Single Final State for NFAs

1

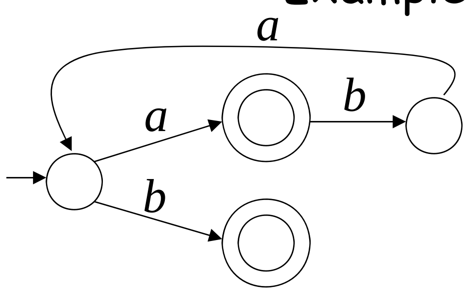
Any NFA can be converted

to an equivalent NFA

with a single final state

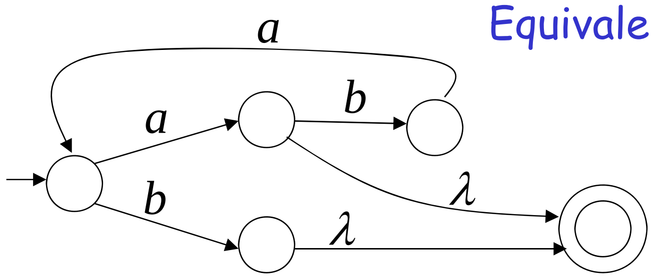
2

Example



NFA

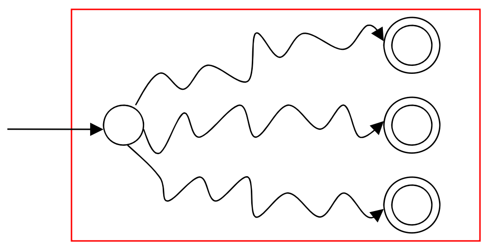
Equivalent NFA



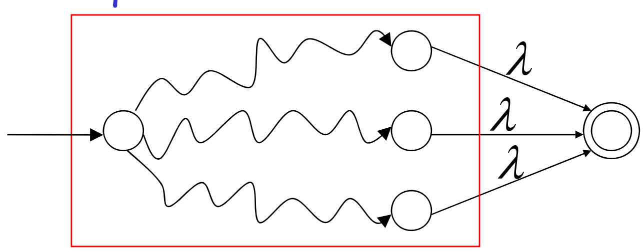
3

In General

NFA



Equivalent NFA

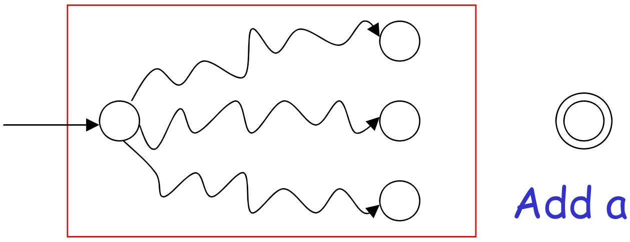


Single
final state

4

Extreme Case

NFA without final state



Add a final state
Without transitions

Properties of Regular Languages

For regular languages L_1 and L_2
we will prove that:

$$\left.\begin{array}{rl} \text{Union:} & L_1 \cup L_2 \\ \text{Concatenation:} & L_1 L_2 \\ \text{Star:} & L_1{}^* \\ \text{Reversal:} & L_1{}^R \\ \text{Complement:} & \overline{L_1} \\ \text{Intersection:} & L_1 \cap L_2 \end{array}\right\}$$

Are regular Languages

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We say: Regular languages are **closed under**

Union: $L_1 \cup L_2$

Concatenation: $L_1 L_2$

Star: $L_1{}^*$

Reversal: $L_1{}^R$

Complement: $\overline{L_1}$

Intersection: $L_1 \cap L_2$

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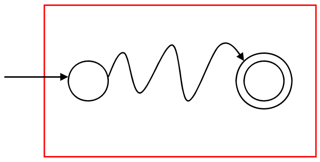
Regular language L_1

$$L(M_1) = L_1$$

NFA M_1

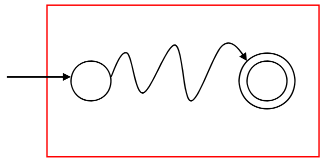


Single final state

Regular language L_2

$$L(M_2) = L_2$$

NFA M_2



Single final state

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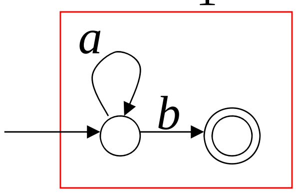
Example

$n \geq 0$

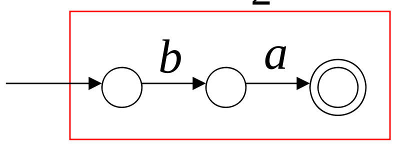
$$L_1 = \{a^n b\}$$

M_1



$$L_2 = \{ba\}$$

M_2



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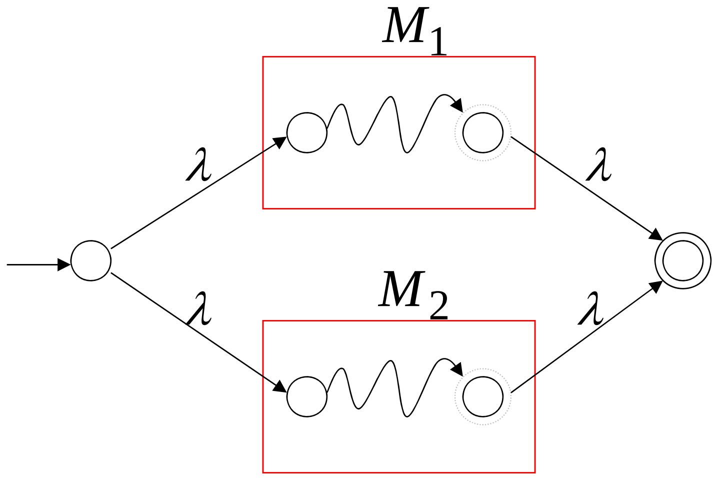
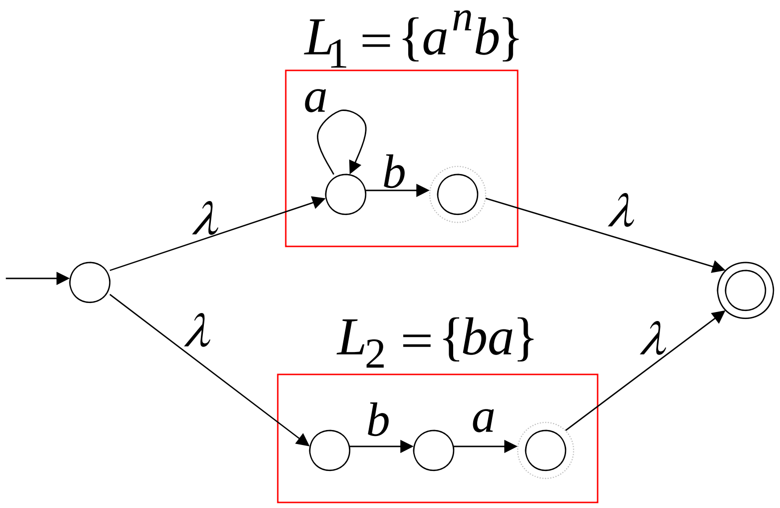
Union

NFA for $L_1 \cup L_2$



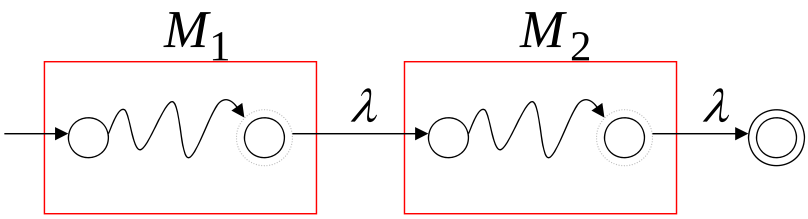
Example

NFA for $L_1 \cup L_2 = \{a^n b\} \cup \{ba\}$



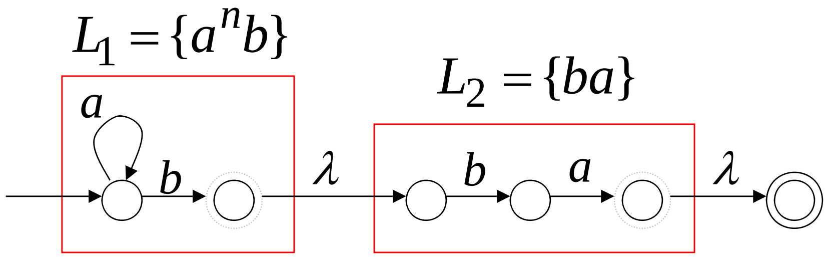
Concatenation

NFA for $L_1 L_2$



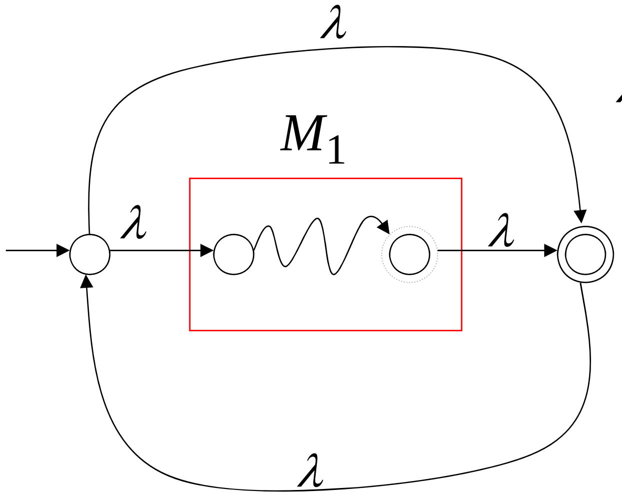
Example

NFA for $L_1 L_2 = \{a^n b\}\{ba\} = \{a^n bba\}$



Star Operation

NFA for L_1*



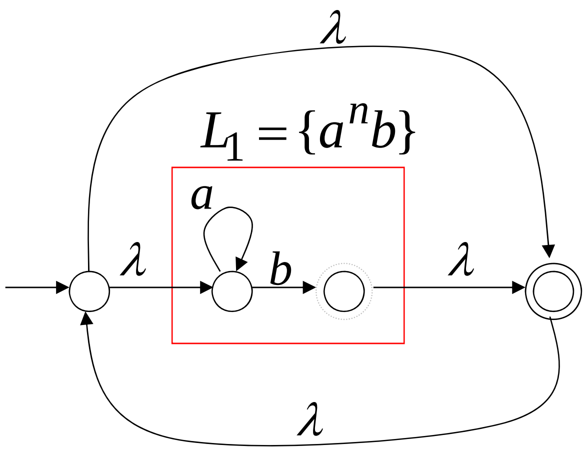
$\lambda \in L_1*$

Example

NFA for $L_1* = \{a^n b\}*$

$w = w_1 w_2 \cdots w_k$

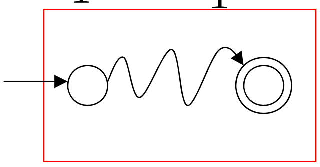
$w_i \in L_1$



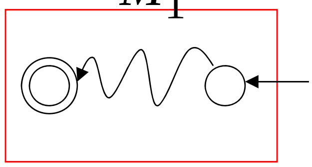
Reverse

NFA for L_1^R

L_1 M_1



M_1'



1. Reverse all transitions

2. Make initial state final state
   and vice versa

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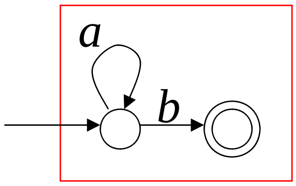
Example

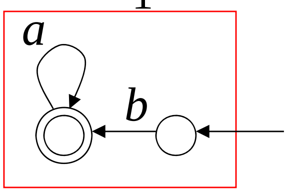
M_1

$L_1 = \{a^n b\}$



M_1'

$L_1^R = \{ba^n\}$



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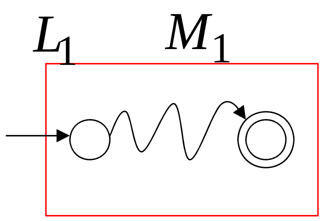
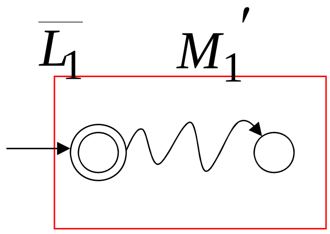
Complement





1. Take the **DFA** that accepts L_1

2. Make final states non-final,
   and vice-versa

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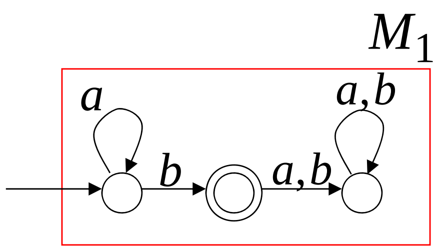
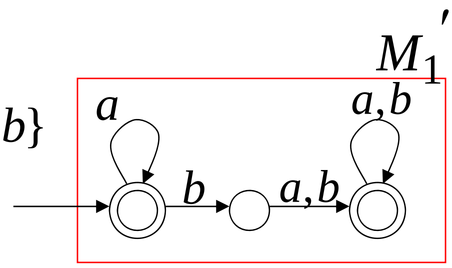
Example

$L_1 = \{a^n b\}$



$\overline{L_1} = \{a,b\}^* - \{a^n b\}$



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Intersection

DeMorgan's Law: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

L_1 , L_2 regular

$\Rightarrow$ $\overline{L_1}$, $\overline{L_2}$ regular

$\Rightarrow$ $\overline{L_1} \cup \overline{L_2}$ regular

$\Rightarrow$ $\overline{\overline{L_1} \cup \overline{L_2}}$ regular

$\Rightarrow$ $L_1 \cap L_2$ regular

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Example

$L_1 = \{a^n b\}$ regular

$L_2 = \{ab, ba\}$ regular

$\Rightarrow$ $L_1 \cap L_2 = \{ab\}$ regular

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Regular language

accept

describes

DFA/NFA

Regular Expressions

Regular Expressions

Regular expressions
describe regular languages

Example: $(a+b\cdot c)*$

describes the language

$$\{a,bc\}^* = \{\lambda, a, bc, aa, abc, bca, ...\}$$

Recursive Definition

Primitive regular expressions: $\emptyset, \quad \lambda, \quad \alpha$

Given regular expressions r_1 and r_2

$$\left.\begin{array}{l} r_1 + r_2 \\ r_1 \cdot r_2 \\ r_1 * \\ (r_1) \end{array}\right\}$$ Are regular expressions

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Examples

A regular expression: $(a + b \cdot c) * \cdot (c + \emptyset)$

Not a regular expression: $(a + b +)$

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Languages of Regular Expressions

$L(r)$: language of regular expression r

Example

$$L((a+b\cdot c)^*)=\{\lambda,a,bc,aa,abc,bca,\ldots\}$$

Definition

For primitive regular expressions:

$$L(\varnothing)=\varnothing$$

$$L(\lambda)=\{\lambda\}$$

$$L(a)=\{a\}$$

Definition (continued)

For regular expressions r_1 and r_2

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

$$L((r_1)) = L(r_1)$$

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Example

Regular expression: $(a + b) \cdot a *$

$$\begin{aligned} L((a+b)\cdot a *) &= L((a+b))\, L(a *) \\ &= L(a+b)\, L(a *) \\ &= (L(a) \cup L(b))\, (L(a))* \\ &= (\{a\} \cup \{b\})\, (\{a\})* \\ &= \{a, b\}\, \{\lambda, a, aa, aaa, \ldots\} \\ &= \{a, aa, aaa, \ldots, b, ba, baa, \ldots\} \end{aligned}$$

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Example

Regular expression $r = (a+b)*(a+bb)$

$$L(r) = \{a, bb, aa, abb, ba, bbb, ...\}$$

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Example

Regular expression $r = (aa)*(bb)*b$

$$L(r) = \{a^{2n}b^{2m}b : \quad n, m \geq 0\}$$

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Example

Regular expression $r = (0+1)^* 00 (0+1)^*$

$L(r)$ = { all strings with at least two consecutive 0 }

Example

Regular expression $r = (1+01)^* (0+\lambda)$

$L(r)$ = { all strings without two consecutive 0 }

Equivalent Regular Expressions

Definition:

Regular expressions r_1 and r_2

are **equivalent** if $L(r_1) = L(r_2)$

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Example

L = { all strings without two consecutive 0 }

$$r_1 = (1+01)^*(0+\lambda)$$

$$r_2 = (1^*011^*)^*(0+\lambda)+1^*(0+\lambda)$$

$L(r_1) = L(r_2) = L$ $\Longrightarrow$ r_1 and r_2 are equivalent regular expr.

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Regular Expressions and Regular Languages

Theorem

{ Languages Generated by Regular Expressions } = { Regular Languages }

DFA/NFA

Theorem - Part 1

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

1. For any regular expression r
   the language $L(r)$ is regular

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Theorem - Part 2

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

2. For any regular language L there is
   a regular expression r with $L(r) = L$

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Proof - Part 1

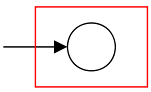
1. For any regular expression r
   the language $L(r)$ is regular

Proof by induction on the size of r

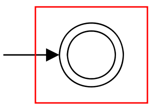
Induction Basis

Primitive Regular Expressions: $\varnothing, \quad \lambda, \quad \alpha$

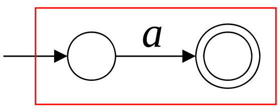
NFAs



$$L(M_1) = \varnothing = L(\varnothing)$$



$$L(M_2) = \{\lambda\} = L(\lambda)$$



$$L(M_3) = \{a\} = L(a)$$

regular languages

Inductive Hypothesis

Assume

for regular expressions r_1 and r_2

that

$L(r_1)$ and $L(r_2)$ are regular languages

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Inductive Step

We will prove:

$$\left.\begin{array}{l} L(r_1 + r_2) \\ \\ L(r_1 \cdot r_2) \\ \\ L(r_1 *) \\ \\ L((r_1)) \end{array}\right\} \text{Are regular Languages}$$

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By definition of regular expressions:

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

$$L((r_1)) = L(r_1)$$

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By inductive hypothesis we know:

$L(r_1)$ and $L(r_2)$ are regular languages

We also know:

Regular languages are closed under:

Union	$L(r_1) \cup L(r_2)$
Concatenation	$L(r_1)\, L(r_2)$
Star	$(L(r_1))*$

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Therefore:

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

Are regular languages

And trivially:

$L((r_1))$ is a regular language

Proof - Part 2

2. For any regular language L there is a regular expression r with $L(r) = L$

Proof by construction of regular expression

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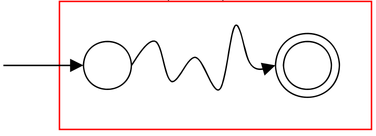
Since L is regular take the NFA M that accepts it

$L(M) = L$



Single final state

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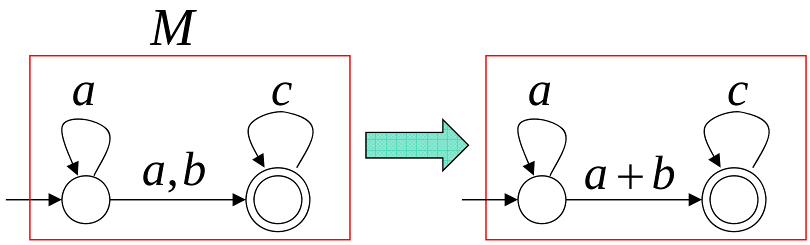
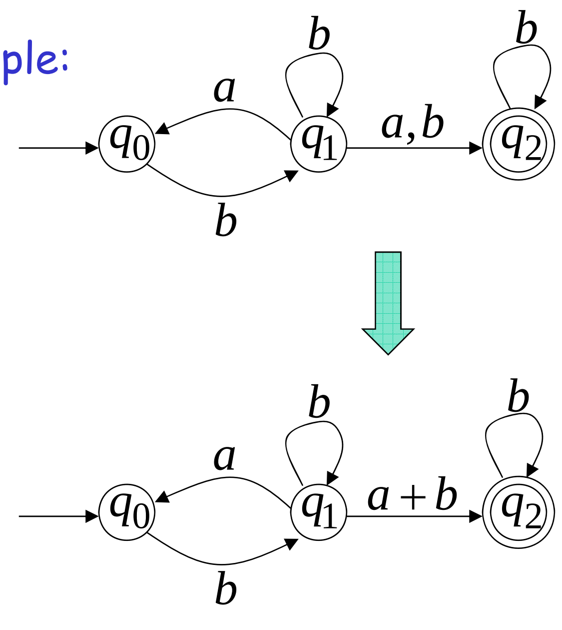
From M construct the equivalent
Generalized Transition Graph
in which transition labels are regular expressions

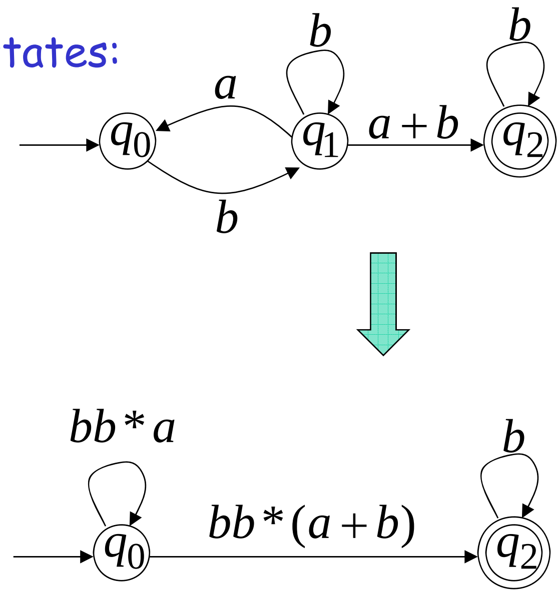
Example:



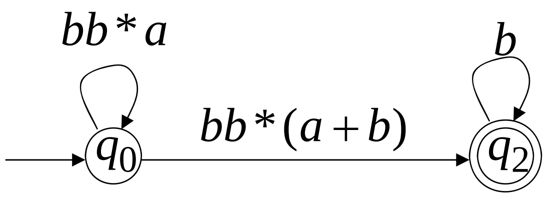
Another Example:



Reducing the states:



Resulting Regular Expression:

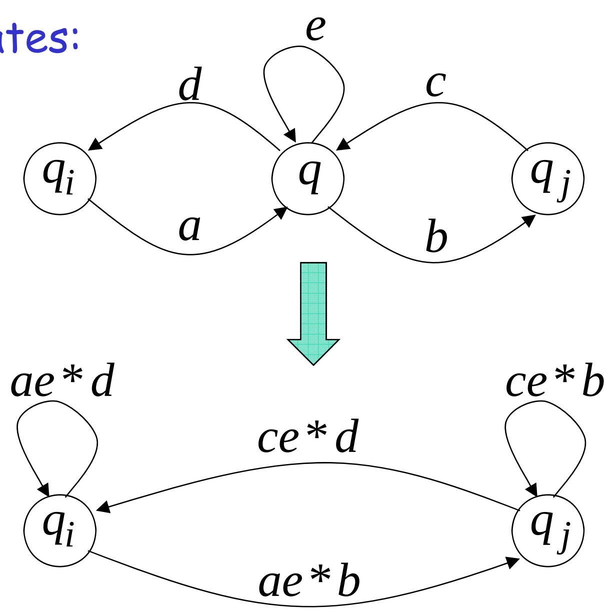


$$r = (bb^*a)^*bb^*(a+b)b^*$$

$$L(r) = L(M) = L$$

In General

Removing states:



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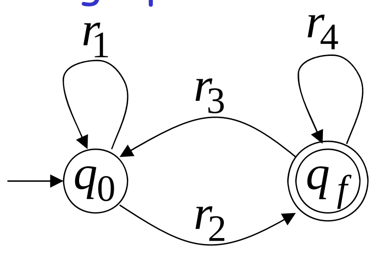
The final transition graph:



The resulting regular expression:

$$r = r_1^* r_2 (r_4 + r_3 r_1^* r_2)^*$$

$$L(r) = L(M) = L$$

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