CS 798 - Algorithmic Spectral Graph Theory, Fall 2015, Waterloo

Lecture 10: Expander graphs

We will see a deterministic construction of expander graphs, and then we will discuss some properties and various applications of expander graphs.

Expander graphs

There are different reasonable ways to define expander graphs.

- Spectrally, expander graphs are graphs with a large spectral gap, i.e. $\lambda_1 - \lambda_2$ is large.
- Combinatorially, expander graphs are graphs with very good connectivity, i.e. $|\delta(S)|/|S|$ is large for all $S \subseteq V$ (edge expansion), or $|N(S)|/|S|$ is large for all $S \subseteq V$ (vertex expansion).
- Probabilistically, expander graphs are graphs for which the random walks mix very rapidly.

These definitions are closely related.

Cheeger's inequality shows that a graph has a large spectral gap if and only if its edge expansion is large.

Also, we have seen that the lazy random walks mix quickly if and only if there is a large spectral gap.

Actually, complete graphs are the best expander graphs in each of the above definitions, but we are interested in sparse expander graphs (only linear number of edges), e.g. d-regular graphs for constant d.

It can be shown that a random d-regular graph is an expander graph with high probability using the combinatorial definition, by standard proof techniques (Chernoff bound + union bound).

Perhaps surprisingly, while almost every graph is an expander graph, it is very difficult to come up with a deterministic construction of expander graphs.

In constructing expander graphs, it turns out that the spectral definition is easier to work with.

We will use the following stronger spectral definition:

Definition Let G be a d-regular graph. Let $d = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq -d$ be the eigenvalues of its adjacency matrix. We say G is an (n,d,ε)-graph if it has n vertices, d-regular and $\max\{\alpha_2, |\alpha_n|\} \leq \varepsilon d$.

The quantity $\alpha := \max\{\alpha_2, |\alpha_n|\}$ is often called the spectral radius of the graph. The smaller is the spectral radius, the stronger it is as an expander.

Combinatorially, $|\alpha_n|$ is small if and only if there is no induced subgraph close to a bipartite component.

Probabilistically, α is small if and only if the non-lazy random walks mix quickly.

It can be shown that the spectral radius of a random d-regular graph is $O(\sqrt{d})$ whp (see [HLW]) and we are interested in deterministic constructions of d-regular graphs with small spectral radius.

Deterministic constructions

While almost all d-regular graphs are expander graphs, it is very surprising that it is hard to construct an expander graph deterministically. One possible explanation is that random graphs have high descriptive complexity, while in deterministic constructions the graphs can be described in a succint way.

There are some explicit constructions of d-regular expander graphs.

- A family of 8-regular graphs G_m for every integer m. The vertex set $V = \mathbb{Z}_m \times \mathbb{Z}_m$. The neighbors of the vertex (x,y) are $(x+y,y)$, $(x-y,y)$, $(x,y+x)$, $(x,y-x)$, $(x+y+1,y)$, $(x-y+1,y)$, $(x,y+x+1)$, $(x,y-x+1)$, where all operations are modulo m. Note that this family is very explicit, meaning that the neighbors of a vertex is very easy to compute, which is very useful in applications like probability amplification. Gabber and Galil proved that $\alpha \le 5\sqrt{2} < 8$. The proof uses Fourier analysis; see Chapter 8 of [HLW]

- A family of 3-regular p-vertex graph for every prime p. The vertex set $V = \mathbb{Z}_p$, and a vertex x is connected to $x+1$, $x-1$, and its multiplicative inverse x^{-1} (for vertex 0 its inverse is 0), where the operations are modulo p. The proof uses some deep results in number theory.

- The main source of explicit deterministic constructions is from Cayley graphs, graphs defined by groups. Some of the strongest expanders, called Ramanujan graphs with spectral radius $\le 2\sqrt{d-1}$, are from Cayley graphs and the proofs require sophisticated mathematical tools. In the last part of this course, we will study a new way to show the existence of Ramanujan graphs using combinatorial and probabilistic methods, whose proofs use interlacing family of polynomials.

We will present a combinatorial construction of expander graphs, known as the zig-zag product, whose proof is simpler and more intuitive, but it is less explicit as the above constructions.

Combinatorial constructions

The idea of the combinatorial constructions is to construct a bigger expander graph from smaller expander graphs.

Let G be an (n, k, ε_1)-graph and H be an (k, d, ε_2)-graph.

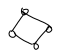
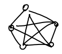
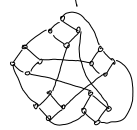
For each vertex $v \in G$, let $e_v^1, \ldots, e_v^k$ be its incident edges.

We will identify the vertices in H to $e_v^1, \ldots, e_v^k$.

The vertex set of the product is $V(G) \times V(H)$, where each vertex in G is replaced by a "cloud" of k vertices $(v,1), \ldots, (v,k)$, one for every edge incident with v.

A natural product to consider is called the replacement product, denoted by $G \circledR H$.

The edges of $G \circledR H$ are simply the union of the edges in G and n copies of the edges in H.

For example, H is  and G is , then $G \circledR H$ is .

Intuitively, $G \circledR H$ is an expander if G and H are.

Consider a set $S \subseteq V(G \circledR H)$.

If S has either large or small intersection with each cloud, then S should have large expansion because of the large expansion of G (i.e. as S is like a set in G).

If S has medium intersection with many clouds, then S should have large expansion because of the large expansion of H (i.e. many crossing edges within each cloud).

However, I don't know of a proof that made the above intuition precise, as there seems to be no clean way to decompose a set's contribution into its contribution in G and its contribution in H.

Actually, the spectral proof that we are going to see soon can be thought of as a linear algebraic way to carry out this idea in a more general setting (than the combinatorial setting).

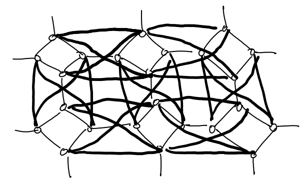
Zig-zag product

The actual construction that we will analyze is slightly more complicated.

The zig-zag product of G and H, denoted by $G \textcircled{z} H$, has the same vertex set $V(G) \times V(H)$ as the replacement product.

The edges of the zig-zag product is obtained from a length three walk in the replacement product: there is an edge $(u_1, v_1) - (u_2, v_2)$ in $G \textcircled{z} H$ iff there exists v_1' such that $(u_1, v_1) - (u_1, v_1')$,

$(u_1, v_1') - (u_2, v_1')$ and $(u_2, v_1') - (u_2, v_2)$ are edges in $G \circledR H$. In other words, each edge in $G \textcircled{z} H$ corresponds to a length three walk in $G \circledR H$, where the first step is within a cloud, the second step is across two clouds, and the third step is within a cloud.

For example, if G is a grid and H is a 4-cycle, then $G \circledR H =$ 

and the edges in $G \textcircled{z} H$ are the thick edges.

The intuition that the zig-zag product is an expander graph is from random walk, each edge in $G \textcircled{z} H$ corresponds to a random step in H, a deterministic step in G, and a random step in H.

So, the first two steps correspond to going to a random neighbor cloud, and the third step corresponds to moving to a random neighbor within the cloud.

Since both G and H are expander and so fast mixing, after a few steps of random walks in $G \textcircled{z} H$, we won't know which clouds we are in and also won't know the location within the cloud, and so $G \textcircled{z} H$ is also fast mixing and hence an expander.

<u>Theorem</u> (Zig-Zag Product) Let G be an (n, k, ε_1)-graph and H be an (k, d, ε_2)-graph. Then $G \textcircled{z} H$ is an $(nk, d^2, \varepsilon_1 + 2\varepsilon_2 + \varepsilon_2^2)$-graph.

Before we see the construction of the zig-zag product, let's first see how it can be combined with powering to construct bigger and bigger expander graphs.

The k-th power $G^k = (V, E')$ is a graph on the same vertex set as G and we add an edge (u,v) to E' for every path (not necessarily simple) of length exactly k in G from u to v.

Note that the adjacency matrix of G^k is A^k where A is the adjacency matrix of G. So, if G is an (n, d, ε) graph then G^k is an (n, d^k, ε^k) graph.

While the spectral gap has improved, the degree also increases significantly.

The idea is to use the zig-zag product to decrease the degree while not losing expansion much.

Let H be a $(d^4, d, 1/16)$-graph. We can prove its existence by probabilistic method. Since d is a constant, we can exhaustively find it in constant time.

Using the building block H, we inductively define G_n by: $G_1 = H^2$ and $G_{n+1} = (G_n)^2 \textcircled{z} H$.

We claim that G_n is a $(d^{4n}, d^2, 1/4)$-graph for all n.

The base case is true. clearly.

Assume G_n is a $(d^{4n}, d^2, 1/4)$-graph, then G_n^2 is a $(d^{4n}, d^4, 1/16)$-graph, and $G_{n+1} = (G_n)^2 \circledz H$ is a $(d^{4(n+1)}, d^2, 1/4)$-graph by the zig-zag theorem.

Proof of zig-zag theorem

It can be checked that G has nk vertices and is d^2-regular.

First we write down the walk matrix Z of the zig-zag product.

Let B be the walk matrix of H and $\hat{B}$ be n copies of B on the diagonal. The matrix $\tilde{B}$ corresponds to the movements within the clouds.

The steps between clouds are deterministic: we move from a vertex (v,i) to the unique vertex (u,j) for which $e_v^i = e_u^j$. The walk matrix is a permutation matrix P with $P_{(v,i),(u,j)} = 1$ if $e_v^i = e_u^j$ and 0 otherwise.

So $Z = \tilde{B} P \tilde{B}$.

The graph $G \circledz H$ is regular, and so $\vec{1}_{nk}$ is an eigenvector of Z with eigenvalue one.

To prove the zig-zag product theorem, we will prove that $|f^T Z f| / \|f\|^2 \le \varepsilon_1 + 2\varepsilon_2 + \varepsilon_2^2$ for all $f \perp \vec{1}_{nk}$, that is, the Rayleigh quotient (with respect to Z) is small for all vectors orthogonal to the first eigenvector.

We decompose f to two vectors to apply the results in G and H.

This is where the power of linear algebra comes from, as in the larger domain of $\mathbb{R}^{nk}$ there is a simple operation for the decomposition, while in the combinatorial domain it is not clear how to decompose.

Define f_G as the average of f on clouds, i.e. $f_G(x,i) = \frac{1}{k} \sum_{j \in [k]} f(x,j)$. Two vertices in the same cloud have the same value in f_G.

Define $f_H = f - f_G$. So f_H sums to zero in each cloud.

Roughly speaking, we will use the result in G to show that $|f_G^T Z f_G| \le \varepsilon_1 \|f_G\|$, and use the result in H to argue that $|f_H^T Z f_H| \le \varepsilon_2 \|f_H\|$.

Note that $|f^T Z f| = |f^T \tilde{B} P \tilde{B} f| = |(f_G + f_H)^T \tilde{B} P \tilde{B} (f_G + f_H)| \le |f_G^T \tilde{B} P \tilde{B} f_G| + 2|f_G^T \tilde{B} P \tilde{B} f_H| + |f_H^T \tilde{B} P \tilde{B} f_H|$.

Since $B \vec{1}_k = \vec{1}_k$, it follows that $\tilde{B} f_G = f_G$ as vertices in the same cloud have the same value.

So, $|f^T Z f| \le |f_G^T P f_G| + 2|f_G^T P \tilde{B} f_H| + |f_H^T \tilde{B} P \tilde{B} f_H|$.

We will use the result of G to bound the first term, use the result of H to bound the third term, and some simple bound for the second term.

Since the spectral radius of B is ε_2, we have $\|Bx\| \le \varepsilon_2 \|x\|$ for any $x \perp \vec{1}_k$. To see this,

write $x = \sum_{i \ge 2}^{k} c_i v_i$ where v_i are the orthonormal eigenvectors of B with eigenvalues λ_i. Note that

$c_1 = 0$ as $v_1 = \vec{1}/\sqrt{k}$ and $x \perp \vec{1}$. Then $\|Bx\|^2 = \|B(\sum_{i\ge 2}^{k} c_i v_i)\|^2 = \|\sum_{i \ge 2}^{k} c_i \lambda_i v_i\|^2 = \sum_{i \ge 2}^{k} c_i^2 \lambda_i^2 \le \varepsilon_2^2 \sum_{i \ge 2}^{k} c_i^2 = \varepsilon_2^2 \|x\|^2$.

This implies that $\|\tilde{B} f_H\| \le \varepsilon_2 \|f_H\|$ as f_H sums to zero in each cloud.

The third term is $|f_H^T \tilde{B} P \tilde{B} f_H| \le \|f_H^T \tilde{B}\| \, \|P \tilde{B} f_H\|$ by Cauchy-Schwarz

$= \|f_H^T \tilde{B}\| \, \|\tilde{B} f_H\|$ as $\|Px\|_2 = \|x\|_2$ since P is a permutation matrix

$\le \varepsilon_2^2 \|f_H\|^2$

Similarly, the second term is $2|f_G^T P \tilde{B} f_H| \le 2\|f_G\| \|P \tilde{B} f_H\| = 2\|f_G\| \|\tilde{B} f_H\| \le 2\varepsilon_2 \|f_G\| \|f_H\|$.

Finally, we will prove in the following claim that $\|f_G^T P f_G\| \le \varepsilon_1 \|f_G\|^2$, and this would imply

that $|f^T Z f| \le \varepsilon_1 \|f_G\|^2 + 2\varepsilon_2 \|f_G\| \|f_H\| + \varepsilon_2^2 \|f_H\|^2 \le (\varepsilon_1 + 2\varepsilon_2 + \varepsilon_2^2) \|f\|^2$, where the last

inequality holds because $\|f\|^2 = \|f_G + f_H\|^2 = \|f_G\|^2 + \|f_H\|^2$ (as $f_G \perp f_H$) and so $\|f\| \ge \|f_G\|$ and $\|f\| \ge \|f_H\|$.

This completes the proof, and so it remains to prove the following claim.

Claim $|f_G^T P f_G| \le \varepsilon_1 \|f_G\|^2$.

Proof In short, this is just the same as the spectral radius of G.

To work with $|f_G^T P f_G|$, we "contract" the matrix P to the walk matrix of G by collapsing each cloud to a vertex.

Define $g: V(G) \to \mathbb{R}$ as $g(v) = \sqrt{k} f_G(v,i)$. Note that $\|g\|^2 = \|f_G\|^2$.

Also, $f_G^T P f_G = g^T W g$ where W is the walk matrix of G, as each edge $(u,i),(v,j)$ contributes $f_G(u,i) f_G(v,j)$ in

$f_G^T P f_G$, while the corresponding edge (u,v) contributes $(\sqrt{k} f_G(u,i))(\frac{1}{k})(\sqrt{k} f_G(v,j)) = f_G(u,i) f_G(v,j)$.

Therefore, $f_G^T P f_G / \|f_G\|^2 = g^T W g / \|g\|^2$.

Since $f \perp \vec{1}$, we have $f_G \perp \vec{1}$, and thus $g \perp \vec{1}$.

As G is an (n,k,ε_1)-graph, we have $f_G^T P f_G / \|f_G\|^2 = g^T W g / \|g\|^2 \le \varepsilon_1$. □

Properties of expander graphs

A useful property of expander graphs is that it behaves like random graphs.

Let $E(S,T) = \{(u,v) : u \in S, v \in T, uv \in E\}$ where an edge with $u \in S \cap T$ and $v \in S \cap T$ is counted twice.

In a random graph where each edge presents with probability d/n, we expect that $|E(S,T)|$ is close to $\frac{d}{n}|S||T|$.

The expander mixing lemma says that in an expander graph $|E(S,T)|$ is close to this number.

Theorem (Expander Mixing Lemma) Let $G=(V,E)$ be a d-regular graph with spectral radius α.

Then for every $S \subseteq V$ and $T \subseteq V$, $\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \alpha \sqrt{|S||T|}$.

Proof Let χ_S and χ_T be the characteristic vectors of S and T, i.e. $\chi_S(i) = 1$ if $i \in S$ and zero otherwise.

Let $v_1, v_2, \ldots, v_n$ be an orthonormal basis of eigenvectors. Recall that $v_1 = \vec{1}/\sqrt{n}$.

Write $\chi_S = \sum_i a_i v_i$ and $\chi_T = \sum_i b_i v_i$. So $a_1 = \langle \chi_S, v_1 \rangle = \frac{|S|}{\sqrt{n}}$ and $b_1 = \langle \chi_T, v_1 \rangle = \frac{|T|}{\sqrt{n}}$.

Note that $|E(S,T)| = \chi_S^T A \chi_T = \sum_{i \geq 1} \alpha_i a_i b_i = d|S||T|/n + \sum_{i \geq 2} \alpha_i a_i b_i$.

So, $\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \left| \sum_{i \geq 2} \alpha_i a_i b_i \right| \leq \sum_{i \geq 2} |\alpha_i||a_i||b_i| \leq \alpha \sum_{i \geq 2} |a_i||b_i|$ by spectral radius

$\leq \alpha \|a\| \|b\|$ by Cauchy-Schwarz

$= \alpha \|\chi_S\| \|\chi_T\| = \alpha \sqrt{|S||T|}$. □

Independent set and chromatic number

Let $X \subseteq V$ be an independent set, i.e. there is no edge between any pair of vertices in X.

By definition, $|E(X,X)| = 0$.

By the expander mixing lemma with $S=T=X$, we have $d|X|^2/n \leq \alpha |X|$, and thus $|X| \leq \alpha n/d$.

For graphs with $\alpha \leq \varepsilon d$, this implies that the maximum size of an independent set is $\leq \varepsilon n$,

and it follows that the minimum chromatic number of such graphs is $\geq \frac{1}{\varepsilon}$.

Diameter

We claim that the diameter of an expander graph is $O(\log n)$, for graphs with spectral radius $\leq \varepsilon d$.

By Cheeger's inequality, $\phi(S) \geq \frac{\lambda_2}{2} = \frac{d-\alpha_2}{2d}$ where λ_2 is the second eigenvalue of the normalized Laplacian

matrix and α_2 is the second largest eigenvalue of A.

This implies that $\frac{|\delta(S)|}{d|S|} \geq \frac{d-\alpha_2}{2d} \geq \frac{d-\varepsilon d}{2d}$, and thus $|\delta(S)| \geq \left(\frac{1-\varepsilon}{2}\right) d|S|$ for all $|S| \leq n/2$.

Let $N(S) := \{ i \in V-S \mid i \sim j \text{ for some } j \in S \}$ be the neighbor set of S.

Then $|N(S)| \geq |\delta(S)|/d \geq \left(\frac{1-\varepsilon}{2}\right)|S|$ for all S with $|S| \leq n/2$.

Let v be a vertex and $B(v,r) := \{ i \in V \mid \text{dist}(i,v) \leq r \}$ be the ball of radius r around v.

Then the above statement implies that $B(v,r) \geq \left(\frac{3-\varepsilon}{2}\right) B(v,r-1) \geq \ldots \geq \left(\frac{3-\varepsilon}{2}\right)^r B(v,1) = \left(\frac{3-\varepsilon}{2}\right)^r$.

Whenever $\varepsilon < 1$, then $B(v,r) > \frac{n}{2}$ for $r = O(\log n)$.

This implies that the distance between any pair of vertices is at most $2r$, hence the diameter is $O(\log n)$.

Vertex Expansion (optional)

One can give a bound on the vertex expansion through edge expansion like what we did above

Here we prove a stronger bound known as the Tanner's theorem.

Theorem (Tanner) Let G be an expander graph with degree d and spectral radius $\leq \varepsilon d$.

Then $|N(S)|/|S| \geq 1/(c(1-\varepsilon^2)+\varepsilon^2)$ for $|S|=cn$ where $0<c\leq\frac{1}{2}$ is a constant.

Proof Let $S\subseteq V$ with $|S|=cn$, χ_S be the characteristic vector of S, and A be the adjacency matrix of G

with eigenvalues $\alpha_1\geq\alpha_2\geq\ldots\geq\alpha_n$.

Write $\chi_S=\sum_{i\geq1} a_i v_i$, where $v_1=\vec{1}/\sqrt{n}$ and $a_1=\langle\chi_S,v_1\rangle=|S|/\sqrt{n}$.

Consider $\|A\chi_S\|_2^2$. Let $N[S]=N(S)\cup S$.

On one hand, $\|A\chi_S\|_2^2=\sum_{v\in N[S]}|S\cap N(v)|^2 \geq \left(\sum_{v\in N[S]}|S\cap N(v)|\right)^2/\left(\sum_{v\in N[S]}1^2\right)$ by Cauchy-Schwartz

$= (d|S|)^2/|N[S]|$.

On the other hand, $\|A\chi_S\|_2^2=\|\sum_{i\geq1}a_i\alpha_i v_i\|_2^2=\sum_{i\geq1}a_i^2\alpha_i^2\leq\frac{d^2|S|^2}{n}+\alpha^2\sum_{i\geq2}a_i^2=\frac{d^2|S|^2}{n}+\alpha^2(\|\chi_S\|_2^2-a_1^2)$

$=\frac{d^2|S|^2}{n}+(\varepsilon d)^2\left(|S|-\frac{|S|^2}{n}\right)=d^2cn\left(c+\varepsilon^2(1-c)\right)$.

Combining, we get $d^2cn|S|/N[S]\leq d^2cn(c+\varepsilon^2(1-c)) \Rightarrow |N[S]|/|S|\geq\frac{1}{c+\varepsilon^2(1-c)}$. □

It shows that when $c\ll\varepsilon^2$, then $|N[S]|/|S|\gtrsim\frac{1}{\varepsilon^2}$ which implies that $|N[S]|$ can be much larger than $|S|$ when $|S|$ is small enough.

Spectral Gap

How large can the spectral gap be?

Let $\alpha=\max\{|\alpha_2|,|\alpha_n|\}$ where α_i is the eigenvalue of the adjacency matrix.

There are graphs, called Ramanujan graphs, that $\alpha\leq2\sqrt{d-1}$.

This is essentially tight, as it is proved that $\alpha\geq2\sqrt{d-1}-o_n(1)$ where $o_n(1)$ tends to zero as n tends to infinity.

We can easily prove that $\alpha\geq\sqrt{d}(1-o_n(1))$.

We know that $nd\leq\operatorname{trace}(A^2)$, as each edge contributes one to $\operatorname{trace}(A^2)$.

On the other hand, we know that $\operatorname{trace}(A^2)=\sum_{i\geq1}\alpha_i^2\leq d^2+(n-1)\alpha^2$.

Therefore $d^2+(n-1)\alpha^2\geq nd$ and thus $\alpha^2\geq d(n-d)/(n-1)$ and hence $\alpha\geq\sqrt{d}\sqrt{\frac{n-d}{n-1}}$.

Random walks in expander graphs (optional)

Let $G=(V,E)$ be a d-regular graph. Let's assume $\alpha = \varepsilon d$ and $\varepsilon \le 1/10$. Let $X \subseteq V$ with $|X| \le |V|/100$.

Let v_0 be the initial random vertex, and $v_1, v_2, \ldots, v_t$ be the vertices produced by the t steps of the random walk.

Let $S = \{ i : v_i \in X \}$. We choose v_0 as a uniformly random vertex, each vertex is of probability $1/n$.

Theorem $\Pr(|S| > t/2) \le \left(\frac{2}{\sqrt{5}}\right)^{t+1}$.

First we set up the matrix formulation of the problem.

The initial distribution is $\pi = \vec{1}/n$.

Let χ_X and $\chi_{\bar{X}}$ be the characteristic vectors of X and $\bar{X}$, where $\bar{X} = V - X$.

Let I_X be the diagonal matrix with a 1 in the i-th diagonal entry if $i \in X$, and similarly $I_{\bar{X}}$.

Let p be a probability distribution. Then $I_X p$ is the probability vector on X.

Then $q = W I_X \pi$ is the probability vector where the initial random vertex is in S, where $W = A/d$.

Then, the probability that the walk is in X at precisely the time steps in S is

$$\vec{1}^T I_{Z_t} W I_{Z_{t-1}} W I_{Z_{t-2}} W \ldots I_{Z_2} W I_{Z_1} W \pi \quad , \text{ where } Z_i = X \text{ if } i \in S \text{ and } Z_i = \bar{X} \text{ if } i \notin S.$$

We will prove that this probability is at most $(1/5)^{|S|}$.

This will imply that $\Pr(|S| > t/2) \le \sum_{|S| > t/2} \Pr(\text{the walk is in } X \text{ at precisely the times in } S)$ by union bound

$$\le 2^{t+1} (1/5)^{(t+1)/2} = (2/\sqrt{5})^{t+1}.$$

Recall that $\|M\| = \max_y \|My\|/\|y\| = \max_y y^T M y / y^T y$ for symmetric M. You can check that $\|I_X\| = \|I_{\bar{X}}\| = \|W\| = 1$.

We will prove that $\|I_X W\| \le 1/5$, and this would imply that the above probability is at most $(1/5)^{|S|}$.

To see this, $\vec{1}^T I_{Z_t} W I_{Z_{t-1}} W \ldots I_{Z_2} W I_{Z_1} W \pi = \vec{1}^T (I_{Z_t} W)(I_{Z_{t-1}} W) \ldots (I_{Z_2} W)(I_{Z_1} W)\pi$

$$\le \|\vec{1}^T\| \, \|(I_{Z_t} W)(I_{Z_{t-1}} W) \ldots (I_{Z_2} W)(I_{Z_1} W)\pi\| \quad \text{Cauchy-Schwarz}$$

$$\le \|\vec{1}^T\| \left(\prod_{i=1}^{t} \|I_{Z_i} W\| \right) \|\pi\|$$

$$\le \|\vec{1}^T\| \, (1/5)^{|S|} \, \|\pi\| \quad \text{as } \|I_{Z_i} W\| \le 1/5 \text{ if } Z_i = X \text{ and } \|I_{Z_i} W\| \le \|I_{Z_i}\|\|W\| = 1 \text{ if } Z_i = \bar{X}$$

$$\le (1/5)^{|S|} \quad \text{as } \|\vec{1}^T\| = \sqrt{n} \text{ and } \|\pi\| = 1/\sqrt{n}.$$

It remains to prove that $\|I_X W\| \le 1/5$.

Let y be any nonzero vector and write $y = c_1 v_1 + \ldots + c_n v_n$ where $v_1 = \vec{1}/\sqrt{n}$ and $c_1 = \langle y, v_1 \rangle = \sum y_i / \sqrt{n}$, where $v_1, \ldots, v_n$ are the orthonormal eigenvectors of W with eigenvalues $1 = \lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n \ge -1$.

$$\|I_X W y\|^2 = \|I_X W(c_1v_1+\dots+c_nv_n)\|^2 \leq \|I_X \sum_{i\geq 1} c_i\lambda_i v_i\|^2 \leq 2\|I_X c_1\lambda_1 v_1\|^2 + 2\|I_X \sum_{i\geq 2} c_i\lambda_i v_i\|^2$$

$$\leq 2\|I_X(\frac{\sum y_i}{n}\vec{1})\|^2 + 2\|I_X\|^2\|\sum_{i\geq 2} c_i\lambda_i v_i\|^2$$

$$= 2|X|\left(\frac{\sum y_i}{n}\right)^2 + 2\sum_{i\geq 2}\|c_i\lambda_i v_i\|^2 \quad \text{by orthogonality}$$

$$\leq 2|X|\cdot\frac{\|y\|^2}{n} + 2\sum_{i\geq 2}\|c_i\lambda_i v_i\|^2 \quad \text{by Cauchy Schwartz as } (\sum y_i)^2 \leq (\sum y_i^2)(\sum 1^2)$$

$$\leq \frac{1}{50}\|y\|^2 + 2\sum_{i\geq 2} c_i^2\lambda_i^2$$

$$\leq \frac{1}{50}\|y\|^2 + 2\varepsilon^2\sum_{i\geq 2} c_i^2 \leq \frac{1}{50}\|y\|^2 + \frac{1}{50}\|y\|^2$$

Thus $\|I_X W y\| \leq \frac{1}{5}\|y\|^2$, finishing the proof.

Application : Probability Amplification (optional)

Suppose we have a randomized algorithm with error probability $1/100$ by reading n random bits.

This means that among the 2^n n-bit strings, only $2^n/100$ are "bad" strings.

To amplify the success probability, one can pick k random n-bit strings, then the error probability is at most $(1/100)^k$ using kn random bits.

We show how to exponentially decrease the error probability while using only $n+ck$ bits for a constant c.

Construct a d-regular expander graph with 2^n vertices and $\varepsilon \leq 1/10$.

In the first step, we use an n-bit random string, with error probability $\leq 1/100$.

In the subsequent steps, instead of picking independent n-bit strings, we do a $(k-1)$-step random walk and use the strings corresponding to the vertices in the random walk as "random" n-bit strings.

After we try the k "random" strings, we use the majority answer as our answer.

What is the error probability of this algorithm?

We output the wrong answer if the wrong answer is the majority.

Since the number of bad strings is at most $2^n/100$, the error probability is at most $(2/\sqrt{5})^k$, by letting X to be the set of bad strings.

The number of random bits used is $n+(k-1)\log_2 d$ since each random neighbor can be chosen with $\log_2 d$ bits.

Note that it works for two-sided error randomized algorithms as well.

This is just one example, expander graphs have many applications in derandomization; see [HLW].

Application : Constructing Efficient Objects (optional)

One can think of a d-regular expander graph as a very efficient object, as it only has very few edges but it achieves very high connectivity.

It is not surprising that expander graphs are useful in constructing efficient communication network.

One example is the construction of superconcentrators, which are directed graphs with n input nodes I and n output nodes, and satisfying the strong connectivity property that there are k vertex disjoint paths for any k input nodes and any k output nodes and for any $k \leq n$.

For instance, a complete bipartite graph satisfies this property, but it has $\Theta(n^2)$ edges.

Using expander graphs one can construct a superconcentrator with the following properties:

- it has $O(n)$ nodes, every node is of constant degree, and thus it has $O(n)$ edges.

A superconcentrator can be used as an efficient switching network. Besides, it can be used to design faster algorithms for computing network coding and computing matrix rank.

Another famous example of using expander graphs is to construct an optimal sorting network, with only $O(n \log n)$ edges and with depth $O(\log n)$.

Application: Undirected connectivity in log space

One important application of the zig-zag product is to solve the s-t connectivity problem in log space, i.e. to determine if there is a path from s to t in an undirected graph using minimal space.

As we will see later, it is easy to solve the problem if we are allowed to use randomness: just doing random walks for $O(n^3)$ steps would do.

There is a simple deterministic algorithm using $O(\log^2 n)$ space.

Reingold discovered a deterministic algorithm using $O(\log n)$ space using zig-zag product.

We briefly discuss the high level ideas here.

Suppose the graph is a d-regular expander graph for a constant d. Then G has diameter $O(\log n)$. Then we can enumerate all paths of length $O(\log n)$ in $O(\log n)$ space, since the graph is of constant degree.

The idea of Reingold's algorithm is to transform any graph G into a constant degree expander graph H such that s, t are connected in G iff s, t are connected in H.

We can reduce G into a constant degree graph by replacing each vertex of high degree by some low degree graph, say a cycle or an expander, just like what we did in the replacement product.

To improve the expansion, we construct the graph $G^8 ⓩ H$, and it can be shown that the spectral gap doubles in the resulting graph. (Some calculations skipped here.)

So, we just need to repeat this construction $O(\log n)$ times to get constant spectral gap, as initially the spectral gap is at least $\frac{1}{n^2}$, which holds for any connected graph.

Now, we run the exhaustive search algorithm on this resulting constant degree expander graph.

Note that we don't construct this graph completely, as it requires too much space.

We just compute each edge on demand. The observation is that there are only $O(\log n)$ recursion levels for the construction, and in each level we just need constant space, since there are only three steps and the degree is constant.

So, to compute each edge, we only need $O(\log n)$ space in total, and we don't need to store the actual edges for the exhaustive search (we just need to store the current vertex, and the "index" of the edges in the path so far, where each index can be stored in constant space, as the graph is of constant degree).

The total space required is thus $O(\log n)$.

It is open whether there is a $O(\log n)$-space algorithm for directed s-t connectivity.

<u>**References**</u> You should be able to find more details of everything in the excellent survey "Expander graphs and their applications", by Hoory, Linial and Wigderson [HLW].