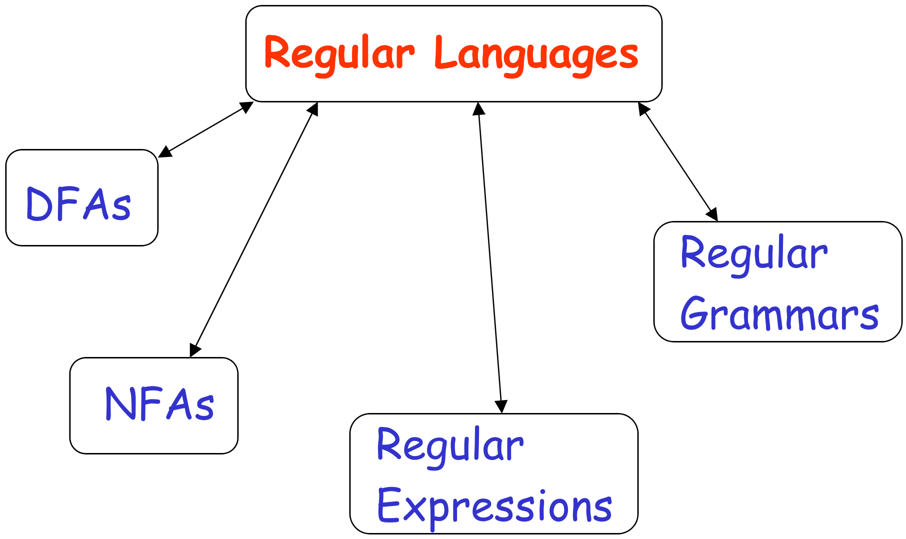
Standard Representations of Regular Languages



When we say: We are given a Regular Language L

We mean: Language L is in a standard representation

Elementary Questions

about

Regular Languages

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Membership Question

Question: Given regular language L
and string w
how can we check if $w \in L$?

Answer: Take the DFA that accepts L
and check if w is accepted

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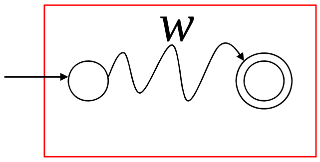
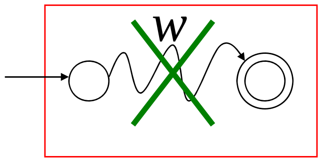
DFA



$w \in L$

DFA



$w \notin L$

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Question: Given regular language L
how can we check
if L is empty: $(L = \emptyset)$?

Answer: Take the DFA that accepts L

Check if there is any path from
the initial state to a final state

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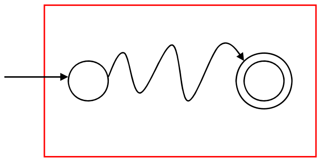
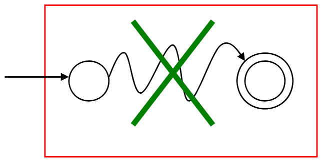
DFA



$L \neq \varnothing$

DFA



$L = \varnothing$

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Question: Given regular language L
how can we check
if L is finite?

Answer: Take the DFA that accepts L

Check if there is a walk with cycle
from the initial state to a final state

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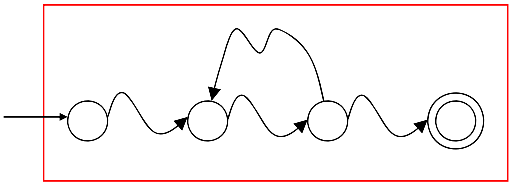
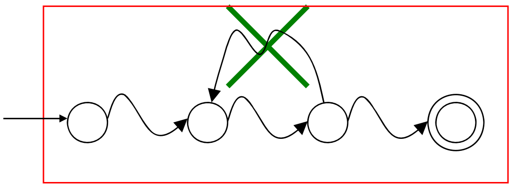
DFA



L is infinite

DFA



L is finite

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Question: Given regular languages L_1 and L_2 how can we check if $L_1 = L_2$?

Answer: Find if $(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \emptyset$

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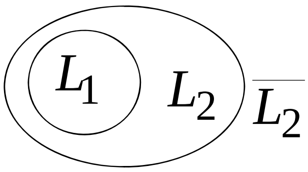
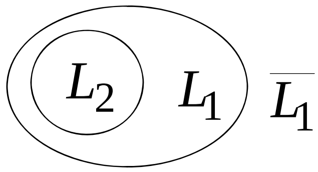
$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \varnothing$$

$L_1 \cap \overline{L_2} = \varnothing$ **and** $\overline{L_1} \cap L_2 = \varnothing$



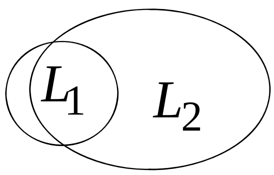


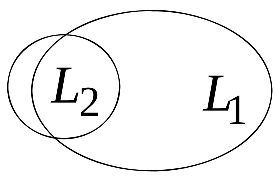
$L_1 \subseteq L_2$ $\qquad$ $L_2 \subseteq L_1$

$$L_1 = L_2$$

$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) \neq \varnothing$$

$L_1 \cap \overline{L_2} \neq \varnothing$ **or** $\overline{L_1} \cap L_2 \neq \varnothing$





$L_1 \not\subset L_2$ $\qquad$ $L_2 \not\subset L_1$

$$L_1 \neq L_2$$

Non-regular languages

Non-regular languages

$$\{a^n b^n : \quad n \geq 0\}$$

$$\{vv^R : \quad v \in \{a,b\}^*\}$$

Regular languages

$$a^*b \qquad b^*c+a$$

$$b+c(a+b)^*$$

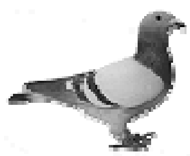
$etc...$

How can we prove that a language L
is not regular?

Prove that there is no DFA that accepts L

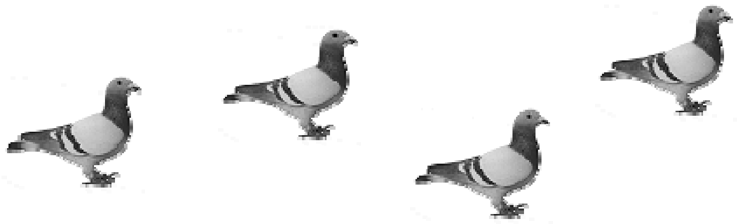
Problem: this is not easy to prove

Solution: the Pumping Lemma !!!

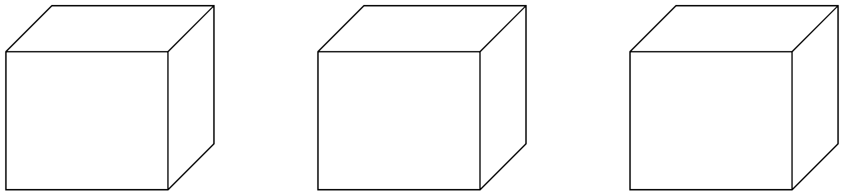


The Pigeonhole Principle

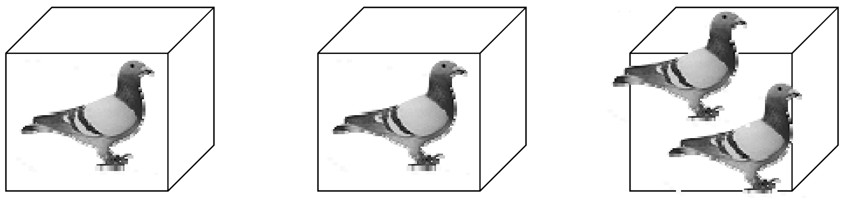
4 pigeons



3 pigeonholes

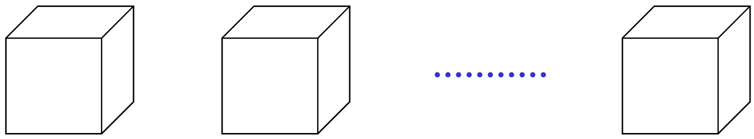


A pigeonhole must
contain at least two pigeons



n pigeons

m pigeonholes $\quad n > m$



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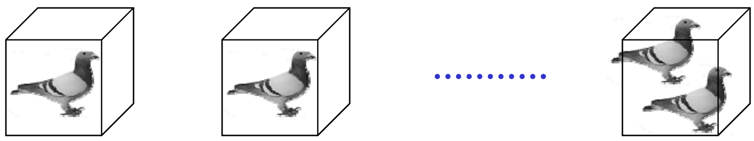
The Pigeonhole Principle

n pigeons

m pigeonholes

$n > m$

There is a pigeonhole with at least 2 pigeons



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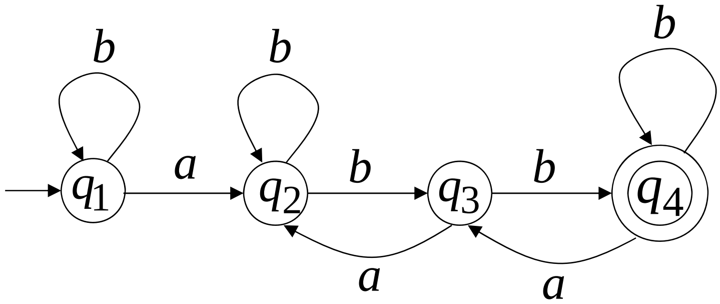
The Pigeonhole Principle

and

DFAs

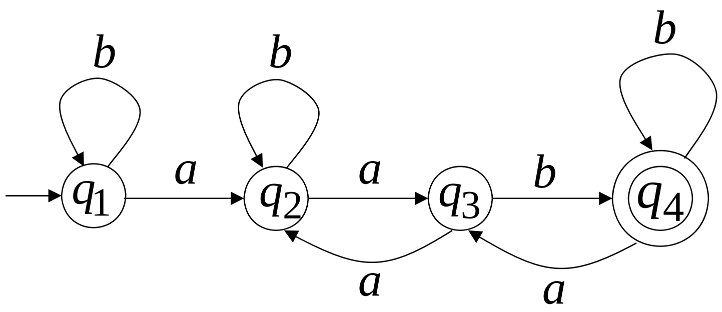
DFA with 4 states



In walks of strings: a no state is repeated

aa

aab

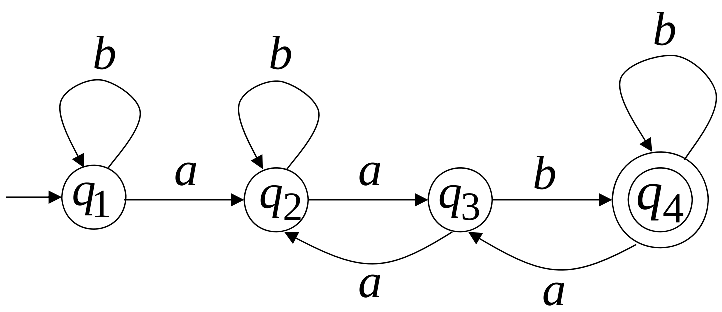


In walks of strings: $aabb$ a state is repeated

$bbaa$

$abbabb$

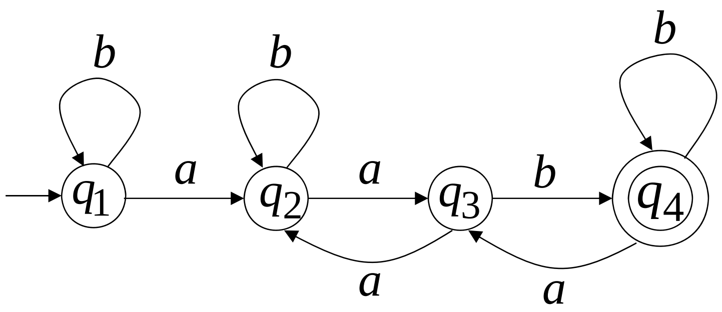
$abbbabbabb...$



If string w has length $|w| \geq 4$:

Then the transitions of string w are more than the states of the DFA

Thus, a state must be repeated



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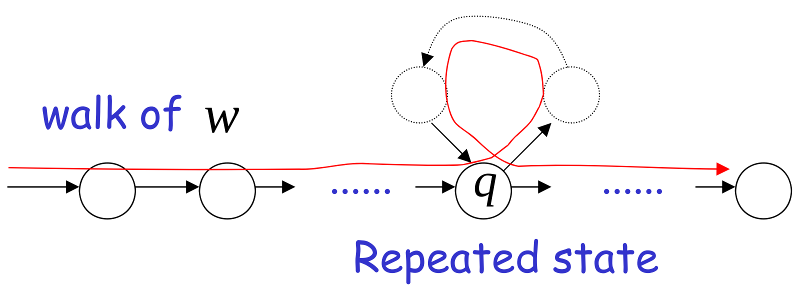
In general, for any DFA:

String w has length $\geq$ number of states

↓

A state q must be repeated in the walk of w



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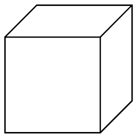
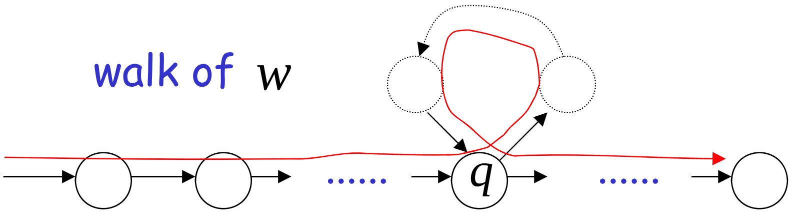
In other words for a string w:

$\xrightarrow{a}$ transitions are pigeons

q states are pigeonholes



walk of w

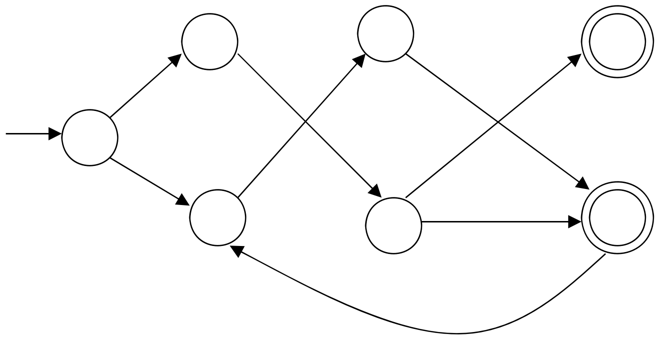


Repeated state

The Pumping Lemma

Take an infinite regular language L

There exists a DFA that accepts L



m states

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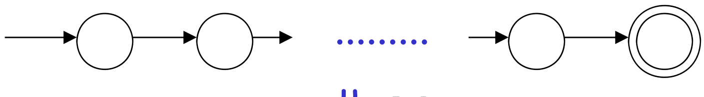
Take string w with $w \in L$

There is a walk with label w:



walk w

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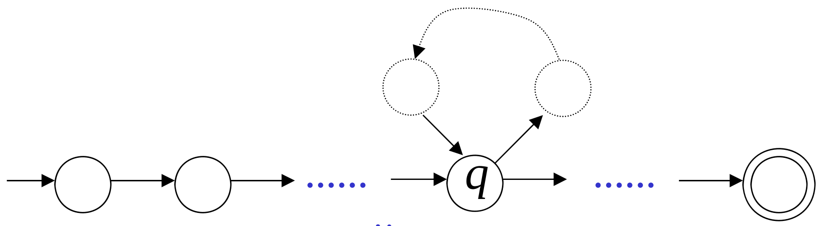
If string w has length $|w| \geq m$ (number of states of DFA)

then, from the pigeonhole principle:

a state is repeated in the walk w



walk w

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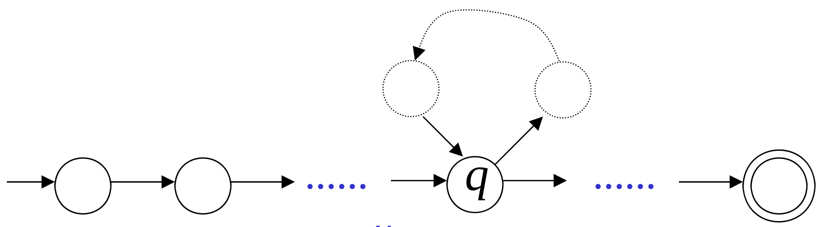
Let q be the first state repeated in the walk of w



walk w

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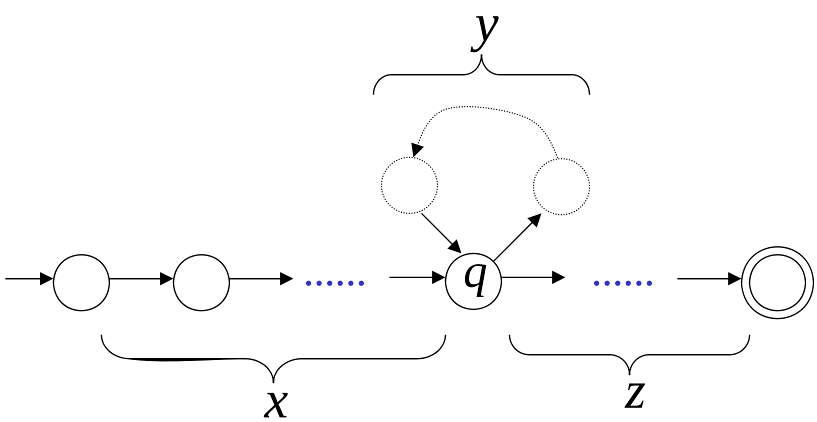
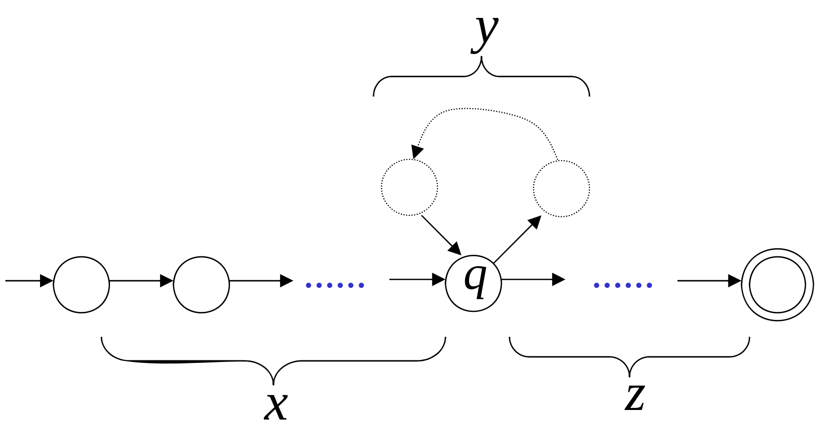
Write $w = x\ y\ z$

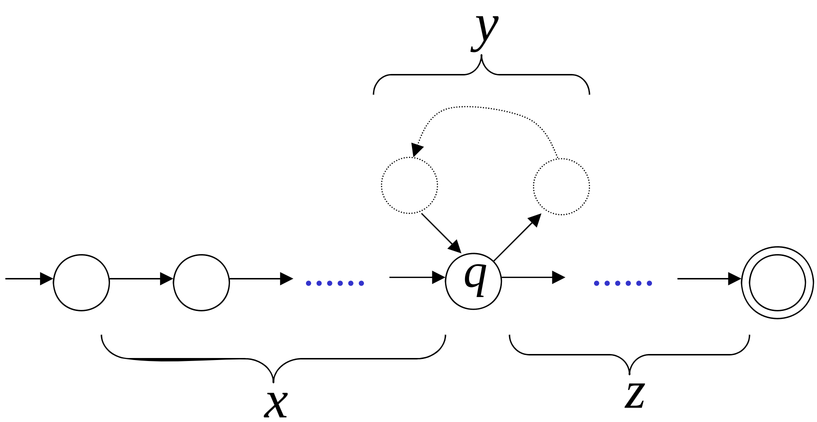


Observations: length $|x\ y| \le m$ number of states of DFA

length $|y| \ge 1$



Observation: The string $x\ z$
is accepted



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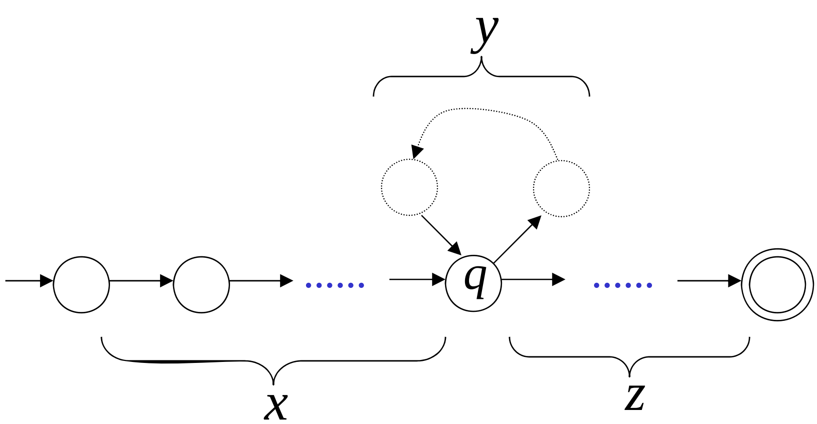
Observation: The string $x\ y\ y\ z$
is accepted



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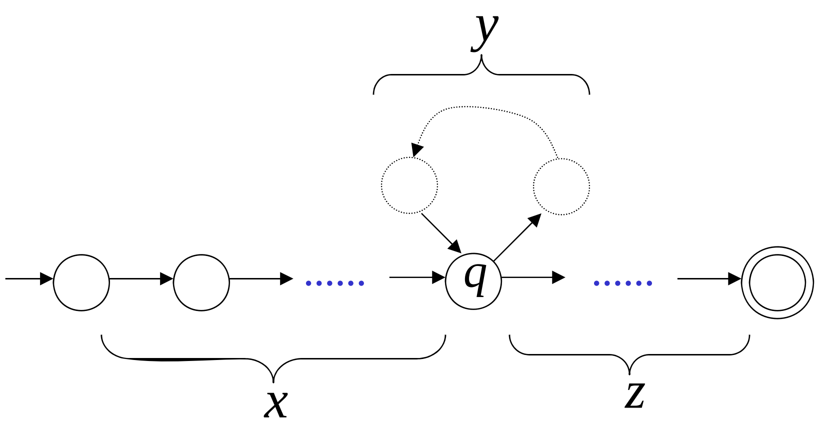
Observation: The string $x\,y\,y\,y\,z$ is accepted



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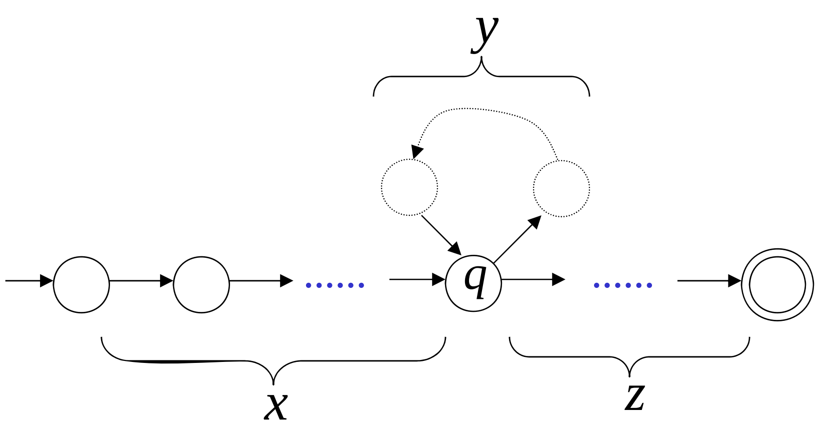
In General: The string $x\,y^i\,z$ is accepted $i = 0, 1, 2, \ldots$



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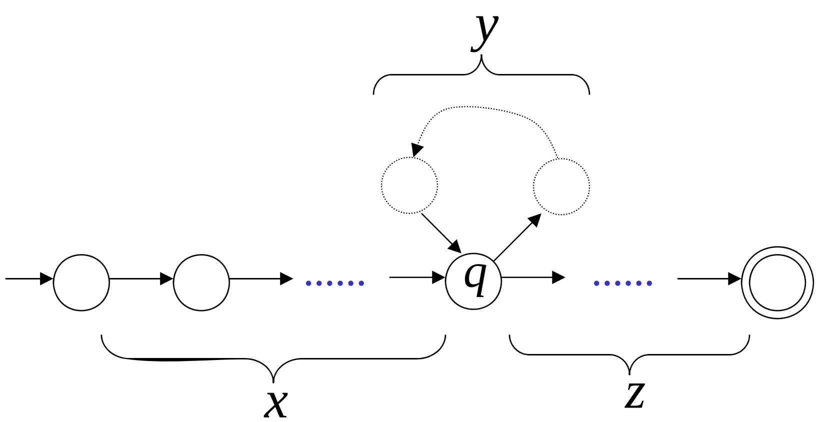
In General: $x\, y^i\, z \in L \qquad i = 0, 1, 2, \ldots$

Language accepted by the DFA



In other words, we described:

The Pumping Lemma !!!

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\, y\, z$
- with $|x\, y| \leq m$ and $|y| \geq 1$
- such that: $x\, y^i\, z \in L \qquad i = 0, 1, 2, \ldots$

Applications

of

the Pumping Lemma

Theorem: The language $L = \{a^n b^n : n \geq 0\}$ is not regular

Proof: Use the Pumping Lemma

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$L = \{a^n b^n : n \geq 0\}$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

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$$L = \{a^n b^n : n \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^m b^m$

Write: $a^m b^m = x\ y\ z$

From the Pumping Lemma

it must be that length $|x\ y| \leq m, \quad |y| \geq 1$

$$xyz = a^m b^m = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...ab...b}_{z}$$

$$\overbrace{a...aa...aa...a}^{m}\overbrace{b...b}^{m}$$

Thus: $y = a^k, \quad k \geq 1$

$x\,y\,z = a^m b^m \qquad y = a^k, \quad k \geq 1$

From the Pumping Lemma: $x\,y^i\,z \in L$

$i = 0, 1, 2, \ldots$

Thus: $x\,y^2\,z \in L$

$x\,y\,z = a^m b^m \qquad y = a^k, \quad k \geq 1$

From the Pumping Lemma: $x\,y^2\,z \in L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}a...a}^{m+k}\overbrace{b...b}^{m} \in L$$

Thus: $a^{m+k}b^m \in L$

$$a^{m+k}b^m \in L \qquad k \geq 1$$

BUT: $L = \{a^n b^n : n \geq 0\}$

$$\Downarrow$$

$$a^{m+k}b^m \notin L$$

CONTRADICTION!!!

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Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

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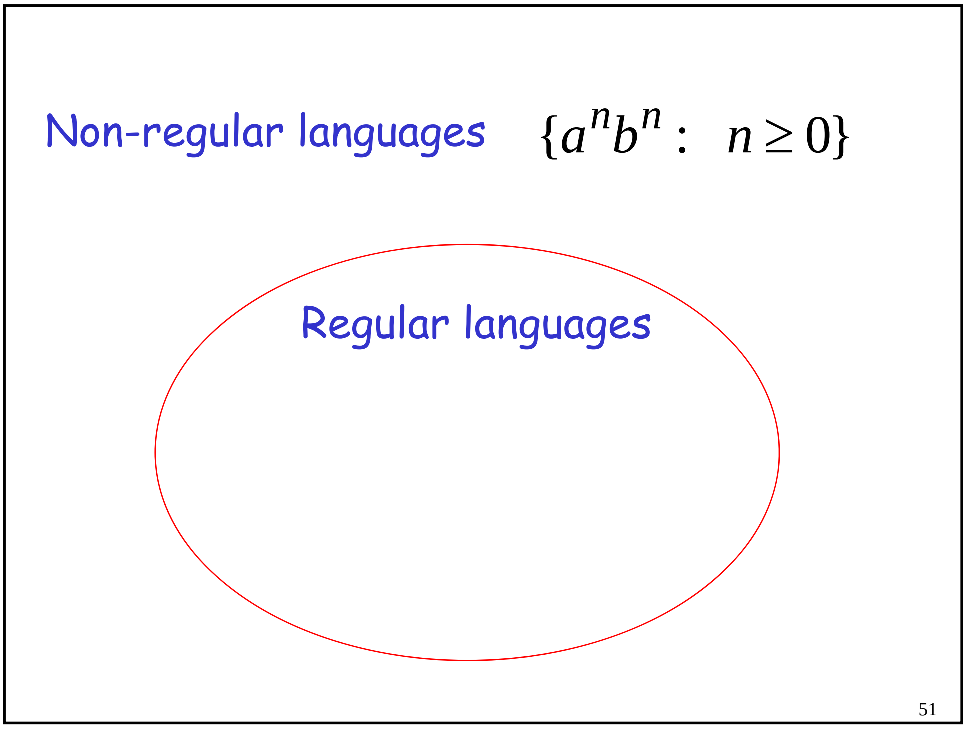
Non-regular languages
$\{a^n b^n : \quad n \geq 0\}$
Regular languages
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