



# PROBABILITY

Probability and Statistics

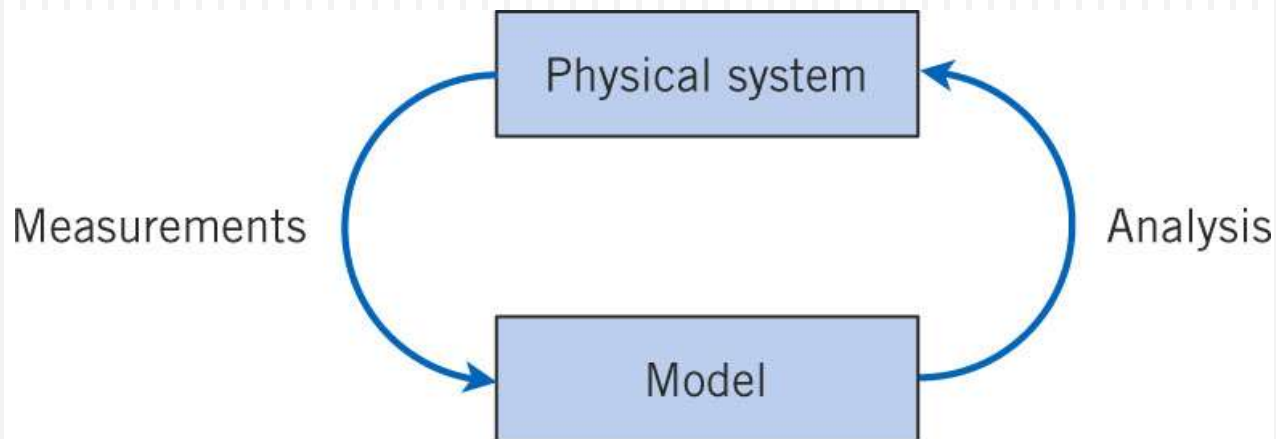
# Outline

- 2-1 Sample Spaces & Events
  - 2-1.1 Random Experiments
  - 2-1.2 Sample Spaces
  - 2-1.3 Events
  - 2-1.4 Count Techniques
- 2-2 Interpretations & Axioms of Probability
- 2-3 Addition Rules
- 2-4 Conditional Probability
- 2-5 Multiplication & Total Probability Rules
- 2-6 Independence
- 2-7 Bayes' Theorem
- 2-8 Random Variables

# Random Experiments

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The goal is to understand, quantify and model the variation affecting a physical system's behavior. The model is used to analyze and predict the physical system's behavior as system inputs affect system outputs. The predictions are verified through experimentation with the physical system.

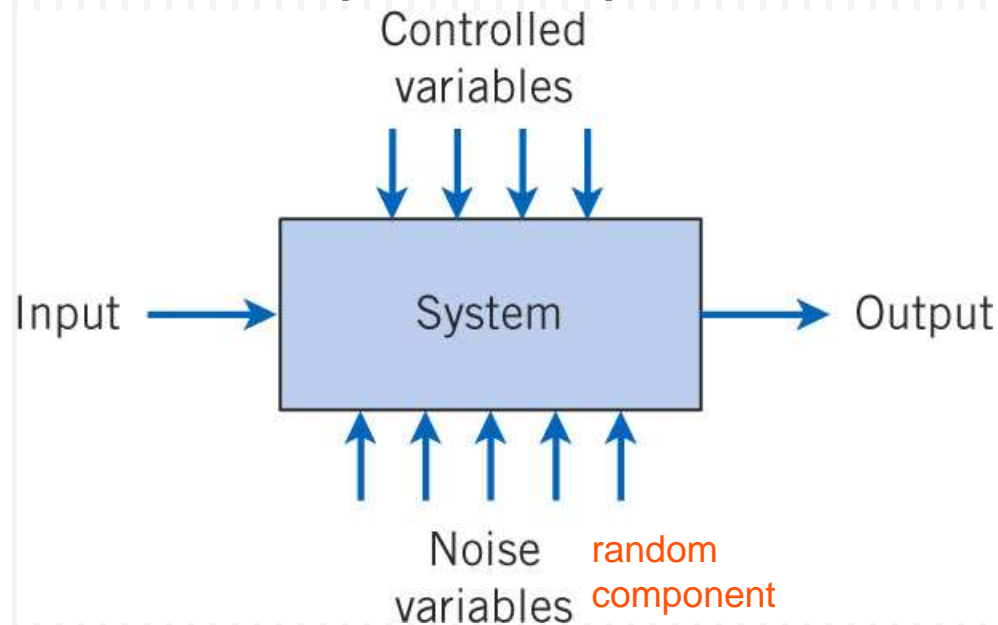


**Figure 2-1** Continuous iteration between model and physical system.

# Noise Produces Output Variation

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Random values of the noise variables cannot be controlled and cause the random variation in the output variables. Holding the controlled inputs constant does not keep the output values constant.



**Figure 2-2** Noise variables affect the transformation of inputs to outputs.

# Random Experiment

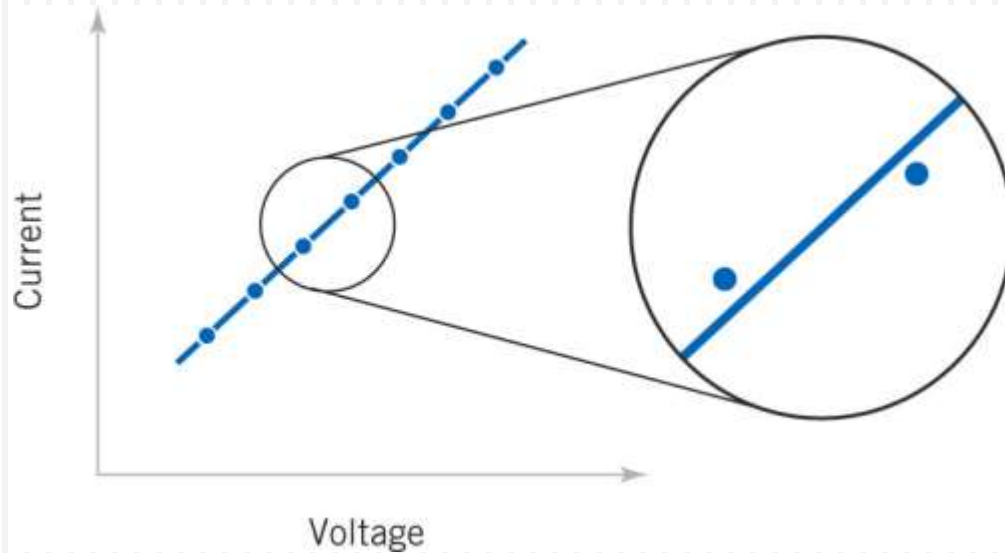
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- An experiment is an operation or procedure, carried out under controlled conditions, executed to discover an unknown result or to illustrate a known law.
- An experiment that can result in different outcomes, even if repeated in the same manner every time, is called a **random experiment**.  
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# Randomness Affects Natural Law

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Ohm's Law current is a linear function of voltage. However, current will vary due to noise variables, even under constant voltage.



**Figure 2-3** A closer examination of the system identifies deviations from the model.

# Randomness Can Disrupt a System

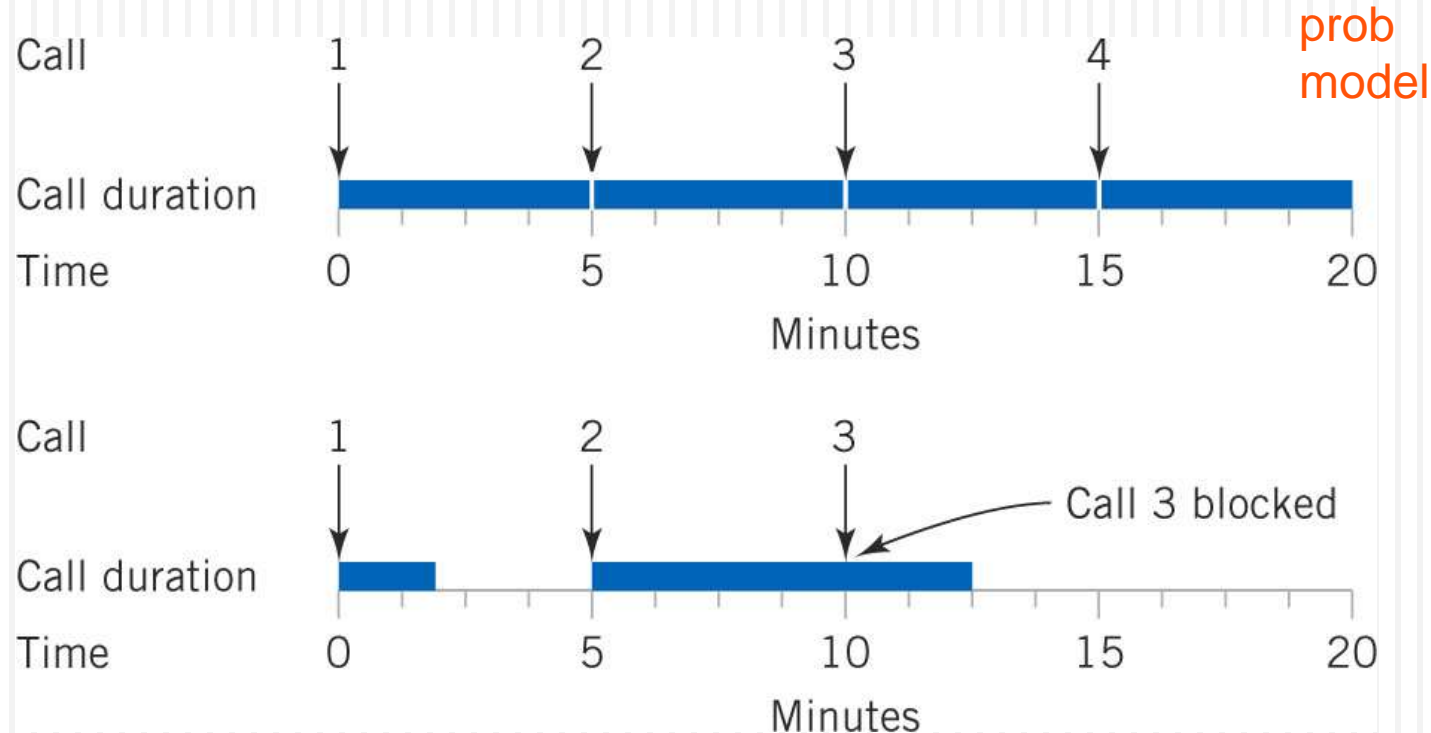
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- Telephone systems must have sufficient capacity (lines) to handle a random number of callers at a random point in time whose calls are of a random duration.
- If calls arrive exactly every 5 minutes and last for exactly 5 minutes, only 1 line is needed – a deterministic system.
- Practically, times between calls are random and the call durations are random. Calls can come into conflict as shown in following slide.
- Conclusion: Telephone system design must include provision for input variation.

# Deterministic & Random Call Behavior

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Calls arrive every 5 minutes. In top system, call durations are all of 5 minutes exactly. In bottom system, calls are of random duration, averaging 5 minutes, which can cause blocked calls, a “busy” signal.



**Figure 2-4** Variation causes disruption in the system.



# Sample Spaces

this is in Discrete  
& Cal

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- Random experiments have unique **outcomes**.
- The set of all possible outcome of a random experiment is called the **sample space**,  $S$ .
- $S$  is **discrete** if it consists of a **finite** or **countable** infinite set of outcomes.
- $S$  is **continuous** if it contains **an interval** (either a finite or infinite width) of **real numbers**.

# Example 2-1: Defining Sample Spaces

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- Randomly select and measure the thickness of a part.  
 $S = R^+ = \{x \mid x > 0\}$ , the positive real line. Negative or zero thickness is not possible.  
 $S$  is continuous.
- It is known that the thickness is between 10 and 11 mm.  
 $S = \{x \mid 10 < x < 11\}$ , continuous.
- It is known that the thickness has only three values.  
 $S = \{low, medium, high\}$ , discrete.
- Does the part thickness meet specifications?  
 $S = \{yes, no\}$ , discrete.

## Example 2-2: Defining Sample Spaces, $n=2$

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- Two parts are randomly selected & measured.

$S = R^+ * R^+$ ,  $S$  is continuous.

- Do the 2 parts conform to specifications? if I can count, it is discrete

$S = \{yy, yn, ny, nn\}$ ,  $S$  is discrete.

- Number of conforming parts?

$S = \{0, 1, 2\}$ ,  $S$  is discrete.

- Parts are randomly selected until a non-conforming part is found.

$S = \{n, yn, yyn, yyyn, \dots\}$ ,  
 $S$  is countably infinite.

# Sample Space Is Defined By A Tree Diagram

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Example 2-3: Messages are classified as on-time or late. 3 messages are classified. There are  $2^3 = 8$  outcomes in the sample space.

$$S = \{ooo, ool, olo, oll, loo, lol, llo, lll\}$$

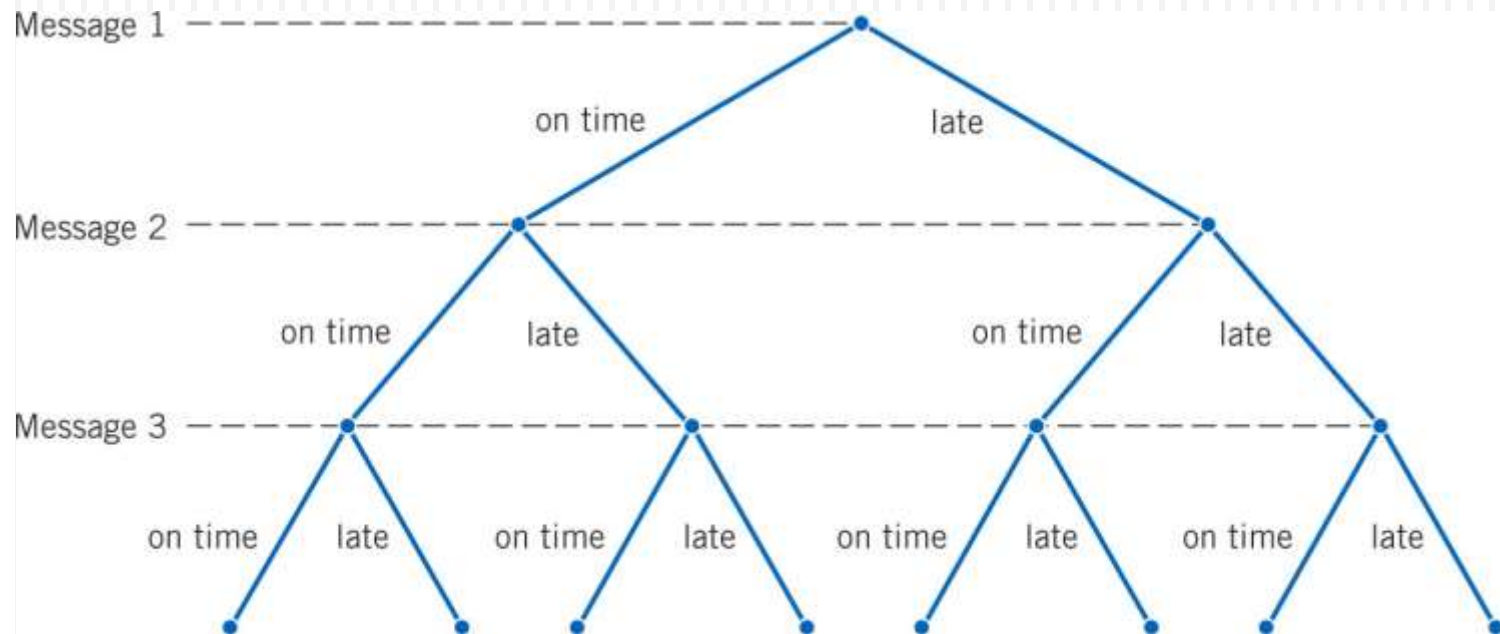


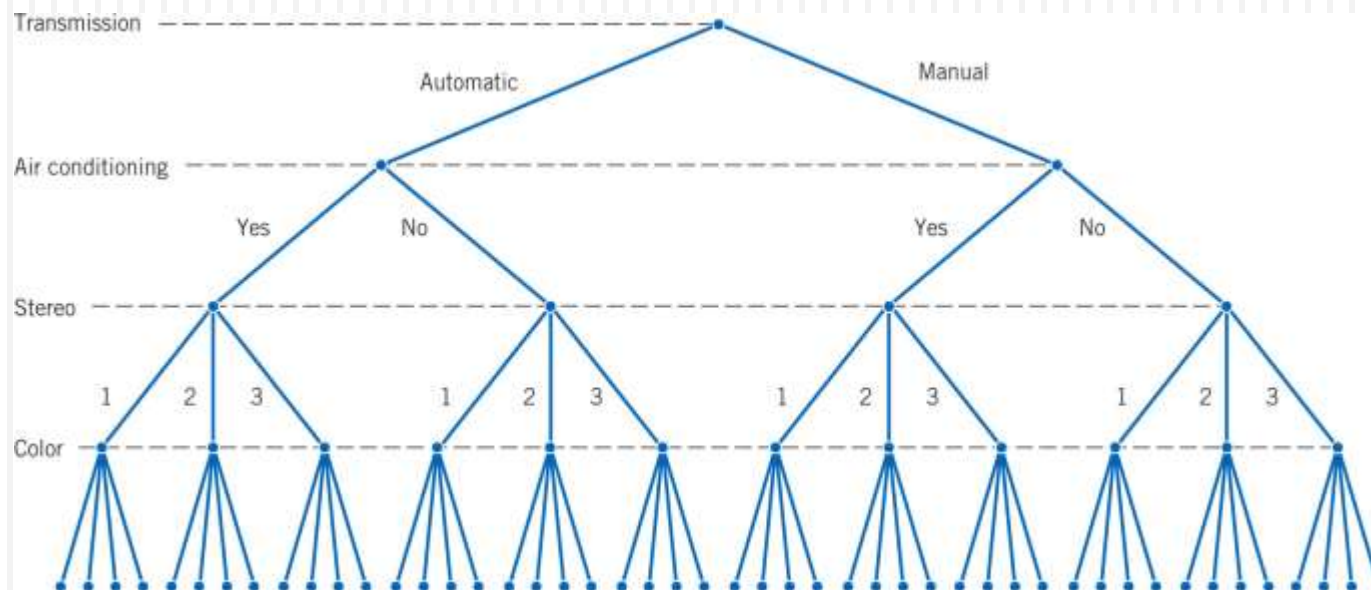
Figure 2-5 Tree diagram for three messages.

# Tree Diagrams Can Fit The Situation

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Example 2-4: New cars can be equipped with selected options as follows:

1. Manual or automatic transmission
2. With or without air conditioning
3. Three choices of stereo sound systems
4. Four exterior color choices



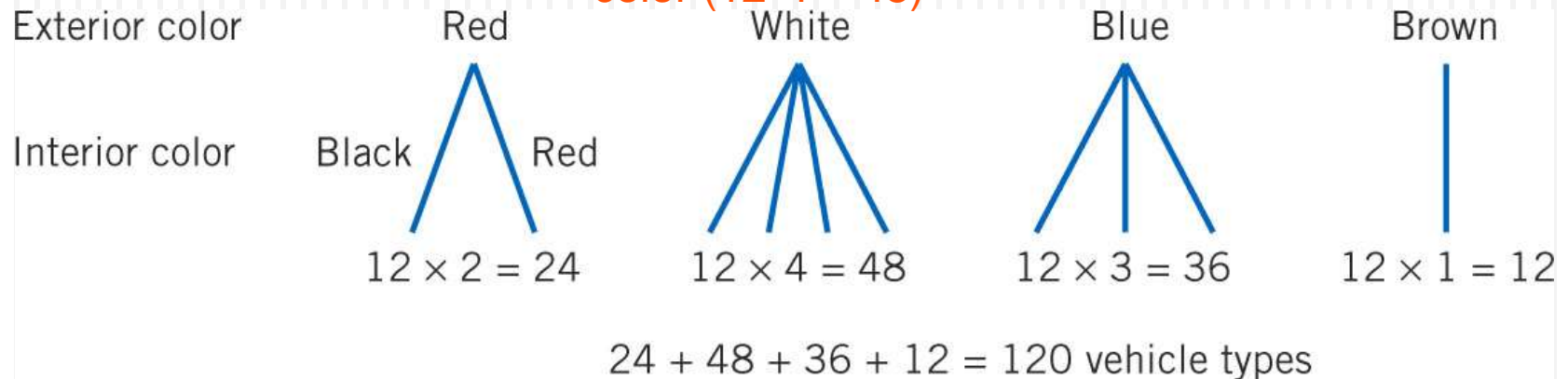
**Figure 2-6** Tree diagram for different configurations of vehicles. Note that  $S$  has  $2 \cdot 2 \cdot 3 \cdot 4 = 48$  outcomes.

# Tree Diagrams Help Count Outcomes

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Example 2-5: The interior car color can depend on the exterior color as shown in the tree diagrams below. There are 12 possibilities without considering color combinations.

12 out of 48 for each  
color ( $12 \times 4 = 48$ )



**Figure 2-7** Tree diagram for different vehicle configurations with interior colors.

- If the Batch consist of three letter A, B, C and our experiment is to select two items **without replacement**, **order matter**, the sample space can be represent as

if the set have  
A,B,C

$S = \{AB, AC, BA, BC, CA, CB\}$

- If the Batch consist of three letter A, B, C and our experiment is to select two items **with replacement**, **order matter**, the sample space can be represent as

if the set have  
A,B,C

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$S = \{AB, AC, BA, BC, CA, CB, AA, BB, CC\}$

- If the Batch consist of three letter A, B, C and our experiment is to select two items **without replacement**, **order doesn't matter**, the sample space can be represent

as if the set have  
A,B,C

$S = \{AB, AC, BC\}$

- If the Batch consist of three letter A, B, C and our experiment is to select two items **with replacement**, **order doesn't matter**, the sample space can be represent as

if the set  
have A,B,C

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$S = \{AA, AB, AC, BB, BC, CC\}$

# Events Are Sets of Outcomes

Sample space = the possible outcome

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- An event (E) is a subset of the sample space of a random experiment, i.e., one or more outcomes of the sample space.
- Event combinations are:
  - **Union** of two events is the event consisting of all outcomes that are contained in either of two events,  $E_1 \cup E_2$ .  
Called  $E_1$  or  $E_2$ .
  - **Intersection** of two events is the event consisting of all outcomes that contained in both of two events,  $E_1 \cap E_2$ .  
Called  $E_1$  and  $E_2$ .
  - **Complement** of an event is the set of outcomes that are not contained in the event,  $E'$  or not  $E$ .



# Example 2-6, Discrete Event Algebra

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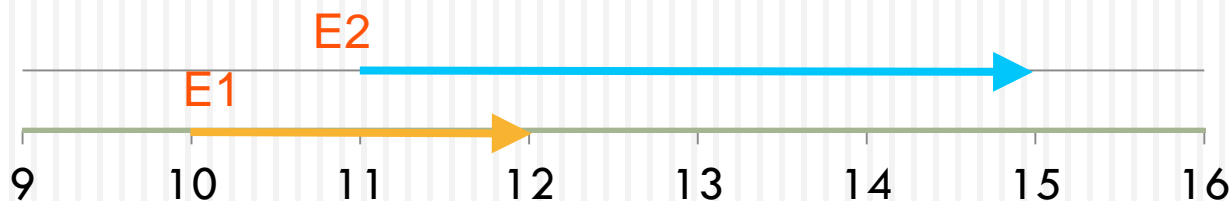
- Recall the sample space from Example 2-2,  $S = \{yy, yn, ny, nn\}$  concerning conformance to specifications.
  - Let  $E_1$  denote the event that at least one part does conform to specifications,  $E_1 = \{yy, yn, ny\}$
  - Let  $E_2$  denote the event that no part conforms to specifications,  $E_2 = \{nn\}$
  - Let  $E_3 = \emptyset$ , the null or empty set.
  - Let  $E_4 = S$ , the universal set.
  - Let  $E_5 = \{yn, ny, nn\}$ , at least one part does not conform.
  - Then  $E_1 \cup E_5 = S$
  - Then  $E_1 \cap E_5 = \{yn, ny\}$
  - Then  $E_1' = \{nn\}$

# Example 2-7, Continuous Event Algebra

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Measurements of the thickness of a part are modeled with the sample space:  $S = \mathbb{R}^+$ .

- Let  $E_1 = \{x \mid 10 \leq x < 12\}$ , show on the real line below.
- Let  $E_2 = \{x \mid 11 < x < 15\}$
- Then  $E_1 \cup E_2 = \{x \mid 10 \leq x < 15\}$
- Then  $E_1 \cap E_2 = \{x \mid 11 < x < 12\}$
- Then  $E_1' = \{x \mid x < 10 \text{ or } x \geq 12\}$
- Then  $E_1' \cap E_2 = \{x \mid 12 \leq x < 15\}$



# Example 2-8, Hospital Emergency Visits

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- This table summarizes the ER visits at 4 hospitals. People may leave without being seen by a physician( LWBS). The remaining people are seen, and may or may not be admitted.

	A	Hospital				
	1	2	3	4	Total	
Total	5,292	6,991	5,640	4,329	22,252	
LWBS	195	270	246	242	953	B
Admitted	1,277	1,558	666	984	4,485	
Not admitted	3,820	5,163	4,728	3,103	16,814	

- Let A be the event of a visit to Hospital 1.
- Let B be the event that the visit is LWBS.
- Find number of outcomes in:

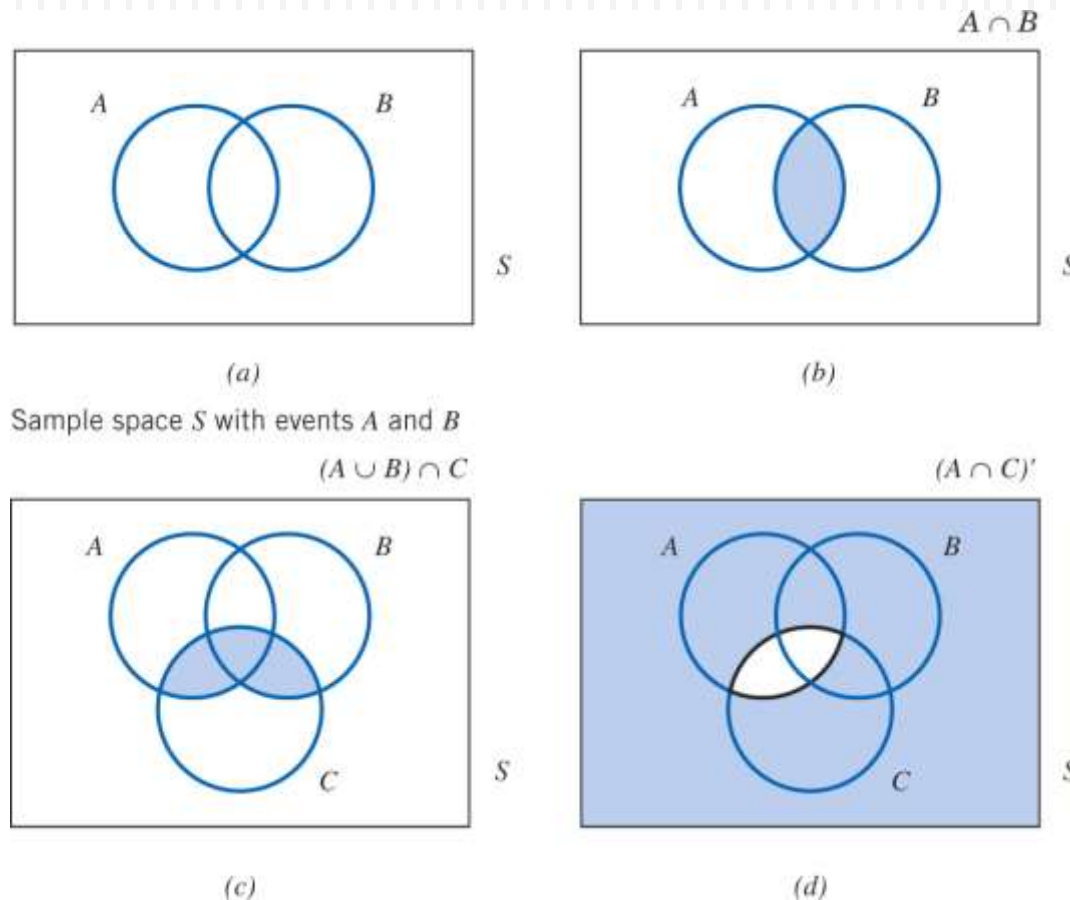
$$A \vee B = A + B - (A \wedge B)$$

- $A \cap B$  195
- $A'$   $22,252 - 5,292 = 16,960$
- $A \cup B$   $5,292 + 953 - 195 = 6,050$

# Venn Diagrams Show Event Relations

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Events  $A$  &  $B$  contain their respective outcomes. The shaded regions indicate the event relation of each diagram.



# Venn Diagram of Mutually Exclusive Events

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- Events A & B are mutually exclusive because they share no common outcomes.
- The occurrence of one event precludes the occurrence of the other.
- Symbolically,  $A \cap B = \emptyset$  ex. A is not admitted  
B is admitted

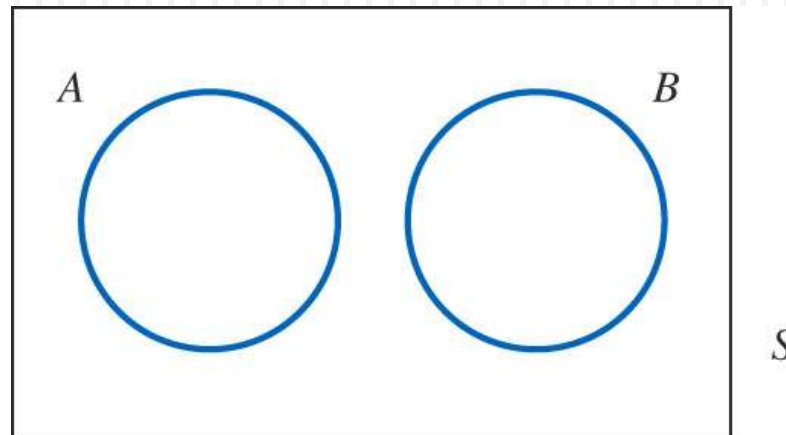


Figure 2-9 Mutually exclusive events

# Event Relation Laws

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- Transitive law (event order is unimportant):
  - $A \cap B = B \cap A$  and  $A \cup B = B \cup A$
- Distributive law (like in algebra):
  - $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
  - $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- DeMorgan's laws:
  - $(A \cup B)' = A' \cap B'$  The complement of the union is the intersection of the complements.
  - $(A \cap B)' = A' \cup B'$  The complement of the intersection is the union of the complements.

# Counting Techniques

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- These are three special rules, or counting techniques, used to determine the number of outcomes in the events and the sample space.
- They are the:
  1. Multiplication rule
  2. Permutation rule
  3. Combination rule
- Each has its special purpose that must be applied properly – the right tool for the right job.

# Counting – Multiplication Rule

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- Multiplication rule:
  - ▣ Let an operation consist of  $k$  steps and
    - $n_1$  ways of completing step 1,
    - $n_2$  ways of completing step 2, ... and
    - $n_k$  ways of completing step  $k$ .
  - ▣ Then, the total number of ways or outcomes are:
    - $n_1 * n_2 * \dots * n_k$



# Example 2-9: Multiplication Rule

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- In the design for a gear housing, we can choose to use among:
  - ▣ 4 different fasteners,
  - ▣ 3 different bolt lengths and
  - ▣ 2 different bolt locations.
- How many designs are possible?

# Counting – Permutation Rule

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- A **permutation** is a unique **sequence of distinct items**.
- If  $S = \{a, b, c\}$ , then there are 6 permutations
  - ▣ Namely: abc, acb, bac, bca, cab, cba (**order matters**)
  - ▣ The # of ways 3 people can be arranged.
- **# of permutations** for a set of  $n$  items **is  $n!$**
- $n!$  (factorial function) =  $n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$
- $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5,040 = \text{FACT}(7)$  in Excel
- By definition:  $0! = 1$

# Counting – Sub-set Permutations

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- To sequence only  $r$  items from a set of  $n$  items:

$$P_r^n = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

$$P_3^7 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7*6*5*4!}{4!} = 7*6*5 = 210$$

In Excel: `permut(7,3)` = 210

## Example 2-10: Circuit Board Designs

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- A printed circuit board has eight different locations in which a component can be placed. If four different components are to be placed on the board, how many designs are possible?
- Answer: order is important, so use the permutation formula with  $n = 8$ ,  $r = 4$ .  $P = 8!/(8-4)! = 8*7*6*5 = 1,680$

# Counting - Similar Item Permutations

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- Used for counting the sequences when not all the items are different.
- The number of permutations of:
  - ▣  $n = n_1 + n_2 + \dots + n_r$  items of which
    - $n_1$  are identical,
    - $n_2$  are identical, ..., and
    - $n_r$  are identical.

- Is calculated as:

$$\frac{n!}{n_1! n_2! \dots n_r!}$$

# Example 2-1 1: Machine Shop Schedule

30

- In a machining operation, a piece of sheet metal needs two identical-diameter holes drilled and two identical-size notched cut. The drilling operation is denoted as d and the notching as n.
- How many sequences are there?

$$4!/(2!*2!) = 6$$

- What is the set of sequences?

$$S = \{d1d2n1n2, dndn, dnnd, nddn, ndnd, nndd\}$$

## Example 2-12: Bar Codes

31

- A part is labeled with 4 thick lines, 3 medium lines, and two thin lines. Each sequence is a different label.
- ▣ How many unique labels can be created?

$$9!/(4!*3!*2!) = 1,260$$

- ▣ In Excel:

$$(FACT(4)*FACT(3)*FAC$$

# Counting – Combination Rule

32

- A combination is a selection of  $r$  items from a set of  $n$  where **order does not matter**.
- If  $S = \{a, b, c\}$ ,  $n = 3$ , then there is 1 combination.  
because  $abc = acb = bac = bca = cab = cba$ , so
  - If  $r = 3$ , there is 1 combination, namely:  $abc$
  - If  $r = 2$ , there are 3 combinations, namely  $ab, ac, bc$
- # of permutations  $\geq$  # of combinations

$$C_r^n = \frac{n!}{r!(n-r)!} \quad n!/(r!*(n-r)!) \quad (2-4)$$



## Example 2-13: Applying the Combination Rule

33

- A circuit board has eight locations in which a component can be placed. If 5 identical components are to be placed on a board, how many different designs are possible?
- The order of the components is not important, so the combination rule is appropriate.

$$8!/(5!(8-5)!) = 56$$

EXCEL: COMBIN(8,5)

## Example 2-14: Sampling w/o Replacement-1

34

- A bin of 50 parts contains 3 defectives & 47 good parts. A sample of 6 parts is selected from the 50 **without** replacement. <sup>ไม่ใส่คืน</sup>
- How many different samples of size 6 are there that contain exactly 2 defective parts?

$$C = 3!/(2!1!) = 3$$

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## Example 2-14: Sampling w/o Replacement-2

35

- Now, how many ways are there of selecting 4 parts from the 47 acceptable parts?

$$\frac{47!}{(4!43!)} = 178,365$$

different ways

## Example 2-14: Sampling w/o Replacement-3

36

- Now, how many ways are there to obtain:
  - ▣ 2 from the 3 defectives, and
  - ▣ 4 from the 47 non-defectives?

$$C(3,2)*C(47,4) = 3!/(2!1!)*47!/(4!43!)$$

## Example 2-14: Sampling w/o Replacement-4

37

- Furthermore, how many ways are there to obtain 6 parts (0-6 defectives) from the set of 50?

$$50!/(6!44!) = 15,890,700$$

- So the ratio of obtaining 2 defectives out of 6 to any number (0-6) defectives out of 6 is:

$$C(3,2)*C(47,4)/C(50,6) = 3*178,365/15,890,700 = 0.034$$

# What Is Probability?

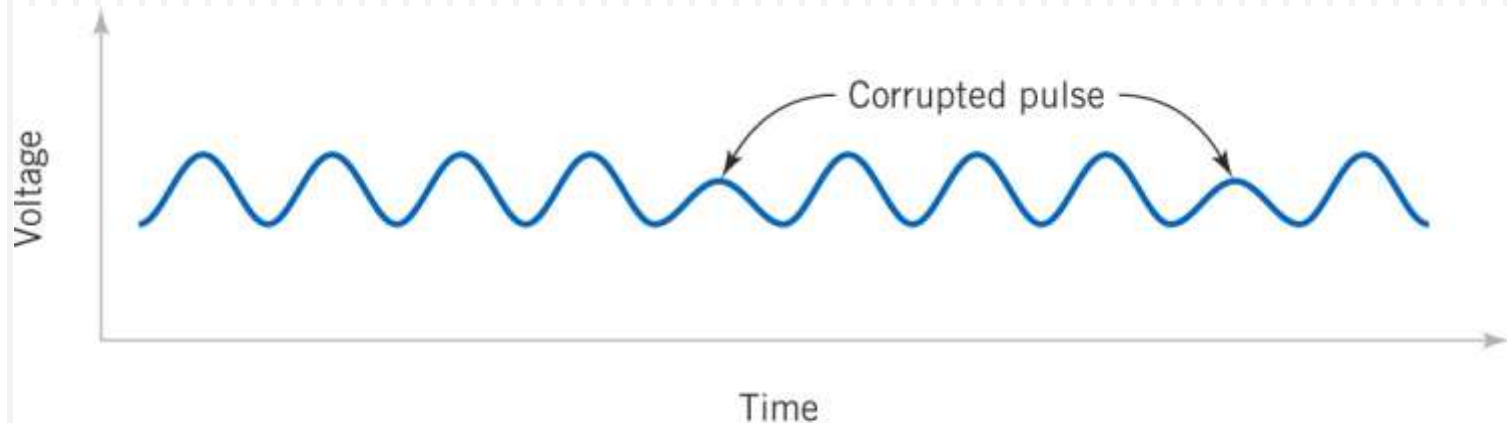
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- Probability is the likelihood or chance that a particular outcome or event from a random experiment will occur.
- Here, only finite sample spaces ideas apply.
- Probability is a number in the  $[0,1]$  interval.
- May be expressed as a:  $\frac{n(E)}{n(S)}$ 
  - ▣ proportion (0.15)
  - ▣ percent (15%)
  - ▣ fraction (3/20)
- A probability of:
  - ▣ 1 means certainty
  - ▣ 0 means impossibility

# Types of Probability

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- **Subjective probability** is a “degree of belief.”
  - “There is a 50% chance that I’ll study tonight.”
- **Relative frequency probability** is based how often an event occurs over a very large sample space.



$$\text{Relative frequency of corrupted pulse} = \frac{2}{10}$$

**Figure 2-10** Relative frequency of corrupted pulses over a communications channel

# Probability Based on Equally-Likely Outcomes

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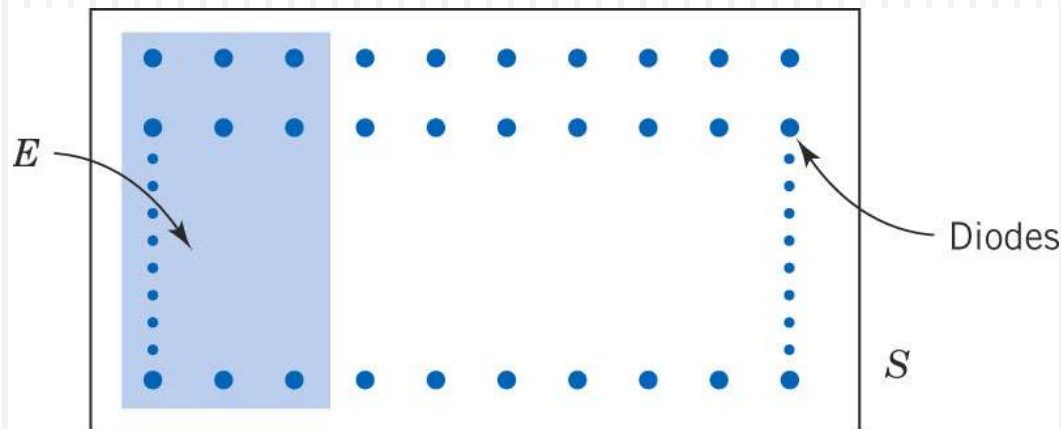
- Whenever a sample space consists of  $N$  possible outcomes that are equally likely, the probability of each outcome is  $1/N$ .
- Example: In a batch of 100 diodes, 1 is colored red. A diode is randomly selected from the batch. Random means each diode has an equal chance of being selected. The probability of choosing the red diode is  $1/100$  or 0.01, because each outcome in the sample space is equally likely.



# Example 2-15: Laser Diodes

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- Assume that 30% of the laser diodes in a batch of 100 meet a customer requirements.
- A diode is selected randomly. Each diode has an equal chance of being selected. The probability of selecting an acceptable diode is 0.30.



$$P(E) = 30(0.01) = 0.30$$

**Figure 2-11** Probability of the event  $E$  is the sum of the probabilities of the outcomes in  $E$ .

# Probability of an Event

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- For a discrete sample space, the *probability of an event  $E$* , denoted by  $P(E)$ , equals the sum of the probabilities of the outcomes in  $E$ .
- The discrete sample space may be:
  - ▣ A finite set of outcomes
  - ▣ A countably infinite set of outcomes.
- Further explanation is necessary to describe probability with respect to continuous sample spaces.

## Example 2-16: Probabilities of Events

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- A random experiment has a sample space  $\{w, x, y, z\}$ . These outcomes are not equally-likely; their probabilities are: 0.1, 0.3, 0.5, 0.1.
- Event  $A = \{w, x\}$ , event  $B = \{x, y, z\}$ , event  $C = \{z\}$ 
  - $P(A) = 0.1 + 0.3 = 0.4$
  - $P(B) = 0.3 + 0.5 + 0.1 = 0.9$
  - $P(C) = 0.1$
  - $P(A') = 0.6$  and  $P(B') = 0.1$  and  $P(C') = 0.9$
  - Since event  $A \cap B = \{x\}$ , then  $P(A \cap B) = 0.3$
  - Since event  $A \cup B = \{w, x, y, z\}$ , then  $P(A \cup B) = 1.0$
  - Since event  $A \cap C = \{\text{null}\}$ , then  $P(A \cap C) = 0.0$

# Example 2-17: Contamination Particles

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- An inspection of a large number of semiconductor wafers revealed the data for this table. A wafer is selected randomly.

- Let  $E$  be the event of selecting a 0 particle wafer.  $P(E) =$
- Let  $E$  be the event of selecting a wafer with 3 or more particles.  $P(E) =$

Number of Contamination Particles	Proportion of Wafers
0	0.40
1	0.20
2	0.15
3	0.10
4	0.05
5 or more	0.10
Total	1.00

# Example 2-18: Sampling w/o Replacement

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- A batch of parts contains 6 parts  $\{a,b,c,d,e,f\}$ . Two are selected at random. Suppose part  $f$  is defective. What is the probability that part  $f$  appears in the sample?
- How many possible samples can be drawn?  $C(6,2) = 6!/(2!4!) = 15$
- How many samples contain part  $f$ ?  $\{af, bf, cf, df, ef\}$
- $P(\text{defective part}) = 5/15$

# Axioms of Probability

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- Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:
  1.  $P(S) = 1$
  2.  $0 \leq P(E) \leq 1$
  3. For each two events  $E_1$  and  $E_2$  with  $E_1 \cap E_2 = \emptyset$ ,
$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$
- These imply that:
  - $P(\emptyset) = 0$  and  $P(E') = 1 - P(E)$
  - If  $E_1$  is contained in  $E_2$ , then  $P(E_1) \leq P(E_2)$ .

# Addition Rules

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- Joint events are generated by applying basic set operations to individual events, specifically:
  - ▣ Unions of events,  $A \cup B$
  - ▣ Intersections of events,  $A \cap B$
  - ▣ Complements of events,  $A'$
- Probabilities of joint events can often be determined from the probabilities of the individual events that comprise it. And conversely.

# Example 2-19: Semiconductor Wafers

48

A wafer is randomly selected from a batch as shown in the table.

- Let  $H$  be the event of high concentrations of contaminants. Then  $P(H) = 358/940$
- Let  $C$  be the event of the wafer being located at the center of a sputtering tool used in manufacture. Then  $P(C) = 626/940$ .

■  $P(H \cap C) = 112/940$

Table 2-1		Location of Tool		
Contamination		Center	Edge	Total
		514	68	582
Low				
High	H	112	246	358
Total		626	314	940

■  $P(H \cup C) = P(H) + P(C) - P(H \cap C) = (358+626-112)/940$

■ This is the **addition rule**.



- The probability of a union:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (2-5)$$

and, as rearranged:

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

- If events A and B are mutually exclusive:

$$P(A \cap B) = \emptyset$$

no intersection part

therefore:

$$P(A \cup B) = P(A) + P(B) \quad (2-6)$$

# Example 2-20: Contaminants & Location

50

Wafers in last example are now classified by degree of contamination per table of proportions.

- $E_1$  is the event that a wafer has 4 or more particles.

$$P(E_1) = 0.05 + 0.10 = 0.15$$

- $E_2$  is the event that a wafer was on edge.

$$P(E_2) = 0.28$$

$$P(E_1 \cap E_2) = 0.01 + 0.03 = 0.04$$

$$P(E_1 \cup E_2) = 0.15 + 0.28 - 0.04 = 0.39$$

Number of Contamination Particles	Table 2-2		
	Location of Tool		Total
	Center	Edge $E_2$	
0	0.30	0.10	0.40
1	0.15	0.05	0.20
2	0.10	0.05	0.15
3	0.06	0.04	0.10
4	0.04	0.01	0.05
5 or more $E_1$	0.07	0.03	0.10
Totals	0.72	0.28	1.00

# Addition Rule: 3 or More Events

51

$$\begin{aligned} P(A \cup B \cup C) = & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C) \end{aligned} \quad (2-7)$$

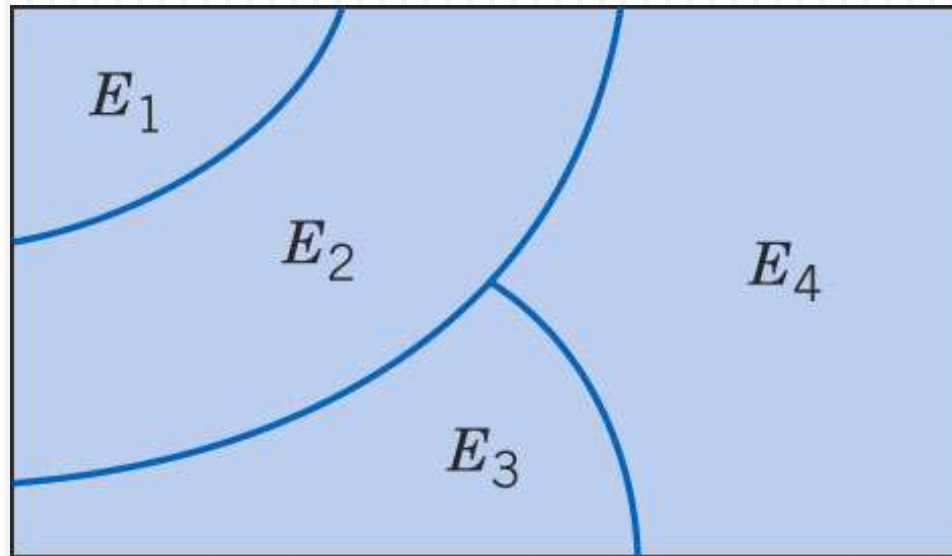
Note the alternating signs.

If a collection of events  $E_i$  is mutually exclusive,  
thus for all pairs:  $E_i \cap E_j = \emptyset$

$$\text{Then: } P(E_1 \cup E_2 \cup \dots \cup E_k) = \sum_{i=1}^k P(E_i) \quad (2-8)$$

# Venn Diagram of Mutually Exclusive Events

52



**Figure 2-12** Venn diagram of four mutually exclusive events. Note that no outcomes are common to more than one event, i.e. all intersections are null.

## Example 2-21: pH

53

- Let  $X$  denote the pH of a sample. Consider the event that  $P(6.5 < X \leq 7.8) =$   
$$P(6.5 < X \leq 7.0) + P(7.0 < X \leq 7.5) + P(7.5 < X \leq 7.8)$$
- The partition of an event into mutually exclusive subsets is widely used to allocate probabilities.

# Conditional Probability

54

- Probabilities should be reevaluated as additional information becomes available.
- $P(B | A)$  is called the probability of event  $B$  occurring, given that event  $A$  has already occurred.
- A communications channel has an error rate of 1 per 1000 bits transmitted. Errors are rare, but do tend to occur in bursts. If a bit is in error, the probability that the next bit is also an error ought to be greater than  $1/1000$ .

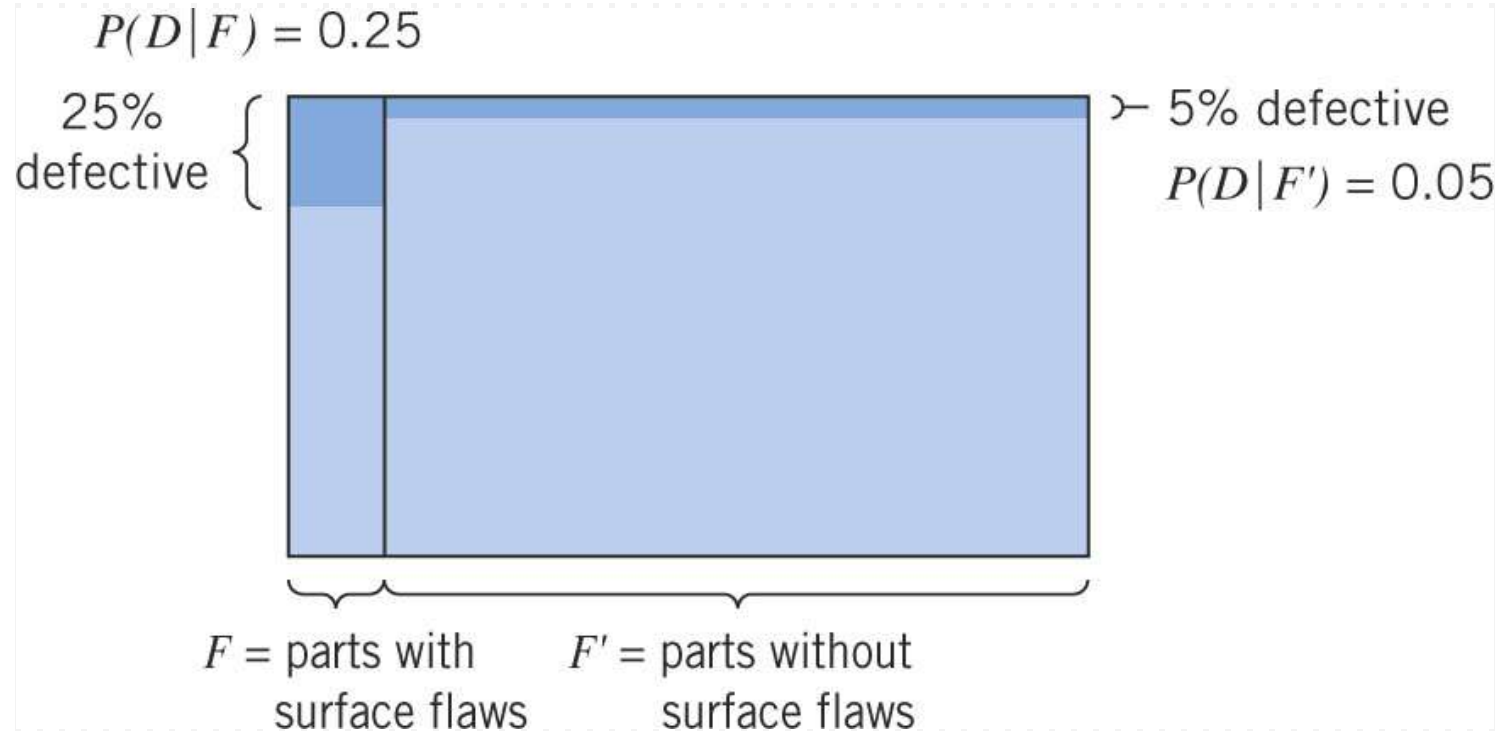
# An Example of Conditional Probability

55

- In a thin film manufacturing process, the proportion of parts that are not acceptable is 2%. However the process is sensitive to contamination that can increase the rate of parts rejection.
- If we know that the plant is having filtration problems that increase film contamination, we would presume that the rejection rate has increased.

# Another Example of Conditional Probability

56



**Figure 2-13** Conditional probability of rejection for parts with surface flaws and for parts without surface flaws. The probability of a defective part is not evenly distributed. Flawed parts are five times more likely to be defective than non-flawed parts, i.e.,  $P(D|F) / P(D|F')$ .



## Example 2-22: A Sample From Prior Graphic

57

- Table 2-3 shows that 400 parts are classified by surface flaws and as functionally defective.

Observe that:

- $P(D|F) = 10/40 = 0.25$       $P(D'|F) = 30/40 = 0.75$
- $P(D|F') = 18/360 = 0.05$       $P(D'|F') = 342/360 = 0.95$

Table 2-3 Parts Classified			
	Surface Flaws		
Defective	Yes ( $F$ )	No ( $F'$ )	Total
Yes ( $D$ )	10	18	28
No ( $D'$ )	30	342	372
Total	40	360	400

$$P(D) = 28/400$$
$$P(D') = 372/400$$

# Conditional Probability Rule

58

- The **conditional probability** of event  $B$  given event  $A$ , denoted as  $P(B | A)$ , is:

$$P(B | A) = P(A \cap B) / P(A) \quad (2-9)$$

for  $P(A) > 0$ .

$$P(A \cap B) = P(B | A) * P(A)$$

$$P(A \cap B) = P(A | B) * P(B)$$

$$P(A | B) = P(A \cap B) / P(B)$$

$$\text{so, } P(B | A) * P(A) = P(A | B) * P(B)$$

- From a relative frequency perspective of  $n$  equally likely outcomes:

- $P(A) = (\text{number of outcomes in } A) / n$

- $P(A \cap B) = (\text{number of outcomes in } A \cap B) / n$

## Example 2-23: More Surface Flaws

59

Refer to Table 2-3 again. There are 4 probabilities conditioned on flaws.

Table 2-3 Parts Classified

	Surface Flaws		Total
	Yes ( $F$ )	No ( $F'$ )	
Defective			
Yes ( $D$ )	10	18	28
No ( $D'$ )	30	342	372
Total	40	360	400

$$P(F) = 40/400 \text{ and } P(D) = 28/400$$

$$P(D|F) = P(D \cap F) / P(F) = \frac{10}{400} / \frac{40}{400} = \frac{10}{40}$$

$$P(D'|F) = P(D' \cap F) / P(F) = \frac{30}{400} / \frac{40}{400} = \frac{30}{40}$$

$$P(D|F') = P(D \cap F') / P(F') = \frac{18}{400} / \frac{360}{400} = \frac{18}{360}$$

$$P(D'|F') = P(D' \cap F') / P(F') = \frac{342}{400} / \frac{360}{400} = \frac{342}{360}$$

## Example 2-23: Tree Diagram

60

Tree illustrates sampling two parts without replacement:

- At the 1<sup>st</sup> stage (flaw), every original part of the 400 is equally likely.
- At the 2<sup>nd</sup> stage (defect), the probability is conditional upon the part drawn in the prior stage.

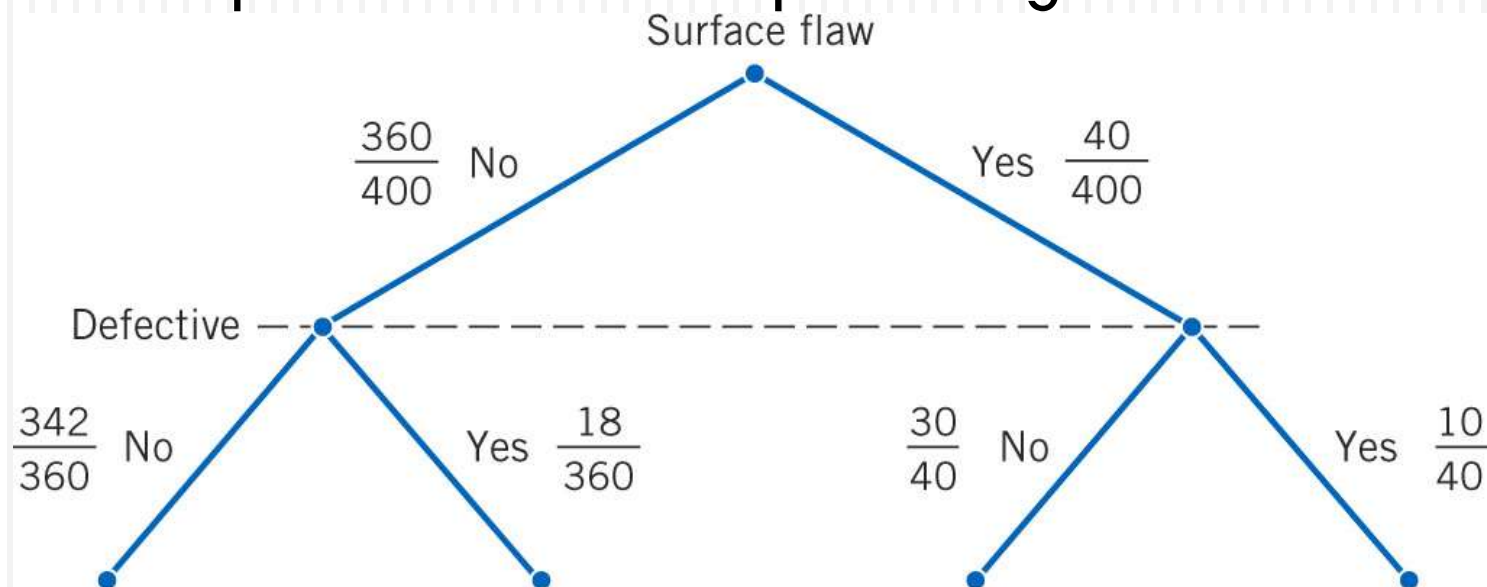


Figure 2-14 Tree diagram for parts classification

# Random Samples & Conditional Probabilities

61

- Random means each item is equally likely to be chosen. If more than one item is sampled, random means that every sampling outcome is equally likely.
- 2 items are taken from  $S = \{a,b,c\}$  without replacement.
- Ordered sample space:  $S = \{ab,ac,bc,ba,bc,cb\}$
- Unordered sample space:  $S = \{ab,ac,bc\}$
- This is done by enumeration – too hard 😞

# Sampling Without Enumeration

62

- Use conditional probability to avoid enumeration. To illustrate: A batch of 50 parts contains 10 made by Tool 1 and 40 made by Tool 2. We take a sample of  $n=2$ .
- What is the probability that the 2<sup>nd</sup> part came from Tool 2, given that the 1<sup>st</sup> part came from Tool 1?
  - $P(1^{\text{st}} \text{ part came from Tool 1}) = 10/50$
  - $P(2^{\text{nd}} \text{ part came from Tool 2}) = 40/49$
  - $P(\text{Tool 1, then Tool 2 part sequence}) = (10/50) * (40/49)$
  - To select randomly implies that, at each step of the sample, the items remaining in the batch are equally likely to be selected.

## Example 2-24: Sampling Without Replacement

63

- A production lot of 850 parts contains 50 defectives. Two parts are selected at random.
- What is the probability that the 2<sup>nd</sup> is defective, given that the first part is defective?
- Let  $A$  denote the event that the 1<sup>st</sup> part selected is defective.  $50/850$
- Let  $B$  denote the event that the 2<sup>nd</sup> part selected is defective.  $49/849$
- Probability desired is  $P(B | A) = 49/849$ .

$$P(B) = P(B \cap A) + P(B \cap A') = 50/850 * 49/849 + 800/850 * 50/849$$

# Example 2-25: Continuing Prior Example

64

- Now, 3 parts are sampled randomly.
- What is the probability that the first two are defective, while the third is not?

		not defect
defect1	defect2	$850 - 50 = 800$

$$P(\text{ddn}) = 50/850 * 49/849 * 800/848 = 0.0032$$



# Multiplication Rule

65

- The conditional probability definition of Equation 2-9 can be rewritten to generalize it as the **multiplication** rule.

$$P(B|A) = P(B \cap A) / P(A)$$

$$P(A|B) \neq P(B|A)$$

- $P(A \cap B) = P(B | A) * P(A) = P(A | B) * P(B)$  (2-10)

- The last expression is obtained by exchanging the roles of  $A$  and  $B$ .

## Example 2-26: Machining Stages

66

- The probability that, a part made in the 1<sup>st</sup> stage of a machining operation passes inspection, is 0.90. The probability that, it passes inspection after the 2<sup>nd</sup> stage, is 0.95.
- What is the probability that the part meets specifications?
- Let A & B denote the events that the 1<sup>st</sup> & 2<sup>nd</sup> stages meet specs.

$$P(A \cap B) = P(B|A) * P(A) = 0.95 * 0.90 = 0.955$$

# Two Mutually Exclusive Subsets

67

- $A$  &  $A'$  are mutually exclusive.
- $A \cap B$  and  $A' \cap B$  are mutually exclusive
- $B = (A \cap B) \cup (A' \cap B)$

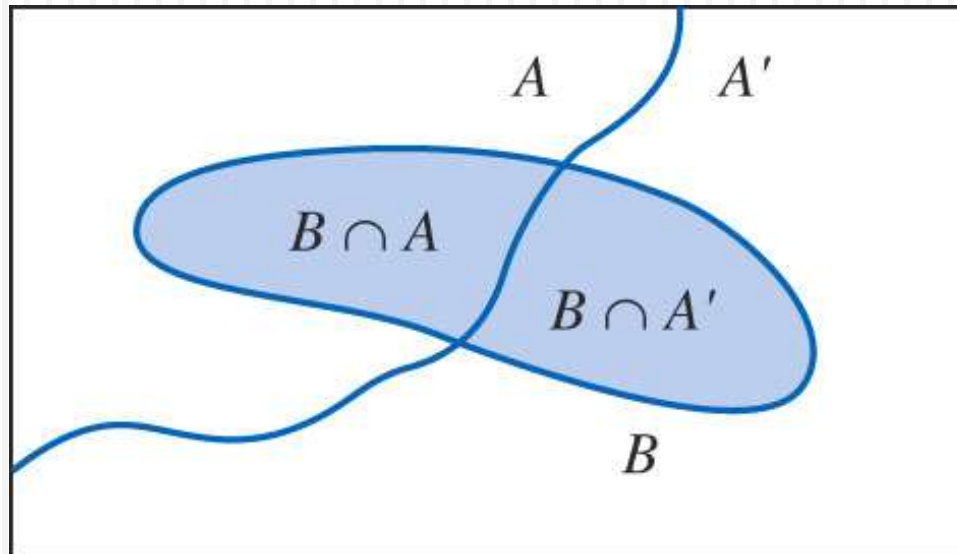


Figure 2-15 Partitioning an event into two mutually exclusive subsets.

# Total Probability Rule

68

For any two events  $A$  and  $B$ :

$$P(B|A) = P(B \cap A) / P(A)$$

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A') \\ &= P(B | A) * P(A) + P(B | A') * P(A') \end{aligned} \quad (2-11)$$

## Example 2-27: Semiconductor Contamination

69

- Information about product failure based on chip manufacturing process contamination.

	Probability of Failure	Level of Contamination	Probability of Level
$P(F H)$	0.100	High	0.2
$P(F H')$	0.005	Not High	0.8

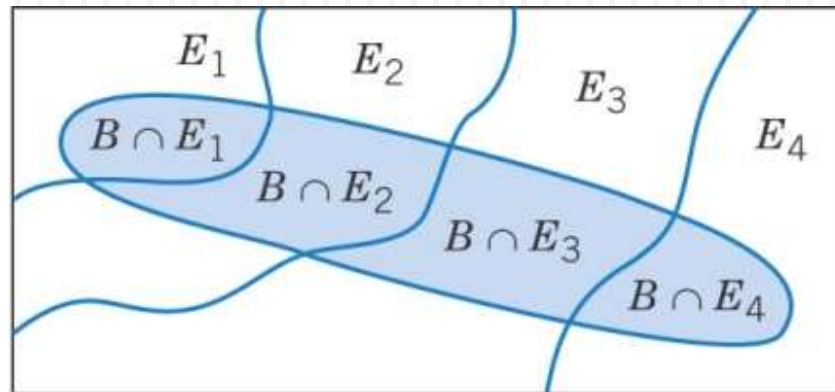
- $F$  denotes the event that the product fails.
- $H$  denotes the event that the chip is exposed to high contamination during manufacture.
- $P(F | H) = 0.100$  &  $P(H) = 0.2$ , so  $P(F \cap H) = 0.02$
- $P(F | H') = 0.005$  and  $P(H') = 0.8$ , so  $P(F \cap H') = 0.004$
- $P(F) = P(F \cap H) + P(F \cap H') = 0.020 + 0.004 = 0.024$

# Total Probability Rule (multiple events)

70

- Assume  $E_1, E_2, \dots, E_k$  are  $k$  mutually exclusive & exhaustive subsets. Then:

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \dots + P(B \cap E_k) \\ &= P(B | E_1) * P(E_1) + P(B | E_2) * P(E_2) + \dots + P(B | E_k) * P(E_k) \quad (2-11) \end{aligned}$$



$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

**Figure 2-16** Partitioning an event into several mutually exclusive subsets.

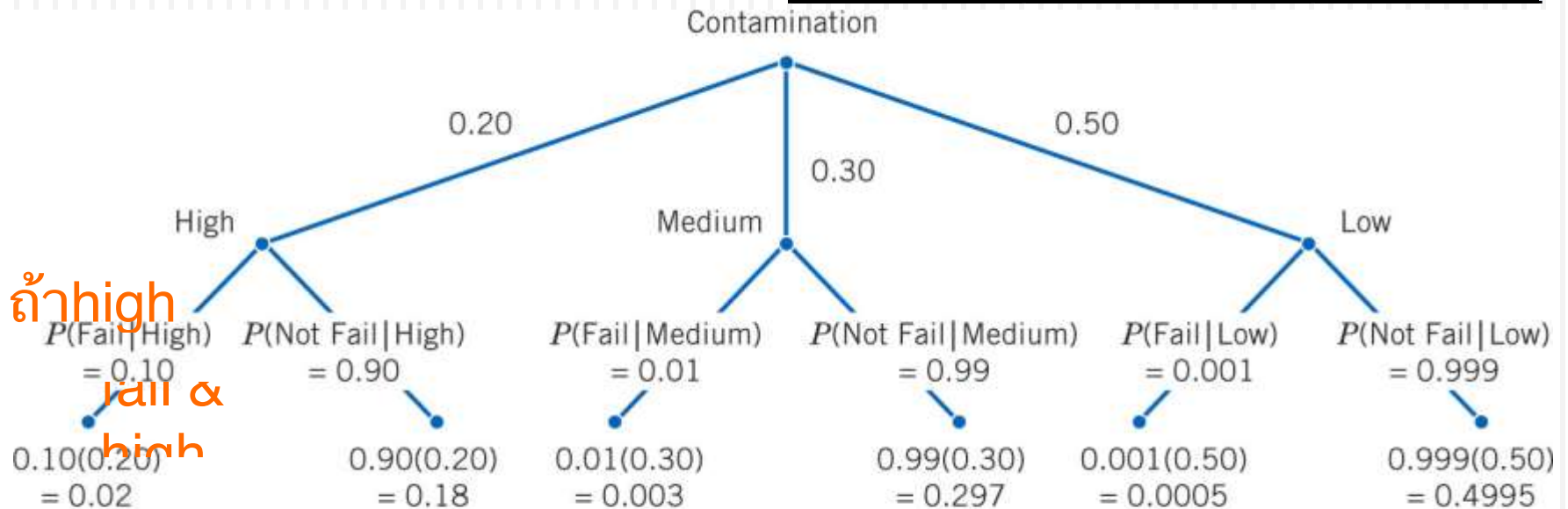
# Example 2-28: Refined Contamination Data

71

Continuing the discussion of contamination during chip manufacture:

Figure 2-17 Tree diagram

Probability of Failure	Level of Contamination	Probability of Level
0.100	High	0.2
0.010	Medium	0.3
0.001	Low	0.5



+P(F^M)

$$P(\text{Fail}) = 0.02 + 0.003 + 0.0005 = 0.0235$$

P(Fail) =

0.0235

# Event Independence

72

- Two events are independent if any one of the following equivalent statements are true:
  1.  $P(B | A) = P(B)$
  2.  $P(A | B) = P(A)$
  3.  $P(A \cap B) = P(A) * P(B)$
- This means that occurrence of one event has no impact on the occurrence of the other event.



## Example 2-29: Sampling With Replacement

73

- A production lot of 850 parts contains 50 defectives. Two parts are selected at random, but the first **is replaced** before selecting the 2<sup>nd</sup>.
- Let  $A$  denote the event that the 1<sup>st</sup> part selected is defective.  $P(A) = 50/850$
- Let  $B$  denote the event that the 2<sup>nd</sup> part selected is defective.  $P(B) = 50/850$
- What is the probability that the 2<sup>nd</sup> is defective, given that the first part is defective? The same.  $P(B | A) = P(B) = 50 / 850$
- Probability that both are defective is:  
$$P(A)*P(B) = 50/850 * 50/850 = 0.0035.$$

# Example 2-30: Flaw & Functions

74

The data shows whether the events are independent.

Table 2-3 Parts Classified				Table 2-4 Parts Classified (data chg'd)			
	Surface Flaws				Surface Flaws		
Defective	Yes (F)	No (F')	Total	Defective	Yes (F)	No (F')	Total
Yes (D)	10	18	28	Yes (D)	2	18	20
No (D')	30	342	372	No (D')	38	342	380
Total	40	360	400	Total	40	360	400
	$P(D F) = 10/40 = 0.25$				$P(D F) = 2/40 = 0.05$		
	$P(D) = 28/400 = 0.07$				$P(D) = 20/400 = 0.05$		
	not same				same		
	Events D & F are <b>dependent</b>				Events D & F are <b>independent</b>		

## Example 2.31: Conditioned vs. Unconditioned

75

- A production lot of 850 parts contains 50 defectives. Two parts are selected at random, **without replacement**.
- Let  $A$  denote the event that the 1<sup>st</sup> part selected is defective.  $P(A) = 50/850$
- Let  $B$  denote the event that the 2<sup>nd</sup> part selected is defective.  $P(B|A) = 49/849$
- Probability that the 2<sup>nd</sup> is defective is:  
$$P(B) = P(B|A)*P(A) + P(B|A')*P(A')$$
$$P(B) = (49/849) * (50/850) +$$
$$(50/850) * (49/849)$$
$$P(B) =$$
$$P(B) =$$
$$P(B) =$$
- Since  $P(B|A) \neq P(B)$ , then  $A$  and  $B$  are dependent.

# Independence with Multiple Events

76

The events  $E_1, E_2, \dots, E_k$  are independent if and only if, for any subset of these events:

$$P(E_1 \cap E_2 \cap \dots \cap E_k) = P(E_1) * P(E_2) * \dots * P(E_k) \quad (2-14)$$

Be aware that, if  $E_1$  &  $E_2$  are independent,  $E_2$  &  $E_3$  may or may not be independent.

## Example 2-32: Series Circuit

77

This circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that the devices fail independently. What is the probability that the circuit operates?



Let  $L$  &  $R$  denote the events that the left and right devices operate. The probability that the circuit operates is:

$$P(L \cap R) = P(L) * P(R) = 0.8 * 0.9 = 0.72.$$

## Example 2-33: Another Series Circuit

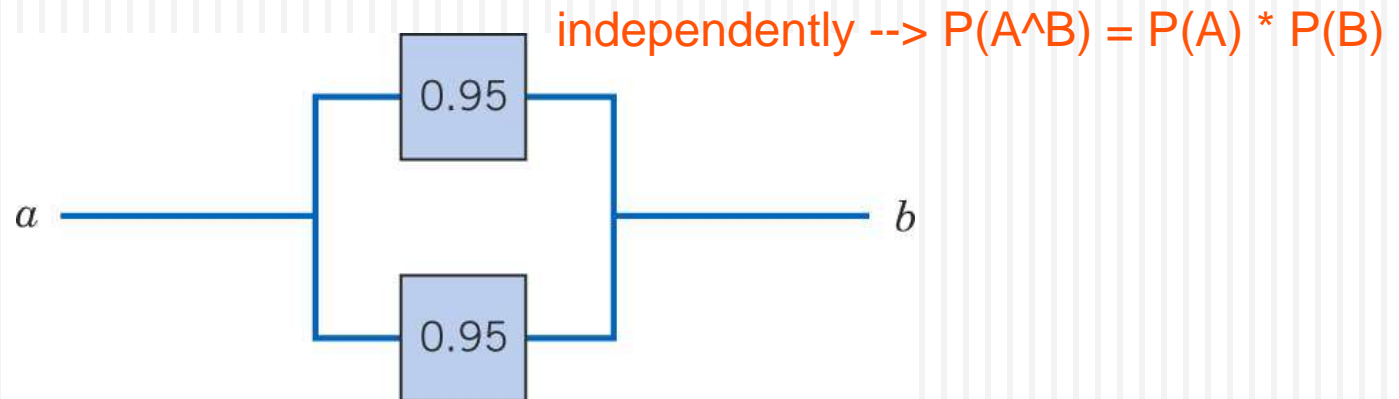
78

- The probability that a wafer contains a large particle of contamination is 0.01. The wafer events are independent.
- $P(E_i)$  denotes the event that the  $i^{\text{th}}$  wafer contain no particles and  $P(E_i) = 0.99$ .
- If 15 wafers are analyzed, what is the probability that no large particles are found?
- $$P(E_1 \cap E_2 \cap \dots \cap E_k) = P(E_1) * P(E_2) * \dots * P(E_k)$$
$$= (0.99)^{15} = 0.86.$$

## Example 2-34: Parallel Circuit

79

This circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown. **Each device fails independently.**



Let T & B denote the events that the top and bottom devices operate. The probability that the circuit operates is:

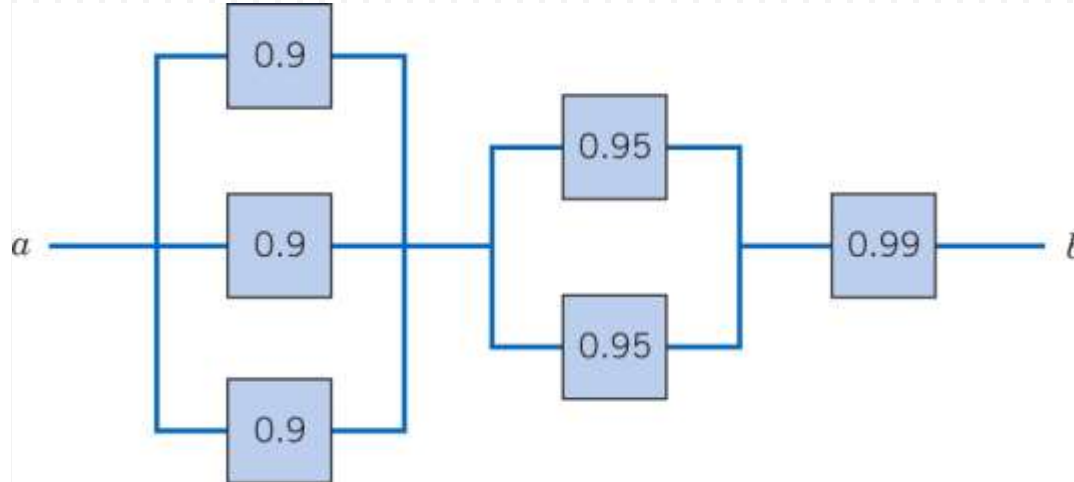
$$P(T \cup B) = 1 - P(T' \cap B') = 1 - P(T') * P(B') = 1 - 0.05^2 = 1 - 0.0025 = 0.9975.$$

( this is 1 minus the probability that they both don't fail)

## Example 2-35: Advanced Circuit

80

This circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown. Each device fails independently.



Partition the graph into 3 columns with L & M denoting the left & middle columns.

$P(L) =$

$P(M) =$

so the probability that the circuit operates is:

( this is a series of parallel circuits).



# Bayes Theorem

81

- Thomas Bayes (1702-1761) was an English mathematician and Presbyterian minister.
- His idea is that we observe conditional probabilities through prior information.

$$\begin{aligned} P(A | B) &= P(A \cap B) / P(B) \\ &= P(B|A) * P(A) / P(B) \end{aligned}$$

- The short formal statement is:

$$P(A | B) = \frac{P(B | A) * P(A)}{P(B)} \quad \text{for } P(B) > 0 \quad (2-15)$$

- Note the reversal of the condition!

## Example 2-36:

82

- From Example 2-27, find  $P(F)$  which is not given:

	Probability of Failure	Level of Contamination	Probability of Level
$P(F H)$	0.100	High	0.2
$P(F H')$	0.005	Not High	0.8

$P(H \wedge F) = P(H|F) \cdot P(F) = P(F|H) \cdot P(H)$

$$\begin{aligned}
 P(H|F) &= P(H \wedge F) / P(F) &= P(H|F)P(F) &= P(F|H)P(H) \\
 &= P(B|A) \cdot P(A) / P(F) &= 0.10 \cdot 0.20 / 0.024 &= 0.83
 \end{aligned}$$

$$\begin{aligned}
 P(F) &= P(F \wedge H) + P(F \wedge H') \\
 &= P(F | H) \cdot P(H) + P(F | H') \cdot P(H') \\
 &= 0.1 \cdot 0.2 + 0.005 \cdot 0.8 = 0.024
 \end{aligned}$$

$$P(E_1|B) = P(E_1 \wedge B)/P(B)$$

# Bayes Theorem with Total Probability

83

- If  $E_1, E_2, \dots, E_k$  are  $k$  mutually exclusive and exhaustive events and  $B$  is any event,

$$P(E_1 | B) = \frac{P(B | E_1) P(E_1)}{P(B | E_1) P(E_1) + P(B | E_2) P(E_2) + \dots + P(B | E_k) P(E_k)}$$

$$P(B) = P(B \wedge E_1) + P(B \wedge E_2) + \dots + P(B \wedge E_k)$$

for  $P(B) > 0$  (2-16)

- Note that the:
  - ▣ Total probability expression of the denominator
  - ▣ Numerator is always one term of the denominator.

# Example 2-37: Medical Diagnostic-1

84

Because a new medical procedure has been shown to be effective in the early detection of a disease, a medical screening of the population is proposed. The probability that the test correctly identifies someone with the disease as positive is 0.99, and probability that the test correctly identifies someone without the disease as negative is 0.95. The incidence of the illness in the general population is 0.0001. You take the test and the result is positive. What is the probability that you have the illness?

Let  $D$  denote the event that you have the disease and let  $S$  denote the event that the test signals positive. Given info is:

- ▣  $P(S' | D') = 0.95$ , so  $P(S | D') = 0.05$ , and  $P(D) = 0.0001$ ,
- ▣  $P(S | D) = 0.99$ . We desire  $P(D | S)$ .

## Example 2-37: Medical Diagnostic-2

85

$$\begin{aligned} &= P(D^S)/P(S) \\ &= P(D^S)/(P(S^D)+P(S^D')) \\ &= P(D^S)/(P(SID)P(D) + P(SID')P(D')) \\ &= 0.99*0.0001/0.99*0.0001+0.05*1-0.0001) \\ &= 1/506 \\ &= 0.002 \end{aligned}$$

Before the test, your chance was 0.0001. After the positive result, your chance is 0.00198. So your risk of having the disease has increased 20 times  $= 0.00198/0.00010$ , but is still tiny.

## Example 2-38: Bayesian Network-1

86

- Bayesian networks are used on Web sites of high-tech manufacturers to allow customers to quickly diagnose problems with products. A printer manufacturer obtained the following probabilities from its database. Printer failures are of 3 types: hardware  $P(H) = 0.3$ , software  $P(S)=0.6$ , and other  $P(O)=0.1$ . Also:
  - ▣  $P(F|H) = 0.9$ ,  $P(F|S) = 0.2$ ,  $P(F|O) = 0.5$ .
- Find the max of  $P(H|F)$ ,  $P(S|F)$ ,  $P(O|F)$  to direct the diagnostic effort.

# Example 2-38: Bayesian Network-2

87

$$P(F) = P(FIH)P(H) + P(FIS)P(S) + P(FIO)P(O) = 0.36$$

$$P(H|F) = P(FIH) * P(H) / P(F) = 0.250$$

$$P(S|F) = P(FIS) * P(S) / P(F) = 0.9 * 0.1 = 0.333$$

$$P(O|F) = P(FIO) * P(O) / P(F) = 0.5 *$$

Note that the conditionals based on Failure add to 1.  
Since the Other category is the most likely cause of the failure, diagnostic effort should be so initially directed.

# Random Variables

88

- A variable that associates a number with the outcome of a random experiment is called a **random variable**.
- A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.
- Particular notation is used to distinguish the random variable (rv) from the real number. The rv is denoted by an uppercase letter, such as  $X$ . After the experiment is conducted, the measured value is denoted by a lowercase letter, such as  $x = 70$ .  $X$  and  $x$  are shown in italics, e.g.,  $P(X=x)$ .

$$P(X=70)$$

$$P(X=80)$$

$$P(X=72.5)$$



# Continuous & Discrete Random Variables

89

- A **discrete** random variable is a rv with a finite (or countably infinite) range. They are usually integer counts, e.g., number of errors or number of bit errors per 100,000 transmitted (rate). The ends of the range of rv values may be finite ( $0 \leq x \leq 5$ ) or infinite ( $x \geq 0$ ).
- A **continuous** random variable is a rv with an interval (either finite or infinite) of real numbers for its range. Its precision depends on the measuring instrument.

# Examples of Discrete & Continuous RVs

90

- Discrete rv's:
  - ▣ Number of scratches on a surface. 1,2,3,4,5,
  - ▣ ~~Proportion of~~ defective parts among 100 tested.
  - ▣ Number of transmitted bits received in error.
  - ▣ Number of common stock shares traded per day.
- Continuous rv's:
  - ▣ Electrical current and voltage.
  - ▣ Physical measurements, e.g., length, weight, time, temperature, pressure.