△ Inverse Kinematics

- Recall: forward kinematics

$T_{IE}(q) = \begin{bmatrix} C_{IE}(q) & {}_I r_{IE}(q) \\ 0 & 1 \end{bmatrix} \in SE(3)$

parametrize

$\chi_e = \begin{pmatrix} r_e(q) \\ \phi_e(q) \end{pmatrix} = f(q)$

$\chi_e = \begin{pmatrix} \chi_{eP} \\ \chi_{eR} \end{pmatrix}$

- Q: $q = f^{-1}(\chi_E)$? — give you χ_E, give me q

△ sol^n

- Analytic: – for 3 intersecting neighboring axes
- Geometric: – use length, theorem... "geometric info"
- Algebraic – use TFs to get joints
- numerical!

△ numerical method: inverse differential kinematics

- recall

$w_e = J_e \dot{q}$ — pseudo inverse

$\dot{q} = J_{e0}^{+} w_e^{*}$ — multiple sol / singular

- singularities
  - occur @ q_s ⇒ $J_e(q_s)$ is column deficient
  - $\dot{q} = J_{e0}^{+} w_e^{*}$ — inf ← column deficient ← small

- categories
  - boundary: when end-arm is @ its limitation; easy to prevent
  - internal: hard to prevent

- damped version of Moore-Penrose pseudo inverse

$\dot{q} = J^{+} w \Leftrightarrow \arg\min_{\dot q} \|J\dot{q} - w\|_2$

$\Leftrightarrow \arg\min_{\dot q} \|J\dot{q} - w\|_2^2 + \lambda^2 \|\dot{q}\|_2^2$

$\dot{q} = J^T (J J^T + \lambda^2 I)^{-1} w$

- Redundancy

$q \in R^n$, $w = J\dot{q}$, $w \in R^m$, $J \in R^{m \times n}$, $m > n$ → redundancy

- $J_{e0}\dot{q} = w_e^{*}$

⇒ $J_{e0}(J_{e0}^{+} w_e^{*} + N\dot{q}_0) = w_e^{*}$

⇒ $\dot{q} = J_{e0}^{+} w_e^{*} + N\dot{q}_0$

$N = \mathcal{N}(J_{e0})$ null space, $J_{e0} N = 0$

- get N?
  - $N = I - J_{e0}^{+} J_{e0}$
  - SVD
  - QR

  end-up w/ different basis that span null space

△ multi-task control

- track desired position & orientation
- break down tasks: $\{J_i, w_i^*\}$

$\dot{q} = \underbrace{\begin{bmatrix} J_1 \\ \vdots \\ J_{n_t} \end{bmatrix}^{+}}_{\bar{J}} \underbrace{\begin{bmatrix} w_1^* \\ \vdots \\ w_{n_t}^* \end{bmatrix}}_{\bar{w}}$

desired velocity in the end-effector's state space; it's just an objective space; $\bar J$ is the mapping matrix

⇒ $\min \|\bar{J}\dot{q} - \bar{w}\|_2$

⇒ $\min \|\dot{q}\|_2^2$ s.t. $\bar{J}\dot{q} = \bar{w}$

- weighting

$\bar{J}^{+W} = (\bar{J}^T W \bar{J})^{-1} \bar{J}^T W$

- Prioritization

recall: $\dot{q} = J_1^{+} w_1^* + N_1 \dot{q}_0$

⇒ $w_2 = J_2 \dot{q}$

$= J_2(J_1^{+} w_1^* + N_1\dot{q}_0)$

⇒ $\dot{q}_0 = (J_2 N_1)^{+}(w_2^* - J_2 J_1^{+} w_1^*)$

⇒ $\dot{q} = J_1^{+} w_1^* + N_1 (J_2 N_1)^{+}(w_2^* - J_2 J_1^{+} w_1^*)$

∴ $\dot{q} = \sum_{i=1}^{n_t} N_i \dot{q}_i$

w/ $\dot{q}_i = (J_i N_i)^{+}(w_i^* - J_i \sum_{k=1}^{i-1} N_k \dot{q}_k)$

△ Back to inverse kinematics

- $\dot{\chi}_e = J_{eA}(q)\dot{q}$ — analytical

$w = J_0 \dot{q}$ — geometric

- now $\Delta\chi_e = J_{eA}(q)\Delta q$

- tracking a point χ_e^*, $q_0 = q^0$

pseudo-code

$q \leftarrow q^0$

while $\|\chi_e^* - \chi_e(q)\| \ge tol$ do

- $J_{eA} \leftarrow \frac{\partial \chi_e}{\partial q}(q)$
- $J_{eA}^{+} \leftarrow (J_{eA})^{+}$
- $\Delta\chi_e \leftarrow \chi_e^* - \chi_e(q)$
- $q \leftarrow q + J_{eA}^{+}\Delta\chi_e$

(usually joint-space bigger than task-space → redundant system)

△ iterative method

- problem 1

$q^{i+1} = q^i + J_{eA}^{+}\Delta\chi$ — if $\Delta\chi$ ↑ & J_{eA}^{+} is ill-conditioned

∴ $q^{i+1} = q^i + k J_{eA}^{+}\Delta\chi$ — slower converge

- problem 2

Jacobian is rank deficient → bad condition

∴ $q^{i+1} = q^i + J_{eA}^T (J_{eA} J_{eA}^T + \lambda^2 I)^{-1}\Delta\chi$

or $q^{i+1} = q^i + \alpha J_{eA}^T \Delta\chi$

△ Orientation

- depends on parameterization
- $\in SO(3)$
- $\Delta\chi_{rot} \neq \Delta\varphi$

$C_{GS}(\Delta\varphi) = C_{GS}(\varphi^*) C_{GS}^T(\varphi^t)$

⇒ $q \leftarrow q + k_{pR} J_{eOR}^{+} \Delta\varphi$ — Lie algebra, geometric one

△ Traj. cont.

- given $r_e^*(t)$, $\dot{r}_e^*(t)$
- feedback

$\Delta r_e^t = r_e^*(t) - r_e(q^t)$

$\dot{q}_P^* = J_{eoP}^{+}(q^t)\cdot(\dot{r}_e^*(t) + k_{pp}\Delta r_e^t)$

$\dot{q}_R^* = J_{eoR}^{+}(q^t)\cdot(w_e^*(t) + k_{pR}\Delta\varphi)$

△ L.A. revisit

△ determinants

- $\det(A) = 0$ — squeeze original state space → lower dimension
- $\det(A) = c$ — Area = 0; if $\mathrm{sgn}(c) = (-)$ then

△ Rank

rank$(A) = n$ then n is the dimension of the spanned space of the output column space

△ Nullspace (Kernel)

$Ax = 0$, $A \in R^{m \times n}$

is a set of vectors, which form a space, get "squeeze" to the origin; note that all A has trivial null space $\{0\}$; null space $\in R^{n-m}$ if $n > m$; related rank

△ ill/well-condition of matrix

$Ax = c$

ill conditioned? well conditioned? } small change in $[A]$ small/large change in $[c]$?

vice-versa → small change in $[c]$ small/large change in $[A]$?

- △ small ΔA → small Δc "well-conditioned"
- △ small ΔA → large Δc "ill-conditioned"

e.g. $[\;][\;] = [\;]$ …

- △ condition number

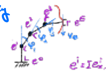
cond$[A] \approx 1.0$ well

cond$[A] > 1.0$ ill

cond$[A] = \|A\|_\infty \|A^{-1}\|_\infty$

Rule of thumb: $\log[\mathrm{cond}[A]]$ = decimals of accuracy lost

△ inverse kinematics case study



(1) given $q_t = [\pi/6, \pi/3, \pi/3]^T$, ${}_0 r_{E,t} = [1, 1]^T$, $e^i = [e_x^i, e_y^i]$

find $\dot{q}$ (generalized velocity)

$w = J\dot{q}$ (assume $l_i = 1$)

△ $r_E = \begin{bmatrix} S\varphi_1 + S(\varphi_1+\varphi_2) + S(\varphi_1+\varphi_2+\varphi_3) \\ 1 + C\varphi_1 + C(\varphi_1+\varphi_2) + C(\varphi_1+\varphi_2+\varphi_3) \end{bmatrix}$

⇒ $J_E = \begin{bmatrix} \frac{\partial r_1}{\partial q_1} & \frac{\partial r_1}{\partial q_2} & \frac{\partial r_1}{\partial q_3} \\ \frac{\partial r_2}{\partial q_1} & \frac{\partial r_2}{\partial q_2} & \frac{\partial r_2}{\partial q_3} \end{bmatrix}$

⇒ w/ $q_t = [\pi/6\ \pi/3\ \pi/3]^T$

$J_E = \frac{1}{2}\begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{3} \\ -4 & -3 & -1 \end{bmatrix}$

⇒ $\dot{q} = J_E^{+} w$

$= J^T(JJ^T)^{-1}[1\ 1]^T$

$= \begin{bmatrix} 0.069 \\ -0.56 \\ -0.595 \end{bmatrix}$

check! $\frac{1}{2}\begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{3} \\ -4 & -3 & -1 \end{bmatrix}\begin{bmatrix} 0.069 \\ -0.56 \\ -0.595 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(2) multi-task

we want $q_1 = q_2 = 0$

$J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- parallel (no priorities)

$\dot{q} = \begin{bmatrix} \frac{1}{2}\begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{3} \\ -4 & -3 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{bmatrix}^{+} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\dot q \in R^{3\times1}$

$[R^{4\times3}]^{+}[R^{4\times1}]$

⇒ $\dot{q} = \begin{matrix} -0.133 \\ -0.067 \\ -1.132 \end{matrix}$

check! $\begin{bmatrix} J_1 \\ J_2 \end{bmatrix}\dot{q} = \begin{bmatrix} 1.039 \\ 0.983 \\ -0.133 \\ -0.067 \end{bmatrix} \approx \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

(3) w/ priority, task 1 most important! "$w = J\dot{q}$"

⇒ $\dot{q} = J_1^{+} w_1 + N_1 \dot{q}_0$, $N_1 = I - J_1^{+}J_1$

$J_1[I - J_1^{+}J_1] = J_1 - J_1 J_1^{+} J_1 = 0$

⇒ $J_2\dot{q} = J_2 J_1^{+} w_1 + J_2 N_1 \dot{q}_0 = w_2$

⇒ $\dot{q}_0 = (J_2 N_1)^{+}(w_2 - J_2 J_1^{+} w_1)$

⇒ $\dot{q} = J_1^{+} w_1 + N_1 (J_2N_1)^{+}(w_2 - J_2 J_1^{+} w_1)$

$= \begin{bmatrix} -0.169 \\ -0.485 \\ -1.070 \end{bmatrix}$

check! $\begin{bmatrix} J_1 \\ J_2 \end{bmatrix}\dot{q} = \begin{bmatrix} 1 \\ 1 \\ -0.169 \\ -0.485 \end{bmatrix}$

△ matrix norm

$[A] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\|A\|_\infty = \max\{|1|+|2|, |0|+|1|\}$ — row sum w/ abs.

$= \max\{3, 1\} = 3$

$\|A\|_1$ = sum of row 1 = 3

$\|A\|_2$ = sum of row 2 = 1

e.g. recall cond$[A] = \|A\|_\infty \|A^{-1}\|_\infty$

$A = \begin{bmatrix} 400 & -201 \\ -801 & 401 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -1.003 & -5.01 \\ -2 & 1 \end{bmatrix}$

⇒ $\|A\|_\infty = |-801| + |401| = 1201$

$\|A^{-1}\|_\infty = |-2| + |-1| = 3$

⇒ cond(A) = 3603

△ more on matrix norm

- recall vector norms

L_1-norm $\|u\|_1 = |u_1| + |u_2|$

L_2-norm. $\|u\|_2 = \sqrt{u_1^2 + u_2^2}$

- matrix norms "based on" vector norm

matrix → linear transformation

∴ matrix norm: "HOW BIG MY MATRIX CAN GET MY OUTPUT VECTOR".

$\|A\|_2 = \max_{\|u\|_2 = 1} \|Au\|_2$

- $\|AB\|_p \le \|A\|_p \|B\|_p$
- $\|A^k\|_p = \max_{\|u\|_p = 1} \|A^k u\|_p$

$A[u_1 \cdots u_i] = \lambda_i [u_1 \cdots u_i]$

$Au = \lambda u$

⇒ $\|Au_i\|_p = |\lambda_i| \|u_i\|_p \le \|A\|_p \|u_i\|_p$

⇒ $\|\lambda_i\| \le \|A\|_p$

⇒ $\max|\lambda_i| \le \|A\|_p$

⇒ $\max|\lambda_i|^k \le \|A^k\|_p$

△ Dynamics

this chapter basically talks about EoM — description of motion; source of motion

- kinematic v dynamic

don't care about the cause, let me know the relation! / what's the source

- cause of motion

τ → $\ddot{q}$

cause → description of motion!

★ $M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau + J_c^T F_c$

- Newton-Euler (free cut, conservation of impulse, angular momentum)
- Projected Newton-Euler (generalized coordinates)
- Lagrange II (energy)

△ Principle of virtual work (D'Alembert's Principle)

- Dynamic equilibrium ⇒ 0 virtual work

$\delta W = \int_B \delta r^T (\ddot{r}\,dm - dF_{ext}) = 0$

- Newton's law for every particle

force $F = ma$

⇒ $F = \frac{d}{dt}(mv) = \dot{p}$, p momentum (linear impulse)

momentum

$r \times F = r \times ma$

$\Gamma = \frac{d}{dt}(r \times mv) = \dot{N}$, N angular momentum

△ Derivation of virtual displacements of Single Rigid bodys

- from rigid body kinematics

$r = r_S + p$, $[\Omega]_\times p = -[p]_\times \Omega$

$\dot{r} = v_S + \Omega \times p = [I_{3\times3}\ -[p]_\times]\begin{bmatrix} v_S \\ \Omega \end{bmatrix}$

$\ddot{r} = a_S + \psi \times p + \Omega \times (\Omega \times p)$

$= [I_{3\times3}\ -[p]_\times]\begin{bmatrix} a_S \\ \psi \end{bmatrix} + [\Omega]_\times [\Omega]_\times p$

$\delta r = \delta r_S + \delta\Phi \times p$

$= [I_{3\times3}\ -[p]_\times]\begin{bmatrix} \delta r_S \\ \delta\Phi \end{bmatrix}$

- Apply principle of virtual work

$\delta W = \int_B \delta r^T(\ddot{r}\,dm - dF_{ext}) = 0$

$= \int_B \underbrace{\begin{pmatrix} \delta r \\ \delta\Phi \end{pmatrix}^T \begin{pmatrix} I_{3\times3} \\ [p]_\times \end{pmatrix}}_{\delta r} \cdot \left(\left\{ [I_{3\times3}\ -[p]_\times]\begin{bmatrix} a_S \\ \psi \end{bmatrix} + [\Omega]_\times[\Omega]_\times p \right\} dm - dF_{ext} \right)$

$= \begin{pmatrix} \delta r \\ \delta\Phi \end{pmatrix}^T \int_B \left(\begin{bmatrix} I_{3\times3}dm & [p]_\times^T dm \\ [p]_\times dm & -[p]_\times^2 dm \end{bmatrix}\begin{pmatrix} a_S \\ \psi \end{pmatrix} + \begin{pmatrix} [\Omega]_\times^2 p\,dm \\ [p]_\times[\Omega]_\times^2 p\,dm \end{pmatrix} - \begin{pmatrix} dF_{ext} \\ [p]_\times dF_{ext} \end{pmatrix} \right)$

w/ $\int_B dm = m$

$\int_B p\,dm = 0$, S = CoG (moment)

$\int_B -[p]_\times^2 dm = \int_B [p]_\times [p]_\times^T dm = \Theta_S$ — moment of inertial

⇒ $0 = \delta W = \begin{pmatrix} \delta r_S \\ \delta\Phi \end{pmatrix}^T \left(\begin{bmatrix} I_{3\times3}m & 0 \\ 0 & \Theta_S \end{bmatrix}\begin{bmatrix} a_S \\ \psi \end{bmatrix} + \begin{pmatrix} 0 \\ [\Omega]_\times \Theta_S \Omega \end{pmatrix} - \begin{pmatrix} F_{ext} \\ T_{ext} \end{pmatrix} \right)$

$\forall \begin{pmatrix} \delta r_S \\ \delta\Phi \end{pmatrix}$ → ★ free body can move in all directions

⇒ means that

$\begin{cases} m a_S - F_{ext} = 0 \\ \Theta_S \psi + [\Omega]_\times \Theta_S \Omega - T_{ext} = 0 \end{cases}$

note that

$\begin{cases} P_S = m v_S & \text{momentum} \\ \dot{P}_S = m a_S\ (f) & \text{change in momentum} \end{cases}$

$\begin{cases} N_S = \Theta_S \Omega_S & \text{angular momentum} \\ \dot{N}_S = \Theta_S \psi + \Omega \times \Theta_S \Omega & \text{change in angular momentum} \end{cases}$

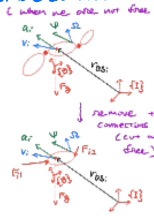
⇒ $\begin{cases} \dot{P}_S = F_{ext} & \text{force} & \text{newton} \\ \dot{N}_S = T_{ext} & \text{torque} & \text{euler} \end{cases}$

"change in linear/angular momentum"

△ some reminder

	linear	rotational
@	x	θ
v	v	ω
a	a	α
motion	$x = vt$	$\theta = \omega t$
	$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
	$x = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
	$v^2 = v_0^2 + 2ax$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
mass	m	I
	$F = ma$	$\tau = I\alpha$
	$P = mv$	$L = I\omega$
work	Fd	$\tau\theta$
	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$
power	Fv	$\tau\omega$

△ More on Newton-Euler method (when we are not free body)



remove the connections (cut all bodies free) → produce constraining forces

- based on this, w/ n rigid bodies
  - 6n equation
  - 5n constrained force

why 5n: if 1 DOF joints you are imposing 5 DOF constraints in each joint

△ projected Newton-Euler (in generalized motion direction)

- recall

$0 = \delta W = \sum_{i=1}^{n_b} \begin{pmatrix} \delta r_{S_i} \\ \delta\Phi_{S_i} \end{pmatrix}^T \left(\begin{bmatrix} \dot{p}_{S_i} \\ \dot{N}_{S_i} \end{bmatrix} - \begin{bmatrix} F_{ext,i} \\ T_{ext,i} \end{bmatrix} \right)$ $\forall \begin{pmatrix} \delta r_{S_i} \\ \delta\Phi_{S_i} \end{pmatrix}$

now, this is limited due to linkages!

- now express in generalized coordinates to represent the joints
- express linear/angular momentum in generalized coordinates

$\begin{cases} \begin{bmatrix} v_S \\ \Omega \end{bmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix}\dot{q} \\ \begin{bmatrix} a_S \\ \psi \end{bmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix}\ddot{q} + \begin{bmatrix} \dot{J}_P \\ \dot{J}_R \end{bmatrix}\dot{q} \end{cases}$

⇒ $\begin{bmatrix} \dot{P}_{S_i} \\ \dot{N}_{S_i} \end{bmatrix} = \begin{bmatrix} m a_{S_i} \\ \Theta_{S_i}\psi_{S_i} + \Omega_{S_i} \times \Theta_{S_i}\Omega_{S_i} \end{bmatrix}$

$= \begin{pmatrix} m J_{S_i} \\ \Theta_{S_i} J_{R_i} \end{pmatrix}\ddot{q} + \begin{pmatrix} m \dot{J}_{S_i}\dot{q} \\ \Theta_{S_i}\dot{J}_{R_i}\dot{q} + J_{R_i}\dot{q} \times \Theta_{S_i} J_{R_i}\dot{q} \end{pmatrix}$

- virtual displacement in generalized coordinates

$\begin{pmatrix} \delta r_S \\ \delta\Phi \end{pmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix}\delta q$

∴ $0 = \delta W = \sum_{i=1}^{n} \begin{pmatrix} \delta r_{S_i} \\ \delta\Phi_{S_i} \end{pmatrix}^T \left(\begin{bmatrix} \dot{P}_{S_i} \\ \dot{N}_{S_i} \end{bmatrix} - \begin{bmatrix} F_{ext,i} \\ T_{ext,i} \end{bmatrix} \right)$

$= \delta q^T \sum_{i=1}^{n} \Big\{ \underbrace{\begin{pmatrix} J_{S_i} \\ J_{R_i} \end{pmatrix}^T \begin{pmatrix} m J_{S_i} \\ \Theta_{S_i} J_{R_i} \end{pmatrix}\ddot{q}}_{M(q)} + \underbrace{\begin{pmatrix} J_{S_i} \\ J_{R_i} \end{pmatrix}^T \begin{pmatrix} m \dot{J}_{S_i}\dot{q} \\ \Theta_{S_i}\dot{J}_{R_i}\dot{q} + J_{R_i}\dot{q} \times \Theta_{S_i} J_{R_i}\dot{q} \end{pmatrix}}_{b(q,\dot q)} \underbrace{- \begin{pmatrix} J_{S_i} \\ J_{R_i} \end{pmatrix}^T \begin{pmatrix} F_{ext,i} \\ T_{ext,i} \end{pmatrix}}_{g(q)} \Big\}$ $\forall \delta q$

- $M(q)\ddot{q} + b(q,\dot{q}) + g(q) = 0$

$M = \sum_{i=1}^{n_b} ({}_A J_{S_i}^T m\, {}_A J_{S_i} + {}_B J_{R_i}^T\, {}_B\Theta_{S_i}\, {}_B J_{R_i})$

$b = \sum_{i=1}^{n_b} ({}_A J_{S_i}^T m\, {}_A \dot{J}_{S_i}\dot{q} + {}_B J_{R_i}^T ({}_B\Theta_{S_i}\, {}_B \dot{J}_{R_i}\dot{q} + {}_B\Omega_{S_i} \times {}_B\Theta_{S_i}\, {}_B\Omega_{S_i}))$

$g = \sum_{i=1}^{n_b} (-{}_A J_{S_i}^T\, {}_A F_{g,i})$

△ Lagrange II

- Lagrangian $\mathcal{L} = T - U$ (kinetic, potential)
- Lagrangian Equation

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = \tau$

- ∵ $U = U(q)$, $T = T(q, \dot{q})$

∴ $\underbrace{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q}}_{\text{inertial force}} + \underbrace{\frac{\partial U}{\partial q}}_{\text{gravity vector}} = \tau$

△ Kinetic energy

- joint space $T = \frac{1}{2}\dot{q}^T M \dot{q}$
- bodies space

$T = \sum_{i=1}^{n_b}\left(\frac{1}{2} m_i\, {}_A\dot{r}_{S_i}^T\, {}_A\dot{r}_{S_i} + \frac{1}{2}\, {}_B\Omega_{S_i}^T\, {}_B\Theta_{S_i}\, {}_B\Omega_{S_i}\right)$

w/ $\dot{r}_{S_i} = J_{S_i}\dot{q}$, $\Omega_{S_i} = J_{R_i}\dot{q}$

∴ $T(q,\dot{q}) = \frac{1}{2}\dot{q}^T \underbrace{\left(\sum_{i=1}^{n_b} (J_{S_i}^T m J_{S_i} + J_{R_i}^T \Theta_{S_i} J_{R_i})\right)}_{M(q)} \dot{q}$

△ potential energy

- gravity

$F_{g_i} = m_i g_I e_z$ ⇒ $U_g = -\sum_{i=1}^{n_b} r_{S_i}^T F_{g_i}$

- spring

$F_S = k_j(\|r_1 - r_0\| - d_0)\frac{r_1 - r_0}{\|r_1 - r_0\|}$

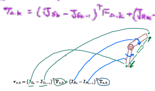
⇒ $U_{S_j} = \frac{1}{2}k_j(d(q) - d_0)^2$

△ External Force

- given $n_{f,ext}$ F_j T_k
- $\tau_{F,ext} = \sum_{j=1}^{n_f} J_{P,j}^T F_j$ — joint space; task space
- $\tau_{T,ext} = \sum_{k=1}^{n_m} J_{R,k}^T T_k$

actuator torques

$\tau_{a,k} = (J_{S_k} - J_{S_{k-1}})^T F_{a,k} + (J_{R_k} - J_{R_{k-1}})^T T_{a,k}$



△ Dynamic Control

how we hv EoM, how do we contl it?

$M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau + J_c^T F_c$

How do we get this

Actuator control $\xrightarrow{VA}$ Actuator torque sensor $\xrightarrow{\tau}$ Robot Dynamics

$q, \dot{q}, \ddot{q}$

$(q^*\ \dot{q}^*)$ τ^*

△ position-based control
- don't care about dynamics
- high gain PID: good point-based tracking
- disturbances are compensated by PID
- cannot control force directly
  ⇒ interaction force can only be controlled w/ compliant surface

△ integrate force-feedback (Dynamic control)
- active regulation of system dynamics
- model-based load compensation
- interaction force control.

△ Joint Impedance Control

- $M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau$
- get desired τ

△ torque as functions of P/V error

$\tau^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$

can think of it as spring force w/ damping

⇒ $M(q)\ddot{q} + b(q,\dot{q}) + g(q) = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$

static offset due to gravity (when static $M\ddot{q}$, $b = 0$, $\dot{q}^* - \dot{q} = 0$)

△ impedance control & gravity compensation

$\tau^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q}) + \underbrace{\hat{g}(q)}_{\text{estimated}}$

configuration dependent eg CoG shifts

△ independent of configuration: inverse dynamics control

- $\tau = \hat{M}(q)\ddot{q}^* + \hat{b}(q,\dot{q}) + \hat{g}(q)$

get $\ddot{q}^*$, & throw into this EoD, and get the final τ.
based on inverse kinematics → inverse dynamics

result in — assume our dynamic modelling error ...
$I\ddot{q} = \ddot{q}^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$
& $\omega = \sqrt{k_p}$ $\quad D = \frac{k_d}{2\sqrt{k_p}}$

- describe from task space

$\dot{w}_e = \begin{pmatrix}\ddot{r}\\ \dot{\omega}\end{pmatrix}_e = J_e\ddot{q} + \dot{J}_e\dot{q}$

∴ $\ddot{q} = J_e^+(\dot{w}_e - \dot{J}_e\dot{q})$

& similarly, multi-task

- $\ddot{q} = \begin{bmatrix}J_1\\ \vdots\\ J_{n_T}\end{bmatrix}^+ \left(\begin{bmatrix}\dot{w}_1\\ \vdots\\ \dot{w}_{n_T}\end{bmatrix} - \begin{bmatrix}\dot{J}_1\\ \vdots\\ \dot{J}_{n_T}\end{bmatrix}\dot{q}\right)$

parallel

- $\ddot{q} = \sum_{i=1}^{n_T} N_i\ddot{q}_i$ w/

w/ $\ddot{q}_i = (J_iN_i)^+\left(\dot{w}_i^* - \dot{J}_i\dot{q} - J_i\sum_{k=1}^{i-1}N_k\ddot{q}_k\right)$

- get $\ddot{q}$, & insert back to E.o.M.

△ task-space dynamics

- recall joint space
  $M(q)\ddot{q} + b(q,\dot{q}) + g(q) = \tau$
- for end-effector
  $\Lambda\dot{w}_e + \mu + p = F_e$
  linking the two — link

$\begin{cases}\tau = J_e^T F_e \\ \dot{w}_e = \begin{pmatrix}\ddot{r}\\ \dot{\omega}\end{pmatrix}_e = J_e\ddot{q} + \dot{J}_e\dot{q}\end{cases}$ $\quad \ddot{q} = M^{-1}(\tau - b - g)$

derivation

⇒ $\dot{w}_e = J_eM^{-1}(\tau - b - g) + \dot{J}_e\dot{q}$

⇒ $\dot{w}_e - \dot{J}_e\dot{q} + J_eM^{-1}b + J_eM^{-1}g = J_eM^{-1}\tau$

⇒ $\underbrace{\dot{w}_e - \dot{J}_e\dot{q} + J_eM^{-1}b}_{\mu} + \underbrace{J_eM^{-1}g}_{p} = \underbrace{J_eM^{-1}J_e^T}_{\Lambda^{-1}}F_e$

⇒ $\Lambda\dot{w}_e + \mu + p = F_e$

→ geometric interpretation: inertial ellipsoid (depend on our configuration)

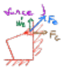
∴ get $\dot{w}_e^* = k_p E(\chi_e^* - \chi_e) + k_d(w_e^* - w_e) + \dot{w}_e(\tau)$ — $\Delta y = E_R(\chi_e)\Delta\chi_R$ — feed-forward

determine desired acceleration → task-level force → Jacobi transformation → joint-space torques

$\tau^* = J^T(\Lambda_e\dot{w}_e^* + \hat{\mu} + \hat{p})$

△ interaction control

operation space control (generalized framework to control motion & force)



- $F_c + \Lambda\dot{w}_e + \mu + p = F_e$

△ selection force

$\hat{\Lambda}S_M\dot{w}_e + S_FF_c + \hat{\mu} + \hat{p} = F_e$

⇒ $\tau^* = J^TF_e$

$\Sigma_p = \begin{bmatrix}\sigma_{px} & & \\ & \sigma_{py} & \\ & & \sigma_{pz}\end{bmatrix}$ $\quad \sigma \in \{0, 1\}$: 1 → can move, 0 → can apply force

$\Sigma_F = I_3 - \Sigma_p$

$\Sigma_R = \begin{bmatrix}\sigma_{Rx} & & \\ & \sigma_{Ry} & \\ & & \sigma_{Rz}\end{bmatrix}$

$\Sigma_{RF} = I_3 - \Sigma_R$

w/ C: transformation between contact local frame & world frame

∴ $S_M = \begin{bmatrix}C^T\Sigma_pC & 0\\ 0 & C^T\Sigma_RC\end{bmatrix}$

$S_F = \begin{bmatrix}C^T(I_3 - \Sigma_p)C & 0\\ 0 & C^T(I_3 - \Sigma_R)C\end{bmatrix}$

∴ inject back to

$\hat{\Lambda}S_M\dot{w}_e + S_FF_c + \hat{\mu} + \hat{p} = F_e$

△ inverse dynamic as QP

1. classic inverse dynamics

recall
$w = J\dot{q}$
$\dot{w} = J\ddot{q} + \dot{J}\dot{q} \Rightarrow \ddot{q} = J^+(\dot{w} - \dot{J}\dot{q})$

∴ $\tau^* = M(q)\ddot{q}^* + b(q,\dot{q}) + g$
$= M(q)[J^+(\dot{w} - \dot{J}\dot{q})] + b(q,\dot{q}) + g$

2. operation space control

$\tau^* = J_e^TF_e$
$= J_e^T(\Lambda\dot{w}_e^* + \mu + p)$

Not only sol^n

∴ $\tau^* = J_e^T(\Lambda\dot{w}_e^* + \mu + p) + \mathcal{N}(J_e^T)\tau_0$ — null space — can change my ...

$\mathcal{N}(J_e^T) = \underbrace{(I - J^T[JM^{-1}J^T]^{-1}JM^{-1})}_{\text{Mass-weighted pseudo inverse}}$ (w.o. changing desired $\dot{w}$) — proof

$\dot{w} = J\ddot{q} + \dot{J}\dot{q}$
$= JM^{-1}(\tau^* - b - g) + \dot{J}\dot{q}$
$= JM^{-1}(J^T(\Lambda\dot{w}_e^* + \mu + p) + \underbrace{(I - J^T(JM^{-1}J^T)^{-1}JM^{-1})\tau_0}_{=0} - b - g) + \dot{J}\dot{q}$

- we can also see:

$\begin{cases}\tau^* = M(q)\ddot{q}^* + b(q,\dot{q}) + g \\ \tau^* = J_e^TF_e = J_e^T(\Lambda\dot{w}_e^* + \mu + p) \\ \quad = J_e^T((J_eM^{-1}J_e^T)^{-1}\dot{w} + \mu + p)\end{cases}$

⇒ $\ddot{q} = M^{-1}(\tau^* - b - g)$
$= M^{-1}(J_e^T(J_eM^{-1}J_e^T)^{-1}\dot{w} + \mu + p) - b - g)$
some sort of mass-matrix weighted pseudo-inverse

3. QP

- $\tau = M\ddot{q} + b + g$
  $J\ddot{q} + \dot{J}\dot{q} = w$
  $\min \|\ddot{q}\|$
  $\min \|\tau\|$
  ↔ classic inverse dynamic

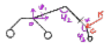
- $[M\ -I]\begin{bmatrix}\ddot{q}\\ \tau\end{bmatrix} + b + g = 0$
  $[J_e\ 0]\begin{bmatrix}\ddot{q}\\ \tau\end{bmatrix} + \dot{J}\dot{q} = \dot{w}^*$

⇓ $\quad x = \begin{bmatrix}\ddot{q}\\ \tau\end{bmatrix}$

multi-task:
$\min_x \left\|\begin{bmatrix}M & -1\\ J_e & 0\end{bmatrix}x - \begin{bmatrix}-b-g\\ \dot{w}_e^* - \dot{J}\dot{q}\end{bmatrix}\right\|_2$

priority:
$\min_x \|[J_e\ 0]x - (\dot{w}_e^* - \dot{J}\dot{q})\|_2$
s.t. $[M\ -I]x - (-b-g) = 0$

△ case study



- $q = (\varphi_1, \varphi_2, \varphi_3)^T$
- $M\ddot{q} + b + g = \tau$
- $J_P \in R^{2\times 3}$
  $J_R \in R^{1\times 3}$

(1) static gravity compensation joint torque?
i.e. $\cancel{M\ddot{q}} + \cancel{b} + g = \tau$

(2) lifting 100 kg, joint torque?
$M\ddot{q} + b + g + J_P^T\begin{pmatrix}0\\ 100\end{pmatrix} = \tau$
⇒ $\tau = g + J_P^T\begin{pmatrix}0\\ 100\end{pmatrix}$

(3) accelerate w/ 1 m/s² upward, φ, x = const, joint torque?

- $J_e = \begin{bmatrix}J_P\\ J_R\end{bmatrix}$ $\quad \dot{w}_e^* = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$ — my desired acceleration.

- we know that $w = J\dot{q}$, $\dot{w} = J\ddot{q} + \dot{J}\dot{q}$

⇒ $\ddot{q}^* = J^+(\dot{w} - \dot{J}\dot{q}) + N\ddot{q}_0$

⇒ $\tau = M\ddot{q}^* + b + g$
$= MJ^+(\dot{w} - \dot{J}\dot{q}) + b + g$

(4) task space: 1 m/s² in y, x, φ = const, joint torque? is the same the same?

- $\dot{w}_e^* = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$ $\quad \Lambda = (J_eM^{-1}J_e^T)^{-1}$, $\mu = \Lambda J_eM^{-1}b - \Lambda\dot{J}_e\dot{q}$, $p = \Lambda J_eM^{-1}g$

$J_e = J_E = \begin{bmatrix}J_P\\ J_R\end{bmatrix}$

⇒ $\tau^* = J_e^T\underbrace{(\Lambda\dot{w}_e^* + \mu + p)}_{F_e}$

(5) interaction control

$\dot{w}_e^* = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$ desired $\quad J_e = \begin{bmatrix}J_P\\ J_R\end{bmatrix}$ — contact force

$\tau^* = J_e^T\underbrace{\left[\Lambda\dot{w}_e^* + \mu + p + \begin{bmatrix}-100\\ 0\\ 0\end{bmatrix}\right]}_{F_e}$

(6) QP problems.

(a) least-square minimal-torque to get $\dot{w}^* = [0\ 1\ 0]^T$?

$x = \begin{bmatrix}\ddot{q}\\ \tau\end{bmatrix}$

- $[M\ -I]x - (-b-g) = 0$ (1)
  $A_1x - b_1 = 0$
- $\begin{bmatrix}J_P & 0\\ J_R & \end{bmatrix}x - \left(\dot{w}^* - \begin{bmatrix}\dot{J}_P\\ \dot{J}_R\end{bmatrix}\dot{q}\right) = 0$ (2) $\quad \{w = J\dot{q},\ \dot{w} = J\ddot{q} + \dot{J}\dot{q}\}$
  $A_2x - b_2 = 0$
- $[0\ I]x = 0$ (minimize torque) (3)
  $A_3x - b_3 = 0$
  as good as possible then solve.

(b) assume $\dot{q}_i = 0$, and $w^* = [\ ]$

$[M\ -I]x - (-b-g) = 0$ (1)

$\begin{bmatrix}J_P & \\ J_R & 0\\ [1,0,0,0] & \end{bmatrix}x - \left(\begin{bmatrix}0\\ 1\\ 0\end{bmatrix} - \begin{bmatrix}\dot{J}_P\\ \dot{J}_R\\ [0,0,0,0]\end{bmatrix}\dot{q}\right) = 0$ (2)

(c) assume $\tau_1 = 0$, $w^* = [\ ]$

$[M\ -I]x - (-b-g) = 0$ (1)

$[0\ [1\ 0\ 0\ 0]]x = 0$ (2)

$\begin{bmatrix}J_P & 0\\ J_R & \end{bmatrix}x - \left(\dot{w}^* - \begin{bmatrix}\dot{J}_P\\ \dot{J}_R\end{bmatrix}\dot{q}\right) = 0$ (3)

△ Least-square Optimization

- $Ax = b \Rightarrow x = A^+b$
  ⇕
  $\min_x \|Ax - b\|_2$
  or
  $\min_x \|x\|_2$ s.t. $Ax = b$

- $A_1x_1 + A_2x_2 = b$
  → $[A_1\ A_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix} = b$ ⇒ $\begin{bmatrix}x_1\\ x_2\end{bmatrix} = [A_1\ A_2]^+b$
  ⇕
  $\min_{x_1,x_2} \left\|[A_1\ A_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix} - b\right\|_2$
  or
  $\min_{x_1,x_2} \left\|\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right\|_2$ s.t. $[A_1\ A_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix} = b$

- $A_1x - b_1 = 0$, $A_2x - b_2 = 0$ → Equal priority
  $x = \begin{bmatrix}A_1\\ A_2\end{bmatrix}^+\begin{bmatrix}b_1\\ b_2\end{bmatrix}$

- Hierarchy
  $x = A_1^+b_1 + \mathcal{N}(A_1)x_0$ $\quad \min \|A_2x - b_2\|_2$ s.t. $\|A_1x - b_1\| = c$
  $A_1(A_1^+b_1 + \mathcal{N}(A_1)x_0) = A_1A_1^+b_1 + 0 = b_1$

⇒ $A_2x - b_2 = 0$
⇒ $A_2(A_1^+b_1 + \mathcal{N}(A_1)x_0) - b_2 = 0$
⇒ $x_0 = [A_2\mathcal{N}(A_1)]^+[b_2 - A_2A_1^+b_1]$ — try to fulfill $A_2x = b_2$

△ solving a set of QPs

- w/ different priorities

methods:
- exploit nullspace of tasks w/ higher priority
- everystep = QP w/ constraints
- iteratively nullspace projection

- pseudo-code

n_T = number of tasks
$x = 0$
$N_1 = I$
for $i = 1 \to n_T$ do
  $x_i = (A_iN_i)^+(b_i - A_ix)$
  $x = x + N_ix_i$
  $N_{i+1} = \mathcal{N}([A_1^T \cdots A_i^T]^T)$
end for

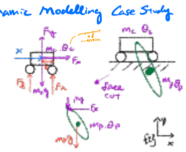
e.g. ① $i = 1$
$x = 0$
$N_1 = I$
$x_1 = (A_1I)^+(b_1 - A_10)$
$= A_1^+b_1$
$x = 0 + IA_1^+b_1$
$N_2 = \mathcal{N}(A_1)$

② $i = 2$
$x = A_1^+b_1$
$N_2 = \mathcal{N}(A_1)$
$x_2 = [A_2\mathcal{N}(A_1)]^+[b_2 - A_2A_1^+b_1]$
$x = A_1^+b_1 + \mathcal{N}(A_1)[A_2\mathcal{N}(A_1)]^+[b_2 - A_2A_1^+b_1]$

use 2 tasks (most of the time)

△ Dynamic Modelling Case Study

(1) NE



△ get linear / angular momentum of cart
$m_c\ddot{x}_c = F_x$
$m_c\ddot{y}_c = F_y + F_L + F_R - m_cg$
$\theta_c\ddot{\varphi}_c = F_Rb - F_Lb$

△ get linear / angular momentum of pendulum
$m_p\ddot{x}_p = -F_x$
$m_p\ddot{y}_p = -F_y - m_pg$
$\theta_p\ddot{\varphi}_p = F_xl\cos(\varphi_p) - F_yl\sin(\varphi_p)$

△ kinematics
$x_c = x$
$y_c = 0$
$\varphi_c = 0$
$[x, \varphi]^T$ ⇒ generalized coordinates
$x_p = x + l\sin(\varphi)$
$y_p = -l\cos(\varphi)$
$\varphi_p = \varphi$

$\ddot{x}_p = \ddot{x} + \ddot{\varphi}l\cos(\varphi) - \dot{\varphi}^2l\sin(\varphi)$
$\ddot{y}_p = \ddot{\varphi}l\sin(\varphi) + \dot{\varphi}^2l\cos(\varphi)$
$\ddot{\varphi}_p = \ddot{\varphi}$

- (7),(10-12) in (1) and (4-6)
  $m_c\ddot{x} = F_x$ (13)
  $m_p(\ddot{x} + \ddot{\varphi}l\cos(\varphi) - \dot{\varphi}^2l\sin(\varphi)) = -F_x$ (14)
  $m_p(\ddot{\varphi}l\sin(\varphi) + \dot{\varphi}^2l\cos(\varphi)) = -F_y - m_pg$ (15)
  $\theta_p\ddot{\varphi} = F_xl\cos(\varphi) + F_yl\sin(\varphi)$ (16)
- From (13) and (14) remove F_x
  $(m_p + m_c)\ddot{x} + m_pl\cos(\varphi)\ddot{\varphi} - \dot{\varphi}^2lm_p\sin(\varphi) = 0$
- Insert (13) and (15) in (16) to remove F_x and F_y
  $(\theta_p + m_pl^2)\ddot{\varphi} + m_pl\cos(\varphi)\ddot{x} + glm_p\sin(\varphi) = 0$

(2) NE - projected

$q = \begin{bmatrix}x\\ \varphi\end{bmatrix}$

- $r_{OS_c} = \begin{bmatrix}x\\ 0\end{bmatrix}$
  $J_{P_c} = \frac{\partial r}{\partial q} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$
  $\dot{J}_{P_c} = \frac{\partial J_{P_c}}{\partial t} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$
- $r_{OS_p} = \begin{bmatrix}x + l\sin\varphi\\ -l\cos\varphi\end{bmatrix}$
  $J_{P_p} = \frac{\partial r}{\partial q} = \begin{bmatrix}1 & l\cos(\varphi)\\ 0 & l\sin(\varphi)\end{bmatrix}$
  $\dot{J}_{P_p} = \frac{\partial J_{P_p}}{\partial t} = \begin{bmatrix}0 & -l\sin(\varphi)\dot{\varphi}\\ 0 & l\cos(\varphi)\dot{\varphi}\end{bmatrix}$
  $\omega_p = \dot{\varphi}$
  $J_{R_p} = \frac{\partial\omega_p}{\partial\dot{q}} = [0\ 1]$

- E.o.M

$\mathbf{M} = \sum \mathbf{J}_{S_i}^T m_i \mathbf{J}_{S_i} + \mathbf{J}_{R_i}^T \Theta_{S_i} \mathbf{J}_{R_i} = \mathbf{J}_{P_c}^T m_c \mathbf{J}_{P_c} + \mathbf{J}_{P_p}^T m_p \mathbf{J}_{P_p} + \mathbf{J}_{R_p}^T \theta_p \mathbf{J}_{R_p} = \begin{bmatrix}m_c + m_p & lm_p\cos(\varphi)\\ lm_p\cos(\varphi) & m_pl^2 + \theta_p\end{bmatrix}$

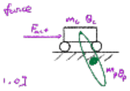
$\mathbf{b} = \sum \mathbf{J}_{S_i}^T m_i \dot{\mathbf{J}}_{S_i}\dot{\mathbf{q}} + \mathbf{J}_{R_i}^T(\Theta_{S_i}\dot{\mathbf{J}}_{R_i}\dot{\mathbf{q}} + \Omega_{S_i} \times \Theta_{S_i}\Omega_{S_i}) = \mathbf{J}_{P_p}^T m_p \dot{\mathbf{J}}_{P_p}\dot{\mathbf{q}} = \begin{pmatrix}-\dot{\varphi}^2lm_p\sin(\varphi)\\ 0\end{pmatrix}$

$\mathbf{g} = \sum -\mathbf{J}_{S_i}^T \mathbf{F}_i^g = -\mathbf{J}_{P_c}^T\begin{pmatrix}0\\ -m_cg\end{pmatrix} - \mathbf{J}_{P_p}^T\begin{pmatrix}0\\ -m_pg\end{pmatrix} = \begin{pmatrix}0\\ m_pgl\sin(\varphi)\end{pmatrix}$

(3) Lagrange II

- Kinematics cart and pendulum
  $r_{OS_c} = \begin{pmatrix}x\\ 0\end{pmatrix}$ $\quad r_{OS_p} = \begin{pmatrix}x + l\sin(\varphi)\\ -l\cos(\varphi)\end{pmatrix}$ $\quad \varphi_c = 0$
  $\dot{r}_{S_c} = \begin{pmatrix}\dot{x}\\ 0\end{pmatrix}$ $\quad \dot{r}_{S_p} = \begin{pmatrix}\dot{x} + l\cos(\varphi)\dot{\varphi}\\ l\sin(\varphi)\dot{\varphi}\end{pmatrix}$ $\quad \varphi_p = \varphi$
- Kinetic and potential energy
  $\mathcal{T} = \sum \frac{1}{2}\dot{r}_i^T m_i \dot{r}_i + \frac{1}{2}\omega_i^T \Theta_i \omega_i = \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_p\dot{x}^2 + \frac{1}{2}m_pl^2\dot{\varphi}^2 + m_p\dot{x}l\cos(\varphi)\dot{\varphi} + \frac{1}{2}\theta\dot{\varphi}^2$
  $\mathcal{U} = -m_pgl\cos(\varphi)$ ← 0-level can be chosen
- Equation of motion
  $\frac{d}{dt}\left(\frac{\partial\mathcal{T}}{\partial\dot{\mathbf{q}}}\right) - \frac{\partial\mathcal{T}}{\partial\mathbf{q}} + \frac{\partial\mathcal{U}}{\partial\mathbf{q}} = 0$
  $\frac{\partial\mathcal{T}}{\partial\dot{\mathbf{q}}} = \begin{pmatrix}m_c\dot{x} + m_p\dot{x} + m_pl\cos(\varphi)\dot{\varphi}\\ m_pl^2\dot{\varphi} + m_p\dot{x}l\cos(\varphi) + \theta\dot{\varphi}\end{pmatrix}$
  $\frac{d}{dt}\left(\frac{\partial\mathcal{T}}{\partial\dot{\mathbf{q}}}\right) = \begin{pmatrix}m_c\ddot{x} + m_p\ddot{x} + m_pl\cos(\varphi)\ddot{\varphi} - m_pl\sin(\varphi)\dot{\varphi}^2\\ m_pl^2\ddot{\varphi} + m_p\ddot{x}l\cos(\varphi) - m_p\dot{x}l\sin(\varphi)\dot{\varphi} + \theta\ddot{\varphi}\end{pmatrix}$
  $\frac{\partial\mathcal{T}}{\partial\mathbf{q}} = \begin{pmatrix}0\\ -m_p\dot{x}l\sin(\varphi)\dot{\varphi}\end{pmatrix}$
  $\frac{\partial\mathcal{U}}{\partial\mathbf{q}} = \begin{pmatrix}0\\ m_pgl\sin(\varphi)\end{pmatrix}$

(4) External force



- $J_c^TF_c$
  where $J_c = [1, 0]$

∴ $\tau = J^TF_c$
$= \begin{bmatrix}1\\ 0\end{bmatrix}F_{ext} = \begin{bmatrix}F_{ext}\\ 0\end{bmatrix}$

△ Eigendecomposition