Ordinary Differential Equation

Let us consider a first order differential equation of the type :
$\dfrac{dy}{dx} = f(x, y)$ The solution of this type of differential equation is given by:

- Euler Method
- Runge-Kutta Method

Euler Method

Consider differential equation:

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

If f is a function of x alone then we can directly write

$$y(x) = \int f(x)dx \tag{2}$$

But here f is the function of x and y. So we apply Taylor Series by expanding y(x) at $x = x_0$.

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$$y(x) = y(x_0) + (x - x_0)y'(x_0) \tag{3}$$

Here the series is truncated at the second term.

$$y(x) = y(x_0) + (x - x_0)f(x_0, y_0) \tag{4}$$

$$y(x) = y(x_0) + hf(x_0, y_0) \tag{5}$$

Where $h = (x - x_0)$ is the size of step.
Similarly:

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$$y(x_1) = y(x_0) + hf(x_0, y_0) \tag{6}$$

$$y(x_2) = y(x_1) + hf(x_1, y_1) \tag{7}$$

So in general,

$$y(x_{i+1}) = y(x_i) + hf(x_i, y_i) \tag{8}$$

This is simple **Euler's Method**.

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The flow chart of this method is as shown below:

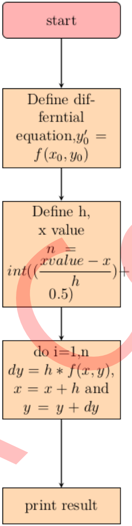

Figure: Showing different values of x and their corresponding functional value

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(Q)Solve the differential equation $\frac{dy}{dx} = \frac{2y}{x}$ by using **Euler's Method** for different values of **x,y,xvalue and h**.
The Fortran code to solve this differential equation is as shown below:

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```
program Eulerdifferential
implicit none
real:: x,y,xvalue,h,dy,fx
integer::n,i
external fx
write(*,*)"give the values of x,y,xvalue,h"
read(*,*)x,y,xvalue,h
n=int((xvalue-x)/h+0.5)
```

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```
do i=1, n
dy=h*fx(x,y)
x=x+h
y=y+dy
write(*,*)i,x,y
enddo
write(*,*)"value of y at x=",x,"is",y
end program
real function fx(x,y)
real::x,y
fx=2.0*y/x
end function
!this programming is used to solve the differential equation
!dy/dx = 2*y/x by using Euler's method
```

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When the same program is run with different h values, the following result is obtained.

e2.png

Figure: Output of the differential equation $\frac{2y}{x}$ at different x=1.0, y=2.0, xvalue=2.0 and h=0.10,0.15

When the value of h changes, the number of iterations required to solve the differential equation changes, as does the output of the differential value.

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Runge-Kutta Method

- The Euler method is a one-step finite difference method of first order. (The total error is of order h if it is of first order)
- In this approach, the total error will be of higher order in h.
- The Euler's Method is an example of general class of approximations of Runge-Kutta Method.
- More popular and improved method is Runge-Kutta Method.

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We notice that all the Euler Methods can be written in the form:

$$y(x) = y(x_0) + h[\alpha f(x_0, y_0) + \beta f(x_0 + \gamma h, y_0 + \delta h f_0)] \tag{9}$$

$$y(x) = y(x_0) + h\alpha f(x_0, y_0) + h\beta[f(x_0, y_0) + h\gamma \frac{df(x_0, y_0)}{dx} + h\delta h f(x_0, y_0) \frac{d}{} \tag{10}$$

This expression agrees with Taylor series through term h^2, such that:

$$\alpha + \beta = 1$$
$$\beta\gamma = \frac{1}{2}$$
$$\beta\delta = \frac{1}{2}$$

Where $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}$, $\gamma = \delta = \frac{3}{4}$
This method can be derived in terms of integral

$$y(x = x_0 + h) = y(x_0) + \int_{x_0}^{x_0+h} f(\tau, y) d\tau \tag{11}$$

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Approximating integral by midpoint rule:

$$y(x = x_0 + h) = y(x_0) + hf(x_0 + \frac{h}{2}, y_{mid}) \qquad (12)$$

But

$$y_{mid} = y_0 + \frac{h}{2} f_0$$

Hence

$$y(x = x_0 + h) = y(x_0) + hf(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f_0) \qquad (13)$$

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Now the most popular Runge-Kutta method is fourth order (RK-4 method) and is written as:

$$m_1 = f_0 = f(x, y)$$
$$m_2 = f_1 = f(x + \frac{h}{2}, y + \frac{hm_1}{2})$$
$$m_3 = f_2 = f(x + \frac{h}{2}, y + \frac{hm_2}{2})$$
$$m_4 = f_3 = f(x + \frac{h}{2}, y + \frac{hm_3}{2})$$

Hence the solution is written as:

$$y(x = x_0 + h) = y(x_0) + \frac{h}{6}(f_0 + 2f_1 + 2f_2 + f_3) \tag{14}$$

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Or,

$$y(x = x_0 + h) = y_i + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4) \tag{15}$$

Here these m_1, m_2, m_3, m_4 notations are used only for our convenience.

(Q)Solve the differential equation $\frac{dy}{dx} = \frac{2y}{x}$ by using **Runge Kutta Method** for different values of **x,y,xvalue and h**.

The code to solve this differential equation using **Runge-Kutta Method** is as shown below:

```
program runge˙kutta
implicit none
real:: x,y,xvalue,h,dy,fx,m1,m2,m3,m4
integer::n,i
external fx
write(*,*)"give the values of x,y,xvalue,h"
read(*,*)x,y,xvalue,h
n=int((xvalue-x)/h+0.5)
```

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```
do i=1, n
m1=fx(x,y)
m2=fx(x+(h/2),y+(m1*h/2))
m3=fx(x+(h/2),y+(m2*h/2))
m4=fx(x+h, y+m3*h)
dy=(m1+2*m2+2*m3+m4)*h/6
```

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```
x=x+h
y=y+dy
write(*,*)i,x,y
enddo
write(*,*)"value of y at x=",x,"is",y
end program
real function fx(x,y)
real::x,y
fx=2.0*y/x
end function
!this programming is used to solve the differential equation
!dy/dx = 2*y/x by using Euler's method
```

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When this program is run the output is obtained as shown below:

runge1.png

Figure: Output of the differential equation $\frac{2y}{x}$ at different x=1.0, y=2.0, xvalue=2.0 and h=0.05

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When the same program is run with different h values, the following result is obtained.

runge2.png

Figure: Output of the differential equation $\frac{2y}{x}$ at different x=1.0, y=2.0, xvalue=2.0 and h=0.10, 0.15

Note: When the value of h changes, the number of iterations required to solve the differential equation changes, as does the output of the differential value.

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Second Order Differential Equation

Let us consider a second order differential equation:

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}) \tag{16}$$

One can solve above equation by using

$$\frac{dy}{dx} = z$$

So the above equation reduces to:

$$\frac{dz}{dx} = f(x, y, z) \tag{17}$$

We can now solve this equation using the RK-4 method.
The following is an example of a second order differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 9 \tag{18}$$

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Algorithms to solve this equation is given below:

- Define differential equation. Divide into two different equation like $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 9$ by substituting $\frac{dy}{dx} = z$(one equation) and $\frac{dz}{dx} + 2z + 5y = 9$(another equation).
- Define function according to RK-4 method.
- Give initial values of x, y, z and h.
- Give final value of x ie. x_f.
- And put

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$$m_1 = f(x_1, y_1, z_1)$$

$$p_1 = g(x_1, y_1, z_1)$$

$$m_2 = f(x_1 + \frac{h}{2}, y_1 + m_1\frac{h}{2}, z_1 + p_1\frac{h}{2})$$

$$p_2 = g(x_1 + \frac{h}{2}, y_1 + m_1\frac{h}{2}, z_1 + p_1\frac{h}{2})$$

$$m_3 = f(x_1 + \frac{h}{2}, y_1 + m_2\frac{h}{2}, z_1 + p_2\frac{h}{2})$$

$$p_3 = g(x_1 + \frac{h}{2}, y_1 + m_2\frac{h}{2}, z_1 + p_2\frac{h}{2})$$

$$m_4 = f(x_1 + h, y_1 + m_3 h, z_1 + p_3 h)$$

$$p_4 = g(x_1 + h, y_1 + m_3 h, z_1 + p_3 h)$$

$$m = (m_1 + 2m_2 + 2m_3 + m_4)/6$$

$$p = (p_1 + 2p_2 + 2p_3 + p_4)/6$$

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- Put

$$y_1 = y_1 + h * m$$

$$z_1 = z_1 + h * p$$

$$x_1 = x_1 + h$$

- Print x_1, y_1, z_1
-

$$f(x, y, z) = z$$

$$g(x, y, z) = 9 - 5y - 2z$$

These codes aid in the solution of the following equation (35):
$\dfrac{d^2y}{dx^2} = f(x, y, \dfrac{dy}{dx}) as$

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```
Program secondorder
implicit none
real:: x1,y1,z1,h,m1,m2,m3,m4,p1,p2,p3,p4,f,g,xf,m,p
integer::i,n
external f ,g
x1=1.0
y1=2.0
z1=3.0
xf=5.0
h=0.05
!you can make to give the values of
!x1,y1,z1,xf and h from the keyboard
n=(xf-1)/h+0.5
!here we defined no of intervals
```

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```
do i=1,n
m1=f(x1,y1,z1)
p1=g(x1,y1,z1)
m2=f(x1+h/2., y1+m1*h/2., z1+p1*h/2.)
p2=g(x1+h/2., y1+m1*h/2., z1+p1*h/2.)
m3=f(x1+h/2., y1+m2*h/2., z1+p2*h/2.)
p3=g(x1+h/2., y1+m2*h/2., z1+p2*h/2.)
```

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```
m4=f(x1+h, y1+m3*h, z1+p3*h)
p4=g(x1+h, y1+m3*h, z1+p3*h)
m=(m1+2.0*m2+2.0*m3+m4)/6.
p=(p1+2.0*p2+2.0*p3+p4)/6.
y1=y1+h*m
z1=z1+h*p
x1=x1+h
end do
write(*,*)x1,y1,z1
end program
```

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```
real function f(x,y,z)
real:: x,y,z
f=z
end function
real function g(x,y,z)
real:: x,y,z
g=9.0-5.0*y-2.0*z
end function
```

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```
!here we have a second order differential equation
!d2y/dx2+2.0*dy/dx + 5.0*y=9.0
!to solve this we substitute dy/dx=z(one equation)
!so that the above second order equation reduces
!to dz/dx +2.0*z+5.0*y=9(another equation)
!we use RK-4 method
```

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When we run this program the output obtained for $x_1 = 1.0, y_1 = 2.0, z_1 = 3.0, x_f = 5.0, h = 0.05$ is as shown below:

diff.png

Figure: Solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 9$ for $x_1 = 1.0, y_1 = 2.0, z_1 = 3.0, x_f = 5.0, h = 0.05$

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