CS 798 - Algorithmic Spectral Graph Theory, Fall 2015, Waterloo

Lecture 21: Weaver's conjecture

We study the proof of Weaver's conjecture using the method of interlacing polynomials.

A multivariate barrier argument is used to bound the maximum root of the mixed characteristic polynomial.

Statements

One of the Kadison-Singer-complete problems is the following:

Weaver's conjecture There exist positive constants α and β so that for every m and n and every set of vectors $u_1,\dots,u_m \in \mathbb{R}^n$ such that $\|u_i\|^2 \le \alpha$ for all i and $\sum_{i=1}^m u_i u_i^T = I$, there exists a partition of $\{1,\dots,m\}$ into two sets S_1 and S_2 so that $\|\sum_{i\in S_j} u_i u_i^T\| < 1-\beta$ for $j\in\{1,2\}$.

Marcus, Spielman and Srivastava proved the following theorem about sum of random rank-one matrices that would imply Weaver's conjecture.

Theorem Let $v_1,\dots,v_m$ be independent random vectors in $\mathbb{R}^n$ with finite support such that

$$\sum_{i=1}^m E[v_i v_i^T] = I \quad \text{and} \quad E\|v_i\|^2 \le \varepsilon \quad \text{for all } 1\le i\le m,$$

$$\text{then} \quad \Pr\left[\ \left\|\sum_{i=1}^m v_i v_i^T\right\| \le (1+\sqrt{\varepsilon})^2\right] > 0.$$

Reduction

Weaver's conjecture is about partitioning and MSS theorem is about sum of random matrices.

There is a simple reduction from the former to the latter.

For each vector $u_i \in \mathbb{R}^n$ in Weaver's problem, we create a random vector $v_i \in \mathbb{R}^{2n}$ with two choices, so that $v_i = \sqrt{2}\begin{pmatrix} u_i \\ 0 \end{pmatrix}$ with probability $\frac{1}{2}$ and $v_i = \sqrt{2}\begin{pmatrix} 0 \\ u_i \end{pmatrix}$ with probability $\frac{1}{2}$,

such that the first choice corresponds to putting u_i into the first group and the second choice corresponds to putting u_i into the second group.

Then, $E[v_i v_i^T] = \frac{1}{2}\left(2\begin{pmatrix} u_i \\ 0\end{pmatrix}\begin{pmatrix} u_i^T & 0\end{pmatrix} + 2\begin{pmatrix} 0 \\ u_i\end{pmatrix}\begin{pmatrix} 0 & u_i^T\end{pmatrix}\right) = \begin{pmatrix} u_i u_i^T & 0 \\ 0 & u_i u_i^T\end{pmatrix}$,

and thus $\sum_{i=1}^m E[v_i v_i^T] = \sum_{i=1}^m \begin{pmatrix} u_i u_i^T & 0 \\ 0 & u_i u_i^T\end{pmatrix} = \begin{pmatrix} \sum_{i=1}^m u_i u_i^T & 0 \\ 0 & \sum_{i=1}^m u_i u_i^T\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} = I$.

Similarly, $E[\|v_i\|^2] = E[v_i^T v_i] = \frac{1}{2}\left(2\begin{pmatrix} u_i^T & 0\end{pmatrix}\begin{pmatrix} u_i \\ 0\end{pmatrix} + 2\begin{pmatrix} 0 & u_i^T\end{pmatrix}\begin{pmatrix} 0 \\ u_i^T\end{pmatrix}\right) = 2\|u_i\|^2 \le 2\alpha$.

By MSS, there exists a choice of v_i such that $\left\|\sum_{i=1}^m v_i v_i^T\right\| \le (1+\sqrt{2\alpha})^2$.

We put i into S_1 if we select the first choice for i, and put i into S_2 otherwise.

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Then the conclusion implies that $\left\| \begin{pmatrix} 2\sum_{i\in S_1} u_iu_i^T & 0 \\ 0 & 2\sum_{i\in S_2} u_iu_i^T \end{pmatrix} \right\| \leq (1+\sqrt{2\alpha})^2$.

and thus $\| \sum_{i \in S_j} u_iu_i^T \| \leq \frac{1}{2}(1+\sqrt{2\alpha})^2$.

So, definitely when α is small enough (say $\alpha \leq \frac{1}{16}$), then the RHS < 1, and Weaver's follows.

It is qualitatively stronger as when α is small enough, we can bound how far it is from $\frac{1}{2}I$, the ideal partitioning, and we will use this bound next time.

Method of interlacing polynomials

We need to prove two steps:

1. $\exists$ a choice of v_i such that $\text{maxroot}\left(\det(xI - \sum_{i=1}^m v_iv_i^T)\right) \leq \text{maxroot}\left(\mathbb{E}_{v_1,\ldots,v_m} \det(xI - \sum_{i=1}^m v_iv_i^T)\right)$.

   To prove this, we prove that the family $\det(xI - \sum_{i=1}^m v_iv_i^T)$ form an interlacing family.

   Last time in L20, we showed the mixed characteristic polynomials of sum of random rank-one matrices form an interlacing family, and it fits exactly in our current setting, and so we are done with this step.

2. To prove the theorem, we need to prove $\text{maxroot}\left(\mathbb{E}_{v_1,\ldots,v_m} \det(xI - \sum_{i=1}^m v_iv_i^T)\right) \leq (1+\sqrt{\varepsilon})^2$.

   In the Ramanujan setting last time, the expected polynomial is the matching polynomial, and we can directly reason about the max root.

   In the Weaver's setting, it is a major technical challenge to bound the maxroot.

   Recall that $\mathbb{E}_{v_1,\ldots,v_m} \det(xI - \sum_{i=1}^m v_iv_i^T) = \prod_{i=1}^m (1 - \frac{\partial}{\partial z_i}) \det(xI + \sum_{i=1}^m z_i \mathbb{E}[v_iv_i^T]) \Big|_{z_1=z_2=\cdots=z_m=0}$.

   It was used to prove that the expected polynomials are real-rooted, and this proves the family is interlacing and established the first step.

   It turns out the formula is also crucial in proving the second step.

   The idea is to show an upper bound of a "maxroot" of the multivariate polynomial, and then maintain a good upper bound after each $(1-\frac{\partial}{\partial z_i})$ differentiation operation is applied.

   They use a barrier argument similar to the one in BSS (L18) to inductively bound the maxroots, but generalized to the multivariated setting.

Multivariate barrier argument

Multivariate barrier argument

To bound $\operatorname{maxroot}\left(\mathbb{E}_{v_1,\dots,v_m} \det\left(xI - \sum_{i=1}^m v_i v_i^T\right)\right)$,

we bound $\operatorname{maxroot}\left(\prod_{i=1}^m \left(1 - \frac{\partial}{\partial z_i}\right) \det\left(xI + \sum_{i=1}^m z_i \mathbb{E}[v_i v_i^T]\right) \Big|_{z_1 = z_2 = \dots = z_m = 0}\right)$.

Since $\sum_{i=1}^m \mathbb{E}[v_i v_i^T] = I$ by the assumption of the theorem, we can rewrite the polynomial as

$$\prod_{i=1}^m \left(1 - \frac{\partial}{\partial z_i}\right) \det\left(\sum_{i=1}^m (x + z_i) \mathbb{E}[v_i v_i^T]\right) \Big|_{z_1 = z_2 = \dots = z_m = 0}$$

$$= \prod_{i=1}^m \left(1 - \frac{\partial}{\partial z_i}\right) \det\left(\sum_{i=1}^m z_i \mathbb{E}[v_i v_i^T]\right) \Big|_{z_1 = z_2 = \dots = z_m = x}$$

Call the matrix $\mathbb{E}[v_i v_i^T] = A_i$. Note that $A_i \succcurlyeq 0$.

Call the polynomial $\prod_{i=1}^m \left(1 - \frac{\partial}{\partial z_i}\right) \det\left(\sum_{i=1}^m z_i A_i\right) = Q(z_1, z_2, \dots, z_m)$.

Our task is to find the smallest x^* such that $Q(x, x, \dots, x) \neq 0$ for all $x \geq x^*$.

This would imply that $\operatorname{maxroot}\left(\prod_{i=1}^m \left(1 - \frac{\partial}{\partial z_i}\right) \det\left(\sum_{i=1}^m z_i A_i\right) \Big|_{z_1 = \dots = z_m = x}\right) < x^*$,

and thus $\operatorname{maxroot}\left(\mathbb{E}_{v_1,\dots,v_m} \det\left(xI - \sum_{i=1}^m v_i v_i^T\right)\right) < x^*$.

We see the polynomial $\prod_{i=1}^m \left(1 - \frac{\partial}{\partial z_i}\right) \det\left(\sum_{i=1}^m z_i A_i\right)$ as the final polynomial of a sequence of polynomials, such that $p_0(z_1, \dots, z_m) = \det\left(\sum_{i=1}^m z_i A_i\right)$ and

$p_k(z_0, \dots, z_m) = \prod_{i=1}^k \left(1 - \frac{\partial}{\partial z_i}\right) \det\left(\sum_{i=1}^m z_i A_i\right)$. Note that $p_m = Q$.

Univariate approach

Initially, since $A_i \succcurlyeq 0$, we have $p_0(x, x, \dots, x) > 0 \quad \forall x > 0$.

One natural strategy is to find some initial value $x_0 > 0$, such that $p_0(x, \dots, x) > 0 \quad \forall x > x_0$.

Then, we set $x_i = x_{i-1} + \gamma$ for some small γ and prove inductively that

$$p_i(x, \dots, x) > 0 \quad \forall x > x_i.$$

Unfortunately, this approach would not work.

To prove the theorem, we need $x_m \leq (1 + \sqrt{\varepsilon})^2$, and so we need to set γ to be as small as $\frac{1}{m}(1 + \sqrt{\varepsilon})^2$.

However, each v_i could have $\|v_i\|^2 = \varepsilon$, and it is not possible to set γ to be so small for the induction hypothesis to go through.

The situation is a bit like that in BSS, where we want to bound the max eigenvalue (max root), but we need to use a more global argument involving all eigenvalues (roots).

Multivariate approach

The correct induction hypothesis should be multivariate, since this polynomial is multivariate, and applying the $1-\frac{\partial}{\partial z_i}$ operation should only have effect on the i-th variable.

We use the following definition to generalize the above idea.

Definition Given a multivariate real-stable polynomial $p(z_1,\ldots,z_m)$, we say a point $z\in\mathbb{R}^m$ is "above the roots" of p if $p(\vec{z}+\vec{t})>0$ for all $\vec{t}=(t_1,\ldots,t_m)$ with $t_i\geq 0\ \forall i$.

Our goal is to prove that $(1+\sqrt{\varepsilon})^2\cdot\vec{1}$ is above the roots of $Q(z_1,\ldots,z_m)$

To do this, we start with a solution $(t,t,\ldots,t)$ for some $t>0$ to be chosen later, and we know that $(t,\ldots,t)$ is above the roots of p_0.

The induction hypothesis is that $(\underbrace{t+\delta,\ldots,t+\delta}_{k \text{ coordinates}}, t,\ldots,t)$ is above the roots of p_k.

and this will imply that $(t+\delta,\ldots,t+\delta)$ is about the roots of $p_m = Q$.

Barrier functions

To prove the induction hypothesis, we consider a generalization of the barrier functions used in BSS.

In BSS, we set an upper bound u of the eigenvalue, and we keep track of a potential function $\Phi_u(A)=\mathrm{Tr}(uI-A)^{-1}$ and maintain the invariant that $\Phi_u(A)\leq 1$, to guarantee that u is a "comfortable" upper bound of all the current eigenvalues.

Recall that $\mathrm{Tr}(uI-A)^{-1}=\sum_{i=1}^{n}\frac{1}{u-\lambda_i}$ where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of A.

Equivalently, we can also think of it as bounding the roots of characteristic polynomials.

We can write $\det(xI-A)=\prod_{i=1}^{m}(x-\lambda_i)$, and we observe that $\mathrm{Tr}(xI-A)^{-1}=\frac{\partial_x\det(xI-A)}{\det(xI-A)}$, as $\partial_x\det(xI-A)=\sum_{i=1}^{m}\prod_{j\neq i}(x-\lambda_j)$ and thus $\frac{\partial_x\det(xI-A)}{\det(xI-A)}=\sum_{i=1}^{m}\frac{1}{x-\lambda_i}$.

Also, note that $\frac{\partial_x\det(xI-A)}{\det(xI-A)}=\partial_x\log\det(xI-A)$.

So, we could understand the potential function $\Phi_x(A)$ as $\partial_x\log\det(xI-A)$.

We want to set x to keep this small, meaning that we want to choose x which is a comfortable upper bound so that the rate of change of $\log\det(xI-A)$ is small.

Multivariate barrier functions

Given a polynomial $p\in\mathbb{R}[z_1,\ldots,z_m]$ and a point z above the roots of p, the barrier function of p in direction i at z is $\Phi_p^i(z) = \dfrac{\partial_{z_i} p(z)}{p(z)} = \partial_{z_i} \log p(z)$.

Equivalently, we can define Φ_p^i by $\Phi_p^i(z) = \dfrac{q_{z,i}'(z_i)}{q_{z,i}(z_i)} = \sum_{j=1}^{d} \dfrac{1}{z_i - \lambda_j}$, where $q_{z,i}(t)$ is

the univariate restriction $q_{z,i}(t) = p(z_1, \dots, z_{i-1}, t, z_{i+1}, \dots, z_m)$ and $\lambda_1, \dots, \lambda_d$ are roots of this.

Note that it is real-rooted as substituting real numbers preserves real stability.

Induction hypothesis

We will find a point x_0 above the roots of p_0 such that $\Phi_{p_0}^i(x_0) \le \phi$ for all $1 \le i \le m$.

Specifically, $x_0 = (t, \dots, t)$ for some $t > 0$.

Then, as stated before, let $x_k = (\underbrace{t+\delta, \dots, t+\delta}_{k \text{ coordinates}}, t, \dots, t)$ and $p_k = \prod_{i=1}^{k} \left(1 - \frac{\partial}{\partial z_i}\right) \det\left(\sum_{i=1}^{m} z_i A_i\right)$,

and the induction hypothesis is to maintain $\Phi_{p_k}^i(x_k) \le \phi$ for all $1 \le i \le m$.

Informally, every time we apply the differential operation $1 - \frac{\partial}{\partial z_i}$, we increase the upper bound in the i-th coordinate so that the potential functions are still all bounded by a small value, so that the point x_k remains a point comfortably above the roots.

Properties of barrier functions

The following properties of the barrier functions are important in the proof.

Lemma Suppose $p \in \mathbb{R}[z_1, \dots, z_m]$ is real stable and z is above the roots of p.

Then, for all $i, j \in [m]$ and $\delta \ge 0$, we have

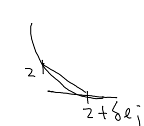
(monotonicity) $\Phi_p^i(z + \delta e_j) \le \Phi_p^i(z)$ where e_j is the j-th standard vector

(convexity) $\Phi_p^i(z + \delta e_j) \le \Phi_p^i(z) + \delta\, \partial_{z_j} \Phi_p^i(z + \delta e_j)$.

The second property can be rewritten as $\dfrac{\Phi_p^i(z+\delta e_j) - \Phi_p^i(z)}{\delta} \le \partial_{z_j} \Phi_p^i(z + \delta e_j)$, saying that

the slope at $z + \delta e_j$ is larger than the line connecting $z + \delta e_j$ to z for all z and δ.

This means that the slope is increasing, equivalent to the function being convex.



When $i = j$, these two properties are quite easy to see.

Recall that $\Phi_p^i(z) = \sum_{j=1}^{d} \dfrac{1}{z_i - \lambda_j}$ where $\lambda_1, \dots, \lambda_d$ are the roots of the univariate polynomial restricted to z_i.

Since z_i is above the roots, $z_i > \lambda_j$ for all j.

So, if we increase z_i, the function Φ_p^i can only decrease, and hence we have monotonicity.

To prove convexity, we just need to show that $\Phi_p^{i\,''}(z) \geq 0$ for all z above the roots.

A direct calculation shows that $\Phi_p^{i\,''}(z) = \sum_{j=1}^{d} \frac{2}{(z_i - \lambda_j)^3} > 0$, as $z_i > \lambda_j$ for all j when z is above roots.

The proof for $i \neq j$ is tricky, and we will come back to it at the end.

For now, we assume this lemma and continue with the inductive proof.

Inductive proof

First, we see that when a point is comfortably above the roots (i.e. $\Phi_p^i(z)$ is small enough), then the monotonicity property implies that z is still above the roots after the operation $1 - \partial_{z_i}$.

Claim Suppose that p is real stable and z is above the roots of p, with the additional property that $\Phi_p^i(z) < 1$ for all i. Then z is also above the roots of $p - \partial_{z_j} p$ $\forall j$.

~~proof~~ Let y be above z. Then, by monotonicity, $\Phi_p^i(y) < 1$.

This implies that $1 > \Phi_p^i(y) = \frac{\partial_{z_i} p(y)}{p(y)}$, and thus $0 < p(y) - \partial_{z_i} p(y) = (1 - \partial_{z_i}) p(y)$,

and so there is still no root above z in $p - \partial_{z_i} p$. □

The claim shows that z is still above the roots after one differentiation operation, but we could not repeat it because the condition $\Phi^i_{p-\partial_{z_j}}(z) < 1$ may no longer hold.

To maintain the invariant, we will increase the upper bound in the j-th coordinate to decrease the potential function.

Lemma Suppose that p is real stable and z is above the roots of p.

Suppose further that $\Phi_p^i(z) \leq 1 - \frac{1}{\delta}$ holds for all i for some $\delta > 0$.

Then, for all i, $\Phi^i_{p - \partial_{z_j} p}(z + \delta e_j) \leq \Phi_p^i(z)$. In particular, $z + e_j$ is above the roots of $p - \partial_{z_j} p$.

~~proof~~ Recall that $\Phi_p^j = \frac{\partial_{z_j} p}{p}$ and so $p - \partial_{z_j} p = p(1 - \Phi_p^j)$.

Taking log, we have $\log(p - \partial_{z_j} p) = \log p + \log(1 - \Phi_p^j)$.

Applying ∂_{z_i}, we have $\Phi^i_{p - \partial_{z_j} p} = \Phi_p^i - \frac{\partial_{z_i} \Phi_p^j}{1 - \Phi_p^j}$

So, $\Phi^i_{p - \partial_{z_j} p}(z + \delta e_j) = \Phi_p^i(z + \delta e_j) - \frac{\partial_{z_i} \Phi_p^j(z + \delta e_j)}{1 - \Phi_p^j(z + \delta e_j)}$.

To prove $\Phi^i_{p - \partial_{z_D}}(z + \delta e_i) \leq \Phi_p^i(z)$, it is equivalent to $\Phi_p^i(z) - \Phi_p^i(z + \delta e_j) \geq - \frac{\partial_{z_i} \Phi_p^j(z + \delta e_j)}{\ }$.

$1-\Phi_p^j(z+\delta e_j)$

By convexity, we know that $\Phi_p^i(z)-\Phi_p^i(z+\delta e_j) \geq -\delta\,\partial_{z_j}\Phi_p^i(z+\delta e_j)$.

So, it would be enough if we could prove $-\delta\,\partial z_j\,\Phi_p^i(z+\delta e_j) \geq \dfrac{-\partial z_i\,\Phi_p^j(z+\delta e_j)}{1-\Phi_p^j(z+\delta e_j)}$.

Note that $\partial z_j\,\Phi_p^i = \partial z_j\,\partial z_i \log p = \partial z_i\,\partial z_j \log p = \partial z_i\,\Phi_p^j$, and so the numerators cancel

and the above is equivalent to $\delta \geq \dfrac{1}{1-\Phi_p^j(z+\delta e_j)}$, because $-\partial z_j\,\Phi_p^i \geq 0$ as

the function is monotonically decreasing.

Our assumption guarantees that $\delta \geq \dfrac{1}{1-\Phi_p^j(z)} \geq \dfrac{1}{1-\Phi_p^j(z+\delta e_j)}$ again by monotonicity,

and thus the conclusion is always satisfied. □

Finishing the proof

If we choose our initial $x_0=(t,\dots,t)$ such that $\Phi_{p_0}^i(x_0) \leq 1-\frac{1}{\delta}$.

Then, by induction, $x_m=(t+\delta, t+\delta, \dots, t+\delta)$ is above the roots of $p_m = Q$.

This would imply that the max root $\left(\underset{v_1,\dots,v_m}{E} \det\left(xI-\sum_{i=1}^m v_i v_i^T\right)\right) \leq t+\delta$.

It remains to optimize t and δ to prove the best upper bound.

Initial value

Recall that $p_0(z)=\det\left(\sum_{i=1}^m z_i A_i\right)$ where $A_i = E[v_i v_i^T]$, and $\sum_{i=1}^m A_i = \sum_{i=1}^m E[v_i v_i^T] = I$.

We need to compute $\Phi_{p_0}^i(z) = \partial z_i \log\det\left(\sum_{i=1}^m z_i A_i\right) = \dfrac{\partial z_i \det\left(\sum_{i=1}^m z_i A_i\right)}{\det\left(\sum_{i=1}^m z_i A_i\right)}$.

Jacobi's formula $\partial_t \det(A+tB) = \det(A+tB)\,\mathrm{Tr}\left((A+tB)^{-1}B\right)$

proof First we consider $\partial_t \det(A+tB)|_{t=0}$.

Note that $\partial_t \det(A+tB) = \det(A)\,\partial_t \det(I+tA^{-1}B) = \det(A)\,\partial_t \prod_i (1+t\lambda_i)$,

where λ_i are the eigenvalues of $A^{-1}B$.

Since the coefficient of the linear term in $\prod_i (1+t\lambda_i)$ is $\sum_i \lambda_i = \mathrm{Tr}(A^{-1}B)$, we have

$\partial_t \det(A+tB)|_{t=0} = \det(A)\,\partial_t \prod_i (1+t\lambda_i)|_{t=0} = \det(A)\,\mathrm{Tr}(A^{-1}B)$, as only

the coefficient of the linear term remains after substituting $t=0$.

To compute $\partial_t \det(A+tB)$, we can compute $\partial_x \det(A+tB+xB)|_{x=0}$,

which is equal to $\det(A+tB)\,\mathrm{Tr}((A+tB)^{-1}B)$ by the above calculation. □

Applying Jacobi's formula, $\Phi_{p_0}^j(z) = \dfrac{\partial z_j \det\left(\sum_{i=1}^m z_i A_i\right)}{\det\left(\sum_{i=1}^m z_i A_i\right)} = \dfrac{\det\left(\sum_{i=1}^m z_i A_i\right)\mathrm{Tr}\left(\left(\sum_{i=1}^m z_i A_i\right)^{-1}A_j\right)}{\det\left(\sum_{i=1}^m z_i A_i\right)} = \mathrm{Tr}\left(\left(\sum_{i=1}^m z_i A_i\right)^{-1}A_j\right)$.

Applying Jacobi's formula, $\Phi_{p_0}^{j}(z) = \frac{\partial z_j \det\left(\sum_{i=1}^m z_i A_i\right)}{\det\left(\sum_{i=1}^m z_i A_i\right)} = \frac{\det\left(\sum_{i=1}^m z_i A_i\right) \operatorname{Tr}\left(\left(\sum_{i=1}^m z_i A_i\right)^{-1} A_j\right)}{\det\left(\sum_{i=1}^m z_i A_i\right)} = \operatorname{Tr}\left(\left(\sum_{i=1}^m z_i A_i\right)^{-1} A_j\right).$

Plugging in $z = x_0 = (t, t, \ldots, t)$ and using $\sum_{i=1}^m A_i = I$, we have $\Phi_{p_0}^{j}(z) = \operatorname{Tr}\left(\left(\frac{I}{t}\right) A_j\right) = \operatorname{Tr}\left(\frac{A_j}{t}\right) \le \frac{\varepsilon}{t}$,

by the assumption that $\mathbb{E}[\|v_i\|^2] \le \varepsilon$ and thus $\operatorname{Tr}(A_j) = \operatorname{Tr}\left(\mathbb{E}[v_i v_i^T]\right) = \mathbb{E}[\|v_i\|^2] \le \varepsilon$.

If we set t so that $\Phi_{p_0}^{j}(z) \le \frac{\varepsilon}{t} \le 1 - \frac{1}{\delta}$, then we will get the final bound $t + \delta$.

So, δ should be $\frac{1}{1 - \varepsilon/t}$, and the final bound is $t + \frac{1}{1 - \varepsilon/t}$.

This is minimized when $t = \sqrt{\varepsilon} + \varepsilon$, so that the final bound is $1 + 2\sqrt{\varepsilon} + \varepsilon = (1 + \sqrt{\varepsilon})^2$,

and the main theorem follows.

Concluding remarks

I decided to skip the proof of monotonicity and convexity and refer to the original paper or the blogpost of Terence Tao for different proofs.

An outstanding open question is whether there is a polynomial time algorithm to find the partitioning whose existence is guaranteed by the theorem.

A more open ended question is whether these techniques would have more combinatorial applications.

We will see a recent one of using these techniques for the traveling salesman problem.

References

Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem,
by Marcus, Spielman and Srivastava.

Real stable polynomials and the Kadison-Singer problem, blogpost by Terence Tao.