- optimization problem

  minimize$_x$ $f_0(x)$

  s.t. $f_i(x) \le 0$

  $h_i(x) = 0$

- feasible

  $x \in \text{dom } f_0$

- infeasible

  $p^* = \infty / -\infty$

- optimal value

  $p^* = \min\{f_0(x) \mid f_i(x) \le 0,\ h_i(x) = 0\}$

  x^* is optimal

- locally optima

  $f_0(x) = \inf\{f_0(z) \mid z \in D,\ \|z-x\|_2 \le R\}$

  $D = \bigcap_{i=0}^{m} \text{dom} f_i \cap \bigcap_{i=0}^{p} \text{dom} h_i$

- feasibility problem

  find x s.t. $f_i(x) \le 0$, $h_i(x) = 0$ $\iff$ minimize$_x$ 0 s.t. $f_i(x) \le 0$, $h_i(x) = 0$

- convex optimization

  minimize$_x$ $f_0(x)$ → convex

  s.t. $f_i(x) \le 0$ → convex

  $a_i^T x = b_i$ → affine

- quasiconvex optimization

  minimize$_x$ $f_0(x)$ ⇝ quasiconvex

  s.t. $f_i(x) \le 0$

  $a_i^T x = b_i$

  $S_\alpha = \{x \in D \mid f(x) \le \alpha\}$

  $\alpha = R$

  S_α = convex

  $f(x) \in$ quasiconvex

- local & global optima

  minimize$_z$ $f_0(z)$

  s.t. $f_i(z) \le 0$

  $h_i(z) = 0$

  $\|z-x\|_2 \le R$

  x is locally optimal when

  $f_0(x) = \inf\{f_0(z) \mid z \text{ feasible}, \|z-x\|_2 \le R\}$

  in convex

  if x is local optimal

  then x is global optimal

  proof:

  △ if $\exists y$ $\left.\begin{array}{l} f_0(y) < f_0(x) \\ \|y-x\|_2 > R \end{array}\right\}$ as if $\|y-x\|_2 < R$ — fall back here

  △ we let $\theta = \dfrac{R}{2\|y-x\|_2} < \frac{1}{2}$

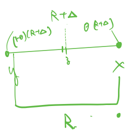
  △ we let

  $z = (1-\theta)x + \theta y$

  convex combination, in convex function

  ∴ $f_0(z) \le (1-\theta) f_0(x) + \theta f_0(y)$

  $0 < \theta < \frac{1}{2}$



  ⇒ from here

  $\|x-z\|_2 = \theta(R+\Delta)$

  $\le \frac{1}{2}(R+\Delta)$

  - we let $\|x-z\|_2 = \frac{1}{2}R$ to fullfil all the condition (we want z to be within $\|x-z\| < R$, s.t. x is local optimal)

  △ w/ assumption $f_0(y) < f_0(x)$

  we also hv

  $\theta f_0(y) + (1-\theta) f_0(x) < f_0(x)$

  ⇒ $f_0(z) = \theta f_0(y) + (1-\theta) f_0(x)$

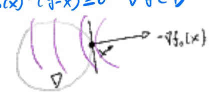
  $< f_0(x)$

  contradicts w/

  $f_0(x) = \inf\{f_0(z) \mid z \in D,\ \|z-x\|_2 \le R\}$

- optimality criterion

  $\nabla f_0(x)^T (y-x) \ge 0 \quad \forall y \in D$



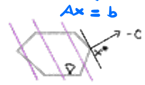
- Convex problems

  - Linear Programming

    minimize$_x$ $c^T x + d$

    s.t. $Gx \preceq h$

    $Ax = b$



  - piecewise-linear minimization

    $f_0(x) = \max_{i=1,\dots,m} (a_i^T x + b_i)$

    minimize$_x$ $f_0(x)$

    $\Updownarrow$

    minimize t

    s.t. $a_i^T x + b_i \le t \quad i = 1, \dots, m$

  - Quadratic Program

    minimize$_x$ $\frac{1}{2} x^T P x + q^T x + r$

    s.t. $Gx \preceq h$

    $Ax = b$

    $P \in S_+^n$

  - Quadratically Constrained Quadratic Program

    minimize$_x$ $\frac{1}{2} x^T P_0 x + q_0^T x + r_0$

    s.t. $\frac{1}{2} x^T P_i x + q_i^T x + r_i \le 0$

    $Ax = b$

    $P_i \in S_+^n$ (cover a feasible region that's intersection of ellipsoids + affine set)

- Second-order Cone Programming

  minimize$_x$ $f^T x$

  $\|A_i x + b_i\|_2 \le c_i^T x + d_i$

  $Fx = g$

  SOCP could be understood:

  $\begin{bmatrix} Ax+b \\ c^T x + d \end{bmatrix} \in R^{k+1}$ lies in SOC

  $(Ax+b,\ c^T x + d) \succeq_Q 0$

  G.I. for cone Q

- e.g. Robust Linear Programming

- Geometric Programming

  - monomial function

    $f: R^n \to R \quad \text{dom} f = R_{++}^n$

    $f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$

    $c > 0,\ a_i \in R$

  - posynomial function

    $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$

    $c_k > 0,\ a_{ik} \in R$

    sum of monomial function

  - transfer to convex problem

    $y_i = \log x_i \quad x_i = e^{y_i}$

    $f(x) = f(e^{y_1}, \dots, e^{y_n})$

    $= c(e^{y_1})^{a_1} \cdots (e^{y_n})^{a_n}$

    $= e^{a^T y + b}$ $(b = \log c)$

    monomial

  original

  minimize $f_0(x)$ — posy

  s.t. $f_i(x) \le 1$

  $h_i(x) = 1$ — mono

  convex equivalence

  minimize $\log\left(\sum_{k=1}^{K} e^{a_{0k}^T y + b_{0k}}\right)$

  s.t. $\log\left(\sum_{k=1}^{K} e^{a_{ik}^T y + b_{ik}}\right) \le 0$

  $Gy + d = 0$

- multicriterion optimization

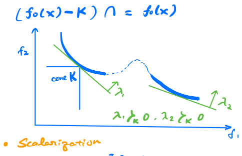
  minimize $f_0(x) = [F_1(x), \dots F_q(x)]$

  s.t. $f_i(x) \le 0$

  $Ax = b$

  - Pareto optimal

    $(f_0(x) - K) \cap = f_0(x)$



  - Scalarization

    minimize $\lambda^T f_0(x)$

    s.t. $f_i(x) \le 0$

    $h_i(x) = 0$

    ⇒ $\forall x^{po}$,

    $\exists \lambda \succ_{K^*} 0$

    s.t. $\tilde{x}$ is an optimal point of the scalarized problem

- Duality

  - The Lagrangian

    minimize $f_0(x)$ → λ_i

    s.t. $f_i(x) \le 0 \quad i = 1, \dots, m$

    $h_i(x) = 0 \quad i = 1, \dots, p$ → ν_i

    $L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$

  - The Lagrange Dual Function

    $g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu)$

    ∴ $\underbrace{g(\lambda, \nu)}_{\text{lower bound}} \le p^* = f_0(\tilde{x})$ → all feasible x

    ($\succeq 0$)

    ∴ $\underbrace{\lambda \succeq 0,\ (\lambda, \nu) \in \text{dom } g}_{\text{dual feasible}}$

  - The Lagrange Dual Problem

    maximize $g(\lambda, \nu)$

    s.t. $\lambda \succeq 0$

- Weak Duality

  $d^* \le p^*$

  best bound, optimal value

  Δ = duality gap

- Strong Duality

  - $d^* = p^*$

  - slater's condition

    $\exists x \in \text{relint } D$

    s.t.

    $f_i(x) < 0,\ Ax = b$

- optimality conditions — complementary slackness

  assume strong duality holds:

  $d^* = p^*$

  $g(\lambda^*, \nu^*) = f_0(x^*)$

  ⇒

  $f_0(x^*) = g(\lambda^*, \nu^*)$

  $= \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x)\right)$

  $\le f_0(x^*) + \sum_{i=1}^{m} \underbrace{\lambda_i^*}_{\succeq 0} \underbrace{f_i(x^*)}_{\le 0} + \sum_{i=1}^{p} \nu_i^* \underbrace{h_i(x^*)}_{= 0}$

  $\le f_0(x^*)$

  $\begin{cases} \text{if } \lambda_i^* > 0 \Rightarrow f_i(x^*) = 0 \\ \text{if } f_i(x^*) < 0 \Rightarrow \lambda_i^* = 0 \end{cases}$

  complementary slackness!

  ∴ $\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = 0$

- KKT optimality conditions

  - let x^* (primal), (λ^*, ν^*) (dual) be optimal points

  - 0-duality gap

  ∴ $x^* = \arg\min_x L(x, \lambda^*, \nu^*)$

  - $\nabla f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^{p} \nu_i^* \nabla h_i(x^*) = 0$

  ∴

  - $f_i(x^*) \le 0$

    $h_i(x^*) = 0$

    $\lambda_i^* \ge 0$

    $\lambda_i^* f_i(x^*) = 0$

  → KKT optimality conditions!

- Perturbation & Sensitivity Analysis

  - recall

    minimize$_x$ $f_0(x)$

    s.t. $f_i(x) \le 0$

    $h_i(x) = 0$

    perturbed by u_i, v_i

    minimize$_x$ $f_0(x)$

    s.t. $f_i(x) \le u_i$

    $h_i(x) = v_i$

  - optimal value (becomes a function) w.r.t. (u, v)

    $p^*(u,v) = \inf\left\{f_0(x) \,\middle|\, \begin{array}{l} \exists x \in D \\ f_i(x) \le u_i \\ h_i(x) = v_i \end{array}\right\}$

- let (λ^*, ν^*) → dual optimal pt. of unperturbed problem

  ⇒ $p^*(0,0) = g(\lambda^*, \nu^*)$

  $\le f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} \nu_i^* h_i(x)$

  $\le f_0(x) + \lambda^{*T} u + \nu^{*T} v$

  ⇒ $p^*(0,0) \le f_0(x) + \lambda^{*T} u + \nu^{*T} v$

  ⇒ $f_0(x) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$

  ⇒ $p^*(u,v) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$

  $(p^*(u,v) \in \{f_0(x) \mid x \in \text{dom} f\})$

  $p^*(u,v) \ge p^*(0,0) - \lambda^{*T} u - \nu^{*T} v$

  perturbation analysis

- local sensitivity

  assume

  - $p^*(u,v)$ differentiable @ $u=0$, $v=0$

  - strong duality holds

  ⇒ $p^*(u,v)$ @ $u=0, v=0$

  $\dfrac{\partial p^*(0,0)}{\partial u_i} = -\lambda_i^*$

  $\dfrac{\partial p^*(0,0)}{\partial v_i} = -\nu_i^*$

- w/ complementary slackness:

  $\lambda_i f_i(x) = 0$

  - $f_i(x) < 0$, $\lambda_i = 0$

    then: value @ $p^*(0,0)$ doesn't change w.r.t. u

  - $f_i(x) = 0$, $\lambda_i^* > 0$

    then: value @ $p^*(0,0)$ changes w.r.t. u

    $\lambda_i^* \uparrow$ → perturb ↑

    $\lambda_i^* \downarrow$ → perturb ↓

  → tells us HOW ACTIVE IT IS

- Sufficiency & Necessity

  if P then Q

  △ Q is necessary for P

  ① truth of Q is guaranteed by truth of P

  ② (falsity of Q guarantees falsity of P)

  ▲ P is sufficient for P

  ③ truth of P guarantees truth of Q

  ④ but falsity of P might still hv truth of Q

  如果"要選總統$_P$"那"需40歲$_Q$"

  e.g.

  ① "人一定是40歲$_Q$"如果"這人選總統$_P$"

  ② 如果"未40歲$_Q$"那"一定非總統$_P$"

  ③ 如果"要選總統$_P$"那"一定40歲$_Q$"

  ④ 如果"不選總統$_P$"那"可能40歲$_Q$"

  sum:

  if P then Q

  △ Q is P's necessary condition

  ▲ P is Q's sufficient condition

- **Numerical Method of Differential Equations**

- **ODE 4 types**

1 separable equations

$$\frac{dy}{dx}=P(x)Q(x)$$

e.g. $y'+2xy=x$

$\Rightarrow \frac{dy}{dx}+2xy=x$

$\Rightarrow \frac{dy}{dx}=x-2xy$

$\Rightarrow \frac{dy}{dx}=x(1-2y)$

$\Rightarrow dy=x(1-2y)dx$

$\Rightarrow \frac{1}{1-2y}dy=x\,dx$

$\Rightarrow \int\frac{1}{1-2y}dy=\int x\,dx$

$\Rightarrow -\frac{1}{2}\ln(1-2y)=\frac{1}{2}x^2+c$

$\Rightarrow e^{-\frac{1}{2}\ln(1-2y)}=e^{-\frac{1}{2}x^2+c}$

$\Rightarrow 1-2y=ce^{-x^2}$

$\Rightarrow y=-\frac{ce^{-x^2}-1}{2}$

2. homogenous method

$$f(kx,ky)=f(x,y)$$

e.g.

$$\frac{dy}{dx}=\frac{x^2+y^2}{xy}$$

check $\frac{k^2x^2+k^2y^2}{kxky}=\frac{x^2+y^2}{xy}$

let $v=\frac{y}{x}$ $\frac{1}{v}=\frac{x}{y}$

$y=vx$

$\frac{dy}{dx}=x\frac{dv}{dv}+v$

- $\frac{dy}{dx}=\frac{\frac{x^2}{x^2}+\frac{y^2}{x^2}}{\frac{xy}{x^2}}=\frac{1+(\frac{y}{x})^2}{\frac{y}{x}}$

$=\frac{1+v^2}{v}$

$\Rightarrow x\frac{dv}{dx}+v=\frac{1+v^2}{v}$

$\Rightarrow x\frac{dv}{dx}+v=\frac{1}{v}+v$

$\Rightarrow x\frac{dv}{dx}=\frac{1}{v}$

$\Rightarrow v\,dv=\frac{1}{x}dx$

$\Rightarrow \int v\,dv=\int\frac{1}{x}dx$

$\Rightarrow \frac{1}{2}v^2=\ln(x)+c$

$\Rightarrow v^2=2\ln|x|+c$

$v=\pm\sqrt{2\ln|x|+c}$

$\frac{y}{x}=\pm\sqrt{2\ln|x|+c}$

$y=\pm x\sqrt{2\ln|x|+c}$

3. Integrating factor

$$\frac{dy}{dx}+P(x)y=Q(x)$$

e.g. $\frac{dy}{dx}+1y=x$ $\quad P(x)=1$

$\Leftrightarrow \frac{dy}{dx}+P(x)y=1$ ①

$\mu(x)=e^{\int P(x)dx}$

$\mu(x)=e^{\int 1dx}$

$=e^x$ ②

$\Rightarrow$ ① & ②

$\mu(x)\left[\frac{dy}{dx}+P(x)y=Q(x)\right]$

$\Rightarrow e^x\left[\frac{dy}{dx}+y=x\right]$

$\Rightarrow e^x\frac{dy}{dx}+ye^x=xe^x$

$\Rightarrow \frac{d}{dx}(e^xy)=xe^x$

$\Rightarrow \int\frac{d}{dx}e^xy=\underbrace{\int xe^xdx}_{IBP}$

$\Rightarrow e^xy=xe^x-e^x+c$

$\Rightarrow y=x-1+\frac{c}{e^x}$

$=x-1+ce^{-x}$

- $\frac{dv}{dt}=g-\frac{c}{m}v$

$$\frac{dv}{dt}+P(t)v=Q(t)$$

$\Rightarrow \frac{dv}{dt}+(\frac{c}{m})v=g$

$\Rightarrow \mu(t)=e^{\int\frac{c}{m}dt}$

$=e^{\frac{c}{m}t}$

$\Rightarrow e^{\frac{c}{m}t}\left(\frac{dv}{dt}+(\frac{c}{m})v=g\right)$

$\Rightarrow e^{\frac{c}{m}t}\frac{dv}{dt}+\frac{c}{m}ve^{\frac{c}{m}t}=ge^{c/m\,t}$

$\Rightarrow \frac{d}{dt}e^{\frac{c}{m}t}v=ge^{c/m\,t}$

$\Rightarrow \int\frac{d}{dt}e^{c/m\,t}v=\int ge^{c/m\,t}dt$

$\Rightarrow e^{c/m\,t}v=\frac{m}{c}ge^{\frac{c}{m}t}+c'$

$\Rightarrow v=e^{-c/m\,t}\left[\frac{m}{c}ge^{\frac{c}{m}t}+c'\right]$

assume $v(0)=0$

$v(0)=+\frac{m}{c}g+c'=0$

$c'=\frac{-m}{c}g$

$\therefore v(t)=e^{-c/m\,t}\left[\frac{m}{c}ge^{-\frac{c}{m}t}-\frac{m}{c}g\right]$

$=e^{-c/m\,t}\frac{m}{c}g\left[e^{-\frac{c}{m}t}-1\right]$

$=\frac{mg}{c}\left[1-e^{-c/m\,t}\right]$

- **linear ODE**
  - Laplace
  - analytically solved
  - $a_0(x)y+a_1(x)y'+\dots+a_n(x)y^{(n)}=b(x)$
  - $Ly=f$

- **Runge - Kutta Methods**

▵ Taylor series

$\frac{dy}{dx}=f(x,y)$

$\frac{d^{(n)}y}{dx^{(n)}}=f^{(n-1)}(x,y)$

$y_{i+1}=\underbrace{y_i+f(x_i,y_i)h}_{\text{Euler's Method}}$

▵ $y_{i+1}=y_i+f(x_i,y_i)h+\frac{f'(x_i,y_i)}{2!}h^2+\dots+\frac{f^{(n-1)}(x_i,y_i)}{n!}h^n+O(h^{n+1})$

$\simeq y_i+\left[f(x_i,y_i)+\frac{f'(x_i,y_i)}{2!}h+\dots+\frac{f^{(n-1)}(x_i,y_i)}{n!}h^{n-1}\right]h$

$=y_i+\phi(x_i,y_i,h)h$

▵ $\phi(x_i,y_i,h)$

$=f(x_i,y_i)+\frac{f'(x_i,y_i)}{2!}h+\dots+\frac{f^{(n-1)}(x_i,y_i)}{n!}h^{n-1}$

▵ Runge-Kutta :

acquire $\phi(x_i,y_i,h)$

w.o/ calculating higher-order term

$y_{i+1}=y_i+\phi(x_i,y_i,h)h$

$\Rightarrow \phi(x_i,y_i,h)$

$=a_1k_1+a_2k_2+\dots+a_{n-1}k_{n-1}+a_nk_n$

$k_1=f(x_i,y_i)$

$k_2=f(x_i+p_1h,\ y_i+q_{11}k_1h)$

$k_3=f(x_i+p_2h,\ y_i+q_{21}k_1h+q_{22}k_2h)$

$\vdots$

$k_n=f(x_i+p_{n-1}h,\ y_i+q_{n-1,1}k_1h+q_{n-1,2}k_2h+q_{n-1,3}k_3h+\dots+q_{n-1,n-1}k_{n-1}h)$

$a,p,q\Rightarrow$ constants

$n=$ order $\Rightarrow$ RK n method!

e.g.

▵ $y=-0.5x^4+4x^3-10x^2+8.5x+1$

$\frac{dy}{dx}=-2x^3+12x^2-20x+8.5$

$f(0)=1$

$f(0.5)=?$

▵ RK4:

$y_{i+1}=y_i+\phi(x_i,y_i,h)h$

$\phi(x_i,y_i,h)$

$=\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$

$k_1=f(x_i,y_i)$

$k_2=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_1h)$

$k_3=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_2h)$

$k_4=f(x_i+h,\ y_i+k_3h)$

▵ $\therefore$

$k_1=f(0,1)=8.5$

$k_2=f(0+0.25,\ 1+0.5\cdot8.5\cdot0.5)=4.21875$

$k_3=f(0+0.25,\ 1+0.5\cdot4.21875\cdot0.5)=4.21875$

$k_4=f(0.5,\ 1+4.21875\cdot0.5)=1.25$

$y(0.5)=1+\frac{0.5}{6}(8.5+2\cdot4.21875+2\cdot4.21875+1.25)$

$=3.21875+O(h^5)$

- **system of equations**

$$\begin{cases}\frac{dy_1}{dx}=f_1(x,y_1,y_2,\dots,y_n)\\ \frac{dy_2}{dx}=f_2(x,y_1,y_2,\dots,y_n)\\ \vdots\\ \frac{dy_n}{dx}=f_n(x,y_1,y_2,\dots,y_n)\end{cases}$$

e.g.

▵ $\begin{cases}\frac{dy_1}{dx}=-0.5y_1\\ \frac{dy_2}{dx}=4-0.3y_2-0.1y_1\end{cases}$

$y_1=4$
$y_2=6$ @ $x=0$

▵ $y'=f(x)$

$$=\begin{bmatrix}\frac{dy_1}{dx}\\ \frac{dy_2}{dx}\end{bmatrix}=\begin{bmatrix}-0.5&0\\-0.1&-0.3\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}+\begin{bmatrix}0\\4\end{bmatrix}$$

▵ RK4

$y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$

$k_1=f(x_i,y_i)$

$k_2=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_1h)$

$k_3=f(x_i+\frac{1}{2}h,\ y_i+\frac{1}{2}k_2h)$

$k_4=f(x_i+h,\ y_i+k_3h)$

▵ $k_{1,1}=f_1(0,4)=-0.5\cdot4=-2$

$k_{1,2}=f_2(0,6)=4-0.3\cdot6-0.1\cdot4=1.8$

▵ $k_{2,1}=f_1(0+\frac{1}{2}\cdot0.5,\ 4+\frac{1}{2}\cdot(-2)\cdot0.5)$

$=-0.5\cdot3.5=-1.75$

$k_{2,2}=f_2(0+\frac{1}{2}0.5,\ 6+\frac{1}{2}\cdot(1.8)\cdot0.5)$

$=4-0.3\cdot6.45-0.1\cdot3.5$

$=1.715$

▵ $k_{3,1}=f_1(0+\frac{1}{2}0.5,\ 4+\frac{1}{2}(-1.75)0.5)$

$=-0.5\cdot3.5625$

$=-1.78125$

$k_{3,2}=f_1(0+\frac{1}{2}\cdot0.25,\ 6+\frac{1}{2}(1.715)\cdot0.5)$

$=4-0.3\cdot6.42875-0.1\cdot3.5625$

$=1.715125$

▵ $k_{4,1}=f_1(0+0.5,\ 4+(-1.78125)\cdot0.5)$

$=-0.5\cdot3.109375$

$=-1.5546875$

$k_{4,2}=f_2(0+0.5,\ 6+(1.715125\cdot0.5))$

$=4-0.3\cdot6.8575625-0.1\cdot3.109375$

$=1.63179375$

▵ $y_{i+1}=y_i+\frac{1}{6}(k_1+2k_2+2k_3+k_4)h$

$y=\begin{bmatrix}4\\6\end{bmatrix}+\frac{0.5}{6}\left(\begin{bmatrix}-2\\1.8\end{bmatrix}+2\begin{bmatrix}-1.75\\1.715\end{bmatrix}+2\begin{bmatrix}-1.78125\\1.715125\end{bmatrix}+\begin{bmatrix}-1.5546875\\1.63179375\end{bmatrix}\right)$

$=\begin{bmatrix}3.115234\\6.875670\end{bmatrix}$ ※

- **solving high-order ODE**

$\begin{cases}\frac{d^2y}{dx^2}+2\frac{dy}{dx}=x+y\\ y=1\quad\frac{dy}{dx}=2\quad\text{when } x=0\end{cases}$

$\begin{cases}\text{let}\quad u=\frac{dy}{dx}\\ \frac{du}{dx}=\frac{d^2y}{dx^2}\end{cases}$

$\begin{cases}\Rightarrow\frac{du}{dx}+2u=x+y\\ \Rightarrow\frac{du}{dx}=x+y-2u\end{cases}$

$\begin{cases}\frac{dy}{dx}=u & y(0)=1\\ \frac{du}{dx}=x+y-2u & u(0)=2\end{cases}$

use this to propagate

$\begin{cases}\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}+3\frac{dy}{dx}=x+y\\ y=1\quad\frac{dy}{dx}=2\quad\frac{d^2y}{dx^2}=3\quad x=0\end{cases}$

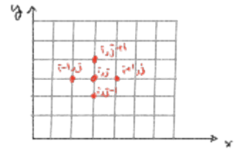
$\begin{cases}\frac{dy}{dx}=u & \frac{d^2y}{dx^2}=\frac{du}{dx}=v\\ \frac{d^2y}{dx^2}=v & \frac{d^3y}{dx^3}=\frac{dv}{dx}\end{cases}$

$\begin{cases}\frac{dv}{dx}+2v+3u=x+y\\ \frac{dy}{dx}=u & y(0)=1\\ \frac{du}{dx}=v & u(0)=2\\ & v(0)=3\\ & x=0\end{cases}$

use this to propagate

- **PDE**
  - Laplace equation

$$\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}=0$$



$$\frac{\partial^2T}{\partial x^2}=\frac{\left(\frac{T_{i+1,j}-T_{i,j}}{\Delta x}\right)-\left(\frac{T_{i,j}-T_{i-1,j}}{\Delta x}\right)}{\Delta x}$$

$$=\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}$$

$$\frac{\partial^2T}{\partial y^2}=\frac{\left(\frac{T_{i,j+1}-T_{i,j}}{\Delta y}\right)-\left(\frac{T_{i,j}-T_{i,j-1}}{\Delta y}\right)}{\Delta y}$$

$$=\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}$$

$$\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}=0$$

$\Rightarrow \frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}=0$

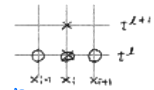
as $\Delta x=\Delta y$

$\Rightarrow T_{i+1,j}+T_{i-1,j}+T_{i,j+1}+T_{i,j-1}-4T_{i,j}=0$

$\Rightarrow T_{i,j}=\frac{T_{i+1,j}+T_{i-1,j}+T_{i,j+1},T_{i,j-1}}{4}$

- Parabolic Equation

$$k\frac{\partial^2T}{\partial x^2}=\frac{\partial T}{\partial t}$$



$k\frac{\partial^2T}{\partial x^2}=\frac{\partial T}{\partial t}$

$$\frac{\partial^2T}{\partial x^2}=\frac{\frac{T^l_{i+1}-T^l_i}{\Delta x}-\frac{T^l_i-T^l_{i-1}}{\Delta x}}{\Delta x}$$

$$\frac{\partial T}{\partial t}=\frac{T^{l+1}_i-T^l_i}{\Delta t}$$

$\Rightarrow k\frac{T^l_{i+1}-2T^l_i+T^l_{i-1}}{\Delta x^2}=\frac{T^{l+1}_i-T^l_i}{\Delta t}$

$\Rightarrow T^{l+1}_i-T^l_i=\frac{k\Delta t}{\Delta x^2}(T^l_{i+1}-2T^l_i+T^l_{i-1})$

$\Rightarrow T^{l+1}_i=T^l_i+\lambda(T^l_{i+1}-2T^l_i+T^l_{i-1})$

$\lambda=\frac{k\Delta t}{\Delta x^2}$

e.g.

$\Delta x=2$ cm

$\Delta t=0.1$ s

$k=0.835$ cm²/s

$\lambda=0.020875$

$\begin{cases}T(0)=100 & @\ \forall t\\ T(10)=50 & \\ T(x)=0 & @\ t=0,\ 0<x<10\end{cases}$ (position)

$T^{l+1}_i=T^l_i+\lambda(T^l_{i+1}-2T^l_i+T^l_{i-1})$

▵ compute T @ $t=0.1$ s, $x=2$ cm

$T^1_1=T^0_1+0.020875(T^0_2-2T^0_1+T^0_0)$ (time, position)

$=0+0.020875(0-2\cdot0+100)$

$=2.0875$

▵ compute T @ $t=0.1$s, $x=4$ cm

$T^1_2=T^0_2+0.020875(T^0_3-2T^0_2+T^0_1)$

$=0+0.020875(0-2\cdot0+0)$

$=0$

▵ compute T @ $t=0.1$s, $x=6$cm

$T^1_3=T^0_3+0.020875(T^0_4-2T^0_3+T^0_2)$

$=0+0.020875(0-2\cdot0+0)$

$=0$

▵ compute T @ $t=0.1$s, $x=8$ cm

$T^1_4=T^0_4+0.020875(T^0_5-2T^0_4+T^0_3)$

$=0+0.020875(50-0+0)$

$=1.0438$

▵ Compute T @ $t=0.2$s, $x=2$cm

$T^2_1=T^1_1+\lambda(T^1_2-2T^1_1+T^1_0)$

$=2.0875+0.020875(0-2\cdot2.0875+100)$

$=4.0878$

- **convergence**

$\lambda=\frac{k\Delta t}{\Delta x^2}\leq\frac{1}{2}\Rightarrow\Delta t\leq\frac{1}{2}\frac{\Delta x^2}{k}$

- $\frac{\partial T}{\partial t}=k\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}\right)$

$$\begin{cases}\frac{\partial^2T}{\partial x^2}=\frac{T^l_{i+1,j}-2T^l_{i,j}+T^l_{i-1,j}}{\Delta x^2}\\ \frac{\partial^2T}{\partial y^2}=\frac{T^l_{i,j+1}-2T^l_{i,j}+T^l_{i,j-1}}{\Delta y^2}\\ \frac{\partial T}{\partial t}=\frac{T^{l+1}_{i,j}-T^l_{i,j}}{\Delta t}\end{cases}$$

$k\left(\frac{T^l_{i+1,j}-2T^l_{i,j}+T^l_{i-1,j}}{\Delta x^2}+\frac{T^l_{i,j+1}-2T^l_{i,j}+T^l_{i,j-1}}{\Delta y^2}\right)$

$=\frac{T^{l+1}_{i,j}-T^l_{i,j}}{\Delta t}$

$\Rightarrow T^{l+1}_{i,j}=T^l_{i,j}+\frac{k\Delta t}{\Delta x^2}(T^l_{i+1,j}-2T^l_{i,j}+T^l_{i-1,j})+\frac{k\Delta t}{\Delta y^2}(T^l_{i,j+1}-2T^l_{i,j}+T^l_{i,j-1})$

let $\Delta x=\Delta y$ $\quad\lambda=\frac{k\Delta t}{\Delta x^2}=\frac{k\Delta t}{\Delta y^2}$

$\Rightarrow T^{l+1}_{i,j}=T^l_{i,j}+\lambda(T^l_{i+1,j}-2T^l_{i,j}+T^l_{i-1,j}+T^l_{i,j+1}-2T^l_{i,j}+T^l_{i,j-1})$

- **convergence**
  - 1-dimension :

$$\lambda=\frac{k\Delta t}{\Delta x^2}\leq\frac{1}{2}$$

  - 2-dimension :

$$\Delta t\leq\frac{1}{8}\frac{\Delta x^2+\Delta y^2}{k}$$

$\Downarrow$

if $\Delta x^2=\Delta y^2$

$$\Delta t\leq\frac{1}{4}\frac{\Delta x^2}{k}$$

- **Linear Programming**
  - Example problem

A_1: origin 1 — costs C_{ij} — terminal 1 b_1, terminal 2 b_2, terminal 3 b_3
A_2: origin 2

$$\underset{x}{\text{minimize}}\ [C_{11}\ C_{12}\ C_{13}]\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\end{bmatrix}+[C_{21}\ C_{22}\ C_{23}]\begin{bmatrix}x_{21}\\x_{22}\\x_{23}\end{bmatrix}$$

s.t. $\begin{cases}x_{11}+x_{12}+x_{13}=a_1\\x_{21}+x_{22}+x_{23}=a_2\end{cases}$ $\begin{cases}x_{11}+x_{21}=b_1\\x_{12}+x_{22}=b_2\\x_{13}+x_{23}=b_3\end{cases}$ $x_{ij}\ge 0$

$\Rightarrow \underset{x}{\text{minimize}}\ [C_{11}\ C_{12}\ C_{13}\ C_{21}\ C_{22}\ C_{23}]\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\\x_{21}\\x_{22}\\x_{23}\end{bmatrix}$

s.t. $\begin{bmatrix}1&1&1&0&0&0\\0&0&0&1&1&1\\1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&0&0&1\end{bmatrix}\begin{bmatrix}x_{11}\\x_{12}\\x_{13}\\x_{21}\\x_{22}\\x_{23}\end{bmatrix}=\begin{bmatrix}a_1\\a_2\\b_1\\b_2\\b_3\end{bmatrix}$, $x\succeq 0$

$\Rightarrow \underset{x}{\text{minimize}}\ c^Tx$ s.t. $Ax=b$, $x\succeq 0$

- transform to standard form

minimize c^Tx s.t. $Ax\preceq b$, $x\succeq 0$ $\Updownarrow$ $\underset{x,y}{\text{minimize}}\ c^Tx$ s.t. $Ax+y=b$, $x\succeq0$, $y\succeq 0$

minimize c^Tx s.t. $Ax\succeq b$, $x\succeq 0$ $\Updownarrow$ $\underset{x,y}{\text{minimize}}\ c^Tx$ s.t. $Ax-y=b$, $x\succeq0$, $y\succeq0$

$\underset{x,y}{\text{minimize}}\ c^Tx+y$ s.t. $Ax+y=b$, $x\succeq 0$ (no $y\succeq0$)
$\Downarrow$ let $y=u-v$
$\underset{x,y,u,v}{\text{minimize}}\ c^Tx+u-v$, $Ax+u-V=b$, $x\succeq0$ $u\succeq0$ $v\succeq0$

$\underset{x,y,z}{\text{minimize}}\ c^Tx+y+z$ s.t. $Ax+y+z=b$, $y+z=d$, $x\succeq0$ $y\succeq0$ (no $z\succeq0$)
$\Downarrow$ let $z=b-Ax-y$
$\underset{x,y}{\text{minimize}}\ c^Tx+y+b-Ax-y$ s.t. $y+b-Ax-y=d$, $x\succeq0$ $y\succeq0$
$\Downarrow$
$\underset{x,y}{\text{minimize}}\ c^Tx+b-Ax$ s.t. $b-Ax=d$, $x\succeq0$ $y\succeq0$

- Solution - Simplex method
  高中線性歸化
- Solution - Basic Feasible solution (BFS)

$\underset{x}{\text{minimize}}\ c^Tx$ s.t. $Ax=b$, $x\succeq0$

let $\tilde{x}$ be
1. $\tilde{x}\in D$, i.e., $A\tilde{x}=b$, $\tilde{x}\succeq0$
2. cols of A corresponding to non-zero of $\tilde{x}$ are linearly independent

Theorem: $\tilde{x}$ is an extreme (rank$(A)=m$) i.f.f it is BFS

I. Basic/non-Basic variables
- for any BFS $\tilde{x}$, we rewrite:
  $\tilde{x}'=\begin{pmatrix}x_B\\x_N\end{pmatrix}$ B: non-zero, N: zero
  $A=(B, N)$
  $c=\begin{pmatrix}c_B\\c_N\end{pmatrix}$
- $Ax=b \Updownarrow Bx_B+Nx_N=b \Updownarrow Bx_B=b$ — rank$(B)=m_b$, $x_B=B^{-1}b$, $x_N=0^{n-m_b}$
- $Bx_B+Nx_N=b$: $\begin{cases}x_B=B^{-1}(b-Nx_N)\succeq 0^{m_b}\\x_N\succeq 0^{n-m_b}\end{cases}$

2. $\underset{x}{\text{minimize}}\ c^Tx$ s.t. $Ax=b$, $x\succeq0^n$ $\Leftrightarrow$ $\underset{x}{\text{minimize}}\ c_B^Tx_B+c_N^Tx_N$ s.t. $Bx_B+Nx_N=b$, $x_B\succeq0^{m_b}$, $x_N\succeq0^{n-m_b}$

$\Leftrightarrow \underset{x}{\text{minimize}}\ c_B^T(B^{-1}(b-Nx_N))+c_N^Tx_N$ s.t. $B^{-1}(b-Nx_N)\succeq0^{m_b}$, $x_N\succeq0^{n-m_b}$

- **Simplex method: the Tableau**

minimize x_1-x_2 s.t. $x_1-x_2\le2$, $x_1+x_2\le6$, $x_1\ge0$, $x_2\ge0$ — $x_1=0$, $x_2=6$, $x_3=8$, $x_4=0$

$\Updownarrow$ minimize $P=x_1-x_2$ s.t. $x_1-x_2+s_1=2$, $x_1+x_2+s_2=6$, $x_1\ge0$ $x_2\ge0$, $s_1\ge0$ $s_2\ge0$

$-x_1+x_2=P$
$-P-x_1+x_2=0$
$\Rightarrow P+x_1-x_2=0$

$$\begin{array}{c}x_1\ \ x_2\ \ s_1\ \ s_2\ \ P\\ \begin{bmatrix}1&-1&1&0&0&2\\1&1&0&1&0&6\\1&-1&0&0&1&0\end{bmatrix}\begin{matrix}-2\\6\\ \end{matrix}\end{array}$$
$\Downarrow$
$$\begin{bmatrix}2&0&1&1&0&8\\1&1&0&1&0&6\\1&-1&0&0&1&0\end{bmatrix}$$
$\Downarrow$
$$\begin{bmatrix}2&0&1&1&0&8\\1&1&0&1&0&6\\2&0&0&1&1&6\end{bmatrix}$$

maximize $P=5x+4y$ s.t. $3x+5y\le78$, $4x+y\le36$, $x,y\ge0$
$\Downarrow$
$\underset{x,y,s_1,s_2}{\text{Maximize}}\ P=5x+4y$, $3x+5y+s_1=78$, $4x+y+s_2=36$, $x,y,s_1,s_2\ge0$

$$\begin{array}{c}x\ \ y\ \ s\ \ s_2\ \ P\\ \begin{bmatrix}3&5&1&0&0&78\\4&1&0&1&0&36\\-5&-4&0&0&1&0\end{bmatrix}\begin{matrix}26\\9\\ \end{matrix}\end{array}$$
$\Downarrow$
$$\begin{bmatrix}3&5&1&0&0&78\\1&1/4&0&1/4&0&9\\-5&-4&0&0&1&0\end{bmatrix}$$
$\Downarrow$
$$\begin{bmatrix}0&17/4&1&-3/4&0&51\\1&1/4&0&1/4&0&9\\0&-11/4&0&5/4&1&45\end{bmatrix}\begin{matrix}12\\36\\ \end{matrix}$$
$\Downarrow$
$$\begin{bmatrix}0&1&4/17&-3/17&0&12\\1&1/4&0&1/4&0&9\\0&-11/4&0&5/4&1&45\end{bmatrix}$$
$\Downarrow$
$$\begin{bmatrix}0&1&4/17&-3/17&0&12\\1&0&-1/17&4/17&0&6\\0&0&11/17&13/17&1&78\end{bmatrix}$$

$x=6$
$y=12$
$P=78$

- **Integer/Mixed-Integer Linear Programming**
  ILP/MILP

$\underset{x}{\text{minimize}}\ c^Tx$
s.t. $Ax=b$, $x\succeq0$, $x\in\{x\mid x_i\in\mathbb{Z}, i=1..n\}$

- Solution - Branch & Bound
  - Branch: partition D into subsets (check objective value is subsets)
  - Bound: acquire a bound w.o./ integer constraints

e.g.
$\underset{x}{\text{maximize}}\ 9x_1+5x_2+6x_3$
s.t. $6x_1+3x_2+5x_3\le10$, $-x_1+x_3\le0$, $x_i\in\{0,1\}$ $i=1,2,3$
$\Updownarrow$
$\underset{x}{\text{maximize}}\ 9x_1+5x_2+6x_3$
s.t. $6x_1+3x_2+5x_3\le10$, $-x_1+x_3\le0$, $x_i\le1$ $i=1,2,3$, $x_i\ge0$ $i=1,2,3$, $x_i\in\mathbb{Z}$ $i=1,2,3$

△ Bound:
- w.o./ $\mathbb{Z}$ constraints: $x_1=1$, $x_2=1$, $x_3=0.2$, $P=15.2$ (UB)
- Rounded down solution: $x_1=1$, $x_2=1$, $x_3=0$, $P=14$ (LB)

△ Branch
UB 15.2 (1,1,0.2)
LB 14 (1,1,0)
$x_3=0$: feasible; $x_3=1$: infeasible
UB 14 (1,1,0)
LB 14 (1,1,0)
no need to check hv UB=LB

- **Solutions to Optimization Problems.**
  - recall Duality

$\underset{x}{\text{minimize}}\ f_0(x)$ s.t. $f_i(x)\le0$, $h_i(x)=0$

△ Lagrangian:
$\mathcal{L}(x,\lambda,\nu)=f_0(x)+\sum_{i=1}^m\lambda_if_i(x)+\sum_{i=1}^p\nu_ih_i(x)$

△ Lagrangian Dual function:
let $g(\lambda,\nu)=\inf_{x\in D}\{\mathcal{L}(x,\lambda,\nu)\}$

$\underset{\lambda,\nu}{\text{maximize}}\ g(\lambda,\nu)$ s.t. $\lambda\succeq0$

△ let λ^*,ν^*, $d^*=g(\lambda^*,\nu^*)$ be optimal
$p^*\ge d^*$

△ strong duality
Slater's condition
$\exists x\in$ relint D s.t. $f_i(x)<h_i$, $h_i(x)=0$

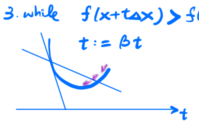
△ complementary slackness
$f_0(x^*)=g(\lambda^*,\nu^*)$
$=\inf_x f_0(x)+\sum_{i=1}^m\lambda_i^*f_i(x)+\sum_{i=1}^p\nu_i^*h_i(x)$
$\le f_0(x^*)+\sum_{i=1}^m\lambda_i^*f_i(x^*)+\sum_{i=1}^p\nu_i^*h_i(x^*)$
$\le f_0(x^*)$

$\lambda_i^*f_i(x^*)=0$
$\Rightarrow\begin{cases}f_i(x^*)<0\rightarrow\lambda_i^*=0\\\lambda_i^*>0\rightarrow f_i(x^*)=0\end{cases}$

△ KKT optimality conditions
$$\begin{cases}f_i(x^*)\le0\\h_i(x^*)=0\\\lambda^*\succeq0\\\lambda_i^*f_i(x^*)=0\\\nabla f_0(x^*)+\sum_{i=1}^m\lambda_i\nabla f_i(x^*)+\sum_{i=1}^p\nu_i\nabla h_i(x^*)=0\end{cases}$$
this will be used later

- **Descent Method**
1. given $x^{(0)}\in$ dom f
2. repeat
   1) descent direction Δx
   2) stepsize $t\ge0$
   3) $x^{k+1}=x^k+t^k\Delta x^k$
3. Stop $\|\Delta f(x)\|_2\le\eta$

- **Line Search**
  - Exact Line Search
    $t=\underset{s\ge0}{\text{argmin}}\ f(x+s\Delta x)$
  - Backtracking Line Search
    1. given Δx^k, $f(x^k)$, $\alpha\in(0,0.5)$, $\beta\in(0,1)$
    2. set $t:=1$
    3. while $f(x+t\Delta x)>f(x)+\alpha t\nabla f(x)^T\Delta x$
       $t:=\beta t$



- **Primal-dual Interior-point method**

$\underset{}{\text{minimize}}\ tf_0(x)+\phi(x)$ s.t. $Ax=b$

KKT:
$\begin{cases}Ax^*(t)=b\\t\nabla f_0(x^*(t))+\nabla\phi(x^*(t))+A^T\hat{\nu}=0\end{cases}$

$\Rightarrow\begin{cases}Ax^*(t)=b\\t\nabla f_0(x^*(t))+\sum_{i=1}^m\frac{1}{-f_i(x^*(t))}\nabla f_i(x^*(t))+A^T\hat{\nu}=0\end{cases}$

$\Rightarrow\begin{cases}Ax=b\\\nabla f_0(x)+\sum_{i=1}^m-\frac{1}{f_i(x)t}\nabla f_i(x)+A^T\nu=0\end{cases}$ $\frac{\hat{\nu}}{t}=\nu$

$\Rightarrow\begin{cases}\nabla f_0(x)+\sum_{i=1}^m\lambda_i\nabla f_i(x)+A^T\nu=0\\-\lambda_if_i(x)=1/t\\Ax=b\end{cases}$

$\Rightarrow r_t(x,\lambda,\nu)=\begin{bmatrix}\nabla f_0(x)+Df(x)^T\lambda+A^T\nu\\-\text{diag}(\lambda)f(x)-1/t\,\mathbf{1}\\Ax-b\end{bmatrix}$ — r_{dual}, r_{cent}, r_{primal}

$\Rightarrow r_t(x+\Delta x,\lambda+\Delta\lambda,\nu+\Delta\nu)=\begin{bmatrix}\nabla^2f_0(x)+\sum_{i=1}^m\lambda_i\nabla^2f_i(x)&Df(x)^T&A^T\\-\text{diag}(\lambda)Df(x)&-\text{diag}(f(x))&0\\A&0&0\end{bmatrix}\begin{bmatrix}\Delta x\\\Delta\lambda\\\Delta\nu\end{bmatrix}=\begin{bmatrix}-r_{dual}\\-r_{cent}\\-r_{pri}\end{bmatrix}$

- **Unconstrained problem**
  - Gradient descent
1. given $x_0\in$ dom f
2. repeat
   - $\Delta x^k=-\nabla f(x)$
   - t: from line search
   - $x^{k+1}=x^k+t\Delta x^k$
3. until $\|\nabla f(x)\|\le\eta$

  - Newton's method
1. given: $x^0\in$ dom f
2. repeat:
   $\Delta x^k=-\nabla^2f(x^k)^{-1}\nabla f(x^k)$
   $\lambda^{k^2}=\nabla f(x^k)\nabla^2f(x^k)\nabla f(x^k)$
   t: from line search
   $x^{k+1}=x^k+t\Delta x$
3. until $\lambda^{k^2}/2\le\epsilon$

on Newton's Method
$f(x)$
$\Rightarrow\hat{f}(x+v)=f(x)+\nabla f(x)^Tv+\frac{1}{2}v^T\nabla^2f(x)v$
$\Rightarrow\nabla\hat{f}(x+v)=0$
$\Rightarrow\nabla f(x)+\nabla^2f(x)v=0$
$\Rightarrow v=-\nabla^2f(x)^{-1}\nabla f(x)$

- **Equality-constrained Problem**

minimize $f(x)$ s.t. $Ax=b$

recall
Slater's condition
$\exists x\in$ relint D s.t. $Ax=b$
KKT optimality condition
$$f_i(x^*)\le0,\quad h_i(x^*)=0,\quad \lambda^*\succeq0,\quad \lambda_i^*f_i(x^*)=0$$
$$\nabla f_0(x^*)+\sum_{i=1}^m\lambda_i^*\nabla f_i(x^*)+\sum_{i=1}^p\nu_i^*\nabla h_i(x^*)=0$$

△ Here @ optimal $\tilde{x},\tilde{\nu}$
$\begin{cases}A\tilde{x}=b\\\nabla f(\tilde{x})+A^T\tilde{\nu}=0\end{cases}$

  - Newton's Method
1. @ x
   addition to x: $x+v$
   $\therefore x+v\in$ dom f
   $A(x+v)=b$
   $\therefore Av=0$ —①
2. $\hat{f}(x+v)$
   $=f(x)+\nabla f(x)v+\frac{1}{2}v^T\nabla^2f(x)v$
   $\therefore$ minimize $f(x)+\nabla f(x)v+\frac{1}{2}v^T\nabla^2f(x)v$ s.t. $Av=0$ —②
③ $\tilde{v}$ is optimal of ②
$\therefore\exists w$ s.t. — Lagrange multiplier of ②
$\nabla f(x)+\nabla^2f(x)\tilde{v}+A^Tw=0$
and recall $A\tilde{v}=0$
from KKT $\mathcal{L}(v,w)=\hat{f}(v)+w(Av)$
$\therefore A\tilde{v}=0$
$\nabla f(x)+\nabla^2f(x)\tilde{v}+A^Tw=0$
$\Rightarrow\begin{bmatrix}\nabla^2f(x)&A^T\\A&0\end{bmatrix}\begin{bmatrix}\tilde{v}\\w\end{bmatrix}=\begin{bmatrix}-\nabla f(x)\\0\end{bmatrix}$
$\therefore\tilde{v}=\Delta x_{nt}.$

- **Inequality-Constrained Problem**
  - Barrier Method

minimize $f_0(x)$ s.t. $f_i(x)\le0$, $Ax=b$
$\Updownarrow$
minimize $tf_0(x)+\phi(x)$ s.t. $Ax=b$
$\phi(x)=-\sum_{i=1}^m\log(-f_i(x))$

1. given $x^0\in$ dom f
   $t^0:=t^0>0$
   $\mu>1$
   $\epsilon>0$
2. repeat
   a. solve
      $\underset{x}{\text{minimize}}\ tf_0(x)+\phi(x)$ s.t. $Ax=b$
      (get x^*, w/ t^k)
   b. $x:=x^*(t)$
   c. until $m/t<\epsilon$
   d. $t:=\mu t$