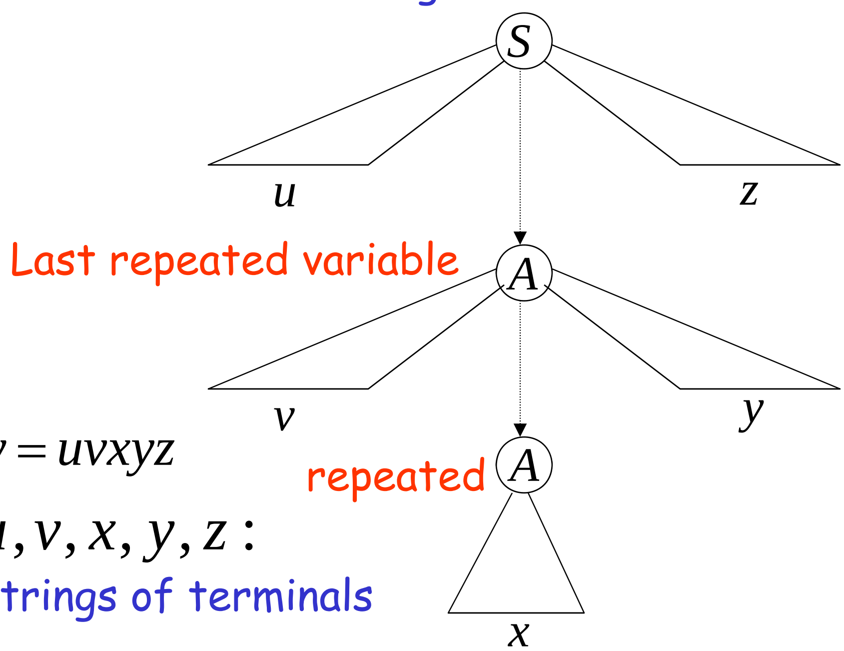
The Pumping Lemma for Context-Free Languages

Derivation tree of string w



$w = uvxyz$

u, v, x, y, z :

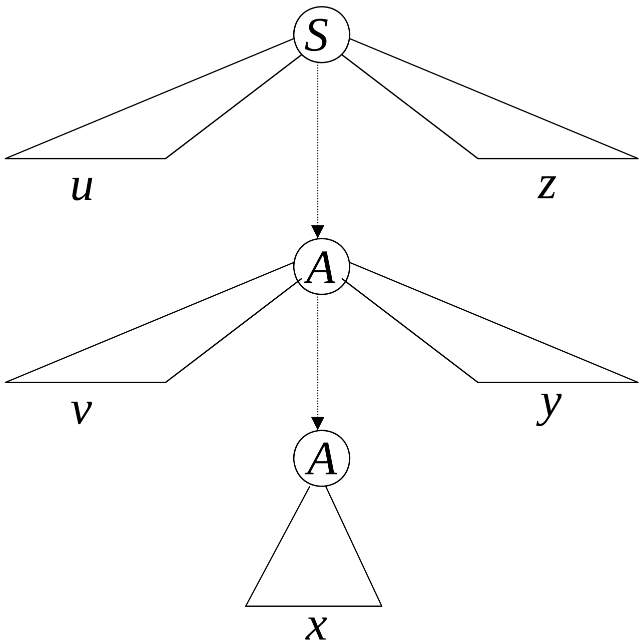
Strings of terminals

Possible derivations:

$$S \overset{*}{\Rightarrow} uAz$$

$$A \overset{*}{\Rightarrow} vAy$$

$$A \overset{*}{\Rightarrow} x$$



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We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uxz$$

$$uv^0xy^0z$$

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We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvxyz$$

The original $w = uv^1xy^1z$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow} uvvxyyz$$

$$uv^2xy^2z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \Rightarrow uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow}$$
$$\overset{*}{\Rightarrow} uvvvAyyyz \overset{*}{\Rightarrow} uvvvxyyyz$$

$$uv^3xy^3z$$

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We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow}$$
$$\overset{*}{\Rightarrow} uvvvAyyyz \overset{*}{\Rightarrow} \ldots$$
$$\overset{*}{\Rightarrow} uvvv\cdots vAy\cdots yyyz \overset{*}{\Rightarrow}$$
$$\overset{*}{\Rightarrow} uvvv\cdots vxy\cdots yyyz$$

$$uv^ixy^iz$$

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Therefore, any string of the form

$$uv^i xy^i z \qquad i \geq 0$$

is generated by the grammar G

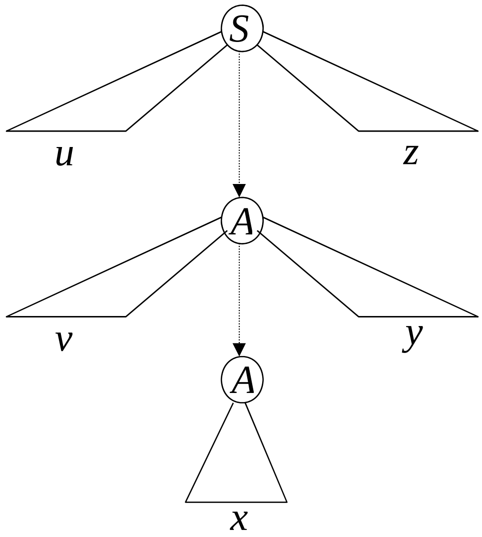
Therefore,

knowing that $uvxyz \in L(G)$

we also know that $uv^i xy^i z \in L(G)$

$L(G) = L - \{\lambda\}$ $\Downarrow$

$$uv^i xy^i z \in L$$



Observation: $|vxy| \leq m$

Since A is the last repeated variable

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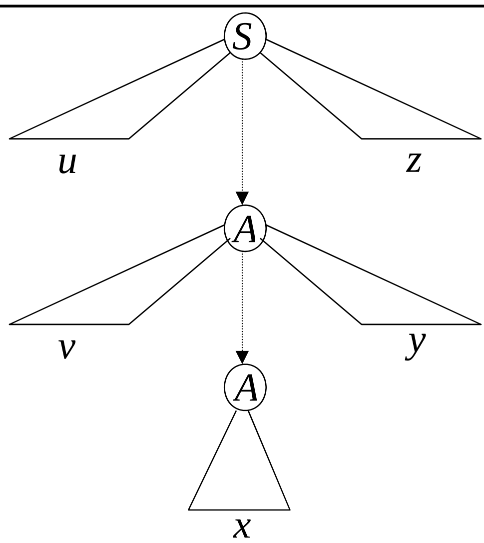

Observation: $|vy| \geq 1$

Since there are no unit or λ-productions

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The Pumping Lemma:

For infinite context-free language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

and it must be:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

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Applications of The Pumping Lemma

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Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$

Context-free languages

$$\{a^n b^n : n \geq 0\}$$

Theorem: The language

$$L = \{a^n b^n c^n : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^n b^n c^n : n \geq 0\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{a^n b^n c^n : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string $w \in L$ with length $|w| \geq m$

We pick: $w = a^m b^m c^m$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

We can write: $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

We examine **all** the possible locations of string vxy in w

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within a^m

$$\overbrace{aaa...aaa}^{m}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{a}_{u}\underbrace{aa...a}_{vxy}\underbrace{aa\ bbb...bbb\ ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 1: v and y consist from only a

$$\underbrace{a}_{u}\underbrace{a}_{vxy}\!\!\!\!\overbrace{aa...aaa}^{m}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{a}_{u}\;\underbrace{a...a}_{vxy}\;\underbrace{a\;bbb...bbb\;ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 1: Repeating v and y

$k \geq 1$

$$\overbrace{aaaaaa...aaaaaa}^{m+k}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{a}_{u}\;\underbrace{aaaaa...aaaaa}_{v^2xy^2}\;\underbrace{a\;bbb...bbb\;ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$k \geq 1$

$$\overbrace{aaaaaa...aaaaaa}^{m+k}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{a}_{u}\underbrace{aaaaa...aaaaa}_{v^2xy^2}\underbrace{a\;bbb...bbb\;ccc...ccc}_{z}$$

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$k \geq 1$

However: $uv^2xy^2z = a^{m+k}b^m c^m \notin L$

Contradiction!!!

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: vxy is within b^m

$$\overbrace{aaa...aaa}^{m}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$u \qquad\qquad vxy \qquad\qquad z$$

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: Similar analysis with case 1

$$\overbrace{aaa...aaa}^{m}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$u \qquad\qquad vxy \qquad\qquad z$$

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 3: vxy is within c^m

$$\underbrace{\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...bbb}^{m}\,c}_{u}\underbrace{cc...c}_{vxy}\underbrace{cc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 3: Similar analysis with case 1

$$\underbrace{\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...bbb}^{m}\,c}_{u}\underbrace{cc...c}_{vxy}\underbrace{cc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: vxy overlaps a^m and b^m

$$\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aa}_{u}\underbrace{a\ldots aaa\,bbb}_{vxy}\underbrace{\ldots bbb\,ccc\ldots ccc}_{z}$$

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 1: v contains only a
y contains only b

$$\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}\,\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aa}_{u}\underbrace{a\ldots aaa\,bbb}_{vxy}\underbrace{\ldots bbb\,ccc\ldots ccc}_{z}$$

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: Possibility 1: v contains only a

$k_1 + k_2 \geq 1$ $\qquad$ y contains only b

$$\overbrace{aaa\ldots aaaaaaa}^{m+k_1}\;\overbrace{bbbbbbb\ldots bbb}^{m+k_2}\;\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaaaaaa\;bbbbbbb.}_{v^2xy^2}\underbrace{..bbb\;ccc\ldots ccc}_{z}$$

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$k_1 + k_2 \geq 1$

$$\overbrace{aaa\ldots aaaaaaa}^{m+k_1}\;\overbrace{bbbbbbb\ldots bbb}^{m+k_2}\;\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaaaaaa\;bbbbbbb.}_{v^2xy^2}\underbrace{..bbb\;ccc\ldots ccc}_{z}$$

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$k_1 + k_2 \geq 1$

However: $uv^2xy^2z = a^{m+k_1} b^{m+k_2} c^m \notin L$

Contradiction!!!

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: Possibility 2: v contains a and b
y contains only b

$$\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...aaa\,bbb}_{vxy}\underbrace{...bbb\,ccc...ccc}_{z}$$

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

However: $\qquad k_1 + k_2 + k \geq 1$

$$uv^2xy^2z = a^m b^{k_1} a^{k_2} b^{m+k} c^m \notin L$$

Contradiction!!!

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: Possibility 3: v contains only a

y contains a and b

$$\overbrace{aaa...aaa}^{m}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{...aaa\ bbb}_{vxy}\underbrace{...bbb\ ccc...ccc}_{z}$$

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$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: Possibility 3: v contains only a
y contains a and b

Similar analysis with Possibility 2

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 5: vxy overlaps b^m and c^m

$$\overbrace{aaa...aaa}^{m}\;\overbrace{bbb...bbb}^{m}\;\overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\;bbb...}_{u}\underbrace{bbb\;ccc...}_{vxy}\underbrace{ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 5: Similar analysis with case 4

$$\underbrace{\overbrace{aaa...aaa}^{m}\,\overbrace{bbb...}^{m}}_{u}\underbrace{bbb\,\overbrace{ccc...}^{m}}_{vxy}\underbrace{ccc}_{z}$$

There are no other cases to consider

(since $|vxy| \leq m$, string vxy cannot

overlap a^m, b^m and c^m at the same time)

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{a^n b^n c^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free