CS 476/676: Assignment 1

Winter 2015

Instructor: Wei Xu	Office: DC 1312	e-mail: *wei.xu@uwaterloo.ca*
Lecture Times :	MWF 11:30-12:20	MC 2054
Office Hours Until Feb 6, 2015:	Monday 1:30-3:00pm (Wei Xu, DC 1312)	
	Wednesday 2:00-4:00pm (Kai Ma, DC3594);	
	Jan. 29th (TH), Feb. 5th (TH) 11:00am-12:00pm(Kai Ma, DC3594)	

All lecture notes are on LEARN system.

Due February 6, 2015 in class

IMPORTANT: In this and in future assignments, most of the marks for programming questions are allocated for explanations of algorithms (e.g. pseudo-code) and discussion of results. If all you hand in is the listing of the "Raw Code" or "Raw Output" by itself, you will get poor marks. All coding should be done in Matlab. **All the plots should be appropriately labeled**. By default, you should submit listings of all Matlab code used in your assignment. **Be sure to document (i.e. add liberal comments) your code**. The TA will take off marks for poor documentation.

Assignment Questions

1. (6 marks) (Binomial Lattice)

Assume that the XYZ stock pays no dividend and is currently priced at $S_0 = \$40$. Assume that, at time $T > 0$, the stock price goes up to uS_0 with probability $0 < p < 1$ and down to dS_0 with probability $1 - p$. We know that $d < 1 < u$ but do not know d or u. Assume that there is no arbitrage and the interest rate is zero. Consider the following three options with the same expiry T on the XYZ stock. Assume that a European call option with strike price \$50 is priced at \$10 while a European call option with strike price \$40 is priced \$13. What is the fair value of a European call option with a strike price of \$35? Explain your answer.

2. (12 marks: total) (Binomial Lattice)

Develop code for pricing European options using a binomial lattice.

Note that there is a Matlab function *binprice* for binomial tree pricing, but it is only for standard American call and put options, and it stores the entire tree.

Your code should take only $O(N)$ storage NOT $O(N^2)$, $N = T/\Delta t$. This can be done efficiently if you store the payoff in an array of size $O(N)$ and index it appropriately.

Table 1: Some typical option parameters

σ	30%
r	1.5 %
Time to expiry	1.0 years
Initial asset price S^0	\$100

(a). (8 marks) Using data in Table 1, test your code for standard European call and put by comparing your results with the exact solutions from the Matlab function *blsprice* . Show tables with the initial option value $V(S_0, 0)$ (both at-the-money puts and calls, i.e., $K = S_0$) as a function of Δt. Start off with a timestep $(\Delta t)^0 = .01$, and show the option value for $(\Delta t)^0/2, (\Delta t)^0/4, \ldots$. You should see your results converging to the *blsprice* value. Your tables should look like Table 2.

Table 2: Convergence Test

Δt	Value	Change	Ratio
.01	V_1		
.005	V_2	$V_2 - V_1$	
.0025	V_3	$V_3 - V_2$	$\frac{V_2 - V_1}{V_3 - V_2}$
...	...	...	...

Let V_0^{exact} be the exact initial price $V(S_0, 0)$ (from the solution to the Black-Scholes PDE), and $V^{tree}(\Delta t)$ be the price from a lattice pricer with the time step Δt, then it can be shown that

$$V_0^{tree}(\Delta t) \quad = \quad V_0^{exact} + \alpha \Delta t + O((\Delta t)^{3/2}) \tag{1}$$

where α is a constant independent of Δt. Computationally determine

$$\lim_{\Delta t \to 0} \frac{V_0^{tree}((\Delta t)/2) - V_0^{tree}(\Delta t)}{V_0^{tree}((\Delta t)/4) - V_0^{tree}((\Delta t)/2)} \ . \tag{2}$$

Does your convergence table agree with the theory in terms of rate of convergence?

(b). (4 mark) A capped call option has the payoff at the expiry T below:

$$\min(\max(S - K, 0), H)$$

where $H \geq 0$ is a given cap. Similarly a capped put has the payoff

$$\min(\max(K - S, 0), H)$$

Compute European capped put/call option values by using the data in Table 1. Using $\Delta t = 0.0025$ to compute the initial values of European capped puts with the expiry $T = 1$, $H = 10$, and the strike $K = 60 : 10 : 100$. Plot the initial value against the strike. How does the capped put value compare to that of a standard put with the same strike K and expiry T? Comment and explain your observations.

Submit Matlab listings, tables, and explain what you see.

3. (4 marks) (Properties of a Standard Brownian Motion)

A standard Brownian motion is a stochastic process Z_t satisfies the following properties

(a) $Z_0 = 0$.

(b) For any $t \geq 0$, Z_t is a normal variable with mean $\mathbf{E}(Z_t) = 0$ and variance $\mathbf{Var}(Z_t) = \mathbf{E}(Z_t^2) = t$.

(c) For any $0 \le t_1 < t_2 \le t_3 < t_4$, $Z_{t_2} - Z_{t_1}$ and $Z_{t_4} - Z_{t_3}$ are independent normals.

Prove that, for any $0 \le s < t$,

$$\begin{aligned}&\mathbf{E}(Z_t - Z_s) = 0\\&\mathbf{Var}(Z_t - Z_s) = t - s\end{aligned}$$

4. (6 marks: total) (Analytical Solution of Special SDEs)

Apply Ito's Lemma to show that

(a). (3 marks) $X_t = e^{Z_t - \frac{1}{2}t}$ solves $dX_t = X_t dZ_t$

(b). (3 marks) $X_t = e^{2Z_t - t}$ solves $dX_t = X_t dt + 2X_t dZ_t$

5. (6 marks) (Binomial Lattice Property)

Consider two European options with payoff $W(S,T)$ and $V(S,T)$ on the same underlying S. Prove that, for a sufficiently small Δt, the binomial lattice price will satisfy the arbitrage inequality below: if

$$W(S^n, T) \quad > \quad V(S^n, T) \quad ; \quad \forall \text{ price } S^n \text{ on a binomial node at time } T$$

then

$$W(S^0, 0) \quad > \quad V(S^0, 0) \ .$$

6. (4 marks) (Binomial Lattice Property)

Show that if there is no arbitrage in the binomial tree model, then $d \le e^{r\triangle t} \le u$.

7. (12 marks: total) (Monte Carlo)

A European down-and-out barrier put option with strike K has the standard put payoff at the expiry T if the underlying price never falls below the specified down-barrier $S_d < S^0$ in the time horizon $[0, T]$. Otherwise, the option pays nothing.

For the computation in this question, assume $K = 110$ and barrier $S_d = 80$. Other parameters are as in Table 1.

(a). (4 marks) The down-and-out European put option value V_{out}, assuming that the barrier has not been reached at time t, is given by the analytic formula

$$\begin{aligned}V_{\text{out}}(S,t) \quad = \quad & Ke^{-r(T-t)}\left(\mathcal{N}_{\text{cdf}}(d_4) - \mathcal{N}_{\text{cdf}}(d_2) - \left(\frac{S_d}{S}\right)^{(-1+\frac{2r}{\sigma^2})}(\mathcal{N}_{\text{cdf}}(d_7) - \mathcal{N}_{\text{cdf}}(d_5))\right)\\ & -S\left(\mathcal{N}_{\text{cdf}}(d_3) - \mathcal{N}_{\text{cdf}}(d_1) - \left(\frac{S_d}{S}\right)^{(1+\frac{2r}{\sigma^2})}(\mathcal{N}_{\text{cdf}}(d_8) - \mathcal{N}_{\text{cdf}}(d_6))\right)\end{aligned}$$

where $\mathcal{N}_{\text{cdf}}(d)$ is the cumulative distribution function for the standard normal and

$$d_1 = \frac{\log(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\log(\frac{S}{K}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_3 = \frac{\log(\frac{S}{S_d}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_4 = \frac{\log(\frac{S}{S_d}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_5 = \frac{\log(\frac{S}{S_d}) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_6 = \frac{\log(\frac{S}{S_d}) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_7 = \frac{\log(\frac{SK}{S_d^2}) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_8 = \frac{\log(\frac{SK}{S_d^2}) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

In Matlab, the cumulative distribution function for normal distribution $\mathcal{N}_{\text{cdf}}(d)$ is **normcdf**. Compute and plot the initial down-and-out European put option value for $S = 80 : 2 : 120$ and $T = 1$.

(b). (8 marks) As we derived in class, for the purposes of pricing options, we can pretend that the asset price S evolves in the risk-neutral world:

$$dS = rSdt + \sigma SdZ \tag{3}$$

where r is the risk free return, σ is the volatility, and dZ is the increment of a standard Brownian motion.

Let the expiry time of an option be T, and let

$$\begin{aligned} N &= \frac{T}{\Delta t} \\ S(n\Delta t) &= S^n \end{aligned} \tag{4}$$

Then, given an initial price S^0, M realizations of the path of a risky asset are generated using the algorithm

$$S^{n+1} = S^n e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\phi^n\sqrt{\Delta t}} \tag{5}$$

where $\{\phi^n\}$ are independent standard normals.

If the mth value of $S^N = S(T)$ is denoted by $(S^N)^m$, then an approximate initial value of the option is given by (assuming that $S(0) = S^0$)

$$\tilde{V}(S^0, 0) = e^{-rT} \frac{\sum_{m=0}^{m=M} \text{ Payoff}((S^N)^m)}{M} \tag{6}$$

For the down-and-out barrier put,

$$\text{Payoff}(S^N) = \begin{cases} \max(K - S^N, 0), & \text{if } S^n > S_d, \text{ for } n = 1, \cdots, N \\ 0 & \text{otherwise.} \end{cases}$$

- The price simulation using (5) has no time discretization error. Does the error in the computed value $\tilde{V}(S^0, 0)$ depend on the time discretization? Explain.

- Code up this algorithm in Matlab to determine the initial value of the **down**-and-out European put.
- For a fixed Δt plot the MC computed option values versus the number of simulations for a number of values of M. Repeat this computation for different Δt. Show the exact price computed from (a) on the plot. Explain what you see. You might start out using a timestep of 5 days (assuming 250 trading days in a year) with $M = 1000$.

Submit a listing of your code, plots, and discussion.

8. (Graduate Student Question) (10 marks)

You can choose **one** of the following papers to read and implment.

(a). Explain why the Monte Carlo method in Problem 6 is slow in obtaining an accurate barrier option value. Read the short paper by K. Moon, *Efficient Monte Carlo Algorithm for Pricing Barrier Options*, Commun. Korean Math. Soc. 23 (2008), No. 2, pp. 285-294. Implement and investigate the New Monte Carlo Method proposed on page 289.

Using the data in Table 1 in this assignment, for the single down barrier $S_d = 80$, generate figures similar to Figure 1 & 2 on page 291, and a table similar to Table 1 on page 292, in Moon's paper. Note that the figures and table in Moon's paper are for double barrier options. You are asked to produce plots and table for the single barrier option specified in this assignment. Note that the exiting probability formula (10) in the paper holds when $D = (B, +\infty)$ as well.

Comment on your observations of the computation results and discuss the improvement of the results compared to your implementation in Problem 6.

(b). For barrier option pricing, the binomial/trinomial tree method shows erratic convergence behavior because of the interaction between the discrete placement of the nodes in the lattice and the location of the barrier. In the paper, *The adaptive mesh model: a new approach to efficient option pricing*, Journal of Financial Economics 53, (1999), 313-351, Figlewski and Gao proposed using a combination of high-resolution lattice near the barrier and low-resolution lattice away from the barrier to obtain smoother convergence behavior and better performance. Implement the proposed method for down-and-out call option using the parameters in Table 3 and produce a table similar to Table 3 but without the RTM columns. In addition, produce the AMM values using the same parameters except using $K = 95$ and $\sigma = 0.3$. Comment on the accuracy and efficiency of the AMM algorithm compared to standard binomial tree. Note that you only need to read Section 4 of the paper for understanding and implementing the algorithm, with the only exception that a few notations are defined in earlier parts of the paper. You are encouraged to read Section 1 and 2 of the paper but this is not necessary as far as the assignment is concerned.