DECISION MAKING FOR A SINGLE SAMPLE

Probability and Statistics

8-1: Statistical Inference

- The field of statistical inference consists of those methods used to make decisions or draw conclusions about a **population.**
- These methods utilize the information contained in a **sample** from the population in drawing conclusions.

8-1: Statistical Inference

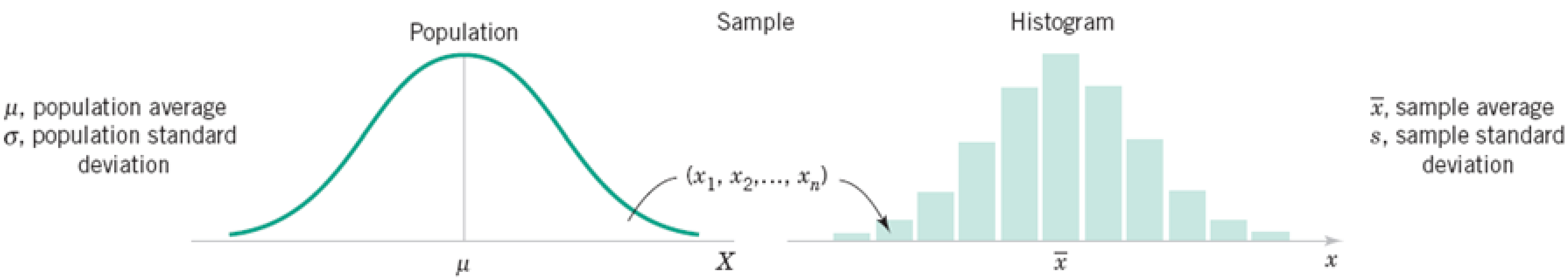


Figure 8-1 Relationship between a population and a sample.

8-2: Point Estimation

A **point estimate** of some population parameter θ is a single numerical value $\hat{\theta}$ of a statistic $\hat{\Theta}$.

Unknown Parameter θ	Statistic $\hat{\Theta}$	Point Estimate $\hat{\theta}$
μ	$\overline{X} = \dfrac{\sum X_i}{n}$	$\overline{x}$
σ^2	$S^2 = \dfrac{\sum(X_i - \overline{X})^2}{n-1}$	s^2
p	$\hat{p} = \dfrac{X}{n}$	$\hat{p}$
$\mu_1 - \mu_2$	$\overline{X}_1 - \overline{X}_2 = \dfrac{\sum X_{1i}}{n_1} - \dfrac{\sum X_{2i}}{n_2}$	$\overline{x}_1 - \overline{x}_2$
$p_1 - p_2$	$\hat{p}_1 - \hat{p}_2 = \dfrac{X_1}{n_1} - \dfrac{X_2}{n_2}$	$\hat{p}_1 - \hat{p}_2$

8-2: Point Estimation

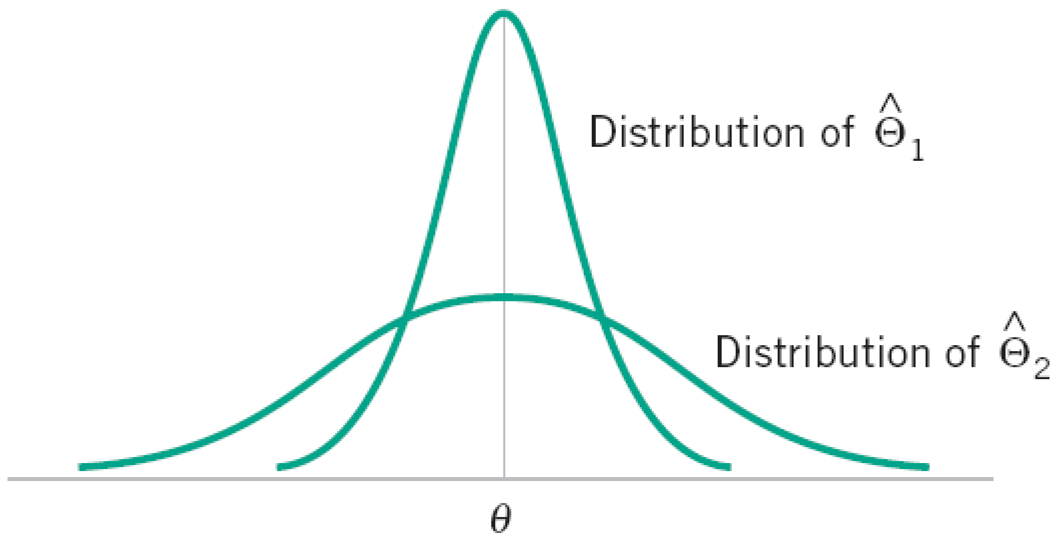


Figure 8-2 The sampling distribution of two unbiased estimators, $\hat{\Theta}_1$ and $\hat{\Theta}_2$.

8-2: Point Estimation

If we consider all unbiased estimators of θ, the one with the smallest variance is called the **minimum variance unbiased estimator** (MVUE).

The **mean square error** of an estimator $\hat{\Theta}$ of the parameter θ is defined as

$$\text{MSE}(\hat{\Theta}) = E(\hat{\Theta} - \theta)^2 \qquad (8\text{-}1)$$

The **standard error** of a statistic is the standard deviation of its sampling distribution. If the standard error involves unknown parameters whose values can be estimated, substitution of these estimates into the standard error results in an **estimated standard error.**

8-3: Hypothesis Testing

8-3.1 Statistical Hypotheses

We like to think of statistical hypothesis testing as the data analysis stage of a **comparative experiment,** in which the engineer is interested, for example, in comparing the mean of a population to a specified value (e.g. mean pull strength).

A **statistical hypothesis** is a statement about the parameters of one or more populations.

8-3: Hypothesis Testing

8-3.1 Statistical Hypotheses

For example, suppose that we are interested in the burning rate of a solid propellant used to power aircrew escape systems.

- Now burning rate is a random variable that can be described by a probability distribution.
- Suppose that our interest focuses on the **mean** burning rate (a parameter of this distribution).
- Specifically, we are interested in deciding whether or not the mean burning rate is 50 centimeters per second.

8-3: Hypothesis Testing

8-3.1 Statistical Hypotheses

Two-sided Alternative Hypothesis

$$H_0 : \mu = 50 cm/s$$
$$H_1 : \mu \neq 50 cm/s$$

One-sided Alternative Hypotheses

textbook style

$$H_0 : \mu = 50 cm/s$$
$$H_0 : \mu \geq 50 cm/s$$
$$H_1 : \mu < 50 cm/s$$

or

$$H_0 : \mu = 50 cm/s$$
$$H_0 : \mu \leq 50 cm/s$$
$$H_1 : \mu > 50 cm/s$$

8-3: Hypothesis Testing

8-3.1 Statistical Hypotheses

Test of a Hypothesis

- A procedure leading to a decision about a particular hypothesis
- Hypothesis-testing procedures rely on using the information in a **random sample from the population of interest.**
- If this information is *consistent* with the hypothesis, then we will conclude that the hypothesis is **true**; if this information is *inconsistent* with the hypothesis, we will conclude that the hypothesis is **false.**

8-3: Hypothesis Testing

8-3.2 Testing Statistical Hypotheses

$$H_0 : \mu = 50 cm/s$$

$$H_1 : \mu \neq 50 cm/s$$

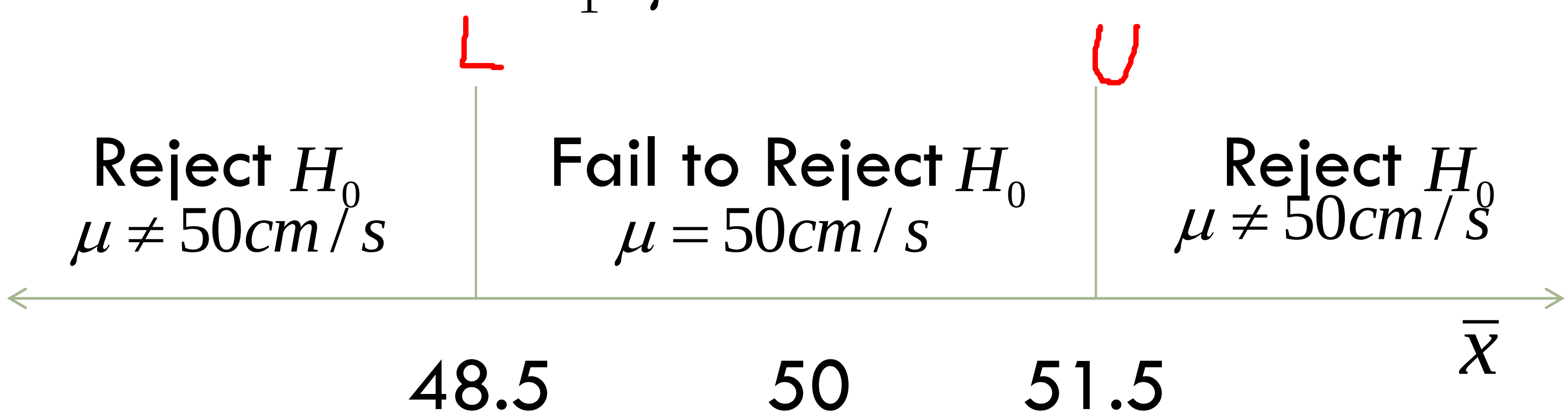


Figure 8-3 Decision criteria for testing $H_0 : \mu = 50 cm/s$ versus $H_1 : \mu \neq 50 cm/s$

8-3 Hypothesis Testing

8-3.2 Testing Statistical Hypotheses

Rejecting the null hypothesis H_0 when it is true is defined as a **type I error.**

Failing to reject the null hypothesis when it is false is defined as a **type II error.**

8-3: Hypothesis Testing

8-3.2 Testing Statistical Hypotheses

Table 8-1 Decisions in Hypothesis Testing

Decision	H_0 Is True	H_0 Is False
Fail to reject H_0	No error	Type II error
Reject H_0	Type I error	No error

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true})$$

Sometimes the type I error probability is called the **significance level**, or the **α-error**, or the **size** of the test.

8-3.2 Tests of Statistical Hypotheses

$\alpha = P(\overline{X} < 48.5$ when $\mu = 50) + P(\overline{X} > 51.5$ when $\mu = 50)$

The z-values that correspond to the critical values 48.5 and 51.5 are

$$z_1 = \frac{48.5 - 50}{0.79} = -1.90 \quad \text{and} \quad z_2 = \frac{51.5 - 50}{0.79} = 1.90$$

Therefore

$\alpha = P(Z < -1.90) + P(Z > 1.90) = 0.028717 + 0.028717 = 0.057434$

8-3: Hypothesis Testing

8-3.2 Testing Statistical Hypotheses

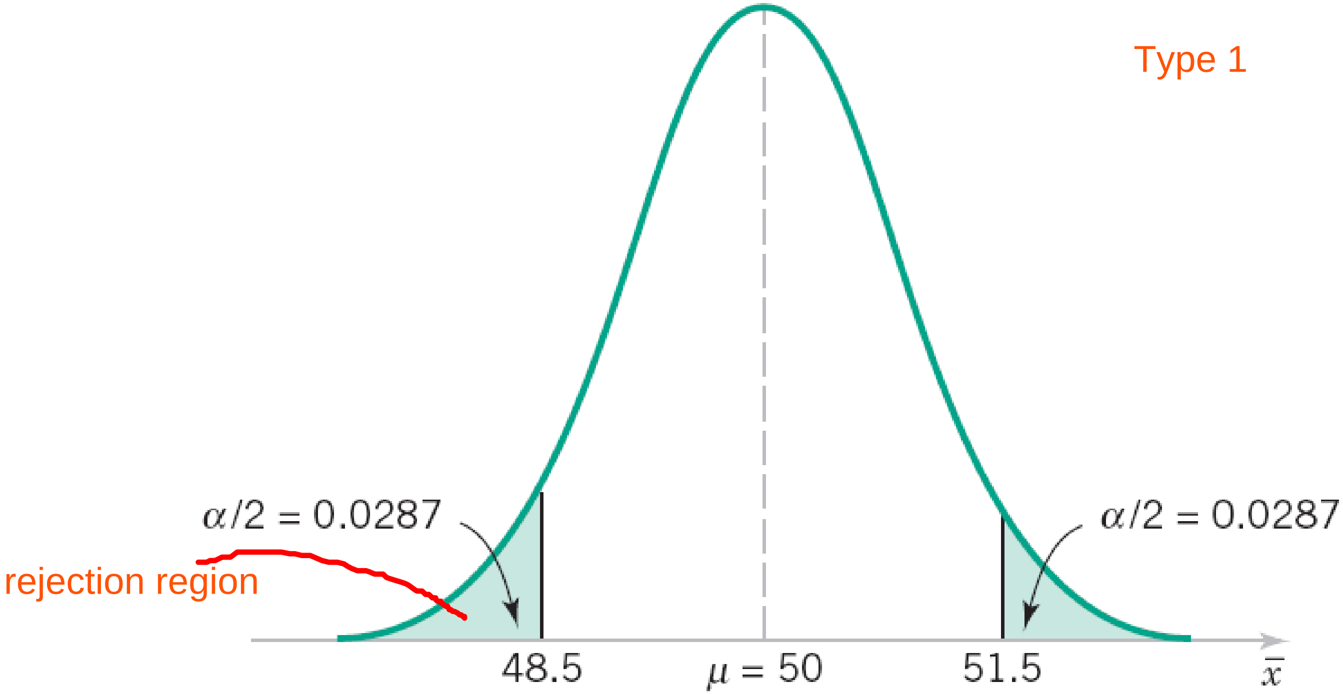


Figure 8-4 The critical region for $H_0: \mu = 50$ versus $H_1: \mu \neq 50$ and $n = 10$.

8-3: Hypothesis Testing

8-3.2 Testing Statistical Hypotheses

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false})$$

Probability density function Normal

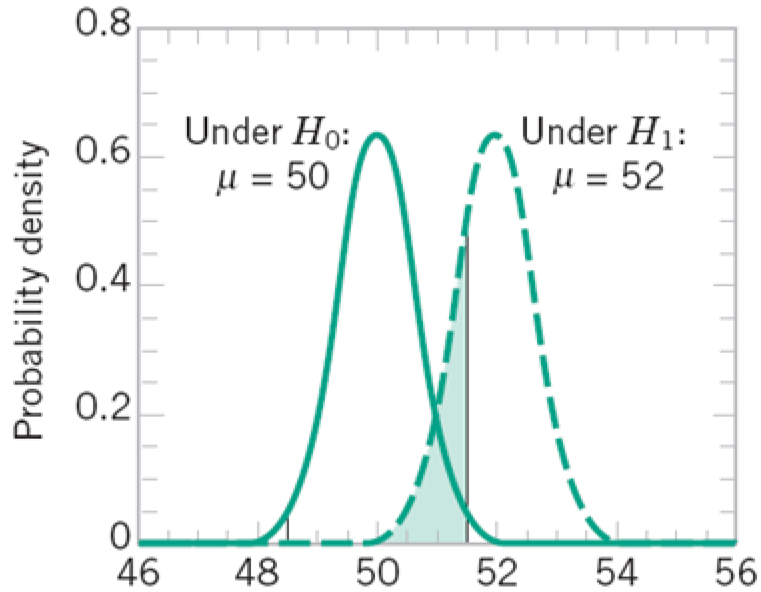


Figure 8-5 The probability of type II error when $\mu = 50$ and $n = 10$.

8-3: Hypothesis Testing

$$\beta = P(48.5 \leq \overline{X} \leq 51.5 \text{ when } \mu = 52)$$

The z-values corresponding to 48.5 and 51.5 when $\mu = 52$ are

$$z_1 = \frac{48.5 - 52}{0.79} = -4.43 \quad \text{and} \quad z_2 = \frac{51.5 - 52}{0.79} = -0.63$$

Therefore

$$\beta = P(-4.43 \leq Z \leq -0.63) = P(Z \leq -0.63) - P(Z \leq -4.43)$$

$$= 0.2643 - 0.0000 = 0.2643$$

8-3: Hypothesis Testing

$$\beta = P(48.5 \leq \overline{X} \leq 51.5 \text{ when } \mu = 50.5)$$

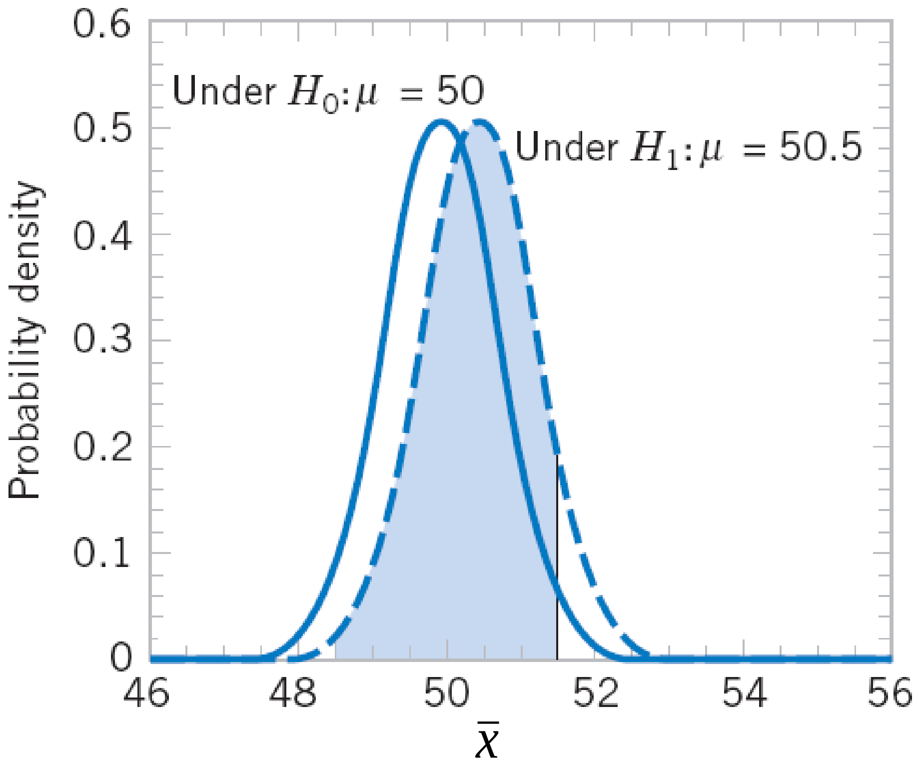


Figure 8-6 The probability of type II error when μ = 50.5 and n = 10.

8-3: Hypothesis Testing

$$\beta = P(48.5 \le \overline{X} \le 51.5 \text{ when } \mu = 50.5)$$

As shown in Fig. 8-4, the z-values corresponding to 48.5 and 51.5 when μ = 50.5 are

$$z_1 = \frac{48.5 - 50.5}{0.79} = -2.53 \quad \text{and} \quad z_2 = \frac{51.5 - 50.5}{0.79} = -1.27$$

Therefore

$$\beta = P(-2.53 \le Z \le -1.27) = P(Z \le -1.27) - P(Z \le -2.53)$$
$$= 0.8980 - 0.0057 = 0.8923$$

8-3: Hypothesis Testing

$$\beta = P(48.5 \le \bar{X} \le 51.5 \text{ when } \mu = 52)$$

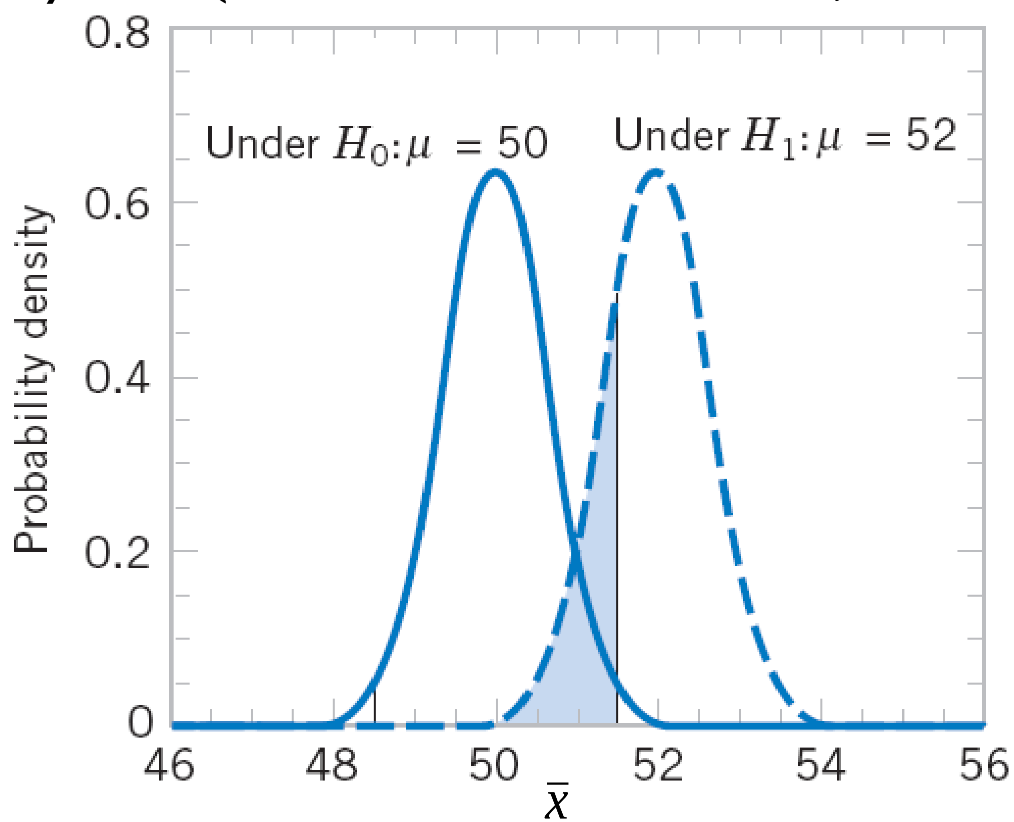


Figure 8-7 The probability of type II error when $\mu = 2$ and $n = 16$.

8-3: Hypothesis Testing

$$\beta = P(48.5 \le \overline{X} \le 51.5 \text{ when } \mu = 52)$$

When n = 16, the standard deviation of $\overline{X}$ is $\sigma / \sqrt{n} = 2.5 / \sqrt{16} = 0.625$, and the z-values corresponding to 48.5 and 51.5 when μ = 52 are

$$z_1 = \frac{48.5 - 52}{0.79} = -5.60 \quad \text{and} \quad z_2 = \frac{51.5 - 52}{0.79} = -0.80$$

Therefore

$$\beta = P(-5.60 \le Z \le -0.80) = P(Z \le -0.80) - P(Z \le -5.60)$$
$$= 0.2119 - 0.0000 = 0.2119$$

8-3 Hypothesis Testing

8-3.2 Testing Statistical Hypotheses

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false})$$

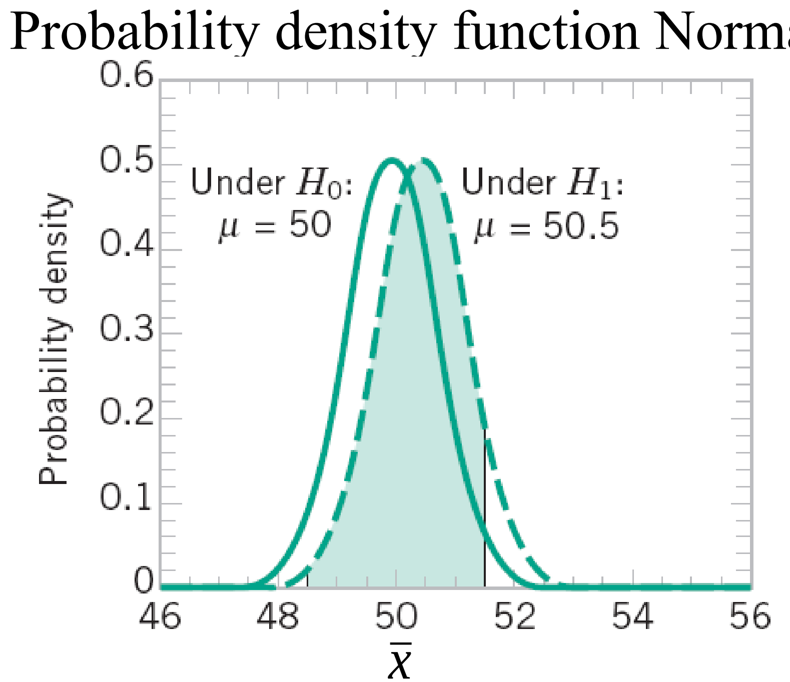


Figure 8-8 The probability of type II error when μ = 52 and μ = 16.

8-3: Hypothesis Testing

8-3.2 Testing Statistical Hypotheses

1. The size of the critical region, and consequently the probability of a type I error , can always be reduced by appropriate selection of the critical values.

has a relationship

Acceptance Region	Sample Size	α	β at μ = 52	β at μ = 50.5
$48.5 < \bar{x} < 51.5$	10	0.0574	0.2643	0.8923
$48 < \bar{x} < 52$	10	0.0114	0.5000	0.9705
$48.5 < \bar{x} < 51.5$	16	0.0164	0.2119	0.9445
$48.42 < \bar{x} < 51.58$	16	0.0114	0.2515	0.9578
$48.81 < \bar{x} < 51.19$	16	0.0574	0.0966	0.8606
$48 < \bar{x} < 52$	16	0.0014	0.5000	0.9918

8-3: Hypothesis Testing

8-3.2 Testing Statistical Hypotheses

2. Type I and type II errors are related. A decrease in the probability of one type of error always results in an increase in the probability of the other, provided that the sample size n does not change. FIX

Acceptance Region	Sample Size	α	β at μ = 52	β at μ = 50.5
$48.5 < \bar{x} < 51.5$	10	0.0574	0.2643	0.8923
$48 < \bar{x} < 52$	10	0.0114	0.5000	0.9705
$48.5 < \bar{x} < 51.5$	16	0.0164	0.2119	0.9445
$48.42 < \bar{x} < 51.58$	16	0.0114	0.2515	0.9578
$48.81 < \bar{x} < 51.19$	16	0.0574	0.0966	0.8606
$48 < \bar{x} < 52$	16	0.0014	0.5000	0.9918

8-3: Hypothesis Testing

8-3.2 Testing Statistical Hypotheses

3. An increase in sample size reduce *beta*, provided that *alpha* is held constant.

Acceptance Region	Sample Size	α	β at μ = 52	β at μ = 50.5
$48.5 < \bar{x} < 51.5$	10	0.0574	0.2643	0.8923
$48 < \bar{x} < 52$	10	0.0114	0.5000	0.9705
$48.5 < \bar{x} < 51.5$	16	0.0164	0.2119	0.9445
$48.42 < \bar{x} < 51.58$	16	0.0114	0.2515	0.9578
$48.81 < \bar{x} < 51.19$	16	0.0574	0.0966	0.8606
$48 < \bar{x} < 52$	16	0.0014	0.5000	0.9918

8-3: Hypothesis Testing

8-3.2 Testing Statistical Hypotheses

4. When the null hypothesis is false, *beta* increases as the true value of the parameter approaches the value hypothesized in the null hypothesis. The value of *beta* decreases as the difference between the true mean and the hypothesized value increases.

Acceptance Region	Sample Size	α at μ = 50	β at μ = 52	β at μ = 50.5
$48.5 < \bar{x} < 51.5$	10	0.0574	0.2643	0.8923
$48 < \bar{x} < 52$	10	0.0114	0.5000	0.9705
$48.5 < \bar{x} < 51.5$	16	0.0164	0.2119	0.9445
$48.42 < \bar{x} < 51.58$	16	0.0114	0.2515	0.9578
$48.81 < \bar{x} < 51.19$	16	0.0574	0.0966	0.8606
$48 < \bar{x} < 52$	16	0.0014	0.5000	0.9918

8-3: Hypothesis Testing

8-3.2 Testing Statistical Hypotheses

The **power** of a statistical test is the probability of rejecting the null hypothesis H_0 when the alternative hypothesis is true.

- The power is computed as 1 - b, and power can be interpreted as *the probability of correctly rejecting a false null hypothesis.* We often compare statistical tests by comparing their **power** properties.
- For example, consider the propellant burning rate problem when we are testing H_0 : μ= 50 centimeters per second against H_1 : μ not equal 50 centimeters per second . Suppose that the true value of the mean is μ = 52. When n = 10, we found that b = 0.2643, so the power of this test is 1 - b = 1 - 0.2643 = 0.7357 when m = 52.

8-3.3 One-Sided and Two-Sided Hypotheses

Two-Sided Test:

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0 \qquad (8\text{-}2)$$

One-Sided Tests:

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0 \qquad (8\text{-}3)$$

or

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0 \qquad (8\text{-}4)$$

8-3: Hypothesis Testing

1 Side

Example 8-1: Propellant Burning Rate

$\bar{x}$

Consider the propellant burning rate problem. Suppose that if the burning rate is less than 50 centimeters per second, we wish to show this with a strong conclusion. The hypotheses should be stated as

$$H_0: \mu = 50 \quad \text{centimeters per second}$$
$$H_1: \mu < 50 \quad \text{centimeters per second}$$

Here the critical region lies in the lower tail of the distribution of $\bar{X}$. Since the rejection of H_0 is always a strong conclusion, this statement of the hypotheses will produce the desired outcome if H_0 is rejected. Notice that, although the null hypothesis is stated with an equals sign, it is understood to include any value of μ not specified by the alternative hypothesis. Therefore, failing to reject H_0 does not mean that μ = 50 centimeters per second exactly, but only that we do not have strong evidence in support of H_1.

8-3: Hypothesis Testing

8-3.3 One-Sided and Two-Sided Hypotheses

The ***P*-value** is the smallest level of significance that would lead to rejection of the null Hypothesis H_0.

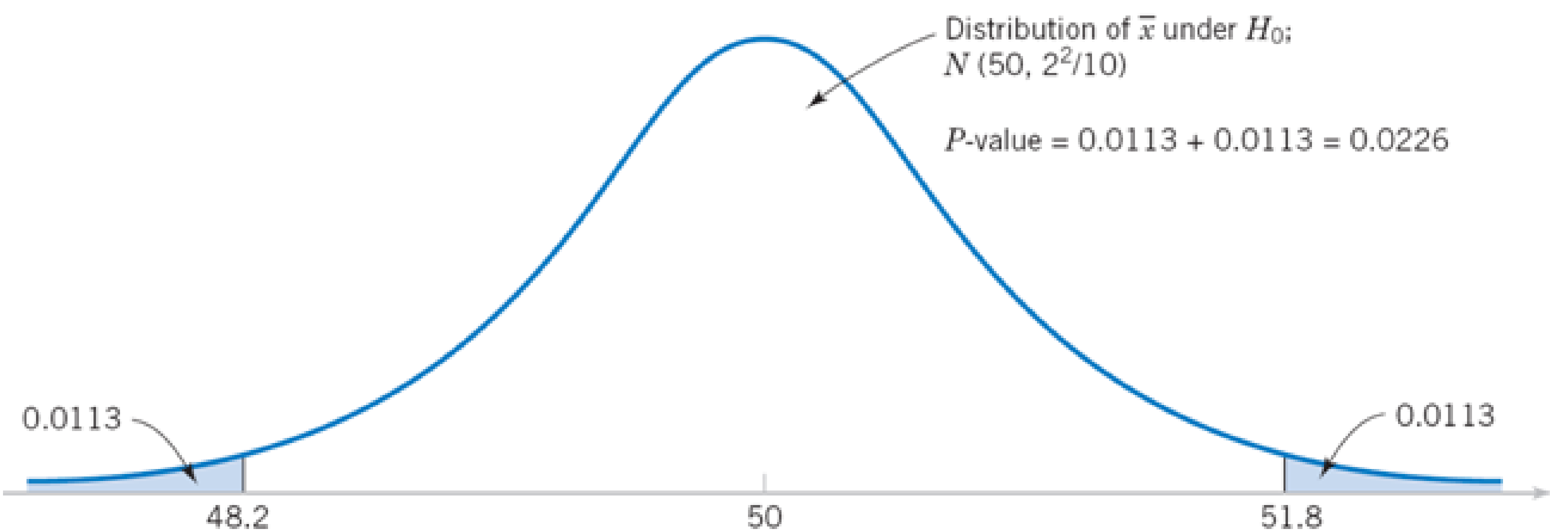


Figure 8-9 Calculation the P-value for the propellant burning rate problem.

8-3: Hypothesis Testing

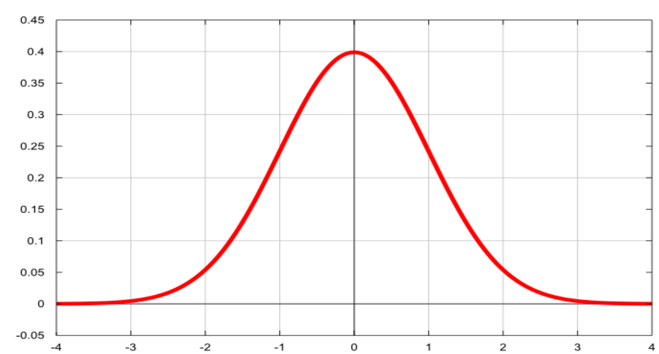
2 side

8-3.4 P-Values in Hypothesis Tests

Consider the two-sided hypothesis test for burning rate

$$H_0: \mu = 50 \qquad H_1: \mu \neq 50$$

with n = 16 and σ = 2.5. Suppose that the observed sample mean is $\bar{X}$ = 51.3 centimeters per second. A critical region for this test with critical values at 51.3 and the symmetric value 48.7. The P-value of the test is the α associated with this critical region. Any smaller value for α expands the critical region and the test fails to reject the null hypothesis when $\bar{X}$ = 51.3. The P-value is easy to compute after the test statistic is observed. In this example.



$$\begin{aligned}
P-value &= 1 - P(48.7 < \bar{X} < 51.3) \\
&= 1 - P\left(\frac{48.7-50}{2.5/\sqrt{16}} < Z < \frac{51.3-50}{2.5/\sqrt{16}}\right) \\
&= 1 - P(-2.08 < Z < 2.08) \\
&= 1 - 0.962 = 0.038
\end{aligned}$$

8-3: Hypothesis Testing

8-3.4 P-Values in Hypothesis Tests

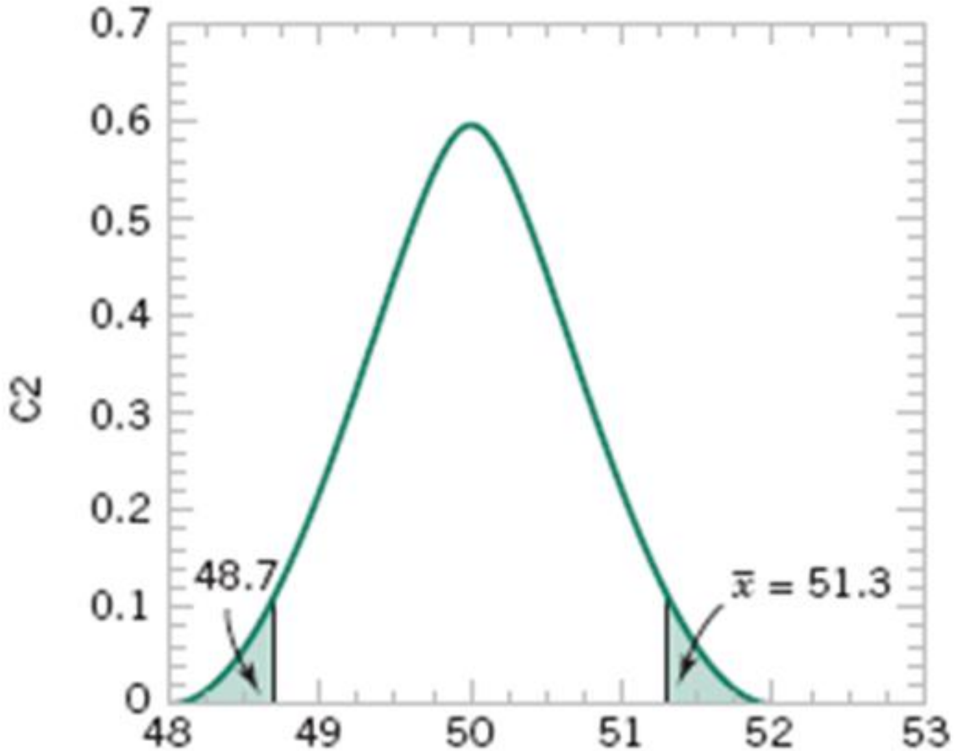


Figure 8-10 P-value is area of shaded region when $\bar{x} = 51.3$.

Imp. in final exam
You need to have at least 6 points

8-3: Hypothesis Testing

8-3.5 General Procedure for Hypothesis Testing

1. Parameter of interest: From the problem context, identify the parameter of interest.
2. Null hypothesis, H_0: State the null hypothesis, H_0.
3. Alternative hypothesis, H_1: Specify an appropriate alternative hypothesis, H_1.
4. Test statistic : State an appropriate test statistic.
5. Reject H_0 if: Define the criteria that will lead to rejection of H_0 .
6. Computation: Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute that value.
7. Conclusions: Decide whether or not H_0 should be rejected and report that in the problem context. This could involve computing a P-value or comparing the test statistic to a set of critical values.

Steps 1-4 should be completed prior to examination of the sample data.

8-4: Inference on the **Mean** of a Population, **Variance Known**

Assumptions

1. $X_1, X_2, \ldots, X_n$ is a random sample of size n from a population.
2. The population is normally distributed, or if it is not, the conditions of the central limit theorem apply.

Under the previous assumptions, the quantity

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \tag{8-5}$$

has a standard normal distribution, $N(0, 1)$.

8-4: Inference on the **Mean** of a Population, **Variance Known**

8-4.1 Hypothesis Testing on the Mean

We wish to test:

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu \neq \mu_0$$

The **test statistic** is:

$$Z_0 = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}} \qquad (8\text{-}6)$$

8-4: Inference on the **Mean** of a Population, **Variance Known**

8-4.1 Hypothesis Testing on the Mean

Reject H_0 if the observed value of the test statistic z_0 is either:

$$z_0 > z_{\alpha/2} \quad \text{or} \quad z_0 < -z_{\alpha/2}$$

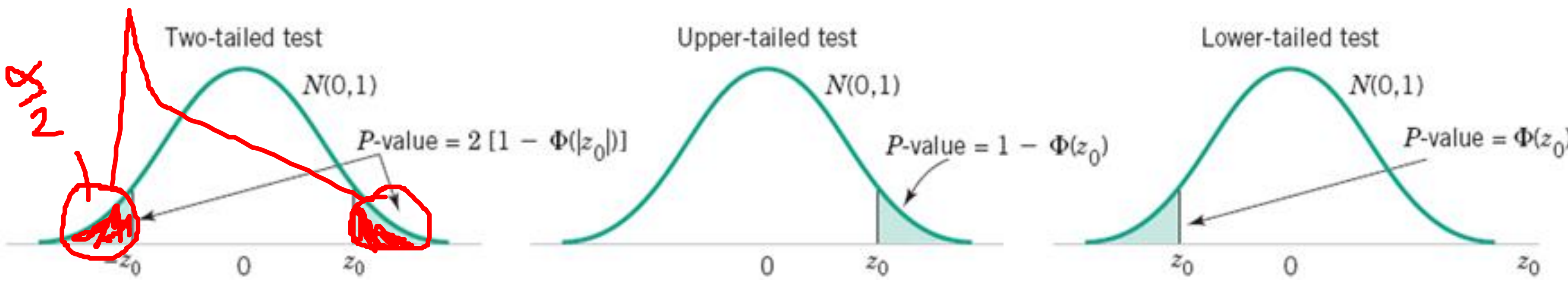
Fail to reject H_0 if

$$-z_{\alpha/2} \leq z_0 \leq z_{\alpha/2}$$

8-4: Inference on the **Mean** of a Population, **Variance Known**

8-4.1 Hypothesis Testing on the Mean

Reject if Z0 fall into these region then it is rejected



(a) (b) (c)

Figure 8-11 The P-value for a z-test. (a) The-sided alternation $H_1: \mu \neq \mu_0$. (b) The one-sided alternative $H_1: \mu > \mu_0$. (c) The one – sided alterative $H_1: \mu < \mu_0$.

8-4: Inference on the **Mean** of a Population, **Variance Known**

Imp. Final

Example 8-2: Propellant Burning Rate (2-sided)

Aircrew escape systems are powered by a solid propellant. The burning rate of this propellant is an important product characteristic. Specification require tat the mean burning rate must be 50 centimeters per second. We know that the standard deviation of burning rate is σ =2 centimeters per second. The experimenter decides to specify a type I error probability or significance level of α =0.05 and selects a random sample of n=25 and obtains a sample average burning rate of $\overline{x}$ = 51.3 centimeters per second. What conclusion should be drawn?

We may solve this program by following the seven-step procedure outlined in Section 8-3.5. This results in

1. **Parameter of interest:** The parameter of interest is μ, the mean burning rate.
2. **Null hypothesis,** $H_0 : \cancel{H_0} : \mu = 50$ centimeters per second
3. **Alternative hypothesis,** $H_1 : \cancel{H_1} : \mu \neq 50$ centimeters per second

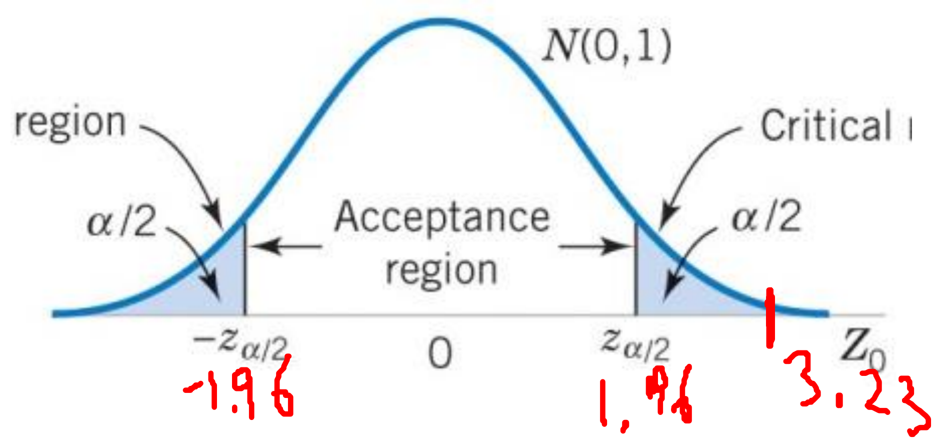
8-4: Inference on the **Mean** of a Population, **Variance Known**

Example 8-2: Propellant Burning Rate

4. **Test statistic :** The test statistic is $z_0 = \dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} = \dfrac{51.3-50}{2/\sqrt{25}} = 3.25$, since $\bar{x}=51.3$ and $\sigma = 2$

5. **Rejection Criteria:** Reject H_0 if the P-value is less than 0.05.

To use a fixed significance level test, the boundaries of the critical region would be $z_o < -z_{\alpha/2}$ and $z_o > z_{\alpha/2}$



8-4: Inference on the **Mean** of a Population, **Variance Known**

Example 8-2: Propellant Burning Rate

Reject / NOT Reject

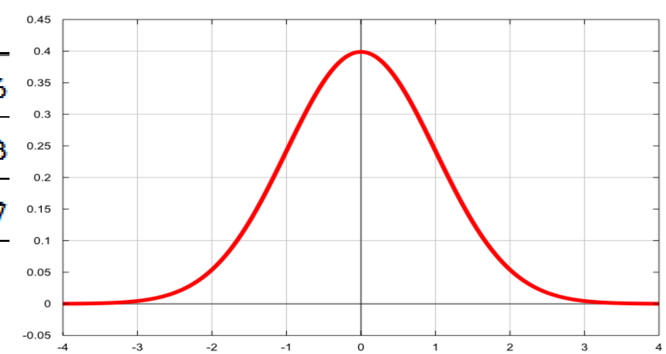
6. **Decision:** Reject $H_0: \mu = 50$ at the 0.05 level of significance

p-value: Since $z_0 = 3.25$,the P-value is $P-value = 2[1-\Phi(3.25)] = \cancel{0.0012}$

0.00116

Fixed level test: $z_0 = 3.25 > z_{0.025} = 1.96$

z - LT	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558
2	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077



7. **Conclusion :** We conclude that the mean burning rate differs from 50 centimeters per second, based on a sample of 25 measurements. In fact, there is strong evidence that the mean burning rate ~~exceeds~~ 50 centimeters per second.

is not equal to

8-4: Inference on the **Mean** of a Population, **Variance Known**

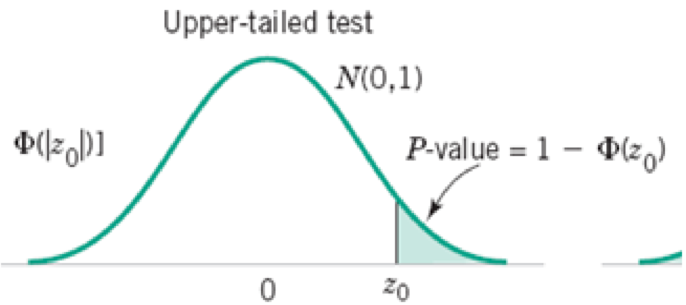
8-4.1 Hypothesis Tests on the Mean (Upper Tailed)

New let's consider the one-sided alternatives. Suppose that we are testing

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$

Once again, suppose that we have a random sample of size n and that the sample mean is $\overline{x}$. We compute the test statistic from Equation 9-8 and obtain z_0. Because the test is an upper-tailed test, only values of $\overline{x}$ that are greater than μ_0 are consistent with the alternative hypothesis.

Therefore, the P-value would be the probability that the standard normal random variable is greater than the value of the statistic z_0. This P-value is computed as



$$P = 1 - \Phi(z_0)$$

8-4: Inference on the **Mean** of a Population, **Variance Known**

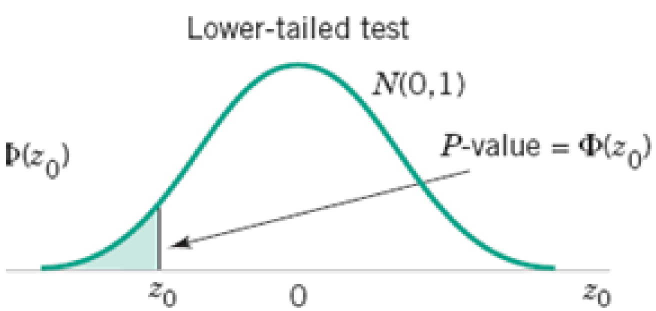
8-4.1 Hypothesis Tests on the Mean (Lower Tailed)

The lower-tailed test involves the hypotheses

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu < \mu_0$$

Similarly, to test the lower-tailed case, we would calculate the test statistic Z_0 and reject H_0 if the value of Z_0 is too small. That is in the lower tail of the standard normal distribution as in Fig. 8-11(c), and we reject H_0 if

$$z_0 < -z_\alpha$$



8-4: Inference on the **Mean** of a Population, **Variance Known**

8-4.1 Hypothesis Testing on the Mean

Testing Hypotheses on the Mean, Variance Known (*z*-Test)

Always = sign

Null hypothesis: $H_0\colon \mu = \mu_0$

Test statistic: $$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$$

Alternative Hypotheses	*P*-Value	Rejection Criterion for Fixed-Level Tests
$H_1\colon \mu \neq \mu_0$	Probability above $\lvert z_0\rvert$ and probability below $-\lvert z_0\rvert$, $P = 2[1-\Phi(\lvert z_0\rvert)]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1\colon \mu > \mu_0$	Probability above z_0, $P = 1-\Phi(z_0)$	$z_0 > z_\alpha$
$H_1\colon \mu < \mu_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

The *P*-values and critical regions for these situations are shown in Figs. 4-9 and 4-10.

8-4: Inference on the **Mean** of a Population, **Variance Known**

P-Values in Hypothesis Tests

The ***P*-value** is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.

$$P = \begin{cases} 2[1-\Phi(|z_0|)] & \text{for a two-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu \neq \mu_0 \\ 1-\Phi(z_0) & \text{for a upper-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu > \mu_0 \\ \Phi(z_0) & \text{for a lower-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu < \mu_0 \end{cases}$$

8-4: Inference on the **Mean** of a Population, **Variance Known**

- Repeat **Example 8-2 Propellant Burning Rate with** Z_0 criteria