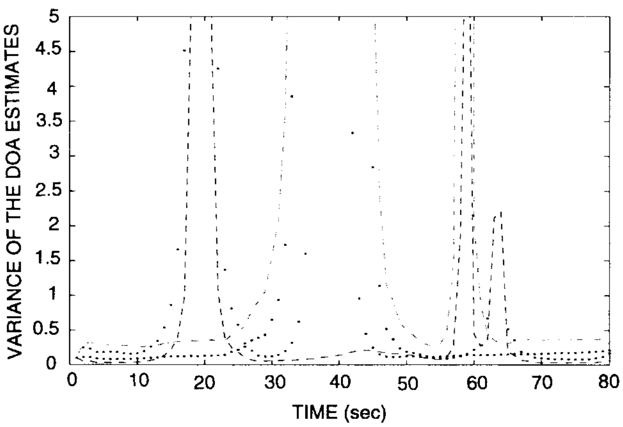

Fig. 3. Theoretical variance of the estimated DOA's according to Stoïca and Nehorai $N = 50$, SNR $= 5$ dB.

ML estimation approach. The study of the behavior of our approach in scenarios where the signals are highly correlated will be the topic of future work.

Fig. 3 exhibits the theoretical asymptotical variance of the estimated DOA's according to (9), which we used in the Kalman part of the algorithm. Except in the crossing areas (which is also the case when using the Cramér–Rao bounds in the Rao's algorithm), the variances are in good agreement with the experimental ones. The actual theoretical performance analysis of the proposed algorithm is the aim of a future work.

V. Conclusion

In this correspondence, we investigated the applicability of the PASTd algorithm in the DOA tracking context. To this end, we proposed an efficient and computationally simple algorithm for tracking the DOA's of multiple moving targets. After a detailed discussion concerning the implementation of our approach, we illustrated its tracking capability via computer sumulations. The obtained results revealed that the proposed algorithm offers an excellent DOA tracking performance in scenarios with crossing targets and low SNR's. Compared with similar DOA tracking algorithms based on ML estimation, our approach is much easier to implement and gives better performance under the same simulation conditions.

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The Extended Kalman Filter as an Exponential Observer for Nonlinear Systems

Konrad Reif and Rolf Unbehauen

***Abstract*— In this correspondence, we analyze the behavior of the extended Kalman filter as a state estimator for nonlinear deterministic systems. Using the direct method of Lyapunov, we prove that under certain conditions, the extended Kalman filter is an exponential observer, i.e., the dynamics of the estimation error is exponentially stable. Furthermore, we discuss a generalization of the Kalman filter with exponential data weighting to nonlinear systems.**

***Index Terms*— Asymptotic stability, Kalman filtering, nonlinear systems, observers.**

I. Introduction

The state estimation of nonlinear stochastic systems corrupted by noise is a widely encountered problem in science and engineering. Although this nonlinear estimation problem is, in general, hard to solve, there are various practical approximation methods available (see, e.g., [12], [19], and [21]). The most useful one is the extended Kalman filter, which has achieved the broadest acceptance

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$\zeta_{n+1} = A_n(I - K_n C_n)\zeta_n + r_n$ (7)

for practical application [1], [5], [6], [10]. A closely related problem is the state estimation for nonlinear deterministic systems, where the system to be observed and the output are noise free. The usual solution is the design of a dynamic state observer, which consists of a model for the system to be observed and an appropriate injection of the outputs. Many possibilities for the construction of observers have been suggested (see, e.g., [13] and [20]). Important relations between the deterministic and the stochastic estimation problem have been clarified in [3], especially the possibility to obtain deterministic observers as asymptotic limits of nonlinear filters. Motivated by these results, we apply the extended Kalman filter as an observer for nonlinear deterministic systems. Using the second method of Lyapunov [9], we prove that it is an exponential observer [8], i.e., the dynamics of the estimation error are exponentially stable. This result includes the convergence of the extended Kalman filter (see also [4]). This property is of particular interest to obtain a nonlinear separation-type property in the context of feedback stabilization of nonlinear systems [22]. In addition to this, we apply the Kalman filter with exponential data weighting [1], [5] to nonlinear systems and prove that the obtained observer has a prescribed degree of stability, i.e., the time constant for the error decay can be assigned in advance. In this correspondence, we focus on the deterministic discrete-time case, the deterministic continuous-time case is treated in [15] and [16], and for the stochastic case, see [17] and [18].

II. The Discrete-Time Extended Kalman Filter

Consider a nonlinear discrete-time system represented by

$$z_{n+1} = f(z_n, x_n) \tag{1a}$$
$$y_n = h(z_n) \tag{1b}$$

where the following abbreviations are used:

- $n \in \mathbf{N}_0$ discrete time;
- $z_n \in \mathbf{R}^q$ state;
- $x_n \in \mathbf{R}^p$ input;
- $y_n \in \mathbf{R}^m$ output.

The functions $f(\cdot, \cdot)$ and $h(\cdot)$ are assumed to be $\mathcal{C}^1$ functions. For this system, we introduce an observer given by

$$\hat{z}_{n+1}^- = f(\hat{z}_n^+, x_n) \tag{2a}$$
$$\hat{z}_n^+ = \hat{z}_n^- + K_n(y_n - h(\hat{z}_n^-)) \tag{2b}$$

where the observer gain K_n is a time-varying $q \times m$ matrix. The estimated states $\hat{z}_n^-$, $\hat{z}_n^+$ are called the *a priori* and *a posteriori* estimate, respectively (see, e.g., [11]). Because $f(\cdot, \cdot)$ and $h(\cdot)$ are $\mathcal{C}^1$ functions, they can be expanded via higher order term

$$f(z_n, x_n) - f(\hat{z}_n^+, x_n) = A_n(z_n - \hat{z}_n^+) + \varphi(z_n, \hat{z}_n^+, x_n) \tag{3}$$
$$h(z_n) - h(\hat{z}_n^-) = C_n(z_n - \hat{z}_n^-) + \chi(z_n, \hat{z}_n^-) \tag{4}$$

with a $q \times q$ matrix A_n and a $m \times q$ matrix C_n given by

$$A_n = \frac{\partial f}{\partial z}(\hat{z}_n^+, x_n) \tag{5a}$$
$$C_n = \frac{\partial h}{\partial z}(\hat{z}_n^-) \tag{5b}$$

respectively. Introducing the estimation error

$$\zeta_n = z_n - \hat{z}_n^- \tag{6}$$

subtracting (2a) from (1a), and using (1b) and (2b)–(4) yields

$$\zeta_{n+1} = A_n(I - K_n C_n)\zeta_n + r_n \tag{7}$$

where r_n is given by

$$r_n = \varphi(z_n, \hat{z}_n^+, x_n) - A_n K_n \chi(z_n, \hat{z}_n^-). \tag{8}$$

For the analysis of the error dynamics (7) we recall the exponential stability of discrete-time systems (cf. [9, Sec. 4.1, Def. 4.1.2, p. 88]) and introduce a discrete-time version of the exponential observer in [8]:

Definition 1: The difference equation (7) has an exponentially stable equilibrium point at 0 if there are positive real numbers ϵ, $\eta > 0$, and $\theta > 1$ such that

$$\|\zeta_n\| \le \eta \|\zeta_0\| \theta^{-n} \tag{9}$$

holds for every $n \ge 0$ and for every solution ζ_n of (7) with $\zeta_0 \in B_\epsilon$, where $B_\epsilon = \{v \in \mathbf{R}^q \mid \|v\| < \epsilon\}$.

Remark: In this correspondence, we consider the *a priori* estimation error given by $z_n - \hat{z}_n^-$. It is also possible to use the *a posteriori* estimation error $z_n - \hat{z}_n^+$ instead. Consequently, then, we have to consider a bound for the *a posteriori* initial error $z_0 - \hat{z}_0^+$.

Definition 2: The observer given by (2a) and (2b) is an exponential observer if the difference equation (7) has an exponentially stable equilibrium at 0.

For the considerations stated below, we use a slightly more general definition of the extended Kalman filter such that the following two cases can be treated at once: the usual extended Kalman filter and the Kalman filter with exponential data weighting.

Definition 3: A (deterministic) discrete-time extended Kalman filter is given by the following coupled difference equations.

Time Update:

$$\hat{z}_{n+1}^- = f(\hat{z}_n^+, x_n) \tag{10a}$$
$$P_{n+1}^- = \alpha^2 A_n P_n^+ A_n^T + Q. \tag{10b}$$

Linearization:

$$A_n = \frac{\partial f}{\partial z}(\hat{z}_n^+, x_n). \tag{11}$$

Measurement Update:

$$\hat{z}_n^+ = \hat{z}_n^- + K_n(y_n - h(\hat{z}_n^-)) \tag{12a}$$
$$P_n^+ = (I - K_n C_n) P_n^-. \tag{12b}$$

Kalman Gain:

$$K_n = P_n^- C_n^T (C_n P_n^- C_n^T + R)^{-1}. \tag{13}$$

Linearization:

$$C_n = \frac{\partial h}{\partial z}(\hat{z}_n^-). \tag{14}$$

The system output y_n is determined by (1a) and (1b), where

- Q symmetric positive definite $q \times q$ matrix;
- R symmetric positive definite $m \times m$ matrix
- $\alpha \ge 1$ real number.

Remarks:

1) For $\alpha = 1$, we obtain the usual extended Kalman filter (see, e.g., [1, Sec. 8.2, p. 195]), and for $\alpha > 1$ the Kalman filter exponential data weighting (see, e.g., [1, Sec. 6.2, pp. 135–138]).
2) Using the Kalman filter in its original sense as an optimal filter for linear stochastic systems, the matrices Q and R are the covariance matrices of the corrupting noise terms. For the application as a nonlinear deterministic observer, Q and R can be selected as arbitrary symmetric positive definite matrices. Although they do not affect the stability of the observer, they have a significant influence on the performance. Similarly for P_0^+, any symmetric positive definite matrix can be chosen if we treat the observation of nonlinear deterministic systems.

3) The measurement update (12b) for the matrix P_n^+ can be reformulated by (see, e.g., [5, Sec, 4.2, p. 108])

$$P_n^+ = (I - K_n C_n) P_n^- (I - K_n C_n)^T + K_n R K_n^T. \tag{15}$$

4) The formula (13) of the Kalman gain can be rewritten by (see, e.g., [5, Sec, 4.2, p. 112])

$$K_n = P_n^+ C_n^T R^{-1}. \tag{16}$$

To prove that the discrete-time extended Kalman filter is an exponential observer, we use the following three preparatory lemmas. First, we establish a bound for the remainder r_n in (7); second, we recapitulate a well-known formula for inverting matrix expressions; last, we verify a useful matrix inequality concerning the solution of (10b) and (12b) or (15).

Lemma 4: Consider real vectors $z,\, \hat{z}^-,\, \hat{z}^+ \in \mathbf{R}^q$, $x \in \mathbf{R}^p$, a $q \times q$ matrix A, a $m \times q$ matrix C, a $q \times m$ matrix K, and nonlinear functions $\varphi(\cdot,\cdot,\cdot)$ and $\chi(\cdot,\cdot)$ such that the following assumptions hold.

1) There are positive real numbers $\overline{a},\, \overline{c},\, \overline{k} > 0$ such that the subsequent bounds are satisfied as

$$\|A\| \le \overline{a} \tag{17a}$$
$$\|C\| \le \overline{c} \tag{17b}$$
$$\|K\| \le \overline{k}. \tag{17c}$$

2) There are positive real numbers $\epsilon_\varphi,\, \epsilon_\chi,\, \kappa_\varphi,\, \kappa_\chi > 0$ such that

higher-order term

$$\|\varphi(z, \hat{z}^+, x)\| \le \kappa_\varphi \|z - \hat{z}^+\|^2 \tag{18a}$$
$$\|\chi(z, \hat{z}^-)\| \le \kappa_\chi \|z - \hat{z}^-\|^2 \tag{18b}$$

hold for $\|z - \hat{z}^+\| \le \epsilon_\varphi$ and for $\|z - \hat{z}^-\| \le \epsilon_\chi$, respectively.

3) $\hat{z}^+$ satisfies

$$\hat{z}^+ = \hat{z}^- + KC(z - \hat{z}^-) + K\chi(z, \hat{z}^-). \tag{19}$$

Let r be defined by

$$r = \varphi(z, \hat{z}^+, x) - AK\chi(z, \hat{z}^-). \tag{20}$$

Then, there are positive real numbers $\epsilon,\, \kappa > 0$ such that

$$\|r\| \le \kappa \|z - \hat{z}^-\|^2 \tag{21}$$

holds for $\|z - \hat{z}^-\| \le \epsilon$.

Proof: From (19), using the triangular inequality as well as (17b), (17c), (18b), we obtain

$$\|z - \hat{z}^+\| \le (1 + \overline{k}\overline{c} + \overline{k}\kappa_\chi \epsilon_\chi)\|z - \hat{z}^-\| \tag{22}$$

for $\|z - \hat{z}^-\| \le \epsilon_\chi$. Consider now (20). Applying the triangular inequality and (17a) (17c) yields

$$\|r\| \le \|\varphi(z, \hat{z}^+, x)\| + \overline{a}\overline{k}\|\chi(\hat{z}^-, x)\|. \tag{23}$$

Choose

$$\epsilon = \min\left(\epsilon_\chi,\, \frac{\epsilon_\varphi}{1 + \overline{k}\overline{c} + \overline{k}\kappa_\chi \epsilon_\chi}\right). \tag{24}$$

For $\|z - \hat{z}^-\| \le \epsilon$, we satisfy $\|z - \hat{z}^-\| \le \epsilon_\chi$ as well as $\|z - \hat{z}^+\| \le \epsilon_\varphi$ if we take into account (22). Using Assumption 2 and (22), we obtain from (23)

$$\|r\| \le \kappa_\varphi (1 + \overline{k}\overline{c} + \overline{k}\kappa_\chi \epsilon_\chi)^2 \|z - \hat{z}^-\|^2 + \overline{a}\overline{k}\kappa_\chi \|z - \hat{z}^-\|^2$$

i.e., (21) with

$$\kappa = \kappa_\varphi (1 + \overline{k}\overline{c} + \overline{k}\kappa_\chi \epsilon_\chi)^2 + \overline{a}\overline{k}\kappa_\chi. \tag{25}$$

□

Lemma 5: Consider two nonsingular $q \times q$ matrices Γ and Δ, and assume that $\Gamma^{-1} + \Delta$ is nonsingular. Then

$$(\Gamma^{-1} + \Delta)^{-1} = \Gamma - \Gamma(\Gamma + \Delta^{-1})^{-1}\Gamma. \tag{26}$$

Proof: See, e.g., [1, Sec. 6.3, p. 139] or [11, App. A.2, p. 347]. □

Lemma 6: Consider symmetric positive definite solutions $P_n^-,\, P_n^+,\, n \ge 0$ of the difference equations (10b) and (12b). Define $\Pi_n^-,\, \Pi_n^+$ by

$$\Pi_n^- = (P_n^-)^{-1} \tag{27a}$$
$$\Pi_n^+ = (P_n^+)^{-1} \tag{27b}$$

and assume that A_n^{-1} and $(I - K_n C_n)^{-1}$ exist for $n \ge 0$. Then

$$\begin{aligned} \Pi_{n+1}^- \le\, & \alpha^{-2} A_n^{-T} (I - K_n C_n)^{-T} \\ & \cdot \left[\Pi_n^- - \Pi_n^- (\Pi_n^+ + \alpha^2 A_n^T Q^{-1} A_n)^{-1} \Pi_n^-\right] \\ & \cdot (I - K_n C_n)^{-1} A_n^{-1}. \end{aligned} \tag{28}$$

Proof: From (15), we have

$$P_n^+ \ge (I - K_n C_n) P_n^- (I - K_n C_n)^T.$$

Taking the inverse and using (27a) and (27b), we obtain

$$\Pi_n^+ \le (I - K_n C_n)^{-T} \Pi_n^- (I - K_n C_n)^{-1}. \tag{29}$$

Inverting (10b), i.e.,

$$P_{n+1}^- = \alpha^2 A_n (P_n^+ + \alpha^{-2} A_n^{-1} Q A_n^{-T}) A_n^T$$

yields

$$\Pi_{n+1}^- = \alpha^{-2} A_n^{-T} (P_n^+ + \alpha^{-2} A_n^{-1} Q A_n^{-T})^{-1} A_n^{-1}.$$

Applying Lemma 5, we get

$$\Pi_{n+1}^- = \alpha^{-2} A_n^{-T} [\Pi_n^+ - \Pi_n^+ (\Pi_n^+ + \alpha^2 A_n^T Q^{-1} A_n)^{-1} \Pi_n^+] A_n^{-1}. \tag{30}$$

Using (29) and two rearranged forms of (12b), i.e.,

$$\Pi_n^+ = \Pi_n^- (I - K_n C_n)^{-1}, \qquad \Pi_n^+ = (I - K_n C_n)^{-T} \Pi_n^-$$

(since Π_n^- and Π_n^+ are symmetric), we obtain the desired result (28). □

With these prerequisites, we can state the main result of Section II.

Theorem 7: Consider a discrete-time extended Kalman filter as stated in Definition 3, and let the following assumptions hold.

1) There are positive real numbers $\overline{a},\, \overline{c},\, \underline{p},\, \overline{p} > 0$ such that the following bounds on various matrices are fulfilled for every $n \ge 0$:

$$\|A_n\| \le \overline{a} \tag{31a}$$
$$\|C_n\| \le \overline{c} \tag{31b}$$
$$\underline{p} I \le P_n^- \le \overline{p} I \tag{32a}$$
$$\underline{p} I \le P_n^+ \le \overline{p} I. \tag{32b}$$

2) A_n is nonsingular for every $n \ge 0$.

3) There are positive real numbers $\epsilon_\varphi,\, \epsilon_\chi,\, \kappa_\varphi,\, \kappa_\chi > 0$ such that the nonlinear functions $\varphi(\cdot,\cdot,\cdot)$, $\chi(\cdot,\cdot)$ in (8) are bounded via

$$\|\varphi(z, \hat{z}^+, x)\| \le \kappa_\varphi \|z - \hat{z}^+\|^2 \tag{33a}$$
$$\|\chi(z, \hat{z}^-)\| \le \kappa_\chi \|z - \hat{z}^-\|^2 \tag{33b}$$

for $z,\, \hat{z}^+,\, \hat{z}^- \in \mathbf{R}^q$, and $x \in \mathbf{R}^p$ with $\|z - \hat{z}^+\| \le \epsilon_\varphi$ and $\|z - \hat{z}^-\| \le \epsilon_\chi$, respectively.

$\kappa \|\epsilon_{k}\| \le \kappa\alpha$

$-\kappa \|\epsilon_{k}\| \ge -\kappa\epsilon$

Then, the extended Kalman filter is an exponential observer. Moreover, the constant θ for the exponential error decay in (9) satisfies $\theta > \alpha$. exponential data weighting

Remarks:

1) Setting $\alpha = 1$ in (10b), we conclude that the usual extended Kalman filter is an exponential observer. Choosing an $\alpha > 1$, it follows from $\theta > \alpha$ that the constant θ in (9) can be assigned in advance, i.e., we obtain an observer with a prescribed degree of stability.

2) In general, the bounds (32a) and (32b) can be checked on-line during the estimation process. Moreover, they are satisfied if the matrices A_n, C_n fulfill the uniform observability condition (cf. [6, Sec. 7.6, pp. 234–243]). Generalizations to a weaker condition, i.e., uniform detectability has been proposed in [2] and [14].

Proof: We consider the difference equation (7) for the estimation error ζ_n. To prove its exponential stability, we select the Lyapunov function

$$V_n(\zeta_n) = \zeta_n^T \Pi_n^- \zeta_n \tag{34}$$

with $\Pi_n^- = (P_n^-)^{-1}$. Because of (32a), we have the bounds for the Lyapunov function

$$\frac{1}{\overline{p}} \|\zeta_n\|^2 \le V_n(\zeta_n) \le \frac{1}{\underline{p}} \|\zeta_n\|^2 \tag{35}$$

which include positive definiteness and decrescence. From (32a) and (32b), it follows that P_n^- and P_n^+ are nonsingular, and because of (12b), the matrix

$$(I - K_n C_n)^{-1} = P_n^- \Pi_n^+$$

exists. Together with Assumption 2, we fulfill the requirements to apply Lemma 6. Estimating $V_{n+1}(\zeta_{n+1})$ with (7) and (28), we get

$$\begin{aligned} V_{n+1}(\zeta_{n+1}) &= \zeta_{n+1}^T \Pi_{n+1}^- \zeta_{n+1} \\ &\le \alpha^{-2} \zeta_n^T [\Pi_n^- - \Pi_n^- (\Pi_n^+ + \alpha^2 A_n^T Q^{-1} A_n)^{-1} \Pi_n^-] \zeta_n \\ &\quad + 2 r_n^T \Pi_{n+1}^- A_n (I - K_n C_n) \zeta_n + r_n^T \Pi_{n+1}^- r_n. \end{aligned} \tag{36}$$

r_n

Denoting the smallest eigenvalue of R by $\underline{r}$, we obtain from (16) and (31b), (32b)

$$\|K_n\| \le \|P_n^+\| \|C_n\| \|R^{-1}\| \le \overline{k} \tag{37}$$

cond 2

cond 1

where $\overline{k} = \overline{p}\,\overline{c}/\underline{r}$. Considering (31a), (31b), (37), (33a), and (33b), we meet the requirements to apply Lemma 4 to (36). Together with (31a)–(32b) and (37), this leads to

$$\begin{aligned} V_{n+1}(\zeta_{n+1}) \le \alpha^{-2} \zeta_n^T \Pi_n^- \zeta_n - \frac{1}{\alpha^2 \overline{p}^2 (\overline{p} + \alpha^2 \overline{a}^2 / \underline{q})} \|\zeta_n\|^2 \\ + 2\kappa \|\zeta_n\|^2 \frac{\overline{a}(1 + \overline{k}\overline{c})}{\underline{p}} \|\zeta_n\| + \kappa \|\zeta_n\|^2 \frac{1}{\underline{p}} \kappa\epsilon \|\zeta_n\| \end{aligned} \tag{38}$$

for $\|\zeta_n\| \le \epsilon$, where $\underline{q} > 0$ is the smallest eigenvalue of the positive definite matrix Q. Defining κ' by

$$\kappa' = \frac{\kappa}{\underline{p}} (2\overline{a}(1 + \overline{k}\overline{c}) + \kappa\epsilon) \tag{39}$$

and using (34), the inequality (38) can be rewritten as

$$\begin{aligned} &V_{n+1}(\zeta_{n+1}) \\ &\quad \le \alpha^{-2} V_n(\zeta_n) - \left(\frac{1}{\alpha^2 \overline{p}^2 (\overline{p} + \alpha^2 \overline{a}^2 / \underline{q})} - \kappa' \|\zeta_n\| \right) \|\zeta_n\|^2. \end{aligned} \tag{40}$$

Introducing

$$\epsilon' = \min \left(\epsilon, \frac{1}{2\alpha^2 \kappa' \overline{p}^2 (\overline{p} + \alpha^2 \overline{a}^2 / \underline{q})} \right) \tag{41}$$

it follows that

$$\begin{aligned} &V_{n+1}(\zeta_{n+1}) - V_n(\zeta_n) \\ &\quad \le -\frac{1}{2\alpha^2 \overline{p}^2 (\overline{p} + \alpha^2 \overline{a}^2 / \underline{q})} \|\zeta_n\|^2 + (\alpha^{-2} - 1) V_n(\zeta_n) \end{aligned} \tag{42}$$

holds for $\|\zeta_n\| \le \epsilon'$, which implies that $V_{n+1}(\zeta_{n+1}) - V_n(\zeta_n)$ is locally negative definite [because of (35) and $\alpha \ge 1$, the second term on the right-hand side is negative semidefinite]. Applying standard results on Lyapunov functions for difference equations (cf. [9, Sec. 4.8, Th. 4.8.3, p. 108]), we conclude that the difference equation (7) has an asymptotically stable equilibrium point at 0. Furthermore, fr

and using (34), the inequality (38) can be rewritten as

$$\begin{aligned} &V_{n+1}(\zeta_{n+1}) \\ &\quad \le \alpha^{-2} V_n(\zeta_n) - \left(\frac{1}{\alpha^2 \overline{p}^2 (\overline{p} + \alpha^2 \overline{a}^2 / \underline{q})} - \kappa' \|\zeta_n\| \right) \|\zeta_n\|^2. \end{aligned}$$

Without loss of generality, we may assume $\overline{p} > 1$, which implies

$$1 - \frac{\underline{p}}{2\overline{p}^2 (\overline{p} + \alpha^2 \overline{a}^2 / \underline{q})} > 0.$$

Using (35) once more, we get, from (44)

$$\|\zeta_n\| \le \sqrt{\overline{p}/\underline{p}} \|\zeta_0\| \left(\alpha \Big/ \sqrt{1 - \frac{\underline{p}}{2\overline{p}^2 (\overline{p} + \alpha^2 \overline{a}^2 / \underline{q})}} \right)^{-n} \tag{45}$$

i.e., (9) with

$$\eta = \sqrt{\overline{p}/\underline{p}} > 0 \tag{46}$$

and

$$\theta = \alpha \Big/ \sqrt{1 - \frac{\underline{p}}{2\overline{p}^2 (\overline{p} + \alpha^2 \overline{a}^2 / \underline{q})}} > \alpha \tag{47}$$

□

III. Conclusions

The purpose of this correspondence is to point out that although the Kalman filter was originally designed for linear stochastic systems, it is very useful in the context of nonlinear deterministic systems. If the extended Kalman filter operates as a deterministic observer, it has very nice properties, i.e., under certain conditions it is an exponential observer. This fact is embodied in Theorem 7. To prove this theorem, we employ a standard Lyapunov-function technique for difference equations (cf. [9]). Although the proof is a little involved, we use exclusively standard estimation techniques and straightforward calculations. A key role plays the close relation between the Lyapunov and Riccati matrix equations: a fact that is widely used in robust control theory (see, e.g., [7] and the references cited therein).

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$\|r\| \leq \kappa \|z - \hat{z}^{-}\|^2$

$\|z - \hat{z}^{-}\| \leq \epsilon.$

From (19), usir

$|r|^2$ $|r|$.

$\propto \|\delta - \hat{\delta}\|^2$ $\propto \|\delta - \hat{\delta}\|^2$

$\Rightarrow \propto \|\delta - \hat{\delta}\|^2 \propto \|\delta - \hat{\delta}\|$ β

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Modified B-Splines for the Sampling of Bandlimited Functions

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***Abstract*—It is shown that there is an optimal finite linear combination of B-splines (denominated modified B-splines) such that a pertinent low frequency condition called M-flatness is satisfied. A profound relationship of the modified B-splines with the Beta distribution implies an asymptotic sampling theorem with exact reconstruction requiring only small oversampling.**

***Index Terms*— Beta distribution, B-splines, oversampling, sampling theorem.**

I. Introduction

Despite its paramount importance, the well-known Shannon sampling theorem [1] for bandlimited functions poses some pertinent theoretical and practical problems. A major theoretical question is whether bandlimited functions physically exist at all since in theory, a bandlimited function cannot be time,limited and vice versa. This difficulty has been addressed by Slepian [2], and his conclusion is that most functions occurring in real situations are both essentially bandlimited *and* time limited. A more practical flaw of the sampling theorem is that the sampling sinc kernel decays too slowly at infinity, making truncation a delicate exercise. This has been addressed in [3], resulting in sampling kernels with a better behavior at infinity.

In this correspondence, we discuss the sampling of a bandlimited function by means of a time limited kernel based on B-splines [4]. B-splines are, in a sense, the most natural interpolants since they are obtained by the successive convolutions of a rectangular pulse (Fourier transform = sinc!). It should therefore not come as a surprise that there is a connection between B-splines and the sampling theorem. Unfortunately, as a consequence of the Paley–Wiener theorem [5], the sampling kernels defined in [6] are not time limited.

We therefore address the construction of a time limited sampling kernel consisting of a finite linear combination of B-splines by imposing a low-frequency M-flat condition on an upper bound for the relative sampling error. Our approach bears a close relationship with recent developments in connection with Strang–Fix theory [7]. It is shown that there is an optimal finite linear combination of B-splines, called modified B-splines, such that M-flatness is satisfied. The profound relationship of the modified B-splines with the Beta distribution [8] then leads to an asymptotic sampling theorem with exact reconstruction requiring only small oversampling with respect to the Shannon rate.

II. A Low-Frequency Sampling Condition

Consider the function $x(t)$ bandlimited in the Fourier domain to $|\omega| \leq B$ and its sampling approximation

$$\tilde{x}(t) = \Delta \sum_{n \in Z} x(n\Delta) u(t - n\Delta) \tag{1}$$

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