

11. APPROXIMATION ALGORITHMS

- ▶ *load balancing*
- ▶ *center selection*
- ▶ *pricing method: weighted vertex cover*
- ▶ *LP rounding: weighted vertex cover*
- ▶ *generalized load balancing*
- ▶ *knapsack problem*

Lecture slides by Kevin Wayne

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<http://www.cs.princeton.edu/~wayne/kleinberg-tardos>

Coping with NP-completeness

Q. Suppose I need to solve an NP-hard problem. What should I do?

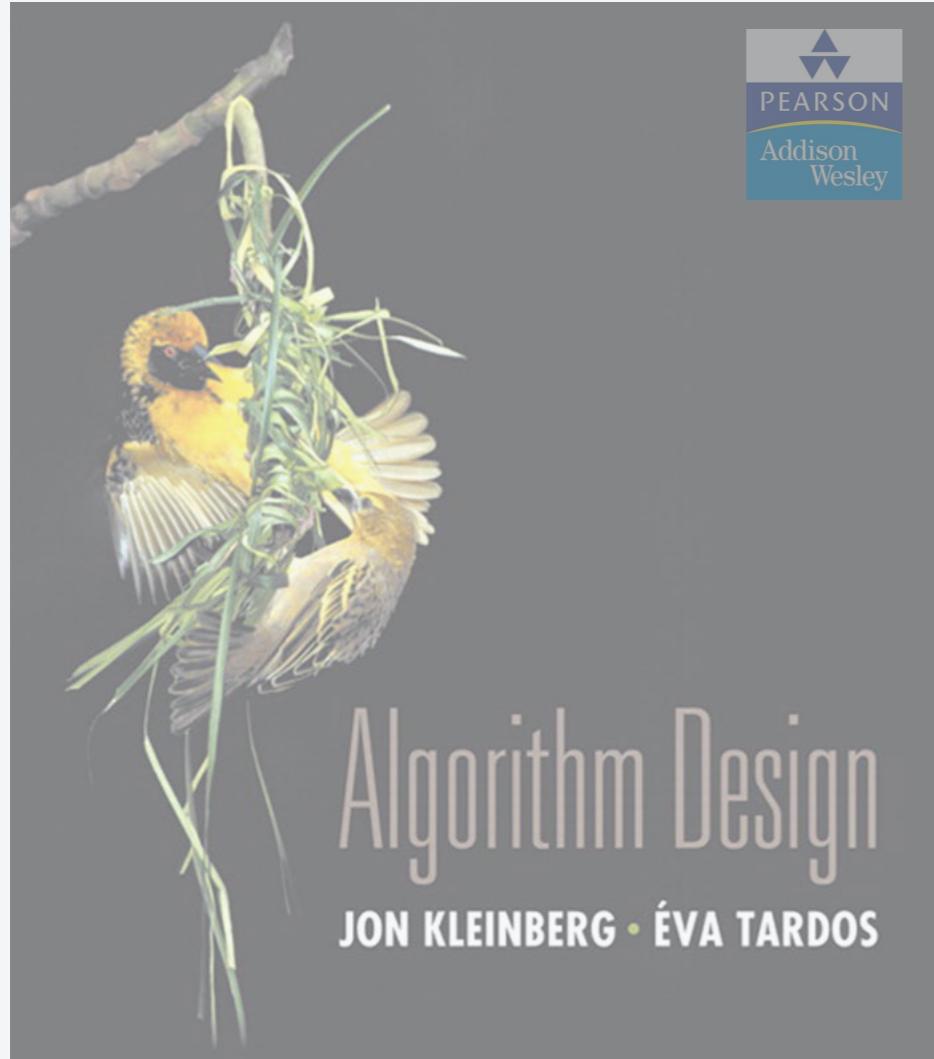
A. Sacrifice one of three desired features.

- i. Solve arbitrary instances of the problem.
- ii. Solve problem to optimality.
- iii. Solve problem in polynomial time.

ρ -approximation algorithm.

- Guaranteed to run in poly-time.
- Guaranteed to solve arbitrary instance of the problem
- Guaranteed to find solution within ratio ρ of true optimum.

Challenge. Need to prove a solution's value is close to optimum,
without even knowing what optimum value is



SECTION 11.1

11. APPROXIMATION ALGORITHMS

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Load balancing

Input. m identical machines; n jobs, job j has processing time t_j .

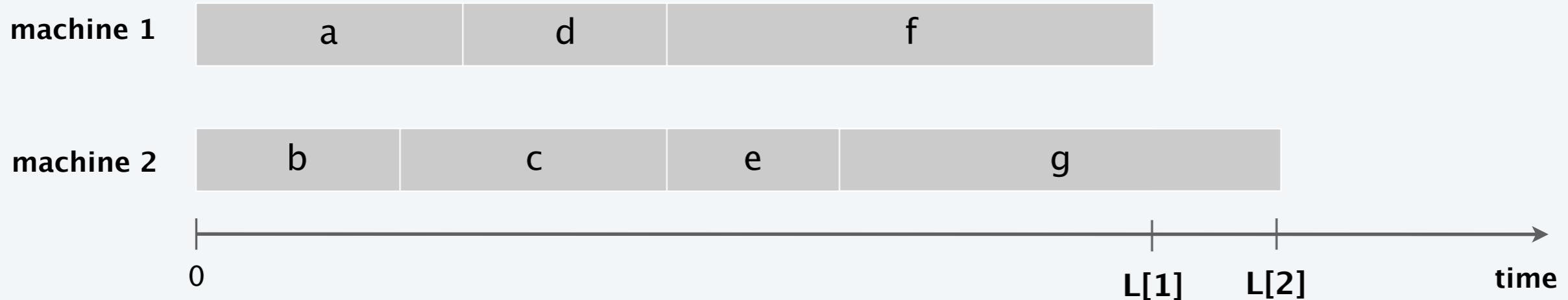
- Job j must run contiguously on one machine.
- A machine can process at most one job at a time.

Def. Let $S[i]$ be the subset of jobs assigned to machine i .

The **load** of machine i is $L[i] = \sum_{j \in S[i]} t_j$.

Def. The **makespan** is the maximum load on any machine $L = \max_i L[i]$.

Load balancing. Assign each job to a machine to minimize makespan.

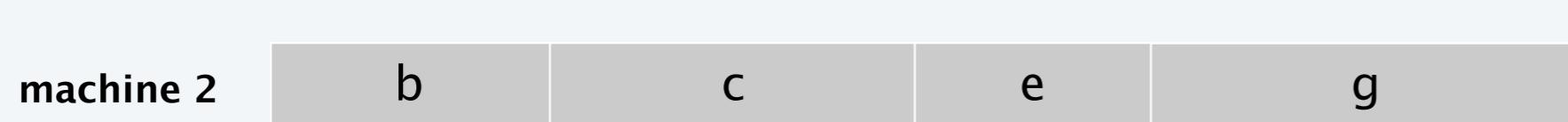
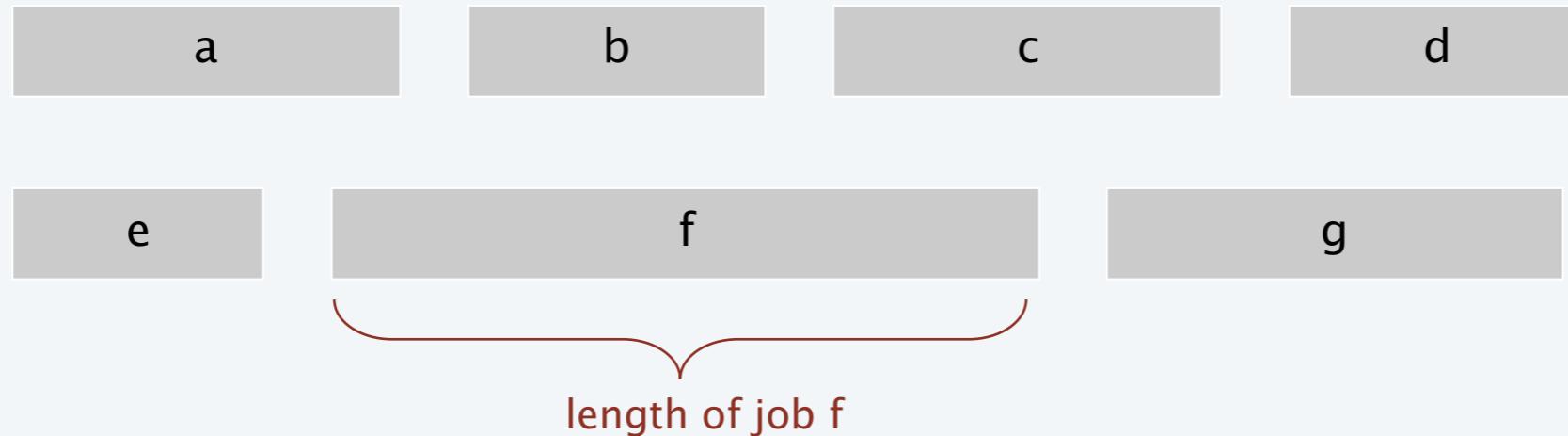


Load balancing on 2 machines is NP-hard

Claim. Load balancing is hard even if $m = 2$ machines.

Pf. PARTITION \leq_p LOAD-BALANCE.

NP-complete by Exercise 8.26



yes



Load balancing: list scheduling

List-scheduling algorithm.

- Consider n jobs in some fixed order.
- Assign job j to machine i whose load is smallest so far.



LIST-SCHEDULING ($m, n, t_1, t_2, \dots, t_n$)

FOR $i = 1$ **TO** m

$L[i] \leftarrow 0.$ \leftarrow load on machine i

$S[i] \leftarrow \emptyset.$ \leftarrow jobs assigned to machine i

FOR $j = 1$ **TO** n

$i \leftarrow \operatorname{argmin}_k L[k].$ \leftarrow machine i has smallest load

$S[i] \leftarrow S[i] \cup \{j\}.$ \leftarrow assign job j to machine i

$L[i] \leftarrow L[i] + t_j.$ \leftarrow update load of machine i

RETURN $S[1], S[2], \dots, S[m].$

Implementation. $O(n \log m)$ using a priority queue for loads $L[k].$

Load balancing: list scheduling analysis

Theorem. [Graham 1966] Greedy algorithm is a 2-approximation.

- First worst-case analysis of an approximation algorithm.
- Need to compare resulting solution with optimal makespan L^* .

Lemma 1. The optimal makespan $L^* \geq \max_j t_j$.

Pf. Some machine must process the most time-consuming job. ▀

Lemma 2. The optimal makespan $L^* \geq \frac{1}{m} \sum_j t_j$.

Pf.

- The total processing time is $\sum_j t_j$.
- One of m machines must do at least a $1/m$ fraction of total work. ▀

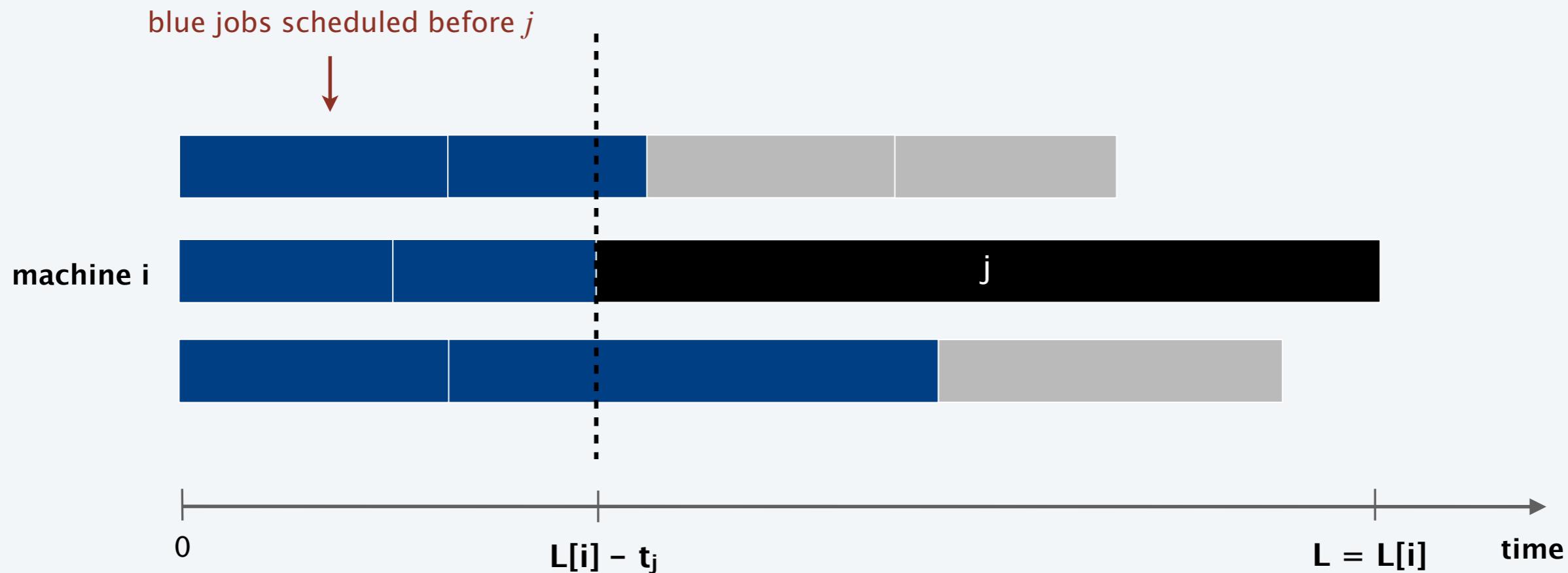
Load balancing: list scheduling analysis

Theorem. Greedy algorithm is a 2-approximation.

Pf. Consider load $L[i]$ of bottleneck machine i . \leftarrow machine that ends up with highest load

- Let j be last job scheduled on machine i .
- When job j assigned to machine i , i had smallest load.

Its load before assignment is $L[i] - t_j \Rightarrow L[i] - t_j \leq L[k]$ for all $1 \leq k \leq m$.



Load balancing: list scheduling analysis

Theorem. Greedy algorithm is a 2-approximation.

Pf. Consider load $L[i]$ of bottleneck machine i . \leftarrow machine that ends up with highest load

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- When job j assigned to machine i , i had smallest load.

Its load before assignment is $L[i] - t_j \Rightarrow L[i] - t_j \leq L[k]$ for all $1 \leq k \leq m$.

- Sum inequalities over all k and divide by m :

$$\begin{aligned} L[i] - t_j &\leq \frac{1}{m} \sum_k L[k] \\ &= \frac{1}{m} \sum_k t_k \\ \text{Lemma 2 } \rightarrow &\leq L^*. \end{aligned}$$

- Now, $L = L[i] = \underbrace{(L[i] - t_j)}_{\leq L^*} + \underbrace{t_j}_{\leq L^*} \leq 2L^*$.

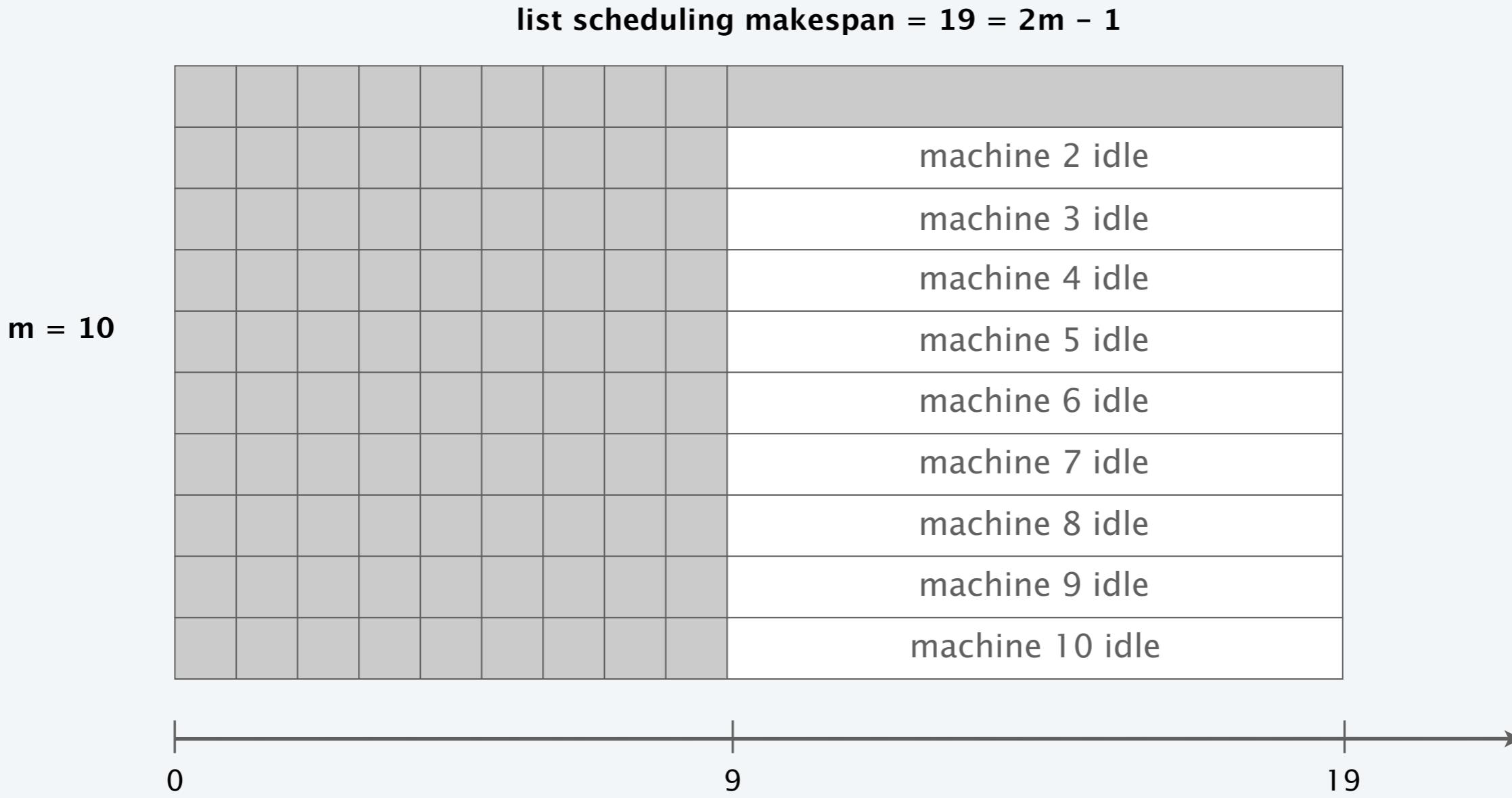
$$\begin{array}{ccc} \overbrace{} & \overbrace{} \\ \leq L^* & \leq L^* \\ \uparrow & \uparrow \\ \text{above inequality} & \text{Lemma 1} \end{array}$$

Load balancing: list scheduling analysis

Q. Is our analysis tight?

A. Essentially yes.

Ex: m machines, $m(m - 1)$ jobs length 1 jobs, one job of length m .

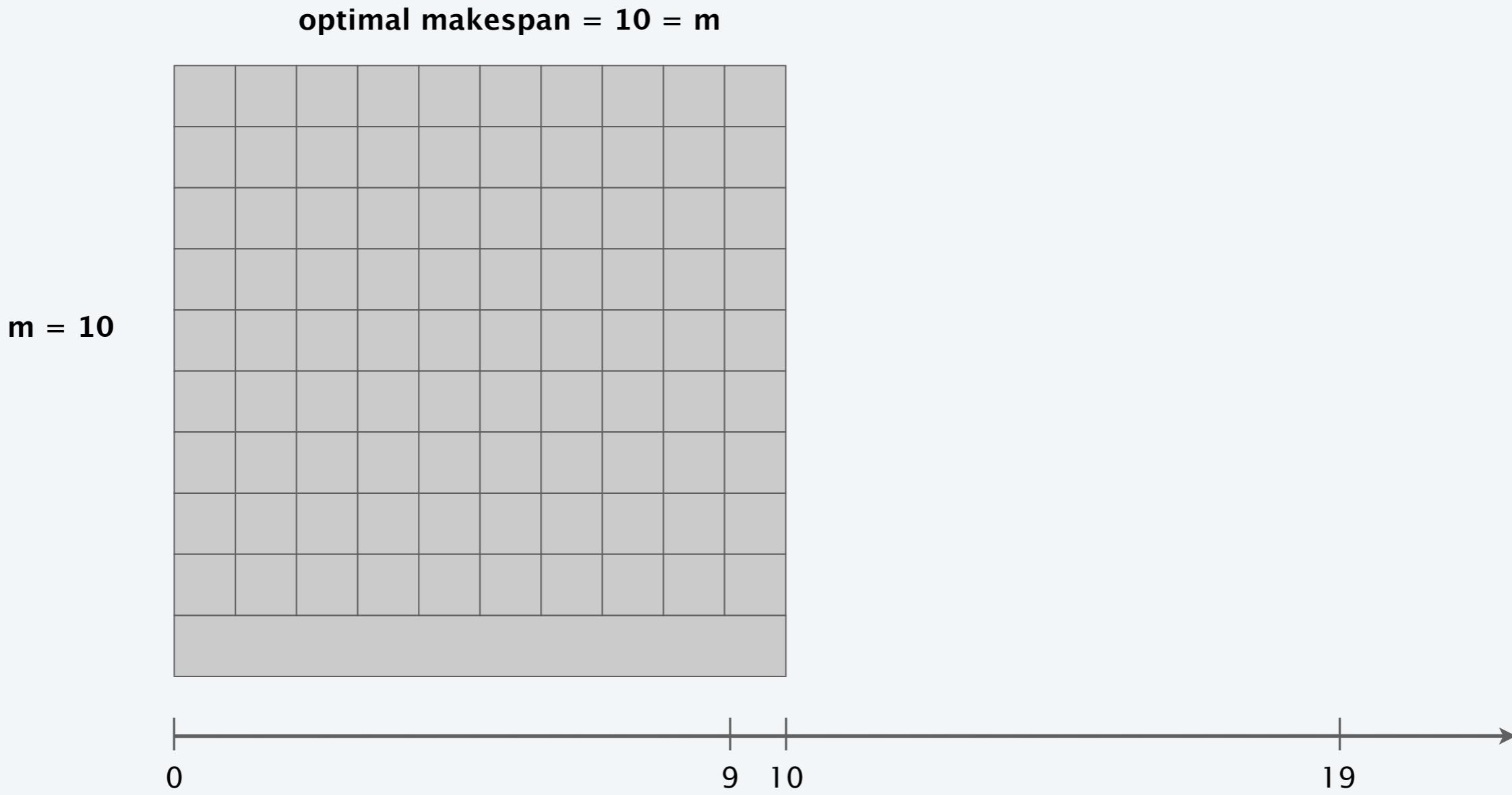


Load balancing: list scheduling analysis

Q. Is our analysis tight?

A. Essentially yes.

Ex: m machines, $m(m - 1)$ jobs length 1 jobs, one job of length m .



Load balancing: LPT rule

Longest processing time (LPT). Sort n jobs in decreasing order of processing times; then run list scheduling algorithm.

LPT-LIST-SCHEDULING ($m, n, t_1, t_2, \dots, t_n$)

SORT jobs and renumber so that $t_1 \geq t_2 \geq \dots \geq t_n$.

FOR $i = 1$ TO m

$L[i] \leftarrow 0$. \leftarrow load on machine i

$S[i] \leftarrow \emptyset$. \leftarrow jobs assigned to machine i

FOR $j = 1$ TO n

$i \leftarrow \operatorname{argmin}_k L[k]$. \leftarrow machine i has smallest load

$S[i] \leftarrow S[i] \cup \{j\}$. \leftarrow assign job j to machine i

$L[i] \leftarrow L[i] + t_j$. \leftarrow update load of machine i

RETURN $S[1], S[2], \dots, S[m]$.

Load balancing: LPT rule

Observation. If bottleneck machine i has only 1 job, then optimal.

Pf. Any solution must schedule that job. ■

Lemma 3. If there are more than m jobs, $L^* \geq 2t_{m+1}$.

Pf.

- Consider processing times of first $m+1$ jobs $t_1 \geq t_2 \geq \dots \geq t_{m+1}$.
- Each takes at least t_{m+1} time.
- There are $m+1$ jobs and m machines, so by pigeonhole principle, at least one machine gets two jobs. ■

Theorem. LPT rule is a $3/2$ -approximation algorithm.

Pf. [similar to proof for list scheduling]

- Consider load $L[i]$ of bottleneck machine i .
- Let j be last job scheduled on machine i . ← assuming machine i has at least 2 jobs, we have $j > m$

$$L = L[i] = (\underbrace{L[i] - t_j}_{\text{as before}}) + t_j \leq \frac{3}{2} L^* .$$

as before → $\leq L^*$ $\leq \frac{1}{2} L^*$ ← Lemma 3 (since $t_j \leq t_{m+1}$)

Load balancing: LPT rule

Q. Is our $3/2$ analysis tight?

A. No.

Theorem. [Graham 1969] LPT rule is a $4/3$ -approximation.

Pf. More sophisticated analysis of same algorithm.

Q. Is Graham's $4/3$ analysis tight?

A. Essentially yes.

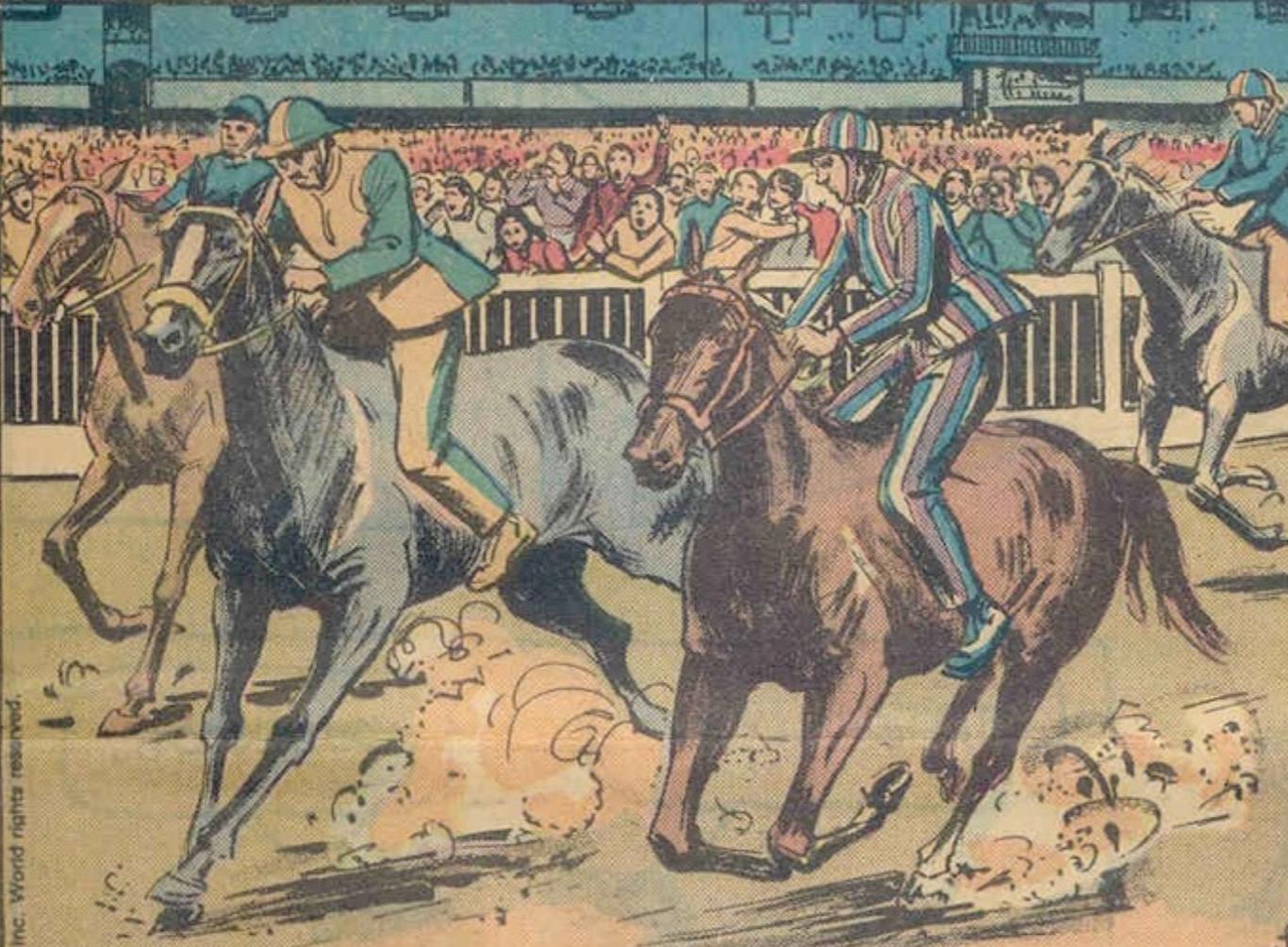
Ex.

- m machines
- $n = 2m + 1$ jobs
- 2 jobs of length $m, m + 1, \dots, 2m - 1$ and one more job of length m .
- Then, $L / L^* = (4m - 1) / (3m)$

Believe it or not

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Believe It or Not!



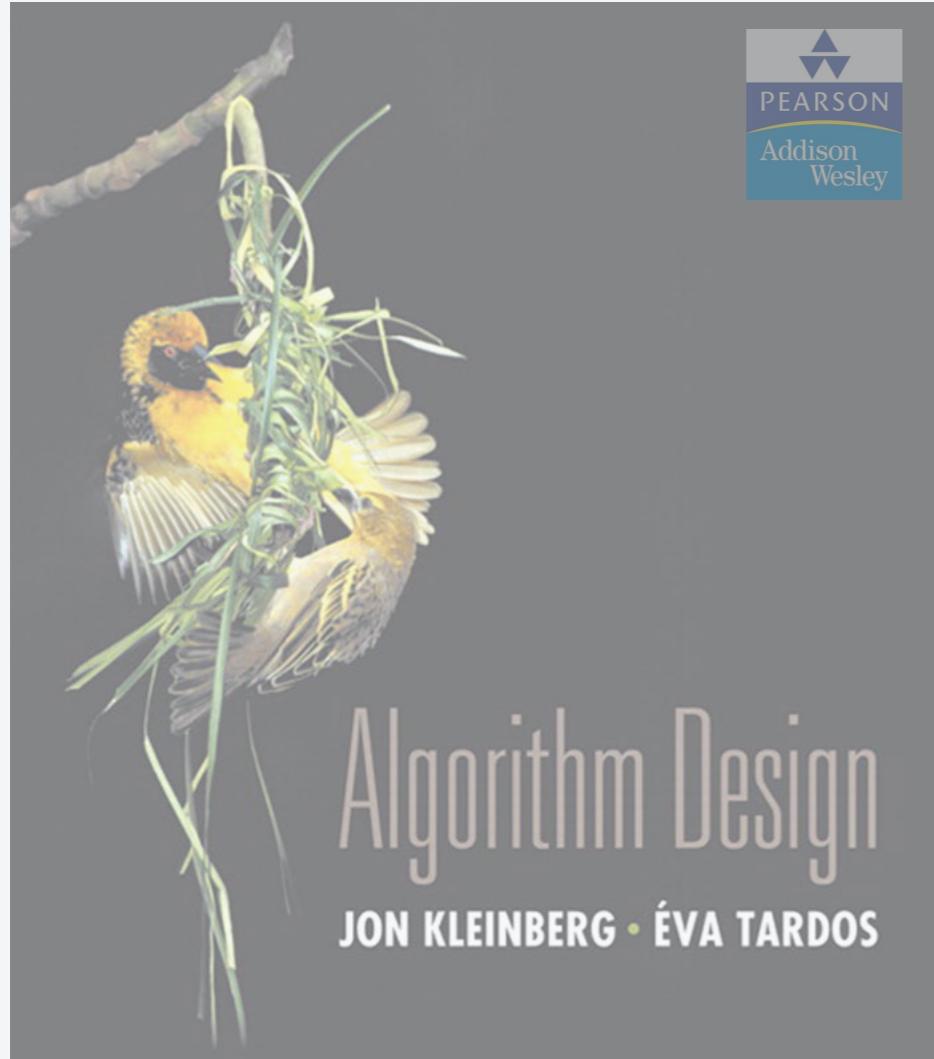
A RACE IN WHICH LOSING IS AKIN TO DEATH

THE PALIO, a horse race held each summer around the main square of Siena, Italy, traditionally ends with the winners holding a MOCK FUNERAL FOR THE LOSERS

4-10



RONALD GRAHAM
head of Bell Laboratories mathematical Studies Center in Murray Hill, N.J., is one of the world's foremost mathematicians, publishes more than 12 math papers a year and is on the editorial boards of 20 math journals — yet is a highly skilled trampolinist and juggler, and has been elected president of the International Jugglers Association



SECTION 11.2

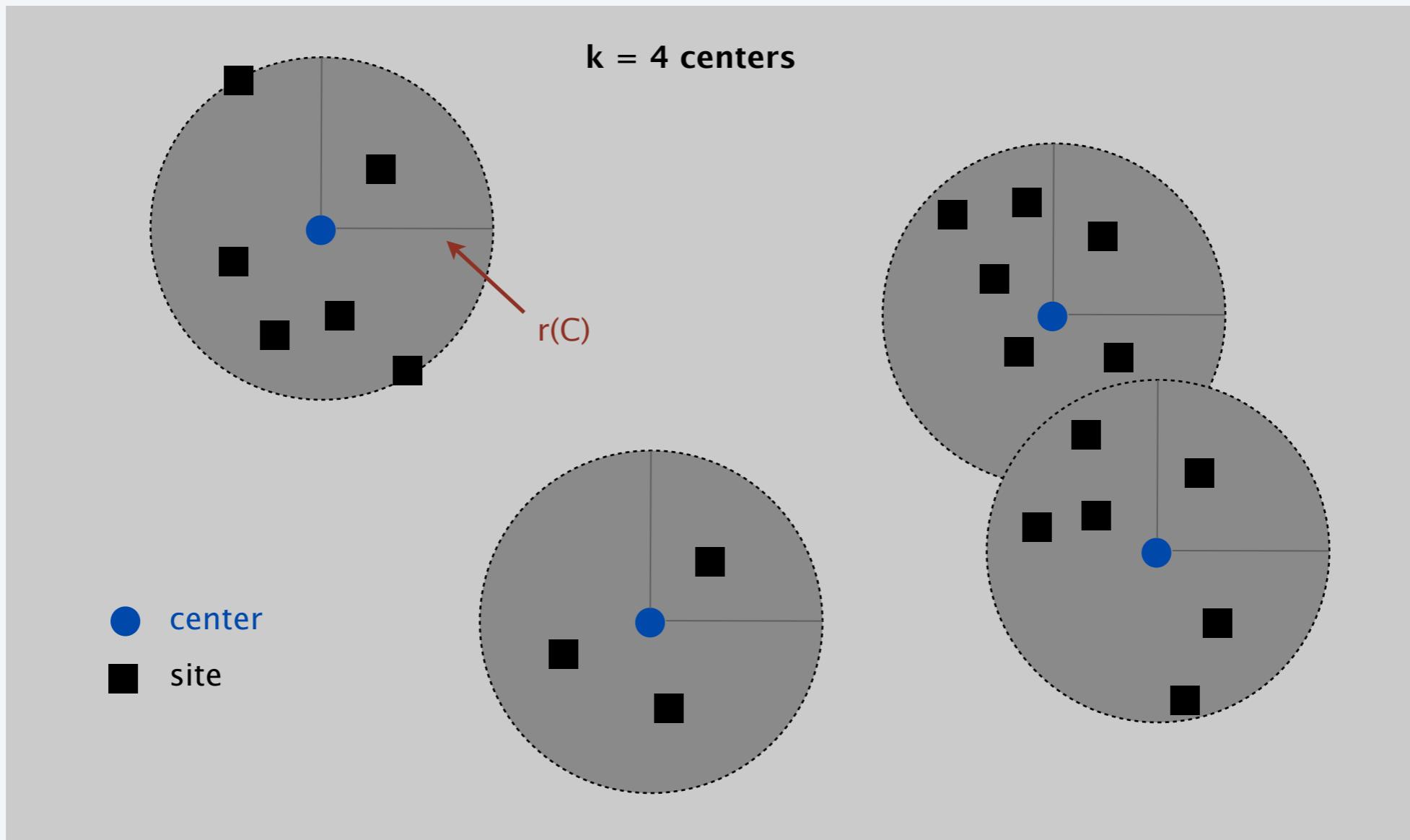
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Center selection problem

Input. Set of n sites s_1, \dots, s_n and an integer $k > 0$.

Center selection problem. Select set of k centers C so that maximum distance $r(C)$ from a site to nearest center is minimized.



Center selection problem

Input. Set of n sites s_1, \dots, s_n and an integer $k > 0$.

Center selection problem. Select set of k centers C so that maximum distance $r(C)$ from a site to nearest center is minimized.

Notation.

- $\text{dist}(x, y) =$ distance between sites x and y .
- $\text{dist}(s_i, C) = \min_{c \in C} \text{dist}(s_i, c) =$ distance from s_i to closest center.
- $r(C) = \max_i \text{dist}(s_i, C) =$ smallest covering radius.

Goal. Find set of centers C that minimizes $r(C)$, subject to $|C| = k$.

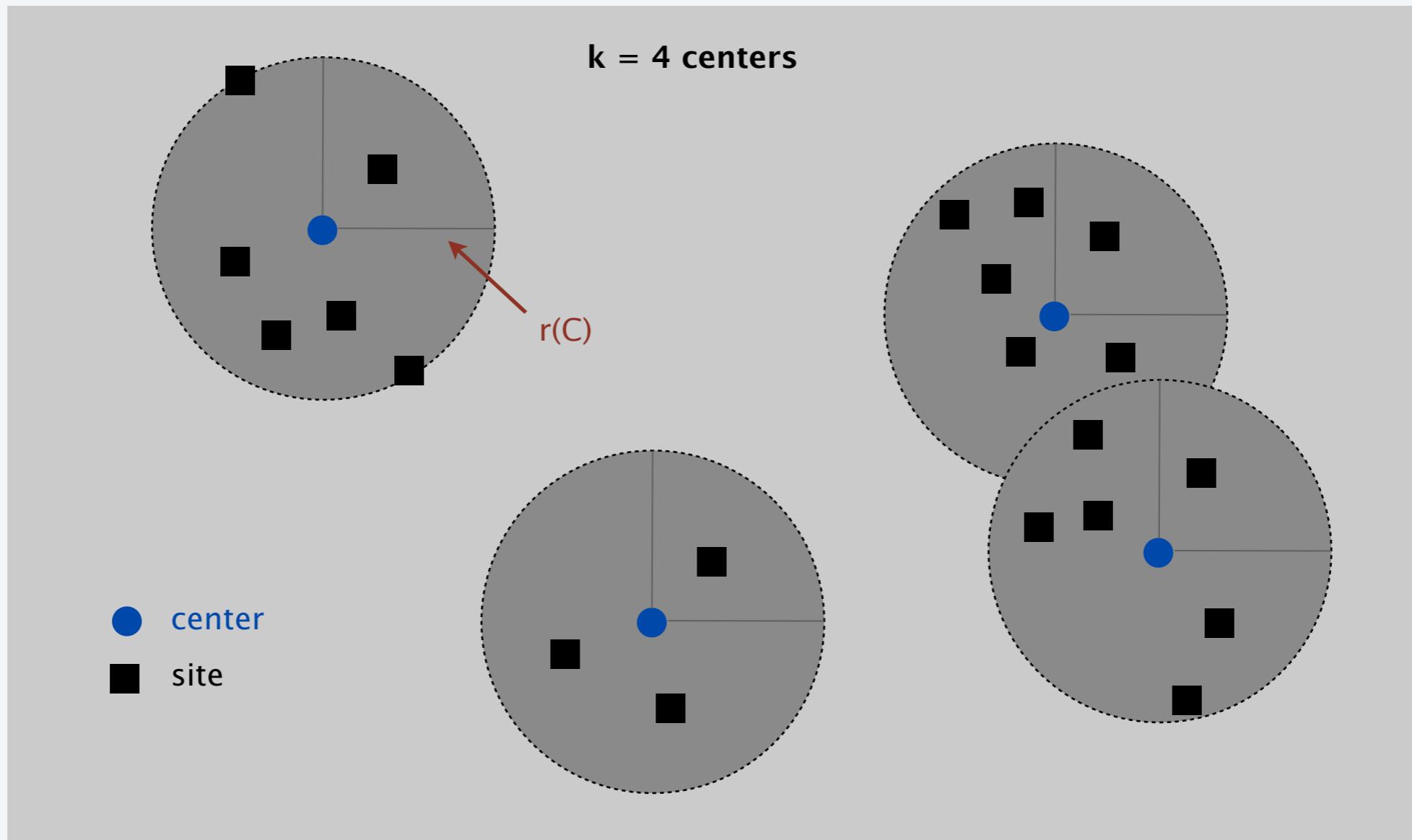
Distance function properties.

- $\text{dist}(x, x) = 0$ [identity]
- $\text{dist}(x, y) = \text{dist}(y, x)$ [symmetry]
- $\text{dist}(x, y) \leq \text{dist}(x, z) + \text{dist}(z, y)$ [triangle inequality]

Center selection example

Ex: each site is a point in the plane, a center can be any point in the plane, $dist(x, y)$ = Euclidean distance.

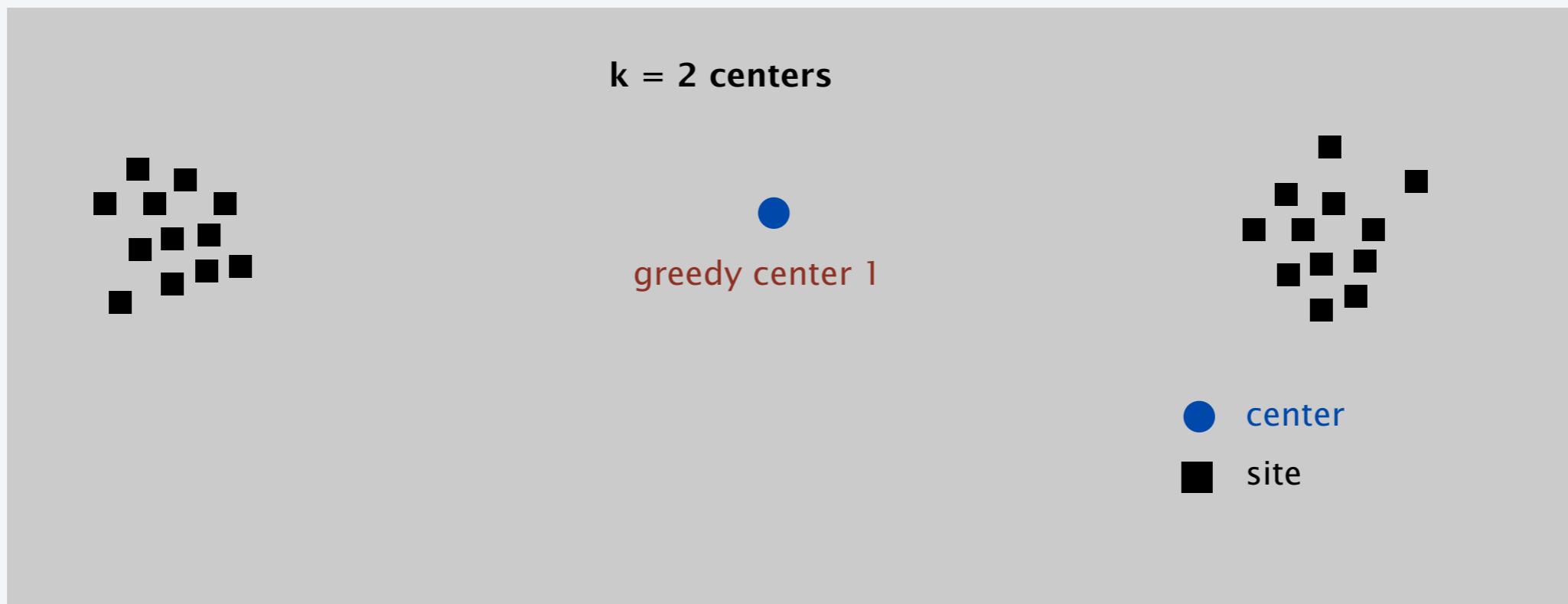
Remark: search can be infinite!



Greedy algorithm: a false start

Greedy algorithm. Put the first center at the best possible location for a single center, and then keep adding centers so as to reduce the covering radius each time by as much as possible.

Remark: arbitrarily bad!



Center selection: greedy algorithm

Repeatedly choose next center to be site **farthest** from any existing center.

GREEDY-CENTER-SELECTION ($k, n, s_1, s_2, \dots, s_n$)

$C \leftarrow \emptyset.$

REPEAT k times

Select a site s_i with maximum distance $\text{dist}(s_i, C)$.

$C \leftarrow C \cup s_i.$

RETURN C .

↑
site farthest
from any center

Property. Upon termination, all centers in C are pairwise at least $r(C)$ apart.

Pf. By construction of algorithm.

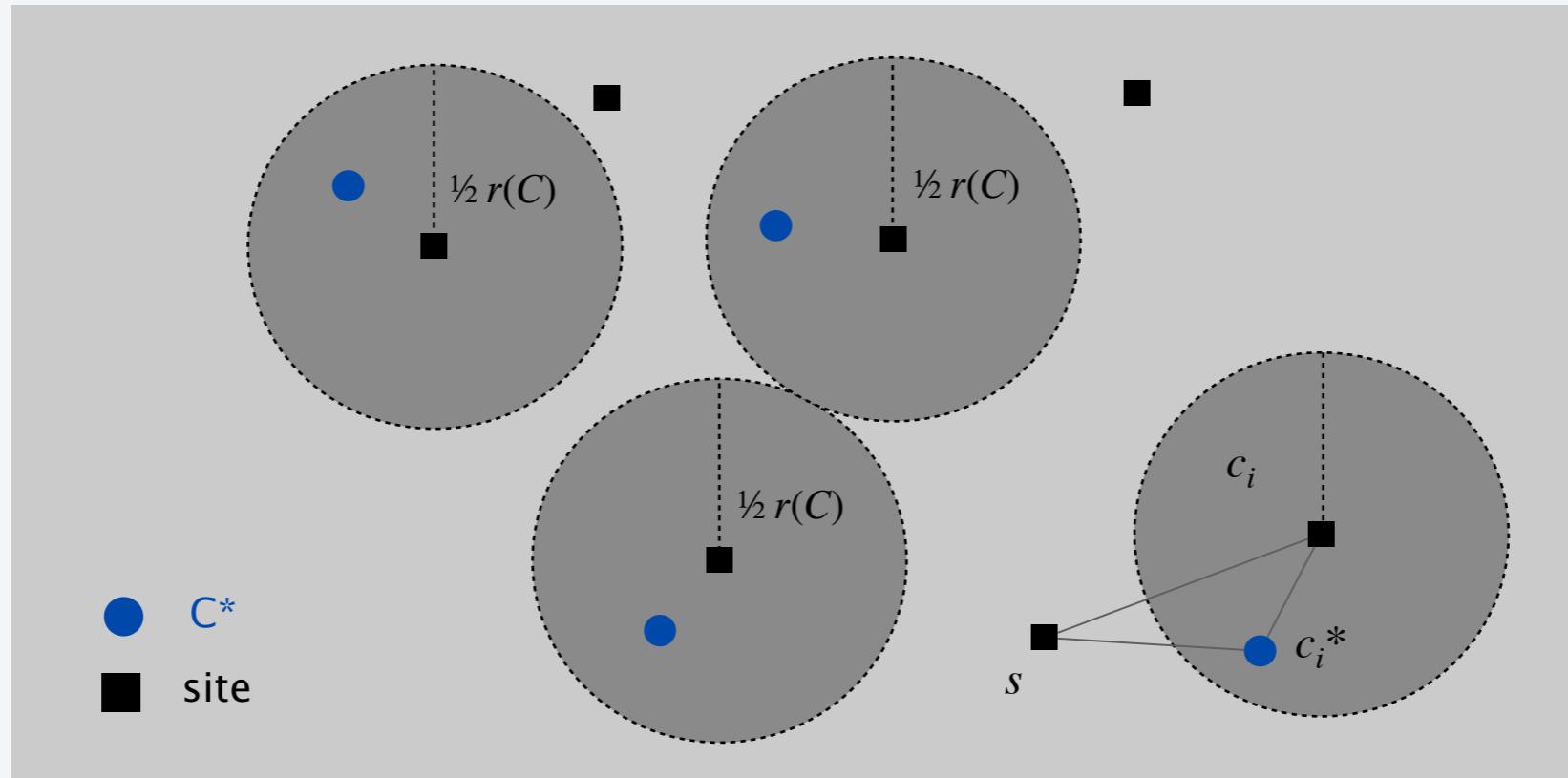
Center selection: analysis of greedy algorithm

Lemma. Let C^* be an optimal set of centers. Then $r(C) \leq 2r(C^*)$.

Pf. [by contradiction] Assume $r(C^*) < \frac{1}{2} r(C)$.

- For each site $c_i \in C$, consider ball of radius $\frac{1}{2} r(C)$ around it.
- Exactly one c_i^* in each ball; let c_i be the site paired with c_i^* .
- Consider any site s and its closest center $c_i^* \in C^*$.
- $dist(s, C) \leq dist(s, c_i) \leq dist(s, c_i^*) + dist(c_i^*, c_i) \leq 2r(C^*)$.

- Thus, $r(C) \leq 2r(C^*)$.
 ↑■ ↑■
 Δ-inequality ≤ r(C^*) since c_i^* is closest center



Center selection

Lemma. Let C^* be an optimal set of centers. Then $r(C) \leq 2r(C^*)$.

Theorem. Greedy algorithm is a 2-approximation for center selection problem.

Remark. Greedy algorithm always places centers at sites, but is still within a factor of 2 of best solution that is allowed to place centers anywhere.

e.g., points in the plane

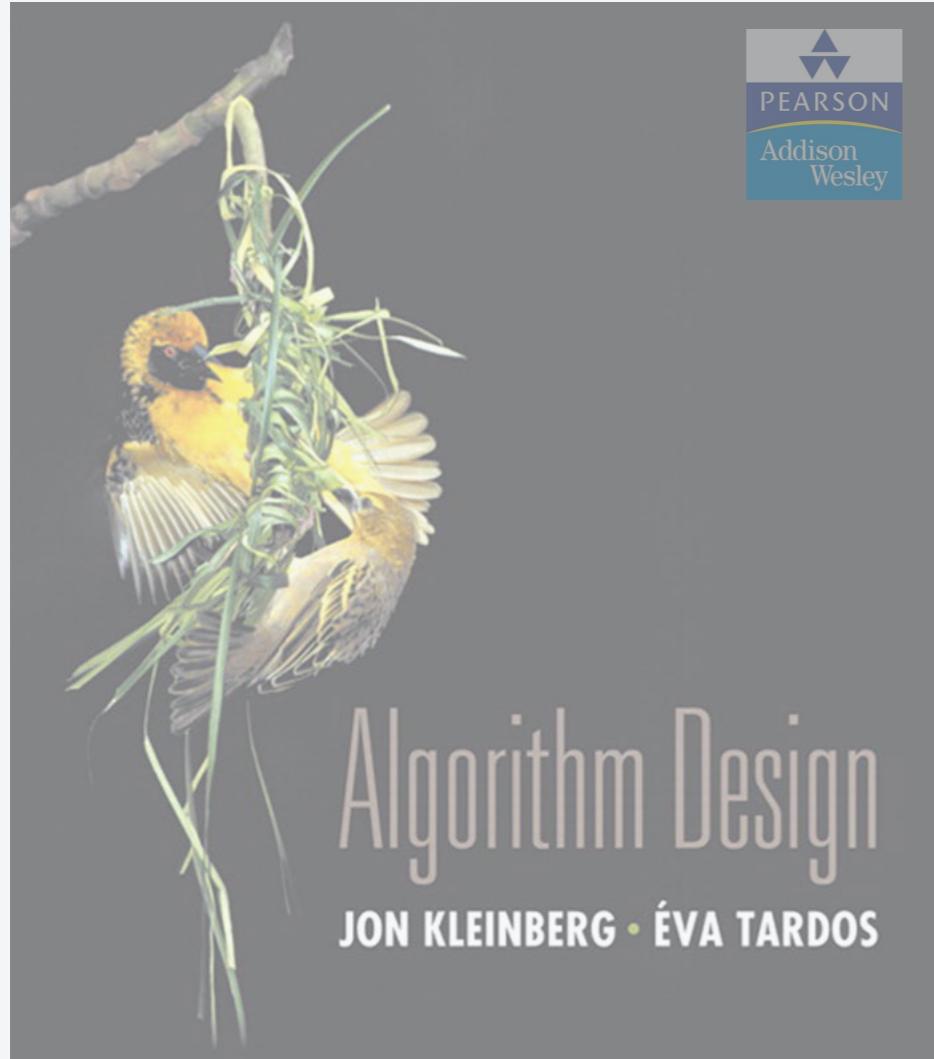
Question. Is there hope of a $3/2$ -approximation? $4/3$?

Dominating set reduces to center selection

Theorem. Unless $P = NP$, there no ρ -approximation for center selection problem for any $\rho < 2$.

Pf. We show how we could use a $(2 - \varepsilon)$ approximation algorithm for CENTER-SELECTION selection to solve DOMINATING-SET in poly-time.

- Let $G = (V, E)$, k be an instance of DOMINATING-SET.
- Construct instance G' of CENTER-SELECTION with sites V and distances
 - $dist(u, v) = 1$ if $(u, v) \in E$
 - $dist(u, v) = 2$ if $(u, v) \notin E$
- Note that G' satisfies the triangle inequality.
- G has dominating set of size k iff there exists k centers C^* with $r(C^*) = 1$.
- Thus, if G has a dominating set of size k , a $(2 - \varepsilon)$ -approximation algorithm for CENTER-SELECTION would find a solution C^* with $r(C^*) = 1$ since it cannot use any edge of distance 2. ▀



SECTION 11.4

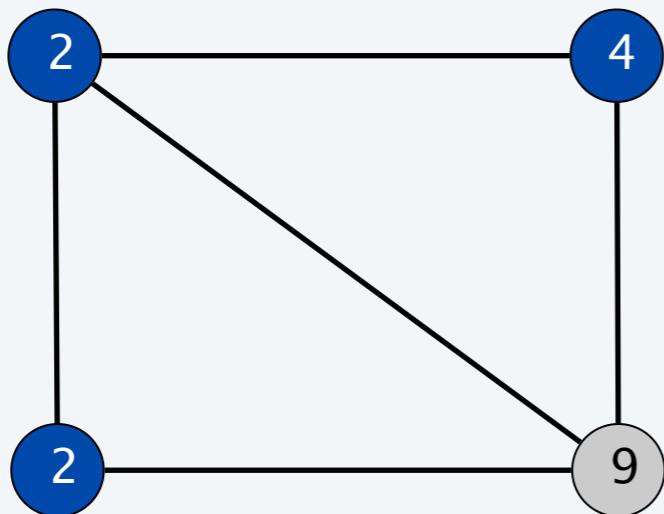
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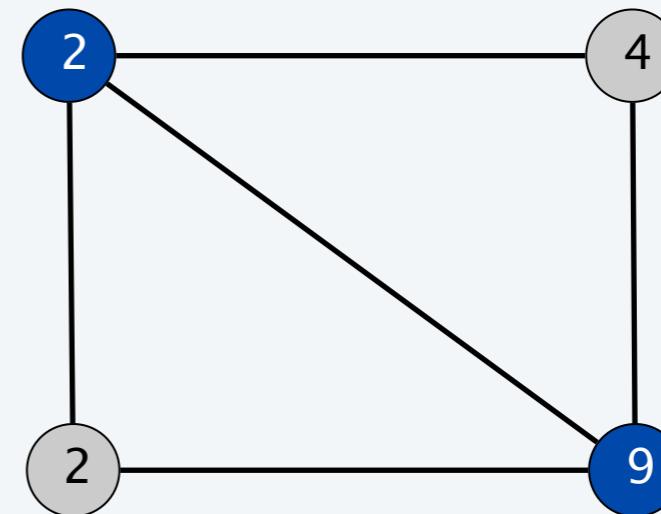
Weighted vertex cover

Definition. Given a graph $G = (V, E)$, a vertex cover is a set $S \subseteq V$ such that each edge in E has at least one end in S .

Weighted vertex cover. Given a graph G with vertex weights, find a vertex cover of minimum weight.



weight = 2 + 2 + 4



weight = 11

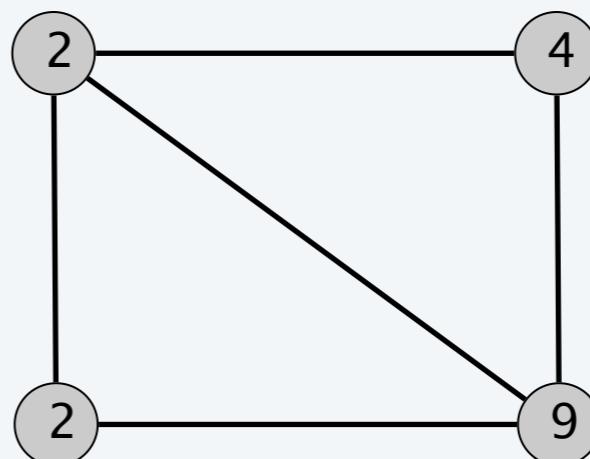
Pricing method

Pricing method. Each edge must be covered by some vertex.

Edge $e = (i, j)$ pays price $p_e \geq 0$ to use both vertex i and j .

Fairness. Edges incident to vertex i should pay $\leq w_i$ in total.

$$\text{for each vertex } i : \sum_{e=(i,j)} p_e \leq w_i$$



Fairness lemma. For any vertex cover S and any fair prices p_e : $\sum_e p_e \leq w(S)$.

$$\text{Pf. } \sum_{e \in E} p_e \leq \sum_{i \in S} \sum_{e=(i,j)} p_e \leq \sum_{i \in S} w_i = w(S). \blacksquare$$

each edge e covered by
at least one node in S

sum fairness inequalities
for each node in S

Pricing method

Set prices and find vertex cover simultaneously.

WEIGHTED-VERTEX-COVER (G, w)

$S \leftarrow \emptyset.$

FOREACH $e \in E$

$p_e \leftarrow 0.$

$$\sum_{e=(i,j)} p_e = w_i$$



WHILE (there exists an edge (i, j) such that neither i nor j is tight)

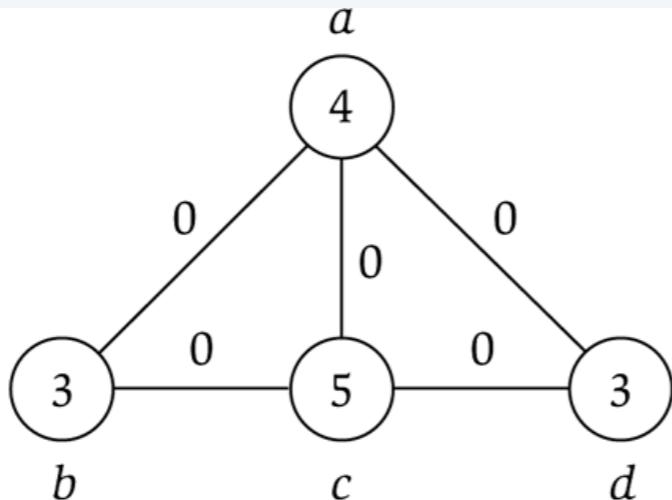
Select such an edge $e = (i, j)$.

Increase p_e as much as possible until i or j tight.

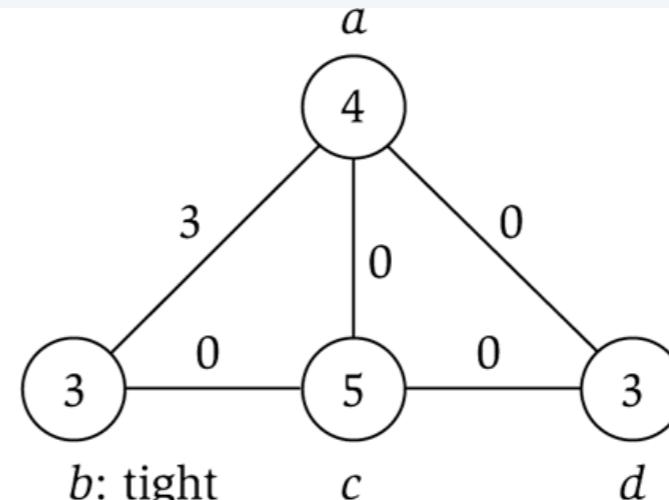
$S \leftarrow$ set of all tight nodes.

RETURN S .

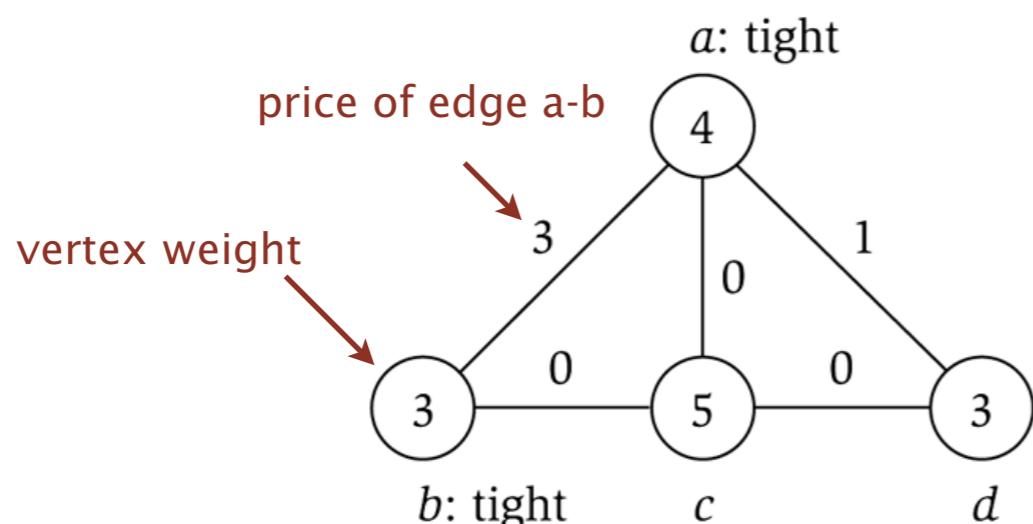
Pricing method example



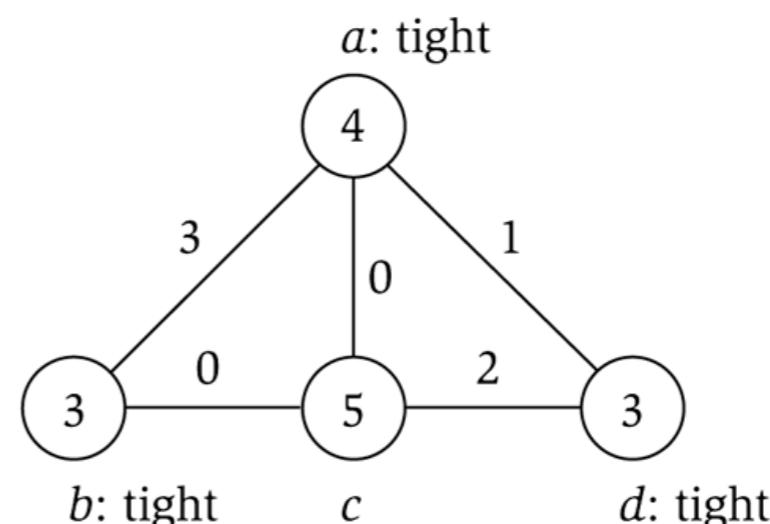
(a)



(b)



(c)



(d)

Pricing method: analysis

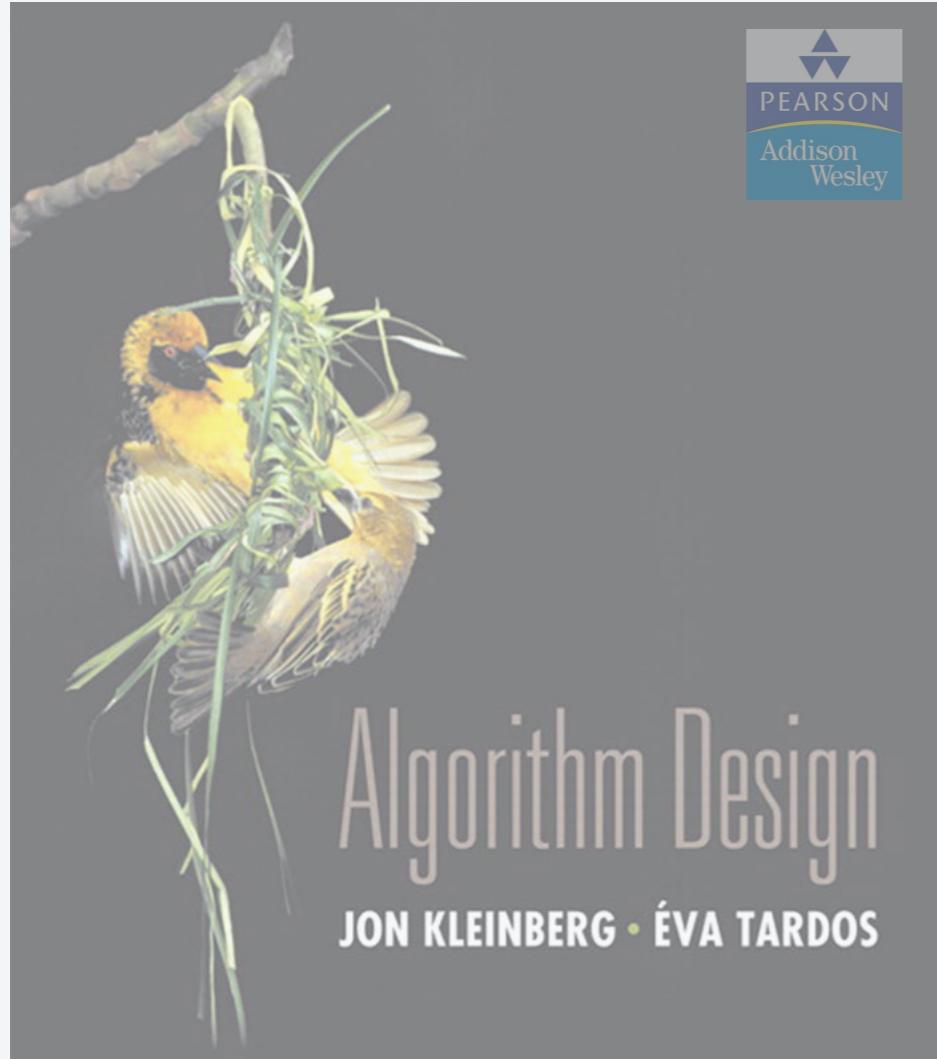
Theorem. Pricing method is a 2-approximation for WEIGHTED-VERTEX-COVER.

Pf.

- Algorithm terminates since at least one new node becomes tight after each iteration of while loop.
 - Let S = set of all tight nodes upon termination of algorithm.
 S is a vertex cover: if some edge (i, j) is uncovered, then neither i nor j is tight. But then while loop would not terminate.
 - Let S^* be optimal vertex cover. We show $w(S) \leq 2 w(S^*)$.

$$w(S) = \sum_{i \in S} w_i = \sum_{i \in S} \sum_{e=(i,j)} p_e \leq \sum_{i \in V} \sum_{e=(i,j)} p_e = 2 \sum_{e \in E} p_e \leq 2w(S^*). \blacksquare$$

↑ all nodes in S are tight ↑ $S \subseteq V$,
 ↑ each edge counted twice ↑ fairness lemma
 ↓ prices ≥ 0



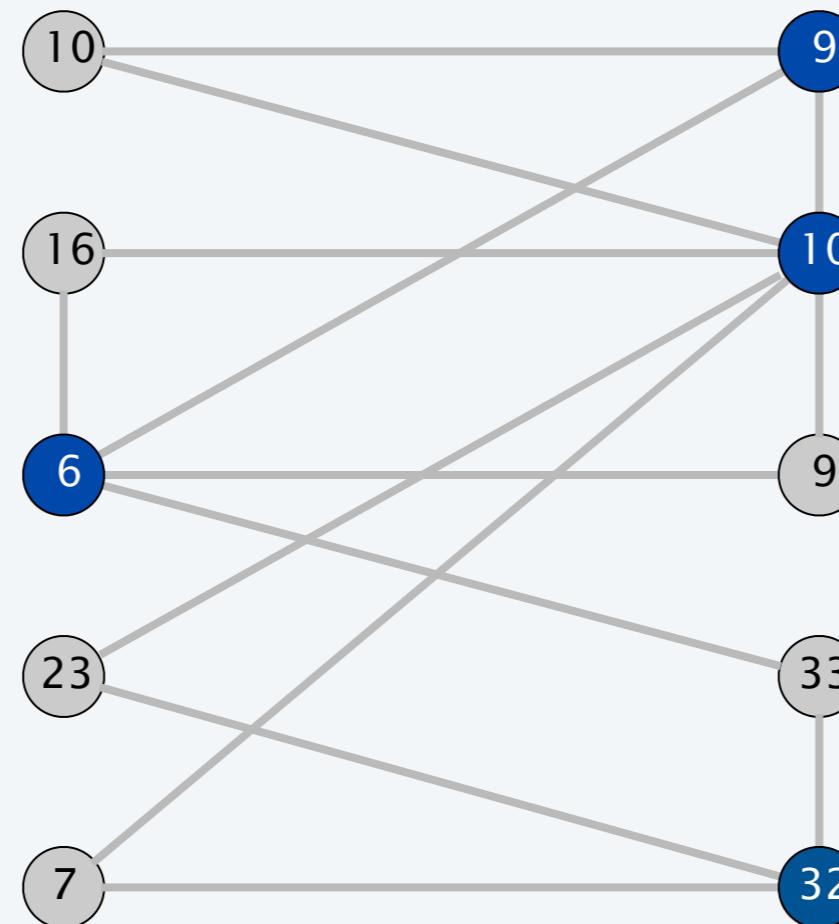
SECTION 11.6

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Weighted vertex cover

Given a graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in S .



total weight = $6 + 9 + 10 + 32 = 57$

Weighted vertex cover: ILP formulation

Given a graph $G = (V, E)$ with vertex weights $w_i \geq 0$, find a min-weight subset of vertices $S \subseteq V$ such that every edge is incident to at least one vertex in S .

Integer linear programming formulation.

- Model inclusion of each vertex i using a 0/1 variable x_i .

$$x_i = \begin{cases} 0 & \text{if vertex } i \text{ is not in vertex cover} \\ 1 & \text{if vertex } i \text{ is in vertex cover} \end{cases}$$

Vertex covers in 1–1 correspondence with 0/1 assignments:

$$S = \{ i \in V : x_i = 1 \}.$$

- Objective function: minimize $\sum_i w_i x_i$.
- For every edge (i, j) , must take either vertex i or j (or both): $x_i + x_j \geq 1$.

Weighted vertex cover: ILP formulation

Weighted vertex cover. Integer linear programming formulation.

$$(ILP) \quad \begin{aligned} & \min \quad \sum_{i \in V} w_i x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad (i, j) \in E \\ & x_i \in \{0, 1\} \quad i \in V \end{aligned}$$

Observation. If x^* is optimal solution to ILP , then $S = \{ i \in V : x_i^* = 1 \}$ is a min-weight vertex cover.

Integer linear programming

Given integers a_{ij} , b_i , and c_j , find integers x_j that satisfy:

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \\ & x \text{ integral}\end{array}$$

$$\begin{array}{lll}\min & \sum_{j=1}^n c_j x_j \\ \text{s.t.} & \sum_{j=1}^n a_{ij} x_j \geq b_i & 1 \leq i \leq m \\ & x_j \geq 0 & 1 \leq j \leq n \\ & x_j \text{ integral} & 1 \leq j \leq n\end{array}$$

Observation. Vertex cover formulation proves that INTEGER-PROGRAMMING is an NP-hard search problem.

Linear programming

Given integers a_{ij} , b_i , and c_j , find **real numbers** x_j that satisfy:

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0\end{array}$$

$$\begin{array}{lll}\min & \sum_{j=1}^n c_j x_j \\ \text{s.t.} & \sum_{j=1}^n a_{ij} x_j \geq b_i & 1 \leq i \leq m \\ & x_j \geq 0 & 1 \leq j \leq n\end{array}$$

Linear. No x^2 , xy , $\arccos(x)$, $x(1-x)$, etc.

Simplex algorithm. [Dantzig 1947] Can solve LP in practice.

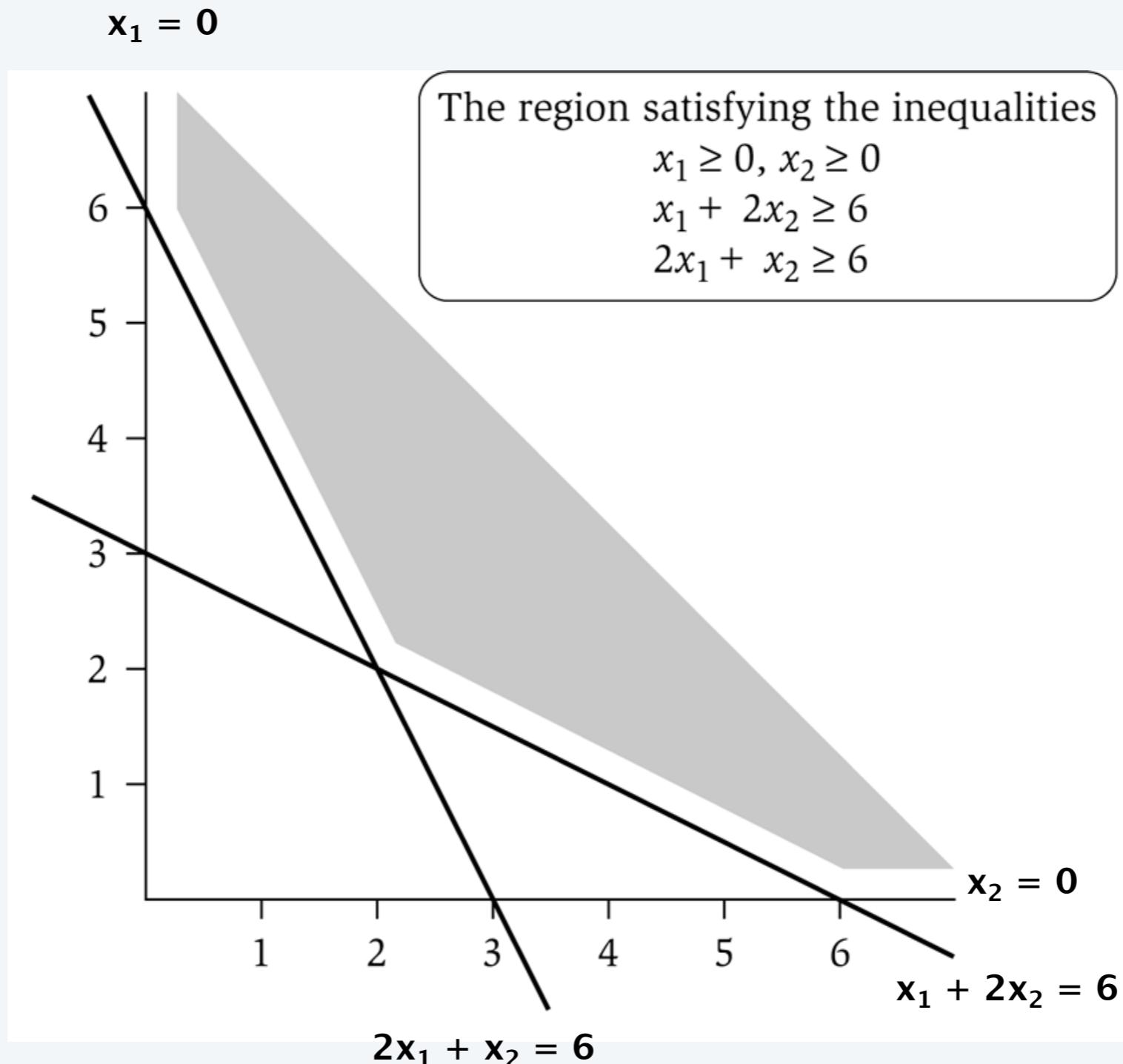
Ellipsoid algorithm. [Khachiyan 1979] Can solve LP in poly-time.

Interior point algorithms. [Karmarkar 1984, Renegar 1988, ...]

Can solve LP both in poly-time and in practice.

LP feasible region

LP geometry in 2D.



Weighted vertex cover: LP relaxation

Linear programming relaxation.

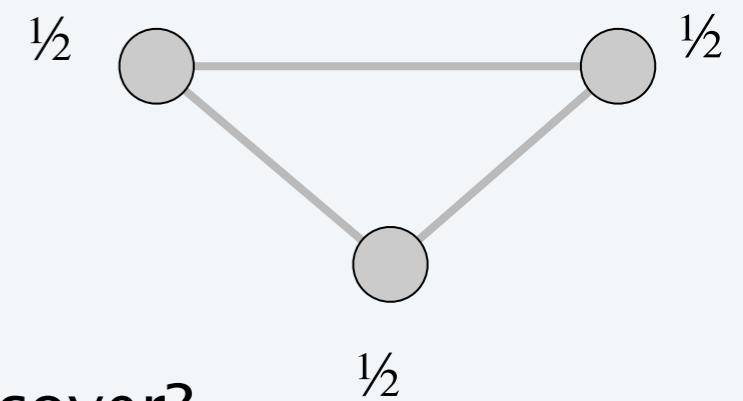
$$(LP) \quad \min \quad \sum_{i \in V} w_i x_i$$
$$\text{s.t.} \quad x_i + x_j \geq 1 \quad (i, j) \in E$$
$$x_i \geq 0 \quad i \in V$$

Observation. Optimal value of LP is \leq optimal value of ILP .

Pf. LP has fewer constraints.

Note. LP is not equivalent to weighted vertex cover.
(even if all weights are 1)

Q. How can solving LP help us find a low-weight vertex cover?
A. Solve LP and round fractional values.



Weighted vertex cover: LP rounding algorithm

Lemma. If x^* is optimal solution to LP , then $S = \{ i \in V : x_i^* \geq \frac{1}{2} \}$ is a vertex cover whose weight is at most twice the min possible weight.

Pf. [S is a vertex cover]

- Consider an edge $(i, j) \in E$.
- Since $x_i^* + x_j^* \geq 1$, either $x_i^* \geq \frac{1}{2}$ or $x_j^* \geq \frac{1}{2}$ (or both) $\Rightarrow (i, j)$ covered.

Pf. [S has desired cost]

- Let S^* be optimal vertex cover. Then

$$\sum_{i \in S^*} w_i \geq \sum_{i \in S} w_i x_i^* \geq \frac{1}{2} \sum_{i \in S} w_i$$

↑ ↑

LP is a relaxation $x_i^* \geq \frac{1}{2}$

Theorem. The rounding algorithm is a 2-approximation algorithm.

Pf. Lemma + fact that LP can be solved in poly-time.

Weighted vertex cover inapproximability

Theorem. [Dinur–Safra 2004] If $P \neq NP$, then no ρ -approximation for WEIGHTED-VERTEX-COVER for any $\rho < 1.3606$ (even if all weights are 1).

On the Hardness of Approximating Minimum Vertex Cover

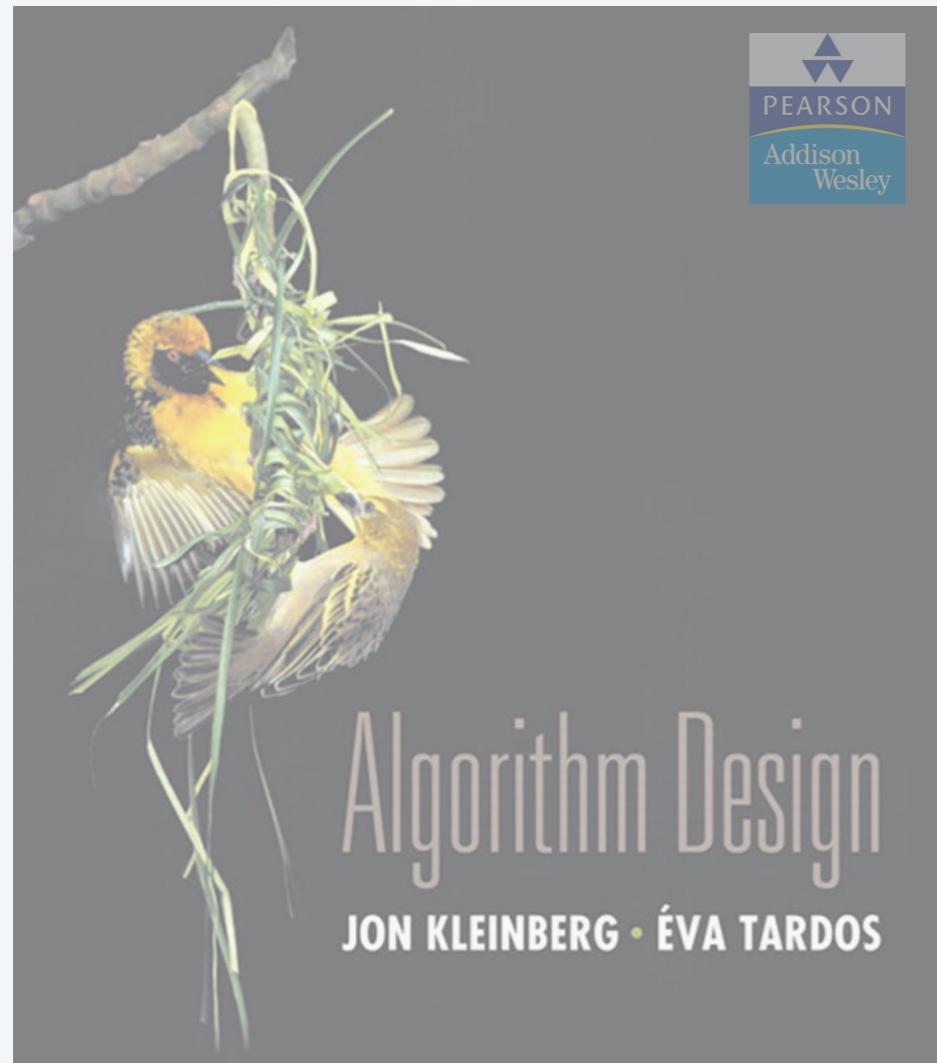
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Abstract

We prove the Minimum Vertex Cover problem to be NP-hard to approximate to within a factor of 1.3606, extending on previous PCP and hardness of approximation technique. To that end, one needs to develop a new proof framework, and borrow and extend ideas from several fields.

Open research problem. Close the gap.



SECTION 11.7

11. APPROXIMATION ALGORITHMS

- ▶ *load balancing*
- ▶ *center selection*
- ▶ *pricing method: weighted vertex cover*
- ▶ *LP rounding: weighted vertex cover*
- ▶ ***generalized load balancing***
- ▶ *knapsack problem*

Generalized load balancing

Input. Set of m machines M ; set of n jobs J .

- Job $j \in J$ must run contiguously on an **authorized machine** in $M_j \subseteq M$.
- Job $j \in J$ has processing time t_j .
- Each machine can process at most one job at a time.

Def. Let J_i be the subset of jobs assigned to machine i .

The load of machine i is $L_i = \sum_{j \in J_i} t_j$.

Def. The makespan is the maximum load on any machine = $\max_i L_i$.

Generalized load balancing. Assign each job to an authorized machine to minimize makespan.

Generalized load balancing: integer linear program and relaxation

ILP formulation. x_{ij} = time machine i spends processing job j .

$$(IP) \quad \begin{aligned} & \min \quad L \\ \text{s. t.} \quad & \sum_i x_{ij} = t_j \quad \text{for all } j \in J \\ & \sum_j x_{ij} \leq L \quad \text{for all } i \in M \\ & x_{ij} \in \{0, t_j\} \quad \text{for all } j \in J \text{ and } i \in M_j \\ & x_{ij} = 0 \quad \text{for all } j \in J \text{ and } i \notin M_j \end{aligned}$$

LP relaxation.

$$(LP) \quad \begin{aligned} & \min \quad L \\ \text{s. t.} \quad & \sum_i x_{ij} = t_j \quad \text{for all } j \in J \\ & \sum_j x_{ij} \leq L \quad \text{for all } i \in M \\ & x_{ij} \geq 0 \quad \text{for all } j \in J \text{ and } i \in M_j \\ & x_{ij} = 0 \quad \text{for all } j \in J \text{ and } i \notin M_j \end{aligned}$$

Generalized load balancing: lower bounds

Lemma 1. The optimal makespan $L^* \geq \max_j t_j$.

Pf. Some machine must process the most time-consuming job. ▀

Lemma 2. Let L be optimal value to the *LP*. Then, optimal makespan $L^* \geq L$.

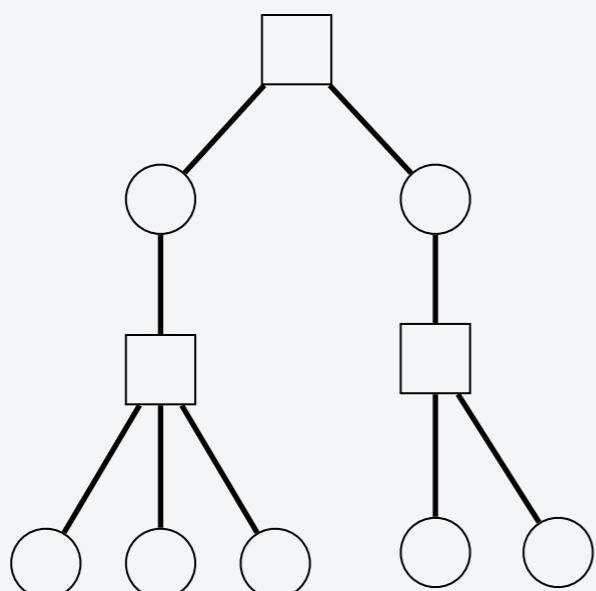
Pf. *LP* has fewer constraints than *ILP* formulation. ▀

Generalized load balancing: structure of LP solution

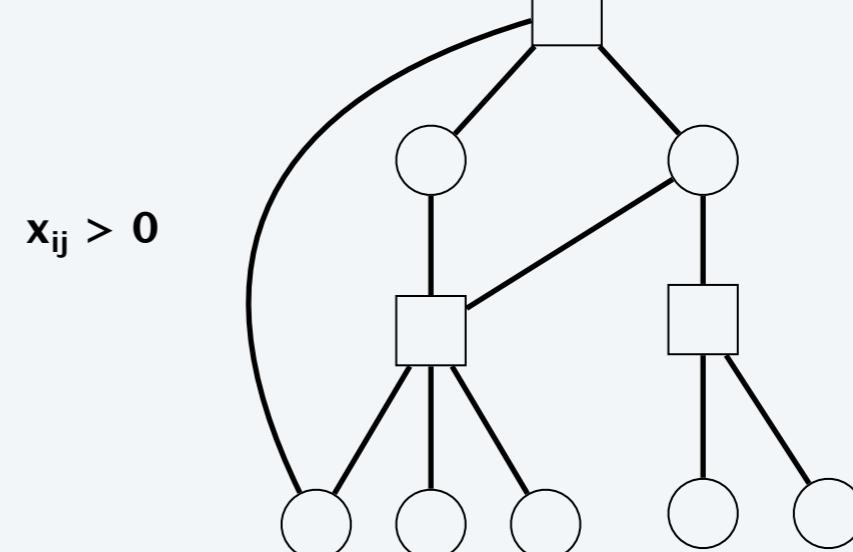
Lemma 3. Let x be solution to LP . Let $G(x)$ be the graph with an edge between machine i and job j if $x_{ij} > 0$. Then $G(x)$ is **acyclic**.

Pf. (deferred)

can transform x into another LP solution where $G(x)$ is acyclic if LP solver doesn't return such an x



$G(x)$ acyclic



$G(x)$ cyclic



job



machine

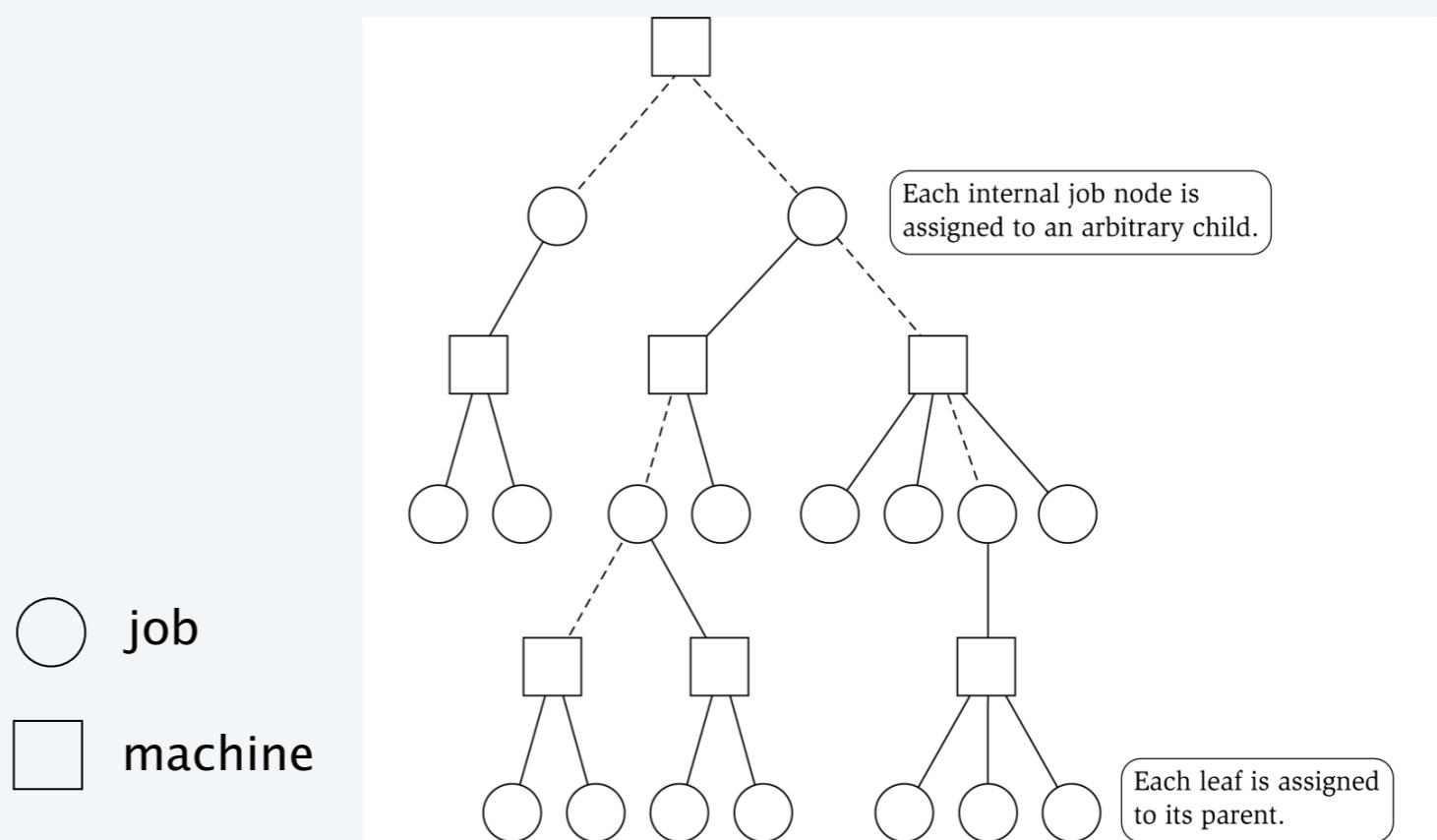
Generalized load balancing: rounding

Rounded solution. Find LP solution x where $G(x)$ is a forest. Root forest $G(x)$ at some arbitrary machine node r .

- If job j is a leaf node, assign j to its parent machine i .
- If job j is not a leaf node, assign j to any one of its children.

Lemma 4. Rounded solution only assigns jobs to authorized machines.

Pf. If job j is assigned to machine i , then $x_{ij} > 0$. LP solution can only assign positive value to authorized machines. ■



Generalized load balancing: analysis

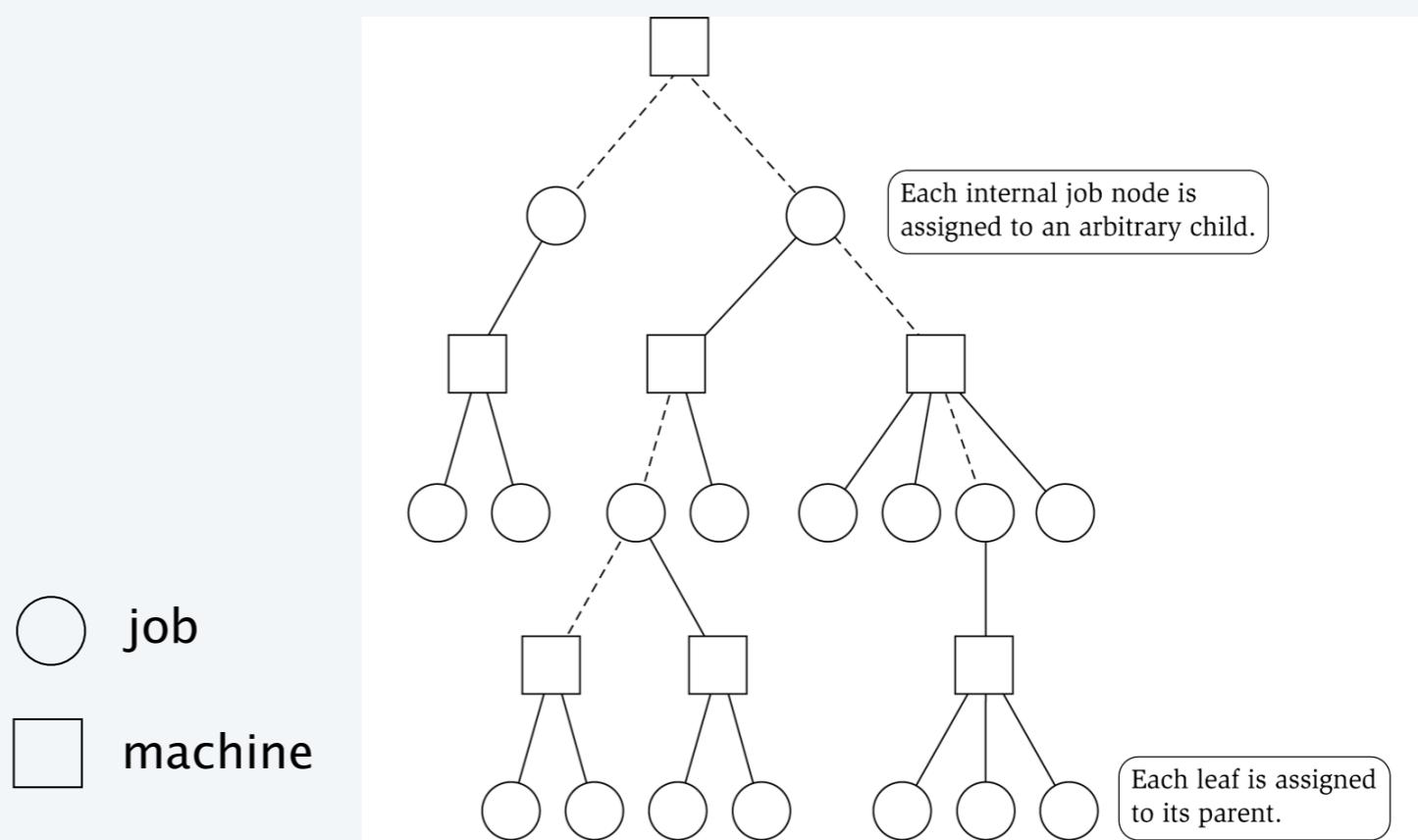
Lemma 5. If job j is a leaf node and machine $i = \text{parent}(j)$, then $x_{ij} = t_j$.

Pf.

- Since i is a leaf, $x_{ij} = 0$ for all $j \neq \text{parent}(i)$.
- LP constraint guarantees $\sum_i x_{ij} = t_j$. ■

Lemma 6. At most one non-leaf job is assigned to a machine.

Pf. The only possible non-leaf job assigned to machine i is $\text{parent}(i)$. ■



Generalized load balancing: analysis

Theorem. Rounded solution is a 2-approximation.

Pf.

- Let $J(i)$ be the jobs assigned to machine i .
- By LEMMA 6, the load L_i on machine i has two components:

- leaf nodes:

Lemma 5

$$\sum_{\substack{j \in J(i) \\ j \text{ is a leaf}}} t_j = \sum_{\substack{j \in J(i) \\ j \text{ is a leaf}}} x_{ij} \leq \sum_{j \in J} x_{ij} \stackrel{\text{LP}}{\leq} L \stackrel{\text{Lemma 2 (LP is a relaxation)}}{\leq} L^*$$

optimal value of LP

Lemma 1

- parent: $t_{\text{parent}(i)} \leq L^*$

- Thus, the overall load $L_i \leq 2L^*$. ■

Generalized load balancing: flow formulation

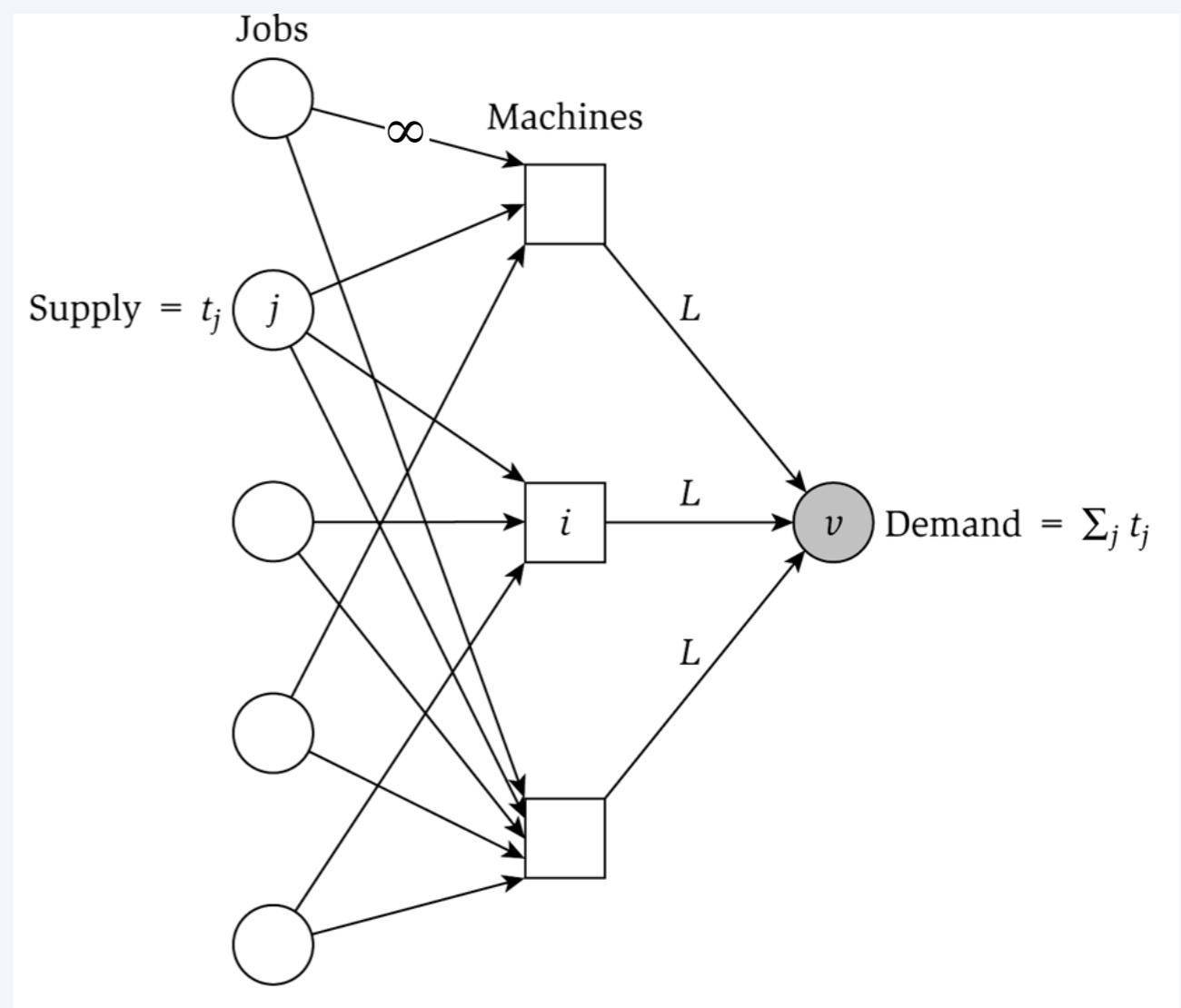
Flow formulation of *LP*.

$$\sum_i x_{ij} = t_j \quad \text{for all } j \in J$$

$$\sum_j x_{ij} \leq L \quad \text{for all } i \in M$$

$$x_{ij} \geq 0 \quad \text{for all } j \in J \text{ and } i \in M_j$$

$$x_{ij} = 0 \quad \text{for all } j \in J \text{ and } i \notin M_j$$



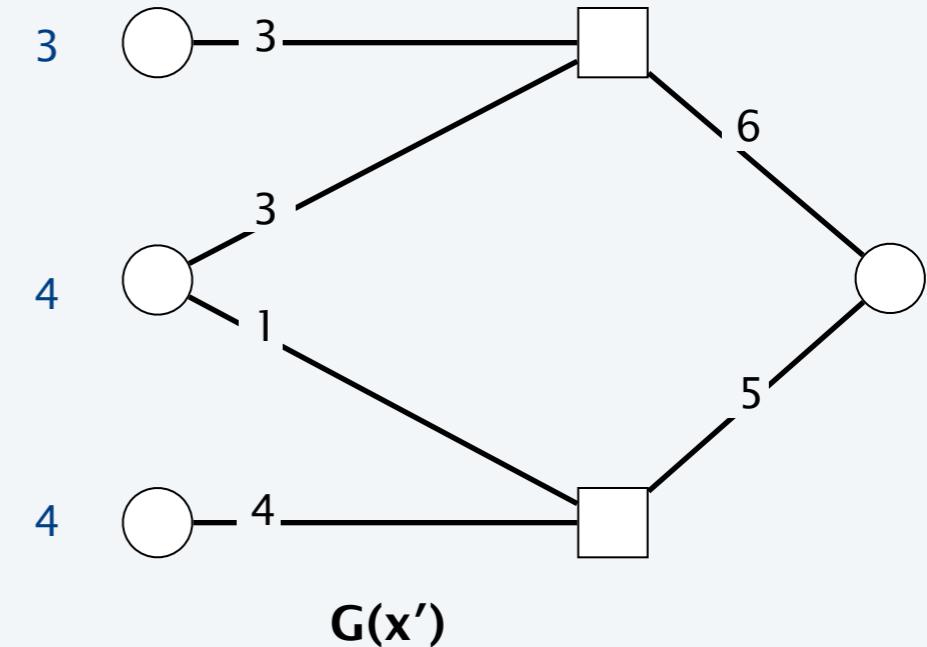
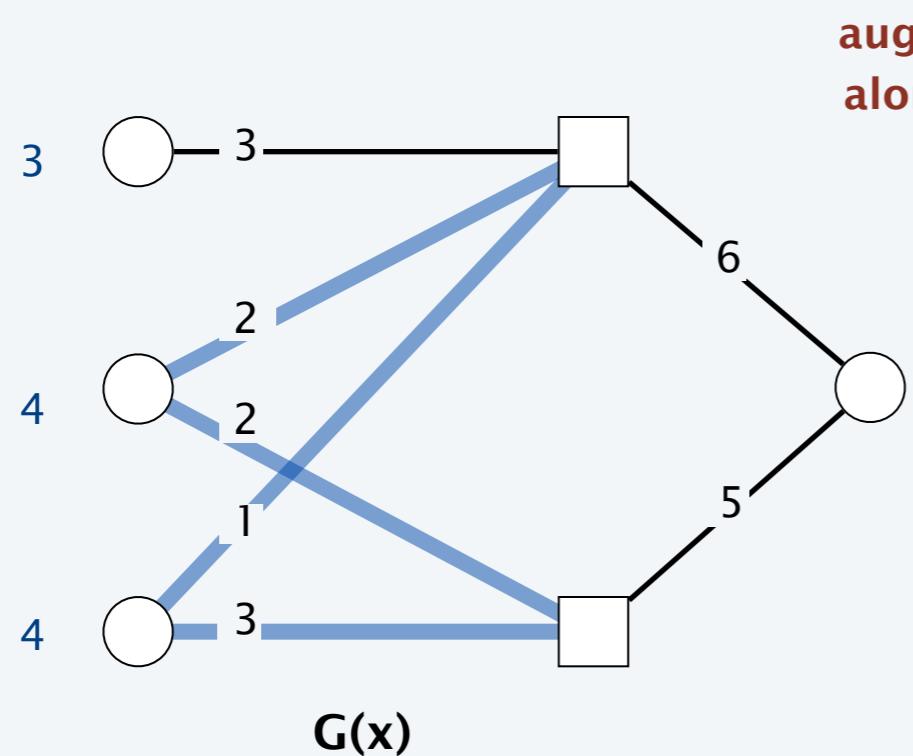
Observation. Solution to feasible flow problem with value L are in 1-to-1 correspondence with *LP* solutions of value L .

Generalized load balancing: structure of solution

Lemma 3. Let (x, L) be solution to LP . Let $G(x)$ be the graph with an edge from machine i to job j if $x_{ij} > 0$. We can find another solution (x', L) such that $G(x')$ is acyclic.

Pf. Let C be a cycle in $G(x)$.

- Augment flow along the cycle C . ← flow conservation maintained
- At least one edge from C is removed (and none are added).
- Repeat until $G(x')$ is acyclic. ■



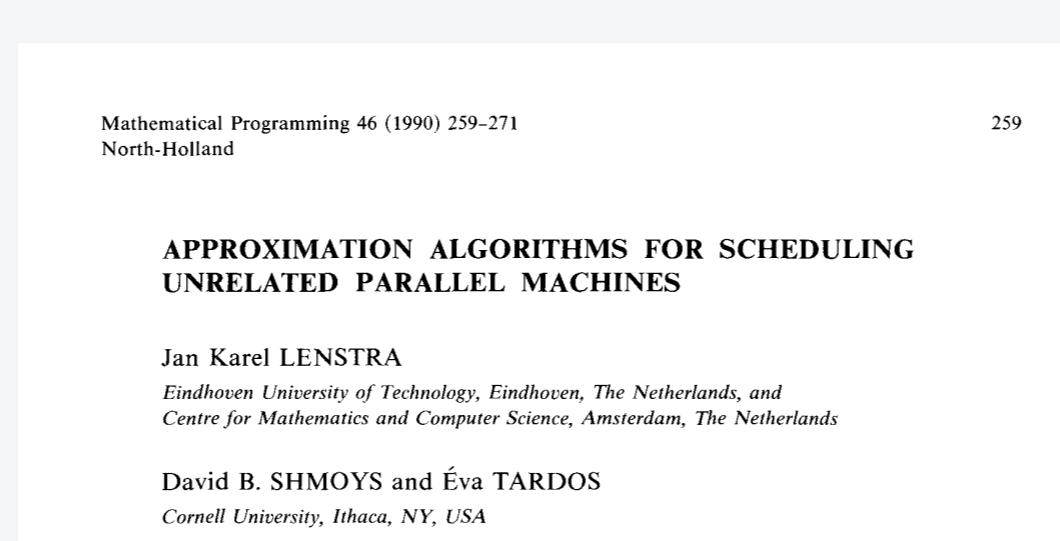
Conclusions

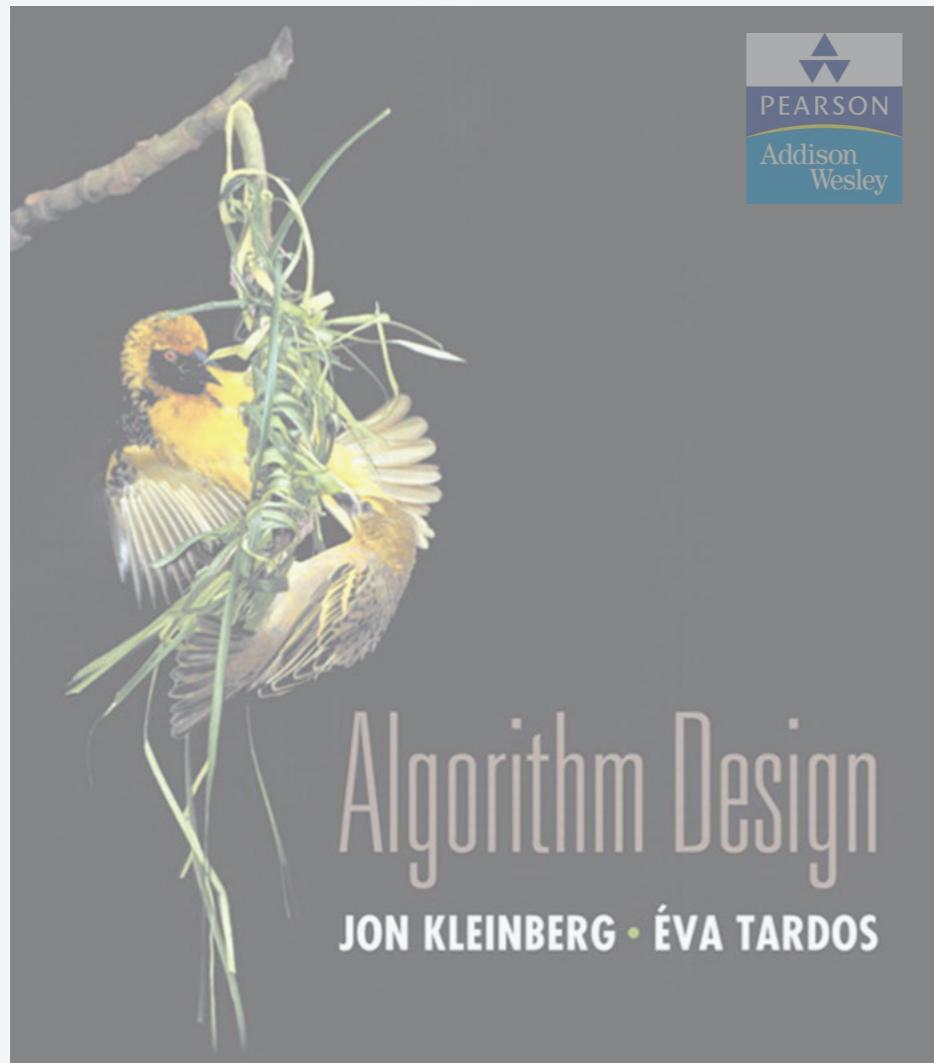
Running time. The bottleneck operation in our 2-approximation is solving one LP with $m n + 1$ variables.

Remark. Can solve LP using flow techniques on a graph with $m+n+1$ nodes: given L , find feasible flow if it exists. Binary search to find L^* .

Extensions: unrelated parallel machines. [Lenstra–Shmoys–Tardos 1990]

- Job j takes t_{ij} time if processed on machine i .
- 2-approximation algorithm via LP rounding.
- If $P \neq NP$, then no ρ -approximation exists for any $\rho < 3/2$.





SECTION 11.8

11. APPROXIMATION ALGORITHMS

- ▶ *load balancing*
- ▶ *center selection*
- ▶ *pricing method: weighted vertex cover*
- ▶ *LP rounding: weighted vertex cover*
- ▶ *generalized load balancing*
- ▶ ***knapsack problem***

Polynomial-time approximation scheme

PTAS. $(1 + \varepsilon)$ -approximation algorithm for any constant $\varepsilon > 0$.

- Load balancing. [Hochbaum–Shmoys 1987]
- Euclidean TSP. [Arora, Mitchell 1996]

Consequence. PTAS produces arbitrarily high quality solution, but trades off accuracy for time.

This section. PTAS for knapsack problem via rounding and scaling.

Knapsack problem

Knapsack problem.

- Given n objects and a knapsack.
- Item i has value $v_i > 0$ and weighs $w_i > 0$. ← we assume $w_i \leq W$ for each i
- Knapsack has weight limit W .
- Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

item	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

original instance ($W = 11$)

Knapsack is NP-complete

KNAPSACK. Given a set X , weights $w_i \geq 0$, values $v_i \geq 0$, a weight limit W , and a target value V , is there a subset $S \subseteq X$ such that:

$$\begin{aligned}\sum_{i \in S} w_i &\leq W \\ \sum_{i \in S} v_i &\geq V\end{aligned}$$

SUBSET-SUM. Given a set X , values $u_i \geq 0$, and an integer U , is there a subset $S \subseteq X$ whose elements sum to exactly U ?

Theorem. SUBSET-SUM \leq_P KNAPSACK.

Pf. Given instance (u_1, \dots, u_n, U) of SUBSET-SUM, create KNAPSACK instance:

$$\begin{aligned}v_i = w_i = u_i &\quad \sum_{i \in S} u_i \leq U \\ V = W = U &\quad \sum_{i \in S} u_i \geq U\end{aligned}$$

Knapsack problem: dynamic programming I

Def. $OPT(i, w)$ = max value subset of items $1, \dots, i$ with weight limit w .

Case 1. OPT does not select item i .

- OPT selects best of $1, \dots, i - 1$ using up to weight limit w .

Case 2. OPT selects item i .

- New weight limit = $w - w_i$.
- OPT selects best of $1, \dots, i - 1$ using up to weight limit $w - w_i$.

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max\{OPT(i-1, w), v_i + OPT(i-1, w-w_i)\} & \text{otherwise} \end{cases}$$

Theorem. Computes the optimal value in $O(nW)$ time.

- Not polynomial in input size.
- Polynomial in input size if weights are small integers.

Knapsack problem: dynamic programming II

Def. $OPT(i, v) = \min$ weight of a knapsack for which we can obtain a solution of value $\geq v$ using a subset of items $1, \dots, i$.

Note. Optimal value is the largest value v such that $OPT(n, v) \leq W$.

Case 1. OPT does not select item i .

- OPT selects best of $1, \dots, i-1$ that achieves value $\geq v$.

Case 2. OPT selects item i .

- Consumes weight w_i , need to achieve value $\geq v - v_i$.
- OPT selects best of $1, \dots, i-1$ that achieves value $\geq v - v_i$.

$$OPT(i, v) = \begin{cases} 0 & \text{if } v \leq 0 \\ \infty & \text{if } i = 0 \text{ and } v > 0 \\ \min \{OPT(i-1, v), w_i + OPT(i-1, v - v_i)\} & \text{otherwise} \end{cases}$$

Knapsack problem: dynamic programming II

Theorem. Dynamic programming algorithm II computes the optimal value in $O(n^2 v_{\max})$ time, where v_{\max} is the maximum of any value.

Pf.

- The optimal value $V^* \leq n v_{\max}$.
- There is one subproblem for each item and for each value $v \leq V^*$.
- It takes $O(1)$ time per subproblem. ▀

Remark 1. Not polynomial in input size!

Remark 2. Polynomial time if values are small integers.

Knapsack problem: polynomial-time approximation scheme

Intuition for approximation algorithm.

- Round all values up to lie in smaller range.
- Run dynamic programming algorithm II on rounded/scaled instance.
- Return optimal items in rounded instance.

item	value	weight
1	934221	1
2	5956342	2
3	17810013	5
4	21217800	6
5	27343199	7

original instance ($W = 11$)

item	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

rounded instance ($W = 11$)

Knapsack problem: polynomial-time approximation scheme

Round up all values:

- $0 < \varepsilon \leq 1$ = precision parameter.
- v_{\max} = largest value in original instance.
- θ = scaling factor = $\varepsilon v_{\max} / 2n$.

$$\bar{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil \theta, \quad \hat{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil$$

Observation. Optimal solutions to problem with \bar{v} are equivalent to optimal solutions to problem with \hat{v} .

Intuition. \bar{v} close to v so optimal solution using \bar{v} is nearly optimal; \hat{v} small and integral so dynamic programming algorithm II is fast.

Knapsack problem: polynomial-time approximation scheme

Theorem. If S is solution found by rounding algorithm and S^* is any other feasible solution, then $(1 + \epsilon) \sum_{i \in S} v_i \geq \sum_{i \in S^*} v_i$

Pf. Let S^* be any feasible solution satisfying weight constraint.

$$\begin{aligned}
 \sum_{i \in S^*} v_i &\leq \sum_{i \in S^*} \bar{v}_i && \text{always round up} \\
 &\leq \sum_{i \in S} \bar{v}_i && \text{solve rounded instance optimally} \\
 &\leq \sum_{i \in S} (v_i + \theta) && \text{never round up by more than } \theta \\
 &\leq \sum_{i \in S} v_i + n\theta && |S| \leq n \\
 &= \sum_{i \in S} v_i + \frac{1}{2} \epsilon v_{\max} && \theta = \epsilon v_{\max} / 2n \\
 &\leq (1 + \epsilon) \sum_{i \in S} v_i && v_{\max} \leq 2 \sum_{i \in S} v_i
 \end{aligned}$$

subset containing only the item of largest value



choosing $S^* = \{ \max \}$

$$\begin{aligned}
 v_{\max} &\leq \sum_{i \in S} v_i + \frac{1}{2} \epsilon v_{\max} \\
 &\leq \sum_{i \in S} v_i + \frac{1}{2} v_{\max} \\
 \text{thus} \quad v_{\max} &\leq 2 \sum_{i \in S} v_i
 \end{aligned}$$

Knapsack problem: polynomial-time approximation scheme

Theorem. For any $\varepsilon > 0$, the rounding algorithm computes a feasible solution whose value is within a $(1 + \varepsilon)$ factor of the optimum in $O(n^3 / \varepsilon)$ time.

Pf.

- We have already proved the accuracy bound.
- Dynamic program II running time is $O(n^2 \hat{v}_{\max})$, where

$$\hat{v}_{\max} = \left\lceil \frac{v_{\max}}{\theta} \right\rceil = \left\lceil \frac{2n}{\epsilon} \right\rceil$$