Horseshoe Vortex Method

Enrique Ortega

e.ortega@upc.edu

Horseshoe Vortex Method

- The Horseshoe Vortex Method (HVM) solves numerically the Prandtl's lifting-line problem, which is the simplest model for analyzing lifting wings.
- Despite the simplicity of the HVM, it allows computing lift, moment, induced drag and spanwise load distribution with satisfactory accuracy. The HVM can also model wing sweep, twist and deflection of control surfaces.
- The HVM does not account for thickness effects and, consequently, it is only suitable for thin wings. In addition, this method is not appropriate for wings having small aspect ratios (e.g. A<5). Such flows require an improved wake model.
- Similarly to the DVM (2D), the Kutta condition in the HVM is accounted for in an implicit manner, through a proper selection of the vortex and control point locations.

1. Geometry discretization

The wing is discretized by *N* horseshoe vortex panels (HVs) *i* in the spanwise direction (Figure 1). The head of each vortex is aligned in the y-direction, and its midpoint lies along the wing's quarter chord line (elementary wings). The panels can be uniformly distributed along the span or concentrated at certain zones of the wing where higher gradients are expected (e.g. wing tips).

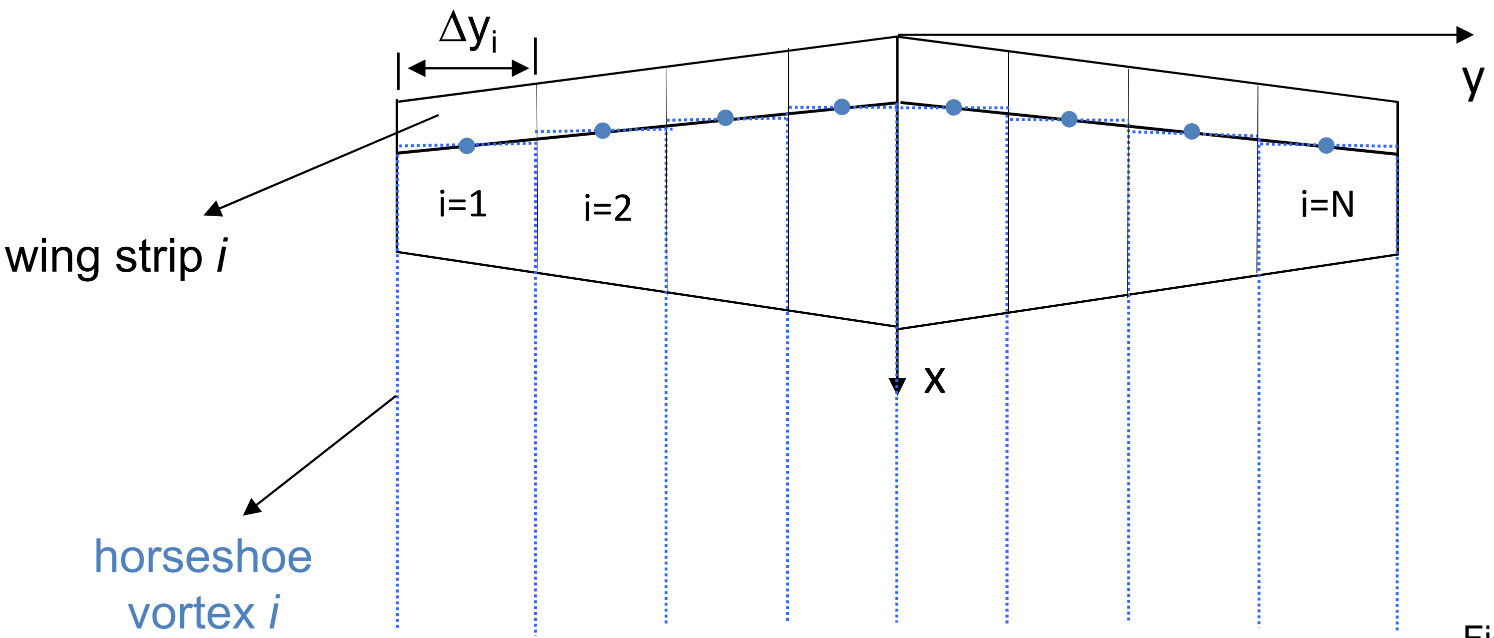


Figure 1

Then, at each wing panel *i* a horseshoe vortex element is defined according to Figure 2 below.

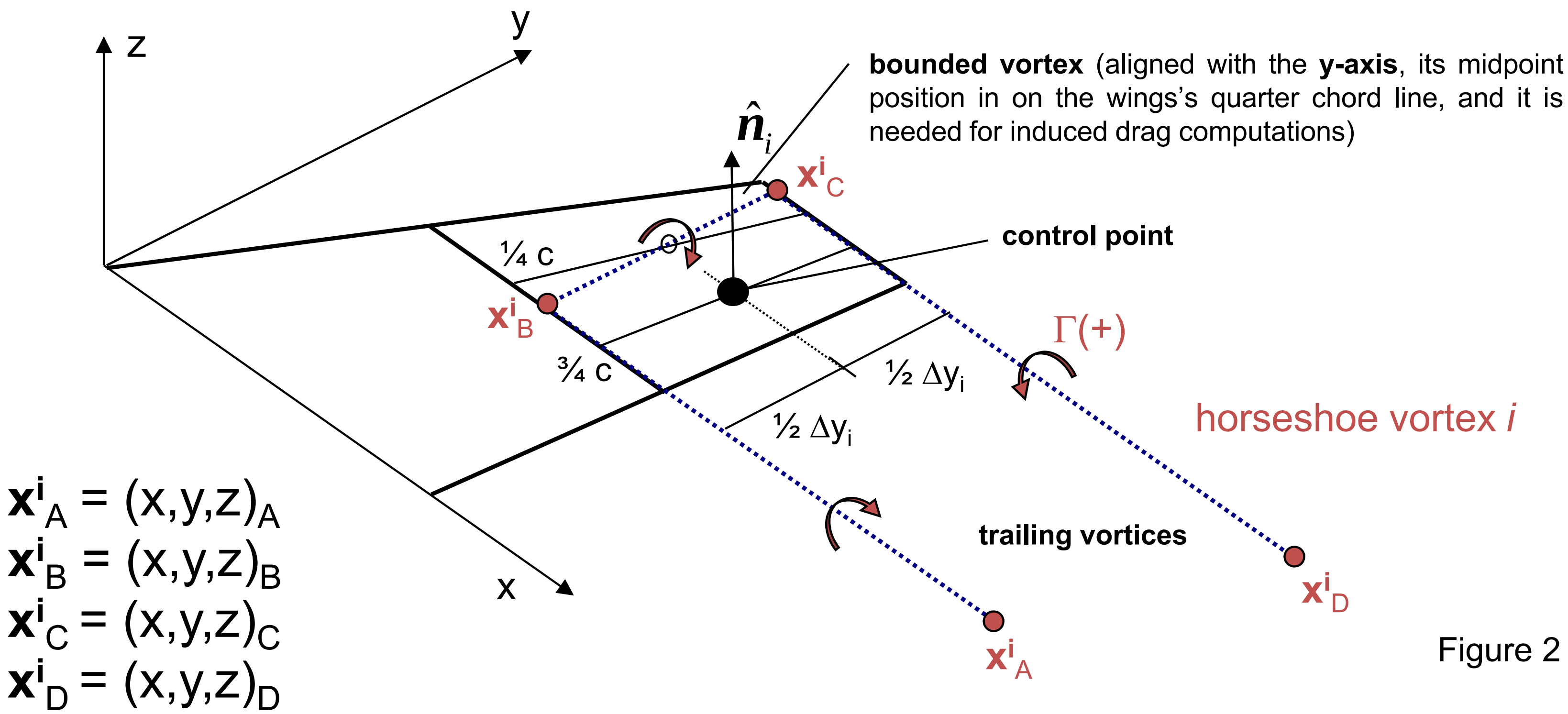


Figure 2

Tip: to define the wing geometry, all the lengths can be adimensionalized with the span (b). In trapezoidal wings, it is only necessary to define the aspect ratio (A), taper ratio (λ) and quarter chord sweep angle (Δ_{25}). Generate the HVs is easier if the locations of the vortex head midpoints along the wing's quarter chord line are calculated first; then points B and C can be then obtained using the panel span.

Each horseshoe vortex *i* is defined by the points $\mathbf{x}^{\mathbf{i}}_A,\mathbf{x}^{\mathbf{i}}_B,\mathbf{x}^{\mathbf{i}}_C,\mathbf{x}^{\mathbf{i}}_D$. The vortex line segment pointing from x_B to x_C is the ***bounded (head) vortex*** and the vortex line segments pointing from $\mathbf{x}_A$ to $\mathbf{x}_B$ and from $\mathbf{x}_C$ to $\mathbf{x}_D$ are the ***trailing vortices***.

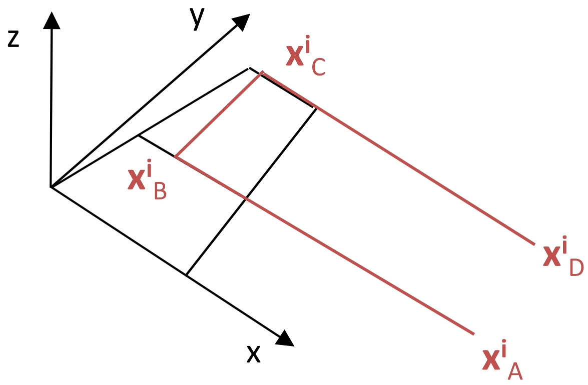
Note that the direction in which the vortex lines are defined corresponds to a positive circulation Γ (right-hand rule).

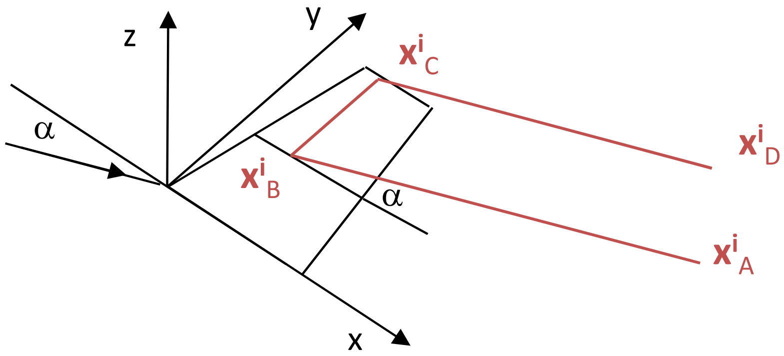
The **bounded vortex** is oriented along the span direction (y-axis) and its midpoint is placed along the line at ¼ of the panel chord. The **control point** (where the solid boundary condition is enforced) is located at the midpoint of the line at ¾ of the panel chord (Weissinger's extended lifting-line model, see NACA 1120, 1947). This definition, similar to that adopted for the 2D DVM method, allows satisfying the Kutta condition at the trailing edge of the panel in an approximate manner. The midpoint location of the bounded vortex is necessary for induced drag computations.

A unit normal vector to the section's zero-lift line (ZLL) at the control point *i* ($\mathbf{n}_i$) is needed to enforce the solid boundary condition. The ZLL is the direction of $\mathbf{V}_\infty$ for which the lift of the section is zero. This definition of the panel normal vectors allows modelling the effects of the zero-lift angle of attack of the section, wing geometric twist, flap or aileron deflection, etc.

Theoretically, trailing vortices cannot end in the fluid and, consequently, they must go to infinite. As a practical approach, a **trailing vortex length of about 20b** seems to be enough. Vortex lines located beyond this distance have a negligible effect on the wing's control points.

The points $\mathbf{x}^i_A$ and $\mathbf{x}^i_D$ can be defined in such a way that the wake is aligned with the plane of the wing (A) or aligned with the freestream velocity vector (B). However, in general, only a minor difference in the numerical results is observed.





(A) $$\begin{cases} x_A = x_D = 20b \\ y_A = y_B \text{ and } y_D = y_C \\ z_A = z_D = 0 \end{cases}$$

(B) $$\begin{cases} x_A = x_D = 20b \\ y_A = y_B \text{ and } y_D = y_C \\ z_A = x_A \tan\alpha \text{ and } z_D = x_D \tan\alpha \end{cases}$$

Note that the approaches (A) and (B) are both inaccurate from a physical point of view. The simplicity of the HVM makes it difficult to account for the wake real behavior, and thus, the model is not suitable for wings having low aspect ratio (where the wake interaction and non-nonlinear effects can be significant).

Summarizing:

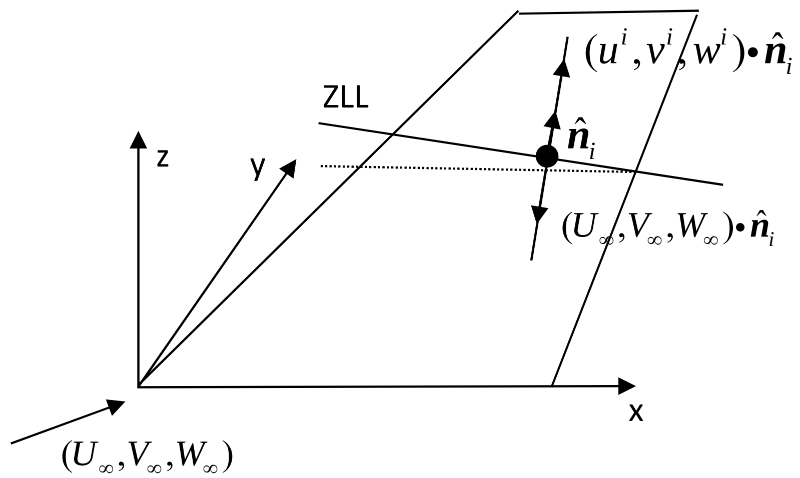
- The wing is discretized by an arrangement of N panels along the span.
- For each panel *i*, a horseshoe vortex element is defined by the points $(x,y,z)^i_A$, $(x,y,z)^i_B$, $(x,y,z)^i_C$ and $(x,y,z)^i_D$, according to Figure 2. Among other options, trailing vortices $A \rightarrow B$ and $C \rightarrow D$ can be placed in the plane of the wing, or aligned with the freestream velocity direction.
- A control point $(x,y,z)^i_{cp}$ is defined at each panel *i* (see Figure 2). In addition, at each control point, a unit normal vector ($\mathbf{n}_i$) to the section zero-lift line is calculated. Also, for each panel i, the midpoint location of the bounded vortex is necessary for induced drag computations.
- At this stage, the area of each wing panel S_i can be also computed. This data is needed in order to calculate the load distribution along the span.

2. The system of equations for Γ

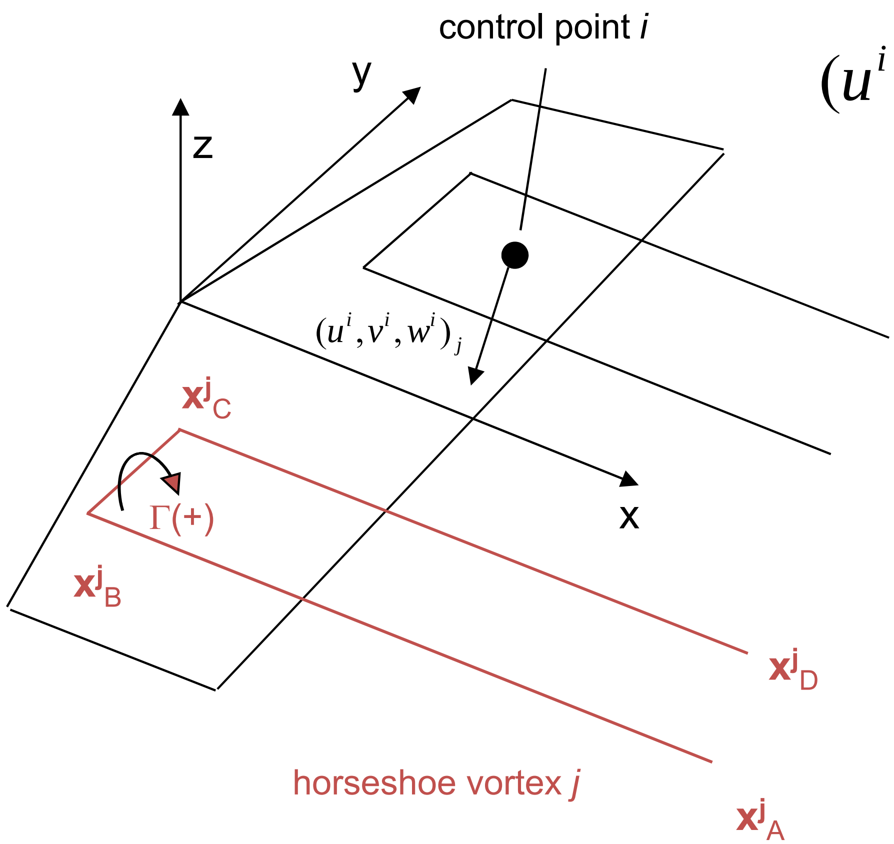
As the horseshoe vortex element satisfies the Laplace equation for the potential, only proper boundary conditions must be applied. Similar to the 2D DVM, in this case the velocity vector is forced (at each control point) to be tangent to the **section's zero-lift line** by canceling its normal component. This results in the following equation for each control point *i*

$$\left[(u^i, v^i, w^i) + (U_\infty, V_\infty, W_\infty)\right] \bullet \hat{\boldsymbol{n}}_i = 0 \quad (3)$$

where (u^i,v^i,w^i) are the components of the velocity induced at the control point *i* by all the horseshoe vortices Γ_j (j=1,N) spanned along the wing.



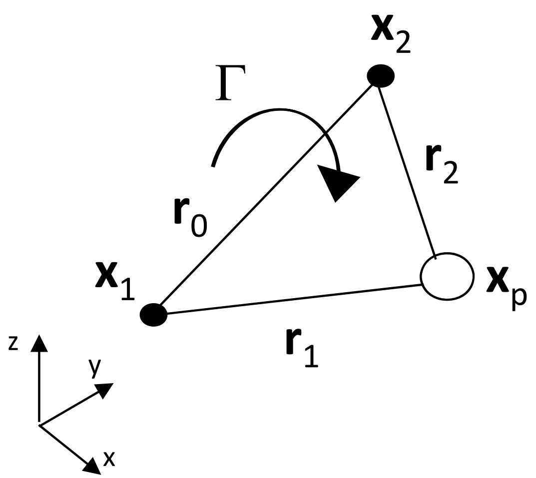
The velocity induced at a control point *i* by a single horseshoe vortex *j*, $(u^i,v^i,w^i)_j$, is obtained by adding the velocities induced by the bounded vortex and the trailing vortices of the horseshoe element *j*, i.e.



$$
\begin{aligned}
(u^i,v^i,w^i)_j = \; & (u^i,v^i,w^i)_{\text{due to segment } \boldsymbol{x}_A^j \rightarrow \boldsymbol{x}_B^j} \\
& + (u^i,v^i,w^i)_{\text{due to segment } \boldsymbol{x}_B^j \rightarrow \boldsymbol{x}_C^j} \\
& + (u^i,v^i,w^i)_{\text{due to segment } \boldsymbol{x}_C^j \rightarrow \boldsymbol{x}_D^j}
\end{aligned}
\qquad (4)
$$

Notice that the orientation of the vortex lines defining the horseshoe element corresponds with a positive circulation Γ.

The velocity induced by a straight vortex line segment is calculated by the Biot-Savart law. According to the latter, the velocity induced at a point $(x,y,z)_p$ due to a vortex line pointing from (x_1,y_1,z_1) to (x_2,y_2,z_2) is



$$(u,v,w)_P = \frac{\Gamma}{4\pi}\left[\frac{\boldsymbol{r}_0}{\left|\boldsymbol{r}_1\times\boldsymbol{r}_2\right|}\cdot\left(\frac{\boldsymbol{r}_1}{r_1}-\frac{\boldsymbol{r}_2}{r_2}\right)\right]\frac{\boldsymbol{r}_1\times\boldsymbol{r}_2}{\left|\boldsymbol{r}_1\times\boldsymbol{r}_2\right|} \quad (5)$$

with

$$\boldsymbol{r}_0 = \left(x_2-x_1\,,y_2-y_1\,,z_2-z_1\right)^{\mathrm{T}}$$

$$\boldsymbol{r}_1 = \left(x_p-x_1\,,y_p-y_1\,,z_p-z_1\right)^{\mathrm{T}}$$

$$\boldsymbol{r}_2 = \left(x_p-x_2\,,y_p-y_2\,,z_p-z_2\right)^{\mathrm{T}}$$

$$r_1 = \left((x_p-x_1)^2+(y_p-y_1)^2+(z_p-z_1)^2\right)^{1/2}$$

$$r_2 = \left((x_p-x_2)^2+(y_p-y_2)^2+(z_p-z_2)^2\right)^{1/2}$$

Important! In order to avoid dividing by zero in Eq. (5), set $(u,v,w)_p = 0$ if $|\mathbf{r}_1 \mathrm{x} \mathbf{r}_2|$ or r_1 or r_2 are smaller than a small constant value.

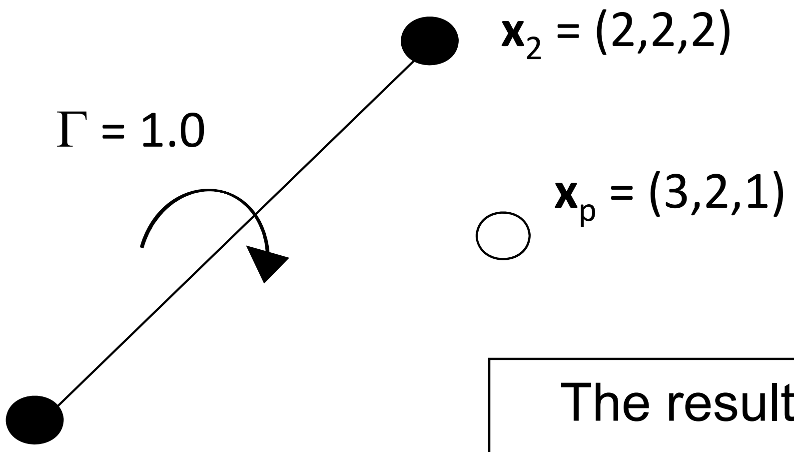
(see [1] for further details)

```fortran
Subroutine VORTXL (x1,y1,z1,x2,y2,z2,xp,yp,zp,gamma,u,v,w)

 Implicit None

 real*8 , intent(in)  :: x1,y1,z1,x2,y2,z2,xp,yp,zp,gamma
 real*8 , intent(out) :: u,v,w

 real*8 :: a,b,c,d,r1,r2,ror1,ror2,com,pi

!   Start calculations

 pi = 3.14159265d0

 a  =  (yp-y1)*(zp-z2) - (zp-z1)*(yp-y2)
 b  = -(xp-x1)*(zp-z2) + (zp-z1)*(xp-x2)
 c  =  (xp-x1)*(yp-y2) - (yp-y1)*(xp-x2)

 d  = a*a + b*b + c*c
 r1 = dsqrt( (xp-x1)*(xp-x1) + (yp-y1)*(yp-y1) + (zp-z1)*(zp-z1) )
 r2 = dsqrt( (xp-x2)*(xp-x2) + (yp-y2)*(yp-y2) + (zp-z2)*(zp-z2) )

 if ( (d.gt.1.0d-6) .and. (r2.gt.1.0d-6) .and. (r1.gt.1.0d-6) ) then
   ror1 = (x2-x1)*(xp-x1)+(y2-y1)*(yp-y1)+(z2-z1)*(zp-z1)
   ror2 = (x2-x1)*(xp-x2)+(y2-y1)*(yp-y2)+(z2-z1)*(zp-z2)
   com  = (gamma/(4.*pi*d))*((ror1/r1)-(ror2/r2))
   u    = a * com
   v    = b * com
   w    = c * com
 else
   u = 0.d0
   v = 0.d0
   w = 0.d0
 endif

 End Subroutine VORTXL
```

Example of a fortran subroutine for computing Eq. (5).

Notice that, in order to compute the velocity induced by a horseshoe element, this routine must be called 3 times, one per vortex line segment composing the horseshoe element. Then, the individual induced velocities due to each vortex segment must be added according to Eq. (4).

In order to check your calculations, consider the vortex line segment with unit strength in the figure below and calculate the components of the velocity induced at the control point $\mathbf{x}_p$ = (3,2,1).



$\mathbf{x}_1$ = (1,1,1)

The result for the velocity components at $\mathbf{x}_p$ is

u = -0.01779

v = 0.03559

w = -0.01779

The system of equations due to Eqs. (3) can be rearranged as

$$\left[\sum_{j=1}^{N}\Gamma_j(\tilde{u}^i,\tilde{v}^i,\tilde{w}^i)_j\right]\cdot\hat{\boldsymbol{n}}_i=-(U_\infty,V_\infty,W_\infty)\cdot\hat{\boldsymbol{n}}_i \quad i=1,N \qquad (6)$$

where $(\tilde{u}^i,\tilde{v}^i,\tilde{w}^i)_j$ are the components of the velocity induced at the control point *i* by a horseshoe vortex *j* having unit strength Γ_j = 1.0 (Eq.4). Remember that we assume Γ_j = 1.0 at this stage because Γ_j is not known in advance.

Now, we define the ***influence coefficient*** $\mathbf{a}^{\mathbf{i}}_{j}$ as the normal velocity component induced at a control point (x_i,y_i,z_i) due to a horseshoe vortex *j* with unit strength

$$a^i_j=(\tilde{u}^i,\tilde{v}^i,\tilde{w}^i)_j\cdot\hat{\boldsymbol{n}}_i \qquad (7)$$

Then, we can write the system of equations (6) as follows

$$\begin{bmatrix} a_1^1 & a_2^1 & \cdots & a_N^1 \\ a_1^2 & a_2^2 & \cdots & a_N^2 \\ \vdots & \vdots & \vdots & \vdots \\ a_1^N & a_2^N & \cdots & a_N^N \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_N \end{bmatrix} = -(U_\infty, V_\infty, W_\infty) \cdot \begin{bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \vdots \\ \hat{\boldsymbol{n}}_N \end{bmatrix} \qquad (8)$$

where

$$a_j^i = (\tilde{u}^i, \tilde{v}^i, \tilde{w}^i)_j \cdot \hat{\boldsymbol{n}}_i$$

$$\begin{aligned} (u^i, v^i, w^i)_j = \quad & (u^i, v^i, w^i)_{\text{due to segment } \boldsymbol{x}_A^j \to \boldsymbol{x}_B^j} \\ & + (u^i, v^i, w^i)_{\text{due to segment } \boldsymbol{x}_B^j \to \boldsymbol{x}_C^j} \\ & + (u^i, v^i, w^i)_{\text{due to segment } \boldsymbol{x}_C^j \to \boldsymbol{x}_D^j} \end{aligned}$$

The assembly of the system of equations (8) can be performed by an outer *do loop* over the control points and an inner *do loop* over the horseshoe vortices.

do i = 1,N loop over the control points

n$_i$ normal vector at the control point *i*

do j = 1,N loop over the horseshoe vortices

horseshoe vortex *j*

$$a^i_j = (\tilde{u}^i, \tilde{v}^i, \tilde{w}^i)_j \cdot \hat{\boldsymbol{n}}_i \qquad (*)$$

end do

$$b^i = -(U_\infty, V_\infty, W_\infty) \cdot \hat{\boldsymbol{n}}_i$$

end do

(*) At this stage it is possible to account for the symmetry of the problem avoiding modeling the whole wing, see for instance [1]. With a similar procedure it is also possible to take into account other effects such as ground proximity.

3. Loads calculation

Once the vortices strength Γ_i are known, the local lift generated by each panel can be computed using the Kutta-Joukowski theorem, i.e.

$$\Delta L^i = \rho U_\infty \Gamma_i \Delta y_i \quad (9)$$

Then, the total wing's lift and pitching moment (about the leading edge) can be obtained by adding all the panel contributions (9). This leads to

$$L = \sum_{i=1}^{N} \Delta L^i \quad (10)$$

$$M_{le} = -\sum_{i=1}^{N} \Delta L^i \left(x_i^\Gamma - x_{le} \right) \cos\alpha \quad (11)$$

where $x^\Gamma_i=\frac{1}{2}(x^i_B+x^i_C)$ is the midpoint location of the bounded vortex (we can consider lift acting at this point) and typically $x_{le}=0$ (root chord leading edge). Since α is small, $\cos\alpha\approx 1$ can be adopted in Eq. (11).

Non-dimensional lift and moment coefficients are obtained from Eqs. (10-11) as

$$C_L = \frac{L}{\frac{1}{2}\rho U_\infty^2 S} = \frac{2}{U_\infty S}\sum_{i=1}^{N}\Gamma_i \Delta y_i \qquad (12)$$

$$C_{M\,le} = \frac{M_{le}}{\frac{1}{2}\rho U_\infty^2 S \bar{\bar{c}}} = -\frac{2}{U_\infty S \bar{\bar{c}}}\sum_{i=1}^{N}\Gamma_i (x_i^\Gamma - x_{le})\Delta y_i \cos\alpha \qquad (13)$$

where S is the wing planform area and $\bar{\bar{c}}$ is the mean aerodynamic chord. The local (spanwise) lift distribution $C_l(y)$ can be computed by

$$C_l(y) = \frac{\Delta L^i}{q_\infty S_i} = \frac{\rho U_\infty}{\frac{1}{2}\rho U_\infty^2 S_i}\Gamma_i \Delta y_i = \frac{2\Gamma_i \Delta y_i}{U_\infty S_i} \qquad (14)$$

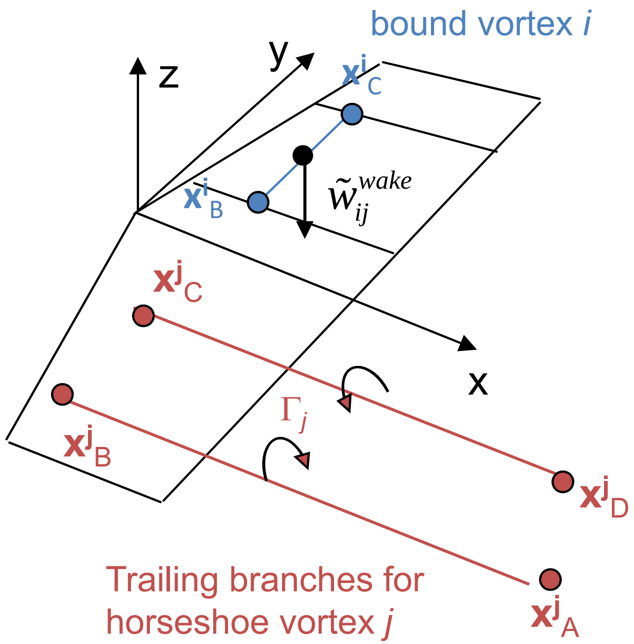
where S_i is the surface area of the wing panel *i*.

Induced drag calculation

The contribution of each panel *i* to the wing's induced drag can be obtained by rotating the local lift vector L^i an angle corresponding to the induced angle of attack α_i^{ind} at the same section. The latter can be obtained by

$$\alpha_i^{ind} \approx \left(\frac{-w^i}{U_\infty}\right) = -\frac{1}{U_\infty}\sum_{j=1}^{N}\Gamma_j \tilde{w}_{ij}^{wake} \qquad (15)$$

where $\tilde{w}_{ij}^{wake}$ is the z-component of the velocity induced at the midpoint of the bounded vortex *i* (we can consider the local lift acting at this point) by the trailing branches of a horseshoe vortex *j*. Note that only the trailing vortices must be taken into account when $\tilde{w}_{ij}^{wake}$ is computed (*).



(*) These velocities can be obtained during the computation of the influence coefficients or once we know the value of the circulation.

Therefore, using Eq. (15), the induced drag for each panel *i* results

$$\Delta D_i^{ind} = \Delta L^i \alpha_i^{ind} = -\Delta L^i \frac{1}{U_\infty} \sum_{j=1}^{N} \Gamma_j \tilde{w}_{ij}^{wake} \qquad j = 1, N \qquad (16)$$

and adding all the contributions, the induced drag for the wing is

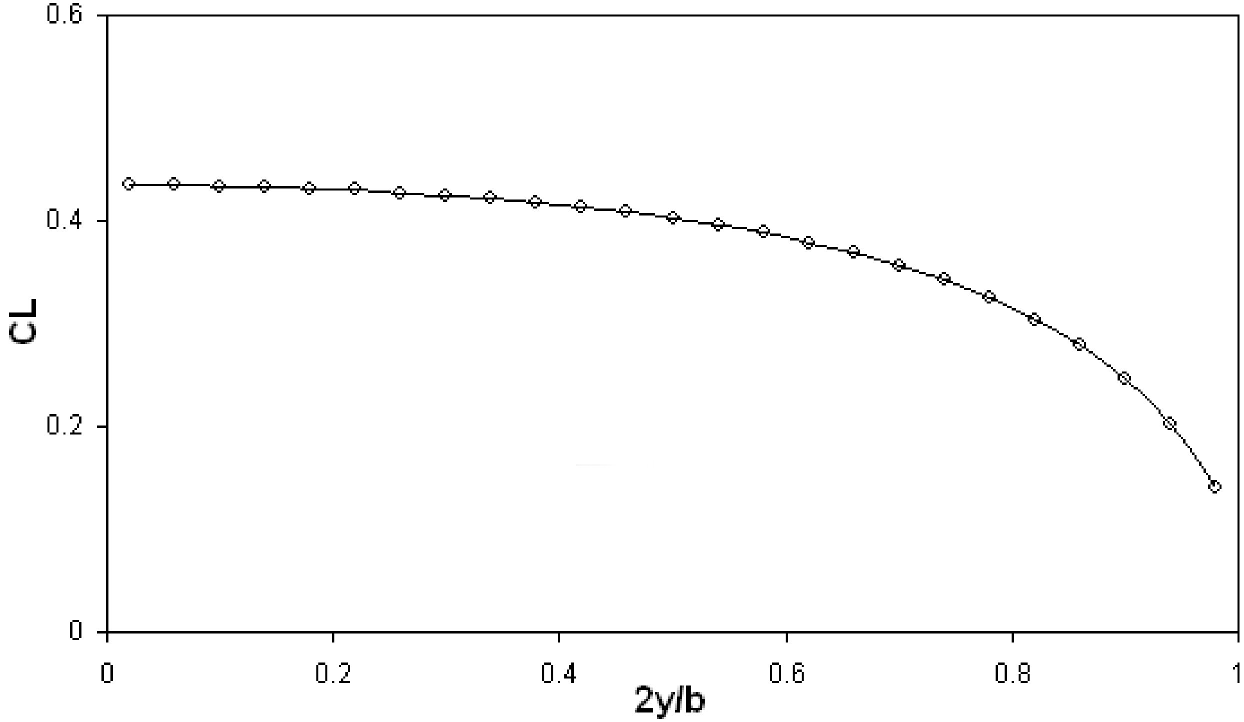
$$D^{ind} = \sum_{i=1}^{N} \Delta D_i^{ind} \qquad i = 1, N \qquad (17)$$

From the expressions above, the non-dimensional induced drag coefficient can be computed as

$$C_D^{ind} = \frac{D^{ind}}{q_\infty S} = -\frac{2}{U_\infty^2 S} \sum_{i=1}^{N} \left[\Gamma_i \Delta y_i \left(\sum_{j=1}^{N} \Gamma_j \tilde{w}_{ij}^{wake} \right) \right] \qquad (18)$$

4. Numerical results

Numerical results computed for a planar rectangular wing with aspect ratio A = 5 are presented below for an angle of attack α = 5º. The discretization employed consists of 25 panels uniformly distributed along the semispan.



Distribution of Cl along the semispan for α=5º.

α = 5º	
CL	0.34620
CM_{le}	-0.08622
CD_i	0.00754

1. Katz J., Plotkin A. Low speed aerodynamics: from wing theory to panel methods. McGraw-Hill series in aeronautical and aerospace engineering (1991).

Enrique Ortega
e.ortega@upc.edu