

Robot Dynamics

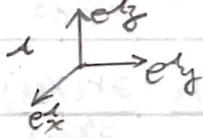
Fundamentals

vector

r / \vec{r}

$B \rightarrow P$

\vec{r}_{BP}

in \mathcal{L} frame \vec{r}_{AB} orthonormal basis of \mathbb{R}^3
 $(\vec{e}_x^1, \vec{e}_y^1, \vec{e}_z^1) :=$ orthonormal basis of \mathbb{R}^3 parameterization of $\vec{r}_{AB} = x \vec{e}_x^1 + y \vec{e}_y^1 + z \vec{e}_z^1$

Cartesian coordinates

$x_{pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\Delta \vec{r}_{AB} = x \vec{e}_x^1 + y \vec{e}_y^1 + z \vec{e}_z^1$

Cylindrical coordinates

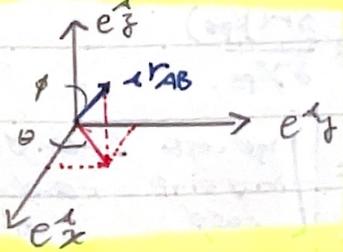
$x_{pz} = \begin{pmatrix} r \\ \theta \\ z \end{pmatrix}$

$\Delta \vec{r}_{AB} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix}$

Spherical Coordinates

$x_{ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$

$\Delta \vec{r}_{AB} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \sin \phi \cos \theta \\ r \sin \phi \sin \theta \\ r \cos \phi \end{pmatrix}$



ex 1-1.

$\vec{r}_{AP} = \vec{r}_{AB} + \vec{r}_{BP}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_{pc} \\ y_{pc} \\ z_{pc} \end{pmatrix} + \begin{pmatrix} x_{pc} \\ y_{pc} \\ z_{pc} \end{pmatrix}$

get $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{cases} x_{pc} = \\ x_{pz} = \\ x_{ps} = \end{cases}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{cases} x_{pc} = \\ x_{pz} = \\ x_{ps} = \end{cases}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{cases} x_{pc} = \\ x_{pz} = \\ x_{ps} = \end{cases}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_{pc} \\ y_{pc} \\ z_{pc} \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix} = \begin{pmatrix} r \sin \phi \cos \theta \\ r \sin \phi \sin \theta \\ r \cos \phi \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos 90^\circ \\ r \sin 90^\circ \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ r \sin 90^\circ \cos 0^\circ \\ r \sin 90^\circ \sin 0^\circ \end{pmatrix} = \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix}$

$x_{pz} = \begin{pmatrix} 0^\circ \\ 0^\circ \\ 0^\circ \end{pmatrix}$

$x_{ps} = \begin{pmatrix} 0^\circ \\ 90^\circ \\ 0^\circ \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} x_{pc} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix} = \begin{pmatrix} r \sin \phi \cos \theta \\ r \sin \phi \sin \theta \\ r \cos \phi \end{pmatrix}$

$\therefore \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos 90^\circ \\ \sin 90^\circ \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \sin 45^\circ \cos 90^\circ \\ \sqrt{2} \sin 45^\circ \sin 90^\circ \\ \sqrt{2} \cos 45^\circ \end{pmatrix}$

$x_{pz} = \begin{pmatrix} 90^\circ \\ 1 \\ 0 \end{pmatrix}$

$x_{ps} = \begin{pmatrix} \sqrt{2} \\ 90^\circ \\ 45^\circ \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x_{pc} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \cos \theta \\ \sqrt{3} \sin \theta \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{3} \sin(\alpha \tan \beta) \cos 45^\circ \\ \sqrt{3} \sin(\alpha \tan \beta) \sin 45^\circ \\ \sqrt{3} \cos(\alpha \tan \beta) \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \cos 45^\circ \\ \sqrt{3} \sin 45^\circ \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ \sqrt{3} \tan \frac{\pi}{4} \\ 1 \end{pmatrix}$

vector derivatives (linear velocity)

 $r = r(\chi)$ vector being represented by specific parameterization

$\dot{r} = \dot{r}(\chi) \dot{\chi}$

$\Rightarrow \dot{r} = \frac{\partial r}{\partial \chi} \dot{\chi}$

$\dot{r} = E_p(\chi) \dot{\chi}$

$E_p(\chi)^{-1} \dot{r} = \dot{\chi}$

$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ Cartesian coordinates

$x_{pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\therefore E_p(x_{pc}) = E_p^{-1}(x_{pc}) = I$

cylindrical coordinates

$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix}$

matrix calculus

vector - by - vector

$\frac{\partial y}{\partial x} (x, y \in \mathbb{R}^n)$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \dots & \dots & \vdots \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \dots & \dots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$$

column picture!

$$r(x_{PQ}) = [e \cos \theta \ e \sin \theta]$$

$$x_{PQ} = [e \ \theta]^T$$

calculus revision

o.g. $\nabla f = \frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1} \ \frac{\partial f}{\partial x_2} \ \frac{\partial f}{\partial x_3} \right]$ (gradient) $\in \mathbb{R}^3$

vector by scalar $y = [y_1, y_2, \dots, y_m]^T, x$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x} \\ \frac{\partial y_2}{\partial x} \\ \vdots \\ \frac{\partial y_m}{\partial x} \end{bmatrix}$$

e.g. position
velocityscalar by vector $y, x = [x_1, x_2, \dots, x_n]^T$

$$\frac{\partial y}{\partial x} = \left[\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \dots, \frac{\partial y}{\partial x_n} \right] \text{ e.g. gradient}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = \left(\frac{\partial f}{\partial x} \right)^T$$

vector by vector

$$y = [y_1, y_2, \dots, y_m]^T$$

$$x = [x_1, x_2, \dots, x_n]^T$$

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \dots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

Jacobian Matrix

$$r = r(x)$$

$$\dot{r} = \frac{\partial r}{\partial x} \dot{x} = E_p(x) \dot{x}$$

Cartesian coordinates, as mentioned.

$$x_{pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$E_{pc}(x_{pc}) = E_{pc}^{-1}(x_{pc}) = I$$

Cylindrical coordinates, from matrix calculus

$$r = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix}$$

$$x_{PQ} = \begin{pmatrix} e \\ \theta \\ z \end{pmatrix}$$

$$E_{PQ}(x_{PQ}) = \frac{\partial r(x_{PQ})}{\partial x_{PQ}} = \begin{bmatrix} \frac{\partial r(x_{PQ})_1}{\partial x_{PQ1}}, \frac{\partial r(x_{PQ})_1}{\partial x_{PQ2}}, \frac{\partial r(x_{PQ})_1}{\partial x_{PQ3}} \\ \frac{\partial r(x_{PQ})_2}{\partial x_{PQ1}}, \frac{\partial r(x_{PQ})_2}{\partial x_{PQ2}}, \frac{\partial r(x_{PQ})_2}{\partial x_{PQ3}} \\ \frac{\partial r(x_{PQ})_3}{\partial x_{PQ1}}, \frac{\partial r(x_{PQ})_3}{\partial x_{PQ2}}, \frac{\partial r(x_{PQ})_3}{\partial x_{PQ3}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Spherical coordinates

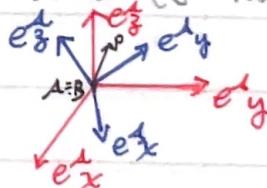
$$r = \begin{pmatrix} r \sin \phi \cos \theta \\ r \sin \phi \sin \theta \\ r \cos \phi \end{pmatrix}$$

$$x_{PS} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$$

$$E_{PS}(x_{PS}) = \frac{\partial r(x_{PS})}{\partial x_{PS}}$$

$$= \begin{bmatrix} \sin \phi \cos \theta & -r \sin \phi \sin \theta & r \cos \phi \cos \theta \\ \sin \phi \sin \theta & r \sin \phi \cos \theta & r \cos \phi \sin \theta \\ \cos \phi & 0 & -r \sin \phi \end{bmatrix}$$

rotation transformation



$${}^A r_{AP} = \begin{pmatrix} {}^A r_{APx} \\ {}^A r_{APy} \\ {}^A r_{APz} \end{pmatrix} \quad {}^B r_{AP} = \begin{pmatrix} {}^B r_{APx} \\ {}^B r_{APy} \\ {}^B r_{APz} \end{pmatrix}$$

$${}^A r_{AP} = {}^A e_x^B {}^B r_{APx} + {}^A e_y^B {}^B r_{APy} + {}^A e_z^B {}^B r_{APz}$$

vector basis of B frame in A frame

$$= [{}^A e_x^B \ {}^A e_y^B \ {}^A e_z^B] {}^B r_{AP}$$

$$= C_{AB} {}^B r_{AP}$$

$$\therefore {}^A r_{AP} = C_{AB} {}^B r_{AP}$$

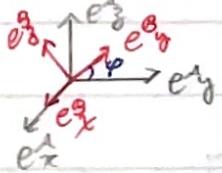
$$\Delta r_{AP} = C_{AB} \otimes r_{Ap}$$

$$C_{AB} = C_{AB}^{-1} = C_{AB}^T \quad (\text{SO}(3))$$

matrix is orthogonal

Above shows passive rotation (different frame)
↓
active (same frame)

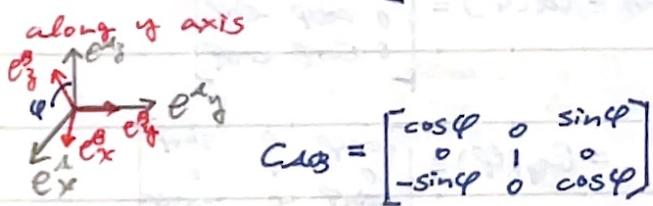
Elementary Rotation = $C_{AB} = [e_x^B \ e_y^B \ e_z^B]$
along X axis



$$C_{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

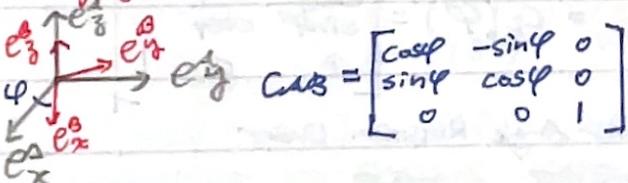
↳ unit vector expressed in other frame (B in A frame)

along y axis



$$C_{AB} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

along z axis



$$C_{AB} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ Homogeneous Transformation = $\begin{matrix} \text{rotation} \\ + \text{translation} \end{matrix}$

$$r_{AP} = r_{AB} + r_{BP}$$

$$\Rightarrow \Delta r_{AP} = \Delta r_{AB} + \Delta r_{BP} = \Delta r_{AB} + C_{AB} \otimes r_{BP}$$

$$\Rightarrow \begin{pmatrix} \Delta r_{AP} \\ 1 \end{pmatrix} = \begin{bmatrix} C_{AB} & \Delta r_{AB} \\ 0_{1 \times 3} & 1 \end{bmatrix} \begin{pmatrix} \otimes r_{BP} \\ 1 \end{pmatrix}$$

$$= C_{AB}^T \Delta r_{AB} + r_{AB}$$

$$= -C_{AB} \Delta r_{AB} \quad T^{-1} = \begin{bmatrix} C_{AB}^T & -C_{AB} \Delta r_{AB} \\ 0_{1 \times 3} & 1 \end{bmatrix}$$

$$= -\otimes r_{AB}$$

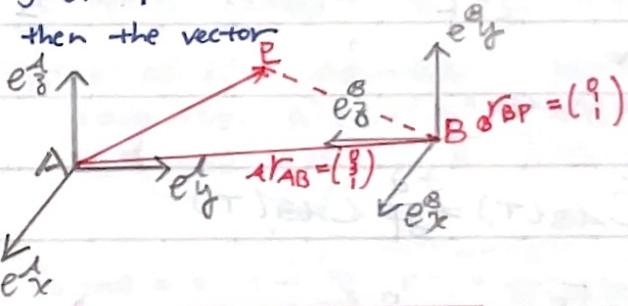
$$= \otimes r_{BA}$$

ex. 1-3

Find Δr_{AP}

find T

then the vectors



rotate along X axis 90° ($\frac{\pi}{2}$)

$$C_{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ \\ 0 & \sin 90^\circ & \cos 90^\circ \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 90^\circ & -\sin 90^\circ & 3 \\ 0 & \sin 90^\circ & \cos 90^\circ & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta r_{AP} = T_{AB} \otimes r_{BP}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\Delta r_{AP} = [0, 2, 1]^T$$

angular velocities

ω_{WAB} := relative rotational velocity of B w.r.t. A in A frame

$$\therefore \omega_{WAB} = -\omega_{WA}$$

given Rotation Matrix C_{AB} (+)

angular velocity is

$$\omega_{WAB} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

where

skew-symmetric matrix

$$[\omega_{WAB}]_x = \begin{bmatrix} 0 & -\omega_y & \omega_z \\ \omega_y & 0 & -\omega_x \\ -\omega_z & \omega_x & 0 \end{bmatrix} = C_{AB} C_{AB}^T$$

$$\omega_{WAB} = C_{AB} \omega_{WAB}$$

ex. 1-4

$$\text{given } C_{AB}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & -\sin(\alpha(t)) \\ 0 & \sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix}$$

determine

$$1 \omega_{AB}$$

$$\dot{C}_{AB}(t) = \frac{\partial}{\partial t} C_{AB}(t)$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin(\alpha(t)) \dot{\alpha}(t) & \cos(\alpha(t)) \dot{\alpha}(t) \\ 0 & \cos(\alpha(t)) \dot{\alpha}(t) & -\sin(\alpha(t)) \dot{\alpha}(t) \end{bmatrix}$$

$$C_{AB}^T(t)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha(t)) & -\sin(\alpha(t)) \\ 0 & \sin(\alpha(t)) & \cos(\alpha(t)) \end{bmatrix}$$

$$C_{AB}(t) \cdot C_{AB}^T(t)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore 1 \omega_{AB} = \begin{pmatrix} -\dot{\alpha}(t) \\ 0 \\ 0 \end{pmatrix}$$

Parameterization of 3D rotation & Quaternion

now we try to parameterize rotation, similar to position (\mathbb{R}^3)

$$\text{Rotation Matrix } C_{AB} = [c_x^B \ c_y^B \ c_z^B]$$

Orthonormality

Parameterization

Euler Angle

Angle Axis

Rotation Vector

Quaternions

} singularity issue
→ non-singular

Euler Angle

Recall: Elementary Rotation

$$C_{AB} = C_x(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{bmatrix}$$

$$= C_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$= C_z(\psi) = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

△ Euler Angle Rotation Order

 ZYZ & ZXZ proper Euler Angle ZYX Tait-Bryan Angle UAV Airplanes
 XYZ Cardan Angle Human Apps.

ZYZ

$$C_{AB} = C_{AB}(Z_1, \text{ Euler-ZYZ})$$

$$= C_{AB}(Z_1) C_{BC}(Y) C_{CD}(Z_2)$$

$$\Rightarrow \omega = C_{AB} \omega$$

CAB

$$= \begin{bmatrix} \cos \beta_1 & -\sin \beta_1 & 0 \\ \sin \beta_1 & \cos \beta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \gamma & 0 & \sin \gamma \\ 0 & 1 & 0 \\ \sin \gamma & 0 & -\cos \gamma \end{bmatrix} \begin{bmatrix} \cos \beta_2 & -\sin \beta_2 & 0 \\ \sin \beta_2 & \cos \beta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos \beta_1 \cos \beta_2 - \sin \beta_1 \sin \beta_2 & -\cos \beta_1 \sin \beta_2 + \cos \beta_2 \sin \beta_1 & \cos \beta_1 \cos \beta_2 + \sin \beta_1 \sin \beta_2 \\ \cos \beta_1 \sin \beta_2 + \cos \beta_2 \sin \beta_1 & \cos \beta_2 \cos \beta_1 - \cos \beta_1 \sin \beta_2 & \cos \beta_1 \sin \beta_2 - \cos \beta_2 \sin \beta_1 \\ -\cos \beta_2 \sin \beta_1 & \sin \beta_2 \cos \beta_1 & \sin \beta_1 \cos \beta_2 \end{bmatrix} \begin{bmatrix} C_3 & S_3 \\ S_3 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \quad (\mathbf{X} \rightarrow \mathbf{C})$$

get rotation from γ

$$\chi_{R, \text{Euler } ZYX} = \begin{pmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_0 \\ \tilde{\gamma}_2 \end{pmatrix} \quad \text{Def } \tilde{\gamma} \text{ from rotation}$$

$$= \begin{pmatrix} \arctan 2(C_{23}, C_{13}) \\ \arctan 2(\sqrt{C_{13}^2 + C_{23}^2} - C_{33}) \\ \arctan 2(-C_{32}, -C_{11}) \end{pmatrix} \quad (C \rightarrow \tilde{\gamma})$$

ZYX

$$C_{AD} = C_{AB}(\tilde{\gamma}) C_{AC}(y) C_{BC}(x)$$

$$= C_{AB}(\tilde{\gamma}) C_{AB}(y) C_{AB}(x)$$

$$= \begin{bmatrix} \cos \tilde{\gamma} & -\sin \tilde{\gamma} & 0 \\ \sin \tilde{\gamma} & \cos \tilde{\gamma} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ \sin y & 0 & \cos y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{bmatrix}$$

$$= \begin{bmatrix} C_0 C_\theta & C_0 S_\theta S_y - C_2 S_\theta & S_x S_\theta + C_x C_\theta S_y \\ C_0 S_\theta & C_0 C_\theta + S_x S_y S_\theta & C_2 S_\theta S_y - C_\theta S_x \\ -S_y & C_y S_x & C_0 C_y \end{bmatrix} \quad \begin{matrix} x \\ y \\ z \end{matrix}$$

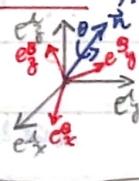
$$\chi_{R, \text{Euler } ZYX} = \begin{pmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_0 \\ \tilde{\gamma}_2 \end{pmatrix} = \begin{pmatrix} \arctan 2(C_{21}, C_{11}) \\ \arctan 2(-C_{31}, \sqrt{C_{22}^2 + C_{32}^2}) \\ \arctan 2(C_{32}, C_{11}) \end{pmatrix} \quad \begin{matrix} x \\ y \\ z \end{matrix}$$

Angle axis

$$\chi_{R, \text{Angle axis}} = (\theta) \quad \begin{matrix} \text{rotation angle} \\ \text{rotation axis} \end{matrix}$$

$$\text{Rotation vector} = \varphi = \theta \cdot n \quad (R^3)$$

a.k.a Euler vectors



$$C_{AB}(\theta, n) = \cos \theta \cdot I_{3 \times 3} - \sin \theta [n]_x + (1 - \cos \theta)n n^T$$

$$\Rightarrow \begin{bmatrix} n_x^2(1 - \cos \theta) + \cos \theta & n_x n_y (1 - \cos \theta) - n_y n_z & n_x n_z (1 - \cos \theta) + n_y n_z \\ n_x n_y (1 - \cos \theta) + n_y n_z & n_y^2 (1 - \cos \theta) + \cos \theta & n_y n_z (1 - \cos \theta) - n_x n_z \\ n_x n_z (1 - \cos \theta) - n_y n_z & n_y n_z (1 - \cos \theta) + n_x n_z & n_z^2 (1 - \cos \theta) + \cos \theta \end{bmatrix}$$

 $(\tilde{\gamma} \rightarrow C)$

$$\theta = \cos^{-1} \left(\frac{C_{11} + C_{22} + C_{33} - 1}{2} \right)$$

$$n = \frac{1}{2 \sin \theta} \begin{pmatrix} C_{32} - C_{23} \\ C_{13} - C_{31} \\ C_{21} - C_{12} \end{pmatrix}$$

singularity

Unit Quaternions

complex numbers in 4D $\tilde{\gamma} = \tilde{\gamma}_0 + \tilde{\gamma}_1 i + \tilde{\gamma}_2 j + \tilde{\gamma}_3 k$ Hamiltonian convention $j^2 = j^2 = k^2 = ijk = -1$

$$\chi_{R, \text{quat}} = \tilde{\gamma} = \begin{pmatrix} \tilde{\gamma}_0 \\ \tilde{\gamma} \end{pmatrix} \in H$$

$$\text{Scalar Real part } \tilde{\gamma}_0 = \cos \frac{\|\varphi\|}{2} = \cos \left(\frac{\theta}{2} \right)$$

$$\text{Imaginary } \tilde{\gamma} = \sin \frac{\|\varphi\|}{2} \cdot \frac{\varphi}{\|\varphi\|} = \sin \left(\frac{\theta}{2} \right) n = \begin{pmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \\ \tilde{\gamma}_3 \end{pmatrix}$$

$$\text{Unitary Constraint } \tilde{\gamma}_0^2 + \tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 + \tilde{\gamma}_3^2 = 1$$

$$\text{Inverse } \tilde{\gamma} \left(\begin{pmatrix} \tilde{\gamma} \end{pmatrix} \right) \leftrightarrow \tilde{\gamma}^{-1} = \begin{pmatrix} \tilde{\gamma}_0 \\ -\tilde{\gamma} \end{pmatrix}$$

$$\text{Identity } \tilde{\gamma} = (1 \ 0 \ 0 \ 0)^T \quad (\theta = \theta)$$

Rotation Matrix

$$C_{AD} = I_{3 \times 3} + 2 \tilde{\gamma}_0 [\tilde{\gamma}]_x + 2 [\tilde{\gamma}]_x^2$$

$$= (2 \tilde{\gamma}_0^2 - 1) I_{3 \times 3} + 2 \tilde{\gamma}_0 [\tilde{\gamma}]_x + 2 \tilde{\gamma} \tilde{\gamma}^T$$

$$\begin{bmatrix} \tilde{\gamma}_0^2 + \tilde{\gamma}_1^2 - \tilde{\gamma}_2^2 - \tilde{\gamma}_3^2 & 2 \tilde{\gamma}_1 \tilde{\gamma}_2 - 2 \tilde{\gamma}_0 \tilde{\gamma}_3 & 2 \tilde{\gamma}_0 \tilde{\gamma}_2 + 2 \tilde{\gamma}_1 \tilde{\gamma}_3 \\ 2 \tilde{\gamma}_0 \tilde{\gamma}_3 + 2 \tilde{\gamma}_1 \tilde{\gamma}_2 & \tilde{\gamma}_0^2 - \tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 - \tilde{\gamma}_3^2 & 2 \tilde{\gamma}_2 \tilde{\gamma}_3 - 2 \tilde{\gamma}_0 \tilde{\gamma}_1 \\ 2 \tilde{\gamma}_1 \tilde{\gamma}_3 - 2 \tilde{\gamma}_0 \tilde{\gamma}_2 & 2 \tilde{\gamma}_2 \tilde{\gamma}_1 + 2 \tilde{\gamma}_3 \tilde{\gamma}_0 & \tilde{\gamma}_0^2 - \tilde{\gamma}_1^2 - \tilde{\gamma}_2^2 + \tilde{\gamma}_3^2 \end{bmatrix}$$

 $\tilde{\gamma} \rightarrow C$

$$\chi_{R, \text{quat}} = \tilde{\gamma} = \frac{1}{2} \begin{pmatrix} C_{11} + C_{22} + C_{33} + 1 \\ \text{sgn}(C_{22} - C_{33}) \sqrt{C_{11} - C_{22} - C_{33} + 1} \\ \text{sgn}(C_{13} - C_{31}) \sqrt{C_{22} - C_{33} - C_{11} + 1} \\ \text{sgn}(C_{21} - C_{12}) \sqrt{C_{33} - C_{11} - C_{22} + 1} \end{pmatrix}$$

 $C \rightarrow \tilde{\gamma}$

Quaternions Algebra

$$q \otimes p = (q_0 + q_1 i + q_2 j + q_3 k)(p_0 + p_1 i + p_2 j + p_3 k)$$

$$= q_0 p_0 + q_0 p_1 i + q_0 p_2 j + q_0 p_3 k$$

$$+ q_1 p_0 + q_1 p_1 i + q_1 p_2 j + q_1 p_3 k$$

$$+ q_2 p_0 + q_2 p_1 i + q_2 p_2 j + q_2 p_3 k$$

$$+ q_3 p_0 + q_3 p_1 i + q_3 p_2 j + q_3 p_3 k$$

$$\begin{pmatrix} i^2 = j^2 = k^2 = ijk = -1 \\ ij = -ji = -ijk^2 = k \\ jk = -kj = i \\ ki = -ik = j \end{pmatrix}$$

$$= q_0 p_0 - q_1 p_1 - q_2 p_2 - q_3 p_3$$

$$+ (q_0 p_1 + q_1 p_0 + q_2 p_3 - q_3 p_2) i$$

$$+ (q_0 p_2 - q_1 p_3 + q_2 p_0 + q_3 p_1) j$$

$$+ (q_0 p_3 + q_1 p_2 - q_2 p_1 + q_3 p_0) k$$

$$= \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$= \begin{bmatrix} q_0 & -\tilde{\gamma}^T \\ \tilde{\gamma} & q_0 I + [\tilde{\gamma}]_x \end{bmatrix} p = m_q(p)$$

$:= m_{\tilde{\gamma}}(p)$

$$= \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{bmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

$:= m_r(p)$

$$= \begin{bmatrix} p_0 & -\tilde{p}^T \\ \tilde{p} & p_0 I - [\tilde{p}]_x \end{bmatrix} \tilde{\gamma} = m_r(p) \tilde{\gamma}$$

$$(W_1 + \vec{V}_1) (W_2 + \vec{V}_2)$$

$$= (W_1 W_2 - \vec{V}_1 \cdot \vec{V}_2, W_1 \vec{V}_2 + W_2 \vec{V}_1 + \vec{V}_1 \times \vec{V}_2)$$

using Quaternion to rotate a vector

$$\mathbf{B}^r = \mathbf{C}_{BI} \mathbf{I}^r$$

write \mathbf{I}^r into Quaternion (part of \mathbf{g})

$$\mathbf{P}(\mathbf{I}^r) = \begin{pmatrix} 0 \\ \mathbf{I}^r \end{pmatrix} \quad \text{unit quaternion}$$

$$\therefore \mathbf{P}(\mathbf{B}^r) = \mathbf{C}_{BI} \otimes \mathbf{P}(\mathbf{I}^r) \otimes \mathbf{C}_{BI}^T$$

$$\mathbf{B}^r = \mathbf{C}_{BI} \mathbf{I}^r$$

$$\mathbf{P}(\mathbf{B}^r) = \mathbf{I} \otimes \mathbf{P}(\mathbf{I}^r) \otimes \mathbf{I}^T$$

$$= m_r(\mathbf{I}) m_r(\mathbf{I}^T) (\mathbf{I}^r)$$

$$\Rightarrow \begin{pmatrix} 0 \\ \mathbf{B}^r \end{pmatrix} = \begin{bmatrix} \mathbf{I}_0 & -\mathbf{I}^T \\ \mathbf{I} & \mathbf{I}_0 \mathbf{I} + [\mathbf{I}]_x \end{bmatrix} \begin{bmatrix} \mathbf{I}_0 & \mathbf{I}^T \\ -\mathbf{I} & \mathbf{I}_0 \mathbf{I} + [\mathbf{I}]_x \end{bmatrix} \begin{pmatrix} 0 \\ \mathbf{I}^r \end{pmatrix}$$

$$= \begin{bmatrix} \mathbf{I}_0^2 + |\mathbf{I}|^2 = 1 & 0 \\ \mathbf{I}_0 \mathbf{I} - \mathbf{I} \mathbf{I}_0 - [\mathbf{I}]_x \mathbf{I} & \mathbf{I}^T \mathbf{I} - \mathbf{I}_0 \mathbf{I}^T - \mathbf{I}^T [\mathbf{I}]_x \end{bmatrix}$$

$$\mathbf{M}_r(\mathbf{I}) = \begin{bmatrix} \mathbf{I}_0 & -\mathbf{I}^T \\ \mathbf{I} & \mathbf{I}_0 \mathbf{I} - [\mathbf{I}]_x \end{bmatrix} \cdot \begin{pmatrix} 0 \\ \mathbf{I}^r \end{pmatrix}$$

$$\begin{aligned} [\mathbf{I}^T]_x &= -[\mathbf{I}]_x & \mathbf{I}^{-1} &= \mathbf{I}^T = \begin{pmatrix} \mathbf{I}_0 \\ -\mathbf{I} \end{pmatrix} \\ & \quad -\mathbf{I}^T [\mathbf{I}]_x & & \quad \begin{aligned} & \quad \mathbf{I}^T [\mathbf{I}]_x & & \quad \mathbf{I}^T - |\mathbf{I}|^2 \mathbf{I} \\ & = -\mathbf{I}^T (-[\mathbf{I}^T]_x) & & = \mathbf{I}^T \mathbf{I} - |\mathbf{I}|^2 \mathbf{I} \\ & = 0 & & = 0 \end{aligned} \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & (\mathbf{I}_0^2 - |\mathbf{I}|^2) \mathbf{I} + 2\mathbf{I}_0 [\mathbf{I}]_x + 2\mathbf{I}^T \mathbf{I} \end{bmatrix}$$

$$\begin{pmatrix} 0 \\ \mathbf{B}^r \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (\mathbf{I}_0^2 - |\mathbf{I}|^2) \mathbf{I} + 2\mathbf{I}_0 [\mathbf{I}]_x + 2\mathbf{I}^T \mathbf{I} \end{bmatrix} \begin{pmatrix} 0 \\ \mathbf{I}^r \end{pmatrix}$$

$$\therefore \mathbf{B}^r = ((\mathbf{I}_0^2 - |\mathbf{I}|^2) \mathbf{I} + 2\mathbf{I}_0 [\mathbf{I}]_x + 2\mathbf{I}^T \mathbf{I}) \mathbf{I}^r$$

$$C(\mathbf{I})$$

Time Derivatives

$$\mathbf{WAD} \Leftrightarrow \dot{\mathbf{x}}_{AD} \quad \frac{d\mathbf{r}}{dt}$$

$$\text{recall: } \dot{\mathbf{r}} = \mathbf{E}_R(\mathbf{x}) \dot{\mathbf{x}}$$

$$\text{Now: } \mathbf{w}_{AB} = \underbrace{\mathbf{E}_R(\mathbf{x}_R)}_{\text{Find this}} \dot{\mathbf{x}}_R$$

$$\mathbf{w}_{AB} = \mathbf{E}_R(\mathbf{x}_R) \dot{\mathbf{x}}_R$$

Take ZYX Euler angle for example
(A)

$$\mathbf{w}_{AD} = \mathbf{w}_{AB} + \mathbf{w}_{BC} + \mathbf{w}_{CD}$$

$$= \mathbf{w}_{AB} + \mathbf{C}_{AB} \mathbf{B} \mathbf{w}_{BC} + \mathbf{C}_{AB} \mathbf{B} \mathbf{C} \mathbf{w}_{CD}$$

$$= \mathbf{C}_{AB} \mathbf{e}_y^A \cdot \dot{\mathbf{z}} + \mathbf{C}_{AB} \mathbf{e}_y^B \cdot \dot{\mathbf{y}} + \mathbf{C}_{AB} \mathbf{e}_y^C \cdot \dot{\mathbf{x}}$$

$$= \left[\mathbf{C}_{AB} \mathbf{e}_y^A \mathbf{e}_y^B \mathbf{e}_y^C \right] \left(\begin{pmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{y}} \\ \dot{\mathbf{x}} \end{pmatrix} \right) \mathbf{E}_R(\mathbf{x}_R)$$

$$\mathbf{C}_{AB} \mathbf{e}_y^A = \begin{bmatrix} \mathbf{C}_0 & -\mathbf{S}_0 & 0 \\ \mathbf{S}_0 & \mathbf{C}_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\mathbf{S}_0 \\ \mathbf{C}_0 \\ 0 \end{pmatrix}$$

$$\mathbf{C}_{AB} \mathbf{e}_y^B = \begin{bmatrix} \mathbf{C}_0 & -\mathbf{S}_0 & 0 \\ \mathbf{S}_0 & \mathbf{C}_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{C}_y & 0 & \mathbf{S}_y \\ 0 & 1 & 0 \\ -\mathbf{S}_y & 0 & \mathbf{C}_y \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{C}_y \mathbf{C}_0 \\ \mathbf{C}_y \mathbf{S}_0 \\ -\mathbf{S}_y \end{pmatrix}$$

$$\therefore \mathbf{E}_R(\mathbf{x}_R) = \begin{bmatrix} 0 & -\mathbf{S}_0 & \mathbf{C}_y \mathbf{C}_0 \\ 0 & \mathbf{C}_0 & \mathbf{C}_y \mathbf{S}_0 \\ 1 & 0 & -\mathbf{S}_y \end{bmatrix}$$

$$\mathbf{w}_{AB} = \begin{bmatrix} 0 & -\mathbf{S}_0 & \mathbf{C}_y \mathbf{C}_0 \\ 0 & \mathbf{C}_0 & \mathbf{C}_y \mathbf{S}_0 \\ 1 & 0 & -\mathbf{S}_y \end{bmatrix} \dot{\mathbf{x}}_R$$

$$\det(\mathbf{E}_R) = -\cos(\theta) \quad (\text{singularity})$$

Angle Axis (\mathbf{n})

$$\mathbf{E}_R, \text{angle axis} = [\mathbf{n} \sin \theta \mathbf{I} \mathbf{a} \mathbf{g} + (1 - \cos \theta) [\mathbf{n}]_x]$$

$$\mathbf{E}_R^{-1}, \text{angle axis} = \left[-\frac{1}{2} \frac{\sin \theta}{1 - \cos \theta} [\mathbf{n}]_x^2 - \frac{1}{2} [\mathbf{n}]_x \right]$$

Rotation Vector ($\varphi = \theta \mathbf{n}$)

$$\mathbf{E}_R, \text{rotation vector} = \left[\mathbf{I}_{2 \times 3} + [\varphi]_x \left(\frac{1 - \cos \|\varphi\|}{\|\varphi\|^2} \right) + [\varphi]_x^2 \left(\frac{1 - \cos \|\varphi\|}{\|\varphi\|^3} \right) \right]$$

$$\mathbf{E}_R^{-1}, \text{rotation vector} = \left[\mathbf{I}_{3 \times 3} - \frac{1}{2} [\varphi]_x + [\varphi]_x^2 \frac{1}{\|\varphi\|^2} \left(1 - \frac{\|\varphi\| - \sin \|\varphi\|}{2(1 - \cos \|\varphi\|)} \right) \right]$$

Quaternion

$$\mathbf{E}_R, \text{quat} = 2H(\mathbf{I})$$

$$\mathbf{E}_R^{-1}, \text{quat} = \frac{1}{2} H(\mathbf{I})^T \quad \text{w/ } H(\mathbf{I}) = \begin{bmatrix} -\mathbf{I}_3 & [\mathbf{I}]_x + \mathbf{I}_0 \mathbf{I}_{2 \times 3} \\ -\mathbf{I}_3 & \mathbf{I}_0 \mathbf{I}_{2 \times 3} \\ -\mathbf{I}_3 & -\mathbf{I}_2 \mathbf{I}_0 \end{bmatrix}$$

$$= \begin{bmatrix} -\mathbf{I}_3 & \mathbf{I}_0 & -\mathbf{I}_3 & \mathbf{I}_2 \\ -\mathbf{I}_3 & \mathbf{I}_0 & \mathbf{I}_0 & -\mathbf{I}_1 \\ -\mathbf{I}_3 & -\mathbf{I}_2 & \mathbf{I}_1 & \mathbf{I}_0 \end{bmatrix}$$

ex. 2-1

Final Rotation Matrix C_{AB}

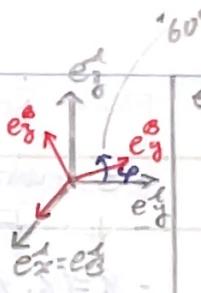
Euler ZYX

Angle Axis

Quaternions

$$C_{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60^\circ & -\sin 60^\circ \\ 0 & \sin 60^\circ & \cos 60^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \times \times$$



Euler ZYX

$$\begin{aligned} \chi_{R, \text{Euler ZYX}} &= \begin{pmatrix} \beta \\ \gamma \\ \alpha \end{pmatrix} = \begin{pmatrix} \arctan 2(C_{21}, C_{11}) \\ \arctan 2(-C_{31}, \sqrt{C_{22}^2 + C_{33}^2}) \\ \arctan 2(C_{22}, C_{33}) \end{pmatrix} \\ &= \begin{pmatrix} \arctan 2(0, 1) \\ \arctan 2(-0, \sqrt{(\sqrt{3}/2)^2 + (1/2)^2}) \\ \arctan 2(\sqrt{3}/2, 1/2) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 60^\circ \end{pmatrix} \times \times \end{aligned}$$

Angle Axis

$$\theta = \cos^{-1} \left(\frac{C_{11} + C_{22} + C_{33} - 1}{2} \right)$$

$$= \cos^{-1} \left(\frac{1 + \cos 60^\circ + \cos 60^\circ - 1}{2} \right)$$

$$= \cos^{-1} (\cos 60^\circ) = 60^\circ \times \times$$

$$n = \frac{1}{2 \sin(\theta)} \begin{pmatrix} C_{22} - C_{33} \\ C_{13} - C_{21} \\ C_{21} - C_{12} \end{pmatrix} = \frac{1}{2 \sin 60^\circ} \begin{pmatrix} \sin 60^\circ & -\sin 60^\circ \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2 \sin 60^\circ} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \times$$

Quaternions

$$\begin{aligned} \chi_{R, \text{Quat}} &= \frac{1}{2} \begin{pmatrix} \sqrt{C_{11} + C_{22} + C_{33} + 1} \\ \operatorname{sgn}(C_{32} - C_{23}) \sqrt{C_{11} - C_{22} - C_{33} + 1} \\ \operatorname{sgn}(C_{13} - C_{31}) \sqrt{C_{22} - C_{33} - C_{11} + 1} \\ \operatorname{sgn}(C_{21} - C_{12}) \sqrt{C_{33} - C_{11} - C_{22} + 1} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \sqrt{1 + \cos 60^\circ + \cos 60^\circ + 1} \\ + \sqrt{1 - \cos 60^\circ - \cos 60^\circ - 1} \\ 0 \cdot \sqrt{\cos 60^\circ - \cos 60^\circ - 1 + 1} \\ 0 \cdot \sqrt{\cos 60^\circ - 1 - \cos 60^\circ + 1} \end{pmatrix} \end{aligned}$$

$$= \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \xi_{AB} \times \times$$

ex. 2-2

Based upon ex. 2-1 $\omega r = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ in A frame
Rotate w/ quaternions to B frame
Rotate directly w/ complex numbers to B frame

$$\begin{aligned} P(Br) &= (\omega r) = \xi_{AB} \otimes P(ar) \otimes \xi_{AB}^T \\ &= M_L(\xi_{AB}) M_R(\xi_{AB}^T) (\omega r) \end{aligned}$$

$$\text{from previous: } \xi_{AB} = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \xi_{AB} = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ -1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \\ 0 \\ 0 \end{pmatrix}$$

$$M_L(\xi) = \begin{bmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{3} & 1 & 0 & 0 \\ -1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & -1 & \sqrt{3} \end{bmatrix}$$

$$\xi_{AB}^T = \xi_{AB}^{-1} = \begin{pmatrix} \xi_0 \\ -\xi_1 \\ -\xi_2 \\ -\xi_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$M_R(\xi) = \begin{bmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & \xi_3 & -\xi_2 \\ \xi_2 & -\xi_3 & \xi_0 & \xi_1 \\ \xi_3 & \xi_2 & -\xi_1 & \xi_0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 & 0 & 0 \\ 1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & -1 & \sqrt{3} \end{bmatrix}$$

$$\begin{aligned} P(Br) &= (\omega r) = \frac{1}{2} \begin{bmatrix} \sqrt{3} & 1 & 0 & 0 \\ -1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & -1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \} \omega r \end{aligned}$$

$$\omega r = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \times \times$$

$$P(Br) = (\omega r) = \xi_{AB} \otimes P(ar) \otimes \xi_{AB}^T \quad \left(\xi_{AB} = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$= \left(\frac{1}{2} (\sqrt{3} - i) \right) (j) \left(\frac{1}{2} (\sqrt{3} + i) \right)$$

$$i^2 = j^2 = k^2 = ijk = -1 \quad = \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right) j \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right)$$

$$ij = -ji = -ijk^2 = k \quad = \left(\frac{\sqrt{3}}{2} j - \frac{i}{2} \right) \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right)$$

$$jk = -kj = 1 \quad = \frac{3}{4} j + \frac{\sqrt{3}}{4} ji - \frac{\sqrt{3}}{4} ij - \frac{ij}{4}$$

$$ki = -ik = j \quad = \frac{3}{4} j - \frac{\sqrt{3}}{4} ij - \frac{ij}{4} - \frac{ji}{4}$$

$$= \frac{3}{4} j - \frac{\sqrt{3}}{2} ij - \frac{j}{4}$$

$$= \frac{1}{2} j - \frac{\sqrt{3}}{2} ij \quad \left(\frac{0}{\sqrt{2}} \right) \times \times$$

Kinematics

Fixed-base system

 n_j joints (revolute, prismatic) $n_l = n_j + 1$ links n_j moving links
1 fixed link

A Robot Arm example

 n moving links: $6n$ parameters n IDoF joints: $5n$ parameters

$6n - 5n = n$ DoFs

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} \in \mathbb{R}^{n_j}$$

complete independent not unique

generalized coordinates

End-effectors

(task space)

$$\boldsymbol{x}_e = \begin{pmatrix} x_{eP} \\ x_{eR} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^m \rightarrow I$$

(kinematic frame)

Operation space coordinates (the "real" DoF) @ end-effectors

$$\boldsymbol{x}_o = \begin{pmatrix} x_{oP} \\ x_{oR} \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \rightarrow \text{DoF @ end-effectors}$$

Ex. 3-1

- 1 Most general robot arm
 2 SCARA robot arm
 3 ANYpulator (4 joints) } get $\boldsymbol{\theta}, m_e, \boldsymbol{x}_e, m_o, \boldsymbol{x}_o$

$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$

$m_e = 6$

$\boldsymbol{x}_e = (x, y, z, \alpha_x, \beta_y, \gamma_z)$

$m_o = 6$

$\boldsymbol{x}_o = (x, y, z, \alpha_x, \beta_y, \gamma_z)$

$\boldsymbol{\theta} = (\alpha, \beta, r, \gamma)$

$m_e = 6$

$\boldsymbol{x}_e = (x, y, z, \alpha_x, \beta_y, \gamma_z)$

$m_o = 4$

$\boldsymbol{x}_o = (x, y, z, \gamma_z)$

$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$

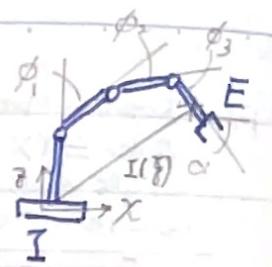
$m_e = 6$

$\boldsymbol{x}_e = (x, y, z, \alpha_x, \beta_y, \gamma_z)$

$m_o = 4$

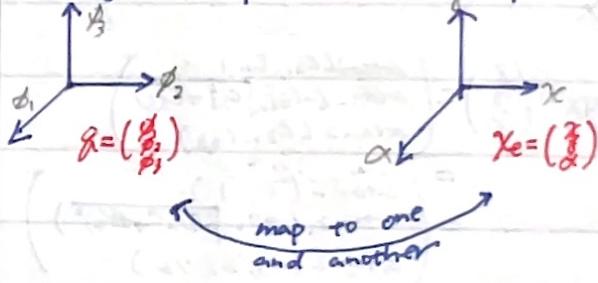
$\boldsymbol{x}_o = \text{hard to pick}$

Planar Robot Arm



- 3 revolute joints
- 1 end-effector
- $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$

- $m_e = 3$
- $\boldsymbol{x}_e = (x, \gamma, \alpha)$

generalize configuration space \leftrightarrow joint space

End-effector

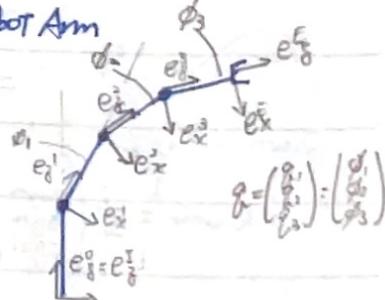
$\boldsymbol{x}_e = \boldsymbol{x}_e(\boldsymbol{\theta}) \in \mathbb{R}^{n_e}$

Transformation Matrix from E to I

$$T_{IE}(\boldsymbol{\theta}) = T_{IO} \cdot \left(\prod_{k=1}^{n_j} T_{k+1 k}(\theta_k) \right) T_{IE}$$

$$= \begin{bmatrix} C_{IE} & I_{IE} \\ 0_{1 \times 3} & I \end{bmatrix}$$

Ex. 3-2

Get the T_{IE} of the Robot Arm RHS.

$T_{IE} = T_{IO} \cdot T_{O1} \cdot T_{12} \cdot T_{23} \cdot T_{3E}$

$T_{IO} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(recall: rotate around y)

$$C_y(\phi) = \begin{bmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{bmatrix}$$

$T_{O1} = \begin{bmatrix} C\phi_1 & 0 & S\phi_1 & 0 \\ 0 & 1 & 0 & 0 \\ -S\phi_1 & 0 & C\phi_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$T_{12} = \begin{bmatrix} C\phi_2 & 0 & S\phi_2 & 0 \\ 0 & 1 & 0 & 0 \\ -S\phi_2 & 0 & C\phi_2 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$T_{23} = \begin{bmatrix} C\phi_3 & 0 & S\phi_3 & 0 \\ 0 & 1 & 0 & 0 \\ -S\phi_3 & 0 & C\phi_3 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$T_{3E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{aligned} l_1 s\phi_1 + l_2 s(\phi_1 + \phi_2) + l_3 s(\phi_1 + \phi_2 + \phi_3) \\ l_1 c\phi_1 + l_2 c(\phi_1 + \phi_2) + l_3 c(\phi_1 + \phi_2 + \phi_3) \end{aligned}$$

(cont'd)

$$\therefore T_{IE} = T_{I0} \cdot T_{01} \cdot T_{12} \cdot T_{23} \cdot T_{3E}$$

$$T_{IE} = \begin{bmatrix} C(\phi_1 + \phi_2 + \phi_3) & 0 & S(\phi_1 + \phi_2 + \phi_3) & x \\ 0 & 1 & 0 & y \\ -S(\phi_1 + \phi_2 + \phi_3) & 0 & C(\phi_1 + \phi_2 + \phi_3) & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = l_1 s\phi_1 + l_2 s(\phi_1 + \phi_2) + l_3 s(\phi_1 + \phi_2 + \phi_3)$$

$$y = 0$$

$$z = l_1 c\phi_1 + l_2 c(\phi_1 + \phi_2) + l_3 c(\phi_1 + \phi_2 + \phi_3)$$

$$\therefore X_{EP}(q) = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} l_1 s\phi_1 + l_2 s(\phi_1 + \phi_2) + l_3 s(\phi_1 + \phi_2 + \phi_3) \\ l_1 c\phi_1 + l_2 c(\phi_1 + \phi_2) + l_3 c(\phi_1 + \phi_2 + \phi_3) \end{pmatrix}$$

$$X_{ER}(q) = (\phi_1 + \phi_2 + \phi_3)$$

Jacobians of a Robot Manipulator

recall Forward Kinematics

$$T_{IE}(q) = \begin{bmatrix} C_{IE}(q) & \Gamma_{IE}(q) \\ 0_{1 \times 3} & 1 \end{bmatrix} \quad X_e = \begin{pmatrix} X_{EP} \\ X_{ER} \end{pmatrix} = V_e(q)$$

Forward Kinematics (Differential)

- Analytic

$$X_e + \delta X_e = X_e(q + \delta q) = X_e(q) + \frac{\partial X_e}{\partial q} \delta q + O(\delta q^2)$$

$$\therefore \delta X_e \approx \frac{\partial X_e}{\partial q} \delta q = J_e(q) \delta q$$

$$J_e = \frac{\partial X_e}{\partial q} = \begin{bmatrix} \frac{\partial X_1}{\partial q_1} & \cdots & \frac{\partial X_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial X_m}{\partial q_1} & \cdots & \frac{\partial X_m}{\partial q_n} \end{bmatrix} \quad \text{recall Matrix Calculus}$$

$$\dot{X}_e = J_e(q) \dot{q}$$

$$R^{m \times n_j}$$

m # end effector state
n_j # joints
(counting hips)

ex. 3-3-1

From last example (ex. 3-2)
determine the analytic Jacobian

$$X_{EP}(q) = (l_1 s\phi_1 + l_2 s(\phi_1 + \phi_2) + l_3 s(\phi_1 + \phi_2 + \phi_3))$$

$$X_{EP}(q) = \phi_1 + \phi_2 + \phi_3$$

$$J_{EP}(q) = \frac{\partial X_{EP}(q)}{\partial q}$$

↑

(cont'd)

$$J_{EP}(q) = \frac{\partial X_{EP}(q)}{\partial q} = \begin{bmatrix} \frac{\partial X_{EP}(q)}{\partial q_1} & \frac{\partial X_{EP}(q)}{\partial q_2} & \frac{\partial X_{EP}(q)}{\partial q_3} \\ \frac{\partial X_{EP}(q)}{\partial q_1} & \frac{\partial X_{EP}(q)}{\partial q_2} & \frac{\partial X_{EP}(q)}{\partial q_3} \\ \frac{\partial X_{EP}(q)}{\partial q_1} & \frac{\partial X_{EP}(q)}{\partial q_2} & \frac{\partial X_{EP}(q)}{\partial q_3} \end{bmatrix}$$

$$J_{ER}(q) = \frac{\partial X_{ER}(q)}{\partial q} = \begin{bmatrix} \frac{\partial X_{ER}(q)}{\partial q_1} & \frac{\partial X_{ER}(q)}{\partial q_2} & \frac{\partial X_{ER}(q)}{\partial q_3} \\ 1 & 1 & 1 \end{bmatrix}$$

$$J_A = \begin{bmatrix} l_1 c\phi_1 + l_2 c(\phi_1 + \phi_2) + l_3 c(\phi_1 + \phi_2 + \phi_3) & l_1 s\phi_1 + l_2 s(\phi_1 + \phi_2) + l_3 s(\phi_1 + \phi_2 + \phi_3) & l_1 c\phi_1 + l_2 c(\phi_1 + \phi_2) + l_3 c(\phi_1 + \phi_2 + \phi_3) \\ l_1 s\phi_1 - l_2 s(\phi_1 + \phi_2) - l_3 s(\phi_1 + \phi_2 + \phi_3) & l_1 c\phi_1 - l_2 c(\phi_1 + \phi_2) - l_3 c(\phi_1 + \phi_2 + \phi_3) & l_1 s\phi_1 - l_2 s(\phi_1 + \phi_2) - l_3 s(\phi_1 + \phi_2 + \phi_3) \end{bmatrix}$$

- Analytic

$$J_A(q) = R^{m \times n_j}$$

depend on parameterization

- Geometric

$$V_e = \begin{pmatrix} V_e \\ W_e \end{pmatrix} = J_{EP}(q) \dot{q}$$

map from generalized to linear & angular velocity

$$J_{EP}(q) = R^{6 \times n_j} \quad \text{always 6!}$$

(recall

$$W_e = E_e(X_e) \dot{X}_e$$

$$J_{EP}(q) = E_e(X_e) J_A(q)$$

$$W_e = (V_e \ W_e) = W_B + W_{BC}$$

$$\left. \begin{array}{l} J_C \dot{q} = J_B \dot{q} + J_{BC} \dot{q} \\ AJ_C = AJ_B + AJ_{BC} \end{array} \right\} \text{algebraic}$$

Derivation of Geometric Jacobian

recall

$$\alpha r_{AP} = \alpha r_{AB} + \alpha r_{BP} = \alpha r_{AB} + C_{AB} \cdot B r_{BP}$$

Differentiate w/ time

$$V_p = \alpha \dot{r}_{AP} = \alpha \dot{r}_{AB} + C_{AB} B \dot{r}_{BP} + C_{AB} \cdot B \dot{r}_{BP}$$

recall

$$[A \omega_{AB}]_x = C_{AB} C_{AB}^T$$

$$C_{BD} = C_{AB}^{-1} = C_{AB}^T$$

$$\therefore C_{AB} = [A \omega_{AB}]_x \cdot (C_{AB}^T)^{-1}$$

$$= [A \omega_{AB}]_x \cdot C_{AB}$$

$$\therefore \alpha \dot{r}_{AP} = \alpha \dot{r}_{AB} + C_{AB} B \dot{r}_{BP} + [A \omega_{AB}]_x C_{AB} B \dot{r}_{BP}$$

$$= \alpha \dot{r}_{AB} + [A \omega_{AB}]_x \alpha r_{BP}$$

$$= \alpha \dot{r}_{AB} + A \omega_{AB} \times \alpha r_{BP}$$

$$\therefore V_p = V_B + \sum L \times r_{BP} \quad \text{in A frame}$$

(continued)

$$v_p = v_B + \sum r_{Bk} \times v_{Bk} \text{ (linear). From above}$$

$$\dot{r}_{Ik} = \dot{r}_{I(k-1)} + \omega_{I(k-1)} \times r_{(k-1)k}$$

omega
of that
body

consecutively

$$\dot{r}_{IE} = \sum_{k=1}^n \omega_{Ik} \times r_{k(n+1)} \text{ (linear)}$$

as for angular

$$\begin{aligned} \omega_{Ik} &= \omega_{I(k-1)} + \omega_{(k-1)k} \\ &= n_k \dot{\theta}_k - \text{velocity of the generalize coordinate} \\ &\quad \text{Normal vector of the joint} \end{aligned}$$

$$\therefore \omega_{IE} = \sum_{i=1}^n n_i \dot{\theta}_i \text{ (angular).}$$

$$\begin{aligned} \therefore \dot{r}_{IE} &= \sum_{k=1}^n \omega_{Ik} \times r_{k(n+1)} \\ &= \sum_{k=1}^n \left\{ \sum_{i=1}^k (n_i \dot{\theta}_i) \times r_{k(n+1)} \right\} \\ &= n_1 \dot{\theta}_1 \times r_{12} \\ &\quad + (n_1 \dot{\theta}_1 + n_2 \dot{\theta}_2) r_{23} \\ &\quad + (n_1 \dot{\theta}_1 + n_2 \dot{\theta}_2 + n_3 \dot{\theta}_3) r_{34} \\ &\quad + \dots \\ &\quad + (n_1 \dot{\theta}_1 + n_2 \dot{\theta}_2 + \dots + n_n \dot{\theta}_n) r_{n(n+1)} \\ &= n_1 \dot{\theta}_1 \times (r_{12} + r_{23} + \dots + r_{n(n+1)}) \\ &\quad + n_2 \dot{\theta}_2 \times (r_{23} + \dots + r_{n(n+1)}) \\ &\quad + \dots \\ &\quad + n_n \dot{\theta}_n \times (r_{n(n+1)}) \end{aligned}$$

$$= \sum_{k=1}^n n_k \dot{\theta}_k \times r_{k(n+1)}$$

$$\dot{r}_{IE} = \sum_{k=1}^n n_k \dot{\theta}_k r_{k(n+1)}$$

velocity of the end effector \rightarrow generalize coordinate \rightarrow could map from $\theta \rightarrow \omega!$

$$\dot{r}_{IE} = [n_1 \times r_{1(n+1)} \ n_2 \times r_{2(n+1)} \ \dots \ n_n \times r_{n(n+1)}] \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{pmatrix}$$

JeoP

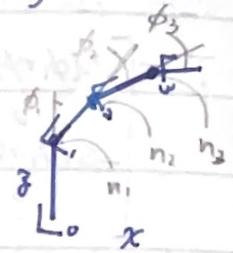
$$\omega_{IE} = \sum_{i=1}^n n_i \dot{\theta}_i = [n_1 \ n_2 \ \dots \ n_n] \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{pmatrix}$$

JeoR

ex. 3-3-2

Determine the Jacobian (geometric) RHS.

$$J_{eo} = \begin{bmatrix} I^{n_1} \times r_{1(n+1)} & I^{n_2} \times r_{2(n+1)} & \dots & I^{n_n} \times r_{n(n+1)} \\ I^{n_1} & I^{n_2} & \dots & I^{n_n} \end{bmatrix}$$



$$J_{eoP} \in \mathbb{R}^{3 \times 3}$$

$$J_{eoR} \in \mathbb{R}^{3 \times 3}$$

$$\begin{aligned} I^{n_1} &= {}^o n_1 = {}^z n_1 = {}^z e_y \\ I^{n_2} &= C_{11} {}^o n_2 = {}^z e_y \\ I^{n_3} &= C_{12} {}^o n_3 = {}^z e_y \end{aligned} \quad \left\{ \begin{array}{l} (0) \\ (0) \end{array} \right.$$

$$I^{r_{IE}} = I^{r_{12}} + I^{r_{23}} + I^{r_{3E}}$$

$$\begin{aligned} &= C_{11} \cdot I^{r_{12}} + C_{12} \cdot I^{r_{23}} + C_{13} \cdot I^{r_{3E}} \\ &= l_1 \begin{pmatrix} \cos \theta_1 \\ 0 \\ \sin \theta_1 \end{pmatrix} + l_2 \begin{pmatrix} \cos(\theta_1 + \theta_2) \\ 0 \\ \sin(\theta_1 + \theta_2) \end{pmatrix} + l_3 \begin{pmatrix} \cos(\theta_1 + \theta_2 + \theta_3) \\ 0 \\ \sin(\theta_1 + \theta_2 + \theta_3) \end{pmatrix} \end{aligned}$$

$$I^{r_{2E}} = I^{r_{23}} + I^{r_{3E}}$$

$$= C_{12} \cdot I^{r_{23}} + C_{13} \cdot I^{r_{3E}}$$

$$= l_2 \begin{pmatrix} \cos(\theta_1 + \theta_2) \\ 0 \\ \sin(\theta_1 + \theta_2) \end{pmatrix} + l_3 \begin{pmatrix} \cos(\theta_1 + \theta_2 + \theta_3) \\ 0 \\ \sin(\theta_1 + \theta_2 + \theta_3) \end{pmatrix}$$

$$I^{r_{3E}} = I^{r_{3E}}$$

$$= l_3 \begin{pmatrix} \cos(\theta_1 + \theta_2 + \theta_3) \\ 0 \\ \sin(\theta_1 + \theta_2 + \theta_3) \end{pmatrix}$$

assume $l_1 = l_2 = l_3 = l$

$$\therefore J_{eo} =$$

$$\begin{bmatrix} l \cos \theta_1 + l \cos(\theta_1 + \theta_2) + l \cos(\theta_1 + \theta_2 + \theta_3) & l \cos(\theta_1 + \theta_2) + l \cos(\theta_1 + \theta_2 + \theta_3) & l \cos(\theta_1 + \theta_2 + \theta_3) \\ 0 & 0 & 0 \\ l \sin \theta_1 - l \sin(\theta_1 + \theta_2) - l \sin(\theta_1 + \theta_2 + \theta_3) & -l \sin(\theta_1 + \theta_2) - l \sin(\theta_1 + \theta_2 + \theta_3) & -l \sin(\theta_1 + \theta_2 + \theta_3) \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ex. 3-3-3

Given

$$\dot{r}_p = \begin{pmatrix} l_1 C_1 \dot{\theta}_1 + l_1 C_2 (\dot{\theta}_1 + \dot{\theta}_2) + l_1 C_{23} (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) \\ -l_1 S_1 \dot{\theta}_1 - l_1 S_{12} (\dot{\theta}_1 + \dot{\theta}_2) - l_1 S_{123} (\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3) \end{pmatrix}$$

$$\omega = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$$

determine Jacobian (geometric)

$$J_{eo} = \begin{bmatrix} J_{eoP} \\ J_{eoR} \end{bmatrix} = \begin{bmatrix} I^{n_1} \times r_{1(n+1)} & I^{n_2} \times r_{2(n+1)} & \dots & I^{n_n} \times r_{n(n+1)} \\ I^{n_1} & I^{n_2} & \dots & I^{n_n} \end{bmatrix}$$

$$I^{n_k} = C_{I(k-1)} \cdot (k-1) n_k$$

