Δ Inverse Kinematics

• Recall: forward kinematics

$T_{IE}(q) = \begin{bmatrix} C_{IE}(q) & {}_I r_{IE}(q) \\ 0 & 1 \end{bmatrix} \in SE(3)$

parametrize

$\chi_e = \begin{pmatrix} r_e(q) \\ \phi_e(q) \end{pmatrix} = f(q)$

$\chi_e = \begin{pmatrix} \chi_{eP} \\ \chi_{eR} \end{pmatrix}$

• Q: $q = f^{-1}(\chi_E)$? — give you χ_E, give me q

Δ sol^n

• Analytic: - for 3 intersecting neighboring axes
• Geometric: - use length, theorem ... "geometric info"
• Algebraic - use TFs to get joints
• numerical!

Δ numerical method: inverse differential kinematics

• recall

$w_e = J_e \dot q$

$\dot q = J_{e0}^+ w_e^*$ — pseudo inverse; multiple sol; singular

• singularities
- occur @ q_s ⇒ $J_e(q_s)$ is column deficient
- $\dot q = J_{e0}^+ w_e^*$ — inf ← column deficient ← small
- categories
  - boundary: when end-arm is @ its limitation; easy to prevent
  - internal: hard to prevent
- damped version of Moore-Penrose pseudo inverse

$\dot q = J^+ w \Leftrightarrow \arg\min_{\dot q} \|J\dot q - w\|_2$

$\Leftrightarrow \arg\min_{\dot q} \|J\dot q - w\|_2^2 + \lambda \|w\|_2^2$

$\dot q = J^T (JJ^T + \lambda^2 I)^{-1} w$

• Redundancy
- $q \in R^n$, $w \in R^m$, $J \in R^{m \times n}$; $w = J\dot q$; $m > n$ → redundancy
- $J_{e0}\dot q = w_e^*$

⇒ $J_{e0}(J_{e0}^+ w_e^* + N \dot q_0) = w_e^*$

⇒ $\dot q = J_{e0}^+ w_e^* + N \dot q_0$

$N = \mathcal{N}(J_{e0})$ null space; $J_{e0} N = 0$

- get N?
  - $N = I - J_{e0}^+ J_{e0}$
  - SVD
  - QR

  end-up w/ different basis that span Null space

Δ multi-task control
- track desired position & orientation, ensure kinematics
- break down tasks: $\{J_i, w_i^*\}$
- $\dot q = \underbrace{\begin{bmatrix} J_1 \\ \vdots \\ J_{n_t} \end{bmatrix}^+}_{\bar J} \underbrace{\begin{bmatrix} w_1^* \\ \vdots \\ w_{n_t}^* \end{bmatrix}}_{\bar w}$

⇒ $\min \|\bar J \dot q - \bar w\|_2$

⇒ $\min \|\dot q\|_2^2$ s.t. $\bar J \dot q = \bar w$

- weighting

$\bar J^{+W} = (\bar J^T W \bar J)^{-1} \bar J^T W$

- Prioritization

recall: $\dot q = J_1^+ w_1^* + N_1 \dot q_0$

⇒ $w_2 = J_2 \dot q = J_2(J_1^+ w_1^* + N_1 \dot q_0)$

⇒ $\dot q_0 = (J_2 N_1)^+ (w_2^* - J_2 J_1^+ w_1^*)$

⇒ $\dot q = J_1^+ w_1^* + N_1 (J_2 N_1)^+ (w_2^* - J_2 J_1^+ w_1^*)$

∴ $\dot q = \sum_{i=1}^{n_t} N_i \dot q_i$

w/ $\dot q_i = (J_i \bar N_i)^+ (w_i^* - J_i \sum_{k=1}^{i-1} N_k \dot q_k)$

Δ Back to inverse Kinematics

- $\dot\chi_e = J_{eA}(q)\dot q$ — analytical
  $w = J_0 \dot q$ — geometric
  always remember → there's a TF between

- now $\Delta\chi_e = J_{eA}(q)\Delta q$

- tracking a point χ_e^*, $q_0 = q^0$

pseudo-code

```
q ← q^0
while ||χe* − χe(q)|| ≥ tol do
  - JeA ← ∂χe/∂q (q)
  - JeA+ ← (JeA)+
  - Δχe ← χe* − χe(q)
  - q ← q + JeA+ Δχe
```

(usually joint-space bigger than task-space → redundant system)

Δ iterative method

• problem 1

$q^{i+1} = q^i + J_{eA}^+ \Delta\chi$

if $\Delta\chi$ ↑ & J_{eA}^+ is ill-condition

∴ $q^{i+1} = q^i + k J_{eA}^+ \Delta\chi$ — slower convergence

• problem 2

Jacobian is rank deficient → bad condition

∴ $q^{i+1} = q^i + J_{eA}^T (J_{eA} J_{eA}^T + \lambda^2 I)^{-1} \Delta\chi$

or $q^{i+1} = q^i + \alpha J_{eA}^T \Delta\chi$

Δ Orientation
- depends on parameterization
- $\in SO(3)$
- $\Delta\chi_{rotate} = \Delta\varphi$

$C_{GS}(\Delta\varphi) = C_{GS}(\varphi^*) C_{GI}^T(\varphi^t)$

⇒ $q \leftarrow q + k_{pR} \underbrace{J_{eoR}^+}_{\text{geometric one}} \Delta\varphi$ ~ Lie algebra

Δ Traj. ctrl.

• given
$r_e^*(t)$
$\dot r_e^*(t)$

• feedback
- $\Delta r_e^t = r_e^*(t) - r_e(q^t)$
- $\dot q_P^* = J_{eoP}^+(q^t) \cdot (\dot r_e^*(t) + k_{pp}\Delta r_e^t)$
- $\dot q_R^* = J_{eoR}^+(q^t) \cdot (w(t)_e^* + k_{pR}\Delta\varphi^t)$

Δ L.A. revisit

Δ determinants
- $\det(A) = 0$ — squeeze original state space to lower dimension
- $\det(A) = c$ — Area = c; if $\mathrm{sgn}(c) = (-)$ then

Δ Rank

$\mathrm{rank}(A) = n$

then n is the dimension of the spanned space of the output (column space)

Δ Nullspace (kernel)

$Ax = 0 \quad A \in R^{m \times n}$

↳ a set of vectors, which form a space, get "squeeze" to the origin;
note that all A has trivial null space $\{0\}$;
null space $\in R^{n-m}$ if $n > m$; reduced rank

Δ ill/well -condition of matrix

$Ax = c$

ill conditioned? / well conditioned? } small change in [A] — small/large change in [c]?

vice-versa → small change in [c] — small/large change in [A]?

Δ small ΔA → small Δc "well-conditioned"

Δ small ΔA → large Δc "ill-conditioned"

e.g. $\begin{bmatrix} 400 & -201 \\ -800 & 401 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ -200 \end{bmatrix}$ $x = \begin{bmatrix} -100 \\ -200 \end{bmatrix}$

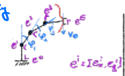
$\begin{bmatrix} 401 & -201 \\ -800 & 401 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ -200 \end{bmatrix}$ $x = \begin{bmatrix} 40{,}000 \\ 79{,}800 \end{bmatrix}$

Δ condition number
cond[A] ≈ 1.0 — well
cond[A] > 1.0 — ill

$\mathrm{cond}[A] = \|A\|_\infty \|A^{-1}\|_\infty$

Rule of thumb: log[cond[A]] = decimals of accuracy lost

Δ inverse kinematics case study



(1) given $q_t = [\pi/6, \pi/3, \pi/3]^T$
${}_0 r_{E,t} = [1, 1]^T$
find $\dot q$ (generalized velocity)

$w = J\dot q$ (assume $l_i = 1$)

Δ $r_E = \begin{bmatrix} S\varphi_1 + S(\varphi_1+\varphi_2) + S(\varphi_1+\varphi_2+\varphi_3) \\ 1 + C\varphi_1 + C(\varphi_1+\varphi_2) + C(\varphi_1+\varphi_2+\varphi_3) \end{bmatrix}$

⇒ $J_E = \begin{bmatrix} \frac{\partial r_1}{\partial q_1} & \frac{\partial r_1}{\partial q_2} & \frac{\partial r_1}{\partial q_3} \\ \frac{\partial r_2}{\partial q_1} & \frac{\partial r_2}{\partial q_2} & \frac{\partial r_2}{\partial q_3} \end{bmatrix}$

= [illegible]

⇒ w/ $q = [\pi/6 \;\; \pi/3 \;\; \pi/3]^T$

$J_E = \frac{1}{2}\begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{3} \\ -4 & -3 & -1 \end{bmatrix}$

⇒ $\dot q = J_E^+ w$
$= J^T(JJ^T)^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$= \begin{bmatrix} 0.069 \\ -0.56 \\ -0.575 \end{bmatrix}$

check! $\frac{1}{2}\begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{3} \\ -4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 0.069 \\ -0.56 \\ -0.575 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(2) multi-task
we want $\dot q_1 = \dot q_2 = 0$

$J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- parallel (no priorities)

$\underset{R^{3\times 1}}{\dot q} = \begin{bmatrix} \frac{1}{2}\begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{3} \\ -4 & -3 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{bmatrix}^+ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$[R^{4\times 3}]^+ [R^{4\times 1}]$

⇒ $\dot q = \begin{matrix} -0.133 \\ -0.067 \\ -1.132 \end{matrix}$

check! $\begin{bmatrix} J_1 \\ J_2 \end{bmatrix}\dot q = \begin{bmatrix} 1.039 \\ 0.933 \\ -0.133 \\ -0.067 \end{bmatrix} \approx \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

(3) w/ priority, task 1 most important! "$w = J\dot q$"

⇒ $\dot q = J_1^+ w_1 + N_1 \dot q_0$ — $N_1 = I - J_1^+ J_1$

$J_1[I - J_1^+ J_1] = J_1 - J_1 J_1^+ J_1 = 0$

⇒ $J_2 \dot q = J_2 J_1^+ w_1 + J_2 N_1 \dot q_0 = w_2$

⇒ $\dot q_0 = (J_2 N_1)^+ (w_2 - J_2 J_1^+ w_1)$

⇒ $\dot q = J_1^+ w_1 + N_1 (J_2 N_1)^+ (w_2 - J_2 J_1^+ w_1)$
$= \begin{bmatrix} -0.169 \\ -0.485 \\ -1.070 \end{bmatrix}$

check! $\begin{bmatrix} J_1 \\ J_2 \end{bmatrix}\dot q = \begin{bmatrix} 1 \\ 1 \\ -0.169 \\ -0.067 \end{bmatrix}$

Δ matrix norm

$[A] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\|A\|_\infty = \max\{|1|+|2|, |0|+|1|\}$ — row sum w/ abs.
$= \max\{3, 1\} = 3$

$\|A\|_1$ = sum of row 1 = 3
$\|A\|_2$ = sum of row 2 = 1

e.g. recall $\mathrm{cond}[A] = \|A\|_\infty \|A^{-1}\|_\infty$

$A = \begin{bmatrix} 400 & -201 \\ -801 & 401 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -1.003 & -5.01 \\ -2 & 1 \end{bmatrix}$

⇒ $\|A\|_\infty = |-801| + |401| = 1201$
$\|A^{-1}\|_\infty = |-2| + |-1| = 3$

⇒ cond(A) = 3603

Δ more on matrix norm
• recall vector norms
L_1-norm $\|u\|_1 = |u_1| + |u_2|$
L_2-norm $\|u\|_2 = \sqrt{u_1^2 + u_2^2}$
• matrix norms "based on" vector norm
matrix → linear transformation
∴ matrix norm:
"HOW BIG MY MATRIX CAN GET MY OUTPUT VECTOR".

$\|A\|_2 = \max_{\|u\|_2 = 1} \|Au\|_2$

• $\|AB\|_p \le \|A\|_p \|B\|_p$
• $\|A^k\|_p = \max_{\|u\|_p = 1} \|A^k u\|_p$

$A[u_1 \; u_2 \cdots u_i] = \lambda_i [u_1 \cdots u_i]$

$Au = \lambda u$

⇒ $\|Au_i\|_p = |\lambda_i| \|u_i\|_p \le \|A\|_p \|u_i\|_p$

⇒ $\|\lambda_i\| \le \|A\|_p$

⇒ $\max|\lambda_i| \le \|A\|_p$

⇒ $\max|\lambda_i|^k \le \|A^k\|_p$

Δ Dynamics

(this chapter basically talks about EoM — description of motion)

- kinematic v dynamic (source of motion)
  kinematic: don't care about the cause, let me know the relation!
  dynamic: what's the source
- cause of motion

$\tau \longrightarrow \ddot q$
cause — description of motion!

★ $M(q)\ddot q + b(q, \dot q) + g(q) = \tau + J_c^T F_c$

- Newton-Euler (free cut, conservation of impulse & angular momentum)
- Projected Newton-Euler (generalized coordinates)
- Lagrange II (energy)

Δ Principle of virtual work (D'Alembert's Principle)

• Dynamic equilibrium ⇒ 0 virtual work

$\delta W = \int_B \delta r^T \cdot (\ddot r\, dm - dF_{ext}) = 0$

• Newton's law for every particle
force $F = ma$

⇒ $F = \frac{d}{dt}(mv) = \dot p$ — p: momentum (linear impulse)

momentum

$r \times F = r \times ma$

$\Gamma = \frac{d}{dt}(r \times mv) = \dot N$ — N: angular momentum

Δ Derivation of virtual displacements of single rigid body

- from rigid body kinematics

$r = r_s + p$ — $[\Omega]_\times p = -[p]_\times \Omega$

$\dot r = v_s + \Omega \times p = [I_{3\times3} \;\; -[p]_\times]\begin{bmatrix} v_s \\ \Omega \end{bmatrix}$

$\ddot r = a_s + \psi \times p + \Omega \times (\Omega \times p)$
$= [I_{3\times3} \;\; -[p]_\times]\begin{bmatrix} a_s \\ \psi \end{bmatrix} + [\Omega]_\times [\Omega]_\times p$

$\delta r = \delta r_s + \delta\Phi \times p$
$= [I_{3\times3} \;\; -[p]_\times]\begin{bmatrix} \delta r_s \\ \delta\Phi \end{bmatrix}$

- Apply principle of virtual work

$\delta W = \int_B \delta r^T (\ddot r\, dm - dF_{ext}) = 0$

$= \int_B \underbrace{\begin{pmatrix} \delta r \\ \delta\Phi \end{pmatrix}^T \begin{pmatrix} I_{3\times3} \\ [p]_\times \end{pmatrix}}_{\delta r} \cdot \Big(\{ [I_{3\times3} \;\; -[p]_\times]\begin{bmatrix} a_s \\ \psi \end{bmatrix} + [\Omega]_\times [\Omega]_\times p \} dm - dF_{ext} \Big)$

$= \begin{pmatrix} \delta r \\ \delta\Phi \end{pmatrix}^T \int_B \Big(\begin{bmatrix} I\,dm & [p]_\times^T dm \\ [p]_\times dm & -[p]_\times^2 dm \end{bmatrix}\begin{pmatrix} a_s \\ \psi \end{pmatrix} + \begin{bmatrix} [\Omega]_\times^2 p\,dm \\ [p]_\times [\Omega]_\times^2 p\,dm \end{bmatrix} \Big) - \begin{pmatrix} dF_{ext} \\ [p]_\times dF_{ext} \end{pmatrix}$

w/ $\int_B dm = m$
$\int_B p\,dm = 0$ — S = COG moment.
$\int_B -[p]_\times^2 dm = \int_B [p]_\times [p]_\times^T dm = \Theta_S$ — moment of inertial

⇒ $0 = \delta W = \begin{pmatrix} \delta r_S \\ \delta\Phi \end{pmatrix}^T \Big(\begin{bmatrix} I_{3\times3} m & 0 \\ 0 & \Theta_S \end{bmatrix}\begin{bmatrix} a_s \\ \psi \end{bmatrix} + \begin{pmatrix} 0 \\ [\Omega]_\times \Theta_S \Omega \end{pmatrix} - \begin{pmatrix} F_{ext} \\ T_{ext} \end{pmatrix} \Big)$

$\forall \begin{pmatrix} \delta r_S \\ \delta\Phi \end{pmatrix}$ ⟶ ★ free body can move in all directions

⇒ means that

$\begin{cases} m a_s - F_{ext} = 0 \\ \Theta_S \psi + [\Omega]_\times \Theta_S \Omega - T_{ext} = 0 \end{cases}$

note that

$\begin{cases} p_S = m v_S & \text{momentum} \\ \dot p_S = m a_S \;(f) & \text{change in momentum} \end{cases}$

$\begin{cases} N_S = \Theta_S \Omega_S & \text{angular momentum} \\ \dot N_S = \Theta_S \psi + \Omega \times \Theta_S \Omega & \text{change in angular momentum} \end{cases}$

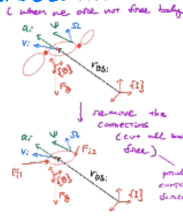
⇒ $\begin{cases} \dot p_S = F_{ext} & \text{force} & \text{newton} \\ \dot N_S = T_{ext} & \text{torque} & \text{euler} \end{cases}$

"change in linear / angular momentum"

Δ some reminder

	linear	rotational
@	x	θ
v	v	ω
a	a	α
motion	$x = vt$	$\theta = \omega t$
	$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
	$x = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
	$v^2 = v_0^2 + 2ax$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
mass	m	I
	$F = ma$	$\tau = I\alpha$
	$P = mv$	$L = I\omega$
work	Fd	$\tau\theta$
	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$
power	Fv	$\tau\omega$

Δ More on Newton-Euler method
(when we are not free body)



• based on this, w/ n rigid bodies
- 6n equation
- 5n constrained force

(why 5n: if 1 DOF joints you are imposing 5 DOF constraints in each joint)

Δ projected Newton-Euler
(in generalized motion direction)

- recall

$0 = \delta W = \sum_{i=1}^{n_b} \begin{pmatrix} \delta r_{S_i} \\ \delta\Phi_{S_i} \end{pmatrix}^T \Big(\begin{bmatrix} \dot p_{S_i} \\ \dot N_{S_i} \end{bmatrix} - \begin{bmatrix} F_{ext,i} \\ T_{ext,i} \end{bmatrix} \Big) \quad \forall \begin{pmatrix} \delta r_{S_i} \\ \delta\Phi_{S_i} \end{pmatrix}$

now, this is limited due to linkages!

- now express in generalized coordinates to represent the joints
- express linear / angular momentum in generalized coordinates

$\begin{cases} \begin{bmatrix} v_S \\ \Omega \end{bmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix}\dot q \\ \begin{bmatrix} a_S \\ \psi \end{bmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix}\ddot q + \begin{bmatrix} \dot J_P \\ \dot J_R \end{bmatrix}\dot q \end{cases}$

⇒ $\begin{bmatrix} \dot p_{S_i} \\ \dot N_{S_i} \end{bmatrix} = \begin{bmatrix} m a_{S_i} \\ \Theta_{S_i}\psi_{S_i} + \Omega_{S_i} \times \Theta_{S_i}\Omega_{S_i} \end{bmatrix}$

$= \begin{pmatrix} m J_{S_i} \\ \Theta_{S_i} J_{R_i} \end{pmatrix}\ddot q + \begin{pmatrix} m \dot J_{S_i}\dot q \\ \Theta_{S_i}\dot J_{R_i}\dot q + J_{R_i}\dot q \times \Theta_{S_i} J_{R_i}\dot q \end{pmatrix}$

- virtual displacement in generalized coordinates

$\begin{pmatrix} \delta r_S \\ \delta\Phi \end{pmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix}\delta q$

∴ $0 = \delta W = \sum_{i=1}^{n} \begin{pmatrix} \delta r_{S_i} \\ \delta\Phi_{S_i} \end{pmatrix}^T \Big(\begin{bmatrix} \dot p_{S_i} \\ \dot N_{S_i} \end{bmatrix} - \begin{bmatrix} F_{ext,i} \\ T_{ext,i} \end{bmatrix} \Big)$

$= \delta q^T \sum_{i=1}^{n} \Big\{ \underbrace{\begin{pmatrix} J_{S_i} \\ J_{R_i} \end{pmatrix}^T \begin{pmatrix} m J_{S_i} \\ \Theta_{S_i} J_{R_i} \end{pmatrix}\ddot q}_{M(q)} + \underbrace{\begin{pmatrix} J_{S_i} \\ J_{R_i} \end{pmatrix}^T \begin{pmatrix} m \dot J_{S_i}\dot q \\ \Theta_{S_i}\dot J_{R_i}\dot q + J_{R_i}\dot q \times \Theta_{S_i} J_{R_i}\dot q \end{pmatrix}}_{b(q, \dot q)} - \underbrace{\begin{pmatrix} J_{S_i} \\ J_{R_i} \end{pmatrix}^T \begin{pmatrix} F_{ext,i} \\ T_{ext,i} \end{pmatrix}}_{g(q)} \Big\} \quad \forall \delta q$

- $M(q)\ddot q + b(q, \dot q) + g(q) = 0$

$M = \sum_{i=1}^{n_b} ({}_A J_{S_i}^T \cdot m \cdot {}_A J_{S_i} + {}_B J_{R_i}^T \cdot {}_B\Theta_{S_i} \cdot {}_B J_{R_i})$

$b = \sum_{i=1}^{n_b} ({}_A J_{S_i}^T \cdot m \cdot {}_A \dot J_{S_i} \cdot \dot q + {}_B J_{R_i}^T ({}_B\Theta_{S_i} \cdot {}_B \dot J_{R_i} \cdot \dot q + {}_B\Omega_{S_i} \times {}_B\Theta_{S_i} \cdot {}_B\Omega_{S_i}))$

$g = \sum_{i=1}^{n_b} (-{}_A J_{S_i}^T \; {}_A F_{g,i})$

Δ Lagrange II

• Lagrangian $\mathcal{L} = T - U$ — T: kinetic, U: potential

• Lagrangian Equation

$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot q}\Big) - \frac{\partial\mathcal{L}}{\partial q} = \tau$

• ∵ $U = U(q)$
$T = T(q, \dot q)$

∴ $\underbrace{\frac{d}{dt}\Big(\frac{\partial T}{\partial\dot q}\Big) - \frac{\partial T}{\partial q}}_{\text{inertial force}} + \underbrace{\frac{\partial U}{\partial q}}_{\text{gravity vector}} = \tau$

Δ Kinetic energy
- joint space $T = \frac{1}{2}\dot q^T M \dot q$
- bodies space

$T = \sum_{i=1}^{n_b} \Big(\frac{1}{2} m_i \, {}_A\dot r_{S_i}^T \, {}_A\dot r_{S_i} + \frac{1}{2} {}_B\Omega_{S_i}^T \, {}_B\Theta_{S_i} \, {}_B\Omega_{S_i} \Big)$

w/ $\dot r_{S_i} = J_{S_i}\dot q$
$\Omega_{S_i} = J_{R_i}\dot q$

∴ $T(q, \dot q) = \frac{1}{2}\dot q^T \underbrace{\Big(\sum_{i=1}^{n_b} (J_{S_i}^T m J_{S_i} + J_{R_i}^T \Theta_{S_i} J_{R_i}) \Big)}_{M(q)} \dot q$

Δ potential energy
- gravity

$F_{g_i} = m_i \, g_I \, e_g \Rightarrow U_g = -\sum_{i=1}^{n_b} r_{S_i}^T F_{g_i}$

- spring

$F_{S_j} = k_j (\|r_1 - r_2\| - d_0) \frac{r_1 - r_2}{\|r_1 - r_2\|}$

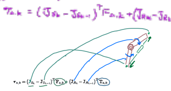
⇒ $U_{S_j} = \frac{1}{2} k_j (d(q) - d_0)^2$

Δ External Force

- given $n_{f,ext}$: F_j, T_k
- $\underset{\text{joint space}}{\tau_{F,ext}} = \sum_{j=1}^{n_f} J_{P,j}^T \underset{\text{task space}}{F_j}$
- $\tau_{T,ext} = \sum_{k=1}^{n_m} J_{R,k}^T T_k$

actuator torques

$\tau_{a,k} = (J_{S_k} - J_{S_{k-1}})^T F_{a,k} + (J_{R_k} - J_{R_{k-1}})^T T_{a,k}$



△ Dynamic Control

now we hv EoM, how do we control it?

$$M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau + J_c^T F_c$$

How do we get this

Actuator control $\xrightarrow{VA}$ Actuator torque sensor $\xrightarrow{\tau}$ Robot Dynamics

$q\ \dot{q}\ \ddot{q}$

$(q^*\ \dot{q}^*)$ τ^*

△ position-based control
- don't care about dynamics
- high gain PID : good point-to-point tracking
- disturbances are compensated by PID
- cannot control force directly
  ⇒ interaction force can only be controlled w/ compliant surface

△ integrate force-feedback (Dynamic control)
- active regulation of system dynamics
- model-based load compensation
- interaction force control.

△ Joint Impedance Control

- $M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau$
- get desired τ

△ torque as functions of P/V error

$$\tau^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$$

can think of it as spring force w/ damping

$$\Rightarrow M(q)\ddot{q} + b(q,\dot{q}) + g(q) = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$$

static offset due to gravity (when static $M\ddot{q}, b = 0$, $\dot{q}^* - \dot{q} = 0$)

△ impedance control & gravity compensation

$$\tau^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q}) + \hat{g}(q)$$

estimated

↳ configuration dependent eg COG shifts

△ independent of configuration
inverse dynamics control

- $\tau = \hat{M}(q)\ddot{q}^* + \hat{b}(q,\dot{q}) + \hat{g}(q)$

get $\ddot{q}^*$, & throw into this EoD, and get the final τ. based on inverse kinematics → inverse dynamics

- assume w/ dynamic modelling error

result in $I\ddot{q} = \ddot{q}^* - k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$ & $\omega = \sqrt{k_p}$ $D = \frac{k_d}{2\sqrt{k_p}}$

- describe from task space

$$\dot{w}_e = \begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix}_e = J_e\ddot{q} + \dot{J}_e\dot{q}$$

$$\therefore \ddot{q} = J_e^+(\dot{w}_e - \dot{J}_e\dot{q})$$

& similarly, multi-task

- $$\ddot{q} = \begin{bmatrix} J_1 \\ \vdots \\ J_{n_T} \end{bmatrix}^+ \left(\begin{bmatrix} \dot{w}_1 \\ \vdots \\ \dot{w}_{n_T} \end{bmatrix} - \begin{bmatrix} \dot{J}_1 \\ \vdots \\ \dot{J}_{n_T} \end{bmatrix} \dot{q} \right)$$

parallel

- $\ddot{q} = \sum_{i=1}^{n_T} N_i \ddot{q}_i$ w/

w/ $\ddot{q}_i = (J_i N_i)^+ \left(\dot{w}_i^* - \dot{J}_i \dot{q} - J_i \sum_{k=1}^{i-1} N_k \ddot{q}_k \right)$

- get $\ddot{q}$, & insert back to E.o.M.

△ task-space dynamics

- recall Joint space

$$M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau$$

- for end-effector

$$\Lambda \dot{w}_e + \mu + p = F_e$$

linking the two — link

$$\begin{cases} \tau = J_e^T F_e \\ \dot{w}_e = \begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix}_e = J_e \ddot{q} + \dot{J}_e \dot{q} \end{cases}$$

$M^{-1}(\tau - b - g)$

derivation

$$\Rightarrow \dot{w}_e = J_e M^{-1}(\tau - b - g) + \dot{J}_e \dot{q}$$

$$\Rightarrow \dot{w}_e - \dot{J}_e\dot{q} + J_e M^{-1} b + J_e M^{-1} g = J_e M^{-1} \tau$$

$$\Rightarrow \dot{w}_e - \underbrace{\dot{J}_e\dot{q} + J_e M^{-1} b}_{\mu} + \underbrace{J_e M^{-1} g}_{p} = \underbrace{J_e M^{-1} J_e^T}_{\Lambda^{-1}} F_e$$

$$\Rightarrow \Lambda \dot{w}_e + \mu + p = F_e$$

↳ geometric interpretation: inertial ellipsoid (depend on our configuration)

$$\therefore \text{get } \dot{w}_e^* = k_p E(\chi_e^* - \chi_e) + k_d(w_e^* - w_e) + \dot{w}_e(\tau)$$

$\Delta \chi = E_R(\chi_e)\Delta\chi_R$ — feed-forward

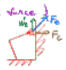
determine desired acceleration → task-level force → Jacobi transformation → joint-space torques

$$\tau^* = \hat{J}^T(\hat{\Lambda}_e \dot{w}_e^* + \hat{\mu} + \hat{p})$$

△ interaction control

operation space control

(generalized framework to control motion & force)



- $F_c + \Lambda \dot{w}_e + \mu + p = F_e$

△ selection force

$$\hat{\Lambda} S_M \dot{w}_e + S_F F_c + \hat{\mu} + \hat{p} = F_e$$

$$\Rightarrow \tau^* = \hat{J}^T F_e$$

$$\begin{cases} \Sigma_p = \begin{bmatrix} \sigma_{px} & & \\ & \sigma_{py} & \\ & & \sigma_{pz} \end{bmatrix} & \sigma \in \{0,1\} \\ \Sigma_F = I_3 - \Sigma_p \end{cases}$$

1 → can move
0 → can apply force

$$\begin{cases} \Sigma_R = \begin{bmatrix} \sigma_{Rx} & & \\ & \sigma_{Ry} & \\ & & \sigma_{Rz} \end{bmatrix} \\ \Sigma_{RF} = I_3 - \Sigma_R \end{cases}$$

w/ C. transformation between contact local frame & world frame

$$\therefore S_M = \begin{bmatrix} C^T \Sigma_p C & 0 \\ 0 & C^T \Sigma_R C \end{bmatrix}$$

$$S_F = \begin{bmatrix} C^T(I_3 - \Sigma_p)C & 0 \\ 0 & C^T(I_3 - \Sigma_R)C \end{bmatrix}$$

∴ inject back to

$$\hat{\Lambda} S_M \dot{w}_e + S_F F_c + \hat{\mu} + \hat{p} = F_e$$

△ inverse dynamic as QP

1. classic inverse dynamics

recall

$w = J\dot{q}$

$\dot{w} = J\ddot{q} + \dot{J}\dot{q} \Rightarrow \ddot{q}^* = J^+(\dot{w} - \dot{J}\dot{q})$

$$\therefore \tau^* = M(q)\ddot{q}^* + b(q,\dot{q}) + g$$
$$= M(q)[J^+(\dot{w} - \dot{J}\dot{q})] + b(q,\dot{q}) + g$$

2. operation space control

$$\tau^* = J_t^T F_e$$
$$= J_t^T(\Lambda \dot{w}_t^* + \mu + p)$$

Not only sol^n

$$\therefore \tau^* = J_t^T(\Lambda \dot{w}_t^* + \mu + p) + N(J_t^T)\tau_0$$

null space — can change my-torque

$N(J_t^T)$

$= (I - J^T[JM^{-1}J^T]^{-1}JM^{-1})$ (w.o. changing desired $\dot{w}$)

Mass, weighted pseudo inverse — proof

$\dot{w} = J\ddot{q} - \dot{J}\dot{q}$
$= JM^{-1}(\tau^* - b - g) + \dot{J}\dot{q}$
$= JM^{-1}(J_t^T(\Lambda \dot{w}_t^* + \mu + p) + (I - J^T(JM^{-1}J^T)^{-1}JM^{-1})\tau_0 - b - g) + \dot{J}\dot{q}$

$= 0$

- we can also see:

$$\begin{cases} \tau^* = M(q)\ddot{q}^* + b(q,\dot{q}) + g \\ \tau^* = J_t^T F_e = J_t^T(\Lambda \dot{w}_t^* + \mu + p) \\ \quad = J_t^T((J_t M^{-1} J_t^T)^{-1}\dot{w} + \mu + p) \end{cases}$$

$$\Rightarrow \ddot{q} = M^{-1}(\tau^* - b - g)$$
$$= M^{-1}(J_t^T(J_t M^{-1} J_t^T)^{-1}\dot{w} + \mu + p) - b - g)$$

some sort of mass-matrix weighted pseudo-inverse

3. QP

- $\left.\begin{array}{l} \tau = M\ddot{q} + b + g \\ J\ddot{q} + \dot{J}\dot{q} = \dot{w} \\ \min \|\ddot{q}\| \\ \min \|\tau\| \end{array}\right\}$ ↔ classic inverse dynamic

- $[M\ -I]\begin{bmatrix} \ddot{q} \\ \tau \end{bmatrix} + b + g = 0$

$[J_e\ 0]\begin{bmatrix} \ddot{q} \\ \tau \end{bmatrix} + \dot{J}\dot{q} = \dot{w}_e^*$

x

⇓

$x = \begin{bmatrix} \ddot{q} \\ \tau \end{bmatrix}$

multi-task:

$$\min_x \left\| \begin{bmatrix} M & -1 \\ J_e & 0 \end{bmatrix} x - \begin{bmatrix} -b-g \\ \dot{w}_e^* - \dot{J}\dot{q} \end{bmatrix} \right\|_2$$

priority:

$$\min_x \|[J_e\ 0]x - (\dot{w}_e^* - \dot{J}\dot{q})\|_2$$
$$\text{s.t. } [M\ -I]x - (-b-g) = 0$$

△ Least-square Optimization

- $Ax = b \Rightarrow x = A^+ b$

⇕

$\min_x \|Ax - b\|_2$

or

$\min_x \|x\|_2$ s.t. $Ax = b$

- $A_1 x_1 + A_2 x_2 = b$

$\Rightarrow [A_1\ A_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [A_1\ A_2]^+ b$

⇕

$\min_{x_1 x_2} \left\| [A_1 A_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - b \right\|_2$

or

$\min_{x_1 x_2} \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_2$ s.t. $[A_1 A_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$

- $A_1 x - b_1 = 0$
  $A_2 x - b_2 = 0$

- Equal priority

$x = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^+ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

- Hierarchy

$x = A_1^+ b_1 + N(A_1)x_0$ — $\min_x \|A_2 x - b_2\|_2$ s.t. $\|A_1 x - b_1\| = c$

$A_1(A_1^+ b_1 + N(A_1)x_0)$
$= A_1 A_1^+ b_1 + 0 = b_1$

$\Rightarrow A_2 x - b_2 = 0$

$\Rightarrow A_2(A_1^+ b_1 + N(A_1)x_0) - b_2 = 0$

$\Rightarrow x_0 = [A_2 N(A_1)]^+ [b_2 - A_2 A_1^+ b_1]$

try to fulfill $A_2 x = b_2$

△ case study

- $q = (\varphi_1\ \varphi_2\ x\ \varphi_3)^T$
- $M\ddot{q} + b + g = \tau$

- $J_P \in R^{2\times4}$
  $J_R \in R^{1\times4}$

(1) static gravity compensation joint torque?

i.e. $\cancel{M\ddot{q}} + \cancel{b} + g = \tau$

(2) lifting 100 kg, joint torque?

$$M\ddot{q} + b + g + J_P^T\begin{pmatrix} 0 \\ 100 \end{pmatrix} = \tau$$

$$\Rightarrow \tau = g + J_P^T\begin{pmatrix} 0 \\ 100 \end{pmatrix}$$

(3) accelerate w/ 1 m/s² upward φ, x = const., joint torque?

- $J_t = \begin{bmatrix} J_P \\ J_R \end{bmatrix}$ $\dot{w}_t^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ might subject to feed back law

my desired acceleration.

- we know that $w = J\dot{q}$
  $\dot{w} = J\ddot{q} + \dot{J}\dot{q}$

$\Rightarrow \ddot{q}^* = J^+(\dot{w} - \dot{J}\dot{q}) + N\ddot{q}_0$

$\Rightarrow \tau = M\ddot{q}^* + b + g$
$= MJ^+(\dot{w} - \dot{J}\dot{q}) + b + g$

(4) task space:

1 m/s² in y, x, φ = const. joint torque? is the sol^n the same?

- $\dot{w}_t^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\Lambda = (J_e M^{-1} J_e^T)^{-1}$
  $J_e = J_t = \begin{bmatrix} J_P \\ J_R \end{bmatrix}$ $\mu = \Lambda J_e M^{-1} b - \Lambda \dot{J}_e \dot{q}$
  $p = \Lambda J_e M^{-1} g$

$$\Rightarrow \tau^* = J_t^T \underbrace{(\Lambda \dot{w}_t^* + \mu + p)}_{F_e}$$

(5) interaction control

$\underbrace{\dot{w}_t^* = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{\text{desired}}$ $J_t = \begin{bmatrix} J_P \\ J_R \end{bmatrix}$ contact force

$$\tau^* = J_t^T \underbrace{\left[\Lambda \dot{w}_t^* + \mu + p + \underbrace{\begin{bmatrix} -1000 \\ -1000 \\ 0 \end{bmatrix}}_{F_c}\right]}_{F_e}$$

(6) QP problems.

(a.) least-square minimal-torque to get $\dot{w}^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

$x = \begin{bmatrix} \ddot{q} \\ \tau \end{bmatrix}$

- $[M\ -I]x - (-b-g) = 0$ (1)
  $A_1\ x - b_1 = 0$

$\begin{cases} w = J\dot{q} \\ \dot{w} = J\ddot{q} + \dot{J}\dot{q} \end{cases}$

- $\left\{\begin{bmatrix} J_P \\ J_R \end{bmatrix} 0\right] x - \left(\dot{w}^* - \begin{bmatrix} \dot{J}_P \\ \dot{J}_R \end{bmatrix}\dot{q}\right) = 0$ (2)
  $A_2 x - b_2 = 0$

- $[0\ I]x = 0$ $\min\|\tau\|$ (minimize torque)
  $A_3 x - b_3 = 0$ (3)

as good as possible

then solve.

(b) assume $\ddot{q}_1 = 0$, and $\dot{w}^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[M\ -I]x - (-b-g) = 0$ (1)

$\begin{bmatrix} J_P & \\ J_R & 0 \\ [1,0,0,0] & \end{bmatrix} x - \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \dot{J}_P \\ \dot{J}_R \\ [0,0,0,0] \end{bmatrix}\dot{q}\right) = 0$ (2)

(c) assume $\tau_1 = 0$, $\dot{w}^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[M\ -I]x - (-b-g) = 0$ (1)

$[0\ [1\ 0\ 0\ 0]]x = 0$ (2)

$\left\{\begin{bmatrix} J_P \\ J_R \end{bmatrix} 0\right] x - \left(\dot{w}^* - \begin{bmatrix} \dot{J}_P \\ \dot{J}_R \end{bmatrix}\dot{q}\right) = 0$ (3)

△ solving a set of QPs

- w/ different priorities

methods:
- exploit Nullspace of tasks w/ higher priority
- every step = QP w/ constraints
- iteratively nullspace projection

- pseudo-code

```
n_T = number of tasks
x = 0
N_1 = I
for i = 1 → n_T do
    x_i = (A_i N_i)^+ (b_i − A_i x)
    x = x + N_i x_i
    N_{i+1} = N([A_1^T ... A_i^T]^T)
end for
```

e.g. @ i = 1

$x = 0$
$N_1 = I$
$x_1 = (A_1 I)^+(b_1 - A_1 0)$
$= A_1^+ b_1$
$x = 0 + I A_1^+ b_1$
$N_2 = N(A_1)$

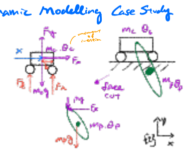
@ i = 2

$x = A_1^+ b_1$
$N_2 = N(A_1)$
$x_2 = [A_2 N(A_1)]^+[b_2 - A_2 A_1^+ b_1]$
x is $A_1^+ b_1 + N(A_1)[A_2 N(A_1)]^+[b_2 - A_2 A_1^+ b_1]$

→ use solver (most of the time)

△ Dynamic Modelling Case Study

(1) NE



△ get linear / angular momentum of cart

$m_c \ddot{x}_c = F_x$
$m_c \ddot{y}_c = F_y + F_L + F_R - m_c g$
$\theta_c \ddot{\varphi}_c = F_R b - F_L b$

△ get linear / angular momentum of pendulum

$m_p \ddot{x}_p = -F_x$
$m_p \ddot{y}_p = -F_y - m_p g$
$\theta_p \ddot{\varphi}_p = F_x l \cos(\varphi_p) - F_y l \sin(\varphi_p)$

△ kinematics

$x_c = x$
$y_c = 0$
$\varphi_c = 0$

$[x, \varphi]^T$ ⇒ generalized coordinates

$x_p = x + l\sin(\varphi)$
$y_p = -l\cos(\varphi)$
$\varphi_p = \varphi$

$\ddot{x}_p = \ddot{x} + \ddot{\varphi} l\cos(\varphi) - \dot{\varphi}^2 l\sin(\varphi)$
$\ddot{y}_p = \ddot{\varphi} l\sin(\varphi) + \dot{\varphi}^2 l\cos(\varphi)$
$\ddot{\varphi}_p = \ddot{\varphi}$

- (7),(10-12) in (1) and (4-6)

$m_c \ddot{x} = F_x$ (13)
$m_p(\ddot{x} + \ddot{\varphi} l\cos(\varphi) - \dot{\varphi}^2 l\sin(\varphi)) = -F_x$ (14)
$m_p(\ddot{\varphi} l\sin(\varphi) + \dot{\varphi}^2 l\cos(\varphi)) = -F_y - m_p g$ (15)
$\theta_p \ddot{\varphi} = F_x l\cos(\varphi) + F_y l\sin(\varphi)$ (16)

- From (13) and (14) remove F_x

$(m_p + m_c)\ddot{x} + m_p l\cos(\varphi)\ddot{\varphi} - \dot{\varphi}^2 l m_p \sin(\varphi) = 0$

- Insert (13) and (15) in (16) to remove F_x and F_y

$(\theta_p + m_p l^2)\ddot{\varphi} + m_p l\cos(\varphi)\ddot{x} + g l m_p \sin(\varphi) = 0$

(2) NE - projected

$q = \begin{bmatrix} x \\ \varphi \end{bmatrix}$

- $r_{OS_c} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$J_{P_c} = \frac{\partial r}{\partial q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\dot{J}_{P_c} = \frac{\partial J_{P_c}}{\partial t} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- $r_{OS_p} = \begin{bmatrix} x + l\sin\varphi \\ -l\cos\varphi \end{bmatrix}$

$J_{P_p} = \frac{\partial r}{\partial q} = \begin{bmatrix} 1 & l\cos(\varphi) \\ 0 & l\sin(\varphi) \end{bmatrix}$

$\dot{J}_{P_p} = \frac{\partial J_{P_p}}{\partial t} = \begin{bmatrix} 0 & -l\sin(\varphi)\dot{\varphi} \\ 0 & l\cos(\varphi)\dot{\varphi} \end{bmatrix}$

$\omega_p = \dot{\varphi}$

$J_{R_p} = \frac{\partial \omega_p}{\partial \dot{q}} = [0\ 1]$

- E.o.M

$$\mathbf{M} = \sum \mathbf{J}_{P_i}^T m_i \mathbf{J}_{P_i} + \mathbf{J}_{R_i}^T \Theta_i \mathbf{J}_{R_i} = \mathbf{J}_{P_c}^T m_c \mathbf{J}_{P_c} + \mathbf{J}_{P_p}^T m_p \mathbf{J}_{P_p} + \mathbf{J}_{R_p}^T \theta_p \mathbf{J}_{R_p} = \begin{bmatrix} m_c + m_p & l m_p \cos(\varphi) \\ l m_p \cos(\varphi) & m_p l^2 + \theta_p \end{bmatrix}$$

$$\mathbf{b} = \sum \mathbf{J}_{P_i}^T m_i \dot{\mathbf{J}}_{P_i} \dot{\mathbf{q}} + \mathbf{J}_{R_i}^T \Theta_i \dot{\mathbf{J}}_{R_i} \dot{\mathbf{q}} + \mathbf{J}_{R_i}^T (\mathbf{J}_{R_i}\dot{\mathbf{q}}) \times \Theta_i \mathbf{J}_{R_i} \dot{\mathbf{q}} = \mathbf{J}_{P_p}^T m_p \dot{\mathbf{J}}_{P_p} \dot{\mathbf{q}} = \begin{pmatrix} -\dot{\varphi}^2 l m_p \sin(\varphi) \\ 0 \end{pmatrix}$$

$=0$ (planar system)

$$\mathbf{g} = \sum_{i=1}^{n} -\mathbf{J}_i^T \mathbf{F}_i^g = -\mathbf{J}_{P_c}^T \begin{pmatrix} 0 \\ -m_c g \end{pmatrix} - \mathbf{J}_{P_p}^T \begin{pmatrix} 0 \\ -m_p g \end{pmatrix} = \begin{pmatrix} 0 \\ m_p g l \sin(\varphi) \end{pmatrix}$$

(3) Lagrange II

- Kinematics cart and pendulum

$\mathbf{r}_{OS_c} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\mathbf{r}_{OS_p} = \begin{pmatrix} x + l\sin(\varphi) \\ -l\cos(\varphi) \end{pmatrix}$ $\varphi_p = \varphi$

$\dot{\mathbf{r}}_{S_c} = \begin{pmatrix} \dot{x} \\ 0 \end{pmatrix}$ $\dot{\mathbf{r}}_{S_p} = \begin{pmatrix} \dot{x} + l\cos(\varphi)\dot{\varphi} \\ l\sin(\varphi)\dot{\varphi} \end{pmatrix}$ $\omega_p = \dot{\varphi}$

- Kinetic and potential energy

$\mathcal{T} = \sum \frac{1}{2}\dot{\mathbf{r}}_{S_i}^T m_i \dot{\mathbf{r}}_{S_i} + \frac{1}{2}\boldsymbol{\omega}_i^T \Theta_i \boldsymbol{\omega}_i = \frac{1}{2}m_c \dot{x}^2 + \frac{1}{2}m_p \dot{x}^2 + \frac{1}{2}m_p l^2 \dot{\varphi}^2 + m_p \dot{x} l\cos(\varphi)\dot{\varphi} + \frac{1}{2}\theta\dot{\varphi}^2$

$\mathcal{U} = -m_p g l\cos(\varphi)$ ← 0-level can be chosen

- Equation of motion

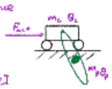
$\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}} = \begin{pmatrix} m_c \dot{x} + m_p \dot{x} + m_p l\cos(\varphi)\dot{\varphi} \\ m_p l^2 \dot{\varphi} + m_p \dot{x} l\cos(\varphi) + \theta\dot{\varphi} \end{pmatrix}$

$\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{T}}{\partial \mathbf{q}} + \frac{\partial \mathcal{U}}{\partial \mathbf{q}} = 0$ $\frac{d}{dt}\left(\frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}}\right) = \begin{pmatrix} m_c \ddot{x} + m_p \ddot{x} + m_p l\cos(\varphi)\ddot{\varphi} - m_p l\sin(\varphi)\dot{\varphi}^2 \\ m_p l^2 \ddot{\varphi} + m_p \ddot{x} l\cos(\varphi) - m_p \dot{x} l\sin(\varphi)\dot{\varphi} + \theta\ddot{\varphi} \end{pmatrix}$

$-\frac{\partial \mathcal{T}}{\partial \mathbf{q}} = \begin{pmatrix} 0 \\ m_p \dot{x} l\sin(\varphi)\dot{\varphi} \end{pmatrix}$

$\frac{\partial \mathcal{U}}{\partial \mathbf{q}} = \begin{pmatrix} 0 \\ m_p g l\sin(\varphi) \end{pmatrix}$

(4) External force



- $J_e^T F_e$

where $J_e = [1, 0]$

$\therefore \tau = J^T F_e$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} F_{ext} = \begin{bmatrix} F_{ext} \\ 0 \end{bmatrix}$