Applied Probability and Stochastic Processes: MGT-484

Introduction to Markov Chains

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Stochastic Processes

Definition: A discrete-time stochastic process is a family of random variables $X_0, X_1, X_2, \ldots$

Examples:

- The daily closing price of a stock
- The yearly gross domestic product of a country
- The daily sales of a retail store
- The weekly temperature of lake Geneva
- The yearly rate of unemployment
- The number of hits of a website every minute
- etc.

Markov Chains

Informal Definition: A Markov chain is a stochastic process whose *"future depends on the past only through the present."*

Put differently, the only part of the history of the process that affects its future evolution is the *current state*.

Example: Let X_t be the closing price of a stock on day t, and assume that the rates of return R_t of this stock from day t to day $t+1$ are independent and identically distributed (iid) and normal. Thus

$$X_{t+1} = X_t(1 + R_t) \quad \text{where} \quad R_t \sim \mathcal{N}(\mu, \sigma^2) \text{ iid.}$$

Note that X_{t+1} depends on the stock price history $X_1, X_2, \ldots, X_t$ only through the current price X_t. This is a Markov chain with a continuous state space (as the stock price can adopt any real number).

Markov Chains

Definition: The state space of a stochastic process is the set of possible values taken by the random variables $X_0, X_1, X_2, \ldots$

In this course we will concentrate mainly on Markov chains with finite or countably infinite state spaces.

Example: Consider a sequence of coin flips and define X_t as the number of heads observed up to time t. If

$$Z_t = \begin{cases} 1 & \text{if the outcome at time } t \text{ is heads,} \\ 0 & \text{otherwise,} \end{cases}$$

then $X_t = Z_0 + Z_1 + \cdots + Z_t = X_{t-1} + Z_t$. Thus, X_t depends on the past coin flips only through the most recent count X_{t-1}.

The (S, s) Inventory Model

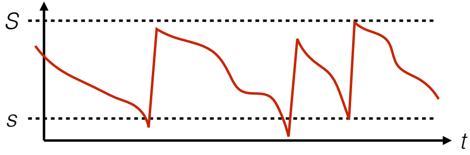
Let X_t denote the inventory level at the end of period t and assume that we face a demand D_t during period t.

Simple ordering policy: Order nothing as long as the inventory exceeds a level s. Otherwise, increase the inventory to a level $S > s$.

Dynamics:[1]

$$X_{t+1} = \begin{cases} (S - D_{t+1})_+ & \text{if } X_t \leq s \\ (X_t - D_{t+1})_+ & \text{if } X_t > s \end{cases}$$

If the demands are independent, X_{t+1} thus depends on the past demands only through the current inventory level X_t.



[1] For any $c \in \mathbb{R}$ we define $c_+ = \max\{c, 0\}$.

Markov Property

Common property of all the examples:

> If you want to know the distribution of the next state given the current state, you gain no additional information if I tell you the entire past history of the process.

Mathematically, this translates to

$$\mathbb{P}(X_{t+1} = x_{t+1} | X_t = x_t) = \mathbb{P}(X_{t+1} = x_{t+1} | X_0 = x_0, X_1 = x_1, \ldots, X_t = x_t),$$

where $x_0, x_1, \ldots, x_t$ is the sequence of observed states (i.e., these are numbers and not random variables).

The above relationship is called the Markov property. It is the defining characteristic of a Markov chain.

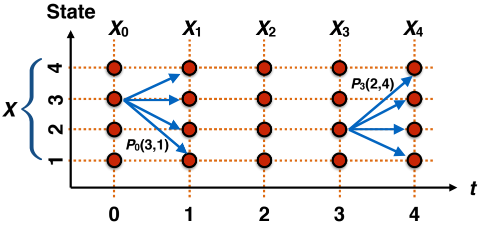
Transition Probabilities

We make the assumption that the state space $\mathbb{X}$ is a subset of $\mathbb{N}$.

Definition: We refer to $P_t(i,j) = \mathbb{P}(X_{t+1} = j | X_t = i)$ for $i, j \in X$ as the transition probabilities. A Markov chain is time-homogeneous if the transition probabilities are independent of time, that is,

$$P_t(i,j) = P_{t'}(i,j) \quad \forall t, t' = 0, 1, 2, \ldots$$

From now on we focus only on time-homogeneous Markov chains.



Markov Chains: Formal Definition

Definition: A process $X_0, X_1, X_2, \ldots$ is a (time-homogeneous) discrete-time Markov Chain with state space $\mathbb{X}$, initial distribution γ and transition matrix P if:

1. X_t is a random variable with values in $\mathbb{X}$ for all t.
2. $X_0 \sim \gamma$, that is, $\mathbb{P}(X_0 = i) = \gamma_i$.
3. The process $\{X_t\}$ satisfies the Markov property.
4. $\mathbb{P}(X_{t+1} = j | X_t = i) = P_{ij}$ for all t.

Remark: P is called a "matrix" even though $\mathbb{X}$ might be infinite. If $\mathbb{X}$ is infinite, then P has also infinitely many entries.

Remark: The relation $\sum_{j \in X} \mathbb{P}(X_{t+1} = j | X_t = i) = 1$ implies that $\sum_{j \in X} P_{ij} = 1$. Thus, all rows of P sum to 1. A matrix with this property is called a stochastic matrix.

Joint State Distribution

The Markov chain is completely described by $\mathbb{X}$, γ and P:

$$
\begin{aligned}
&\mathbb{P}(X_t = x_t, X_{t-1} = x_{t-1}, \dots, X_0 = x_0) \\
\overset{(i)}{=}\ &\mathbb{P}(X_0 = x_0)\,\mathbb{P}(X_1 = x_1 | X_0 = x_0)\,\mathbb{P}(X_2 = x_2 | X_1 = x_1, X_0 = x_0) \cdots \\
&\quad \cdots \mathbb{P}(X_t = x_t | X_{t-1} = x_{t-1}, \dots, X_0 = x_0) \\
\overset{(ii)}{=}\ &\mathbb{P}(X_0 = x_0)\,\mathbb{P}(X_1 = x_1 | X_0 = x_0)\,\mathbb{P}(X_2 = x_2 | X_1 = x_1) \cdots \\
&\quad \cdots \mathbb{P}(X_t = x_t | X_{t-1} = x_{t-1}) \\
\overset{(iii)}{=}\ &\gamma_{x_0}\, P_{x_0 x_1}\, P_{x_1 x_2} \cdots P_{x_{t-1} x_t}
\end{aligned}
$$

(i) as $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A|B \cap C)\mathbb{P}(B \cap C) = \mathbb{P}(A|B \cap C)\mathbb{P}(B|C)\mathbb{P}(C)$

(ii) due to the Markov property

(iii) by the definition of the initial distribution and the transition matrix

Sequence of Coin Flips

Let X_t be the number of heads in t coin flips and set $X_0 = 0$.

State space: $\mathbb{X} = \{0, 1, 2, \ldots\}$

Initial distribution: $\gamma_0 = 1$, $\gamma_i = 0$ for all $i > 0$

Transition matrix: If $X_{t-1} = x_{t-1}$, then X_t can take the values x_{t-1} or $x_{t-1} + 1$ with probability $\frac{1}{2}$ each. Thus,

$$\mathbb{P}(X_t = x_t | X_{t-1} = x_{t-1}) = \begin{cases} \frac{1}{2} & \text{if } x_t = x_{t-1} + 1, \\ \frac{1}{2} & \text{if } x_t = x_{t-1}, \\ 0 & \text{otherwise.} \end{cases}$$

We can write P as a matrix:

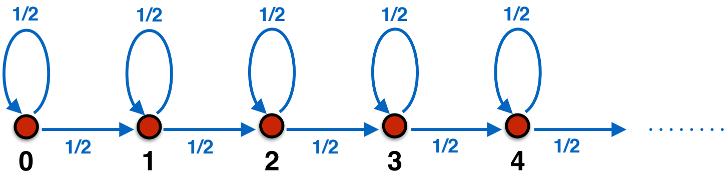
$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Graphical Representation

Any Markov chain has a graphical representation:

1. The nodes of the graph are the elements of $\mathbb{X}$.
2. Draw an arc from i to j if $P_{ij} > 0$.
3. Label the arc with the value P_{ij}.

The sequence of coin flips:



Describing Markov Chains

We have seen three ways of describing a Markov chain:

1. Specify the recursion that expresses X_t in terms of X_{t-1}.
2. Specify the state space $\mathbb{X}$, the initial distribution γ and the transition matrix P.
3. Use the graphical representation of the Markov chain.

A Queuing Model

Example:

- Customers arrive to a waiting room according to the stochastic process $A_0, A_1, \ldots$, where A_t represents the number of customers arriving in period t.
- The server (e.g., a receptionist) can process D_t people in period t.

Denote by X_t the total number of customers in the waiting room at the end of period t:

$$X_{t+1} = \begin{cases} 0 & \text{if } X_t + A_{t+1} \leq D_{t+1} \\ X_t + A_{t+1} - D_{t+1} & \text{otherwise} \end{cases}$$

$$= [X_t + A_{t+1} - D_{t+1}]_+$$

$\Longrightarrow$ $\{X_t\}$ follows a Markov chain if the arrivals are independent.

A Queuing Model

Example (extended):

- A security guard controls access to the building.
- If more than K people are in the waiting room, then any new arrivals are turned away.
- The guard observes the number of people in the room with a delay of 1 period.

The number of people in the room now satisfies the recursion

$$X_{t+1} = \begin{cases} [X_t + A_{t+1} - D_{t+1}]_+ & \text{if } X_{t-1} \le K, \\ [X_t - D_{t+1}]_+ & \text{if } X_{t-1} > K. \end{cases}$$

$\Longrightarrow$ $\{X_t\}$ is not a Markov chain.

Increase the state dimension: If we define $X'_t = (X_t, X_{t-1})$ for all t, then $\{X'_t\}$ is again a Markov chain!

State Space Explosion

This example shows that many (if not most) stochastic processes can be converted to Markov chains if the state space is enlarged.

A key to good Markov chain models is to control the state space explosion!

State Distributions

If $\{X_t\}$ be a Markov chain with initial distribution γ and transition matrix P, then the law of total probability implies:

$$\mathbb{P}(X_1 = j) = \sum_{i \in X} \mathbb{P}(X_1 = j | X_0 = i)\, \mathbb{P}(X_0 = i) = \sum_{i \in X} P_{ij}\gamma_i$$

Denote the distribution of X_t by p_t, i.e., $p_t(j) = \mathbb{P}(X_t = j)$ for all $j \in \mathbb{X}$. Thus, $p_0 = \gamma$, and $p_1 = \gamma P$ (distributions = row vectors).

More generally:

$$p_t = p_{t-1}P = p_{t-2}PP = \cdots = \gamma P^t$$

The (i, j) entry of P^t is the probability of transitioning from i to j in t periods, and the ith row of P^t is the distribution of X_t given $X_0 = i$.

$\Longrightarrow$ Computing state distributions is tantamount to computing powers of the transition matrix.

Sequence of Coin Flips

Let X_t be the number of heads in t coin flips and set $X_0 = 0$.

Question: What is the probability $\mathbb{P}(X_t = j)$ of j heads in t tosses?

Answer: Binomial distribution:

$$\mathbb{P}(X_t = j) = \binom{t}{j}\left(\frac{1}{2}\right)^j\left(\frac{1}{2}\right)^{t-j} = \binom{t}{j}\left(\frac{1}{2}\right)^t \quad \text{for} \quad 0 \le j \le t$$

Alternatively, we can compute the powers of P by hand, e.g.,

$$P^2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \cdots \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The first row contains the numbers $\mathbb{P}(X_t = j | X_0 = 0)$ for $0 \le j < \infty$. This is consistent with the binomial formula above.

Recipe for Computing Powers of P

Input: $P \in \mathbb{R}^{N \times N}$ diagonizable, $t \in \mathbb{N}$; **Output:** P^t.

1. Find a diagonal matrix Λ and an invertible matrix R with

$$P = R\Lambda R^{-1} = R \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} R^{-1},$$

   where $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of P, and the columns of R represent the corresponding eigenvectors.

2. $$P^t = (R\Lambda R^{-1})^t = R\Lambda \underbrace{R^{-1}R}_{=I} \Lambda R^{-1} \cdots R\Lambda R^{-1} = R\Lambda^t R^{-1}$$

$$= R \begin{pmatrix} \lambda_1^t & & \\ & \ddots & \\ & & \lambda_N^t \end{pmatrix} R^{-1}$$

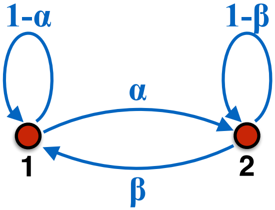
(These calculations are easy to do in MATLAB.)

The Simplest Possible Chain

Given: $\alpha > 0$, $\beta > 0$. Then,

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

has eigenvalues 1 and $1 - \alpha - \beta$.



Task: Find $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $P = R\begin{pmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{pmatrix}R^{-1}$, recalling that $R^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. A direct calculation yields:

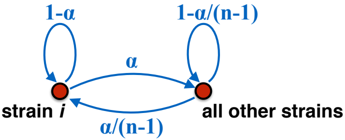
$$R = \begin{pmatrix} 1 & \frac{-\alpha}{\alpha+\beta} \\ 1 & \frac{\beta}{\alpha+\beta} \end{pmatrix}, \quad P^t = \begin{pmatrix} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^t & \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^t \\ \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^t & \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^t \end{pmatrix}$$

Virus Mutation

Consider a virus with $n > 1$ possible strains. In each period, the virus mutates with probability α, in which case it changes randomly to any of the remaining $n-1$ strains. What is $\mathbb{P}(X_t = i | X_0 = i)$?

Method 1: Construct a Markov chain with $\mathbb{X} = (1, 2, \ldots, n)$.

Method 2: Use the previous example with $\beta = \frac{\alpha}{n-1}$:



$$\implies \quad \mathbb{P}(X_t = i | X_0 = i) = \frac{1}{n} + \frac{n-1}{n}\left(1 - \frac{n}{n-1}\alpha\right)^t$$

A Trick for Computing P^t

Assume that we computed the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $P \in \mathbb{R}^{n \times n}$ and that they are all different (the case of degenerate eigenvalues is more intricate but can be handled using the Jordan normal form).

The computation of R and R^{-1}, which may be hard, can be avoided!

Note that $P^t = R\Lambda^t R^{-1}$ is a linear combination of $\lambda_1^t, \ldots, \lambda_n^t$, that is, there must be $n \times n$-matrices $A_1, \ldots, A_n$ with

$$P^t = A_1\lambda_1^t + \cdots A_n\lambda_n^t \quad \forall t = 0, 1, 2, \ldots \quad (\star)$$

Explicitly calculating P^t for $t = 0, \ldots, n-1$ allows us to interpret $(\star)$ as a system of n^3 linear equations for the n^3 entries of the matrices $A_1, \ldots, A_n$.

Expected Reward

Assume that we receive a reward $r(i)$ in state i.

Example: Let X_t be the number of customers in a queue at time t.To calculate the fraction of time the server is busy, assign a reward 1 to states where $X_t > 0$ and a reward 0 to states where $X_t = 0$.

The expected reward at time t is

$$\begin{aligned}\mathbb{E}[r(X_t)] &= \sum_{j \in X} \mathbb{E}[r(X_t)|X_t = j]\,\mathbb{P}(X_t = j) \\ &= \sum_{j \in X} r(j)\,\mathbb{P}(X_t = j) = \gamma P^t r,\end{aligned}$$

where $r = [r(1), r(2), \ldots]^\top$ is the (column-) vector of rewards. Recall that γP^t is the distribution of X_t. Thus, we take the expectation of r w.r.t. this distribution.