CS 475/675 - Spring 2015: Assignment 4

Due July 27, 2015. Written component: 5:00pm. Programming component: Midnight.
Written/analytical work should be submitted to the physical assignment box on 4th floor MC (Q1-4). MATLAB code and any associated output and commentary should be submitted to the DropBox on LEARN (Q5).

For full marks be sure to show all your work!

1. (10 marks) Determine SVDs of the following matrices (by hand calculation):
   (a) $\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, (e) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

2. (5 marks) Suppose you are given the SVD of A, i.e. $A = U\Sigma V^T$. Assume that A is of size $m \times n$ and has full rank. How do you use the U, Σ, V factors to:
   - Solve the linear system $Ax = b$ where $m = n$?
   - Solve the least squares problem $Ax = b$ where $m > n$?

3. (10 marks) Let A be a matrix of size $m \times n$, $m \geq n$, not necessarily square. Consider

$$H = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}.$$

   Determine the eigenvalue decomposition of H; i.e. find Q such that $HQ = Q\Lambda$, Λ is a diagonal matrix. How are Q and Λ related to the singular vectors and singular values of A? Explain. (Hint: Consider

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} V & V & 0 \\ \hat{U} & -\hat{U} & \sqrt{2}\hat{U}_{m-n} \end{bmatrix},$$

   where the SVD of A is given by $U\Sigma V^T$ and $U = [\hat{U} \;\; \hat{U}_{m-n}]$.)

4. (10 marks) Let A be a matrix of size $m \times n$ and w is an $n \times 1$ vector. Define

$$B = \begin{bmatrix} A \\ w^T \end{bmatrix}.$$

   (a) Show that $\sigma_1(B) \leq \sqrt{\|A\|_2^2 + \|w\|_2^2}$. (Hint: Make use of the definition, $\|B\|_2 = \max_{\|x\|_2=1} \|Bx\|_2$.)

   (b) Show that $\sigma_n(B) \geq \sigma_n(A)$. (Hint: Let $\tilde{B} = \begin{bmatrix} \tilde{A} \\ \tilde{w}^T \end{bmatrix}$ be the best rank $n-1$ approximation of B. Form $B - \tilde{B}$ and again make use of the matrix 2-norm definition. You will also need to exploit the low rank approximation theorem to complete the argument.)

5. (20 marks) Spectral clustering image segmentation.

   (a) Create the MATLAB function:

```
NL = CreateImageGraph(U)
```

where `U` is an $m \times n$ image and `NL` is the corresponding (sparse) normalized graph Laplacian matrix of size $mn \times mn$. Use $\sigma_{dist}^2 = 100$ for the distance weighting, and $\sigma_{int}^2 = 0.001$ for the intensity weighting. Consider the 8 surrounding neighbour pixels when determining the weighting. Submit your code.

(b) The incomplete MATLAB program `CellSegment.m` reads a block from a cell image file, computes the normalized graph Laplacian matrix (using `CreateImageGraph` above), performs normalized spectral clustering image segmentation, and displays the final segmentation results. You should complete the missing part of the program as indicated in the code, i.e. implement normalized spectral clustering. This will involve extracting the appropriate eigenvectors from NL, normalizing the rows of the eigenvector matrix, and performing K-means to cluster them.

You may use the Matlab command `eigs` to extract the required (subset of) eigenvectors (note that the default behavior returns the eigenpairs for a few of the *largest* magnitude eigenvalues). Similarly, you should perform K-means clustering using the Matlab command `kmeans`, to determine the cluster indices for the output variable `index` found in the code. (For `kmeans`, use the additional parameter 'replicates' set to 20, to run K-means multiple times and take the best result.)

Determine an effective $K \in [2, 20]$ for K-means by trial-and-error.

Submit all your code. Also submit a copy of Figure 1 generated by the program (i.e. containing the original image and the segmentation results). Modify the title of Figure 1 to state your K value.