- Motivation
  - true-state
  - nominal-state
  - error-state

△ true-state = nominal-state ⊕ error state

△ nominal-state → takes no account in ~~noise~~ model imperfection

Definitions

Magnitude	True	Nominal	Error	Composition	Measured	Noise
Full state (1)	$\mathbf{x}_t$	$\mathbf{x}$	$\delta\mathbf{x}$	$\mathbf{x}_t = \mathbf{x} \oplus \delta\mathbf{x}$		
Position	$\mathbf{p}_t$	$\mathbf{p}$	$\delta\mathbf{p}$	$\mathbf{p}_t = \mathbf{p} + \delta\mathbf{p}$		
Velocity	$\mathbf{v}_t$	$\mathbf{v}$	$\delta\mathbf{v}$	$\mathbf{v}_t = \mathbf{v} + \delta\mathbf{v}$		
Quaternion (2,3)	$\mathbf{q}_t$	$\mathbf{q}$	$\delta\mathbf{q}$	$\mathbf{q}_t = \mathbf{q} \otimes \delta\mathbf{q}$		
Rotation matrix (2,3)	$\mathbf{R}_t$	$\mathbf{R}$	$\delta\mathbf{R}$	$\mathbf{R}_t = \mathbf{R}\,\delta\mathbf{R}$		
Angles vector (4)			$\delta\boldsymbol{\theta}$	$\delta\mathbf{q} = e^{\delta\boldsymbol{\theta}/2}$, $\delta\mathbf{R} = e^{[\delta\boldsymbol{\theta}]_\times}$		
Accelerometer bias	$\mathbf{a}_{bt}$	$\mathbf{a}_b$	$\delta\mathbf{a}_b$	$\mathbf{a}_{bt} = \mathbf{a}_b + \delta\mathbf{a}_b$		$\mathbf{a}_w$
Gyrometer bias	$\boldsymbol{\omega}_{bt}$	$\boldsymbol{\omega}_b$	$\delta\boldsymbol{\omega}_b$	$\boldsymbol{\omega}_{bt} = \boldsymbol{\omega}_b + \delta\boldsymbol{\omega}_b$		$\boldsymbol{\omega}_w$
Gravity vector	$\mathbf{g}_t$	$\mathbf{g}$	$\delta\mathbf{g}$	$\mathbf{g}_t = \mathbf{g} + \delta\mathbf{g}$		
Acceleration	$\mathbf{a}_t$				$\mathbf{a}_m$	$\mathbf{a}_n$
Angular rate	$\boldsymbol{\omega}_t$				$\boldsymbol{\omega}_m$	$\boldsymbol{\omega}_n$

R: rotation matrix from body to inertial frame

4

- Dynamic

p_t $\quad \dot{\mathbf{p}}_t = \mathbf{v}_t$

v_t $\quad \dot{\mathbf{v}}_t = \mathbf{a}_t$

q_t $\quad \dot{\mathbf{q}}_t = \frac{1}{2}\mathbf{q}_t \otimes \boldsymbol{\omega}_t$

a_{bt} $\quad \dot{\mathbf{a}}_{bt} = \mathbf{a}_w$

w_{bt} $\quad \dot{\boldsymbol{\omega}}_{bt} = \boldsymbol{\omega}_w$

g_t $\quad \dot{\mathbf{g}}_t = 0$

- measurement

$$\mathbf{a}_m = \mathbf{R}_t^\top(\mathbf{a}_t - \mathbf{g}_t) + \mathbf{a}_{bt} + \mathbf{a}_n$$
$$\boldsymbol{\omega}_m = \boldsymbol{\omega}_t + \boldsymbol{\omega}_{bt} + \boldsymbol{\omega}_n$$

Sensor noise

Note: a_m, ω_m are measuremed in the body-fixed frame
a_t, ω_t are exoressed in the inertial frame

True State Dynamics

$$\dot{\mathbf{p}}_t = \mathbf{v}_t \qquad (235a)$$
$$\dot{\mathbf{v}}_t = \mathbf{R}_t(\mathbf{a}_m - \mathbf{a}_{bt} - \mathbf{a}_n) + \mathbf{g}_t \qquad (235b)$$
$$\dot{\mathbf{q}}_t = \frac{1}{2}\mathbf{q}_t \otimes (\boldsymbol{\omega}_m - \boldsymbol{\omega}_{bt} - \boldsymbol{\omega}_n) \qquad (235c)$$
$$\dot{\mathbf{a}}_{bt} = \mathbf{a}_w \qquad (235d)$$
$$\dot{\boldsymbol{\omega}}_{bt} = \boldsymbol{\omega}_w \qquad (235e)$$
$$\dot{\mathbf{g}}_t = 0 \qquad (235f)$$

true state

noise term

$\dot{x}_t = f_t(x_t, u, w)$

input

$$\mathbf{a}_t = \mathbf{R}_t(\mathbf{a}_m - \mathbf{a}_{bt} - \mathbf{a}_n) + \mathbf{g}_t \qquad (233)$$
$$\boldsymbol{\omega}_t = \boldsymbol{\omega}_m - \boldsymbol{\omega}_{bt} - \boldsymbol{\omega}_n. \qquad (234)$$

Nominal State Kinematics

$$x = \begin{bmatrix} p \\ v \\ q \\ a_b \\ w_b \\ g \end{bmatrix}$$

$$\dot{\mathbf{p}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = \mathbf{R}(\mathbf{a}_m - \mathbf{a}_b) + \mathbf{g}$$
$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{q} \otimes (\boldsymbol{\omega}_m - \boldsymbol{\omega}_b) \qquad (237)$$
$$\dot{\mathbf{a}}_b = 0$$
$$\dot{\boldsymbol{\omega}}_b = 0$$
$$\dot{\mathbf{g}} = 0.$$

Error State Kinematics

$$\boxed{\mathbf{p}_t = \mathbf{p} + \delta\mathbf{p}}$$

$\dot{\delta p}$ =(235)-(237)

a_n
w_n
a_w
w_w

$$\dot{\delta\mathbf{p}} = \delta\mathbf{v}$$
$$\dot{\delta\mathbf{v}} = -\mathbf{R}\,[\mathbf{a}_m - \mathbf{a}_b]_\times\,\delta\boldsymbol{\theta} - \mathbf{R}\delta\mathbf{a}_b + \delta\mathbf{g} - \mathbf{R}\mathbf{a}_n$$
$$\dot{\delta\boldsymbol{\theta}} = -[\boldsymbol{\omega}_m - \boldsymbol{\omega}_b]_\times\,\delta\boldsymbol{\theta} - \delta\boldsymbol{\omega}_b - \boldsymbol{\omega}_n$$
$$\dot{\delta\mathbf{a}}_b = \mathbf{a}_w$$
$$\dot{\delta\boldsymbol{\omega}}_b = \boldsymbol{\omega}_w$$
$$\dot{\delta\mathbf{g}} = 0.$$

Gaussian noises

Discrete-time Nominal States

Taking the integration of (237) yields the discrete-time form as

$$\mathbf{p} \leftarrow \mathbf{p} + \mathbf{v}\,\Delta t + \frac{1}{2}(\mathbf{R}(\mathbf{a}_m - \mathbf{a}_b) + \mathbf{g})\,\Delta t^2 \qquad (260a)$$
$$\mathbf{v} \leftarrow \mathbf{v} + (\mathbf{R}(\mathbf{a}_m - \mathbf{a}_b) + \mathbf{g})\,\Delta t \qquad (260b)$$
$$\mathbf{q} \leftarrow \mathbf{q} \otimes \mathbf{q}\{(\boldsymbol{\omega}_m - \boldsymbol{\omega}_b)\,\Delta t\} \qquad (260c)$$
$$\mathbf{a}_b \leftarrow \mathbf{a}_b \qquad (260d)$$
$$\boldsymbol{\omega}_b \leftarrow \boldsymbol{\omega}_b \qquad (260e)$$
$$\mathbf{g} \leftarrow \mathbf{g}\,, \qquad (260f)$$

Discrete-time Error States

Taking the integration of (238) yields the discrete-time form as

$$\delta\mathbf{p} \leftarrow \delta\mathbf{p} + \delta\mathbf{v}\,\Delta t \qquad (261a)$$
$$\delta\mathbf{v} \leftarrow \delta\mathbf{v} + (-\mathbf{R}\,[\mathbf{a}_m - \mathbf{a}_b]_\times\,\delta\boldsymbol{\theta} - \mathbf{R}\delta\mathbf{a}_b + \delta\mathbf{g})\Delta t + \mathbf{v_i} \qquad (261b)$$
$$\delta\boldsymbol{\theta} \leftarrow \mathbf{R}^\top\{(\boldsymbol{\omega}_m - \boldsymbol{\omega}_b)\Delta t\}\delta\boldsymbol{\theta} - \delta\boldsymbol{\omega}_b\Delta t + \boldsymbol{\theta}_\mathbf{i} \qquad (261c)$$
$$\delta\mathbf{a}_b \leftarrow \delta\mathbf{a}_b + \mathbf{a_i} \qquad (261d)$$
$$\delta\boldsymbol{\omega}_b \leftarrow \delta\boldsymbol{\omega}_b + \boldsymbol{\omega}_\mathbf{i} \qquad (261e)$$
$$\delta\mathbf{g} \leftarrow \delta\mathbf{g}\,. \qquad (261f)$$

$$\mathbf{V_i} = \sigma_{\tilde{\mathbf{a}}_n}^2\,\Delta t^2\mathbf{I} \quad [m^2/s^2] \qquad (262)$$
$$\boldsymbol{\Theta}_\mathbf{i} = \sigma_{\tilde{\boldsymbol{\omega}}_n}^2\,\Delta t^2\mathbf{I} \quad [rad^2] \qquad (263)$$
$$\mathbf{A_i} = \sigma_{\mathbf{a}_w}^2\,\Delta t\mathbf{I} \quad [m^2/s^4] \qquad (264)$$
$$\boldsymbol{\Omega}_\mathbf{i} = \sigma_{\boldsymbol{\omega}_w}^2\,\Delta t\mathbf{I} \quad [rad^2/s^2] \qquad (265)$$

Integration of covariance matrices

ESKF v.s. EKF

ESKF

$$\hat{\delta\mathbf{x}} \leftarrow \mathbf{F_x}(\mathbf{x}, \mathbf{u}_m)\cdot\hat{\delta\mathbf{x}} \qquad (268)$$
$$\mathbf{P} \leftarrow \mathbf{F_x}\,\mathbf{P}\,\mathbf{F_x}^\top + \mathbf{F_i}\,\mathbf{Q_i}\,\mathbf{F_i}^\top\,, \qquad (269)$$

$$\mathbf{K} = \mathbf{P}\mathbf{H}^\top(\mathbf{H}\mathbf{P}\mathbf{H}^\top + \mathbf{V})^{-1} \qquad (274)$$
$$\hat{\delta\mathbf{x}} \leftarrow \mathbf{K}(\mathbf{y} - h(\hat{\mathbf{x}}_t)) \qquad (275)$$
$$\mathbf{P} \leftarrow (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P} \qquad (276)$$

EKF

$A = \frac{\partial f}{\partial x}|_{\hat{x}_{k-1},u_{k-1}}$ and $C = \frac{\partial g}{\partial x}|_{\hat{x}_k}$

$$\hat{x}_k^- = f(\hat{x}_{k-1}, u_{k-1})$$
$$P_k^- = A_k P_{k-1} A_k^T + Q$$

$$\mathcal{K}_k = P_k^- C^T (C P_k^- C^T + R)^{-1}$$
$$\hat{x}_k = \hat{x}_k^- + \mathcal{K}_k(y_k - \hat{y}_k)$$
$$P_k = (I - \mathcal{K}_k C)P_k^-$$

Injection of the Observed Error into the Nominal State

6.2 Injection of the observed error into the nominal state

After the ESKF update, the nominal state gets updated with the observed error state using the appropriate compositions (sums or quaternion products, see Table 3),

$$\mathbf{x} \leftarrow \mathbf{x} \oplus \hat{\delta\mathbf{x}}\,, \qquad (282)$$

that is,

$$\mathbf{p} \leftarrow \mathbf{p} + \hat{\delta\mathbf{p}} \qquad (283a)$$
$$\mathbf{v} \leftarrow \mathbf{v} + \hat{\delta\mathbf{v}} \qquad (283b)$$
$$\mathbf{q} \leftarrow \mathbf{q} \otimes \mathbf{q}\{\hat{\delta\boldsymbol{\theta}}\} \qquad (283c)$$
$$\mathbf{a}_b \leftarrow \mathbf{a}_b + \hat{\delta\mathbf{a}}_b \qquad (283d)$$
$$\boldsymbol{\omega}_b \leftarrow \boldsymbol{\omega}_b + \hat{\delta\boldsymbol{\omega}}_b \qquad (283e)$$
$$\mathbf{g} \leftarrow \mathbf{g} + \hat{\delta\mathbf{g}} \qquad (283f)$$

Error State Reset

Let us call the error reset function $g()$. It is written as follows,

$$\delta\mathbf{x} \leftarrow g(\delta\mathbf{x}) = \delta\mathbf{x} \ominus \hat{\delta\mathbf{x}}\,, \qquad (284)$$

where $\ominus$ stands for the composition inverse of $\oplus$. The ESKF error reset operation is thus,

$$\delta\mathbf{x} \leftarrow 0 \qquad (285)$$
$$\mathbf{P} \leftarrow \mathbf{G}\,\mathbf{P}\,\mathbf{G}^\top\,. \qquad (286)$$

where $\mathbf{G}$ is the Jacobian matrix defined by,

$$\mathbf{G} \triangleq \left.\frac{\partial g}{\partial \delta\mathbf{x}}\right|_{\hat{\delta\mathbf{x}}}\,. \qquad (287)$$

Similarly to what happened with the update Jacobian above, this Jacobian is the identity on all diagonal blocks except in the orientation error. We give here the full expression and proceed in the following section with the derivation of the orientation error block, $\partial\delta\boldsymbol{\theta}^+/\partial\delta\boldsymbol{\theta} = \mathbf{I} - \left[\frac{1}{2}\hat{\delta\boldsymbol{\theta}}\right]_\times$,

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_6 & 0 & 0 \\ 0 & \mathbf{I} - \left[\frac{1}{2}\hat{\delta\boldsymbol{\theta}}\right]_\times & 0 \\ 0 & 0 & \mathbf{I}_9 \end{bmatrix}\,. \qquad (288)$$