ME564 L6

- Review
  - high order linear ODEs
  - characteristic eq.^{n}
  - matrix system of 1st order ODE $\dot{y} = Ay$
- Example
- Special case $\dot{y} = Dy$ - D diagonal
- Derive eigenvalue equation to diagonalize any system

Higher Order ODE (linear)

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_3 \dddot{x} + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0$$

try $x(t) = e^{\lambda t} \rightarrow x^{n} = \lambda^{n} x$

$$\underbrace{(a_n \lambda^{n} + a_{n-1} \lambda^{n-1} + \dots + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0)}_{= 0} \underbrace{x(t)}_{e^{\lambda t}} = 0$$

characteristic polynomial

$\downarrow$

in German

$\downarrow$

eigen!

In general, n solutions : $\lambda_1, \lambda_2 \dots \lambda_n$

☆ $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_n e^{\lambda_n t} \longrightarrow$ general solution

use $\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \vdots \\ x^{n-1}(0) \end{bmatrix}$ to solve $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

(cont'd)

$$x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \dots + a_3 \dddot{x} + a_2 \ddot{x} + a_1 \dot{x} + a_0 x = 0$$

$$x_1 = x \qquad \dot{x}_1 = x_2$$

$$x_2 = \dot{x} \qquad \dot{x}_2 = x_3$$

$$x_3 = \ddot{x} \qquad \dot{x}_3 = x_4$$

$$\vdots \qquad \vdots$$

$$x_n = x^{(n-1)} \qquad \dot{x}_n = -a_0 x_1 - a_1 x_2 - a_2 x_3 - \dots - a_{n-1} x_n$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$\dot{y} = A y$$ General

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e.g.

$$\ddot{x} + 3\dot{x} + 2x = 0$$

$$\Downarrow$$

$$\left.\begin{aligned} \dot{x} &= v \\ \dot{v} &= -2x - 3\dot{x} \\ &= -2x - 3v \end{aligned}\right\} \quad \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\dot{y} = A \quad y$$

eigenvalues of A are roots of characteristic polynomials

$\det(A - \lambda I) = 0$

$$\det\left(\overset{A}{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}} - \overset{\lambda I}{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}} \right) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -2 & -\lambda-3 \end{bmatrix} \right) = \lambda^2 + 3\lambda + 2 = 0$$

we want to solve general systems $\dot{x} = Ax$

Case 1 : Uncoupled Dynamics (only depends themselves)

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \vdots \\ \cdots & & \ddots & & \\ 0 & \cdots & & & \lambda_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

e.g. zoo animal reproducing

$\Downarrow$

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 \\ \dot{x}_2 &= \lambda_2 x_2 \\ &\vdots \\ \dot{x}_n &= \lambda_n x_n \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1(t) &= e^{\lambda_1 t} x_1(0) \\ x_2(t) &= e^{\lambda_2 t} x_2(0) \\ &\vdots \\ x_n(t) &= e^{\lambda_n t} x_n(0) \end{aligned}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & \ddots & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix} \Rightarrow \boxed{x(t) = e^{At} x(0)}$$

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we need a coord transformation $x = Tz$

so that $\dot{z} = Dz$ → Diagonal

eigenvectors of A
→ define a new coordinate system
→ where on that vector, everything will be decoupled

$$Tz = x$$

$$\Rightarrow T\dot{z} = \dot{x} = Ax = ATz$$

$$\Rightarrow T\dot{z} = ATz$$

$$\Rightarrow \dot{z} = \underbrace{T^{-1}AT}_{= D \text{ (we hope)}} z$$

$$T^{-1}AT = D$$

$$TT^{-1}AT = TD$$

$$\star \ \boxed{AT = TD}$$

eigenvalue equation

eigenvector

$$\underset{A}{\begin{bmatrix} & A & \end{bmatrix}} \underset{T}{\begin{bmatrix} | & | & & | \\ t_1 & t_2 & \cdots & t_n \\ | & | & & | \end{bmatrix}} = \underset{T}{\begin{bmatrix} | & | & & | \\ t_1 & t_2 & \cdots & t_n \\ | & | & & | \end{bmatrix}} \underset{D}{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}$$

$$\begin{aligned} At_1 &= \lambda t_1 \\ At_2 &= \lambda t_2 \\ &\vdots \\ At_n &= \lambda t_n \end{aligned}$$

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>>[T, D] = eig(A);
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$$D = T^{-1}AT$$

$$A = TDT^{-1}$$

$$A^2 = TD^2T^{-1}$$

$$A^3 = TD^3T^{-1}$$

$$e^{At} = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots = Te^{Dt}T^{-1}$$