# Positive Properties of Context-Free languages



## Union

Context-free languages
are closed under: **Union**

$L_1$ is context free

$L_2$ is context free

$\Rightarrow$ $L_1 \cup L_2$ is context-free

## Example

**Language** | **Grammar**

$L_1 = \{a^n b^n\}$ $\qquad$ $S_1 \rightarrow aS_1b \mid \lambda$

$L_2 = \{ww^R\}$ $\qquad$ $S_2 \rightarrow aS_2a \mid bS_2b \mid \lambda$

**Union**

$L = \{a^n b^n\} \cup \{ww^R\}$ $\qquad$ $S \rightarrow S_1 \mid S_2$



## In general:

For context-free languages $L_1, L_2$
with context-free grammars $G_1, G_2$
and start variables $S_1, S_2$

The grammar of the **union** $L_1 \cup L_2$
has new start variable $S$
and additional production $S \rightarrow S_1 \mid S_2$

## Concatenation

Context-free languages
are closed under: **Concatenation**

$L_1$ is context free

$L_2$ is context free

$\Rightarrow$ $L_1 L_2$ is context-free

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## Example

**Language**

$L_1 = \{a^n b^n\}$

$L_2 = \{ww^R\}$

**Grammar**

$S_1 \to aS_1 b \mid \lambda$

$S_2 \to aS_2 a \mid bS_2 b \mid \lambda$

**Concatenation**

$L = \{a^n b^n\}\{ww^R\}$

$S \to S_1 S_2$

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In general:

For context-free languages $L_1,\ L_2$
with context-free grammars $G_1,\ G_2$
and start variables $S_1,\ S_2$

The grammar of the **concatenation** $L_1L_2$
has new start variable $S$
and additional production $S \to S_1S_2$

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## Star Operation

Context-free languages
are closed under: **Star-operation**

$L$ is context free $\Longrightarrow$ $L^*$ is context-free

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## Example

Language

$$L = \{a^n b^n\}$$

Grammar

$$S \rightarrow aSb \mid \lambda$$

**Star Operation**

$$L = \{a^n b^n\}*$$

$$S_1 \rightarrow SS_1 \mid \lambda$$

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In general:

For context-free language $L$
with context-free grammar $G$
and start variable $S$

The grammar of the **star operation** $L*$
has new start variable $S_1$
and additional production $S_1 \rightarrow SS_1 \mid \lambda$

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# Negative Properties of Context-Free Languages



## Intersection

Context-free languages are **not** closed under: **intersection**

$L_1$ is context free

$L_2$ is context free

$\Rightarrow$ $L_1 \cap L_2$ **not** necessarily context-free

## Example

$L_1 = \{a^n b^n c^m\}$

Context-free:

$S \to AC$

$A \to aAb \mid \lambda$

$C \to cC \mid \lambda$

$L_2 = \{a^n b^m c^m\}$

Context-free:

$S \to AB$

$A \to aA \mid \lambda$

$B \to bBc \mid \lambda$

**Intersection**

$L_1 \cap L_2 = \{a^n b^n c^n\}$ NOT context-free



## Complement

Context-free languages are **not** closed under: **complement**

$L$ is context free ⇒ $\overline{L}$ not necessarily context-free

## Example

$$L_1 = \{a^n b^n c^m\}$$

Context-free:

$$S \to AC$$
$$A \to aAb \mid \lambda$$
$$C \to cC \mid \lambda$$

$$L_2 = \{a^n b^m c^m\}$$

Context-free:

$$S \to AB$$
$$A \to aA \mid \lambda$$
$$B \to bBc \mid \lambda$$

**Complement**

$$\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2 = \{a^n b^n c^n\}$$

**NOT** context-free



# Intersection of Context-free languages and Regular Languages

The intersection of

a context-free language and

a regular language

is a context-free language

$L_1$ context free

$L_2$ regular

$\Rightarrow$ $L_1 \cap L_2$ context-free

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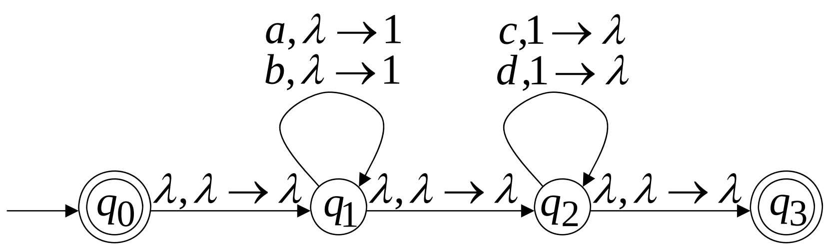
**Example:**

context-free

$$L_1 = \{w_1 w_2 : |w_1| = |w_2|, w_1 \in \{a,b\}^*, w_2 \in \{c,d\}^*\}$$

NPDA $M_1$

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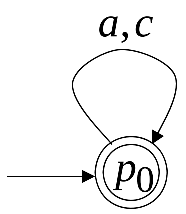
regular

$$L_2 = \{a,c\}^*$$

DFA $M_2$

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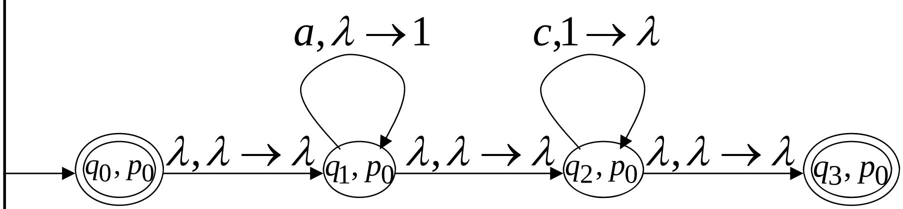
context-free

Automaton for: $L_1 \cap L_2 = \{a^n c^n : n \geq 0\}$

NPDA $M$

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# Applications of Regular Closure



The intersection of
a context-free language and
a regular language
is a context-free language

Regular Closure

$L_1$ context free
$L_2$ regular
$\Rightarrow$ $L_1 \cap L_2$ context-free

# An Application of Regular Closure

Prove that: $L = \{a^n b^n : \quad n \neq 100, n \geq 0\}$

is context-free



We know:

$\{a^n b^n : n \geq 0\}$ is context-free

We also know:

$$L_1 = \{a^{100}b^{100}\}$$ is regular

$$\overline{L_1} = \{(a+b)^*\} - \{a^{100}b^{100}\}$$ is regular



$\{a^n b^n\}$ context-free

$\overline{L_1} = \{(a+b)^*\} - \{a^{100}b^{100}\}$ regular

(regular closure) $\{a^n b^n\} \cap \overline{L_1}$ context-free

$$\{a^n b^n\} \cap \overline{L_1} = \{a^n b^n : \quad n \neq 100, n \geq 0\} = L$$

is context-free

## Another Application of Regular Closure

Prove that: $L = \{w: \; n_a = n_b = n_c\}$

is **not** context-free

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If $L = \{w: \; n_a = n_b = n_c\}$ is context-free

(regular closure)

Then $L \cap \{a^*b^*c^*\} = \{a^n b^n c^n\}$

context-free     regular     context-free

**Impossible!!!**

Therefore, $L$ is **not** context free

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