

MES64 L27

① - Start w/ PDE
(Laplace's Eqⁿ)

② - Get a vector field $\vec{v}(x, y)$

③ Use \vec{v} as RHS for particle trajectory

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} v_1(x, y) \\ v_2(x, y) \end{bmatrix}$$

2D potential flow $\varphi = \varphi(x, y)$

$$\vec{v} = \nabla \varphi \quad \text{where} \quad \left. \begin{array}{l} \nabla \cdot \vec{v} = 0 \\ \nabla \times \vec{v} = 0 \end{array} \right\} \text{iff} \quad \underline{\nabla^2 \varphi = 0}$$

φ is a very special function (potential)

Another special function ψ (stream function)

$$\Phi(z) = \varphi(x, y) + i \psi(x, y) \quad \underline{\nabla^2 \psi = 0}$$

$$z = x + iy$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \partial \varphi / \partial x \\ \partial \varphi / \partial y \end{bmatrix} = \nabla \varphi = \begin{bmatrix} \partial \psi / \partial y \\ -\partial \psi / \partial x \end{bmatrix}$$

$$\Phi(z) = z^n$$

building block

(one function could satisfy Laplace (when n>N) but why?)

$$z^2 = x^2 - y^2 + 2ixy$$

$$\varphi(x, y) = x^2 - y^2$$

$$\psi(x, y) = 2xy$$

1. φ & ψ satisfy

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 2 - 2 = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 - 0 = 0$$

Laplace.

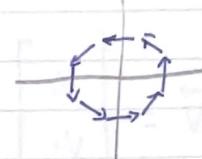
(cont'd)

compute \vec{V}

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$$

$$\begin{matrix} \partial\varphi/\partial x & \partial\varphi/\partial y \\ \partial\varphi/\partial y & -\partial\varphi/\partial x \end{matrix}$$

$$\Phi = \frac{1}{8} \text{ or } \log(8)$$

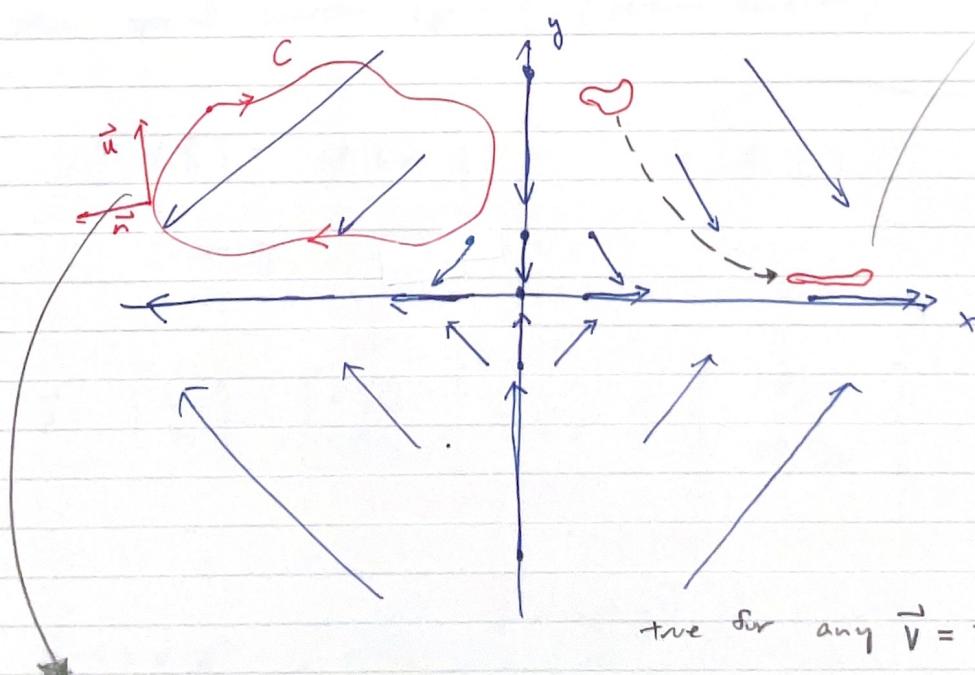
3. \vec{V} is incompressible & irrotational

$$\nabla \cdot \vec{V} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} \cdot \begin{bmatrix} 2x \\ -2y \end{bmatrix} = 0$$

$$\nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -2y & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

plot

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$$



with $t \uparrow$
it maneuvers,
stretched in x
compressed in y
yet volume doesn't change
→ incompressible, irrotational

$$\dot{x} = V_1(x, y)$$

$$\dot{y} = V_2(x, y)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

ODE

induced by
my velocity field!true for any $\vec{V} = \nabla\varphi$

Circulation: $\oint_C \vec{V} \cdot \vec{n} ds = \iint_{\text{inside}} \nabla \times \vec{V} dA = 0$ Stokes.

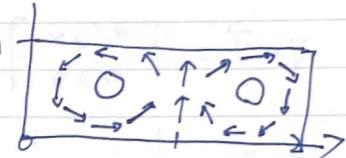
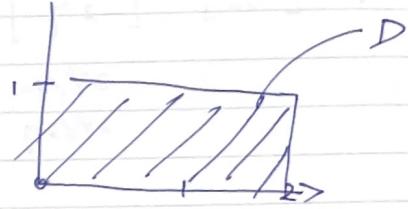
Flux: $\oint_C \vec{V} \cdot \vec{n} ds = \iint_C \vec{V} \cdot \vec{n} dA = 0$

true for $\vec{V} = \nabla\varphi$ & $\nabla^2\varphi = 0$

(stream function.)

$$\psi(x, y) = \sin(\pi x) \sin(\pi y) \text{ on } D = [0, 2] \times [0, 1]$$

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi}{\partial y} \\ -\frac{\partial \psi}{\partial x} \end{bmatrix} = \begin{bmatrix} \pi \sin(\pi x) \cos(\pi y) \\ -\pi \cos(\pi x) \sin(\pi y) \end{bmatrix}$$



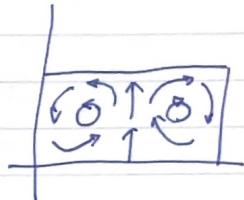
$$[\vec{v}] = [\vec{v}] = \vec{v}$$

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① Double Gyre Flow \rightarrow time-invariant

$$\psi(x, y) = \sin(\pi x) \sin(\pi y) \quad \text{on } D = [0, 1] \times [0, 1]$$

$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \partial \psi / \partial y \\ -\partial \psi / \partial x \end{bmatrix} = \begin{bmatrix} \pi \sin(\pi x) \cos(\pi y) \\ -\pi \cos(\pi x) \sin(\pi y) \end{bmatrix}$$



$$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{aligned} \dot{x} &= V_1(x, y) \\ \dot{y} &= V_2(x, y) \end{aligned} \quad \left\{ \begin{array}{l} \text{ODEs, pplane.} \end{array} \right.$$

② Double Gyre Flow \rightarrow time-variant

$$\begin{aligned} \psi(x, y) &= \sin(\pi x) \sin(\pi y) \\ \psi(x, y) &= \sin(\pi f(x, t)x) \sin(\pi y) \end{aligned}$$

$$f(x, t) = ax^2 + bx \quad a(t) = \epsilon \sin(\omega t)$$

$$b = 1 - 2a$$



(varies in space
& time).

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \partial \psi / \partial y \\ -\partial \psi / \partial x \end{bmatrix}$$