ \rho\sin\theta \ \ z]$

$x_{pz} = [\rho \ \ \theta \ \ z]^T$

calculus revision

e.g. $\nabla f = \frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1} \ \ \frac{\partial f}{\partial x_2} \ \ \frac{\partial f}{\partial x_3}\right]$ (gradient)

$\mathbb{R}^3$

vector by scalar $y = [y_1, y_2 \cdots y_m]^T$, x

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} \\ \frac{\partial y_2}{\partial x} \\ \vdots \\ \frac{\partial y_m}{x} \end{bmatrix}$$

e.g. position $\downarrow$ velocity

scalar by vector y, $x = [x_1\, x_2 \cdots x_n]^T$

$$\frac{\partial y}{\partial x} = \left[\frac{\partial y}{\partial x_1}, \ \frac{\partial y}{\partial x_2} \ \cdots \ \frac{\partial y}{\partial x_n}\right]$$ e.g. gradient

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = \left(\frac{\partial f}{\partial x}\right)^T$$

vector by vector $y = [y_1, y_2 \cdots y_m]^T$
$x = [x_1, x_2 \cdots x_n]^T$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$$

Jacobian Matrix

$r = r(\chi)$

$$\dot{r} = \frac{\partial r}{\partial \chi}\dot{\chi} = E_p(\chi)\dot{\chi}$$

Cartesian coordinates, as mentioned.

$$\chi_{pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$E_{pc}(\chi_{pc}) = E_{pc}^{-1}(\chi_{pc}) = I$$

Cylindral coordinates, from matrix calculus

$$r = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ z \end{pmatrix}$$

$$\chi_{pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}$$

$$E_{pz}(\chi_{pz}) = \frac{\partial r(\chi_{pz})}{\partial \chi_{pz}}$$

$$= \begin{bmatrix} \frac{\partial r(\chi_{pz})_1}{\partial \chi_{pz1}} & \frac{\partial r(\chi_{pz})_1}{\partial \chi_{pz2}} & \frac{\partial r(\chi_{pz})_1}{\partial \chi_{pz3}} \\ \frac{\partial r(\chi_{pz})_2}{\partial \chi_{pz1}} & \frac{\partial r(\chi_{pz})_2}{\partial \chi_{pz2}} & \frac{\partial r(\chi_{pz})_2}{\partial \chi_{pz3}} \\ \frac{\partial r(\chi_{pz})_3}{\partial \chi_{pz1}} & \frac{\partial r(\chi_{pz})_3}{\partial \chi_{pz2}} & \frac{\partial r(\chi_{pz})_3}{\partial \chi_{pz3}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

spherical coordinates

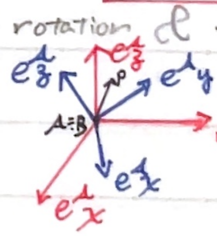
$$r = \begin{pmatrix} r\sin\phi\cos\theta \\ r\sin\phi\sin\theta \\ r\cos\phi \end{pmatrix}$$

$$\chi_{ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$$

$$E_{ps}(\chi_{ps}) = \frac{\partial r(\chi_{ps})}{\partial \chi_{ps}}$$

$$= \begin{bmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}$$

rotation & transformation



$${}_A r_{AP} = \begin{pmatrix} {}_A r_{AP_x} & e_x^A \\ {}_A r_{AP_y} & e_y^A \\ {}_A r_{AP_z} & e_z^A \end{pmatrix}$$

$${}_B r_{AP} = \begin{pmatrix} {}_B r_{AP_x} & e_x^B \\ {}_B r_{AP_y} & e_y^B \\ {}_B r_{AP_z} & e_z^B \end{pmatrix}$$

$${}_A r_{AP} = {}_A e_x^B \, {}_B r_{APx} + {}_A e_y^B \, {}_B r_{APy} + {}_A e_z^B \, {}_B r_{APz}$$

vector basis of B frame in A frame

vector in B frame

$$= [{}_A e_x^B \ \ {}_A e_y^B \ \ {}_A e_z^B] \, {}_B r_{AP}$$

$$= C_{AB} \, {}_B r_{AP}$$

$$\therefore {}_A r_{AP} = C_{AB} \, {}_B r_{AP}$$

what's this ? orthonormal group SO(3)

No.
Date

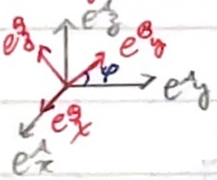
$_{\mathcal{A}}r_{AP} = C_{\mathcal{AB}} \; _{\mathcal{B}}r_{AP}$

$C_{\mathcal{BA}} = C_{\mathcal{AB}}^{-1} = C_{\mathcal{AB}}^{T}$ (SO(3))
matrix is orthogonal

Above shows passive rotation (different frame)
$\Updownarrow$
active (same frame)

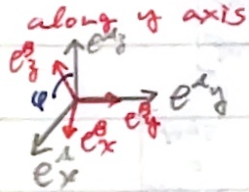
Elementary Rotation $= C_{\mathcal{AB}} = \left[{}_{\mathcal{A}}e_x^{\mathcal{B}} \; {}_{\mathcal{A}}e_y^{\mathcal{B}} \; {}_{\mathcal{A}}e_z^{\mathcal{B}} \right]$
↳ unit vector expressed in other frame ($\mathcal{B}$ in $\mathcal{A}$ frame)

along x axis



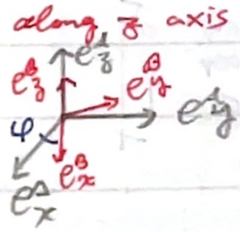
$$C_{\mathcal{AB}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

along y axis



$$C_{\mathcal{AB}} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$

along z axis



$$C_{\mathcal{AB}} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ Homogeneous Transformation = rotation + translation

$r_{AP} = r_{AB} + r_{BP}$

$\Rightarrow \; _{\mathcal{A}}r_{AP} = {}_{\mathcal{A}}r_{AB} + {}_{\mathcal{A}}r_{BP} = {}_{\mathcal{A}}r_{AB} + C_{\mathcal{AB}} \; _{\mathcal{B}}r_{BP}$

$$\Rightarrow \begin{pmatrix} {}_{\mathcal{A}}r_{AP} \\ 1 \end{pmatrix} = \underbrace{\begin{bmatrix} C_{\mathcal{AB}} & {}_{\mathcal{A}}r_{AB} \\ 0_{1\times3} & 1 \end{bmatrix}}_{T} \begin{pmatrix} {}_{\mathcal{B}}r_{BP} \\ 1 \end{pmatrix}$$

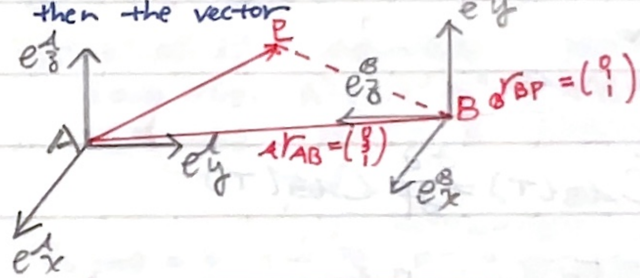
$\Updownarrow$

$$T^{-1} = \begin{bmatrix} C_{\mathcal{AB}}^T & -C_{\mathcal{AB}}^T \; {}_{\mathcal{A}}r_{AB} \\ 0_{1\times3} & 1 \end{bmatrix}$$

$-C_{\mathcal{AB}}^T \; {}_{\mathcal{A}}r_{AB}$
$= -C_{\mathcal{BA}} \; {}_{\mathcal{A}}r_{AB}$
$= -{}_{\mathcal{B}}r_{AB}$
$= {}_{\mathcal{B}}r_{BA}$

ex. 1-3

find ${}_{\mathcal{A}}r_{AP}$
find T
then the vector



rotate along x axis $90°$ ($\frac{\pi}{4}$)

$$C_{\mathcal{AB}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° \\ 0 & \sin 90° & \cos 90° \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° & 3 \\ 0 & \sin 90° & \cos 90° & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}_{\mathcal{A}}r_{AP} = T_{\mathcal{AB}} \; {}_{\mathcal{B}}r_{BP}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

${}_{\mathcal{A}}r_{AP} = [0, 2, 2]^T$

angular velocities

${}_{\mathcal{A}}\omega_{\mathcal{AB}} :=$ relative rotational velocity of $\mathcal{B}$ w.r.t. $\mathcal{A}$ in $\mathcal{A}$ frame

$\therefore \; {}_{\mathcal{A}}\omega_{\mathcal{AB}} = -{}_{\mathcal{A}}\omega_{\mathcal{BA}}$

given Rotation Matrix $C_{\mathcal{AB}}(t)$
angular velocity is

$${}_{\mathcal{A}}\omega_{\mathcal{AB}} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

where
skew-symmetric matrix

$$[{}_{\mathcal{A}}\omega_{\mathcal{AB}}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \dot{C}_{\mathcal{AB}} C_{\mathcal{AB}}^T$$

$${}_{\mathcal{B}}\omega_{\mathcal{AB}} = C_{\mathcal{BA}} \; {}_{\mathcal{A}}\omega_{\mathcal{AB}}$$

No.

Date

ex. 1-4

given $C_{AB}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & \sin(\alpha(t)) \\ 0 & -\sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix}$

determine

$_A\omega_{AB}$

$$\dot{C}_{AB}(t) = \frac{\partial}{\partial t} C_{AB}(t)$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin(\alpha(t))\dot{\alpha}(t) & \cos(\alpha(t))\dot{\alpha}(t) \\ 0 & -\cos(\alpha(t))\dot{\alpha}(t) & -\sin(\alpha(t))\dot{\alpha}(t) \end{bmatrix}$$

$$C_{AB}^T(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & -\sin(\alpha(t)) \\ 0 & \sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix}$$

$$\dot{C}_{AB}(t) \cdot C_{AB}^T(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dot{\alpha}(T) \\ 0 & -\dot{\alpha}(T) & 0 \end{bmatrix} \quad \therefore\ _A\omega_{AB} = \begin{pmatrix} -\dot{\alpha}(t) \\ 0 \\ 0 \end{pmatrix}$$

Parameterization of 3D rotation & Quaternion

now we try to parameterize rotation, similar to position (R^3)

Rotation Matrix $C_{AB} = [\,_A e_x^B \ \ _A e_y^B \ \ _A e_z^B\,]$
$R^{3\times3}$
Orthonormality

Parameterization

- Euler Angle
- Angle Axis
- Rotation Vector

} singularity issue

- Quaternions → non-singular

Euler Angle

Recall: Elementary Rotation

$$C_{AB} = C_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

$$= C_y(\varphi) = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$

$$= C_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

△ Euler Angle Rotation Order

ZYZ & ZXZ — proper Euler Angle

ZYX — Tait-Bryan Angle — UAV Airplanes

XYZ — Cardan Angle — nautic Apps.

$\boxed{ZYZ}$

$$C_{AD} = C_{AD}(\chi_{R,\,Euler\text{-}ZYZ})$$
$$= C_{AB}(z_1)\, C_{BC}(y)\, C_{CD}(z_2)$$

$\Rightarrow\ _A r = C_{AD}\ _D r$

$\Rightarrow C_{AD}$

$$= \begin{array}{ccc} C_z(\varphi) & C_y(\varphi) & C_z(\varphi) \\ \begin{bmatrix} \cos z_1 & -\sin z_1 & 0 \\ \sin z_1 & \cos z_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ \sin y & 0 & \cos y \end{bmatrix} & \begin{bmatrix} \cos z_2 & -\sin z_2 & 0 \\ \sin z_2 & \cos z_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} c_y c_{z_1} c_{z_2} - s_{z_1} s_{z_2} & -c_{z_2} s_{z_1} - c_y c_{z_1} s_{z_2} & c_{z_1} s_y \\ c_{z_1} s_{z_2} + c_y c_{z_2} c_{z_1} & c_{z_1} c_{z_2} - c_y s_{z_1} s_{z_2} & s_y s_{z_1} \\ -c_{z_2} s_y & s_y s_{z_2} & c_y \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \quad (\chi \to C)$$

get rotation from χ

No.

Date

$\chi_{R,\,Euler\,ZYZ} = \begin{pmatrix} z_1 \\ y \\ z_2 \end{pmatrix}$ get χ from rotation

$= \begin{pmatrix} \mathrm{atan2}(c_{23}, c_{13}) \\ \mathrm{atan2}(\sqrt{c_{13}^2+c_{23}^2},\ c_{33}) \\ \mathrm{atan2}(c_{32}, -c_{31}) \end{pmatrix} \quad (C \to \chi)$

$\boxed{ZYX}$

$$C_{AD} = C_{AD}(\chi_{R,\,Euler\,ZYX}) = C_{AB}(z)\,C_{AB}(y)\,C_{AB}(x)$$

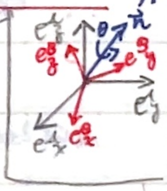
$$= \begin{bmatrix} \cos z & -\sin z & 0 \\ \sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{bmatrix}$$

$$= \begin{bmatrix} c_y c_z & c_z s_x s_y - c_x s_z & s_x s_z + c_x c_z s_y \\ c_y s_z & c_x c_z + s_x s_y s_z & c_x s_y s_z - c_z s_x \\ -s_y & c_y s_x & c_x c_y \end{bmatrix} \quad \chi \downarrow C$$

$$\chi_{R,\,Euler\,ZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} \mathrm{atan2}(c_{21}, c_{11}) \\ \mathrm{atan2}(-c_{31}, \sqrt{c_{32}^2+c_{33}^2}) \\ \mathrm{atan2}(c_{32}, c_{33}) \end{pmatrix} \quad C \downarrow \chi$$

Angle axis

$\chi_{R,\,Angle\,axis} = \begin{pmatrix} \theta \\ n \end{pmatrix}$ rotation angle, rotation axis $\hookrightarrow R^3$



Rotation vector $= \varphi = \theta \cdot n \ (R^3)$

a.k.a. Euler vectors

$$C_{AB}(\theta, n) = \cos\theta \cdot I_{3\times3} - \sin\theta\,[n]_\times + (1-\cos\theta)\,n n^T$$

$$\Rightarrow \begin{bmatrix} n_x^2(1-c_\theta)+c_\theta & n_x n_y(1-c_\theta)-n_z s_\theta & n_x n_z(1-c_\theta)+n_y s_\theta \\ n_x n_y(1-c_\theta)+n_z s_\theta & n_y^2(1-c_\theta)+c_\theta & n_y n_z(1-c_\theta)-n_x s_\theta \\ n_x n_z(1-c_\theta)-n_y s_\theta & n_y n_z(1-c_\theta)+n_x s_\theta & n_z^2(1-c_\theta)+c_\theta \end{bmatrix}$$

$(\chi \to C)$

$$\theta = \cos^{-1}\left(\frac{c_{11}+c_{22}+c_{33}-1}{2}\right)$$

$$n = \frac{1}{2\sin\theta}\begin{pmatrix} c_{32}-c_{23} \\ c_{13}-c_{31} \\ c_{21}-c_{12} \end{pmatrix}$$

singularity

Unit Quaternions

complex numbers in 4D $\xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k$

Hamiltonian convention $i^2 = j^2 = k^2 = ijk = -1$

$\chi_{R,\,quat} = \xi = \begin{pmatrix} \xi_0 \\ \check{\xi} \end{pmatrix} \in H$

Scalar — Real part $\xi_0 = \cos\frac{\|\varphi\|}{2} = \cos\left(\frac{\theta}{2}\right)$

vector — Imaginary $\check{\xi} = \sin\frac{\|\varphi\|}{2} \cdot \frac{\varphi}{\|\varphi\|} = \sin\left(\frac{\theta}{2}\right) n = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$

Unitary Constraint $\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$

Inverse $\xi\begin{pmatrix} \xi_0 \\ \check{\xi} \end{pmatrix} \leftrightarrow \xi^{-1} = \begin{pmatrix} \xi_0 \\ -\check{\xi} \end{pmatrix}$

Identity $\xi = (1\ 0\ 0\ 0)^T \quad (\theta = 0)$

Rotation Matrix

$$C_{AD} = I_{3\times3} + 2\xi_0[\check{\xi}]_\times + 2[\check{\xi}]_\times^2$$

$$= (2\xi_0^2 - 1) I_{3\times3} + 2\xi_0[\check{\xi}]_\times + 2\check{\xi}\check{\xi}^T$$

$$\begin{bmatrix} \xi_0^2+\xi_1^2-\xi_2^2-\xi_3^2 & 2\xi_1\xi_2-2\xi_0\xi_3 & 2\xi_0\xi_2+2\xi_1\xi_3 \\ 2\xi_0\xi_3+2\xi_1\xi_2 & \xi_0^2-\xi_1^2+\xi_2^2-\xi_3^2 & 2\xi_2\xi_3-2\xi_0\xi_1 \\ 2\xi_1\xi_3-2\xi_0\xi_2 & 2\xi_0\xi_1+2\xi_2\xi_3 & \xi_0^2-\xi_1^2-\xi_2^2+\xi_3^2 \end{bmatrix}$$

$\chi \to C$

$$\chi_{R,\,quat} = \xi_{AD} = \frac{1}{2}\begin{pmatrix} \sqrt{c_{11}+c_{22}+c_{33}+1} \\ \mathrm{sgn}(c_{32}-c_{23})\sqrt{c_{11}-c_{22}-c_{33}+1} \\ \mathrm{sgn}(c_{13}-c_{31})\sqrt{c_{22}-c_{33}-c_{11}+1} \\ \mathrm{sgn}(c_{21}-c_{12})\sqrt{c_{33}-c_{11}-c_{22}+1} \end{pmatrix}$$

$C \to \chi$

Quaternions Algebra

$$q \otimes p = (q_0 + q_1 i + q_2 j + q_3 k)(p_0 + p_1 i + p_2 j + p_3 k)$$

$$= q_0p_0 + q_0p_1 i + q_0p_2 j + q_0p_3 k$$
$$+ q_1p_0 i + q_1p_1 ii + q_1p_2 ij + q_1p_3 ik$$
$$+ q_2p_0 j + q_2p_1 ji + q_2p_2 jj + q_2p_3 jk$$
$$+ q_3p_0 k + q_3p_1 ki + q_3p_2 kj + q_3p_3 kk$$

$i^2 = j^2 = k^2 = ijk = -1$

$ij = -ji = -ijk^2 = k$

$jk = -kj = i$

$ki = -ik = j$

$$= q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3$$
$$+ (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\,i$$
$$+ (q_0p_2 - q_1p_3 + q_2p_0 + q_3p_1)\,j$$
$$+ (q_0p_3 + q_1p_2 - q_2p_1 + q_3p_0)\,k$$

$$= \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$= \begin{bmatrix} q_0 & -\check{q}^T \\ \check{q} & q_0 I + [\check{q}]_\times \end{bmatrix} p = M_l(q)\,p$$

$:= M_l(q)$

$$= \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{bmatrix}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

$:= M_r(p)$

$$= \begin{bmatrix} p_0 & -\check{p}^T \\ \check{p} & p_0 I - [\check{p}]_\times \end{bmatrix} q = M_r(p)\,q$$

$$(w_1 + \vec{v}_1)(w_2 + \vec{v}_2) = (w_1 w_2 - \vec{v}_1 \cdot \vec{v}_2,\ w_1\vec{v}_2 + w_2\vec{v}_1 + \vec{v}_1 \times \vec{v}_2)$$

using Quaternion to rotate a vector

$${}_B r = C_{BI}\ {}_I r$$

write ${}_I r$ into Quaternion (as the imaginary part of ξ)

$$p({}_I r) = \begin{pmatrix} 0 \\ {}_I r \end{pmatrix} \quad \to \text{unit quaternion}$$

$$\therefore \ p({}_B r) = \xi_{BI} \otimes p({}_I r) \otimes \xi_{BI}^T$$

$$\Updownarrow$$

$${}_B r = C_{BI}\ {}_I r$$

$$p({}_B r) = \xi \otimes p({}_I r) \otimes \xi^T = M_l(\xi) M_r(\xi^T) \begin{pmatrix} 0 \\ {}_I r \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 \\ {}_B r \end{pmatrix} = \begin{bmatrix} \xi_0 & -\check{\xi}^T \\ \check{\xi} & \xi_0 I + [\check{\xi}]_\times \end{bmatrix} \begin{bmatrix} \xi_0 & \check{\xi}^T \\ -\check{\xi} & \xi_0 I + [\check{\xi}]_\times \end{bmatrix} \begin{pmatrix} 0 \\ {}_I r \end{pmatrix}$$

$$= \begin{bmatrix} \xi_0^2 + |\check{\xi}|^2 = 1 & \cancel{\xi_0 \check{\xi}^T} - \cancel{\xi_0 \check{\xi}^T} - \check{\xi}^T [\check{\xi}]_\times \\ \cancel{\xi_0 \check{\xi}} - \cancel{\xi_0 \check{\xi}} - [\check{\xi}]_\times \check{\xi} \ (=0) & \check{\xi}\check{\xi}^T + \xi_0^2 I + 2\xi_0 [\check{\xi}]_\times + [\check{\xi}]_\times [\check{\xi}]_\times \end{bmatrix} \cdot \begin{pmatrix} 0 \\ {}_I r \end{pmatrix}$$

$$M_r(\xi) = \begin{bmatrix} \xi_0 & -\check{\xi}^T \\ \check{\xi} & \xi_0 I - [\check{\xi}]_\times \end{bmatrix}$$

$$[\check{\xi}^T]_\times = -[\check{\xi}]_\times \qquad \boxed{\xi^{-1} = \xi^T = \begin{pmatrix} \xi_0 \\ -\check{\xi} \end{pmatrix}}$$

$$-\check{\xi}^T [\check{\xi}]_\times = -\check{\xi}^T(-[\check{\xi}^T]_\times) = 0$$

$$[\check{\xi}]_\times [\check{\xi}]_\times = \check{\xi}\check{\xi}^T - |\check{\xi}|^2 I$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & (\xi_0^2 - |\check{\xi}|^2) I + 2\xi_0 [\check{\xi}]_\times + 2\check{\xi}\check{\xi}^T \end{bmatrix}$$

$$\begin{pmatrix} 0 \\ {}_B r \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (\xi_0^2 - |\check{\xi}|^2) I + \xi_0 [\check{\xi}]_\times + 2\check{\xi}\check{\xi}^T \end{bmatrix} \begin{pmatrix} 0 \\ {}_I r \end{pmatrix}$$

$$\therefore \ {}_B r = \underbrace{\left((\xi_0^2 - |\check{\xi}|^2) I + \xi_0 [\check{\xi}]_\times + 2\check{\xi}\check{\xi}^T \right)}_{C(\xi)} {}_I r$$

Time Derivatives

$\omega_{AD} \Leftrightarrow \dot{\chi}_{AD}$ $\quad \frac{\partial r}{\partial x}$

recall: $\dot{r} = E_P(x)\dot{\chi}$

Now: $${}_A \omega_{AB} = \underbrace{E_R(\chi_R)}_{\text{Find this}} \dot{\chi}_R$$

$$\boxed{{}_A \omega_{AB} = E_R(\chi_R)\dot{\chi}_R}$$

Take ZYX Euler angle for example $\begin{pmatrix} z \\ y \\ x \end{pmatrix}$

$${}_A \omega_{AD} = {}_A \omega_{AB} + {}_A \omega_{BC} + {}_A \omega_{CD}$$

$$= {}_A \omega_{AB} + C_{AB}\ {}_B \omega_{BC} + C_{AB} C_{BC}\ {}_C \omega_{CD}$$

$$= {}_A e_z^A \cdot \dot{z} + C_{AB}\ {}_B e_y^B \cdot \dot{y} + C_{AB} C_{BC}\ {}_C e_x^C \cdot \dot{x}$$

$$= \underbrace{\left[{}_A e_z^A \quad C_{AB}\ {}_B e_y^B \quad C_{AB} C_{BC}\ {}_C e_x^C \right]}_{E_R(\chi_R)} \begin{pmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{pmatrix}$$

$$C_{AB}\ {}_B e_y^B = \begin{bmatrix} c_z & -s_z & 0 \\ s_z & c_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -s_z \\ c_z \\ 0 \end{pmatrix}$$

$$C_{AB} C_{BC}\ {}_C e_x^C = \begin{bmatrix} c_z & -s_z & 0 \\ s_z & c_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_y & 0 & s_y \\ 0 & 1 & 0 \\ -s_y & 0 & c_y \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c_y c_z \\ c_y s_z \\ -s_y \end{pmatrix}$$

$$\therefore \ E_R(\chi_R) = \begin{bmatrix} 0 & -s_z & c_y c_z \\ 0 & c_z & c_y s_z \\ 1 & 0 & -s_y \end{bmatrix}$$

$${}_A \omega_{AB} = \begin{bmatrix} 0 & -s_z & c_y c_z \\ 0 & c_z & c_y s_z \\ 1 & 0 & -s_y \end{bmatrix} \dot{\chi}_R$$

$\det(E_R) = -\cos(y)$ (singularity)

Angle Axis $\begin{pmatrix} \theta \\ n \end{pmatrix}$

$$E_{R,\text{angle axis}} = \left[n \quad \sin\theta\, I_{3\times3} + (1-\cos\theta)[n]_\times \right]$$

$$E_{R,\text{angle axis}}^{-1} = \begin{bmatrix} n^T \\ -\frac{1}{2}\frac{\sin\theta}{1-\cos\theta}[n]_\times^2 - \frac{1}{2}[n]_\times \end{bmatrix}$$

Rotation Vector $(\varphi = \theta n)$

$$E_{R,\text{rotation vector}} = \left[I_{3\times3} + [\varphi]_\times \left(\frac{1-\cos\|\varphi\|}{\|\varphi\|^2} \right) + [\varphi]_\times^2 \left(\frac{\|\varphi\| - \sin\|\varphi\|}{\|\varphi\|^3} \right) \right]$$

$$E_{R,\text{rotation vector}}^{-1} = \left[I_{3\times3} - \frac{1}{2}[\varphi]_\times + [\varphi]_\times^2 \frac{1}{\|\varphi\|^2} \left(1 - \frac{\|\varphi\|}{2} \frac{\sin\|\varphi\|}{1-\cos\|\varphi\|} \right) \right]$$

Quaternion

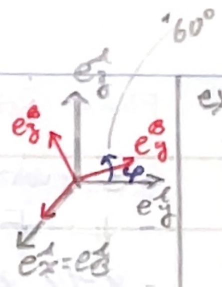
$$E_{R,\text{quat}} = 2H(\xi)$$

$$E_{R,\text{quat}}^{-1} = \frac{1}{2}H(\xi)^T \quad \text{w/ } H(\xi) = \left[-\check{\xi} \quad [\check{\xi}]_\times + \xi_0 I_{3\times3} \right] \in \mathbb{R}^{3\times4}$$

$$= \begin{bmatrix} -\xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ -\xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ -\xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix}$$

ex. 2-1

Find Rotation Matrix C_{AB}
Euler ZYX
Angle Axis
Quaternions



$$C_{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos60^\circ & -\sin60^\circ \\ 0 & \sin60^\circ & \cos60^\circ \end{bmatrix} \quad \begin{matrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} ※$$

Euler ZYX

$$\chi_{R,\,Euler\,ZYX} = \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} \operatorname{atan2}(C_{21}, C_{11}) \\ \operatorname{atan2}(-C_{31}, \sqrt{C_{32}^2 + C_{33}^2}) \\ \operatorname{atan2}(C_{32}, C_{33}) \end{pmatrix}$$

$$= \begin{pmatrix} \operatorname{atan2}(0, 1) \\ \operatorname{atan2}(-0, \sqrt{(\sqrt{3}/2)^2 + (1/2)^2}) \\ \operatorname{atan2}(\sqrt{3}/2, 1/2) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 60^\circ \end{pmatrix} ※$$

Angle Axis

$$\theta = \cos^{-1}\left(\frac{C_{11} + C_{22} + C_{33} - 1}{2}\right)$$

$$= \cos^{-1}\left(\frac{1 + \cos60^\circ + \cos60 - 1}{2}\right)$$

$$= \cos^{-1}(\cos60^\circ) = 60^\circ ※$$

$$n = \frac{1}{2\sin(\theta)}\begin{pmatrix} C_{32} - C_{23} \\ C_{13} - C_{31} \\ C_{21} - C_{12} \end{pmatrix} = \frac{1}{2\sin60^\circ}\begin{pmatrix} \sin60^\circ - (-\sin60^\circ) \\ 0 - 0 \\ 0 - 0 \end{pmatrix}$$

$$= \frac{1}{2\sin60^\circ}\begin{pmatrix} 2\sin60^\circ \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ※$$

Quaternions

$$\chi_{R,\,Quat} = \frac{1}{2}\begin{pmatrix} \sqrt{C_{11} + C_{22} + C_{33} + 1} \\ \operatorname{sgn}(C_{32} - C_{23})\sqrt{C_{11} - C_{22} - C_{33} + 1} \\ \operatorname{sgn}(C_{13} - C_{31})\sqrt{C_{22} - C_{33} - C_{11} + 1} \\ \operatorname{sgn}(C_{21} - C_{12})\sqrt{C_{33} - C_{11} - C_{22} + 1} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \sqrt{1 + \cos60^\circ + \cos60^\circ + 1} \\ +\sqrt{1 - \cos60^\circ - \cos60^\circ + 1} \\ 0 \cdot \sqrt{\cos60^\circ - \cos60^\circ - 1 + 1} \\ 0 \cdot \sqrt{\cos60^\circ - 1 - \cos60^\circ + 1} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \xi_{AB} ※$$

ex. 2-2

Based upon ex. 2-1 ${}_A r = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ in A frame
Rotate w/ quaternions to B frame
Rotate directly w/ complex numbers to B frame

$$P({}_B r) = \begin{pmatrix} 0 \\ {}_B r \end{pmatrix} = \xi_{BA} \otimes P({}_A r) \otimes \xi_{BA}^T$$

$$= M_L(\xi_{BA}) M_R(\xi_{BA}^T) \begin{pmatrix} 0 \\ {}_A r \end{pmatrix}$$

from previous : $\xi_{AB} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \xi_{BA} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \cos\frac{\theta}{2} \\ n\sin\frac{\theta}{2} \end{matrix}$$

$$M_L(\xi) = \begin{bmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 & 0 & 0 \\ -1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & -1 & \sqrt{3} \end{bmatrix}$$

$$\xi_{BA}^T = \xi_{BA}^{-1} = \begin{pmatrix} \xi_0 \\ -\xi \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$M_R(\xi) = \begin{bmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & \xi_3 & -\xi_2 \\ \xi_2 & -\xi_3 & \xi_0 & \xi_1 \\ \xi_3 & \xi_2 & -\xi_1 & \xi_0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 & 0 & 0 \\ 1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & -1 & \sqrt{3} \end{bmatrix}$$

$$P({}_B r) = \begin{pmatrix} 0 \\ {}_B r \end{pmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 & 0 & 0 \\ -1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & -1 & \sqrt{3} \end{bmatrix} \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 & 0 & 0 \\ 1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & -1 & \sqrt{3} \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \Big\} {}_B r$$

$${}_B r = \begin{pmatrix} 0 \\ 1/2 \\ -\sqrt{3}/2 \end{pmatrix} ※$$

$$P({}_B r) = \begin{pmatrix} 0 \\ {}_B r \end{pmatrix} = \xi_{BA} \otimes P({}_A r) \otimes \xi_{BA}^T \quad \left(\xi_{BA} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \\ 0 \\ 0 \end{pmatrix}\right)$$

$i^2 = j^2 = k^2 = ijk = -1$
$ij = -ji = -ijk^2 = k$
$jk = -kj = i$
$ki = -ik = j$

$$= \left(\frac{1}{2}(\sqrt{3} - i)\right)(j)\left(\frac{1}{2}(\sqrt{3} + i)\right)$$

$$= \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) j \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= \left(\frac{\sqrt{3}}{2}j - \frac{ij}{2}\right)\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= \frac{3}{4}j + \frac{\sqrt{3}}{4}ji - \frac{\sqrt{3}}{4}ij - \frac{iji}{4}$$

$$= \frac{3}{4}j - \frac{\sqrt{3}}{4}ij - \frac{\sqrt{3}}{4}ij - \frac{iji}{4}$$

$$= \frac{3}{4}j - \frac{\sqrt{3}}{2}ij - \frac{ki}{4}$$

$$= \frac{3}{4}j - \frac{\sqrt{3}}{2}ij - \frac{j}{4}$$

$$= \frac{1}{2}j - \frac{\sqrt{3}}{2}ij \quad \begin{pmatrix} 0 \\ 1/2 \\ -\sqrt{3}/2 \end{pmatrix} ※$$

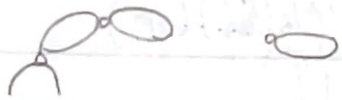
No.

Date

Kinematics

Fixed-base system
n_j joints (revolute, prismatic)
$n_l = n_j + 1$ links.
n_j moving links
1 fixed link

A Robot Arm example



n moving links: 6n parameters
n 1DoF joints: 5n parameters
$6n - 5n = n$ DoFs

$$q = \begin{pmatrix} q_1 \\ \vdots \\ q_{n_j} \end{pmatrix} \in \mathbb{R}^{n_j}$$ { complete, independent, not unique } generalized coordinates

End-effectors (task space)

$$x_e = \begin{pmatrix} x_{eP} \\ x_{eR} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$$ to I (inertial frame)

Operation space coordinates (the "real" DoF @ end-effectors)

$$x_o = \begin{pmatrix} x_{oP} \\ x_{oR} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{m_o} \end{pmatrix}$$ → DoF @ end-effectors

ex. 3-1

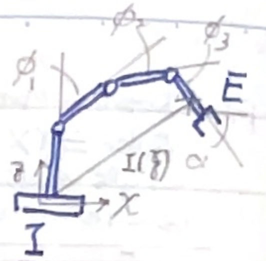
1 Most general robot arm
2 SCARA robot arm
3 ANYpulator (4 joints)
} get q, m_e, x_e, m_o, x_o

1
$q = (q_1\ q_2\ q_3\ q_4\ q_5\ q_6)$
$m_e = 6$
$x_e = (x, y, z, \alpha_x, \beta_y, \gamma_z)$
$m_o = 6$
$x_o = (x, y, z, \alpha_x, \beta_y, \gamma_z)$

2. $q = (\alpha, \beta, \gamma, \zeta)$
$m_e = 6$
$x_e = (x, y, z, \alpha_x, \beta_y, \gamma_z)$
$m_o = 4$
$x_o = (x, y, z, \gamma_z)$

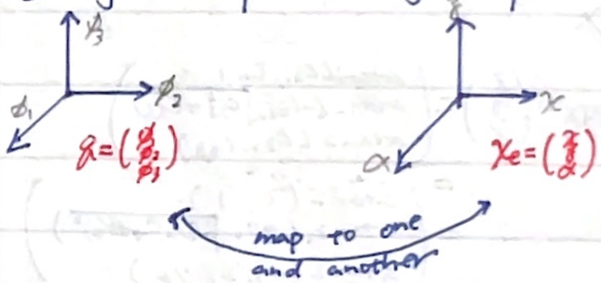
3 $q = (q_1\ q_2\ q_3\ q_4)$
$m_e = 6$
$x_e = (x, y, z, \alpha_x, \beta_y, \gamma_y)$
$m_o = 4$
x_o = hard to pick

Planar Robot Arm



- 3 revolute joints
- 1 end-effector
- $q = (q_1, q_2, q_3) \in \mathbb{R}^3$
- $m_e = 3$
- $x_e = (x, z, \alpha)$

configuration space ⟷ joint space (generalize)

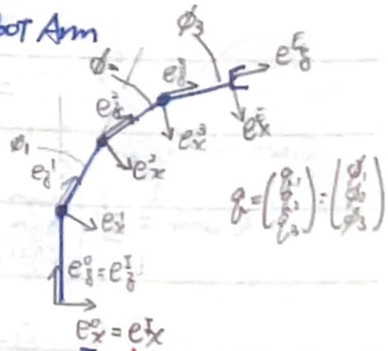


End-effector
$$x_e = x_e(q) \in \mathbb{R}^{n_e}$$

Transformation Matrix from E to I

$$T_{IE}(q) = T_{I0} \left(\prod_{k=1}^{n_j} T_{k-1\,k}(q_k) \right) T_{n_j E}$$
$$= \begin{bmatrix} C_{IE} & {}_I r_{IE} \\ 0_{1\times3} & 1 \end{bmatrix}$$

ex. 3-2
Get the T_{IE} of the Robot Arm RHS.



$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$

$T_{IE} = T_{I0} \cdot T_{01} \cdot T_{12} \cdot T_{23} \cdot T_{3E}$

$$T_{I0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

recall: rotate around y
$$C_y(\varphi) = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$

$$T_{01} = \begin{bmatrix} c\phi_1 & 0 & s\phi_1 & 0 \\ 0 & 1 & 0 & 0 \\ -s\phi_1 & 0 & c\phi_1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_{12} = \begin{bmatrix} c\phi_2 & 0 & s\phi_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s\phi_2 & 0 & c\phi_2 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} c\phi_3 & 0 & s\phi_3 & 0 \\ 0 & 1 & 0 & 0 \\ -s\phi_3 & 0 & c\phi_3 & l_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad T_{3E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No.

Date

$l_1 s\phi_1 + l_2 s(\phi_1+\phi_2) + l_3 s(\phi_1+\phi_2+\phi_3)$
$l_0 + l_1 c\phi_1 + l_2 c(\phi_1+\phi_2) + l_3 c(\phi_1+\phi_2+\phi_3)$

(cont'd)

$\therefore\ T_{IE} = T_{I0} \cdot T_{01} \cdot T_{12} \cdot T_{23} \cdot T_{3E}$

$$T_{IE} = \begin{bmatrix} c(\phi_1+\phi_2+\phi_3) & 0 & s(\phi_1+\phi_2+\phi_3) & x \\ 0 & 1 & 0 & y \\ -s(\phi_1+\phi_2+\phi_3) & 0 & c(\phi_1+\phi_2+\phi_3) & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x = l_1 s\phi_1 + l_2 s(\phi_1+\phi_2) + l_3 s(\phi_1+\phi_2+\phi_3)$
$y = 0$
$z = l_0 + l_1 c\phi_1 + l_2 c(\phi_1+\phi_2) + l_3 c(\phi_1+\phi_2+\phi_3)$

$$\therefore\ \chi_{eP}(q) = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} l_1 s\phi_1 + l_2 s(\phi_1+\phi_2) + l_3 s(\phi_1+\phi_2+\phi_3) \\ l_0 + l_1 c\phi_1 + l_2 s(\phi_1+\phi_2) + l_3 c(\phi_1+\phi_2+\phi_3) \end{pmatrix}$$

$\chi_{eR}(q) = (\phi_1+\phi_2+\phi_3)$

Jacobians of a Robot Manipulator

recall Forward Kinematics

$$T_{IE}(q) = \begin{bmatrix} C_{IE}(q) & {}_I r_{IE}(q) \\ 0_{1\times3} & 1 \end{bmatrix} \quad \chi_e = \begin{pmatrix} \chi_{eP} \\ \chi_{eR} \end{pmatrix} = \chi_e(q)$$

Forward Kinematics (Differential)

- Analytic

joint states

$$\chi_e + \delta\chi_e = \chi_e(q+\delta q) = \chi_e(q) + \frac{\partial \chi_e(q)}{\partial q}\delta q + \cancel{O(\delta q^2)}$$

$$\therefore\ \delta\chi_e \cong \frac{\partial \chi_e(q)}{\partial q}\delta q = J_{eA}(q)\,\delta q$$

recall Matrix Calculus

$$J_{eA} = \frac{\partial \chi_e}{\partial q} = \begin{bmatrix} \frac{\partial \chi_1}{\partial q_1} & \cdots & \frac{\partial \chi_1}{\partial q_{n_j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \chi_m}{\partial q_1} & \cdots & \frac{\partial \chi_m}{\partial q_{n_j}} \end{bmatrix}$$

$$\boxed{\dot{\chi}_e = \underbrace{J_{eA}(q)}_{R^{m_e \times n_j}}\dot{q}}$$

m_e # end-effector state
n_j # joints (generalized coords)

ex.3-3-1

From last example (ex.3-2) determine the analytic Jacobian

$$\chi_{eP}(q) = \begin{pmatrix} l_1 s\phi_1 + l_2 s(\phi_1+\phi_2) + l_3 s(\phi_1+\phi_2+\phi_3) \\ l_0 + l_1 c\phi_1 + l_2 c(\phi_1+\phi_2) + l_3 c(\phi_1+\phi_2+\phi_3) \end{pmatrix}$$

$\chi_{eR}(q) = \phi_1+\phi_2+\phi_3$

$$J_{eAP}(q) = \frac{\partial \chi_{eP}(q)}{\partial q}$$

(cont'd)

$$J_{eAP}(q) = \frac{\partial \chi_{eP}(q)}{\partial q} = \begin{bmatrix} \frac{\partial \chi_{eP}(q)_1}{\partial q_1} & \frac{\partial \chi_{eP}(q)_1}{\partial q_2} & \frac{\partial \chi_{eP}(q)_1}{\partial q_3} \\ \frac{\partial \chi_{eP}(q)_2}{\partial q_1} & \frac{\partial \chi_{eP}(q)_2}{\partial q_2} & \frac{\partial \chi_{eP}(q)_2}{\partial q_3} \end{bmatrix}$$

$$= \begin{bmatrix} l_1 c\phi_1 + l_2 c(\phi_1+\phi_2) + l_3 c(\phi_1+\phi_2+\phi_3) & l_2 c(\phi_1+\phi_2) + l_3 c(\phi_1+\phi_2+\phi_3) & l_3 c(\phi_1+\phi_2+\phi_3) \\ -l_1 s\phi_1 - l_2 s(\phi_1+\phi_2) - l_3 s(\phi_1+\phi_2+\phi_3) & -l_2 s(\phi_1+\phi_2) - l_3 s(\phi_1+\phi_2+\phi_3) & -l_3 s(\phi_1+\phi_2+\phi_3) \end{bmatrix}$$

$$J_{eAR}(q) = \frac{\partial \chi_{eR}(q)}{\partial q} = \begin{bmatrix} \frac{\partial \chi_{eR}(q)_1}{\partial q_1} & \frac{\partial \chi_{eR}(q)}{\partial q_2} & \frac{\partial \chi_{eR}(q)_1}{\partial q_3} \end{bmatrix}$$

$$= [1 \quad 1 \quad 1]$$

$$J_{eA} = \begin{bmatrix} l_1 c\phi_1 + l_2 c(\phi_1+\phi_2) + l_3 c(\phi_1+\phi_2+\phi_3) & l_2 c(\phi_1+\phi_2) + l_3 c(\phi_1+\phi_2+\phi_3) & l_3 c(\phi_1+\phi_2+\phi_3) \\ -l_1 s\phi_1 - l_2 s(\phi_1+\phi_2) - l_3 s(\phi_1+\phi_2+\phi_3) & -l_2 s(\phi_1+\phi_2) - l_3 s(\phi_1+\phi_2+\phi_3) & -l_3 s(\phi_1+\phi_2+\phi_3) \\ 1 & 1 & 1 \end{bmatrix}$$

- Analytic

$J_{eA}(q) = R^{m_e \times n_j}$ depend on parameterization

- Geometric

$$\boxed{w_e = \begin{pmatrix} v_e \\ \omega_e \end{pmatrix} = J_{e0}(q)\dot{q}}$$ map from generalize to linear & angular velocity

$J_{e0}(q) = R^{6 \times n_j}$ always 6!

(recall
$w_e = E_e(\chi_e)\dot{\chi}_e$)

$J_{e0}(q) = E_e(\chi)\,J_{eA}(q)$

$w_c = \begin{pmatrix} v_c \\ \omega_c \end{pmatrix} = w_B + w_{BC}$
$J_c\dot{q} = J_B\dot{q} + J_{BC}\dot{q}$
${}_A J_c = {}_A J_B + {}_A J_{BC}$

algebraic

Derivation of Geometric Jacobian

(recall
${}_A r_{AP} = {}_A r_{AB} + {}_A r_{BP} = {}_A r_{AB} + C_{AB} \cdot {}_B r_{BP}$)

Differentiate w/ time

$v_P = {}_A\dot{r}_{AP} = {}_A\dot{r}_{AB} + C_{AB}\,{}_B\dot{r}_{BP} + \dot{C}_{AB} \cdot {}_B r_{BP}$

(recall
$[{}_A\omega_{AB}]_\times = \dot{C}_{AB} C_{AB}^T$
$C_{BA} = C_{AB}^{-1} = C_{AB}^T$)

$\therefore\ \dot{C}_{AB} = [{}_A\omega_{AB}]_\times (C_{AB}^T)^{-1}$
$= [{}_A\omega_{AB}]_\times \cdot C_{AB}$

$\therefore\ {}_A\dot{r}_{AP} = {}_A\dot{r}_{AB} + C_{AB}\underbrace{{}_B\dot{r}_{BP}}_{=0} + [{}_A\omega_{AB}]_\times C_{AB}\,{}_B r_{BP}$

$= {}_A\dot{r}_{AB} + [{}_A\omega_{AB}]_\times {}_A r_{BP}$

$= {}_A\dot{r}_{AB} + {}_A\omega_{AB} \times {}_A r_{BP}$

$\therefore\ v_P = v_B + \Omega \times r_{BP}$ in A frame

(cont'd)

KOKUYO

No.

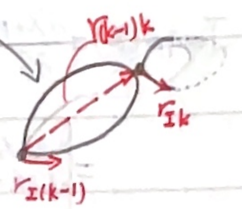
Date

(cont'd)

$v_P = v_B + \Omega \times r_{BP}$ (linear). from above

$\dot r_{Ik} = \dot r_{I(k-1)} + \omega_{I(k-1)} \times r_{(k-1)k}$

omega of that body



consecutively

$\dot r_{IE} = \sum_{k=1}^{n} \omega_{I,k} \times r_{k(k+1)}$ (linear)

as for angular

$\omega_{I(k)} = \omega_{I(k-1)} + \omega_{(k-1)(k)}$

$= \underbrace{n_k}_{\text{normal vector of the joint}} \dot q_k$ — velocity of the generalize coordinate

$\therefore \omega_{Ik} = \sum_{i=1}^{k} n_i \dot q_i$ (angular).

$\therefore \dot r_{IE} = \sum_{k=1}^{n} \omega_{Ik} \times r_{k(k+1)}$

$= \sum_{k=1}^{n} \left\{ \sum_{i=1}^{k} (n_i \dot q_i) \times r_{k(k+1)} \right\}$

$= n_1 \dot q_1 \times r_{12}$

1 $+ (n_1 \dot q_1 + n_2 \dot q_2) r_{23}$

2 $+ (n_1 \dot q_1 + n_2 \dot q_2 + n_3 \dot q_3) r_{34}$

3 $+ \cdots$

$+ (n_1 \dot q_1 + n_2 \dot q_2 + \cdots + n_n \dot q_n) r_{n(n+1)}$

$= n_1 \dot q_1 \times (r_{12} + r_{23} + \cdots + r_{n(n+1)})$ $r_{1(n+1)}$

$+ n_2 \dot q_2 \times (r_{23} + \cdots + r_{n(n+1)})$ $r_{2(n+1)}$

$+ \cdots$

$+ n_n \dot q_n \times (r_{n(n+1)})$ $r_{n(n+1)}$

$= \sum_{k=1}^{n} n_k \dot q_k \times r_{k(n+1)}$

$\therefore \boxed{\dot r_{IE} = \sum_{k=1}^{n} n_k \dot q_k r_{k(n+1)}}$

velocity of the end-effector → generalize coordinate → could map from q to ω!

$\dot r_{IE} = \underbrace{[n_1 \times r_{1(n+1)} \;\; n_2 \times r_{2(n+1)} \cdots n_n \times r_{n(n+1)}]}_{J_{eoP}} \begin{pmatrix} \dot q_1 \\ \dot q_2 \\ \vdots \\ \dot q_n \end{pmatrix}$

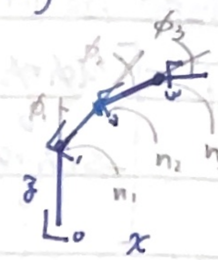
$\omega_{IE} = \sum_{i=1}^{n} n_i \dot q_i = \underbrace{[n_1 \;\; n_2 \cdots n_n]}_{J_{eoR}} \begin{pmatrix} \dot q_1 \\ \dot q_2 \\ \vdots \\ \dot q_n \end{pmatrix}$

$\boxed{{}_I J_{eo} = \begin{bmatrix} J_{eoP} \\ J_{eoR} \end{bmatrix} = \begin{bmatrix} {}_I n_1 \times {}_I r_{1(n+1)} & {}_I n_2 \times r_{2(n+1)} & \cdots & {}_I n_n \times {}_I r_{n(n+1)} \\ {}_I n_1 & {}_I n_2 & \cdots & {}_I n_n \end{bmatrix}}$

${}_I n_k = C_{I(k-1)} \cdot {}_{(k-1)} n_k$

ex. 3-3-2

Determine the Jacobian (geometric) RHS.



${}_I J_{eo} = \begin{bmatrix} {}_I n_1 \times {}_I r_{1(n+1)} & \cdots & {}_I n_n \times {}_I r_{n(n+1)} \\ {}_I n_1 & \cdots & {}_I n_n \end{bmatrix}$

${}_I J_{oP} \in R^{3\times3}$

${}_I J_{oR} \in R^{3\times3}$

$\left.\begin{aligned} {}_I n_1 &= {}_0 n_1 = {}_1 n_1 = {}_I e_y \\ {}_I n_2 &= C_{I1}\, {}_1 n_2 = {}_I e_y \\ {}_I n_3 &= C_{I2}\, {}_2 n_3 = {}_I e_y \end{aligned}\right\} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

${}_I r_{1E} = {}_I r_{12} + {}_I r_{23} + {}_I r_{3E}$

$= C_{I1} \cdot {}_1 r_{12} + C_{I2} \cdot {}_2 r_{23} + C_{I3} \cdot {}_3 r_{3E}$

$= l_1 \begin{pmatrix} s\phi_1 \\ 0 \\ c\phi_1 \end{pmatrix} + l_2 \begin{pmatrix} s(\phi_1+\phi_2) \\ 0 \\ c(\phi_1+\phi_2) \end{pmatrix} + l_3 \begin{pmatrix} s(\phi_1+\phi_2+\phi_3) \\ 0 \\ c(\phi_1+\phi_2+\phi_3) \end{pmatrix}$

${}_I r_{2E} = {}_I r_{23} + {}_I r_{3E}$

$= C_{I2} \cdot {}_2 r_{23} + C_{I3} \cdot {}_3 r_{3E}$

$= l_2 \begin{pmatrix} s(\phi_1+\phi_2) \\ 0 \\ c(\phi_1+\phi_2) \end{pmatrix} + l_3 \begin{pmatrix} s(\phi_1+\phi_2+\phi_3) \\ 0 \\ c(\phi_1+\phi_2+\phi_3) \end{pmatrix}$

${}_I r_{3E} = {}_I r_{3E}$

$= l_3 \begin{pmatrix} s(\phi_1+\phi_2+\phi_3) \\ 0 \\ c(\phi_1+\phi_2+\phi_3) \end{pmatrix}$

assume $l_1 = l_2 = l_3 = l$

$\therefore {}_I J_{eo} =$

$$\begin{bmatrix} lc\phi_1 + lc(\phi_1+\phi_2) + lc(\phi_1+\phi_2+\phi_3) & lc(\phi_1+\phi_2) + lc(\phi_1+\phi_2+\phi_3) & lc(\phi_1+\phi_2+\phi_3) \\ 0 & 0 & 0 \\ -ls\phi_1 - ls(\phi_1+\phi_2) - ls(\phi_1+\phi_2+\phi_3) & -ls(\phi_1+\phi_2) - ls(\phi_1+\phi_2+\phi_3) & -ls(\phi_1+\phi_2+\phi_3) \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

ex. 3-3-3

Given $r_P = \begin{pmatrix} l_1 c_1 \dot q_1 + l_1 c_{12}(\dot q_1 + \dot q_2) + l_1 c_{123}(\dot q_1 + \dot q_2 + \dot q_3) \\ -l_1 s_1 \dot q_1 - l_1 s_{12}(\dot q_1 + \dot q_2) - l_1 s_{123}(\dot q_1 + \dot q_2 + \dot q_3) \end{pmatrix}$

$\omega = \dot q_1 + \dot q_2 + \dot q_3$

determine Jacobian (geometric)

Campus

(cont'd)

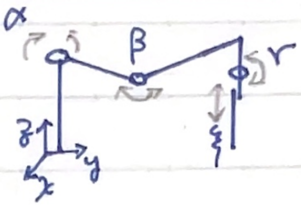
$$\mathcal{W}_e = \begin{pmatrix} v_e \\ \omega_e \end{pmatrix} = J_{eo}(q)\,\dot{q}$$

$$\begin{pmatrix} \dot{r}_p \\ \omega \end{pmatrix} = J_{eo}(q) \cdot \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix}$$

$$\therefore J_{eo}(q)$$

$$= \begin{bmatrix} l_1C_1 + l_1C_{12} + l_1C_{123} & l_1C_{12} + l_1C_{123} & l_1C_{123} \\ -l_1S_1 - l_1S_{12} - l_1S_{123} & -l_1S_{12} - l_1C_{123} & -l_1S_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

SCARA Robot Arm



generalized coordinate

$$\vec{p} = [\alpha\ \beta\ \gamma\ \xi]^T$$

geometric rotation Jacobian

$$\varphi = \overset{\text{offset}}{c} + \alpha + \beta + \gamma$$

$$\dot{\varphi} = \dot{\alpha} + \dot{\beta} + \dot{\gamma} \qquad J_{EOR} = [1\ 1\ 1\ 0]$$

geometric position Jacobian

$$\vec{x}_{EP} = \begin{pmatrix} \cos\alpha + \cos(\alpha+\beta) \\ \sin\alpha + \sin(\alpha+\beta) \\ c - \xi \end{pmatrix}$$

$$\dot{\vec{x}}_{EP} = \begin{pmatrix} -\dot{\alpha}\sin\alpha - (\dot{\alpha}+\dot{\beta})\sin(\alpha+\beta) \\ \dot{\alpha}\cos\alpha + (\dot{\alpha}+\dot{\beta})\cos(\alpha+\beta) \\ -\dot{\xi} \end{pmatrix}$$

$$\therefore J_{EOP} = \begin{bmatrix} -\sin(\alpha) - \sin(\alpha+\beta) & -\sin(\alpha+\beta) & 0 & 0 \\ \cos(\alpha) + \cos(\alpha+\beta) & \cos(\alpha+\beta) & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$