PROBABILITY DISTRIBUTION (DISCRETE DISTRIBUTION)

Probability and Statistics

5-1: Discrete Random Variables

Many physical systems can be modeled by the same or similar random experiments and random variables. The distribution of the random variable involved in each of these common systems can be analyzed, and the results can be used in different applications and examples.

In this chapter, we present the analysis of several random experiments and discrete random variables that frequently arise in applications.

We often omit a discussion of the underlying sample space of the random experiment and directly describe the distribution of a particular random variable.

5-1: Discrete Random Variables

Example 5-1: Voice Lines

- A voice communication system for a business contains 48 external lines. At a particular time, the system is observed, and some of the lines are being used.
- Let X denote the number of lines in use. Then, X can assume any of the integer values 0 through 48.
- The system is observed at a random point in time. If 10 lines are in use, then $x = 10$.

5-1: Discrete Random Variables

Example 5-2: Wafers

In a semiconductor manufacturing process, 2 wafers from a lot are sampled. Each wafer is classified as *pass* or *fail*. Assume that the probability that a wafer passes is 0.8, and that wafers are independent.

The sample space for the experiment and associated probabilities are shown in Table 5-1. The probability that the 1st wafer passes and the 2nd fails, denoted as *pf* is $P(pf) = 0.8 * 0.2 = 0.16$.

The random variable X is defined as the number of wafers that pass.

Table 5-1 Wafer Tests

Outcome				
Wafer #				
1	2	Probability	x	y
Pass	Pass	0.64	2	0
Fail	Pass	0.16	1	1
Pass	Fail	0.16	1	1
Fail	Fail	0.04	0	2
		1.00		

if Y denoted the num of wafer that fail the insprction

5-1: Discrete Random Variables

Example 5-3: Particles on Wafers

- Define the random variable X to be the number of contamination particles on a wafer. Although wafers possess a number of characteristics, the random variable X summarizes the wafer only in terms of the number of particles. The possible values of X are the integers 0 through a very large number, so we write $x \geq 0$.
- We can also describe the random variable Y as the number of chips made from a wafer that fail the final test. If there can be 12 chips made from a wafer, then we write $0 \leq y \leq 12$. (changed)

5-1: Discrete Random Variables

5-1.1 Probability Distribution

- A random variable *X* associates the outcomes of a random experiment to a number on the number line.
- The probability distribution of the random variable *X* is a description of the probabilities with the possible numerical values of *X*.
- A probability distribution of a discrete random variable can be:
  1. A list of the possible values along with their probabilities.
  2. A formula that is used to calculate the probability in response to an input of the random variable's value.

5-1.2: Probability Distributions

- A random variable X associates the outcomes of a random experiment to a number on the number line.
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  2. A formula that is used to calculate the probability in response to an input of the random variable's value.

5-1: Discrete Random Variables

Example 5-4: Digital Channel

- There is a chance that a bit transmitted through a digital transmission channel is received in error.
- Let *X* equal the number of bits received in error of the next 4 transmitted.
- The associated probability distribution of *X* is shown as a graph and as a table.

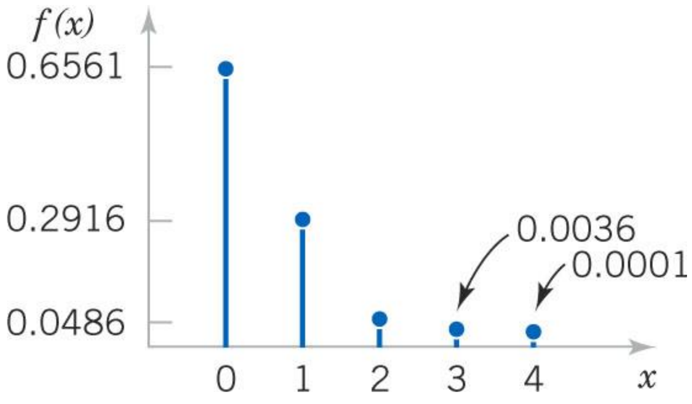


Figure 5-1 Probability distribution for bits in error.

$P(X=0) =$	0.6561
$P(X=1) =$	0.2916
$P(X=2) =$	0.0486
$P(X=3) =$	0.0036
$P(X=4) =$	0.0001
	1.0000

5-1: Discrete Random Variables

5-1.2 Probability Mass Function

Suppose a loading on a long, thin beam places mass only at discrete points. This represents a probability distribution where the beam is the number line over the range of x and the probabilities represent the mass. That's why it is called a probability mass function.

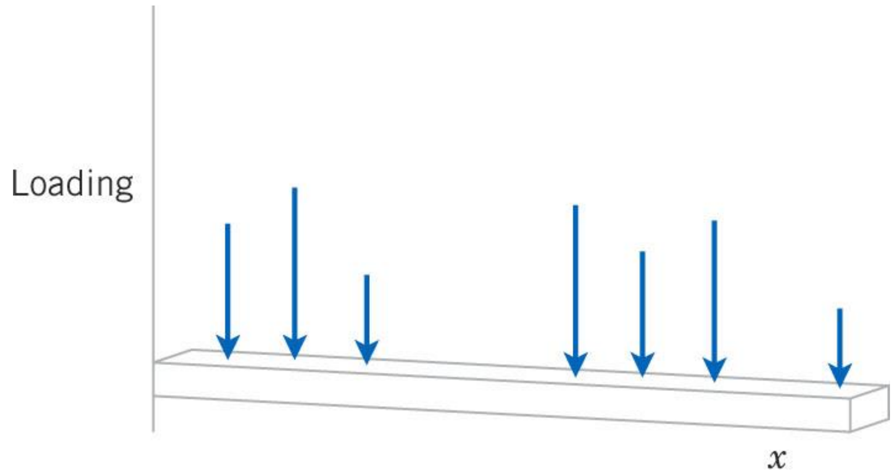


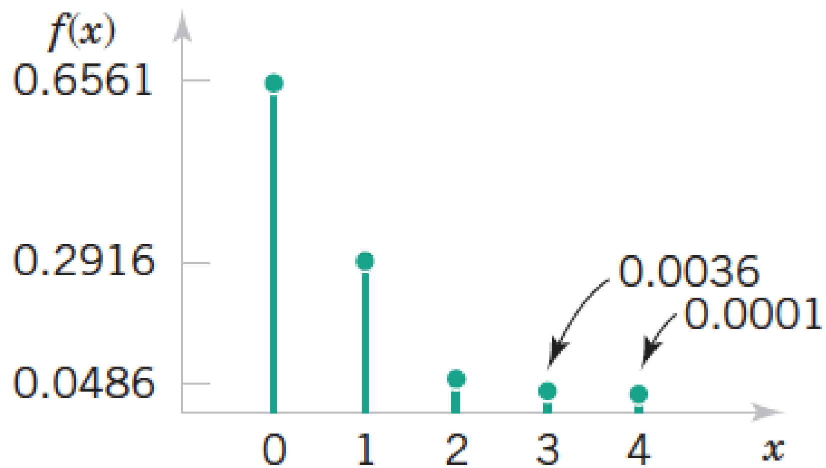
Figure 5-2 Loading at discrete points on a long, thin beam.

5-1: Discrete Random Variables

5-1.2 Probability Mass Function

For a discrete random variable X with possible values $x_1, x_2, \ldots, x_n$, the **probability mass function** (or pmf) is

$$f(x_i) = P(X = x_i) \tag{5-1}$$



P(X = 4) = 0.0001

Figure 5-3 Probability distribution

5-1: Discrete Random Variables

5-1.3 Probability Mass Function Properties

For a discrete random variable X with possible values $x_1, x_2, \ldots x_n$, a probability mass function is a function such that:

(1) $f(x_i) \geq 0$

(2) $\sum_{i=1}^{n} f(x_i) = 1$

(3) $f(x_i) = P(X = x_i)$

5-1: Discrete Random Variables

Example 5-5: Wafer Contamination

- Let the random variable *X* denote the number of wafers that need to be analyzed to detect a large particle. Assume that the probability that a wafer contains a large particle is ~~0.01~~ 0.1, and that the wafers are independent. Determine the probability distribution of *X*.
- Let *p* denote a wafer for which a large particle is **p**resent & let *a* denote a wafer in which it is **a**bsent.
- The sample space is:

$$S = \{p, ap, aap, aaap, \ldots\}$$

- The range of the values of *X* is:

$$x = 1, 2, 3, 4, \ldots$$

Probability Distribution		
P(*X*=1) =	0.1	0.1
P(*X*=2) =	$(0.9)*0.1$	0.09
P(*X*=3) =	$(0.9)^2*0.1$	0.081
P(*X*=4) =	$(0.9)^3*0.1$	0.0729
		0.3439

5-1: Discrete Random Variables

5-1.4 Cumulative Distribution Functions

Example 5-6: Digital Channel (cont.)

- From Example 5-4, we can express the probability of three or fewer bits being in error, denoted as $P(X \leq 3)$.
- The event $(X \leq 3)$ is the union of the mutually exclusive events: $(X=0)$, $(X=1)$, $(X=2)$, $(X=3)$.
- From the table:

x	$P(X=x)$	$P(X \leq x)$
0	0.6561	0.6561
1	0.2916	0.9477
2	0.0486	0.9963
3	0.0036	0.9999
4	0.0001	1.0000
	1.0000	

P(X = 0) + P(X = 1)

ex. at least 3 --> P(X >= 3) = P(X = 3) + P(X = 4) + ... = 1 - P(X < 3)

at most 3 --> P(X <= 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1 - P(X > 3)

5-1: Discrete Random Variables

5-1.4 Cumulative Distribution Functions

The **cumulative distribution function** of a discrete random variable X is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i) \qquad (5\text{-}2)$$

5-1: Discrete Random Variables

5-1.5 Cumulative Distribution Function Properties

The cumulative distribution function is built from the probability mass function and vice versa.

The cumulative distribution function of a discrete random variable X, denoted as $F(x)$, is:

$$F(x) = F(X = x) = \sum_{x_i \le x} x_i$$

For a discrete random variable X, $F(x)$ satisfies the following properties:

(1) $F(x) = P(X \le x) = \sum_{x_i \le x} f(x_i)$

(2) $0 \le F(x) \le 1$

(3) If $x \le y$, then $F(x) \le F(y)$

because it's nondecreasing function

5-1: Discrete Random Variables

Example 5-7: Cumulative Distribution Function

- Determine the probability mass function of X from this cumulative distribution function:

$F(x) =$	0.0	$x < -2$
	0.2	$-2 \le x < 0$
	0.7	$0 \le x < 2$
	1.0	$2 \le x$

PMF		
$f(-2) =$	0.2	0.2 - 0
$f(0) =$	0.5	0.7 - 0.2
$f(2) =$	0.3	1.0 - 0.7

CDF <--> PMF

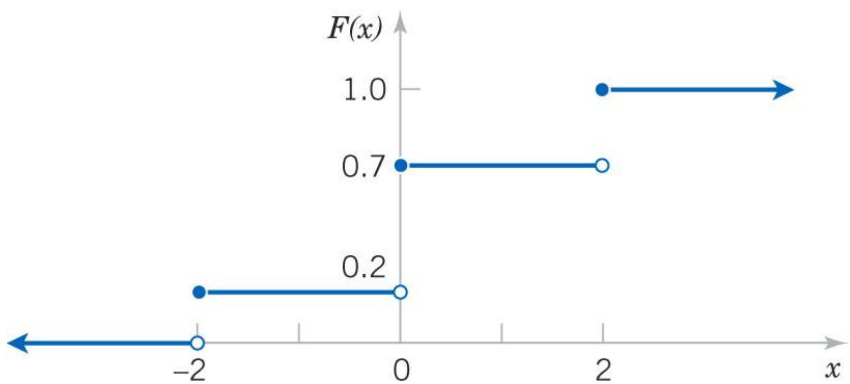


Figure 5-4 Graph of the CDF

5-1: Discrete Random Variables

Example 5-8: Sampling without Replacement

A day's production of 850 parts contains 50 defective parts. Two parts are selected at random without replacement. Let the random variable *X* equal the number of defective parts in the sample. Create the CDF of X.

$$P(X=0)=\frac{800}{850}\cdot\frac{799}{849}=0.886$$ or $\binom{800}{2}/\binom{850}{2}$

2 cases : 1st or 2nd part is defected

$$P(X=1)=2\cdot\frac{800}{850}\cdot\frac{50}{849}=0.111$$

$$P(X=2)=\frac{50}{850}\cdot\frac{49}{849}=0.003$$

Therefore,

$$F(0)=P(X\le 0)=0.886$$

$$F(1)=P(X\le 1)=0.997$$

$$F(2)=P(X\le 2)=1.000$$

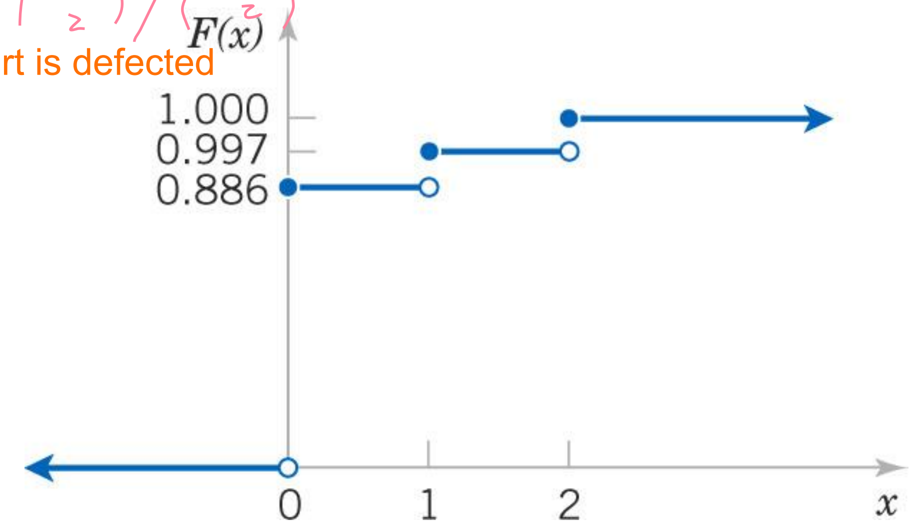


Figure 5-5 CDF. Note that F(x) is defined for all x, -∞ <x < ∞, not just 0, 1 and 2.

Summary Numbers of a Probability Distribution

- The mean is a measure of the center of a probability distribution.
- The variance is a measure of the dispersion or variability of a probability distribution.
- The standard deviation is another measure of the dispersion. It is the square root of the variance.

5-2: Mean and Variance of Discrete Random Variables

ex. 1,1,2,3,4

f(x) = 2/5 , x = 1

= 1/5 , x = 2 ....

M = (1 + 1 + 2 + 3 + 4) / 5

Mean and Variance

or from the formula : 1(2/5) + 2(1/5) + 3(1/5) + 4(1/5)

Let the possible values of the random variable X be denoted as $x_1, x_2, \ldots, x_n$. The pmf of X is $f(x)$, so $f(x_i) = P(X = x_i)$.

The **mean** or **expected value** of the discrete random variable X, denoted as μ or $E(X)$, is

population mean

$$\mu = E(X) = \sum_{i=1}^{n} x_i f(x_i) \tag{5-3}$$

The **variance** of X, denoted as σ^2 or $V(X)$, is

population variance

$$\sigma^2 = V(X) = E(X-\mu)^2 = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i) = \sum_{i=1}^{n} x_i^2 f(x_i) - \mu^2$$

The **standard deviation** of X is σ.

5-2: Mean and Variance of Discrete Random Variables

5-2.1 Mean Defined

The mean or expected value of the discrete random variable X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \sum_x x \cdot f(x)$$

- The mean is the weighted average of the possible values of X, the weights being the probabilities where the beam balances. It represents the center of the distribution. It is also called the arithmetic mean.
- If $f(x)$ is the probability mass function representing the loading on a long, thin beam, then $E(X)$ is the fulcrum or point of balance for the beam.
- The mean value may, or may not, be a given value of x.

5-2: Mean and Variance of Discrete Random Variables

5-2.2 Variance Defined

- The variance is the measure of dispersion or scatter in the possible values for X.
- It is the average of the squared deviations from the distribution mean.

The variance of X, denoted as σ^2 or $V(X)$, is

$$\sigma^2 = V(X) = E(X-\mu)^2 = \sum_x (x-\mu)^2 \cdot f(x) = \sum_x x^2 \cdot f(x) - \mu^2$$

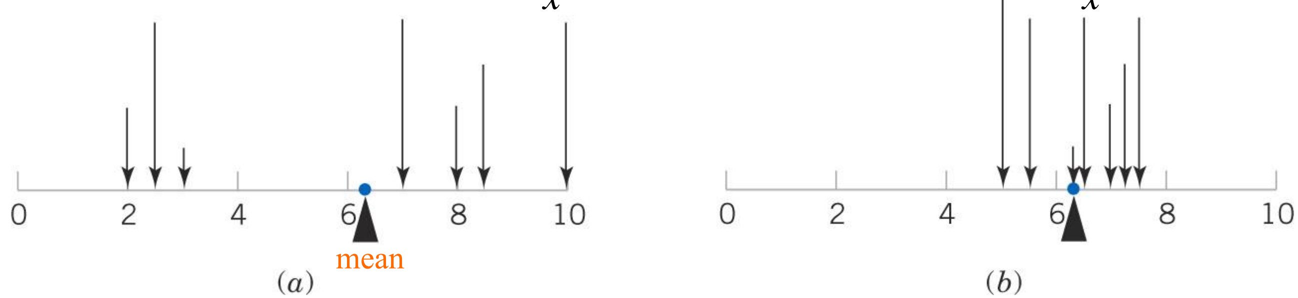


Figure 5-6 The mean is the balance point. Distributions (a) & (b) have equal mean, but (a) has a larger variance.

5-2: Mean and Variance of Discrete Random Variables

5-2.2 Variance Formula Derivations

$$V(X) = \sum_x (x-\mu)^2 f(x) \text{ is the definitional formula}$$

$$= \sum_x \left(x^2 - 2\mu x + \mu^2\right) f(x)$$

$$= \sum_x x^2 f(x) - 2\mu \underbrace{\sum_x x f(x)}_{\mu} + \mu^2 \sum_x f(x)$$

h(x)

$$= \sum_x x^2 f(x) - 2\mu^2 + \mu^2$$

$$= \underbrace{\sum_x x^2 f(x)}_{E(h(x)) = E(x^2)} - \mu^2 \text{ is the computational formula}$$

The computational formula is easier to calculate manually.

Different Distributions Have Same Measures

These measures do not uniquely identify a probability distribution – different distributions could have the same mean & variance.

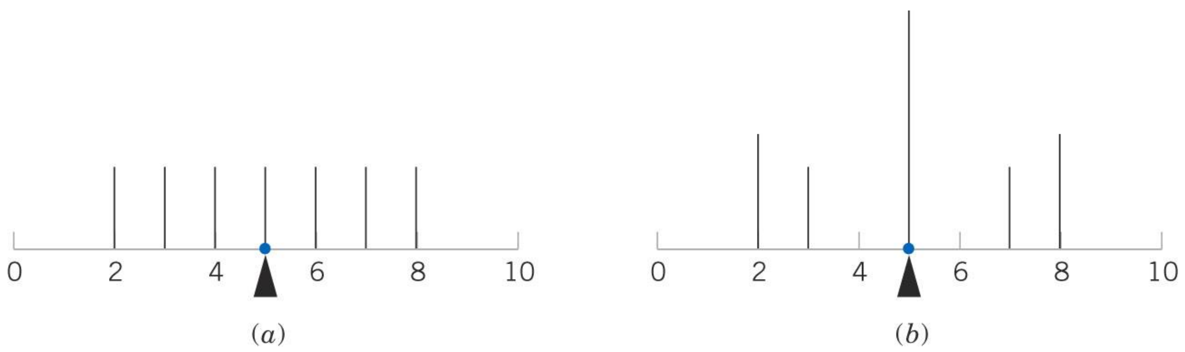


Figure 5-7 These probability distributions have the same mean and variance measures, but are very different in shape.

5-2.3 Functions of Random Variables

If X is a discrete random variable with probability mass function $f(x)$,

$$E\left[h(X)\right]=\sum_{x}h(x)f(x) \qquad (5\text{-}4)$$

If $h(x)=(X-\mu)^2$, then its expectation is the variance of X.

E(2x) = 2E(x) = 2(5) = 10

E(2) = 2

E(x-2)^2 = E(x^2- 2x +4)

= E(x^2) - 2E(x) +E(4)

= sigma xi^2 f(x)

5-2: Mean and Variance of Discrete Random Variables

$$Y = X + c$$

expected value

E(Y) = E(X+c)

= E(X) + E(c)

$$E(Y) = E(X) + c = \mu + c \qquad (5\text{-}5)$$

$$V(Y) = V(X) + 0 = \sigma^2 \qquad (5\text{-}6)$$

variance

$$Y = cX$$

$$E(Y) = E(cX) = cE(X) = c\mu \qquad (5\text{-}7)$$

$$V(Y) = V(cX) = c^2V(X) = c^2\sigma^2 \qquad (5\text{-}8)$$

5-2: Mean and Variance of Discrete Random Variables

5-2.4 Linear Combinations of Independent Random Variables

The mean and variance of the linear function of **independent** random variables are

$$Y = c_0 + c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

$$E(Y) = c_0 + c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n \qquad (5\text{-}9)$$

and

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2 \qquad (5\text{-}10)$$

5-2: Mean and Variance of Discrete Random Variables

Example 5-9:

Two new product designs are to be compared on the basis of revenue potential. Marketing feels that the revenue from design A can be predicted quite accurately to be \$3 million. The revenue potential of design B is more difficult to assess. Marketing concludes that there is a probability of 0.3 that the revenue from design B will be \$7 million, but there is a 0.7 probability that the revenue will be only \$2 million. Which design would you choose?

Solution. Let X denote the revenue from design A. Because there is no uncertainty in the revenue from design A, we can model the distribution of the random variable X as \$3 million with probability one. Therefore, $E(X) = \$3$ million.

Let Y denote the revenue from design B. The expected value of Y in millions of dollars is

y1 = 7f(y1) = 0.3

y2 = 2f(y2) = 0.7

$$E(Y) = \$7(0.3) + \$2(0.7) = \$3.5$$

Because $E(Y)$ exceeds $E(X)$, we might choose design B. However, the variability of the result from design B is larger. That is,

$$\sigma^2 = (7 - 3.5)^2(0.3) + (2 - 3.5)^2(0.7) = 5.25 \text{ millions of dollars squared}$$

$$\sigma = \sqrt{5.25} = 2.29 \text{ millions of dollars}$$

5-2: Mean and Variance of Discrete Random Variables

Example 5-10: Messages

The number of messages sent per hour over a computer network has the following distribution. Find the mean & standard deviation of the number of messages sent per hour.

x	$f(x)$	$x*f(x)$	$x^2*f(x)$
10	0.08	0.80	8
11	0.15	1.65	18.15
12	0.30	3.60	43.2
13	0.20	2.60	33.8
14	0.20	2.80	39.2
15	0.07	1.05	15.75
	1.00	12.50	158.10
		$=E(X)$	$=E(X^2)$

Mean = 12.5 = 10(0.08) + 11(0.15) + ..+ 15(0.07)

Variance = 158.10^2 - 12.5^2 = 1.85

Standard deviation = 1.36

Note that : E(X^2) != [E(X)]^2

V(X) = E(X mx)^2

= E(x^2) - [E(X)]^2

= 10^2(0.08) …..

or V(X) = (10-12.5)^2(0.08) + (11-12.5)^2(0.15) + …+ (15-12.5)^2(0.07)

5-2: Mean and Variance of Discrete Random Variables

Example 5-11: Digital Channel

In Example 5-6, *X* is the number of bits in error in the next four bits transmitted. What is the expected value of the square of the number of bits in error?

x	$f(x)$	$x^2 * f(x)$
0	0.6561	0.0000
1	0.2916	0.2916
2	0.0486	0.1944
3	0.0036	0.0324
4	0.0001	0.0016
	1.0000	0.5200
		$= E(x^2)$

5-3: Important Discrete Distributions

5-3.1 Discrete Uniform Distribution

- Simplest discrete distribution.
- The random variable X assumes only a finite number of values, each with equal probability.
- A random variable X has a discrete uniform distribution if each of the n values in its range, say $x_1, x_2, \ldots, x_n$, has equal probability.

$$f(x_i) = 1/n \qquad (5\text{-}11)$$

5-3: Important Discrete Distributions

5-3.1 Discrete Uniform Distribution

Example 5-12: Discrete Uniform Random Variable

The first digit of a part's serial number is equally likely to be the digits 0 through 9. If one part is selected from a large batch & X is the 1st digit of the serial number, then X has a discrete uniform distribution as shown.

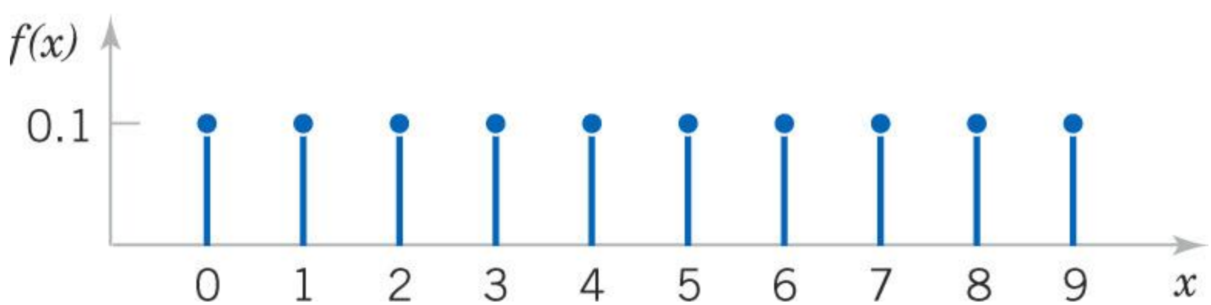


Figure 5-7 Probability mass function, $f(x) = 1/10$ for $x = 0, 1, 2, \ldots, 9$

5-3: Important Discrete Distributions

5-3.1 Discrete Uniform Distribution

- Let X be a discrete uniform random variable from a to b for $a < b$. There are b – (a-1) values in the inclusive interval. Therefore:

$$f(x) = 1/(b-a+1)$$

- Its measures are:

(b + a) / 2

$$\mu = E(x) = \cancel{1/(b-a)}$$

$$\sigma^2 = V(x) = [(b-a+1)^2-1]/12 \qquad (5\text{-}12)$$

Note that the mean is the midpoint of a & b.

5-3: Important Discrete Distributions

5-3.1 Discrete Uniform Distribution

Example 5-13: Number of Voice Lines

Per Example 5-1, let the random variable X denote the number of the 48 voice lines that are in use at a particular time. Assume that X is a discrete uniform random variable with a range of 0 to 48. Find $E(X)$ & $SD(X)$.

Answer:

$E(X) = (48 + 0) / 2 = 24$

5-3: Important Discrete Distributions

5-3.1 Discrete Uniform Distribution

Example 5-14: Proportion of of Voice Lines

Let the random variable Y denote the proportion of the 48 voice line that are in use at a particular time & X as defined in the prior example. Then $Y = X/48$ is a proportion. Find $E(Y)$ & $V(Y)$.

Answer:

E(Y) = E(X)/48 = 24/48 = 0.5

V(Y) = V(X)/48^2 = 14.142^2/48^2

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution

- A trial with only two possible outcomes is used so frequently as a building block of a random experiment that it is called a **Bernoulli trial**.

result of later trial not depend on previous trial

- It is usually assumed that the trials that constitute the random experiment are **independent**. This implies that the outcome from one trial has no effect on the outcome to be obtained from any other trial.

- Furthermore, it is often reasonable to assume that the **probability of a success on each trial is constant**.

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution

- Consider the following random experiments and random variables.
  - Flip a coin 10 times. Let X = the number of heads obtained.
  - Of all bits transmitted through a digital transmission channel, 10% are received in error. Let X = the number of bits in error in the next 4 bits transmitted.

Do they meet the following criteria:

1. Does the experiment consist of **Bernoulli trials**?
2. **Are the trials that constitute the random experiment are independent?**
3. **Is probability of a success on each trial is constant?**

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution

A random experiment consisting of n repeated trials such that

1. the trials are independent,
2. each trial results in only two possible outcomes, labeled as *success* and *failure*, and
3. the probability of a success on each trial, denoted as p, remains constant

is called a *binomial experiment*.

The random variable X that equals the number of trials that result in a *success* has a **binomial distribution** with parameters p and n where $0 \le p \le 1$ and $n = \{1, 2, 3, \ldots\}$. The pmf of X is

total number of experiment

toss 10 coins

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n \qquad (5\text{-}13)$$

p(h=2) = ?

f(2) = C(10,2) (0.5)^2 (1.05)^(10-2)

$$f(x) = C_x^n p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, \ldots n$$

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution

shape depends on n & p

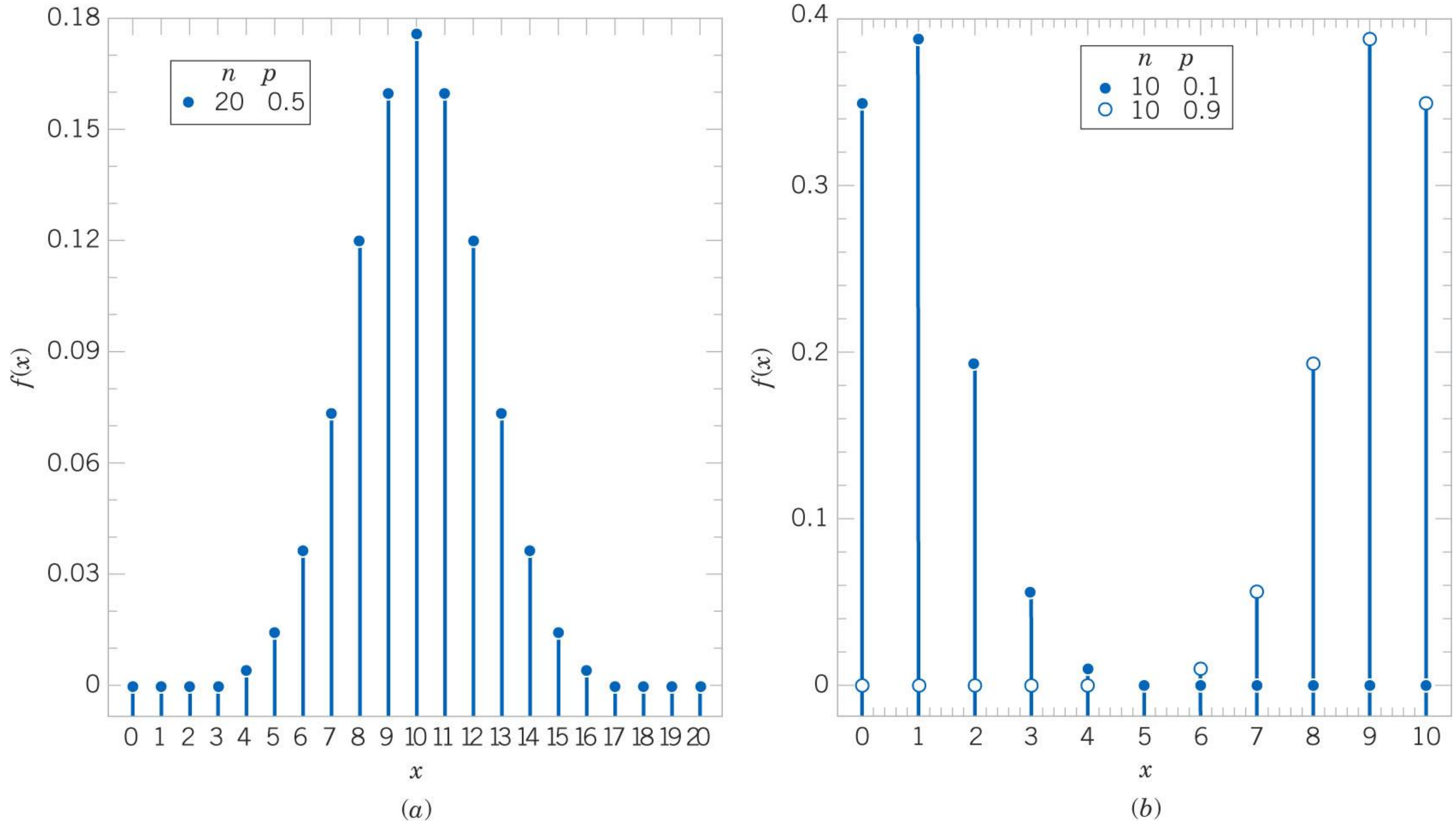


Figure 5-7: Binomial Distributions for selected values of *n* and *p*. Distribution (a) is symmetrical, while distributions (b) are skewed. The skew is right if *p* is small.

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution

Examples of Binomial Random Variables

1. Flip a coin 10 times. X = # heads obtained.
2. A worn tool produces 1% defective parts. X = # defective parts in the next 25 parts produced.
3. A multiple-choice test contains 10 questions, each with 4 choices, and you guess. X = # of correct answers.
4. Of the next 20 births, let X = # females.

These are binomial experiments having the following characteristics:

1. Fixed number of trials (n).
2. Each trial is termed a success or failure. X is the # of successes.
3. The probability of success in each trial is constant (p).
4. The outcomes of successive trials are independent.

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution

Example 5-15: Digital Channel

The chance that a bit transmitted through a digital transmission channel is received in error is 0.1. Assume that the transmission trials are independent. Let X = the number of bits in error in the next 4 bits transmitted. Find $P(X=2)$.

Let E denote a bit in error

Let O denote an OK bit.

Sample space & x listed in table.

6 outcomes where $x = 2$.

Prob of each is $0.1^2 * 0.9^2 = 0.0081$

Prob(X=2) = 6*0.0081 = 0.0486

$$P(X=2) = C_2^4 (0.1)^2 (0.9)^2$$

Outcome	x	Outcome	x
OOOO	0	EOOO	1
OOOE	1	EOOE	2
OOEO	1	EOEO	2
OOEE	2	EOEE	3
OEOO	1	EEOO	2
OEOE	2	EEOE	3
OEEO	2	EEEO	3
OEEE	3	EEEE	4

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution

Example 5-16: Organic Pollution

Each sample of water has a 10% chance of containing ~~a~~ non particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant. Find the probability that, in the next 18 samples, exactly 2 contain the pollutant.

Answer: Let X denote the number of samples that contain the pollutant in the next 18 samples analyzed. Then X is a binomial random variable with $p = 0.1$ and $n = 18$

P(x=2) = c(18,2) (0.1)^2 (0.9)^16 = 153(0.1)^2(0.9)^16 = 0.2835

0.2835	=BINOMDIST(2,18,0.1,FALSE)

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution

Example 5-17: Organic Pollution (Cont.)

Determine the probability that at least 4 samples contain the pollutant.

Answer:

P(x>=4) = zigma(x=4 to 18) C(18,x) (0.1)^x (0.9)^(18-x)

= 1- P(x<4)

= 1 - zigma(x=0 to 3) C(18,x) (0.1)^x (0.9)^18-x

= 0.098

0.0982	=1 - BINOMDIST(3,18,0.1,TRUE)

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution

Table II Cumulative Binomial Probabilities $P(X \leq x)$

		P										
n	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
1	0	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000	0.0500	0.0100
2	0	0.8100	0.6400	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100	0.0025	0.0001
	1	0.9900	0.9600	0.9100	0.8400	0.7500	0.6400	0.5100	0.3600	0.1900	0.0975	0.0199
3	0	0.7290	0.5120	0.3430	0.2160	0.1250	0.0640	0.0270	0.0080	0.0010	0.0001	0.0000
	1	0.9720	0.8960	0.7840	0.6480	0.5000	0.3520	0.2160	0.1040	0.0280	0.0073	0.0003
	2	0.9990	0.9920	0.9730	0.9360	0.8750	0.7840	0.6570	0.4880	0.2710	0.1426	0.0297
4	0	0.6561	0.4096	0.2401	0.1296	0.0625	0.0256	0.0081	0.0016	0.0001	0.0000	0.0000
	1	0.9477	0.8192	0.6517	0.4752	0.3125	0.1792	0.0837	0.0272	0.0037	0.0005	0.0000
	2	0.9963	0.9728	0.9163	0.8208	0.6875	0.5248	0.3483	0.1808	0.0523	0.0140	0.0006
	3	0.9999	0.9984	0.9919	0.9744	0.9375	0.8704	0.7599	0.5904	0.3439	0.1855	0.0394
5	0	0.5905	0.3277	0.1681	0.0778	0.0313	0.0102	0.0024	0.0003	0.0000	0.0000	0.0000
	1	0.9185	0.7373	0.5282	0.3370	0.1875	0.0870	0.0308	0.0067	0.0005	0.0000	0.0000
	2	0.9914	0.9421	0.8369	0.6826	0.5000	0.3174	0.1631	0.0579	0.0086	0.0012	0.0000
	3	0.9995	0.9933	0.9692	0.9130	0.8125	0.6630	0.4718	0.2627	0.0815	0.0226	0.0010
	4	1.0000	0.9997	0.9976	0.9898	0.6988	0.9222	0.8319	0.6723	0.4095	0.2262	0.0490

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution

20	0	0.1216	0.0115	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.3917	0.0692	0.0076	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.6769	0.2061	0.0355	0.0036	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.8670	0.4114	0.1071	0.0160	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.9568	0.6296	0.2375	0.0510	0.0059	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.9887	0.8042	0.4164	0.1256	0.0207	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.9976	0.9133	0.6080	0.2500	0.0577	0.0065	0.0003	0.0000	0.0000	0.0000	0.0000
	7	0.9996	0.9679	0.7723	0.4159	0.1316	0.0210	0.0013	0.0000	0.0000	0.0000	0.0000
	8	0.9999	0.9900	0.8867	0.5956	0.2517	0.0565	0.0051	0.0001	0.0000	0.0000	0.0000
	9	1.0000	0.9974	0.9520	0.7553	0.4119	0.1275	0.0171	0.0006	0.0000	0.0000	0.0000
	10	1.0000	0.9994	0.9829	0.8725	0.5881	0.2447	0.0480	0.0026	0.0000	0.0000	0.0000
	11	1.0000	0.9999	0.9949	0.9435	0.7483	0.4044	0.1133	0.0100	0.0001	0.0000	0.0000
	12	1.0000	1.0000	0.9987	0.9790	0.8684	0.5841	0.2277	0.0321	0.0004	0.0000	0.0000
	13	1.0000	1.0000	0.9997	0.9935	0.9423	0.7500	0.3920	0.0867	0.0024	0.0000	0.0000
	14	1.0000	1.0000	1.0000	0.9984	0.9793	0.8744	0.5836	0.1958	0.0113	0.0003	0.0000
	15	1.0000	1.0000	1.0000	0.9997	0.9941	0.9490	0.7625	0.3704	0.0432	0.0026	0.0000
	16	1.0000	1.0000	1.0000	1.0000	0.9987	0.9840	0.8929	0.5886	0.1330	0.0159	0.0000
	17	1.0000	1.0000	1.0000	1.0000	0.9998	0.9964	0.9645	0.7939	0.3231	0.0755	0.0010
	18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9924	0.9308	0.6083	0.2642	0.0169
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9885	0.8784	0.6415	0.1821

5-3.2 Binomial Distribution

		p										
	c	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95
n = 18	0	0.397	0.150	0.018	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	1	0.774	0.450	0.099	0.014	0.001	0.000	0.000	0.000	0.000	0.000	0.000
	2	0.942	0.734	0.271	0.060	0.008	0.001	0.000	0.000	0.000	0.000	0.000
	3	0.989	$\boxed{0.902}$	0.501	0.165	0.033	0.004	0.000	0.000	0.000	0.000	0.000
	4	0.998	0.972	0.716	0.333	0.094	0.015	0.001	0.000	0.000	0.000	0.000
	5	1.000	0.994	0.867	0.534	0.209	0.048	0.006	0.000	0.000	0.000	0.000
	6	1.000	0.999	0.949	0.722	0.374	0.119	0.020	0.001	0.000	0.000	0.000
	7	1.000	1.000	0.984	0.859	0.563	0.240	0.058	0.006	0.000	0.000	0.000
	8	1.000	1.000	0.996	0.940	0.737	0.407	0.135	0.021	0.001	0.000	0.000
	9	1.000	1.000	0.999	0.979	0.865	0.593	0.263	0.060	0.004	0.000	0.000

5-3.2 Binomial Distribution

Example 5-18: Organic Pollution (Cont.)

Now determine the probability that $3 \le X \le 7$.

Answer:

P(3 <= X <= 7) = sigma(x=3 to 7) C(18,x)(0.1)^x(0.9)(18-x)
= P(X <= 7) - P(X <= 2)
= 0.265

Appendix A, Table II (pg. 705) is a cumulative binomial table for selected values of *p* and *n*.

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution – Mean and Variance

If X is a binomial random variable with parameters p and n,

$$\mu = E(X) = np \quad \text{and} \quad \sigma^2 = V(X) = np(1-p) \tag{5-14}$$

5-3: Important Discrete Distributions

5-3.2 Binomial Distribution – Mean and Variance

Example 5-19: Bit Transmission Error: Binomial Mean and Variance

For the number of transmitted bit received in error in Example 5-15, n = 4 and p = 0.1. Find the mean and variance of the binomial random variable.

Answer:

μ = E(X) = np = 4 * 0.1 = 0.4
σ^2 = V(X) = np(1 - p) = 4 * 0.1 * 0.9 = 3.6
σ = SD(X) = 1.9

5-3: Important Discrete Distributions

Example 5-20: Bit Transmission Error: Geometric Distn.

The probability that a bit, sent through a digital transmission channel, is received in error is 0.1. Assume that the transmissions are independent. Let X denote the number of bits transmitted until the 1st error.

$P(X=5)$ is the probability that the 1st four bits are transmitted correctly and the 5th bit is in error.

$P(X=5) = P(OOOOE) = 0.9^4 0.1 = 0.0656.$

x is the total number of bits sent.

This illustrates the geometric distribution.

5-3: Important Discrete Distributions

don't fix the number of experiment , keep running until find the 1st occur

5-3.3 Geometric Distribution

- Similar to the binomial distribution – a series of Bernoulli trials with fixed parameter p.
- Binomial distribution has:
  - Fixed number of trials. n=10
  - Random number of successes. P(X=1) P(X=5)
- Geometric distribution has reversed roles:
  - Random number of trials. P(X=1) = P(obtain it at first trial) P(X=5) = P(obtain it at fifth trial)
  - Fixed number of successes, in this case 1.
- $f(x) = p(1-p)^{x-1}$ where: (5-15)
  - $x = 1, 2, \dots$ □, the number of failures until the 1[st] success.
  - $0 < p < 1$, the probability of success.

5-3: Important Discrete Distributions

5-3.3 Geometric Distribution

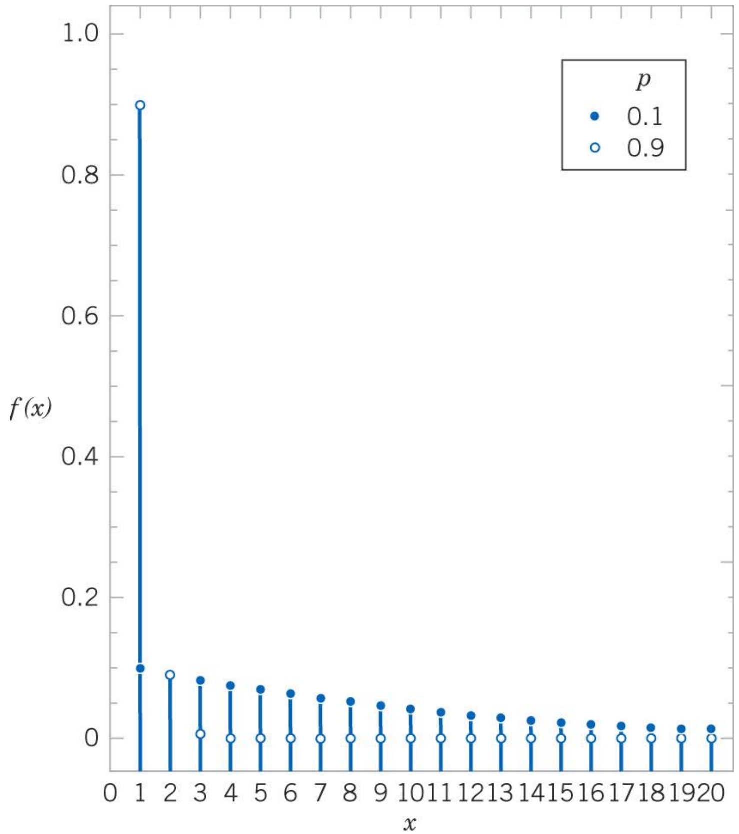


Figure 5-8 Geometric distributions for parameter p values of 0.1 and 0.9. The graphs coincide at $x = 2$.

5-3: Important Discrete Distributions

5-3.3 Geometric Distribution

Example 5-21: Particle in Wafer

The probability that a wafer contains a large particle of contamination is 0.01. Assume that the wafers are independent. What is the probability that exactly 125 wafers need to be analyzed before a particle is detected?

Answer:

Let X denote the number of samples analyzed until a large particle is detected.
Then X is a geometric random variable with parameter p = 0.01

$P(X = 125) = (0.99)^{124}(0.01) = 0.00288$

5-3.3 Geometric Distribution – Mean and Variance

- If X is a geometric random variable with parameter p,

$$\mu = E(X) = \frac{1}{p} \quad \text{and} \quad \sigma^2 = V(X) = \frac{(1-p)}{p^2} \qquad (5\text{-}16)$$

5-3.3 Geometric Distribution

Example 5-22: Particle in Wafer (Cont.)

Consider the transmission of bits in Exercise 5-20. Here, $p = 0.1$. Find the mean and standard deviation.

Answer:

Mean $= \mu = E(X) = 1 / p = 1 / 0.1 = 10$

Variance $= \sigma^2 = V(X) = (1-p) / p^2 = 0.9 / 0.01 = 90$

Standard deviation = sqrt(99) = 9.487

5-3: Important Discrete Distributions

5-3.3 Geometric Distribution

Lack of memory property

- For a geometric random variable, the trials are independent. Thus the count of the number of trials until the next success can be started at any trial without changing the probability.
- The probability that the next bit error will occur on bit 106, given that 100 bits have been transmitted, is the same as it was for bit 006.
- Implies that the system does not wear out!

4-5.4 Exponential Distribution

- Lack of memory property

For an exponential random variable X,

$$P(X < t_1 + t_2 | X > t_1) = P(X < t_2) \tag{4-16}$$

- Areas $A+B+C+D=1$
- $A = P(X < t_2)$
- $A+B+C = P(X<t_1+t_2)$
- $C = P(X<t_1+t_2 \cap X>t_1)$
- $C+D = P(X>t_1)$
- $C/(C+D) = P(X<t_1+t_2 | X>t_1)$
- $A = C/(C+D)$

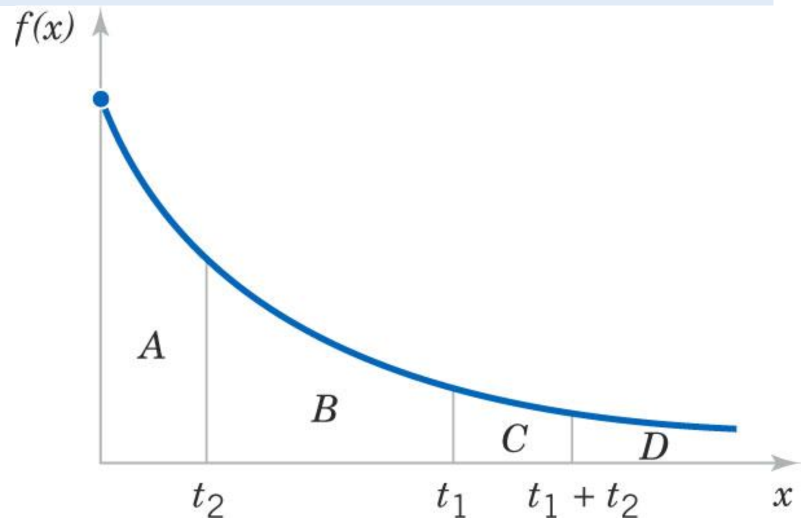
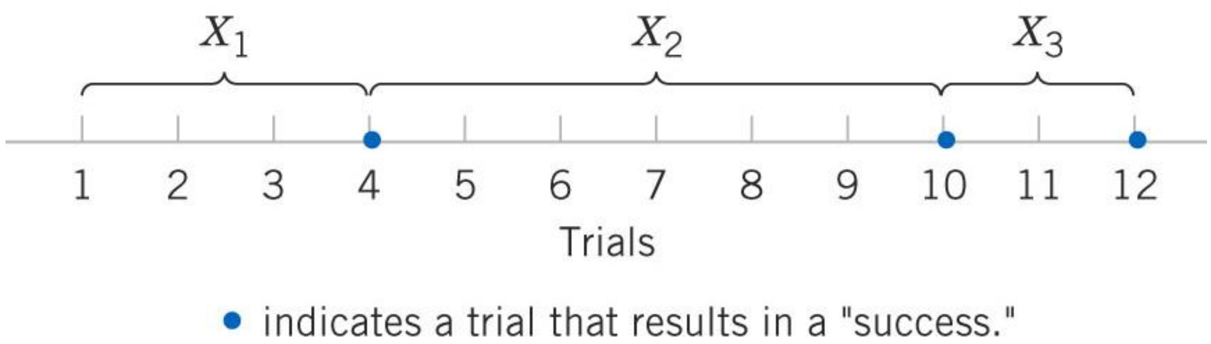


Figure 4-27 Lack of memory property of an exponential distribution.

5-3: Important Discrete Distributions

5-3.3 Geometric Distribution

- Lack of memory property



•Let X_1 denote the number of trials to the 1st success.

•Let X_2 denote the number of trials to the 2nd success, since the 1st success.

•Let X_3 denote the number of trials to the 3rd success, since the 2nd success.

$$P\left(X = t_1 + t_2 \middle| X > t_1\right) = P\left(X = t_2\right) \qquad (5-17)$$

$$P\left(X \geq t_1 + t_2 \middle| X \geq t_1\right) = P\left(X \geq t_2\right) \qquad (5-18)$$

5-3.3 Geometric Distribution

Example 5-22: Bit Transmission Error: Lack of Memory

In Example 5-20, the probability that a bit is transmitted in error is 0.1. Suppose 50 bits have been transmitted, What is the probability that the next error occur at the 60th bit?

Answer:

$$P(X=60 \mid X>50)=\frac{P(X=60)}{P(X>50)}=\frac{f(60)}{1-F(50)}=\frac{0.0002}{1-0.9948}=0.0387$$

$$P(X=10)=f(10)=0.0387$$

5-3.3 Geometric Distribution

Example 5-23: Bit Transmission Error: Lack of Memory

In Example 5-20, the probability that a bit is transmitted in error is 0.1. Suppose 50 bits have been transmitted, What is the probability that the next error occur after the 60th bit?

Answer:

$$P\left(X \geq 61 \middle| X \geq 51\right) = \frac{P\left(X \geq 61\right)}{P\left(X \geq 51\right)} = \frac{1-F\left(61\right)}{1-F\left(51\right)} = \frac{0.0016}{0.0046} = 0.3486$$

$$P\left(X \geq 10\right) = 1 - F\left(10\right) = 1 - 0.6513 = 0.3486.$$

5-3.4 Negative Binomial Distribution

Example 5-24: Bit Transmission Error (cont.)

The probability that a bit, sent through a digital transmission channel, is received in error is 0.1. Assume that the transmissions are independent. Let X denote the number of bits transmitted until the 4th error.

P(X=10) is the probability that 3 errors occur over the first 9 trials, then the 4th success occurs on the 10th trial.

$$\text{3 errors occur over the first 9 trials} = C_3^9 p^3 (1-p)^6$$

$$\text{4th error occurs on the 10th trial} = C_3^9 p^4 (1-p)^6$$

5-3: Important Discrete Distributions

5-3.4 Negative Binomial Distribution

- In a series of independent trials with constant probability of success, let the random variable X denote the number of trials until r successes occur. Then X is a negative binomial random variable with parameters $0 < p < 1$ and r = 1, 2, 3, ...
- The probability mass function is:

$$f(x) = C_{r-1}^{x-1} p^r (1-p)^{x-r} \quad \text{for } x = r, r+1, r+2 \ldots \qquad (5\text{-}19)$$

- From the prior example for $f(X=10|r=4)$:
  - x-1 = 9
  - r-1 = 3

5-3.4 Negative Binomial Distribution

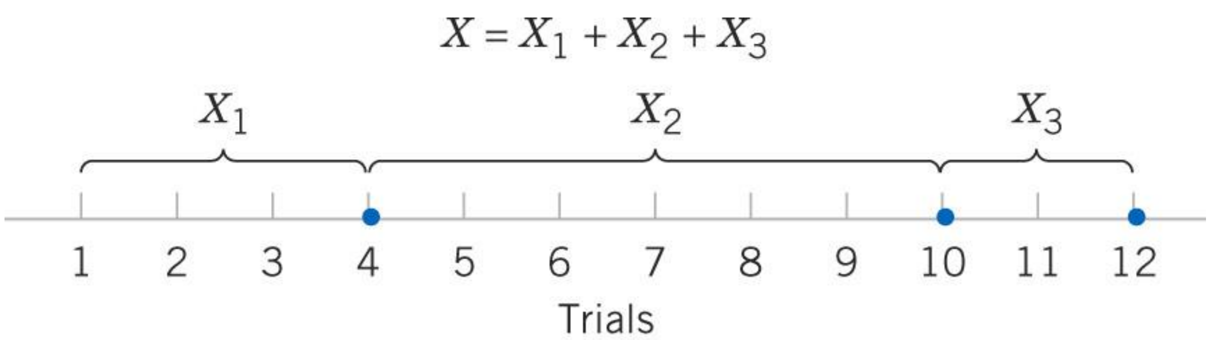


Figure 5-9 • indicates a trial that results in a "success."

- Let X_1 denote the number of trials to the 1st success.
- Let X_2 denote the number of trials to the 2nd success, since the 1st success.
- Let X_3 denote the number of trials to the 3rd success, since the 2nd success.
- Let the X_i be geometric random variables – independent, so without memory.
- Then $X = X_1 + X_2 + X_3$
- Therefore, X is a negative binomial random variable, a sum of three geometric rv's.

5-3: Important Discrete Distributions

5-3.4 Negative Binomial Distribution

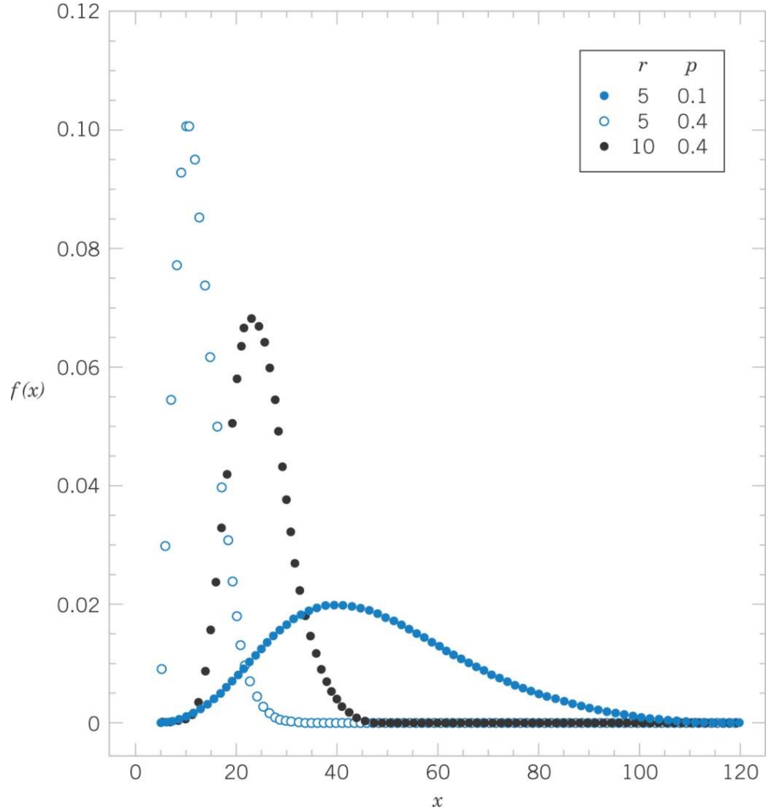


Figure 5-10 Negative binomial distributions for 3 different parameter combinations.

5-3: Important Discrete Distributions

5-3.4 Negative Binomial Distribution – Mean and Variance

- If X is a negative binomial random variable with parameters p and r,

$$\mu = E(X) = \frac{r}{p} \quad \text{and} \quad \sigma^2 = V(X) = \frac{r(1-p)}{p^2} \qquad (5\text{-}20)$$

5-3: Important Discrete Distributions

5-3.4 Negative Binomial Distribution

Example 5-25: Web Server

A Web site contains 3 identical computer servers. Only one is used to operate the site, and the other 2 are spares that can be activated in case the primary system fails. The probability of a failure in the primary computer (or any activated spare) from a request for service is 0.0005. Assume that each request represents an independent trial. What is the mean number of requests until failure of all 3 servers?

5-3.4 Negative Binomial Distribution

Example 5-26: Web Server (cont.)

What is the probability that all 3 servers fail within 5 requests? ($X = 5$)

Answer:

P(X<=5) = P(X = 3) + P(X = 4) + P(X = 5)
= 0.005^3 + C(3,2) * 0.0005^3 * 0.9995 + C(4,2) * 0.00005^3 * 0.99995^2

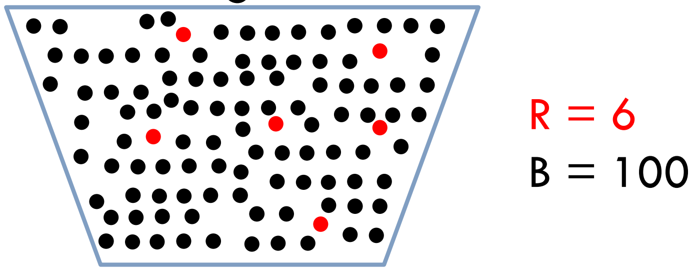
Note that Excel uses a different definition of X; # of failures before the r^{th} success, not # of trials.

What's In A Name?

- Binomial distribution:
  - Fixed number of trials (*n*).
  - Random number of successes (*x*).
- Geometric distribution:
  - Random number of trials (*x*).
  - Single success (*1*).
- Negative binomial distribution:
  - Random number of trials (*x*).
  - Fixed number of successes (*r*).
- Because of the reversed roles, a negative binomial can be considered the opposite or negative of the binomial.

5-3.5 Hypergeometric Distribution

- Applies to sampling without replacement.
- Trials are not independent & a tree diagram used.
- A set of *N* objects contains:
  - *K* objects classified as success
  - *N* - *K* objects classified as failures



- A sample of size *n* objects is selected without replacement from the *N* objects, where:
  - $K \leq N$ *and* $n \leq N$
- Let the random variable *X* denote the number of successes in the sample. Then *X* is a hypergeometric random variable.

$$f(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \quad \text{where } x = \max(0, n+K-N) \text{ to } \min(K, n) \qquad (5\text{-}21)$$

5-3: Important Discrete Distributions

5-3.5 Hypergeometric Distribution

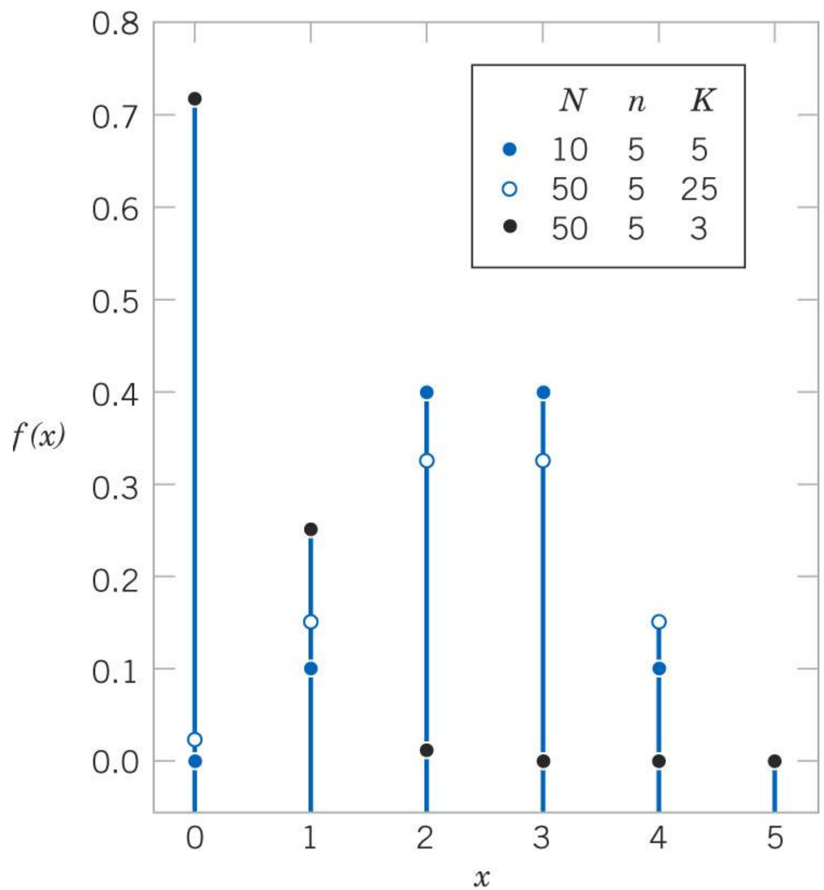


Figure 5-11 Hypergeometric distributions for 3 parameter sets of *N*, *K*, and *n*.

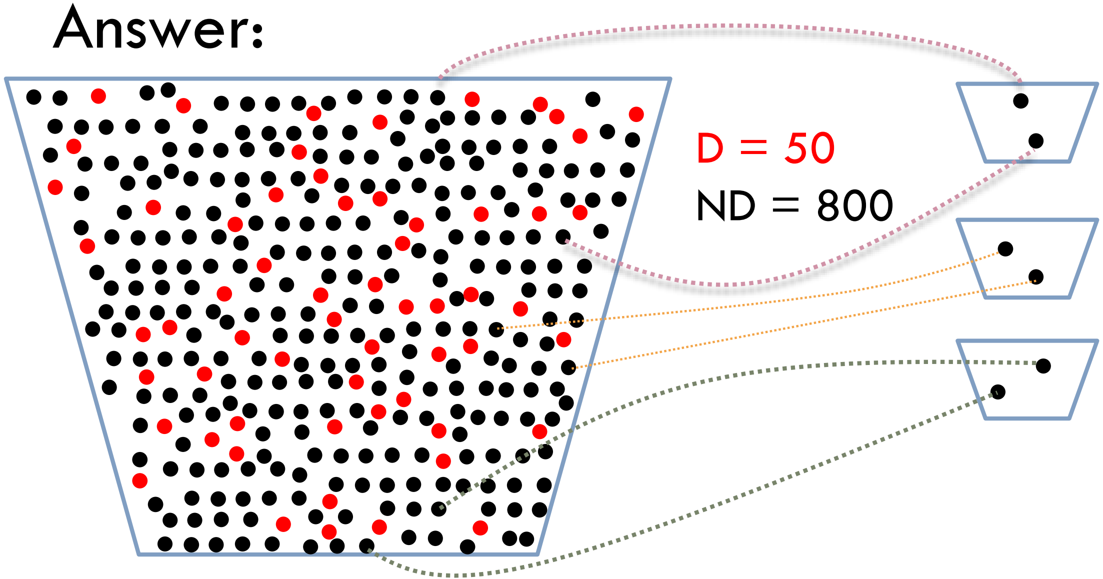
5-3: Important Discrete Distributions

5-3.5 Hypergeometric Distribution

Example 5-27: Sampling without Replacement

From an earlier example, 50 parts are defective on a lot of 850. Two are sampled. Let X denote the number of defectives in the sample. Use the hypergeometric distribution to find the probability distribution.

Answer:



$$P(X=0)=\frac{\binom{50}{0}\binom{800}{2}}{\binom{850}{2}}$$

$$=\frac{319,660}{360,825}=0.88$$

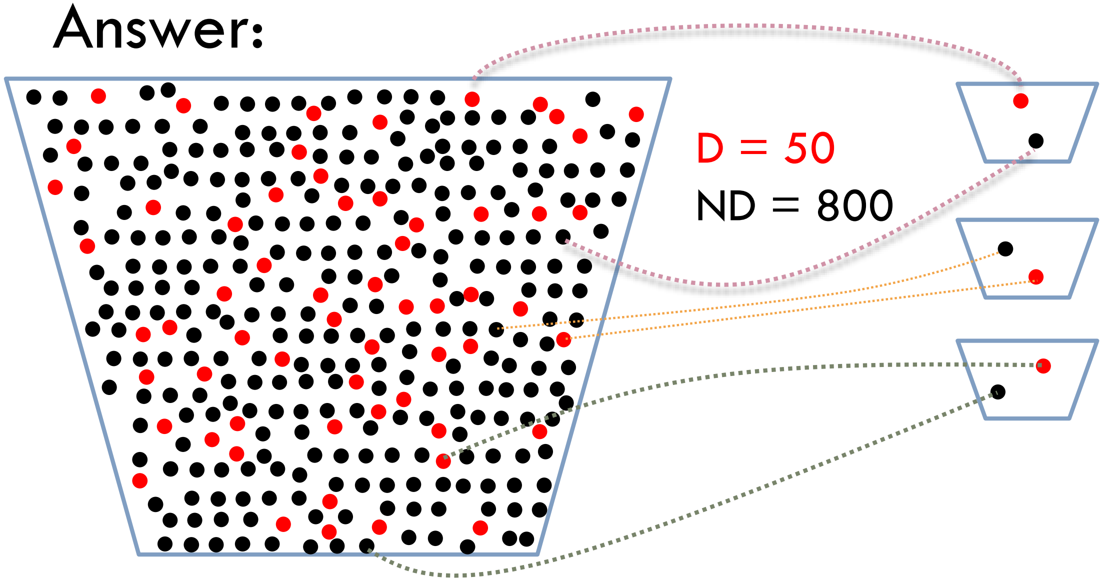
5-3: Important Discrete Distributions

5-3.5 Hypergeometric Distribution

Example 5-27: Sampling without Replacement

From an earlier example, 50 parts are defective on a lot of 850. Two are sampled. Let X denote the number of defectives in the sample. Use the hypergeometric distribution to find the probability distribution.

Answer:



$$P(X=1)=\frac{\binom{50}{1}\binom{800}{1}}{\binom{850}{2}}$$

$$=\frac{40,000}{360,825}=0.111$$

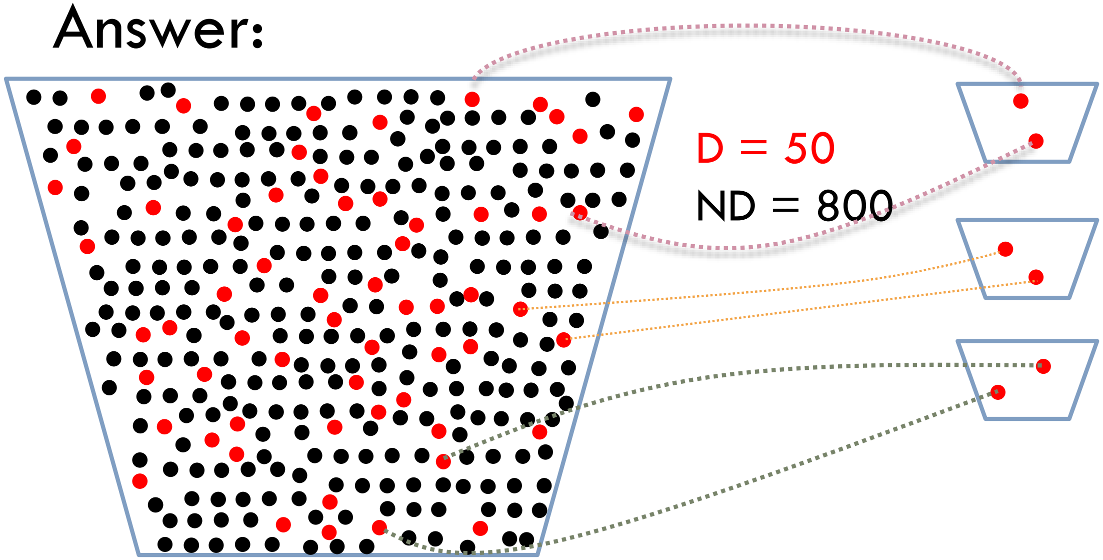
5-3: Important Discrete Distributions

5-3.5 Hypergeometric Distribution

Example 5-27: Sampling without Replacement

From an earlier example, 50 parts are defective on a lot of 850. Two are sampled. Let X denote the number of defectives in the sample. Use the hypergeometric distribution to find the probability distribution.

Answer:



$$P(X=2)=\frac{\binom{50}{2}\binom{800}{0}}{\binom{850}{2}}$$

$$=\frac{1,225}{360,825}=0.003$$

5-3: Important Discrete Distributions

5-3.5 Hypergeometric Distribution

Example 5-28: Parts from Suppliers

A batch of parts contains 100 parts from supplier A and 200 parts from Supplier B. If 4 parts are selected randomly, without replacement, what is the probability that they are all from Supplier A?

Answer: Let X = the number of parts in the sample from Supplier A

P(X = 4) = (100,4)(200,0) / (300,4) = 0.0119

5-3: Important Discrete Distributions

5-3.5 Hypergeometric Distribution

Example 5-28: Parts from Suppliers (cont.)

What is the probability that two or more parts are from Supplier A?

Answer: $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$

$$= \frac{\binom{100}{2}\binom{200}{2}}{\binom{300}{4}} + \frac{\binom{100}{3}\binom{200}{1}}{\binom{300}{4}} + \frac{\binom{100}{4}\binom{200}{0}}{\binom{300}{4}}$$

$$= 0.298 + 0.098 + 0.0119 = 0.408$$

5-3: Important Discrete Distributions

5-3.5 Hypergeometric Distribution

Example 5-28: Parts from Suppliers (cont.)

What is the probability that at least one part is from Supplier A?

Answer:

P(X >= 1) = 1 - P(X = 0) = 1 - (100,0)(200,4) / (300,4)

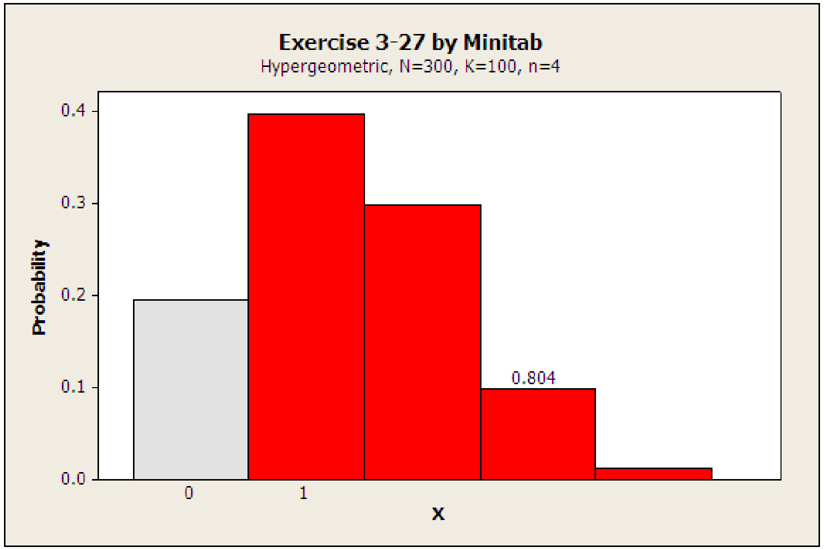


Figure 5-12

5-3.5 Hypergeometric Distribution – Mean and Variance

- If X is a hypergeometric random variable with parameters N, K, and n, then

$$\mu = E(X) = np \quad \text{and} \quad \sigma^2 = V(X) = np(1-p)\left(\frac{N-n}{N-1}\right) \qquad (5\text{-}22)$$

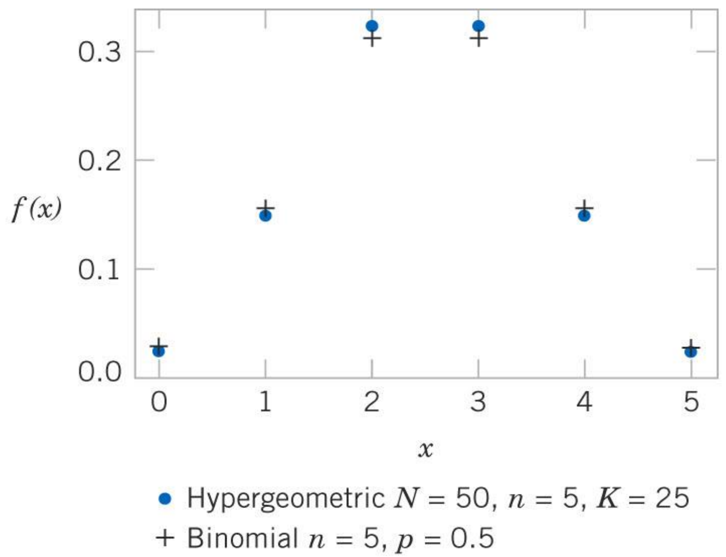
where $p = K/N$

and $\left(\frac{N-n}{N-1}\right)$ is the finite population correction factor.

σ^2 approaches the binomial variance as n/N becomes small.

5-3: Important Discrete Distributions

5-3.5 Hypergeometric Distribution – Mean and Variance



	0	1	2	3	4	5
Hypergeometric probability Binomial probability	0.025 0.031	0.149 0.156	0.326 0.312	0.326 0.312	0.149 0.156	0.025 0.031

Figure 5-13 Comparison of hypergeometric and binomial distributions.

5-3.5 Hypergeometric Distribution

Example 5-29: Customer Sample

A listing of customer accounts at a large corporation contains 1,000 accounts. Of these, 700 have purchased at least one of the company's products in the last 3 months. To evaluate a new product, 50 customers are sampled at random from the listing. What is the probability that more than 45 of the sampled customers have purchased in the last 3 months?

Let X denote the number of customers in the sample who have purchased from the company in the last 3 months. Then X is a hypergeometric random variable with *N* = 1,000, *K* = 700, *n* = 50. This a lengthy problem! ☹

$$P(X > 45) = \sum_{x=46}^{50} \frac{\binom{700}{x}\binom{300}{50-x}}{\binom{1,000}{50}}$$

5-3: Important Discrete Distributions

5-3.5 Hypergeometric Distribution – Binomial Approx.

Example 5-29: Customer Sample (cont.)

Since n/N is small, the binomial will be used to approximate the hypergeometric. Let $p = K/N = 0.7$

700 / 1000

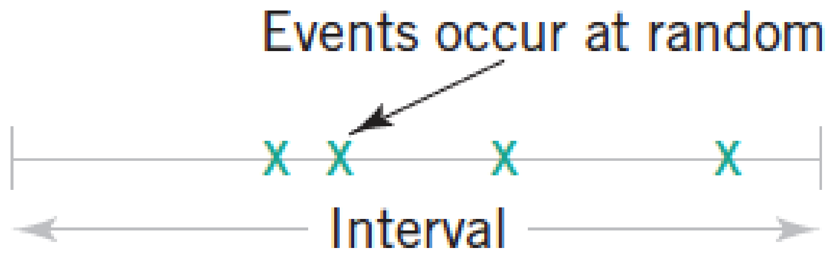
$$P(X > 45) = \sum_{x=46}^{50} \binom{50}{x} 0.7^x (1-0.7)^{50-x} = 0.00017$$

$$P(X > 45) = \sum_{x=46}^{50} \frac{\binom{700}{x}\binom{300}{50-x}}{\binom{1,000}{50}} = 0.000166$$

$$\boxed{\frac{n}{N} < 0.1}$$

5-3.6 Poisson Process

Figure 5-14 In a Poission process, events occur at random in an interval.



In general, consider an interval T of real numbers partitioned into subintervals of small length Δt and assume that as Δt tends to zero,

(1) the probability of more than one event in a subinterval tends to zero,

(2) the probability of one event in a subinterval tends to $\lambda\Delta t/T$,

(3) the event in each subinterval is independent of other subintervals.

A random experiment with these properties is called a **Poisson process.**

5-3: Important Discrete Distributions

5-3.6 Examples of Poisson Process

In general, the Poisson random variable *X* is the number of events (counts) per interval.

1. Particles of contamination per wafer.
2. Flaws per roll of textile.
3. Calls at a telephone exchange per hour.
4. Power outages per year.
5. Atomic particles emitted from a specimen per second.
6. Flaws per unit length of copper wire.

5-3.6 Poisson Distribution

As the number of trials (*n*) in a binomial experiment increases to infinity while the binomial mean (*np*) remains constant, the binomial distribution becomes the Poisson distribution.

$$\text{Let } \lambda = np = E(x), \text{ so } p = \lambda/n$$

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\lim_{n\to\infty} P(X = x) = \lim_{n\to\infty}\left[\binom{n}{x}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}\right] = \frac{e^{-\lambda}\lambda^x}{x!}$$

5-3.6 Poisson Distribution

Example 5-30: Limit of Bit Errors

Consider the transmission of n bits over a digital communication channel. Let the random variable X equal the number of bits in error. When the probability that a bit is in error is constant and the transmissions are independent, X has a binomial distribution. Let p denote the probability that a bit is in error. Then $E(X) = pn$. Now suppose that the number of bits transmitted increases and the probability of an error decreases exactly enough that pn remains equal to a constant—say, λ. That is, n increases and p decreases accordingly, such that $E(X)$ remains constant. Then

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x\, x!}(np)^x(1-p)^n(1-p)^{-x}$$

$P(X = x)$ is written as the product of four terms and with some work it can be shown that the four terms converge to $1/x!$, λ^x, $e^{-\lambda}$, and 1, respectively. Therefore,

$$\lim_{n \to \infty} P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots$$

Also, because the number of bits transmitted tends to infinity, the number of errors can equal any nonnegative integer. Therefore, the possible values for X are the integers from zero to infinity.

5-3.6 Poisson Distribution

- The random variable X that equals the number of events in a Poisson process is a Poisson random variable with parameter λ > 0, and the probability mass function is:

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad \text{for} \quad x = 0, 1, 2, 3, \ldots \infty \qquad (5\text{-}23)$$

5-3: Important Discrete Distributions

5-3.6 Poisson Distribution

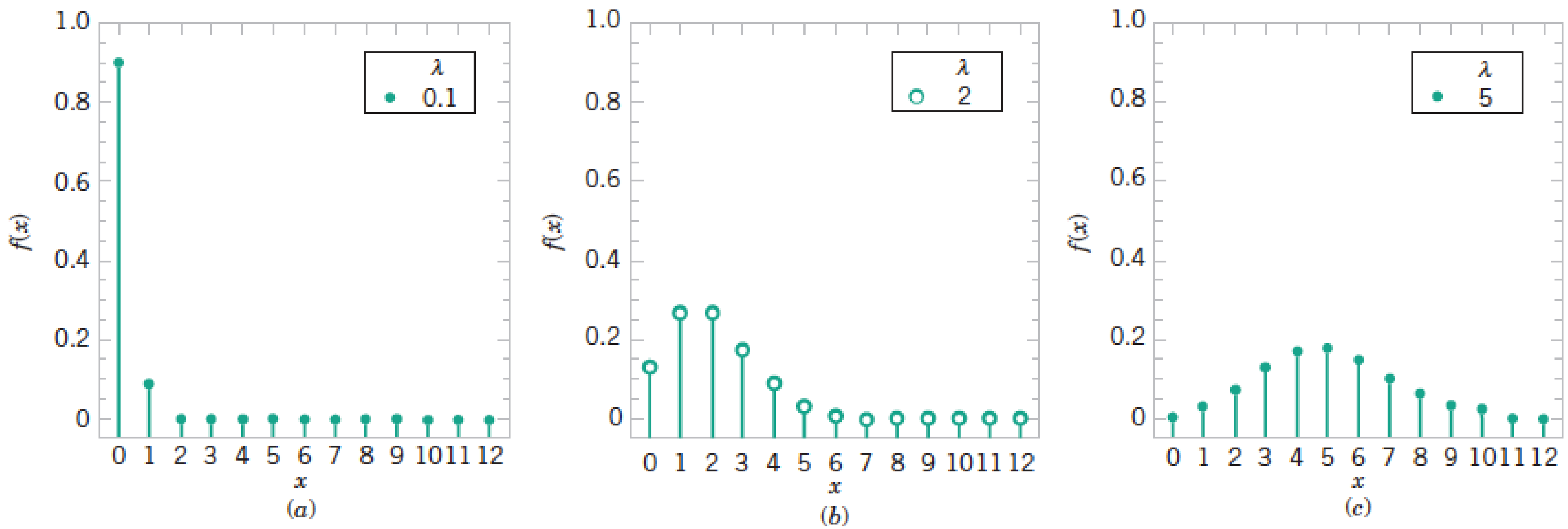


Figure 5-15 Poisson distribution for selected values of the parameter λ.

5-3: Important Discrete Distributions

5-3.6 Poisson Distribution

Example 5-31: Wire Flaws

Flaws occur at random along the length of a thin copper wire. Let X denote the random variable that counts the number of flaws in a length of L mm of wire. Suppose the average number of flaws in L is λ.

Partition L into n subintervals (1 µm) each. If the subinterval is small enough, the probability that more than one flaw occurs is negligible.

Assume that the:

- Flaws occur at random, implying that each subinterval has the same probability of containing a flaw.
- Probability that a subinterval contains a flaw is independent of other subintervals.

X is now binomial. $E(X) = np = \lambda$ and $p = \lambda/n$

As n becomes large, p becomes small and a Poisson process is created.

5-3: Important Discrete Distributions

5-3.6 Poisson Distribution - Poisson Requires Consistent Units

It is important to use consistent units in the calculation of Poisson:

- Probabilities
- Means
- Variances

- Example of unit conversions:
  - Average # of flaws per mm of wire is 3.4.
  - Average # of flaws per 10 mm of wire is 34.
  - Average # of flaws per 20 mm of wire is 68.

5-3.6 Poisson Distribution

Example 5-32: Wire Flaws (cont.)

For the case of the thin copper wire, suppose that the number of flaws follows a Poisson distribution of 2.3 flaws per mm. Let X denote the number of flaws in 1 mm of wire. Find the probability of exactly 2 flaws in 1 mm of wire.

Answer:

$$P(X=2)=\frac{e^{-2.3}\,2.3^2}{2!}=0.265$$

5-3.6 Poisson Distribution

Example 5-33: Wire Flaws (cont.)

Determine the probability of 10 flaws in 5 mm of wire. Now let X denote the number of flaws in 5 mm of wire.

Answer:

$$E(X) = \lambda = 5\text{ mm} \cdot\ 2.3\text{ flaws/mm} = 11.5\text{ flaws/5mm}$$

$$\mathrm{P}(X = 10) = e^{-11.5}\frac{11.5^{10}}{10!} = 0.113$$

5-3.6 Poisson Distribution

Example 5-34: Wire Flaws (cont.)

Determine the probability of at least 1 flaw in 2 mm of wire. Now let X denote the number of flaws in 2 mm of wire. Note that $P(X \geq 1)$ requires □ terms. ☹

Answer:

5-3.6 Poisson Distribution

Example 5-35: Contamination on Optical Disks

Contamination is a problem in the manufacture of optical storage disks. The number of particles of contamination that occur on an optical disk has a Poisson distribution, and the average number of particles per centimeter squared of media surface is 0.1. The area of a disk under study is 100 squared centimeters. Determine the probability that 12 particles occur in the area of a disk under study.

Solution. Let X denote the number of particles in the area of a disk under study. Because the mean number of particles is 0.1 particles per cm^2,

$$\begin{aligned} E(X) &= 100 \text{ cm}^2 \times 0.1 \text{ particles/cm}^2 \\ &= 10 \text{ particles} \end{aligned}$$

Therefore,

$$P(X = 12) = \frac{e^{-10}10^{12}}{12!} = 0.095$$

5-3: Important Discrete Distributions

5-3.6 Poisson Distribution

Example 5-35: Contamination on Optical Disks (cont.)

Determine the probability that zero particles occur in the area of the disk under study.

Solution. Now, $P(X = 0) = e^{-10} = 4.54 \times 10^{-5}$.

Determine the probability that 12 or fewer particles occur in the area of a disk under study.

Solution. This probability is

$$P(X \le 12) = P(X = 0) + P(X = 1) + \cdots + P(X = 12)$$
$$= \sum_{i=0}^{12} \frac{e^{-10} 10^i}{i!}$$

Because this sum is tedious to compute, many computer programs calculate cumulative Poisson probabilities. From Minitab, we obtain $P(X \le 12) = 0.7916$. ■

5-3: Important Discrete Distributions

5-3.6 Poisson Distribution – Mean and Variance

If X is a Poisson random variable with parameter λ, then:

$$\mu = E(X) = \lambda \quad \text{and} \quad \sigma^2 = V(X) = \lambda \qquad (5\text{-}24)$$

The mean and variance of the Poisson model are the same. If the mean and variance of a data set are not about the same, then the Poisson model would not be a good representation of that set.

The derivation of the mean and variance is shown in the text.

5-3: Important Discrete Distributions

5-3.6 Poisson Process – see Exponential Distribution

• The discussion of the Poisson distribution defined a random variable to be the number of flaws along a length of copper wire. The distance between flaws is another random variable that is often of interest.

• Let the random variable X denote the *length* from any starting point on the wire until a flaw is detected.

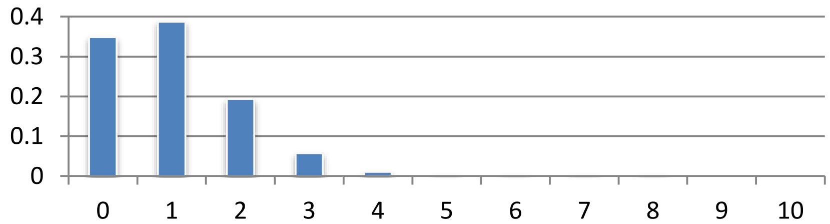
• As you might expect, the distribution of X can be obtained from knowledge of the distribution of the number of flaws. The key to the relationship is the following concept:

> The distance to the first flaw exceeds 3 millimeters if and only if there are no flaws within a length of 3 millimeters—simple, but sufficient for an analysis of the distribution of X.

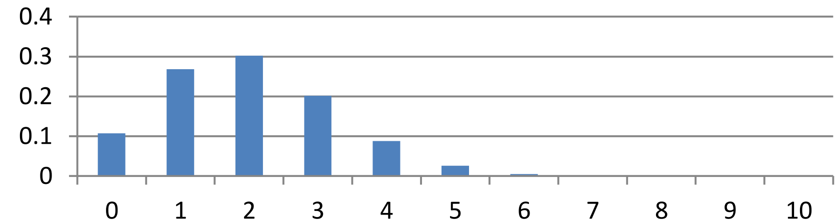
5-4: Normal Approximation to the Binomial, Hypergeometric, and Poisson Distributions

5-4.1 Binomial (n = 10)

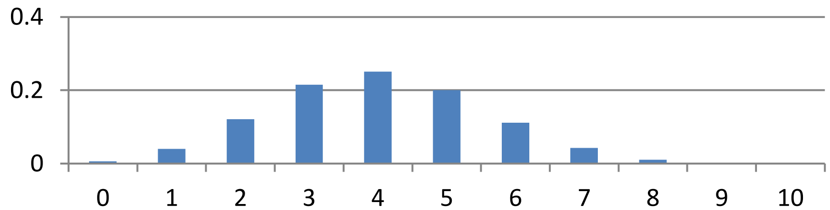
p=0.1, n=10



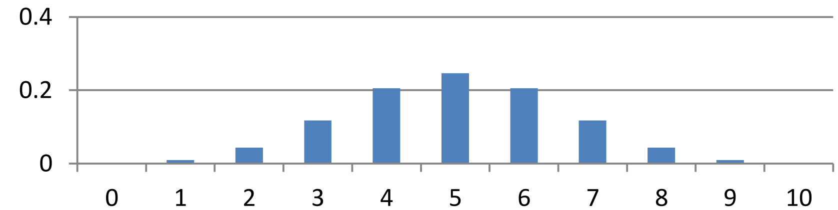
p = 0.2, n = 10



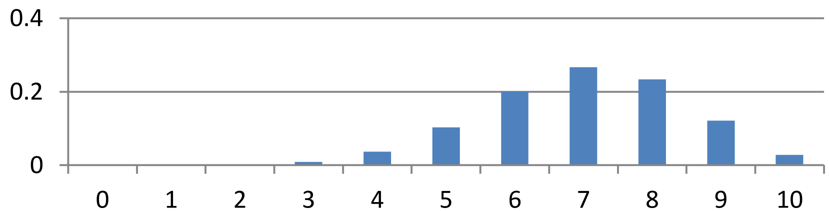
p = 0.4, n = 10



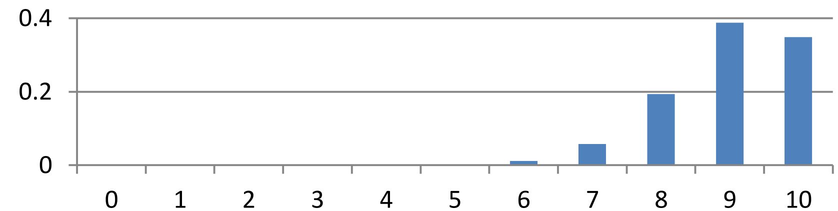
p = 0.5, n = 10



p = 0.7, n = 10



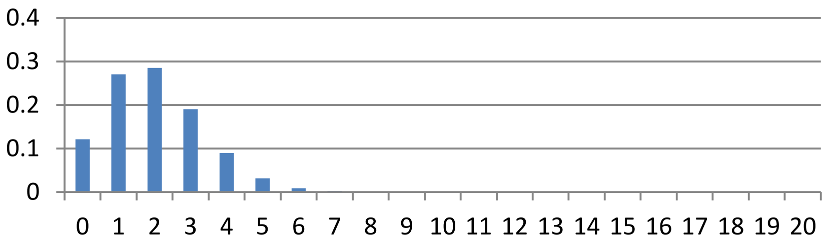
p = 0.9, n = 10



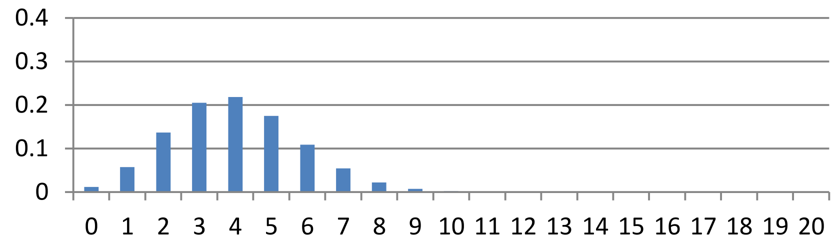
5-4: Normal Approximation to the Binomial, Hypergeometric, and Poisson Distributions

5-4.1 Binomial (n = 20)

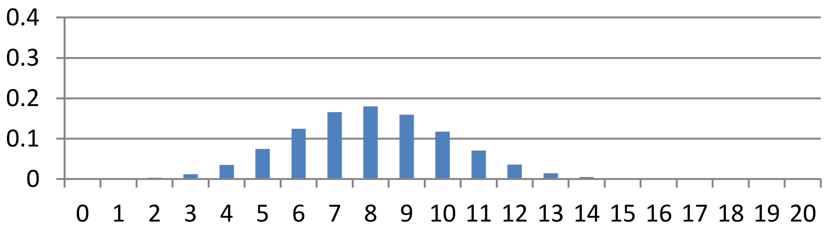
p = 0.1, n = 20



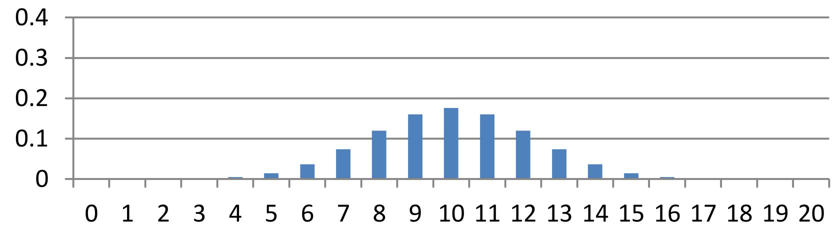
p = 0.2, n = 20



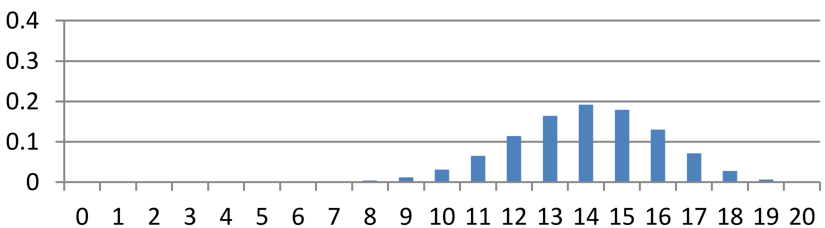
p = 0.4, n = 20



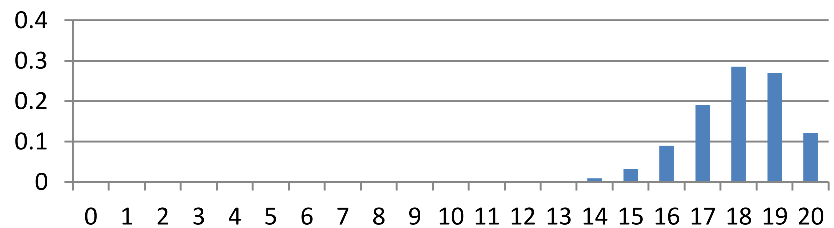
p = 0.5, n = 20



p = 0.7, n = 20



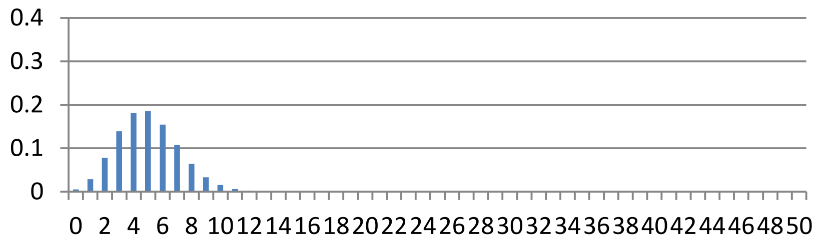
p = 0.9, n = 20



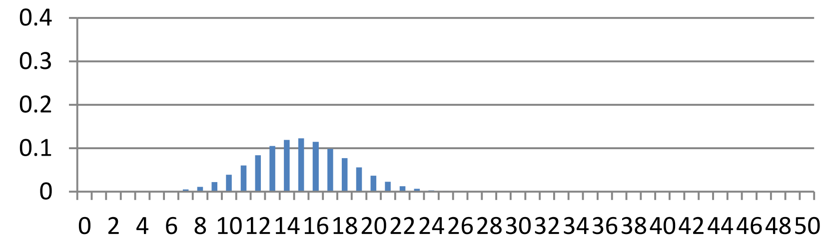
5-4: Normal Approximation to the Binomial, Hypergeometric, and Poisson Distributions

5-4.1 Binomial (n = 50)

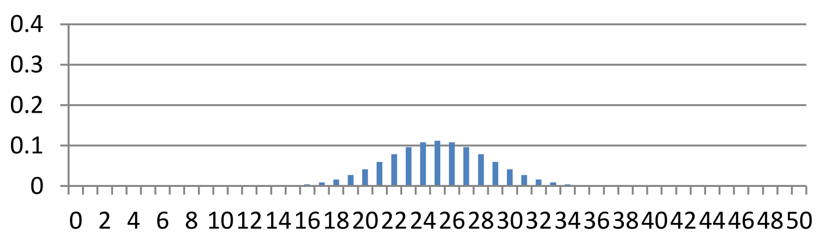
p = 0.1, n = 50



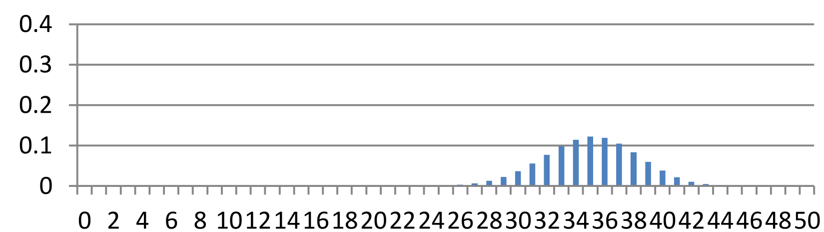
p = 0.3, n = 50



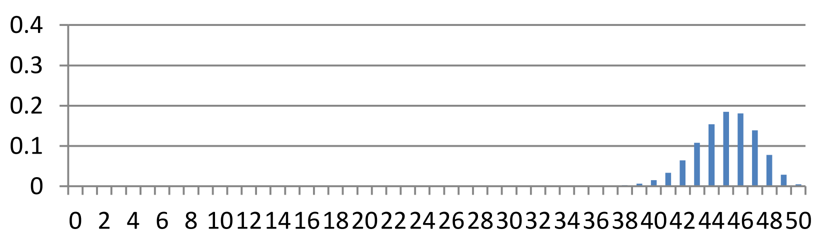
p = 0.5, n = 50



p = 0.7, n = 50



p = 0.9, n = 50



5-4: Normal Approximation to the Binomial, Hypergeometric, and Poisson Distributions

5-4.1 Normal Approximation to the Binomial

If X is a binomial random variable,

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \qquad (5\text{-}25)$$

is approximately a standard normal random variable. Consequently, probabilities computed from Z can be used to approximate probabilities for X.

5-4: Normal Approximation to the Binomial, Hypergeometric, and Poisson Distributions

5-4.1 Normal Approximation to the Binomial

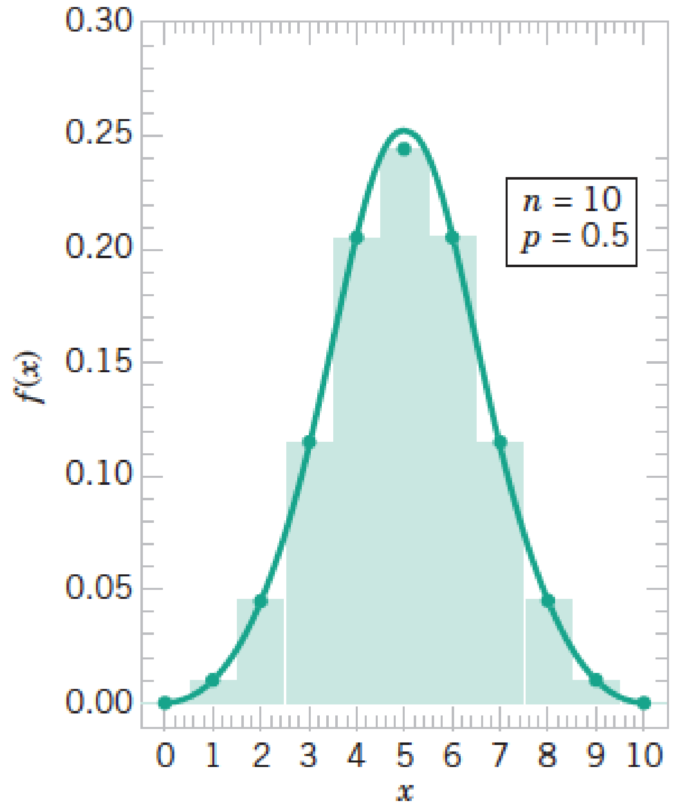


Figure 5-16 Normal approximation to the binomial distribution.

5-4.1 Normal Approximation to the Binomial

If X is a binomial random variable with parameters n and p,

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \tag{5-25}$$

is approximately a standard normal random variable. To approximate a binomial probability with a normal distribution, a continuity correction is applied as follows:

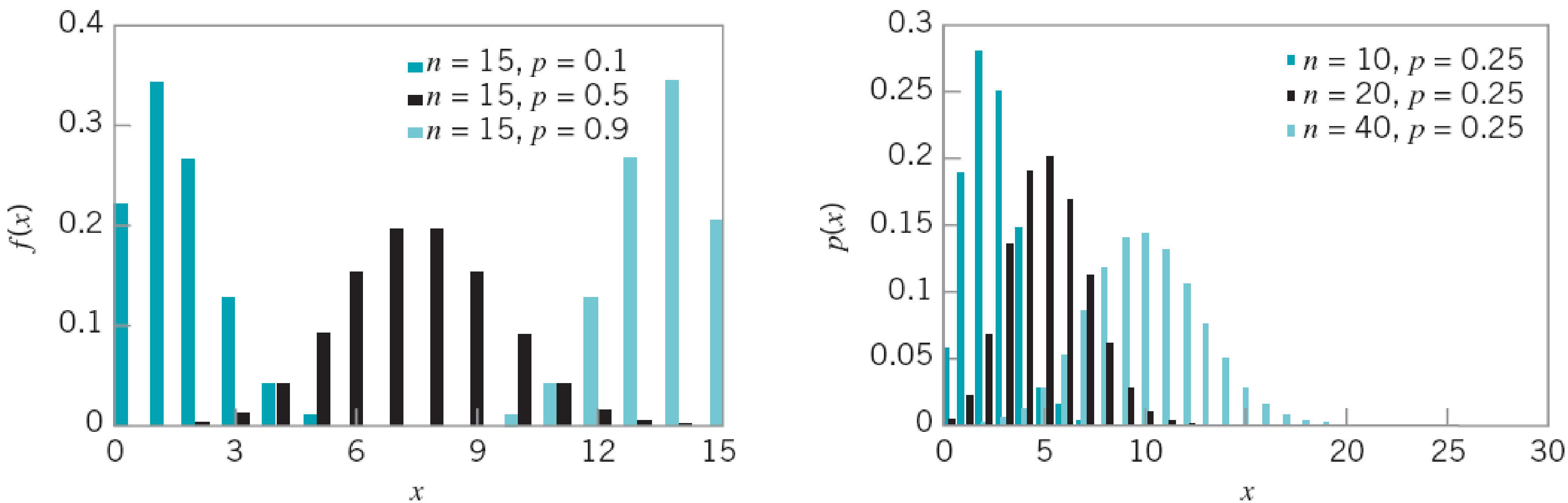
$$P(X \le x) = P(X \le x + 0.5) = P\left(Z \le \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

and

$$P(X \ge x) = P(X \le x - 0.5) = P\left(Z \le \frac{x - 0.5 - np}{\sqrt{np(1-p)}}\right)$$

The approximation is good for $np > 5$ and $n(1-p) > 5$. Refer to Figure 4-19 to see the rationale for adding and subtracting the 0.5 continuity correction.

5-4.1 Normal Approximation to the Binomial



(a) Binomial distributions for different values of p with $n = 15$.

(b) Binomial distributions for different values of n with $p = 0.25$.

Figure 5-17 Binomial distributions for selected values of n and p.

$np > 5$ and $n(1-p) > 5$

5-4.1 Normal Approximation to the Binomial

Example 5-36: Contamination on Optical Disks (cont.)

The digital comm problem in the previous example is solved using the normal approximation to the binomial as follows:

$$P(X \le 150) = P(X \le 150.5)$$

$$= P\left(\frac{X - 160}{\sqrt{160\left(1 - 10^{-5}\right)}} \le \frac{150.5 - 160}{\sqrt{160\left(1 - 10^{-5}\right)}}\right)$$

$$= P\left(Z \le \frac{-9.5}{12.6491}\right) = P(-0.75104) = 0.2263$$

5-4: Normal Approximation to the Binomial, Hypergeometric, and Poisson Distributions

5-4.1 Normal Approximation to the Binomial

Example 5-37: Bit Transmission Error (cont.)

Again consider the transmission of bits. To judge how well the normal approximation works, assume $n = 50$ bits are transmitted and the probability of an error is $p = 0.1$. The exact and approximated probabilities are:

$$P(X \leq 2) = C_0^{50} 0.9^{50} + C_1^{50} 0.1\left(0.9^{49}\right) + C_2^{50} 0.1^2\left(0.9^{48}\right) = 0.112$$

$$P(X \leq 2) = P\left(\frac{X-5}{\sqrt{50(0.1)(0.9)}} < \frac{2.5-5}{\sqrt{50(0.1)(0.9)}}\right)$$

$$= P(Z < -1.18) = 0.119$$

5-4.1 Normal Approximation to the Binomial

Example 5-37: Bit Transmission Error (cont.)

Using Excel	
0.1849	= BINOMDIST(5,50,0.1,FALSE)
0.1863	= NORMDIST(5.5, 5, SQRT(5*0.9), TRUE) - NORMDIST(4.5, 5, SQRT(5*0.9), TRUE)

5-4: Normal Approximation to the Binomial, Hypergeometric, and Poisson Distributions

5-4.1 Normal Approximation to the Binomial

The np > 5 and n(1-p) > 5 approximation rule is needed to keep the tails of the normal distribution from getting out-of-bounds.

As the binomial mean approaches the endpoints of the range of x, the standard deviation must be small enough to prevent overrun.

Figure 4-20 shows the asymmetric shape of the binomial when the approximation rule is not met.

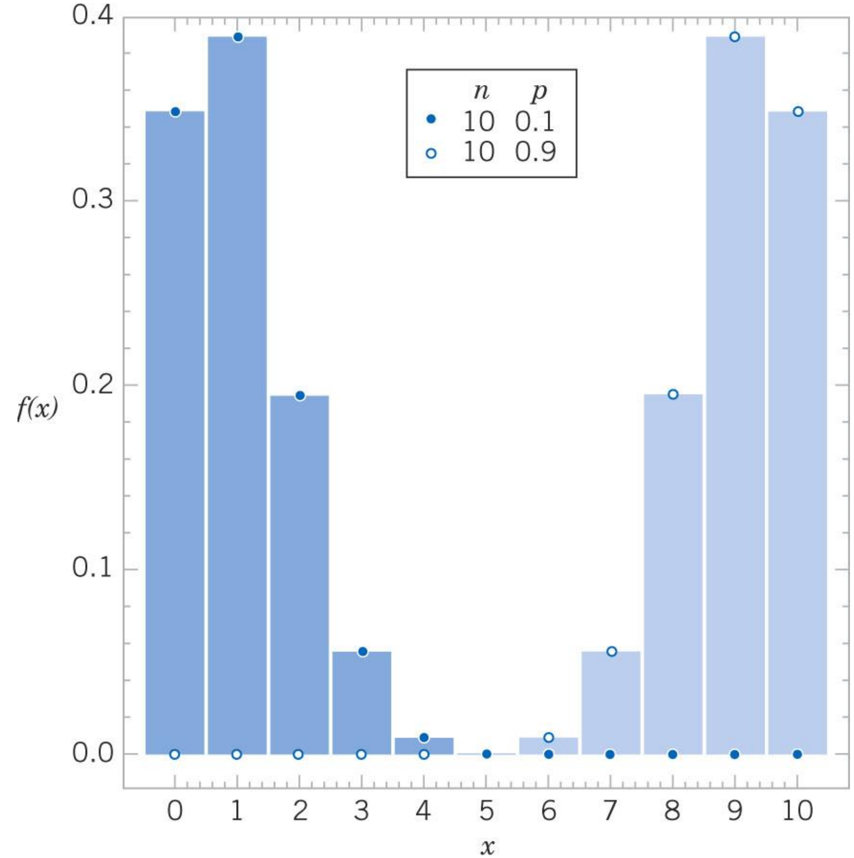


Figure 5-18 Binomial distribution is not symmetric as *p* gets near 0 or 1.

5-4.2 Normal Approximation to the Hypergeometric

Recall that the hypergeometric distribution is similar to the binomial such that $p = K / N$ and when sample sizes are small relative to population size.

Thus the normal can be used to approximate the hypergeometric distribution also.

hypergeometric	approximate $\approx$	binomial	$\approx$	normal
distribution		distribution		distribution
	$n / N < 0.1$		$np > 5$	
			$n(1\text{-}p) > 5$	

Figure 5-19 Conditions for approximatine hypergeometric and binomial with normal probabilities

5-4: Normal Approximation to the Binomial, Hypergeometric, and Poisson Distributions

5-4.1 Normal Approximation to the Poisson

If X is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}} \qquad (5\text{-}26)$$

is approximately a standard normal random variable.

The approximation is good for $\lambda \geq 5$.

5-4.1 Normal Approximation to the Poisson

Example 5-38: Asbestos particles

Assume that the number of asbestos particles in a square meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a square meter of dust is analyzed, what is the probability that 950 or fewer particles are found?

Using Excel	
0.0578	= POISSON(950,1000,TRUE)
0.0588	= NORMDIST(950.5, 1000, SQRT(1000), TRUE)
1.6%	= (0.0588 - 0.0578) / 0.0578 = percent error

$$P(X \le 950) = \sum_{x=0}^{950} \frac{e^{-1000} 1000^x}{x!} \quad \text{... too hard manually!}$$

$$\approx P(X < 950.5) = P\left(Z < \frac{950.5 - 1000}{\sqrt{1000}}\right)$$

$$= P(Z < -1.57) = 0.058$$