

↳ Let's continue on Lyapunov

$|NFA| \leq |A|$

Recall:

- $\dot{x}(x) > 0$
- $\dot{V}(x) < 0$
- Linear neural network
 $\forall x_i \rightarrow$ linear program
- Quadratic forms implies
 $\forall x_i \rightarrow \forall x_i^T x_i$
- e.g. $\dot{x} = Ax$
- $\begin{cases} V(x) = x^T P x \\ V(x) = x^T P x + b^T x \end{cases}$ (positive)
- SOP:
 $\text{find } P \geq 0, P \succeq 0$ (linear programming)
- $P = A^T P A + P$ (Cholesky)

Some of squares

• given a function $f(x)$, is $f(x) \geq 0 \forall x$?

$$\text{def } f(x) = \sum_i x_i P_i x_i \geq 0$$

$\Leftrightarrow \sum_i x_i P_i x_i \geq 0 \forall x$

some of squares (SOS)

characterization of the problem

when we can know $f(x) \geq 0 \forall x$

• special case: Polytopes

$$P(x) = \min_{\lambda} \lambda_1 x_1 + \dots + \lambda_n x_n$$

$\Leftrightarrow P(x) = \lambda^T x \geq 0$

simplex basis

$\text{vec}(x) \in \text{monomial basis}$

e.g. $x_1, x_1^2, x_1 x_2, \dots$

$1, x_1, x_2, x_1^2, x_1 x_2, \dots$

e.g. $2+4x+5x^2$

$$= [1 \ x_1 \ x_2] \begin{bmatrix} P_0 & P_1 & P_2 \\ P_1 & P_2 & P_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

↓ find $P \geq 0$

$$\text{s.t. } P_0 = 2, P_1 = 4, P_2 = 5$$

SOP again!

↳ linear objective

↳ linear constraints

↳ PSD constraints

↳ SOS

↳ Quadratic SOS

↳ SOS LQR

Six-hump camel

$$P(x,y) = 4x^2 + xy - 4y^2 + 2(x^2 + y^2)^2$$



does it look like?

$$\min_x -T \quad \text{s.t. } P(x,y) - T \leq 0$$

RP & Ps

?

$\dot{x}_1 = \nabla V(x)$ (NLEs)

↳ Back to Lyapunov

- Change $V(x)$ to be polynomial

- assume $\dot{x} = f(x)$ is required to be polynomial

$\Rightarrow \dot{V}(x) = \frac{\partial}{\partial x} f(x)$ is still required

$\{V(x) \geq 0 \rightarrow V(x) \text{ is SOS}$

$\dot{V}(x) < 0 \rightarrow \dot{V}(x) \text{ is SOS}$

Now constraint!

bug: $\dot{x}_1 = -x_1 - 2x_2^2$

$$\dot{x}_2 = -x_2 - x_1 x_2 - 2x_2^3$$

$$V(x) = x_1^2 + 2x_2^2$$

$-\dot{V}(x)$ is SOS? $\forall x$

$$\Rightarrow -\dot{V}(x) = m^2(x) P_m(x)$$

↳ decision variable

↳ SOS

↳ SOS LQR

↳ SOS LMI

↳ SOS

↳ SOS