CS 798: Algorithmic Spectral Graph Theory **Last Updated:** Sep 30, 2015
Homework 1 **Due Date: Oct 21, 2015 (in class)**

You are allowed to discuss with others but not allowed to use any references other than the course notes. Please list your collaborators for each question. This will not affect your marks. In any case, you must write your own solutions.

There are totally 80 marks (not including the bonus), and the full mark is 70. This homework is counted 20% of the course.

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1. **Cycles**

   (10 marks) Compute the Laplacian spectrum of C_n (the cycle with n vertices). Use it to conclude that the improved Cheeger's inequality is tight up to a constant factor for any $k \geq 2$.

2. **Hypercubes**

   (15 marks + bonus) A hypercube of n-dimension is an undirected graph with 2^n vertices. Each vertex corresponds to a string of n bits. Two vertices have an edge if and only if their corresponding strings differ by exactly one bit.

   (a) Given two undirected graphs $G = (V, E)$ and $H = (U, F)$, we define $G \times H$ as the undirected graph with vertex set $V \times U$ and two vertices (v_1, u_1), (v_2, u_2) have an edge if and only if either (1) $v_1 = v_2$ and $u_1 u_2 \in F$ or (2) $u_1 = u_2$ and $v_1 v_2 \in E$. Let x be an eigenvector of the Laplacian of G with eigenvalue α, and let y be an eigenvector of the Laplacian of H with eigenvalue β. Prove that we can use x and y to construct an eigenvector of the Laplacian of $G \times H$ with eigenvalue $\alpha + \beta$.

   (b) Use (a) to compute the spectrum of the hypercube of n dimension.

   (c) (bonus 10 marks) Show that the spectral partitioning algorithm may return a set S of conductance $\Omega(\sqrt{\phi(S^*)})$ in a hypercube, where S^* is the set with minimum conductance in the hypercube and $\phi(S^*)$ is the conductance of S^*.

3. **Bipartite Graphs**

   (10 marks) Consider the adjacency matrix A of a connected undirected graph G. Let $\alpha_1 \geq \ldots \geq \alpha_n$ be the eigenvalues of A. Prove that $\alpha_1 = -\alpha_n$ if and only if G is bipartite.

   You may use the Perron-Frobenius theorem to assume that all the entries of the first eigenvector is positive.

4. **Spanning Trees**

   (15 marks) Let $G = (V, E)$ be an undirected graph.

   (a) Let $V = \{1, \ldots, n\}$, $e = ij$, and b_e be the n-dimensional vector with $+1$ in the i-th entry and -1 in the j-th entry and 0 otherwise. Let B be an $n \times m$ matrix where the columns are b_e and m is the number of edges in G. Prove that the determinant of any $(n-1) \times (n-1)$ submatrix of B is ± 1 if and only if the $n-1$ edges corresponding to the columns form a spanning tree of G.

(b) Let L be the Laplacian matrix of G and let L' be the matrix obtained from L by deleting the last row and last column. Use (a) to prove that $\det(L')$ is equal to the number of spanning trees in G. You can use the Cauchy-Binet formula (see wikipedia) to solve this problem.

5. **Local Cheeger's Inequality**

(10 marks) In this question, we study the relation between "local eigenvalues" and "local conductance". Let $G = (V, E)$ be an undirected d-regular graph and $\mathcal{L}$ be its normalized Laplacian matrix. Let $S \subseteq V$ be a subset of vertices with $|S| \leq |V|/2$.

First we define local eigenvalues. Let $\mathcal{L}_S$ be the $|S| \times |S|$ submatrix of $\mathcal{L}$ with rows and columns restricted to those indexed by vertices in S. Let λ_S be the smallest eigenvalue of $\mathcal{L}_S$. We say λ_S is the smallest local eigenvalue of S.

Next we define local conductance. Let $\phi(S)$ be the conductance of S in G, and let $\phi^*(S) = \min_{S' \subseteq S} \phi(S')$. We say $\phi^*(S)$ is the local conductance of S.

Prove that $\phi^*(S) \geq \lambda_S \geq (\phi^*(S))^2/2$.

6. **Bipartiteness Ratio**

(10 marks) Let $G = (V, E)$ be a general (non-regular) undirected graph, and $\mathcal{A} = D^{-1/2} A D^{-1/2}$ be its normalized adjacency matrix. Let

$$\beta_n = \min_x \frac{x^T (I + \mathcal{A}) x}{x^T x}$$

as defined in L04 and let

$$\beta'(G) = \min_{y \in \{-1,0,+1\}^n} \frac{\sum_{ij \in E} (y_i + y_j)^2}{\sum_{i \in V} d_i y_i^2}.$$

Prove that $\beta_n \leq \beta'(G) \leq \sqrt{2\beta_n}$.

7. **Higher-Order Cheeger's Inequality**

(10 marks) Let $G = (V, E)$ be a d-regular undirected graph and $\mathcal{L}$ be its normalized Laplacian matrix with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n \leq 2$. Let $\psi_1, \psi_2, \ldots, \psi_k : V \to \mathbb{R}^n$ be disjointly supported vectors, i.e. $\text{supp}(\psi_i) \cap \text{supp}(\psi_j) = \emptyset$ for $1 \leq i \neq j \leq k$ where $\text{supp}(\psi_i) := \{v \mid x_v \neq 0\}$.

Prove that $\lambda_k \leq 2 \max_i R(\psi_i)$ where the Rayleigh quotient $R(x)$ of a vector $x \in \mathbb{R}^n$ is defined as $R(x) := x^T \mathcal{L} x / x^T x$.