The Discrete Vortex Method

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The lumped vortex element

For a symmetrical airfoil, the thin-airfoil theory adopts a distribution of circulation that satisfies the problem boundary conditions and is given by

$$\gamma(\theta) = 2V_{\infty}\alpha \frac{1+\cos\theta}{\sin\theta} \qquad (1)$$

where $0 \leq \theta \leq \pi$ is the transformed x variable along the airfoil chord.

From a far-field point of view, we can replace the circulation density along the airfoil (Eq.(1)) by a **single vortex that generates the same total circulation**, i.e.

$$\Gamma = \int_0^c \gamma(x)dx \qquad (2)$$

In this way, the original problem is simplified because the airfoil can be seen as a flat plate having only a single discrete vortex with lumped circulation Γ. In order to work with this *"lifting element"*, we need to determine the location of the vortex and how to apply the boundary conditions.

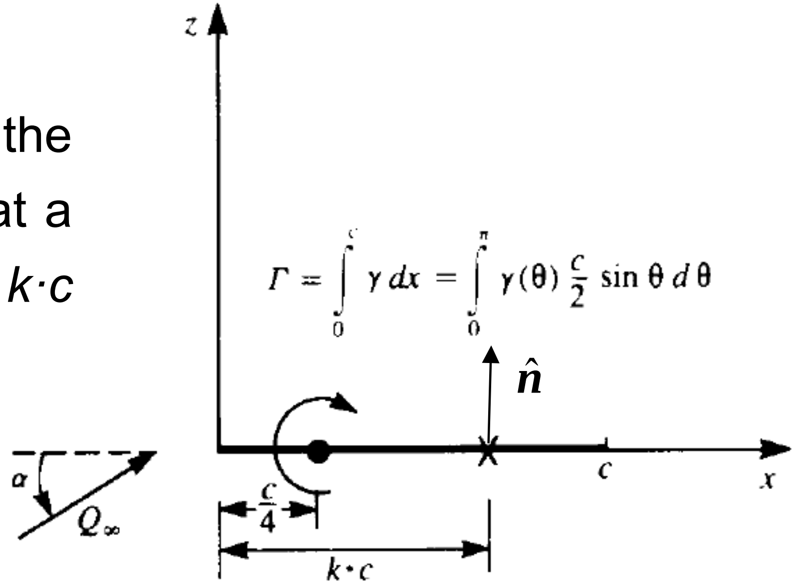
Location of the lumped vortex

Since the center of pressure of a symmetrical airfoil is located at the quarter-chord point, the lift generated can be considered to act on the same point ($C_n \approx C_l$). Consequently, the discrete lumped vortex must be also located at c/4 ($l=\rho_\infty U_\infty \Gamma$).

Boundary conditions

According to the discrete method proposed, the **streamline boundary condition** is also enforced at a single point. Therefore, at an unknown location $x = k \cdot c$ we can state

$$V_\Gamma^n + V_\infty^n = -\frac{\Gamma}{2\pi(kc - c/4)} + U_\infty \alpha = 0 \quad (3)$$



Then, assuming that $\Gamma = \pi c U_\infty \alpha$ (thin-airfoil theory), the parameter ***k = 3/4*** is readily determined from Eq. (3). The point x = 3/4c is usually named *collocation point*.

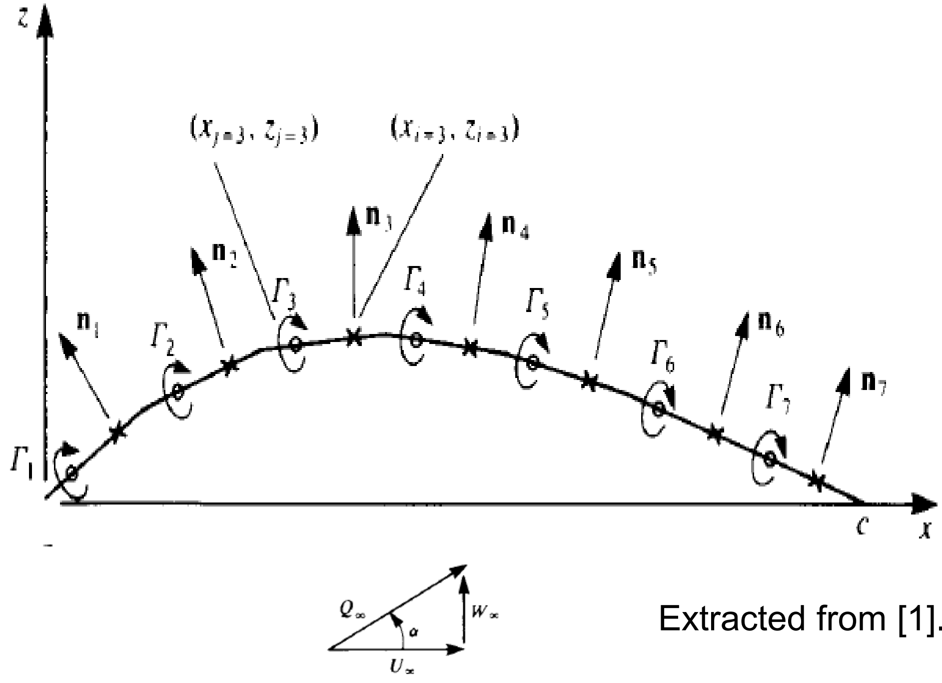
Note that the definition adopted for Γ satisfies the **Kutta condition** at the trailing edge. Thus, this property is also inherited in the discrete problem.

Discrete Vortex Method

- The Discrete Vortex Method (DVM) provides a numerical (approximated) solution for the fundamental equation of the thin-airfoil problem.
- In the DVM the airfoil's camber line is discretized by consecutive flat panels (lumped-vortex elements). The streamline boundary condition (slip condition for the velocity) is enforced at each panel's control point. The Kutta condition is fulfilled in an implicit manner.
- The DVM allows computing the airfoil lift and moment characteristics, and the ΔCp distribution across the mean line.
- The DVM is only suitable for thin airfoils and accounts for the effects of the angle of attack and camber.
- The DVM also allows the study of airfoils interactions (ground effect, multi-element airfoils, etc.) in a very simple way.

1. Geometry discretization

The airfoil's camber line is discretized by *N* points and *M=N-1* flat panels (lumped-vortex elements). In order to refine the discretization near the leading and trailing edges (where higher gradients are expected) a ***full cosine*** distribution can be used, i.e.

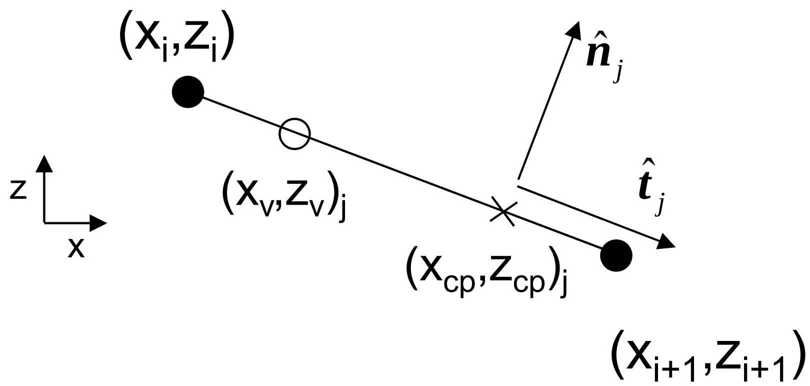


Extracted from [1].

$$x_i = \frac{c}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right) \quad i = 1, \ldots, N \qquad (4)$$

The airfoil coordinates can be obtained from analytical expressions or tabulated data. Recall that NACA 4 and 5-digits airfoils have analytical expressions defining the mean line and the thickness distribution (see for instance [1]).

Once the discretization of the camber line is performed, some geometrical data must be computed for each panel *j*, namely: unit tangent and normal vectors, the coordinates of the vortex and the control points, and the panel length (c_i). This information can be obtained for each panel as follows



$$\Delta x = x_{i+1} - x_i \quad , \quad \Delta z = z_{i+1} - z_i$$

$$c_j = \sqrt{\Delta x^2 + \Delta z^2} \quad \text{panel length} \tag{5}$$

$$\hat{\boldsymbol{n}}_j = [\frac{-\Delta z}{c_j}, \frac{\Delta x}{c_j}]^{\mathrm{T}} \quad \hat{\boldsymbol{t}}_j = [\frac{\Delta x}{c_j}, \frac{\Delta z}{c_j}]^{\mathrm{T}}$$

At each panel, the point vortex $(x_v,z_v)_j$ is located at the quarter panel chord and the control point $(x_{cp},z_{cp})_j$ at ¾ of the panel chord. As seen before, these choices allow obtaining the exact solution of the thin-airfoil problem (see Eq. (3)) at each lumped-vortex element.

2. The system of equations for the vortex strengths

In order to satisfy the streamline boundary condition, at each control point the normal velocity must be set to zero. This yields a set of M linear equations

$$\left[U_\infty\left(\cos\alpha,\sin\alpha\right)+\sum_{j=1}^{m}\Gamma_j\left(\tilde{u}_j^i,\tilde{w}_j^i\right)\right]\cdot\left(n_x^i,n_z^i\right)=0\ \text{ for }\ i=1,M \qquad (6)$$

where $\left(\tilde{u}_j^i,\tilde{w}_j^i\right)$ is the velocity induced at a control point *i* by a vortex of ***unit*** intensity located at the quarter chord point of a panel *j*, and Γ_j are the unknowns sought. The induced velocities can be calculated as

$$\tilde{u}(x,z)=\frac{1}{2\pi}\frac{z-z_0}{r^2}\quad;\quad\tilde{w}(x,z)=-\frac{1}{2\pi}\frac{x-x_0}{r^2} \qquad (7)$$

$$\text{with}\quad r^2=(x-x_0)^2+(z-z_0)^2$$

Note that in Eqs. (7) (x,z) are the coordinates of the point *i* where the velocity is evaluated and (x_0,z_0) is the position of the point vortex *j*.

The Eqs. (3) can be rearranged moving the freestream values to the RHS, i.e.

$$\sum_{j=1}^{m}\Gamma_j\left(\tilde{u}_j^i,\tilde{w}_j^i\right)\cdot\left(n_x^i,n_z^i\right)=-U_\infty\left(\cos\alpha,\sin\alpha\right)\cdot\left(n_x^i,n_z^i\right) \qquad (8)$$

The coefficients of the LHS are called the influence coefficients and they are functions of the geometry only. These result

$$A_j^i=\tilde{u}_j^i\cdot n_x^i+\tilde{w}_j^i\cdot n_z^i \qquad (9)$$

which is the normal component of the velocity induced at control point *i* by a point vortex of unit strength located at panel *j*. The terms on the RHS vector are functions of the geometry and the flight conditions(*)

$$\mathrm{RHS}_i=-U_\infty\cdot(\cos\alpha\cdot n_x^i+\sin\alpha\cdot n_z^i) \qquad (10)$$

Using Eqs. (9) and (10), the system of equations can be written in matrix form as

$$\mathbf{A}\cdot\boldsymbol{\Gamma}=\mathbf{RHS} \qquad (11)$$

(*) Small angles can be assumed, i.e. $\sin\alpha\approx\alpha$ (in rad) and $\cos\alpha\approx 1$.

The assembly of the system of equations (8) can be performed by an outer *do loop* over the control points and an inner *do loop* over the panels, i.e.

do i = 1,M loop over the control points

(x_i,z_i) coordinates of the control point for panel *i*

$\mathbf{n}_i$ unit normal vector at the control point *i*

do j = 1,M loop over the panels

a. Compute the induced velocity at (x_i,z_i) due to a lumped vortex at (x_j,z_j) (Eqs. (7))

b. Compute the influence coefficient (Eq. (9))

$$a(i, j) = (u^i, w^i)_j \cdot \hat{\boldsymbol{n}}_i$$

end do

$$RHS\,(i) = -U_\infty(\cos\alpha, \sin\alpha)\cdot\hat{\boldsymbol{n}}_i \qquad \text{Eq. (10)}$$

end do

3. Loads calculation

Once the Γ distribution is known, the panel load can be computed by the Kutta-Joukowsky theorem. Hence, the lift force generated by a panel *j* (per unit span) is

$$\Delta l_j = \rho U_\infty \Gamma_j \qquad (12)$$

and summing the contributions of each panel, the total airfoil lift results

$$l = \sum_{j=1}^{M} \Delta l_j = \rho U_\infty \sum_{j=1}^{M} \Gamma_j \;\rightarrow\; C_l = \frac{2}{U_\infty c} \sum_{j=1}^{M} \Gamma_j \qquad (13)$$

The aerodynamic moment can be obtained by

$$M_{ref} = -\sum_{j=1}^{M} \Delta l_j (x_j - x_{ref}) \cos\alpha \;\rightarrow\; C_{m_{ref}} = -\frac{2}{U_\infty c^2} \sum_{j=1}^{M} \Gamma_j (x_j - x_{ref}) \cos\alpha \qquad (14)$$

where x_j is the location of the vortex *j* and x_{ref} is the reference point adopted for computing the moment.

Finally, using the elemental lift generated at each panel (Eq.(12)), the pressure jump across a panel (per unit span) can be obtained by

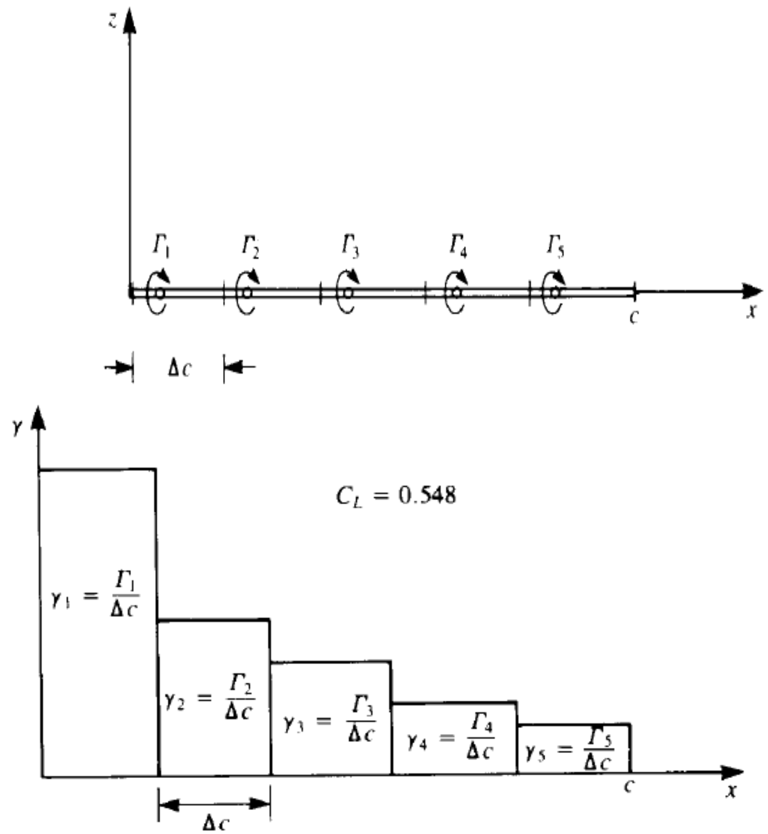
$$\Delta p_j = \frac{\Delta l_j}{c_j} \quad \rightarrow \quad \Delta C p_j = \frac{2}{U_\infty} \frac{\Gamma_j}{c_j} \qquad (15)$$

where c_j is the panel chord. Notice that we can assume $\cos(\alpha) \cong 1$ in Eq. (14) because the angle of attack must be small for the potential analysis to be valid.

Application example (extracted from [1])

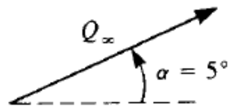
Flat plate with a uniform 5 panels discretization ($\Delta c = c/5$). The system of equations (6) results

$$\frac{1}{\pi\,\Delta c}\begin{pmatrix} -1 & 1 & \frac{1}{3} & \frac{1}{5} & \frac{1}{7} \\ -\frac{1}{3} & -1 & 1 & \frac{1}{3} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{3} & -1 & 1 & \frac{1}{3} \\ -\frac{1}{7} & -\frac{1}{5} & -\frac{1}{3} & -1 & 1 \\ -\frac{1}{9} & -\frac{1}{7} & -\frac{1}{5} & -\frac{1}{3} & -1 \end{pmatrix}\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \end{pmatrix} = -Q_\infty \sin\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$



and the solution of the system gives

$$\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \end{pmatrix} = \pi \Delta c Q_\infty \sin\alpha \begin{pmatrix} 2.46092 \\ 1.09374 \\ 0.70314 \\ 0.46876 \\ 0.27344 \end{pmatrix}$$



Piecewise constant representation for the distribution of vorticity at $\alpha = 5º$.

Important! Grid convergence must be evaluated.

1. Katz J., Plotkin A. Low speed aerodynamics: from wing theory to panel methods. McGraw-Hill series in aeronautical and aerospace engineering (1991).
2. Abbot I. H., Doenhoff A. E. Theory of wing sections. Dover Publicatoins Inc., New York (1959)

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