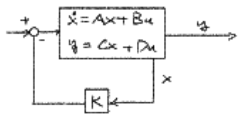
△ LQR



① in Linear system
we let $u = -Kx$
s.t.
$\dot{x} = Ax + Bu$
$\Rightarrow \dot{x} = Ax - BKx$
$= \underline{(A-BK)}\,x$

set K so that A−BK is Hurwitz

② LQR, we optimize K

△ LQR formulation

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt$$

$\Rightarrow u = -Kx$

$\Rightarrow \dot{x} = Ax + Bu$
$y = Cx + Du$

$Q \succ 0, \quad R \succ 0$

△ Ricatti equation

minimize $J = \int_0^\infty (x^T Q x + u^T R u)\, dt$
s.t. $\dot{x} = Ax + Bu$
`>> [K,P] = lqr(A,B,Q,R)`

<u>sol^n</u>

1. Brute-force
2. Learning Algorithms (gradient descent)
3. Analytic approach
   - introduce $P = P^T$

$$J = x_0^T P x_0 - x_0 P^T x_0 + \int_0^\infty (x^T Q x + u^T R u)\, dt$$

$$\Rightarrow J = x_0^T P x_0 + \int_0^\infty \left[\underline{\tfrac{d}{dt}(x^T P x)} + x^T Q x + u^T R u\right] dt$$

$\left[x^T P x\right]_0^\infty = 0 - x_0 P^T x_0$
$\frac{d}{dt}(x^T P x) = \dot{x}^T P x + x^T P \dot{x}$
$= (Ax+Bu)^T P x + x^T P (Ax + Bu)$

$$\Rightarrow J = x_0^T P x_0 + \int_0^\infty \left[(Ax+Bu)^T P x + x^T P (Ax+Bu) + x^T Q x + u^T R u\right] dt$$

$$= x_0^T P x_0 + \int_0^\infty \left[x^T (A^T P + PA + Q) x + \underline{u^T R u + x^T P B u + u^T B^T P x}\right] dt$$

$u^T R u + x^T P B u + u^T B^T P x$
$\Leftrightarrow (u + R^{-1} B^T P x)^T R (u + R^{-1} B^T P x) - x^T (P B R^{-1} B^T P) x$

$$= x_0^T P x_0 + \int_0^\infty \Big[x^T (A^T P + PA + Q - PBR^{-1}B^T P) x + \underbrace{(u + R^{-1} B^T P x)^T R (u + R^{-1} B^T P x)}_{\text{get an "u" s.t. the } \int \text{ is minimized}} \Big] dt$$

$\Rightarrow u = -R^{-1} B^T P x$
$u = -K x$ $\quad\therefore K = R^{-1} B^T \underline{P}$

what is this

let
$A^T P + PA + Q - PBR^{-1}B^T P = 0$
(Algebraic Riccati Equation - ARE)

- find P that solves ARE
- Use P to get K
  (stable one)

△ Tracking Trajectories

$x \to x_d$

let $e = x - x_d$
$v = u - u_d$
$f(x,u) = f(x) + g(x)u$
$\dot{x} = f(x,u)$

$\Rightarrow \dot{e} = \dot{x} - \dot{x}_d$
$= f(x) + g(x)u - (f(x_d) + g(x_d)u_d)$
$= f(e + x_d) - f(x_d)$
$+ g(e + x_d)(v + u_d) - g(x_d)u_d$
$= F(e, v, x_d(t), u_d(t))$

$\Rightarrow$ linearize around $e = 0$
$v = Ke = K(x - x_d)$

$\Rightarrow u = K(x - x_d) + u_d$

△ TL;DR of OCP

- general OCP

$$\underset{u(\cdot)}{\text{minimize}} \int_0^\infty \ell(x(t), u(t))\, dt$$

s.t.
$$\left.\begin{aligned} & x(0) = x_0 \\ & \dot{x}(t) = f(x(t), u(t)) \\ & x(t) \in \mathcal{X} \\ & u(t) \in \mathcal{U} \end{aligned}\right\} \forall t \in [0, \infty)$$

- LQR

$$\underset{u(\cdot)}{\text{minimize}} \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt$$

s.t.
$$\left.\begin{aligned} & x(0) = x_0 \\ & \dot{x}(t) = Ax(t) + Bu(t) \\ & x(t) \in \mathcal{X} \\ & u(t) \in \mathcal{U} \end{aligned}\right\} \forall t \in [0, \infty)$$

- MPC

$$\underset{u(\cdot)}{\text{minimize}} \int_t^{t+N} \ell(x(t), u(t))\, dt$$

s.t.
$$\left.\begin{aligned} & x(t) = x_0 \\ & \dot{x}(t) = f(x(t), u(t)) \\ & x(t) \in \mathcal{X} \\ & u(t) \in \mathcal{U} \end{aligned}\right\} \forall t \in [t, t+N]$$

△ Standard formulation

$$\ell(x, u) = \|x_u - x^r\|_Q^2 + \|u - u^r\|_R^2$$

$$J_N(x, u) = \sum_{k=0}^{N-1} \ell(x_u(k), u(k))$$

$$\Rightarrow \underset{u}{\text{minimize}}\ J_N(x_0, u)$$

s.t. $x_u(k+1) = f(x_u(k), u(k))$
$x_u(0) = x_0$
$u(k) \in \mathcal{U} \quad \forall k \in [0, N-1]$
$x(k) \in \mathcal{X} \quad \forall k \in [0, N]$

△ mobile robot as an example

$x = [x, y, \theta]^T$

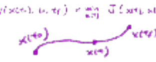
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \frac{r}{2} \begin{bmatrix} (\dot{\phi}_R + \dot{\phi}_L)\cos\theta \\ (\dot{\phi}_R + \dot{\phi}_L)\sin\theta \\ (\dot{\phi}_R - \dot{\phi}_L)/D \end{bmatrix}$$

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \frac{r}{2}(\dot{\phi}_R + \dot{\phi}_L) \\ \frac{r}{2D}(\dot{\phi}_R - \dot{\phi}_L) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

△ Optimal Nonlinear Control

- $\frac{d}{dx} x = f(x(t), u(t), t)\, dt$
- $J(x(t), u(t), t_0, t_f)$
  $= Q(x(t_f), t_f) + \int_{t_0}^{t_f} \ell(x(t), u(t))\, dt$
- $V(x(t_0), t_0, t_f) = \min_{u(t)} J(x(t), u(t), t_0, t_f)$



$V(x(t_0), t_0, t_f)$
$= V(x(t_0), t_0, t) + V(x(t), t, t_f)$

- $\underline{-\frac{\partial V}{\partial t} = \min_{u(t)} \left\{ \left(\frac{\partial V}{\partial x}\right)^T f(x(t), u(t)) + \ell(x(t), u(t)) \right\}}$

Hamilton Jacobi Bellman (HJB) eqn