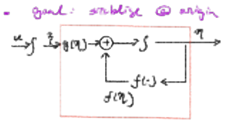
△ Backstepping Control

- $\dot{\eta} = f(\eta) + g(\eta)\xi$ · f, g smooth in D
  $\dot{\xi} = u$ · f, g are known
- Goal: stabilize @ origin



△ Step 1

$\dot{\eta} = f(\eta) + g(\eta)\xi$

- consider ξ as input
- find "smooth" feedback control law

$\xi = \phi(\eta)$
$\phi(0) = 0$

s.t. $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$

has asymptotically stable @ origin

- $V(\eta)$ Lyapunov func.

$\dot{V}(\eta) = \frac{\partial V}{\partial \eta}\frac{\partial \eta}{\partial t}$

$= \frac{\partial V}{\partial \eta}\left(f(\eta) + g(\eta)\phi(\eta) \right)$

$\leq -W(\eta)$

△ Step 2

$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) + g(\eta)[\xi - \phi(\eta)]$
$\dot{\xi} = u$

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let $z = \xi - \phi(\eta)$

$\Rightarrow \dot{\eta} = f(\eta) + g(\eta)\phi(\eta) + g(\eta) z$
$\dot{z} = \dot{\xi} - \dot{\phi}(\eta)$
$= u - \dot{\phi}(\eta)$

$\Rightarrow$ let $u - \dot{\phi}(\eta) = v$

$\Rightarrow \dot{\eta} = f(\eta) + g(\eta)\phi(\eta) + g(\eta) z$
$\dot{z} = v$ ← find this s.t. η is asym. stable

Define $V_c = V(\eta) + \frac{1}{2} z^2$

$\Rightarrow \dot{V_c} = \frac{\partial V}{\partial \eta}\dot{\eta} + z\dot{z}$

$= \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta) + g(\eta)z] + zv$

$= \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] + z[v + \frac{\partial V}{\partial \eta} g(\eta)]$

$\leq -W(\eta) + z[v + \frac{\partial V}{\partial \eta} g(\eta)]$

when v let $\dot{V_c} \leq 0$?

$\therefore v = -\frac{\partial V}{\partial \eta} g(\eta) - kz$

$\Rightarrow \leq -W(\eta) - kz^2$

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recall $u - \dot{\phi}(\eta) = v$

$\Rightarrow u = v + \dot{\phi}(\eta)$

$\Rightarrow u = -\frac{\partial V}{\partial \eta} g(\eta) - kz + \dot{\phi}(\eta)$

$\Rightarrow u = -\frac{\partial V}{\partial \eta} g(\eta) - kz + \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]$

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e.g. $\dot{x}_1 = x_1^2 - x_1^3 + x_2$
$\dot{x}_2 = u$

① $\dot{x}_1 = x_1^2 - x_1^3 + x_2$
$x_2 = \phi(x_1) = -x_1^2 - x_1$

$\Rightarrow \dot{x}_1 = -x_1 - x_1^3$

$\Rightarrow V(x_1) = \frac{1}{2}x_1^2$

$\dot{V}(x_1) = x_1\dot{x}_1 = -x_1^2 - x_1^4 < 0$

$\therefore \dot{V}(\eta) \leq -W(\eta)$

② $z = \xi - \phi(\eta)$
$= x_2 - (-x_1^2 - x_1)$
$= x_2 + x_1^2 + x_1 \Rightarrow x_2 = z - x_1^2 - x_1$

$\Rightarrow \cdot\ \dot{x}_1 = x_1^2 - x_1^3 + x_2$
$= x_1^2 - x_1^3 + z - x_1^2 - x_1$
$= -x_1 - x_1^3 + z$

$\cdot\ \dot{z} = \dot{\xi} - \dot{\phi}(\eta)$
$= \dot{x}_2 - \dot{\phi}(\eta)$
$= u - \dot{\phi}(x_1)$

$\Rightarrow \dot{x}_1 = -x_1 - x_1^3 + z$
$\dot{z} = v$

③ $V_c = \frac{1}{2}x_1^2 + \frac{1}{2}z^2$

$\therefore u = -\frac{\partial V}{\partial \eta} g(\eta) - kz + \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]$

$\therefore u = -\frac{\partial V}{\partial x_1} \cdot 1 - kz + \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2)$

$\frac{\partial V}{\partial x_1} = \frac{\partial}{\partial x_1}(\frac{1}{2}x_1^2) = x_1$ $\quad$ $\frac{\partial \phi}{\partial x_1} = \frac{\partial}{\partial x_1}(-x_1^2 - x_1) = -2x_1 - 1$

$= -x_1 - kz + (-2x_1 - 1)(x_1^2 - x_1^3 + x_2)$
$z = x_2 + x_1^2 + x_1$

$= -x_1 - k(x_2 + x_1^2 + x_1) + (-2x_1 - 1)(x_1^2 - x_1^3 + x_2)$

△ Backstepping : general case

recall
$\begin{cases} \dot{\eta} = f(\eta) + g(\eta)\xi \\ \dot{\xi} = u \end{cases}$ ← let this state become the input of subsystem

① find $\xi = \phi(\eta)$ w/ $\phi(0) = 0$

s.t. $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) \rightarrow$ asym. stable

find $V(\eta)$ s.t.

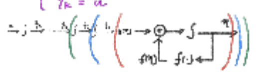
$\frac{\partial V}{\partial \eta}\{ f(\eta) + g(\eta)\xi \} \leq -W(\eta)$

② $z = \xi - \phi(\eta)$
$V_c = V(\eta) + \frac{1}{2}z^2$

③ $u = -\frac{\partial V}{\partial \eta} g(\eta) - kz + \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]$

△ more than 2 states

$\begin{cases} \dot{\eta} = f(\eta) + g(\eta)\xi_1 \\ \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_k = u \end{cases}$



e.g. $\begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u \end{cases}$

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① $\dot{x}_1 = x_1^2 - x_1^3 + \underset{\sim}{x_2}$
$x_2 = \phi_1(x_1) = -x_1^2$

$\dot{x}_1 = -x_1^3$

$V_1(x_1) = \frac{1}{2}x_1^2$

$\dot{V}_1 = -x_1^4 < 0$

② $z_2 = x_2 - \phi_1(x_1) = x_2 + x_1^2$

$\dot{x}_1 = x_1^2 - x_1^3 + x_2$
$= x_1^2 - x_1^3 + z_2 - x_1^2$
$= -x_1^3 + z_2$

$\dot{z}_2 = \dot{x}_2 - \dot{\phi}_1(x_1)$
$= x_3 - \dot{\phi}_1(x_1)$

$V_2(x_1, z_2) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$ $\quad$ $\phi_1(x) = -x_1^2$

$u = -\frac{\partial V}{\partial \eta} g(\eta) - kz + \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]$

$\Rightarrow x_3 = -x_1 - k(x_2 + x_1^2) - 2x_1(x_1^2 - x_1^3 + x_2)$

$\Rightarrow x_3 = -x_1 - x_2 - x_1^2 - 2x_1(x_1^2 - x_1^3 + x_2)$

③ $x_3 = \phi_2(x_1, x_2)$

$\therefore \dot{\eta} = \begin{cases} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = x_3 \end{cases}$

$\dot{\xi} = \{ \dot{x}_3 = u$

$\therefore z_3 = x_3 - \phi_2$

$\therefore \dot{x}_1 = x_1^2 - x_1^3 + x_2$
$\dot{x}_2 = \phi_2 + z_3$
$\dot{z}_3 = \dot{x}_3 - \dot{\phi}_2$
$= u - \dot{\phi}_2$

$\therefore V_3 = V_2 + \frac{1}{2}z_3^2$

$u = -\frac{\partial V}{\partial \eta} g(\eta) - kz + \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]$

$\frac{\partial V_2}{\partial \eta} g(\eta) = [\cancel{\partial V_2/\partial x_1} \quad \cancel{\partial V_2/\partial x_2}] \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$V_2 = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$
$= \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1^2)^2$

$= -\frac{\partial V_2}{\partial \eta} g(\eta) - kz_3 + \frac{\partial \phi_2}{\partial \eta}\{ f(\eta) + g(\eta)\xi \}]$

$= -\frac{\partial V_2}{\partial x_2} - kz_3 + [\frac{\partial \phi_2}{\partial x_1} \quad \frac{\partial \phi_2}{\partial x_2}] \begin{bmatrix} x_1^2 - x_1^3 + x_2 \\ x_3 \end{bmatrix}$

$= -(x_2 + x_1^2) - kz_3 + (-1 - 2x_1 - 6x_1^2 + 8x_1^3 - 2x_2)(x_1^2 - x_1^3 + x_2)$

$+ (-1 - 2x_1)x_3$

△ Sliding-mode control

$\dot{x}_1 = x_2$
$\dot{x}_2 = h(x) + g(x)u$

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sliding surface

$s \triangleq ax_1 + x_2 = 0 \qquad a > 0$

$\Rightarrow x_2 = -ax_1$

$\Rightarrow \dot{x}_1 = -ax_1$

$\Rightarrow x_1(t) = x_1(0)e^{-at}$
$x_2(t) = -ax_1(t)$

- order reduced
- exponential convergence rate
- independent of g(x) h(x)

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let $u = -\beta(x)\,\mathrm{sgn}(s)$

$\mathrm{sgn}(s) = \begin{cases} 1 & s > 0 \\ 0 & s = 0 \\ -1 & s < 0 \end{cases}$

$\beta(x) \geq \rho(x) + \beta_0$

$\rho(x) \geq \left| \frac{ax_2 + h(x)}{g(x)} \right|$

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proof

w/ $\dot{x}_1 = x_2$
$\dot{x}_2 = h(x) + g(x)u$
$s = ax_1 + x_2$
$\dot{s} = a\dot{x}_1 + \dot{x}_2$

$u = -\beta(x)\,\mathrm{sgn}(s)$
$\beta(x) \geq \rho(x) + \beta_0$
$\rho(x) \geq \left| \frac{ax_2 + h(x)}{g(x)} \right|$
$g(x) \geq g_0 \geq 0$

$\dot{s} = ax_2 + h(x) + g(x)u$

$V = \frac{1}{2}s^2$

$\dot{V} = s\dot{s}$

$\dot{V} = s[ax_2 + h(x) + g(x)u]$
$= s[ax_2 + h(x)] + sg(x)u$

$\dot{V} \leq |s|\,|ax_2 + h(x)| + sg(x)u$
$\leq |s|\,\rho(x)g(x) + sg(x)u$
$\leq |s|\,\rho(x)g(x) - sg(x)\beta(x)\,\mathrm{sgn}(s)$
$\leq |s|\,\rho(x)g(x) - |s|g(x)(\rho(x) + \beta_0)$
$\leq -\beta_0\, g(x)|s|$
$\leq -\beta_0 g_0 |s|$

$V = \frac{1}{2}s^2$

$\dot{V} \leq -\beta_0 g_0 |s|$

$\frac{dV}{dt} \leq -\beta_0 g_0 \sqrt{2V}$

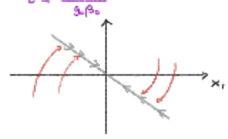
$\frac{1}{\sqrt{V}}dV \leq -\beta_0 g_0 \sqrt{2}\, dt$

$\sqrt{V(s(t))} \leq -\frac{1}{\sqrt{2}}\beta_0 g_0 t + \sqrt{V(s(0))}$

$|s(t)| \leq -\beta_0 g_0 t + |s(0)|$

$0 \leq -\beta_0 g_0 t + |s(0)|$

$t = \frac{|s(0)|}{g_0\beta_0}$



△ Converging Time

△ finite-time stability
converges within
a finite time interval $\forall x_0$

△ fixed-time stability
converges within
a time interval w/ upper bound $\forall x_0$

△ prescribed-time stability
uniformly pre-specifiable
convergence time ⟷
independent to the x_0