Atlas : mechanics + optimization
Anymal : RL-based
manipulation : Imitation-learning

△ underactuated

$$\begin{cases} \dot{x} = f(x,u) \\ y = g(x) \end{cases} \qquad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

△ second-order nonlinear system

$$\ddot{q} = f(q, \dot{q}, u)$$

or

$$\dot{x} = \tilde{f}(x,u) = \begin{bmatrix} \dot{q} \\ f(q,\dot{q},u) \end{bmatrix}$$

△ "control affine" nonlinear systems

$$\ddot{q} = f_1(q,\dot{q}) + f_2(q,\dot{q})u \quad (1)$$

based on this, define underactuated

def

(1) is fully actuated in $q, \dot{q}$ i.f.f. $f_2(q,\dot{q})$ is full row rank matrix

i.e. $\dim(q) = m$, $\dim(u) = n$, $f_2(q,\dot{q}) \in \mathbb{R}^{m \times n}$, $\text{rank}[f_2(q,\dot{q})] = m$

controllable : can choose some inputs over time to achieve some goal

actuable : can choose some inputs to cause acceleration

(1) is underactuated iff in $q, \dot{q}$: $\text{rank}[f_2(q,\dot{q})] < m$

$\forall q, \dot{q}$, $\text{rank}[f_2(q,\dot{q})] < m$

⇔ "system" is underactuated

△ Feedback equivalence (fully-actuated)

given

$$\ddot{q} = f_1(q,\dot{q}) + f_2(q,\dot{q})u$$

$\ddot{q}^d$

⇒ then $u = f_2^{-1}(q,\dot{q})[\ddot{q}^d - f_1(q,\dot{q})]$

⇒ $\ddot{q} = \ddot{q}^d$

give me an "acceleration", I give u an "u"

⇒ feedback equivalent to $\ddot{q} = u$ (double integrator which we kw optimal ctrl fun)

△ feedback equivalence be broken:

input saturations
state constraints
model uncertainty

△ Manipulator Eqs

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu \quad (*)$$

mass matrix · cori olis terms · gravity · torque input · actuator selection Matrix

$M \succ 0$

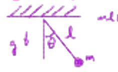
⇒ $\ddot{q} = M^{-1}(q)[-C(q,\dot{q})\dot{q} + \tau_g(q) + Bu]$

$\ddot{q} = f_1(q,\dot{q}) + f_2(q,\dot{q})u$ is in this form!

∴! whether $f_2(q,\dot{q})$ is full rank relies on "B"

as $\underbrace{M^{-1}(q)}_{\text{full rank}}\underbrace{B}_{\text{not sure!}}$

△ Dynamic Programming



$$ml\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$$

△ Control as an optimization

- given trajectory $x(\cdot), u(\cdot)$ ($\forall t$ $x(t)$ $t \in [0, T_{max}]$)
- Assign a score (scalar, reward, cost)
  e.g. time, avg distance
- subject to constraints

e.g. minimum time for double integrator

$\ddot{q} = u$

goal: drive to $q = \dot{q} = 0$ in minimum time from any initial condition

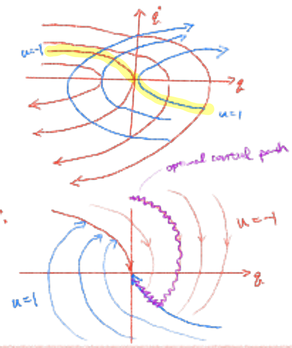
△ intuition: "bang-bang" policy (accelerate as much as possible) (unsmooth)

△ intuition #2:

$\ddot{q} = u$ $u = -1$ $\qquad u = 1$

$\dot{q}(t) = \dot{q}(0) - t$

$q(t) = q(0) + \dot{q}(0)t + \frac{1}{2}at^2$



Q: How to generalize?

△ Dynamic Programming

minimum-time ≅ shortest path problems

- weighted shortest path
- DP

discrete states $s_i \in S$
discrete actions $a_i \in A$
discrete time $s[n+1] = f(s[n], a[n])$
"edge cost" $g(s,a)$
total cost $\sum g(s,a)$

Key idea: Additive cost
e.g. time-to-goal $g(s,a) = \begin{cases} 1 & \forall s \neq s_{goal} \\ 0 & \text{o.w.} \end{cases}$

$\int g(s,a)\,dt$

△ DP

- DP is a recursive algorithm
- solve backwards from the goal
- cost-to-go

$$J^*(s_i) = \min_{a[\cdot]} \sum_{n=0}^{\infty} g(s[n], a[n])$$

watch for convergence

$s[0] = s_i$

hard to search

$$\Leftrightarrow J^*(s_i) = \min_a [g(s,a) + J^*(f(s,a))]$$

single

condition to certify optimality

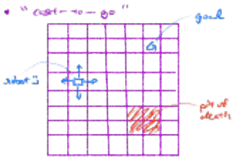
- Algorithm of DP (= value iteration)

$\hat{J}^*$ ← estimate of optimal cost-to-go

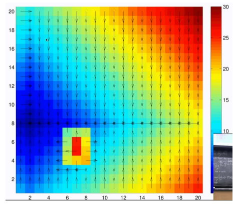
$\forall i$ $\hat{J}^*(s_i) \leftarrow \min_a [g(s,a) + \hat{J}^*(f(s,a))]$

$\hat{J}^* \longrightarrow J^* + c$

- "cost-to-go"



$$g(s,a) = \begin{cases} 0 & @ \text{ goal} \\ 10 & @ \text{ pit} \\ 1 & @ \text{ o.w.} \end{cases}$$



- Caveats
  - accuracy (discretization errors)
  - scalability (dimensions ~5) curse of the dimensionality
  - model is known
  - cost function
  - assume "full state" feedback

△ Lim DP ($\Delta x, \Delta u, \Delta t \to 0$)

- e.g. minimum-time double integrator

$\ddot{q} = u$, $|u| \le 1$

optimal policy is "bang-bang"

- we want "discrete" → "continuous"

△ optimal cost-to-go

- discrete

$s[n+1] = f(s[n], a[n])$

- cost

$\sum \ell(s[n], a[n])$

$\forall s$ $J^*(s) = \min_a [\ell(s,a) + J^*(f(s,a))]$ (cost-to-go function)

∴ $\pi^*(s) = \arg\min_a [\ell(s,a) + J^*(f(s,a))]$ (optimal controller/policy)

△ discrete v continuous

	discrete	continuous
states	s	x
actions	a	u
time	n	t

△ informal derivation

$x[n+1] \approx x[n] + h f_c(x[n], u[n])$

euler-integration, $t_{n+1} - t_n$

$\ell_d(x,u) = h\,\ell_c(x,u)$

∴ $J^*(x) = \min_u [h\ell_c(x,u) + J^*(x + hf_c(x,u))]$

$J^*(x) + \frac{\partial J^*}{\partial x} h f_c(x,u)$

$$\Rightarrow 0 = \min_u \left[\ell_c(x,u) + \frac{\partial J^*}{\partial x} f_c(x,u)\right]$$

Hamilton-Jacobi-Bellman Eq. HJB

e.g. infinite-time Grid world

$J^*(x) = c$

$\frac{dJ}{dt} = \frac{\partial J}{\partial x} f(x,u)$

for u^* cost-to-go

$0 = \ell(x,u^*) + \frac{\partial J}{\partial x}\frac{\partial x}{\partial t}$ ⇒ $\frac{\partial J}{\partial t} = -\ell(x,u^*)$ running cost

⇒ if you are following optimal policy your cost ↓

HJB sufficiency theorem

e.g. double integrator / Quadratic cost

$\ddot{q} = u$ $\quad \ell(x,u) = q^2 + \dot{q}^2 + u^2$ (no constraints)

claim:

$\pi^*(x) = -q - \sqrt{3}\dot{q}$

$J^*(x) = \sqrt{3}q^2 + 2q\dot{q} + \sqrt{3}\dot{q}^2$ $\left(x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}\right)$

proof:

(1) $0 = \min_u [q^2 + \dot{q}^2 + u^2 + \frac{\partial J}{\partial x} f(x,u)]$

$\left(\frac{\partial J}{\partial x} f(x,u) = \left[\frac{\partial J}{\partial q}\ \frac{\partial J}{\partial \dot{q}}\right]\begin{bmatrix} \dot{q} \\ u \end{bmatrix}\right)$

$= \min_u [q^2 + \dot{q}^2 + u^2 + \frac{\partial J}{\partial q}\dot{q} + \frac{\partial J}{\partial \dot{q}}u]$

$= \min_u [0]$

(2) $u = \arg\min_u [u^2 + \frac{\partial J}{\partial \dot{q}}u]$

⇒ $\frac{\partial}{\partial u}(\cdot) = 2u + \frac{\partial J}{\partial \dot{q}} = 0$

⇒ $u = -\frac{1}{2}\frac{\partial J}{\partial \dot{q}}$

$= -\frac{1}{2}(2q + 2\sqrt{3}\dot{q})$

$= -q - \sqrt{3}\dot{q}$

$J^* = x^T \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix} x$

△ LQR

- $\dot{x} = Ax + Bu$

$\ell(x,u) = x^TQx + u^TRu$

$Q \succeq 0$ $\quad R \succ 0$

- inf horizon

$J^*(x) = x^TSx$ (cost-to-go) (can capture the dynamics)

$S \succeq 0$

$\frac{\partial J}{\partial x} = 2x^TS$

$0 = \min_u [x^TQx + u^TRu + \frac{\partial J}{\partial x}(Ax + Bu)]$

⇒ $\frac{\partial [\cdot]}{\partial u} = 0$, $u^* = -R^{-1}B^TSx$

$= -Kx$

Riccati Equation

$0 = Q - SBR^{-1}B^TS + A^TS + SA$

- insights:

$-\frac{\partial J}{\partial x} = -2x^TS$ (where we want our cost-to-go func. to decrease)

$u = -R^{-1}B^TSx$ (gradient)

(underactuated B matrix to adjust Sx)

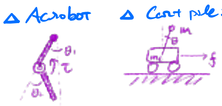
(weighting on B^TSx)

△ DP so far

- Tabular (discrete → graph search)
- Linear dynamics + convex objective (LQR)

↳ here, use acrobot, cartpole, grad/race to demonstrate DP

△ Acrobot △ Cart pole:



△ Equations of motion:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$$

- Acrobot	- Cart pole
$q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$	$q = \begin{bmatrix} x_{cart} \\ \theta_{pole} \end{bmatrix}$
$u = [\tau]$	$u = [f_{cart}]$
$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\lvert u \rvert \le u_{max}$	$\lvert u \rvert \le u_{max}$, $\lvert x \rvert \le x_{max}$

↳ target: swing up + balance

soln 1. tabular value iteration? yes, not elegant

2. LQR? Beautiful solution for balancing (need more for swing-up)

△ LQR for nonlinear systems

$\dot{x} = f(x,u) \Leftrightarrow \dot{x} = Ax + Bu$

⇒ choose nominal x_0, u_0 (want on fixed pt)

⇒ $\dot{x} \approx f(x_0,u_0) + \underbrace{\frac{\partial f}{\partial x}\Big|_{x=x_0, u=u_0}}_{A}(x - x_0) + \underbrace{\frac{\partial f}{\partial u}\Big|_{x=x_0, u=u_0}}_{B}(u - u_0)$

let

$\bar{x} = x - x_0$

$\dot{\bar{x}} = \dot{x} - \dot{x}_0$

$\bar{u} = u - u_0$

⇒ $\dot{\bar{x}} = A\bar{x} + B\bar{u}$

e.g. simple pendulum

- $ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = \tau$

$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ $\quad u = [\tau]$

- linearize about $\theta = \pi$, $\dot{\theta} = 0$, $u = 0$
- $\dot{x} = \begin{bmatrix} \dot{\theta} \\ \frac{1}{ml^2}[\tau - b\dot{\theta} - mgl\sin\theta] \end{bmatrix}$
- check fixed pt ✓

$\dot{x}_0 = \begin{bmatrix} 0 \\ \frac{1}{ml^2}[0 - b\cdot 0 - mgl\sin\pi] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- linearization

$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta & -\frac{b}{ml^2} \end{bmatrix}\Big|_{\theta=\pi} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix}$

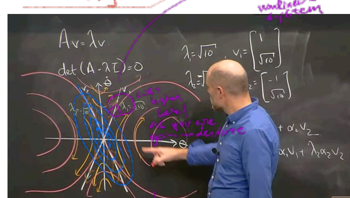
$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$

let $m = l = 1$ $\quad g = 10$

$A = \begin{bmatrix} 0 & 1 \\ 10 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

orange: approximate of a nonlinear system



red: nonlinear system

- LQR Design

$$\begin{cases} \dot{\bar{x}} = A\bar{x} + B\bar{u} \\ J = \int_0^\infty \bar{x}^TQ\bar{x} + \bar{u}^TR\bar{u}\,dt \\ [K,S] = lqr(A,B,Q,R) \\ u = -Kx = -R^{-1}B^TSx \\ J^* = \bar{x}^TS\bar{x} \end{cases}$$

⇒ stable

$\dot{x} = \underbrace{(A - BK)}_{\text{Hurwitz}}x$

△ Is (A,B) controllable?

underactuated but controllable

△ controllable: given $x(0)$ can I find $u(t)$ $\forall t \in [t_0, t_f]$ s.t. $x(t_f) = 0$

Is (A,B) stablizable

△ stablizable: given $x(0)$ can I find $u(t)$ $\forall t \in [t_0, \infty)$ s.t. $\lim_{t\to\infty} x(t) = 0$ (no control input)

△ Nonlinear Dynamics

$T = \frac{1}{2}ml^2\dot{\theta}^2$ (kinetic energy) $\quad v_{tan} = l\dot{\theta}$

(potential energy)

$U = -mgl\cos\theta$

Lagrangian mechanics

$L = K - U$

$K = \frac{1}{2}mv^2$
$= \frac{1}{2}ms^2$
$= \frac{1}{2}ml^2\dot{\theta}^2$

$\theta = \frac{s}{l}$

$U = mgy$
$= -mgl\cos\theta$

∴ $L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

$\frac{\partial L}{\partial \theta} = ml^2\ddot{\theta} + mgl\sin\theta = Q$ generalized force

$Q = \underbrace{-b\dot{\theta}}_{\text{friction}} + \underbrace{u}_{\text{torque}}$

∴ $ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$

⇕

$M(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_g(q) + Bu$

⇕

$\ddot{\theta} = f(\theta, \dot{\theta}, u)$

⇓

$\theta(t) = ?$

△ Nonlinear Dynamics Questions

- What is $\lim_{t\to\infty} \theta(t)$?
- Will my robot fall down?

$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$ (N·m)

$kg\frac{m^2}{s}\frac{1}{s}$ $\quad kg\,m^2\frac{1}{s^2}$ $\quad kg\frac{m^2}{s^2}$

case: $b\dot{\theta} \gg ml^2$

$b\sqrt{\frac{l}{g}} \gg ml^2$ (heavily damped region)

⇒ linear approximation!

⇒ $b\dot{\theta} = u - mgl\sin\theta$

$b\dot{x} = u - mgl\sin x$ $\quad x \in \mathbb{R}$

stable fixed point, fixed pt, u=0, region of attraction basin

△ Stability

- local stability (do not leave a ball)
  - in the sense of Lyapunov (i.s.L.)
    $\begin{cases} \epsilon, \delta, \text{ small positive constants} \\ \forall \epsilon, \exists \delta, \text{ s.t. } \lVert x(0) - x^* \rVert < \delta \\ \quad \Rightarrow \forall t \lVert x(t) - x^* \rVert < \epsilon \end{cases}$
  - locally attractive — get to the region
    $\left\{ \lim_{t\to\infty} x(t) \longrightarrow x^* \right.$
  - asymptotically stable — i.s.L + attractive
    { i.s.L + attractive
  - exponentially stable — get to the region w/ rate
    $\begin{cases} \forall t \\ \lVert x(t) - x^* \rVert < Ce^{-\alpha t} \\ C, \alpha > 0 \end{cases}$

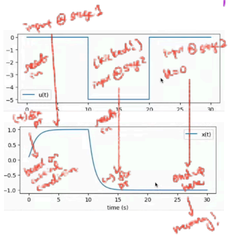
△ Simple RNN

u, w, $\tanh(\sum w_i u)$, Autapse

±b offset

$\dot{x} = -x + \tanh(wx + b)$

$\tanh(wx)$, $-x$, $\dot{x} = -x + \tanh(x)$



△ 2nd-order system

$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = u$

$\dot{x} = f(x,u)$

$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \frac{1}{ml^2}[u - b\dot{\theta} - mgl\sin\theta] \end{bmatrix}$

$b=0$, $u=0$

homoclinic, damped ones

✻ Essence of controls for underactuated robotics:

we kw all these dynamics

→ plug my u, s.t. phase portraits behave sth we want!!!

△ Lyapunov Analysis

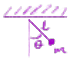
recall DP:
- Tabular — easy to explode
- LQR — only for linear case
- Approximate DP (NN) — works quite well (but solve a different DP, e.g. discounting)

↳ all are trying to get "cost-to-go" function $J^*(x)$

△ now Lyapunov (easy to find) ⟷ optimal value (hard to find)
good enough — very good

might replace the original optimal val.

△ Example: stability analysis of simple pendulum



$ml^2\ddot\theta + mgl\sin\theta = -b\dot\theta$

△ $\theta(t)$ ⟶ hard to solve, cannot be analysis

△ Lyapunov instead!!!

△ $E = K + U$

$= \frac{1}{2}ml^2\dot\theta^2 - mgl\cos\theta$

△ $\frac{d}{dt}E(x)$, $x = \begin{bmatrix}\theta\\ \dot\theta\end{bmatrix}$

$= \frac{\partial E}{\partial x}\frac{\partial x}{\partial t}$

$= \frac{\partial E}{\partial \theta}\dot\theta + \frac{\partial E}{\partial \dot\theta}\ddot\theta$

$= mgl\sin\theta\,\dot\theta + ml^2\dot\theta\ddot\theta$

$= \dot\theta(mgl\sin\theta + ml^2\ddot\theta)$

$= -b\dot\theta^2 \le 0$ if $b > 0$

△ General Energy function

△ given $\dot x = f(x)$ no u

→ want to prove stability of $x^* = 0$

→ construct a differentiable function $V(x)$, s.t.

$\begin{cases} V(0)=0, & V(x)>0,\ x\ne 0 \quad \text{PD} \\ \dot V(0)=0, & \dot V(x)\le 0,\ x\ne 0 \quad \text{NSD} \end{cases}$

sufficient condition

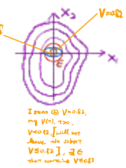
→ then, x^* is stable i.s.L.

△ i.s.L def

$\forall \epsilon > 0,\ \exists \delta > 0$

s.t. $\|x(0) - x^*\| < \delta$

$\forall t\ \|x(t) - x^*\| < \epsilon$



e.g. pendulum

$V(x) = E + mgl$

△ Asymptotically Stable

$\begin{cases} V(0)=0, & V(x)>0,\ x\ne 0 \quad \text{PD} \\ \dot V(0)=0, & \dot V(x) < 0,\ x\ne 0 \quad \text{NSD} \end{cases}$

"<" not "≤"

otherwise $\dot V(x) = 0$, → limit cycles.

△ Global Stability

Global Asymptotically stability (GAS)

$\begin{cases} V(0)=0, & V(x)>0,\ x\ne 0 \quad \text{PD} \\ \dot V(0)=0, & \dot V(x) < 0,\ x\ne 0 \quad \text{NSD} \end{cases}$

+

$\lim_{\|x\|\to\infty} V(x) = \infty$ "radially unbounded"

△ Regional Stability

$\begin{cases} V(0)=0, & V(x)>0,\ x\ne 0 \quad \text{PD} \\ \dot V(0)=0, & \dot V(x) < 0,\ x\ne 0 \quad \text{NSD} \end{cases}$

$\forall x \in D \subset R^n$

△ Exponential Stability

$\begin{cases} V(0)=0, & V(x)>0,\ x\ne 0 \quad \text{PD} \\ \dot V(0)=0, & \dot V(x)\le -\alpha V(x),\ x\ne 0 \quad \text{NSD} \end{cases}$

⇓

$V(x(t)) \le V(x(0))e^{-\alpha t}$

e.g. $\dot x = -x$

$V(x) = x^2$

$\dot V(x) = \frac{\partial V}{\partial x}\dot x$

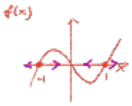
$= 2x(-x)$

$= -2x^2 < 0$ { i.s.L ✓, asymptotic ✓

$\le 2V(x)$ { exponential ✓

$\lim_{|x|\to\infty} V(x) = \infty$ { global stable

e.g. $\dot x = -x + x^3 = f(x)$



$x^* = 0$ is f.p.
$x \in (-1, 1)$ R.O.A.

$\begin{cases} V(x) = x^2 \\ \dot V(x) = 2x(-x+x^3) \end{cases}$

$= 2x^2(x^2-1)$

$= \begin{cases} 0 & x=0, \pm 1, -1 \\ > 0 & |x|>1 \\ < 0 & |x|<1, x\ne 0 \end{cases}$

• sublevel set of V → invariant set

$V(x) \le P$ (inside the R.O.A. of x^*)

△ General form of R.O.A.

if $V(x) > 0$, $\dot V(x) < 0$

$\forall x \in \{x \mid V(x) < P,\ P > 0\}$ invariant set

then $V(x(0)) < P$

⇔ $\lim_{t\to\infty} V \to 0$, $x \to 0$

and $\{x \mid V(x) < P\}$ is inside R.O.A.

△ LaSalle's Theorem

△ Lyapunov ⟶ Cost-to-go func.

HJB:

$0 = \min_u [\,l(x,u) + \frac{\partial J}{\partial x} f(x,u)\,]$

$u^* = \pi^*(x)$

⇒ $0 = l(x,u^*) + \frac{\partial J^*}{\partial x} f(x,u^*)$

$= l(x,u^*) + \frac{\partial J^*}{\partial t}$

⇒ $\dot J^*(x) = -l(x,u^*)$

"cost-to-go" function's derivative has to be decreasing!!!

⇕ relaxation

$\dot V(x) < 0$ ⟶ way more easy!!!

HJB discrete

$J^*(x) = \min_u [\,l(x,u) + J^*(f(x,u))\,]$

△ $J^*(x) = J^*(f(x,u)) - J^*(x)$

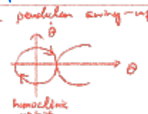
$= -l(x,u^*)$

⇕

△ $V(x) \le 0$

△ Lyapunov-based controller

e.g. pendulum swing-up



homoclinic orbit

• $E^d = mgl$

$V(x) = \frac{1}{2}(E(x) - E^d(x))^2$

$= \frac{1}{2}\tilde E(x)^2$

• $ml^2\ddot\theta + mgl\sin\theta = u$

$\dot E = u\dot\theta$

$\dot{\tilde E} = \dot E - \dot E^d = u\dot\theta$

• let $u = -k\dot\theta\tilde E$, $k > 0$

making

$\dot{\tilde E} = -k\dot\theta^2\tilde E$

△ Promise: find V(x) should be easier

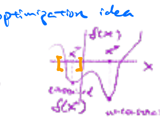
- Global Lyapunov analysis for the pendulum
- input: pendulum dynamics
  parametric family of poly/poly fn. Lyapunov (polynomial over 1, cosθ, sinθ, θ̇)
  output: coefficient of the polynomials
  + certificate that Lyapunov conditions satisfied ∀x ?

→ Lyapunov analysis w/ convex optimization

△ Some basic optimization idea

$\min_x f(x)$

s.t. $g_i(x) \le 0$



constrained $f(x)$ — unconstrained $f(x^*)$

• convex optimization

iff $f(x)$ is convex function

$\{g_i(x)\}$ is a convex set

LP → QP → SOCP → SDP

easy to solve ⟶ hard to solve

△ SDP

linear objective, linear constraints

PSD matrix constraints.

△ Q! How do we compute Lyapunov func?

1. Parameterize Lyapunov candidate

let's say

$\dot x = f(x)$ (linear NN)

$V(x) = \sum_i \alpha_i \phi_i(x) = \alpha^T\phi(x)$

nonlinear basis

2. Use convex optimization search over α to satisfy Lyapunov condition.

find α: $V(0) = 0$, $\forall x_i\ V(x_i) > \epsilon x_i^T x_i$

⇒ $V(x_i) > \epsilon x_i^T x$ — linear constraint in α

$\dot V(0) = 0$, $\forall x_i\ \dot V(x_i) = \frac{\partial V}{\partial x} f(x_i)$

$= \frac{\partial \alpha^T \phi}{\partial x} f(x_i)$

$< -\epsilon x_i^T x$ — linear constraint

How do we make sure for all x?

$\min_\alpha \sum_i |\dot V(x_i) + 1|$

△ $\dot x = Ax$ is it stable?

$\begin{cases} V(x) = x^T P x, & P \succ 0 \\ \dot V(x) = 2x^T P A x \\ \quad = x^T P A x + x A^T P x \end{cases}$ Lyapunov for Linear System

$\dot V(x) < 0$ if $(PA + A^T P) \prec 0$

find P s.t. $P = [\ldots]$

⇕ $P \succ 0$, $PA + A^T P \prec 0$

pick $Q \succ 0$ & $PA + A^T P = -Q$ (Lyapunov eq.)

find α, β: $P_1 \succ 0$, $P_2 \succ 0$

s.t. $PA + A^T P_2 - P_2$

$V(x) = \sum_i \alpha_i \phi(x)$, $\alpha_i \ge 0$

$V(x) = \sum \phi(x)^T G \phi(x)$, $G \succ 0$

$\dot V(x) = \frac{\partial V}{\partial x} f(x)$

→ construct $-\phi^T(x) G \phi(x)$

→ let $\dot V(x) = -\phi^T G \phi$

△ Robust stability analysis

$\dot x = Ax$, $A \in [A, A^+]$ uncertain

⇒ $A = \sum \beta_i A_i$, $\beta_i \ge 0$, $\beta_i = 1$

⇕

find $P \succ 0$ s.t. $\forall i$ $PA_i + A_i^T P < 0$

⇒ implies $P(\sum \beta_i A_i) + (\sum \beta_i A_i)^T P < 0$

△ DP sum

$\frac{d}{dt}x = f(x(t), u(t), x)\,dt$

• $J(x(t), u(t), T_0, t_f)$

$= Q(x(t_f), T_f) + \int_0^{t_f} l(x(\tau), u(\tau))\,d\tau$

• $V(x(T_0), T_0, T_f)$ (cost-to-go)

$= \min_{u(t)} J(x(t), u(t), T_0, T_f)$

• $-\frac{\partial V}{\partial t} = \min_{u(t)} \left(\left(\frac{\partial V}{\partial x}\right)^T f(x(t), u(t)) + l(x(t), u(t)) \right)$

⇔ $\min_{u(t)} \left(\left(\frac{\partial J^*}{\partial x}\right)^T f(x(t), u(t)) + l(x(t), u(t)) \right)$

Hamilton-Jacobi-Bellman (HJB)

• $V(x(T_0), T_0, T_f)$

$= V(x(T_0), T_0, t) + V(x(t), t, T_f)$

Bellman Optimality

⇔ $J^*(s_i) = \min_a [\,l(s,a) + J^*(f(s,a))\,]$

proof — total derivative

$\frac{d}{dt}V(x(t), t, T_f) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right)^T \frac{\partial x}{\partial t}$

$= \frac{d}{dt} \min_{u(t)} J(x(t), u(t), t, T_f)$

$= \frac{d}{dt} \min_{u(t)} Q(x(T_f), T_f) + \int_t^{T_f} l(x(\tau), u(\tau))\,d\tau$

$= \min_{u(t)} \frac{d}{dt} \left[Q(x(T_f), T_f) + \int_t^{T_f} l(x(\tau), u(\tau))\,d\tau \right]$

$= \min_{u(t)}\ 0 + \frac{d}{dt}\left([\mathcal{L}(x(\tau), u(\tau))]_t^{T_f} \right)$

$= \min_{u(t)} \frac{d}{dt}\left(\mathcal{L}(x(T_f), u(T_f)) - \mathcal{L}(x(t), u(t)) \right)$

$= \min_{u(t)} -l(x(t), u(t))$

⇕

$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial x}\right)^T \frac{\partial x}{\partial t} = \min_{u(t)} -l(x(t), u(t))$

⇕

$-\frac{\partial V}{\partial t} = \min_{u(t)} \left(\left(\frac{\partial V}{\partial x}\right)^T \frac{\partial x}{\partial t} + l(x,u) \right)$

$= \min_{u(t)} \left(\frac{\partial V^T}{\partial x} f(x,u) + l(x,u) \right)$

★ Get V ⟶ Get u
- two point bounded value problem
- shooting method

△ Discrete-Time HJB

• $x_{k+1} = F(x_k, u_k)$

• $J(x_0, \{u_k\}_{k=0}^n, n)$

$= Q(x_n, t_n) + \sum_{k=0}^n l(x_k, u_k)$

• $V(x_0, n)$

$= \min_{\{u_k\}_{k=0}^n} J(x_0, \{u_k\}_{k=0}^n, n)$

• $V(x_0, n)$

$= V(x_0, k) + V(x_k, n-k)\ \forall k \in (0, n$

• $V(x_k, n)$

$= \min_{u_k} \left[l(x_k, u_k) + V(\underbrace{x_{k+1}}_{x_{k+1} = F(x_k, u_k)}, n-1) \right]$

⇔ vs $J(s) = \min_a [\,l(s,a) + J^*(f(s,a))\,]$

Discrete HJB

⇕

$V(x_k, n)$

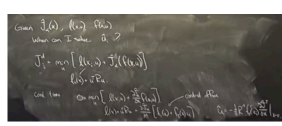
$= \min_{u_k} \left(l(x_k, u_k) + V(F(x_k, u_k), n-1) \right)$

⇕

$V(x) = \min_u \left(l(x,u) + V(F(x,u)) \right)$

↓

$\pi(x) = \arg\min_u \left(l(x,u) + V(F(x,u)) \right)$



- use optimal u: recall CT LQR: $u^* = -K^* x = -R^{-1}B^T S^* x$; DT LQR: $u^* = -K^* x = -(R + B^T S B)^{-1} B^T S A x$
- instead of sampling u_i, only sample $x_i \in X_s \subset X$

⇒ $J_i^d = x_i^T Q x_i + \hat u_i^T R \hat u_i + \hat J_\alpha(Ax_i + Bu_i)$

w/ optimal u w/ current $\hat S$

△ DP (cont'd)

key idea

1) LQR (linear optimal control)
2) local linearization
3) $Q = \begin{bmatrix} 10 & & & \\ & 10 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$ $R = [1]$

@ balancing

Q: How do we swing up?

"value iteration"

⇒ Approximate DP

△ Value iteration on a mesh for Pendulum Swing-up

△ mesh q 51 bins — odd for "0"

$\dot q$ 51 bins

u 9 bins

△ Acrobot

51^4 bins
9 bins ↗ explode!!!

△ NN! Approximating Value Functions

$\hat J_\alpha(x)$

└ parameters + bias

- value iteration update
  - recall:

$\forall s_i \in S$

$\hat J(s_i) = \min_a [\,l(s_i, a) + \hat J(f(s_i, a))\,]$

$s[n+1] = f(s[n], a[n])$

$\sum_{n=0}^{\infty} l(s[n], a[n])$

- Draw samples

$x_i \in X_s \subset X$

$u_j \in U_s \subset U$

$\forall i,j$ $\dot x_{ij} = f(x_i, u_j)$ next state samples

$l_{ij} = l(x_i, u_j)$

desired J @ i — $J_i^d = \min_j [\,l_{ij} + \hat J_\alpha(x'_{ij})\,]$

⇒ $\min_\alpha$ loss $\sum_i [\,\hat J_\alpha(x_i) - J_i^d\,]^2$

⇕

$\min_\alpha$ loss $\sum_i \left[\hat J_\alpha(x_i) - \min_j [\,l_{ij} + \hat J_\beta(x'_{ij})\,] \right]^2$

"target network"

"here we do the training based on the samples. To hv a better sampling, RL kicks in, & guide the state space, to a subset, that is closer to the optima."

Q: when does it work? does it converge

↓ △ linear approximator

△ works @ special case: linear NN

• $\hat J_\alpha(x) = \alpha^T \phi(x)$

nonlinear basis "input features" (fixed)

$= \sum_j \alpha_j \phi_j(x)$

⇒ $\min_\alpha \sum_i [\,\alpha^T\phi(x) - J_i^d\,]^2$

linear least squares has closed-form soln

⇒ SGD is guaranteed to converge to global minima

△ Back to LQR; connection to approximator; w/ discrete LQR

$J^*(x) = x^T \underbrace{S^*}_{NN} x$ ⟶ linear approximator

loss $\sum_i [\,x_i^T \hat S x_i - J_i^d\,]^2$

$\hat S \leftarrow \hat S - \eta \frac{\partial loss}{\partial S}$

Based on this, loss explodes↑

why? LQR tries to solve inf horizon, cost cannot go to inf.

How does normal LQR avoid inf?

$x \to 0$, $u \to 0$ (exact 0)

How about here, when we do sampling? popularly seen in RL

- discounting $\sum_{n=0}^{\infty} \gamma^n l(x[n], u[n])$, $0 \le \gamma \le 1$

optimal: LQR$(\sqrt{\gamma}A, B, Q, \frac{1}{\gamma}R)$

pretend that is more stable

$x[n+1] = Ax[n] + Bu[n]$

$|\text{eig}(A)| \le 1$.

$|\nabla f A| \le |A|$

△ Let's continue on Lyapunov

Recall:

$V_\alpha(x) > 0$

$\dot{V}_\alpha(x) < 0$

- Linear neural network
  $\forall x_i \longrightarrow$ Linear Program
- Question: from samples
  $\forall x_i \longrightarrow \forall x \in \mathbb{R}^n$

e.g. $\dot{x} = Ax$

$$\begin{cases} V(x) = x^T P x & V(x) > 0 \quad P \succ 0 \\ \dot{V}(x) = x^T P A x + x^T A^T P x < 0 & (PA + A^T P) \prec 0 \end{cases}$$

SDP:

find $P \succeq 0$ $P_2 \succeq 0$ — Linear Matrix Inequality

$PA + A^T P = -P_2$ (linear constraints)

△ Sums of squares

- given a function $f(x)$, is $f(x) > 0$ $\forall x$?

let $f(x) = \phi^T(x) P \phi(x)$ $P \succeq 0$

$= \sum \alpha_i \phi_i^2(x)$ $\alpha_i \ge 0$

sums of squares (SOS)
decomposition of the problem

then we can know $f(x) > 0$ $\forall x$

- special case Polynomials

$P_\alpha(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots$

$= \underbrace{\phi^T(x)}_{\text{basis functions}} P \phi(x)$ $P \succ 0$

e.g. $P_\alpha(x) = \alpha^T \begin{bmatrix} 1 \\ x^2 \end{bmatrix}$ ← gets

$\therefore \phi(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}$ → expand $\phi^T P \phi$

simplest basis

$m(x) \equiv$ monomial basis
eg $1, x, x^2, x^4, \dots$
$1, x, y, xy, x^2, y^2$

e.g. $2 - 4x + 5x^2$

$$= \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}$$

$\Updownarrow$

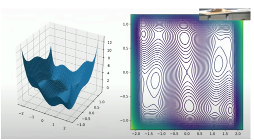
$$\begin{cases} \text{find } P \succeq 0 \\ \text{s.t. } P_{11} = 2 \quad 2P_{12} = -4 \quad P_{22} = 5 \end{cases}$$

SDP again!
- linear objective
- linear constraints
- PSD constraints

SOS $P_\alpha(x)$ is SOS
→ formulate SDP & solve
$\forall x \in \mathbb{R}^n$ $P_\alpha(x) \ge 0$

△ Six-hump camel

$p(x, y) = 4x^2 + xy - 4y^2 - 2.1x^4 + 4y^4 + \frac{x^6}{3}$



SOS kick in!

$\min_\gamma -\gamma$ s.t. $p(x,y) - \gamma$ is SOS

γ & P_2

?

$f(x) = NN^T(x) NN(x)$

△ Back to Lyapunov

- Choose $V(x)$ to be polynomial
- assume $\dot{x} = f(x)$ restricted to polynomial

$\Rightarrow \dot{V}(x) = \frac{\partial V}{\partial x} f(x)$ is still polynomial

$$\begin{cases} V(x) > 0 \longrightarrow V(x) \text{ is SOS} \\ \dot{V}(x) < 0 \longrightarrow -\dot{V}(x) \text{ is SOS} \end{cases}$$

New criterions!

e.g. $\dot{x}_1 = -x_1 - 2x_2^2$
$\dot{x}_2 = -x_2 - x_1 x_2 - 2x_2^3$

$V(x) = x_1^2 + 2x_2^2$

$-\dot{V}(x)$ is S.O.S? $\forall x$

→ $-\dot{V}(x) = m^T(x) P_\alpha m(x)$ — decision variable

inversely: find Lyapunov function

$V(x) = (S_{2,2} + 2x_0x_1S_{1,0} + 2x_0S_{2,1} + x_0^2S_{1,1} + 2x_1S_{2,0} + x_1^2S_{0,0})$

Solution:

$V(x) = (1.000000x_0^2 + 2.000057x_1^2)$

△ R.O.A again

so basically:

- oracle answers T/F $\forall x \in \mathbb{R}^n$
  ↓
  $p(x) \ge 0$

- New Q: $\forall x \in D \subseteq \mathbb{R}^n$ is $p(x) \ge 0$
  How:

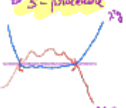
define $D = \{x \mid g(x) \le 0\}$ — polynomial

$$\begin{cases} \text{find}_\alpha \text{ s.t. } p(x) + \lambda_\alpha^T(x) g(x) \text{ is SOS} \\ \lambda_\alpha(x) \text{ is SOS} \end{cases}$$

adding potential negative

→ S-procedure

multiplier multiplier



λg

$p(x)$

- if $p(x) - \lambda g$ is SOS
  $p(x)$ is def. SOS
- if $g(x) \ge 0$, outside the region, $p(x) - \lambda g$ ↑ positive

$g(x) \le 0 \Rightarrow p(x) \ge 0$

△ Trajectory Optimization

SO FAR:
- Dynamic Programming / optimal control
  - tabular (mesh-based)
  - LQR
  - NN (function approximator) (need sampling distributions)
- Lyapunov (simpler DP)
  - sums of squares opt. (10~20 dim) (more w/ SOS approximation)

for higher dimension?

$\forall x \longrightarrow$ NO

just

$x[0] = x_0$ (1 initial condition)

→ motivation

△ Traj opt w/ linear discrete

Choose finite horizon N

$$\min_{\substack{x[\cdot], u[\cdot] \\ 0 \dots N \quad 0 \dots N-1}} \sum_{n=0}^{N-1} \ell(x[n], u[n]) + \underbrace{\ell_f(x[N])}_{\text{terminal cost}}$$

s.t.

$$\forall n \in [0, N-1] \begin{cases} x[n+1] = Ax[n] + Bu[n] \\ x[0] = x_0 \\ -1 \le u[n] \le 1 \end{cases}$$

Choose

$\ell(x, u) = x^T Q x + u^T R u$
$Q \succeq 0$, $R \succeq 0$

→ QP (L-MPC)

but why? this is LQR!

- LQR v QP
  - inf horizon — finite horizon
  - no constraints — can add constraints!

- horizon N ⟶ limiting? affecting sol?

idea: choose $\ell_f(x) = x^T S x$ — LQR

$$\underbrace{\sum_{n=0}^{N-1} \ell(x, u)}_{\text{constraints imposed on here}} + \underbrace{\ell_f(x)}_{N \sim \inf}$$

- can restrict your state!

$$\min_{x[\cdot], u[\cdot]} \sum_{n=0}^{N-1} |u[n]|^2$$

s.t. $x[0] = x_0$ $x[N] = x_f$
$x[n+1] = Ax[n] + Bu[n]$ $\forall n$
$|u[n]| \le 1$

- min time?
  $\ddot{q} = u$, $|u| \le 1$

set N big to be feasible,
then do $N-1$ recursively

△ Categories

- Direct transcription
$$\begin{cases} \text{decision variables} & x[\cdot],\ u[\cdot] \\ \text{constraints} & x[n+1] = Ax[n] + Bu[n] \end{cases}$$
- Direct "shooting"
$$\begin{cases} \text{decision variables} & x[0],\ u[\cdot] \\ \text{constraints (linear)} & x[n] = A^n x_0 + \sum_{k=0}^{n-1} A^{n-1-k} B u[k] \end{cases}$$
linear in $u[k]$

△ MPC

- estimate current state $\hat{x}$
- solve linear traj opt w/ $x[0] = \hat{x}$
- execute $u[0]$
- repeat — "Recursive Feasibility"

★ when ever we are using it in real, we are actually looping, (w/ predefined traj)
$x_f \longrightarrow 0$, where we are optimizing u w.r.t. error state.

△ Nonlinear Traj. Opt.

$x[n+1] = f(x[n], u[n])$ (possibly non-convex)

$\Rightarrow$ transcript / direct } could work: NOT CONVEX

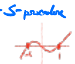
SQP

SQP (second-order method):
1. take big steps (convergence rate)
2. nature to handle constraints

△ R.O.A. using SOS + S-procedure

e.g. $\dot{x} = -x + x^3$

$V(x) = x^2$



we know R.O.A is $\in [-1, 1]$

$\therefore V(x) \le 1$

$\Rightarrow \text{find}_\alpha$ s.t. $-\dot{V}(x) + \lambda_\alpha(x)(1 - V(x))$ is SOS
& $\lambda_\alpha(x)$ is SOS

↑

$D = \{x \mid g(x) = 0\}$

find_α $p(x) + \lambda_\alpha^T(x) g(x)$ is SOS

$\lambda_\alpha(x)$ can be arbitrary.