Quadrotor Trajectory Planning

Minimum Snap/Jerk w/ Bezier Curve

Why use Bezier Curve to parameterize trajectory?
↳ parameterized w/ Bernstein Basis, controlled by control points

1. endpoint interpolation property
   will not pass thru middle points
2. Convex Hull properties
   if all control points stay within the corridor → safety ensurance
3. Hodograph
   derivatives

↳ These could ensure the trajectory stays within constraints

why minimum snap, should refer to differential flatness, a concept that is used in non-linear control.

Differential Flatness (see later section for quadrotor flatness mapping)

for dynamical system

$$\dot{x} = f(x) + g(x)u \qquad \begin{matrix} f: \mathbb{R}^n \to \mathbb{R}^n & x \in \mathbb{R}^n \\ g: \mathbb{R}^n \to \mathbb{R}^n & u \in \mathbb{R}^m \\ & \mathrm{rank}(g) = m \end{matrix}$$

we say that the system is differential flat if there exists

$$\zeta \in \mathbb{R}^m \quad \text{which can be determined by } x \in \mathbb{R}^n \ u \in \mathbb{R}^m$$

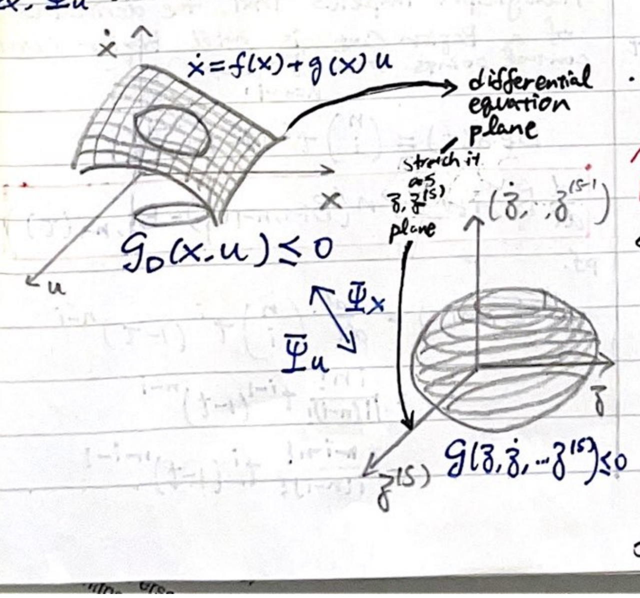
↓

differential outputs

$\zeta \in \mathbb{R}^m$, could determine all state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ through:

$$x = \Psi_x(\zeta, \dot{\zeta}, \cdots, \zeta^{(s-1)}),$$
$$u = \Psi_u(\zeta, \dot{\zeta}, \cdots, \zeta^{s})$$

Ψ_x, Ψ_u are transformation processes.



for quadrotor

$$\zeta = (r, \psi) \in \mathbb{R}^3 \times SO(2)$$
$$\downarrow \Psi_x \ \Psi_u$$
$$x = \{r, v, R, \omega\} \in \mathbb{R}^3 \times \mathbb{R}^3 \times SO(3) \times \mathbb{R}^3$$
$$u = \{f, T\} \in \mathbb{R}_{\ge 0} \times \mathbb{R}^3$$

detailed in later section

∴ so we can then plan a trajectory without differential constraints

Minimum Snap/Jerk

represent the trajectory as piecewise polynomial

$$f_j(t) = \sum_{i=0}^{n} p_i t^i \qquad j = 0 \ldots m$$

- by harnessing flatness, we optimize the trajectory
- optimize it w/ snap$^{(4)}$ || jerk$^{(3)}$

$$f^{(4)}(t) = \sum_{i \ge 4} i(i-1)(i-2)(i-3)\, t^{i-4} p_i$$

$$J(t) = \int_{T_{j-1}}^{T_j} \left(f^{(4)}(t)\right)^2 dt$$

$$= p_i p_l \sum_{\substack{i \ge 4 \\ l \ge 4}} \left[\begin{matrix} \vdots \\ \cdots \frac{i(i-1)(i-2)(i-3)\, l(l-1)(l-2)(l-3)}{i+l-7} t \\ \vdots \end{matrix} \right]_{T_{j-1}^{(i+l-7)}}^{T_j^{(i+l-7)}}$$

$$= p_i^T Q p_l$$

where $Q = \left[\begin{matrix} & \vdots & \\ \cdots & \frac{i(i-1)(i-2)(i-3)\, l(l-1)(l-2)(l-3)}{i+l-7} T^{i+l-7} & \cdots \\ & \vdots & \end{matrix} \right]$

$$\therefore \quad \min J(t) \quad \text{s.t.} \quad \begin{matrix} Ap = b & \text{(continuous constraints)} \\ Ap \le b & \text{(limitation constraints)} \end{matrix}$$

Minimum Snap/Jerk w/ Constraints.

Bernstein polynomials

from monomial basis → Bernstein basis

$$b_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$P_j(t) = c_j^0 b_n^0(t) + c_j^1 b_n^1(t) + \cdots + c_j^n b_n^n(t)$$

$$= \sum_{i=0}^{n} \underbrace{c_j^i}_{\text{control points}} \underbrace{b_n^i(t)}_{\text{bernstein basis}}$$

for n+1 points, polynomial has n order

No.

Date

Berstein Polynomials (cont'd)

from above, any polynomials:

monomial basis $\rightarrow$ Bernstein basis

$$P = MC$$

recall min. snap $\quad \min J = \min P^T Q P$

here $\quad = \min C^T M^T Q M C$

$$P_j(t) = C_j^0 b_n^0(t) + C_j^1 b_n^1(t) + \cdots + C_j^n b_n^n(t)$$

$$= \sum_{i=0}^{n} C_j^i \, b_n^i(t) \qquad b_n^i(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

let $f_u(t)$ be the whole trajectory

$$f_u(t) = \begin{cases} s_1 \sum_{i=0}^{n} C_{u_1}^i b_n^i\left(\frac{t-T_0}{s_1}\right), & t \in [T_0, T_1] \\ s_2 \sum_{i=0}^{n} C_{u_2}^i b_n^i\left(\frac{t-T_1}{s_2}\right), & t \in [T_1, T_2] \\ \vdots \\ s_m \sum_{i=0}^{n} C_{u_m}^i b_n^i\left(\frac{t-T_{m-1}}{s_m}\right), & t \in [T_{m-1}, T_m] \end{cases}$$

m^{th} peice trajectory

$$\tau = \frac{t - T_j}{s_{j+1}} \quad \text{normalization}$$

$$\tau = [0, 1]$$

$$J = \sum_{u \in \{x,y,z\}} \int_0^T \left(\frac{d^k f_u(t)}{dt^k}\right)^2 dt$$

on u axis, j^{th} trajectory

$$s\,d\tau = dt$$

$$J_{u_j} = \int_0^{s_j} \left(\frac{d^k f_{u_j}(t)}{dt^k}\right)^2 dt$$

$$= \int_0^1 \left(\frac{s_j \, d^k g_{u_j}(\tau)}{s_j^k \, d\tau}\right)^2 s_j \, d\tau$$

$$= \int_0^1 \frac{s_j^2}{s_j^{2k}} \cdot s_j \left(\frac{d^k g_{u_j}(\tau)}{d\tau}\right) d\tau$$

$$= \int_0^1 s_j^{3-2k} \frac{d^k g_{u_j}(\tau)}{d\tau} d\tau$$

e.g. for $t \in [0, T]$

$$J = \int_0^T \left(\frac{d^k \left(s \cdot \sum C_i b^i\left(\frac{t}{s}\right)\right)}{dt^k}\right)^2 dt$$

$$= \int_0^1 \left(\frac{d^k \left(s \cdot \sum C_i b^i(\tau)\right)}{s^k d\tau^k}\right)^2 s \, d\tau$$

$$= \int_0^1 s^{3-2k} \left(\frac{d^k \sum C_i b^i(\tau)}{d\tau^k}\right)^2 d\tau$$

$$= \int_0^1 s^{3-2k} \left(\frac{d^k \sum P_i \tau^i}{d\tau^k}\right)^2 d\tau$$

get Q from $P^T Q P$ or $C^T M^T Q M C$

$$Q = \begin{bmatrix} & \vdots & \\ \cdots & \frac{i(i-1)(i-2)(i-3)\,l(l-1)(l-2)(l-3)}{i+l-7} s^{-2k+3} & \cdots \\ & \vdots & \end{bmatrix}$$

get M from $C^T M^T Q M C$

pascal triangle

now we have Q & M

minimize objectives: $\quad \min J = C^T M^T Q M C$

minimize conditions: $\quad Ac = b$

$\quad Ac \le b$

$$C \in \mathbb{R}^{(n\_order+1) \times m}$$

Bezier Curve Hodograph

Hodograph implies that the derivatives of a Bezier Curve is still Bezier curve

control points from $\underset{i=0\sim n+1}{P_i} \rightarrow \underset{i=0\sim n}{Q_i}$

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$\boxed{\frac{d}{dt} B_{i,n}(t) = n\left(B_{i-1,n-1}(t) - B_{i,n-1}(t)\right)}$$

pf.

$$\frac{d}{dt} B_{i,n}(t) = \frac{d}{d\tau} \binom{n}{i} \tau^i (1-\tau)^{n-i}$$

$$= \frac{i\,n!}{i(n-i)!} t^{i-1} (1-t)^{n-i}$$

$$- \frac{(n-i)\,n!}{i(n-i)!} t^i (1-t)^{n-i-1}$$

$$\frac{d}{dt}B_{i,n}(t) = \frac{i\,n!}{i!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{(n-i)\,n!}{i!(n-i)!}t^{i}(1-t)^{n-i-1}$$
$$= \frac{n(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{n(n-1)!}{i!(n-k-1)!}t^{i}(1-t)^{n-i-1}$$
$$= n\left[\frac{(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{(n-1)!}{i!(n-k-1)!}t^{i}(1-t)^{n-i-1}\right]$$
$$= n\left(B_{i-1,n-1}(t) - B_{i,n-1}(t)\right) \#$$

$$F(t) = \sum_{i=0}^{n} B_{i,n}(t)P_i$$

$$\boxed{\frac{d}{dt}F(t) = \sum_{i=0}^{n-1} B_{i,n-1}(t)Q_i \quad Q_i = n(P_{i+1} - P_i)}$$

pf.

let $n=2$

$$F(t) = \sum_{i=0}^{2} B_{i,n}(t)P_i$$
$$= B_{0,2}(t)P_0 + B_{1,2}(t)P_1 + B_{2,2}(t)P_2$$

$$\frac{d}{dt}F(t) = 2(\underbrace{\cancel{B_{-1,1}(t)}}_{=0} - B_{0,1}(t))P_0 + 2(B_{0,1}(t) - B_{1,1}(t))P_1 + 2(B_{1,1}(t) - \underbrace{\cancel{B_{2,1}(t)}}_{=0})P_2 \quad -①$$

when $n=1$, derivative of $F(t)$, $n=2$

$$F(t) = \sum_{i=0}^{1} B_{i,n}(t)Q_i = B_{0,1}(t)Q_0 + B_{1,1}(t)Q_1 \quad -②$$

from ①

$$\frac{d}{dt}F(t) = (-2P_0 + 2P_1)B_{0,1}(t) + (-2P_1 + 2P_2)B_{1,1}(t)$$
$$= 2(P_1 - P_0)B_{0,1}(t) + 2(P_2 - P_1)B_{1,1}(t)$$

from②

$$= Q_0 B_{0,1}(t) + Q_1 B_{1,1}(t)$$

$$Q_0 = 2(P_1 - P_0)$$
$$Q_1 = 2(P_2 - P_1)$$

$$Q_i = n(P_{i+1} - P_i) \#$$

e.g.

$P_0 \; P_1 \; P_2 \; P_3 \; P_4$ $\quad n=4$ $\quad$ pts X5

$\frac{d}{dt}$ ↳ $n(P_1-P_0), n(P_2-P_1), n(P_3-P_2), n(P_4-P_3)$ pts X4

$\frac{d}{dt}$ ↳ $(n-1)\begin{pmatrix} n(P_2-P_1) - n(P_1-P_0) \\ n(P_3-P_2) - n(P_2-P_1) \\ n(P_4-P_3) - n(P_3-P_2) \end{pmatrix}$ pts X3

Equality Condition

starting condition
ending condition
continuous condition

$$Ac = b$$

$c \in R_{(n\_order+1)\times m}$

n_order: highest order of polynomial
m: no. of trajectories

e.g. n_order $= 7$, $m = 5$

$c \in R_{(7+1)\times 5} = R_{40}$

$A \in R_{d\times(n\_order+1)\times m} = R_{d\times 40}$

$b \in R_d \rightarrow$ Xd constraints

starting condition, ending condition $\begin{matrix} j=0 \\ j=f \end{matrix}$

$a^0_{i,j} = c_0 \quad (i,j) \quad a^1_{i,j} = c_1 \quad a^2_{i,j} = c_2$

$P = a_{i,j}(s_j)'$

$V = \frac{d}{dt}a^0_{i,j}(s_j)^0 = n(a^1_{i,j} - a^0_{i,j})$

$a = \frac{d}{dt^2}a^0_{i,j}(s_j)^{-1} = n(n-1)[(a^2_{i,j} - a^1_{i,j}) \cdot (a^1_{i,j} - a^0_{i,j})]$ from above

↳ write everything into matrix

e.g. n_order $= 7$
$m = 5$
$s_j = t$

$$Ac = b$$
$$\begin{bmatrix} 1t & 0 & 0 & \\ -7t^0, & 7t^0 & 0 & \\ 42t^{-1}, & -84t^{-1}, & 42t^{-1} & 0 \\ & 0 & & \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix} = \begin{bmatrix} P \\ V \\ a \\ \vdots \\ 0 \end{bmatrix}$$

continuous condition

$P_{n\_order+1,j} = P_{0,j+1}$ $\quad j = 0,1,2,\dots m$

$V_{n\_order+1,j} = V_{0,j+1}$ $\quad j = 0,1,2,\dots m$

$a_{n\_order+1,j} = a_{0,j+1}$ $\quad j = 0,1,2,\dots m$

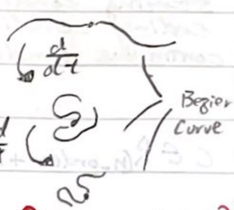
$$\therefore \begin{cases} P_{n\_order+1,j} - P_{0,j+1} = 0 \\ V_{n\_order+1,j} - V_{0,j+1} = 0 \\ a_{n\_order+1,j} - a_{0,j+1} = 0 \\ j \\ \vdots \\ s_{i,i} \end{cases} = 0 \longrightarrow$$

No.

Date

Equality condition (cont'd)

$$\begin{cases} P_{n\_order+1,j} - P_{0,j+1} = 0 \\ V_{n\_order+1,j} - V_{0,j+1} = 0 \\ a_{n\_order+1,j} - a_{0,j+1} = 0 \end{cases}$$



$$\Rightarrow \begin{cases} a^{n\_order+1}_{\mu_j} - a^{0}_{\mu_{j+1}} = 0 \\ \frac{d}{dt} a^{n\_order+1}_{\mu_j} - \frac{d}{dt} a^{0}_{\mu_{j+1}} = 0 \\ \frac{d^2}{dt^2} a^{n\_order+1}_{\mu_j} - \frac{d^2}{dt^2} a^{0}_{\mu_{j+1}} = 0 \end{cases}$$

$$\begin{Bmatrix} \text{first pt of } j+1 \\ \shortparallel \\ \text{last pt of } j \end{Bmatrix}$$

$$\Rightarrow \begin{cases} a^{n\_order+1}_{\mu_j} - a^{0}_{\mu_{j+1}} = 0 \\ n(a^{n\_order+1}_{\mu_j} - a^{n\_order}_{\mu_j}) - n(a^{1}_{\mu_{j+1}}, a^{0}_{\mu_{j+1}}) = 0 \\ n(n-1)\left[(a^{n\_order+1}_{\mu_j} - a^{n\_order}_{\mu_j}) - (a^{n\_order}_{\mu_j} - a^{n\_order-1}_{\mu_j})\right] \\ -n(n-1)\left[(a^{2}_{\mu_{j+1}} - a^{1}_{\mu_{j+1}}) - (a^{1}_{\mu_{j+1}} - a^{0}_{\mu_{j+1}})\right] = 0 \end{cases}$$

↳ write everything into matrix

e.g.

n_order = 7

m = 5 — 4 connection conditions

condition @ j = 1, 2

↓

12 conditions

$$\underbrace{\begin{bmatrix} \cdots & & & 1 & -1 & & \cdots \\ \cdots & -7 & 7 & 7 & -7 & & \cdots \\ 42 & -84 & 42 & -42 & 84 & -42 & \\ \cdots & & & & & & \cdots \end{bmatrix}}_{R_{12\times 40}} \underbrace{\begin{bmatrix} C_0 \\ \vdots \\ C_8 \\ \vdots \\ C_{40} \end{bmatrix}}_{R_{40}} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}_{R_{12}}$$

w, w/o t^{*n} does not matter as $AC = 0$

$$\therefore \; Ac = b$$

$A \in R_{d\times 40}$

$c \in R_{40}$

$b \in R_d$

d: no. of conditions starting ending continuous

Inequality condition (cont'd)

safety corridor

dynamic constraints

$$Ac \leq b$$

- main reason for us to use bezier curve is that we can CONFINE all control points within our desired range
- all expression should be in the form of $Ac \leq b$

$A \in R_{d\times[(n\_order+1)\times m]}$ — no. of trajectories

$C \in R_{(n\_order+1)\times m}$ — no. of ctrl pts per traj

$b \in R_d$ — $d \Rightarrow$ no. of conditions

safety corridor

let $a \leq x \leq b$

$c \leq y \leq d$

$e \leq z \leq f$

$$\Rightarrow \begin{cases} x \leq b \\ -x \leq -a \\ y \leq d \\ -y \leq -c \\ z \leq f \\ -z \leq -e \end{cases}$$

write everything in matrix

$$AC \leq b$$

express everything in terms of control points

$$\Rightarrow t^1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ -1 & & & \\ & -1 & & \\ & & -1 & \ddots \end{bmatrix} \begin{bmatrix} C_0 \\ \vdots \\ C_{(n\_order+1)\times m} \end{bmatrix} \leq b$$

ieq_P

$$\Rightarrow t^0 \begin{bmatrix} -7 & 7 & & & \\ & -7 & 7 & & \\ & & & \ddots & \\ 7 & -7 & & & \\ & 7 & -7 & & \\ & & & \ddots & \end{bmatrix} \begin{bmatrix} C_0 \\ \vdots \\ C_{(n\_order+1)\times m} \end{bmatrix} \leq b$$

ieq_V

$$\Rightarrow t^{-1} \begin{bmatrix} 42 & -84 & 42 & & \\ & 42 & -84 & 42 & \\ & & & & \ddots \\ -42 & 84 & -42 & & \\ & -42 & 84 & -42 & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} C_0 \\ \vdots \\ C_{(n\_order+1)\times m} \end{bmatrix} \leq b$$

ieq_V

e.g. n_order = 7

m = 5

no. of P ctrl_pts = 80

V = 70

a = 60

$\therefore A \in R_{210\times 40}$

$C \in R_{40}$

$b \in R_{210}$