Pergamon

Transpn Res.-B, Vol. 31, No. 2, pp. 83–102, 1997

0191-2615/97 $17.00 + 0.00

PII: S0191-2615(96)00017-3

A CONTINUUM THEORY OF TRAFFIC DYNAMICS FOR FREEWAYS WITH SPECIAL LANES

CARLOS F. DAGANZO

Department of Civil Engineering and Institute of Transportation Studies, University of California, Berkeley, CA 94720, U.S.A.

(Received 14 July 1995; in revised form 24 March 1996)

Abstract—This paper presents a generalized theory of kinematic waves for freeways with two vehicle types and a set of lanes reserved for one of the vehicle classes. The theory is not restricted to freeways on which the special lanes are clearly identified by signs and pavement markings; e.g. for high occupancy vehicles. It may also apply if the restrictions are self-imposed, such as would occur on a freeway segment upstream of a busy off-ramp where the existing traffic naturally avoids the 'far-side' lanes. Of particular interest are oversaturated time periods because the original theory of kinematic waves proposed by Lighthill and Whitham [*Proceedings Royal Society, A* **229**, 281–345 (1955)] and Richards (1956) does not recognize that different traffic conditions (queues and speeds) may arise on the two sets of lanes, and that these may affect the two vehicle classes in different ways. It should be intuitive that whether a single coalesced queue forms on both sets of lanes or separate queues form on each set should depend on the traffic composition by vehicle class. In an attempt to furnish a reasonable depiction of these phenomena, a pair of conservation-type partial differential equations in the densities of the two vehicle types is used to model the freeway. In their simplest form, the equations only require the introduction of one additional parameter (representing the fraction of lanes allocated to each vehicle type) over the basic kinematic wave theory. The model is attractive because the nature of its solution can be described in complete physical detail by means of simple intuitive diagrams that show how the simple kinematic wave model is improved. The paper also introduces an exact solution method for problems with piece-wise constant initial data that can either be applied graphically or numerically. For general (piece-wise smooth) data, the numerical technique can be used to approximate the true solution as closely as desired in a way that does not deteriorate with time. A less precise finite-difference approach that always moves vehicles forward is also presented. © 1997 Elsevier Science Ltd. All rights reserved

1. INTRODUCTION

The subject of our analysis is a long homogeneous freeway on which two vehicle types travel: l-vehicles, which can use all the lanes, and 2-vehicles, which are restricted to a subset of lanes usually located on the right side of the freeway. We assume that the (priority) 1-vehicles can move freely across the boundary between the two sets of lanes and that they will exploit their mobility to place themselves in the lanes that move faster; 2-vehicles cannot do the same.

Our goal is to develop a coarse but general theory of traffic dynamics for the above system that will describe the development and dissipation of moving queues on both sets of lanes in response to time-dependent conditions such as variable input flows and downstream capacities. In addition to efficient numerical prediction schemes, this paper will describe qualitatively the nature of the solution for all possible forms of initial conditions. (This is important because if the theory were to provide unreasonable results for some conditions it would not deserve to be field-tested [*Operations Research* 4, 42–51 (1956)]).

The theory will extend the basic continuum model of Lighthill and Whitham (1955) and Richards (1956) (the LWR model) by keeping track of the two vehicle types separately. As such, this paper is one more in a family of extensions of the LWR theory, where more than one unknown function is sought (e.g. Bick and Newell, 1960; Gazis *et al.*, 1962; Munjal and Pipes, 1971; Michalopoulos and Beskos, 1984). From a mathematical standpoint, the distinguishing feature seems to be that the equations of the proposed theory can be formulated as a pair of integral conservation laws of physically conserved quantities.*

The paper is organized as follows: Section 2 describes the theory in its general form. Section 3 introduces simplifications that reduce the theory's data requirements and facilitate the solution. Section 4 describes an exact solution method for initial value problems with piece-wise constant initial data. Section 5 shows how the method can be used to solve other well-posed problems with piece-wise smooth data; an example is presented. Finally, Section 6 discusses less precise numerical approaches using finite differences.

2. GENERAL FORMULATION

As is customary in the related literature we will use the symbols k, q, and v to denote respectively the density, flow, and speed of traffic. Although these variables are functions of time, t, and space, x, this dependence will be omitted from the notation whenever possible.

To avoid subscripts as much as possible we will use upper case letters to denote the variables pertaining to 1-vehicles and lower case letters for 2-vehicles. Boldfaced letters will denote pairs of variables, e.g. $\mathbf{K} = (K, k)$. We can then write the integral conservation relation for the two vehicle types as follows:

$$\frac{\partial}{\partial t} \int_{x_0}^{x_1} \mathbf{K}(x, t)\, \mathrm{d}x = \mathbf{Q}(x_0, t) - \mathbf{Q}(x_1, t). \tag{1}$$

This relation simply indicates that the change in the number of vehicles (of both types) in the road section between x_0 and x_1 should be equal to the difference between the number of vehicles entering and leaving the section. The expression assumes that traffic flows in the direction of increasing x.

If one then postulates that the relationship between $\mathbf{Q}$ and $\mathbf{K}$ that might exist under equilibrium conditions (independent of x and t) also applies in the dynamic case for all (x, t), one may write a second equation:

$$\mathbf{Q}(x, t) = \mathbf{Q}(\mathbf{K}(x, t)), \tag{2}$$

which combined with eqn (1) may yield $\mathbf{Q}$ and $\mathbf{K}$ for all (x, t), given suitable data.

It turns out from the theory of partial differential equations that this is mathematically possible, i.e. that our problem is 'well-posed', if the $\mathbf{Q}(\mathbf{K})$ relation satisfies certain regularity conditions as explained in the remainder of this section. An acceptable functional form with few parameters and an intuitive physical interpretation is then introduced in Section 3.

The theory of partial differential equations plays a role because on letting $x_1 \to x_0$, and using the subscripts t and x to denote partial differentiation with respect to time and space, eqn (1) reduces to:

$$\mathbf{K}_t = -\mathbf{Q}_x, \tag{1a}$$

which becomes:

$$\mathbf{K}_t + \mathbf{Q}(\mathbf{K})_x = 0 \tag{3a}$$

after substitution of (2) in the right side of (1a). Here and thereafter, the numeral zero can denote either a scalar or a vector of zeros with dimensions determined from the context. Alternatively we can write:

$$\mathbf{K}_t + \mathbf{Q}' \cdot \mathbf{K}_x = 0 \tag{3b}$$

where vectors $\mathbf{K}_t$ and $\mathbf{K}_x$ are defined to be in column form, and $\mathbf{Q}'$ is the Jacobian matrix of the flow–density relation $\mathbf{Q}(\mathbf{K})$:

*Overviews of conservation laws and their treatment can be found in Lax (1973), Whitham (1974) and LeVeque (1992). A more general treatment of partial differential equations relevant to our work is given in Garabedian (1986).

$$\mathbf{Q}' = \begin{bmatrix} \dfrac{\partial Q}{\partial K} & \dfrac{\partial Q}{\partial k} \\ \dfrac{\partial q}{\partial K} & \dfrac{\partial q}{\partial k} \end{bmatrix}. \tag{4}$$

Equations (3a) and (3b) define a pair of partial differential equations that will be hyperbolic if $\mathbf{Q}'$ has real and distinct eigenvalues. A sufficient condition for this to happen is that the off-diagonal terms of eqn (4) be of the same sign.

Hyperbolicity is desirable because it implies that the initial value problem, where $\mathbf{K}(x, 0)$ is given for all x, is well-posed if $\mathbf{Q}(\mathbf{K})$ and $\mathbf{K}(x, 0)$ satisfy some mild regularity conditions. Although the solution will usually contain shocks (i.e. curves in the (t,x) plane where $\mathbf{K}$ may be discontinuous) the location of these in the 'well-posed' solution that depends continuously on the initial data is uniquely determined by a 'jump condition' for the slope of the shock, $s = dx/dt$:

$$s[\mathbf{K}] = [\mathbf{Q}] \tag{5}$$

where the brackets denote the change in the enclosed quantity across the discontinuity. Equation (5) is a direct consequence of eqn (1)* which complements the differentiated conservation law (1a) where the latter does not apply.

Although one could define a very general form of the flow–density relation and then proceed to show how problems could be solved with a suitable finite-difference approximation scheme, this is not our goal. Instead we will attempt to define the simplest possible form of $\mathbf{Q}(\mathbf{K})$ that can capture the phenomena of interest, while introducing little need for additional field data; and then we will develop a convergent numerical solution method for this choice of $\mathbf{Q}(\mathbf{K})$†.

We recall that the objective of the theory is to depict the evolution of queues on the two sets of lanes, whether separate or coalesced, recognizing that priority vehicles will choose to be on the lanes that move faster. A good illustration of what we are trying to do arises when an incident completely blocks the regular lanes of a heavily traveled freeway so that a queue of 2-vehicles forms in the regular lanes behind the obstruction. If the flow of priority vehicles in the approaching stream exceeds the capacity of the 1-lanes, then a queue of 1-vehicles will also have to form directly upstream of the 2-vehicle queue. Logically, such a queue would span all lanes and would entrap 2-vehicles. After all, 1-vehicles cannot be expected to queue voluntarily on the priority lanes while the regular lanes are under utilized. Of course, in between the two queuing regions there should be a transition zone where the 1-vehicles abandon the regular lanes and the 2-vehicles join the 2-queue. Although the proposed theory will not describe the traffic dynamics in the transition zone, it will yield its approximate location and the location of all queues at various times. Of course, this will not just be possible for this example but for any 'well-posed' problem. The next section describes the assumptions.

3. A FLOW–DENSITY RELATION

We shall describe traffic in the region with the 2-vehicle queue as being in a '2-pipe' regime because the two sets of lanes can then be viewed as two pipes carrying separate fluids with different speeds. Likewise, we describe the coalesced queue upstream as a '1-pipe' regime because a mixture of the two fluids is carried by the single pipe with a unique speed.

We postulate in general that traffic can only be in one of the two regimes (1-pipe; 2-pipe) or transitioning between them, and that the transitions between regimes occur

*If $\mathcal{L}$ is a smooth curve enclosing a region of the (t,x) plane, a more general form of eqn (1) is given by the line integral relation: $\int_{\mathcal{L}} \mathbf{Q}\,dx - \mathbf{K}\,dt = 0$. Equation (5) is obtained when this line integral is evaluated around a thin rectangle straddling the shock.

†Although successful in practice, finite-difference schemes have not been proven to converge for systems like eqns (1) and (2) (LeVeque, 1992).

sharply so that their spatial dimension can be ignored. Furthermore, if traffic is in the 2-pipe regime, with vehicles segregated to their own lanes, then we assume that $V \geq v$; and if it is in a 1-pipe regime that $V = v$. This means that traffic on the regular lanes can never pass that on the special lanes, which is reasonable since otherwise 1-vehicles in the slow lanes would have an incentive to move onto the regular lanes until the advantage of the latter was negated.

We also assume that in the 2-pipe regime the prevailing speed on either set of lanes is a non-increasing function of the traffic density per lane on the relevant pipe, independent of the traffic status on the neighboring pipe. This relationship is assumed to be the same for the two sets of lanes, and also to apply to the whole freeway when it is in the 1-pipe regime. The situation would arise if drivers adopted spacings for a given speed that are the same for all lanes and vehicle types, unaffected by the relative speed on neighboring lanes. Although this assumption can be relaxed within the context of the model of Section 2, this requires a more complicated theory which does not seem worth exploring until the simpler one has been compared with data.

If we let u be the function that relates the 1-pipe speed ($V = v$) to the combined density for all the freeway lanes ($K + k$), we can write:

$$V = v = u(K + k) \quad \text{for the 1-pipe regime;} \tag{6a}$$

and letting γ_1 and γ_2 denote the fraction of special and regular lanes on the freeway ($\gamma_1 + \gamma_2 = 1$), we can also write:

$$v = u(k/\gamma_2) \quad \text{and} \quad V = u(k/\gamma_1) \quad \text{for the 2-pipe regime.} \tag{6b}$$

Because the 2-pipe regime arises if $V > v$, we see from eqn (6b) (and the fact that u is non-increasing) that the system must be in the 2-pipe regime if $K/\gamma_1 < k/\gamma_2$. Conversely, the condition $K/\gamma_1 > k/\gamma_2$ implies that the system is in the l-pipe regime. When

$$K/\gamma_1 = k/\gamma_2, \tag{7}$$

both sets of formulas yield the same result ($V = v$). Thus, we can write

$$v = V = u(K + k), \quad \text{if } \frac{K}{\gamma_1} \succ \frac{k}{\gamma_2} \tag{8a}$$

and

$$v = u(k/\gamma_2) \text{ and } V = u(K/\gamma_1), \quad \text{if } K/\gamma_1 \leq k/\gamma_2, \tag{8b}$$

which define a continuous mapping yielding the two speeds $\mathbf{V} = (V, v)$ for each $\mathbf{K}$. Figure 1 depicts in the (K, k)-plane the (generic) iso-speed curves $V = v_0$ and $v = v_0$ defined by the mapping, the regions corresponding to the two traffic regimes, and the boundary, eqn (7), that separates them.

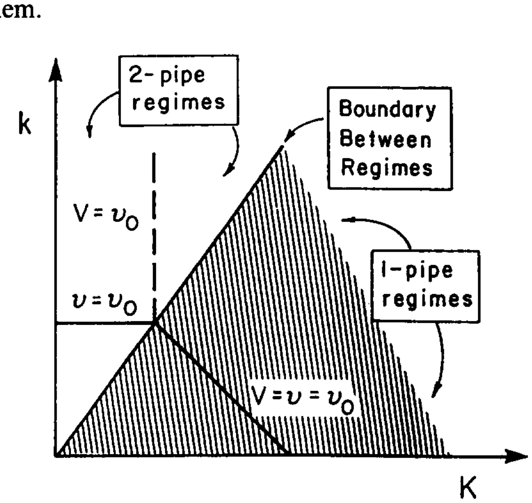


Fig. 1. Iso-speed curves in the (K, k)-plane; shaded area represents 1-pipe regimes.

Together with the relations $Q = KV$ and $q = kv$, eqn (8) defines the (continuous) mapping **Q(K)**. Furthermore, since eqn (8) satisfies $(\partial V/\partial k, \partial v/\partial K) \leq 0$ we can see that the off-diagonal terms of the **Q(K)** relation are of the same sign and that the Jacobian of **Q(K)** has real and distinct eigenvalues, i.e. that our problem is hyperbolic.* Therefore we can be confident that the assumptions so far made suffice to describe the evolution of the system from any set of initial conditions.

Since the function $u(K)$ is usually known (at least coarsely) for a typical facility, implementation of the proposed theory doesn't require much additional data; it only requires the introduction of constants γ_1 and $\gamma_2 = 1 - \gamma_1$, which we have already stated may be assumed to be proportional to the number of lanes of each type.

The model of Fig. 1 and eqn (8) is attractive because for any system which is known to evolve over time always in the same regime (1-pipe or 2-pipe) the LWR theory can be used to predict the future. The only complications arise when both regimes coexist at some time during the study period; and this issue will be the focus of the next sections.

We advance at this point that the transitions between regimes will be curves in the (t,x)-plane which can be treated as a 'moving' boundary condition, whose path can be found from the information carried by the characteristics incident on it.†

With 'well-posed' data we should expect to find a unique set of paths for any given problem. The advantage of formulation (8) is that it confines the complex non-linear behavior of our pair of partial differential equations to a dimensionless portion of the (t,x)-plane (i.e. the moving boundary), which simplifies both the solution methods and the interpretation of the results.

In the remainder of the paper it will be assumed that the scalar speed–density curve $u(K)$ leads to a triangular scalar flow–density relation with free-flow speed v^f, backward wave speed w, 'optimum' density K_o and 'jam' density K^j, as shown in Fig. 2. Newell (1993b) shows how the solution to the LWR problem can be greatly simplified if the actual relation is approximated by Fig. 2; the properties of the simplified q–k relation will also be useful now. Our choice for $u(K)$ leads to a particular form of the **Q(K)** relation which is examined next. The relation is described both in terms of formulas and graphically, as a vector field.

It should be clear that **Q(K)** is continuous, and that in each of the four regions of the (K,k)-plane (labeled 'A' through "D" in Fig. 3) **Q(K)** is differentiable. Region 'A' includes all the traffic states where both car types are freely flowing; in it $\mathbf{Q}/v^f = \mathbf{K}$, as shown by the arrow in Fig. 3. In regions 'C' and 'D' neither car type is freely flowing. In region 'D'

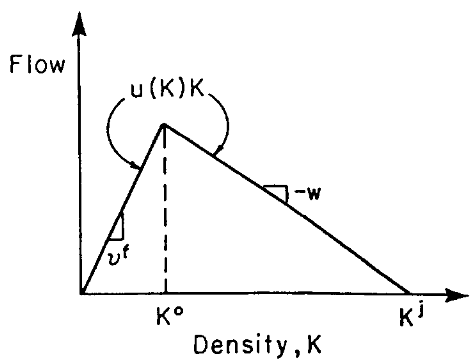


Fig. 2. Simplified flow–density relation with a maximum density, K^j, optimum density, K°, free-flow speed v^f and backward wave speed, w.

*This is not strictly true because the partial derivatives of **V** and **Q** are not defined when **K** satisfies eqn (7). However, eqn (8) can be viewed as the limit of a sequence of models with regular Jacobians, which is also a satisfactory condition in most cases.

†This treatment will be reasonable for problems that only require a coarse level of description if the real world transition zones are stable in size, like a shock in fluid dynamics or in the LWR theory. In the absence of empirical evidence to the contrary, it seems premature to develop a more detailed model with lane changing in the transition zones.

both car types travel at the same speed, $v^D = [K^j - (K + k)]w/(K + k)$, and the flow vector is $\mathbf{Q} = v^D\mathbf{K}$. Graphically this means that the rescaled flow vector $\mathbf{Q}/w$ is in the same direction as $\mathbf{K}$, reaching from the point in question on the (K,k)-plane to the edge of the feasible region as shown in the figure. In region 'C' the 1-vehicles move faster and the flows are obtained from the congested flow relations, $Q = w[\gamma_1 K^j - K]$ and $q = w[\gamma_2 K^j - k]$; this result can be depicted graphically by a rescaled flow vector $\mathbf{Q}/w$ that reaches from the point in question to the corner point of the feasible region $(\gamma^1 K^j, \gamma^2 K^j)$. In region 'B' traffic is in a semi-congested state where 1-vehicles are freely flowing but 2-vehicles are not; i.e. $Q = v^f K$ and $q = w[\gamma^2 K^j - k]$. Again, this can be depicted graphically by a rescaled flow vector $(-Q/v^f, q/w)$ that points to the top left corner of the feasible region.

4. SOLUTION OF THE INITIAL VALUE PROBLEM WITH PIECE-WISE CONSTANT DATA

The objective of this section is finding $\mathbf{K}(x,t)$ for all $t > 0$, given $\mathbf{K}(x,0)$ for all x, when $\mathbf{K}(x,0)$ is piece-wise constant; i.e. when $\mathbf{K}(x,0) = \mathbf{K}_i$ for $x \in (x_{i-1}, x_i)$, where $\{x_i\}$ is an increasing sequence of positions from $-\infty$ to ∞. The basic building block of the procedure will be the solution of a problem with a single discontinuity (termed the "Riemann problem" in the literature). Riemann solutions can always be pieced together to form a composite solution of the original problem in a finite time interval $(0, \tau_1)$, although the composite solution may no longer be correct for $t > \tau_1$. With our choice of $\mathbf{Q}(\mathbf{K})$, however, we shall see that $\mathbf{K}(x,\tau_1)$ is itself piece-wise constant with respect to x. Therefore, it is possible to repeat the original procedure on the new data; and to iterate further, stepping through time to construct the complete exact solution.

4.1. Riemann solutions

For general hyperbolic systems the Riemann problem does not always have a solution, and thus it is important in applications to explore all possible combinations of initial data. In our case we will use the superscripts 'u' and 'd' to denote the upstream and downstream states, respectively, and we will assume without loss of generality that the discontinuity is at $x = 0$.

We note that both the Riemann data and the problem formulation, eqns (3) and (5), are invariant to scaling transformations in which the dimensions of vehicular quantity, t and x, are changed by the same ratio. This means that the Riemann solution must be one of similarity, where a picture of the solution at any time t also represents the solution at another time, βt, if the space dimension in the picture is rescaled by the factor β.

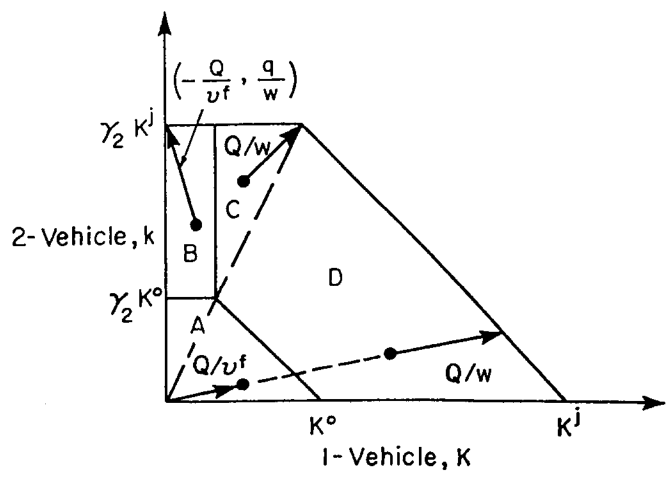


Fig. 3. Graphical depiction of the flow–density relation as a vector field. Q and q are obtained by multiplying the components of a particular vector by one of the following: v^f if in region A, w if in regions C or D, and $(-v^f,w)$ if in region B.

This also means that any discontinuities in the solution (e.g. the path of any boundary separating 1-pipe and 2-pipe regimes) must be a ray in the (x,t)-plane emanating from the origin.

For the formulation of eqn (8), the solution to the Riemann problem is obvious if the initial states are in the same regime (1- pipe or 2-pipe) because the evolution of the system within each regime can be described with the existing LWR theory. Although these results will be presented here for completeness, the focus of the discussion will be on cases where the inter-regime boundary is crossed.

Before this can be done it is necessary to introduce pairs of states that can be neighboring, albeit separated by a discontinuity. By virtue of eqn (5), we see that the vector of flow changes across neighboring states must be proportional to the vector of density changes, and the diagram of Fig. 3 can be useful to check this property for any pair of points. In this way one can construct the locus of points in the (K,k)-plane that can be connected to any given point.

The result of this somewhat tedious process is summarized in Fig. 4: two states can be neighboring if the segment joining them in the (K,k)-plane is of the type shown in the figure. The figure for example indicates that states in region 'C' can be neighboring to any other state in region C, to any states in region B with the same k, to states in region D with the same $k + K$, and to any states in region A such that points 'C', 'A' and the intersection point of the four dashed lines fall in a straight line. Similarly, states in region B can be neighbors to any other states not in region D* that either have the same k or the same K.

Some of the pairings of Fig. 4 could not arise spontaneously, and if they were specified as part of the initial data they would disintegrate immediately. For example if $\mathbf{K}^u$ is a point in D and $\mathbf{K}^d = 0$ (an allowable point in A), one could not reasonably expect the situation to persist, because traffic's acceleration into the empty road ahead would eliminate the boundary between the two original states. Similar considerations would lead one to conclude that the pairs listed in Fig. 4 are also unstable and could not arise spontaneously. This is also consistent with the theory of hyperbolic partial differential equations for in those cases the discontinuity prevents every point in the (t,x)-plane to exhibit conditions that depend continuously on its original data.†

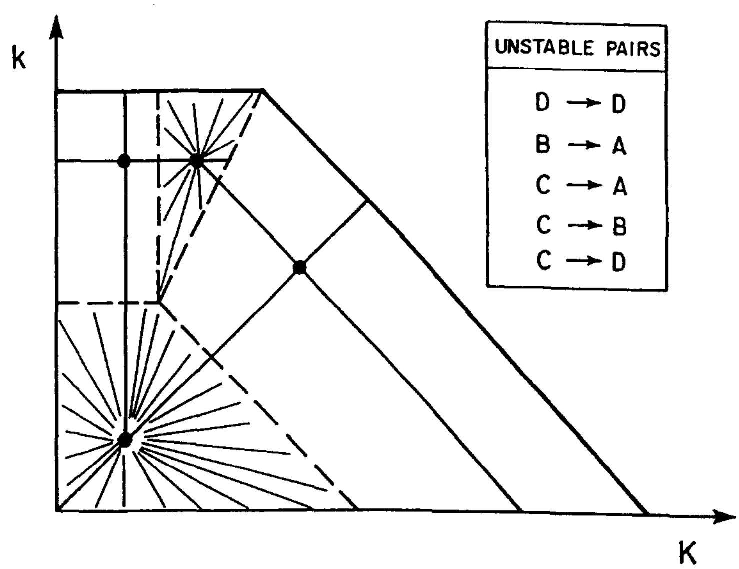
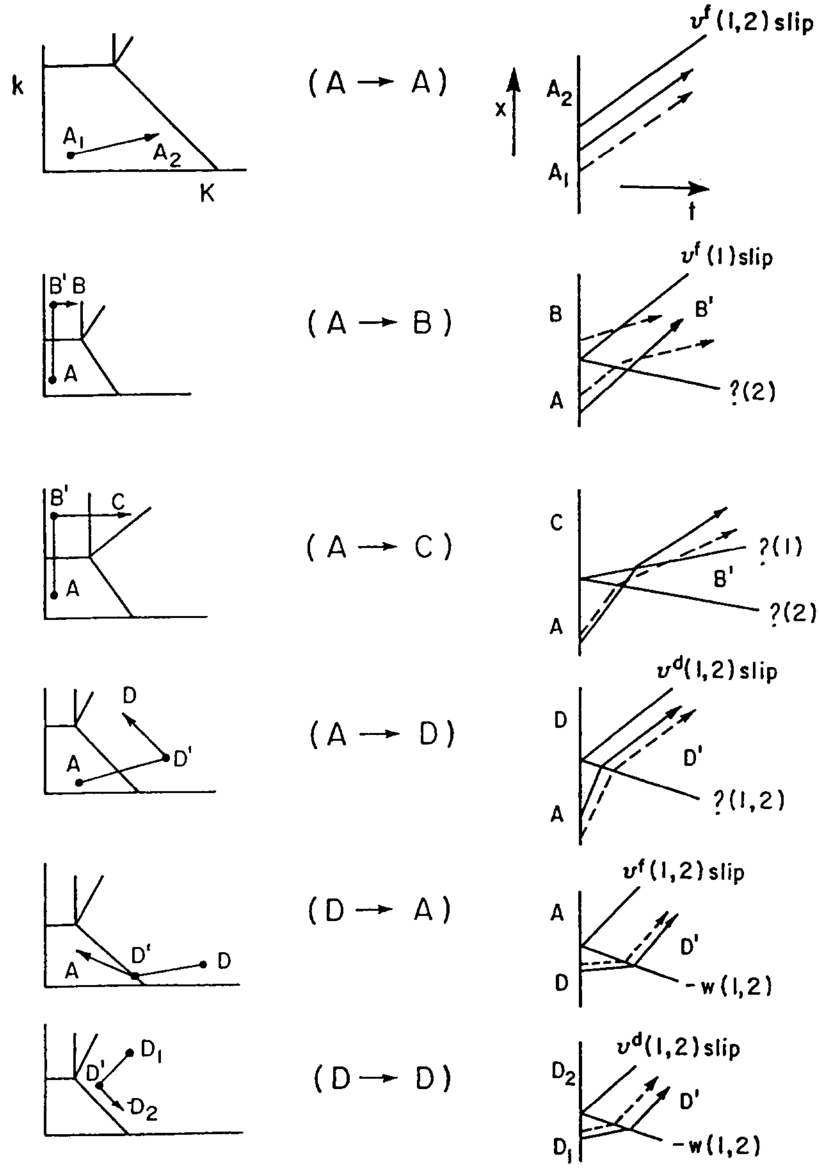


Fig. 4. Pairs of states that can be separated by a discontinuity satisfying eqn (5).

*Particular state pairs in regions B and D may also satisfy eqn (5). Because these pairs play no role in our theory, they are not included in Fig. 4.

†A pertinent discussion of the subject can be found in Garabedian (1992).

Figures 5–7 display the mathematically unique solution to all the possible Riemann problems.* As such, the figures can be viewed as a summary of the proposed theory. In all three figures, each solution is characterized by a (t,x)-diagram displaying the rays of discontinuities (interfaces) between constant traffic states in the solution,† and the trajectories of two or three vehicles. The discontinuities are represented by dark-solid lines. Interfaces denoting a boundary between the 1-pipe and 2-pipe are identified by a double solid line. The 1- and 2-vehicle trajectories are represented, respectively, by solid and dotted lines ended with arrow-heads. The parenthetical numbers by each interface identify the vehicle types whose density is discontinuous across the interface. The velocity of the interface is given to the left of the parenthesis, if it is independent of $\mathbf{K}^u$ and $\mathbf{K}^d$. Otherwise it can be found by applying eqn (5) to the states separated by the interface,



(a)

Fig. 5(a). Solution to Riemann problems with no boundary crossings: cases where $\mathbf{K}^u$ is in regions A or D.

*Figure 7 does not include the two cases in which $\mathbf{K}^d$ is in region A, because they are identical to the 1st and 5th cases of Fig. 5(a).

†This is possible because for our choice of $\mathbf{Q}(\mathbf{K})$ there are no expanding (rarefaction) waves.

and a question mark is used. By analogy to fluid mechanics, the label 'slip' refers to interfaces that are discontinuous in density with no change in speed; in those cases the trajectories of the vehicle family with a discontinuous density run parallel with the interface. Each figure also includes a companion state-space sketch which indicates uniquely the intermediate states that arise. These states, in conjunction with (5) and either Fig. 3 or eqn (8), may be used to determine the interface velocities not given in the figure. (Note that all the line segments appearing in the phase–plane sketches belong to the family of segments displayed in Fig. 4.)

Before giving a brief physical explanation for each one of the figures we note that vehicle trajectories are only affected by interfaces reaching them from the front. This means that a vehicle can only be affected by something that happened ahead of them at an earlier time, and this is a reasonable property of a theory intended to represent how people drive. Note as well that in agreement with our postulates of driver behavior 2-trajectories never overtake the (priority) 1-trajectories.

Figures 5(a,b) display all the cases where the boundary between regimes is not crossed. Since there is no coupling between the two vehicle types in the 2-pipe regime, every solution

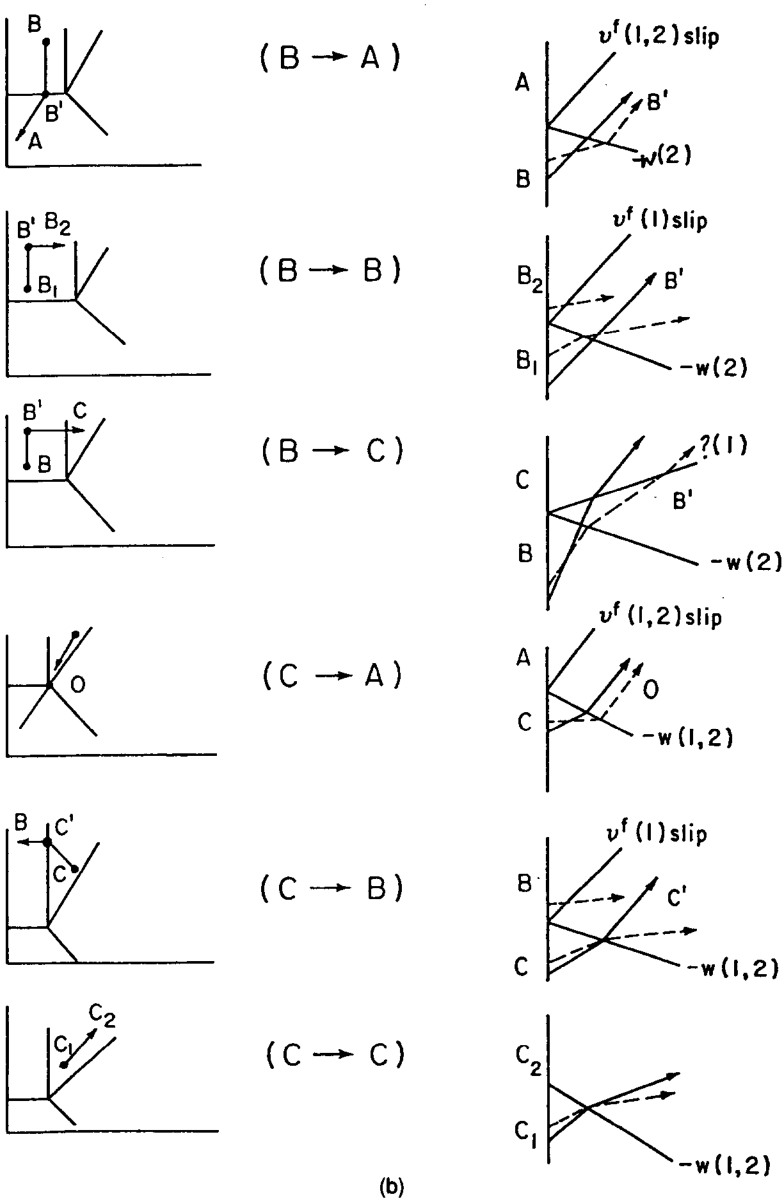


Fig. 5(b). Solution to Riemann problems with no boundary crossings: cases where $\mathbf{K}^u$ is in regions B or C.

in Fig. 5 where the initial conditions are in the 2-pipe regime can be developed from application of the LWR theory to each one of the pipes separately. Where the initial conditions are in the 1-pipe regime the solutions follow from the LWR theory and the recognition that both vehicle types travel at the same speed.

The cases displayed in Fig. 6, where $\mathbf{K}^u$ is a 2-pipe regime and $\mathbf{K}^d$ is either a 1-pipe regime or a 2-pipe free-flow regime (region 'A'), can also be physically related to the scalar LWR model. If one recognizes that vehicles upstream of the discontinuity cannot affect those downstream (in states A or D),* it then becomes apparent that the trajectories of vehicles starting downstream of the discontinuity must be parallel lines with the speed $v^d = V^d$, and that the last trajectory must act like a 'lead vehicle' for upstream traffic. The evolution of traffic in each of the pipes should thus be described by the 'lead vehicle problem' of LWR theory, with the same lead vehicle speed on both pipes. Every case shown in Fig. 6 is consistent with this physical description.

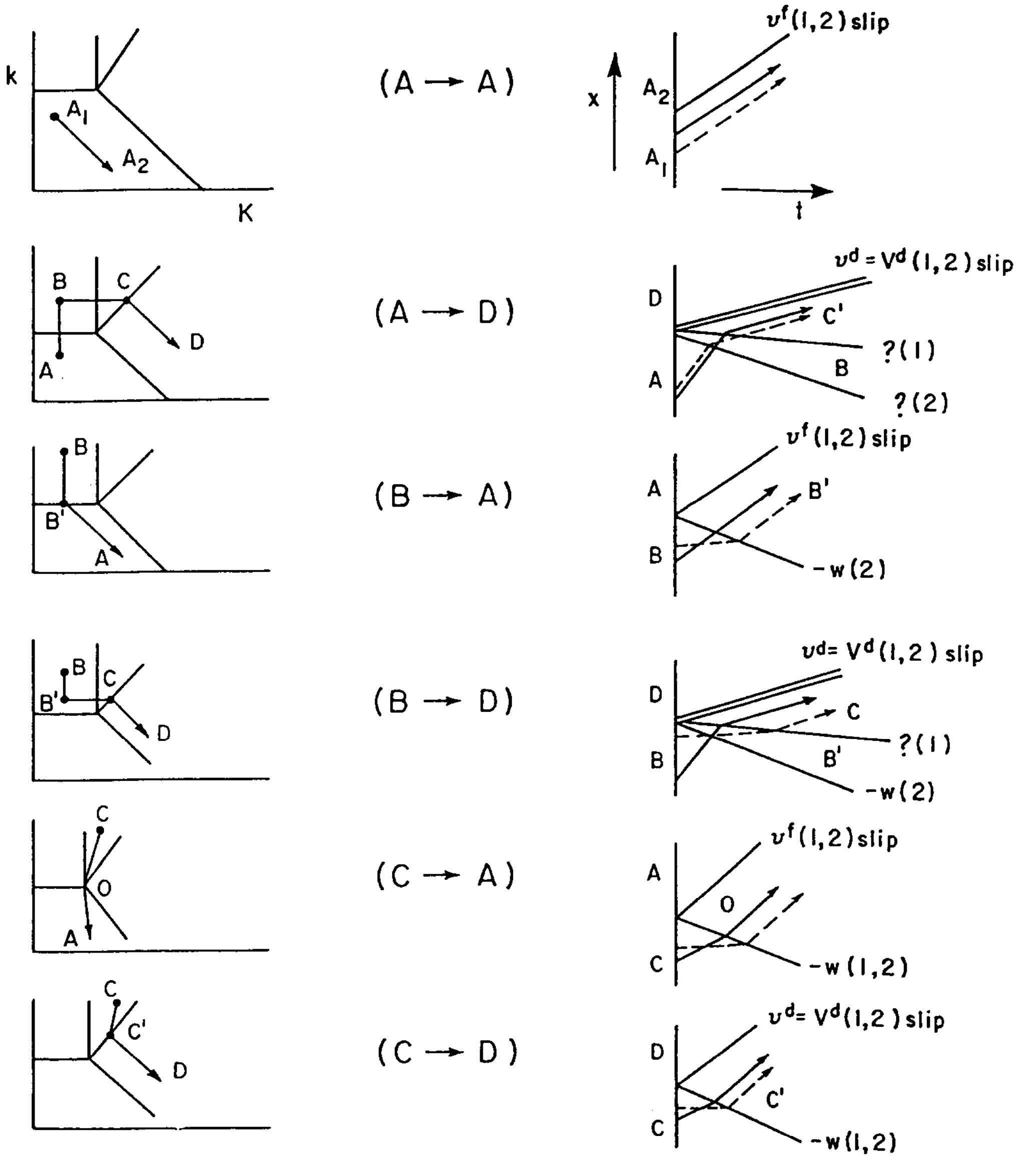


Fig. 6. Solution to Riemann problems of the 'lead vehicle' type: $\mathbf{K}^u$ is in the 2-pipe regime and Kd is in regions A or D.

*This should be clear because the upstream vehicles cannot push their way through those ahead, which are either traveling at the maximum speed or in the (no-passing) 1-pipe regime.

Figure 7 depicts the remaining four non-trivial cases* with transitions into the congested (C) or semi-congested (B) 2-pipe regimes from 1-pipe regimes (A and D). In these cases the 2-vehicles can be treated independently because they can join the 2-queue without any interference from 1-vehicles, which should be competing instead for position in the faster moving 1-lanes. The trajectory of the 2-shock (determined with the LWR theory for 2-vehicles alone) acts then as a 'moving bottleneck' which dictates the 1-vehicle behavior. The LWR theory can then be used to determine what happens to 1-vehicles as well. The results of Fig. 7 are consistent with this characterization of driver behavior.

Figure 8 depicts intuitive 'snapshots' taken at some time $t > 0$ for four basic cases that cover all the situations of Fig. 7. In the figure, darkened rectangles represent 1-vehicles,

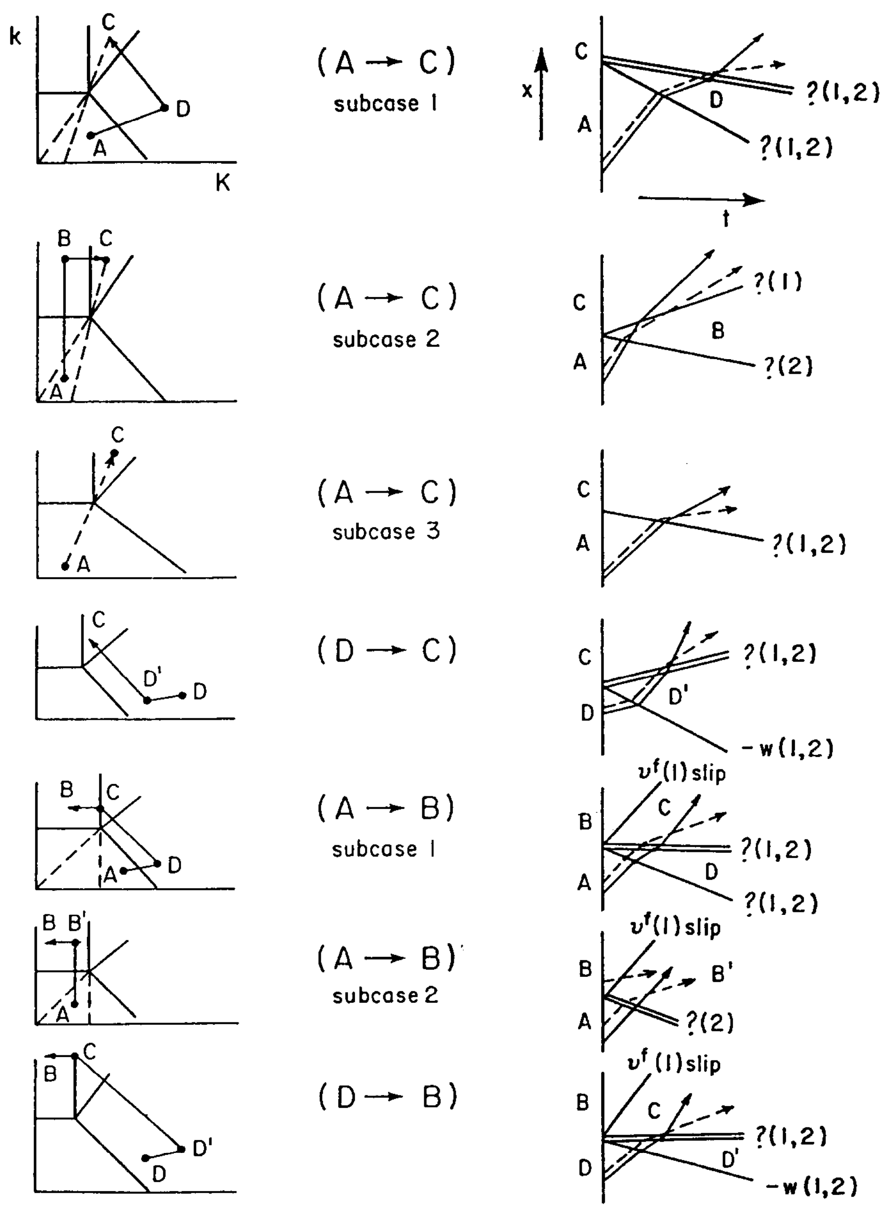


Fig. 7. Solution to Riemann problems of the 'moving bottleneck' type: K^u is in regions A or D and K^d is in regions B or C.

*Transitions from 1-pipe regimes (A and D) into the uncongested 2-pipe regime (A) are trivial and are not included in the figure. From A to A the diagram is as the very top ones of Figs 5(a) and 6 and from D to A it is as the penultimate one of Fig. 5(a). In essence these cases can be treated as if the downstream state was also in the 1-pipe regime.

vertical lines represent changes in traffic density (and/or) speed on the lanes spanned by the line. A vertical double line represents a boundary between traffic regimes. The speed of traffic is also indicated. In cases III and IV the speeds satisfy $v^d \le v \le V^d$ so that on crossing the inter-regime boundary 1-vehicles accelerate while 2-vehicles decelerate; note as well that if $v^u < v^f$ (upstream state is congested) then the upstream interface travels with velocity: $s = -w$.

Cases I and II in the figure correspond to the second and sixth cases of Fig. 7. These are situations where the proportion of 1-vehicles in the upstream traffic is so low that a semi-congested intermediate state (B) arises. In these cases the 2-vehicles join a queue that is bypassed by the 1-vehicles; these may (diagram I) or may not (diagram II) join a faster-moving queue further downstream.

Cases III and IV depict the remaining situations where the approaching stream is rich in 1-vehicles. They first involve a general adjustment in speed and density for the stream as a whole, with no change in the mix of vehicles, followed by a boundary crossing that entails no change in density but an adjustment in the mix. The diagram is consistent with our earlier assertion that 1-drivers crossing this boundary would be competing to squeeze into the faster flowing lanes (recall that $V^d \ge v \ge v^d$), while the 2-drivers simply

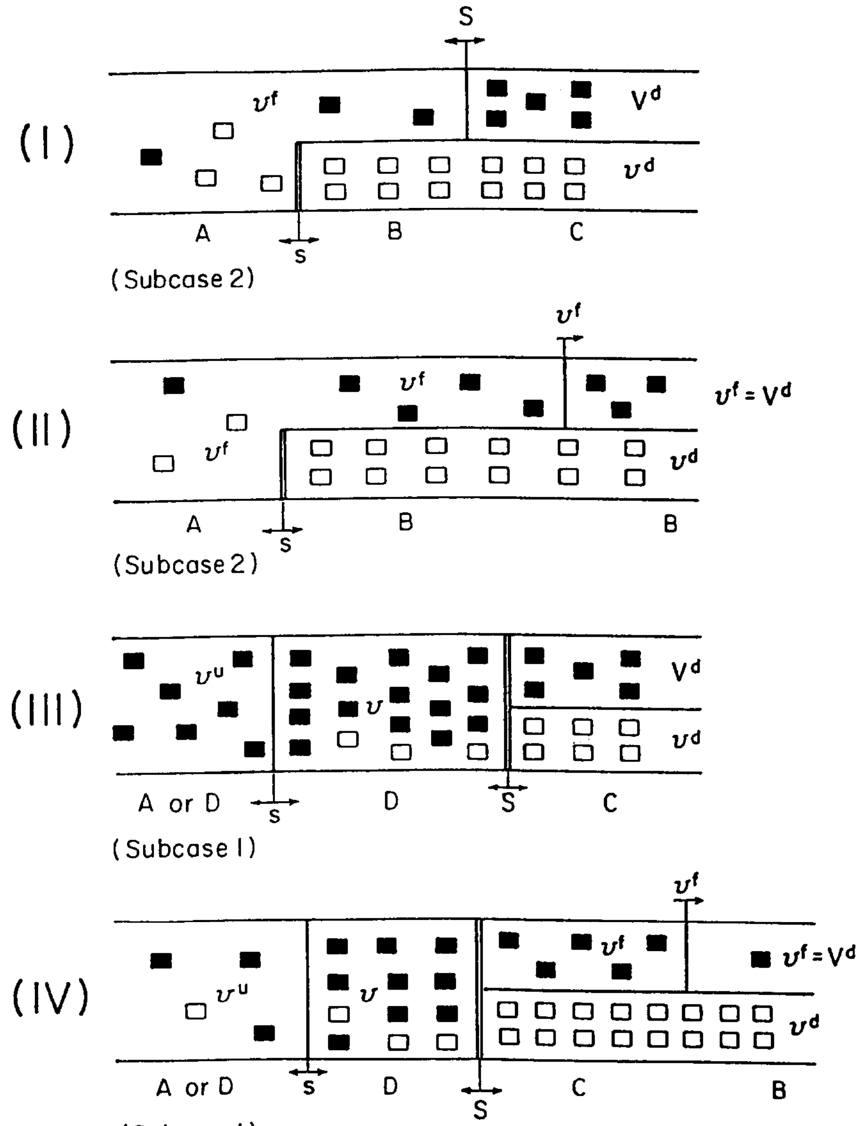


Fig. 8. Four possible end states for Riemann problems of the 'lead vehicle' type.

would join the queue. This jockeying has the effect of reducing the proportion of 1-vehicles downstream of the boundary in the appropriate way.

It should be emphasized that the solutions of Figs 5–7 are also the unique ones that would be obtained from the theory of partial differential equations (hyperbolic conservation laws) through an analysis of the characteristics.* Our prior discussion simply illustrates that the solution of the conservation hyperbolic laws (1) and (2) is consistent with what one might expect drivers to do. This is important for a field-like traffic flow theory where experimental evidence is noisy and hard to obtain.

Although our theory could have been developed from these considerations alone, without relying on systems of conservation laws, a characterization as such is useful in that it establishes the relevance of existing numerical solution methods for conservation laws. This will be exploited in a later section of this paper.

4.2. Exact solution with piece-wise constant data

Given some initial piece-wise constant data at time $t = 0$, $D_0 = \{\mathbf{K}_i, x_i\}$ where $\mathbf{K}(x,0) = k_i$ if $x_{i-1} < x \leq x_i$, it should be clear that (for small τ) $\mathbf{K}(x,\tau)$ will also be piece-wise constant, with new states introduced in the neighborhood of each x_i by the fans of interfaces issued at x_i as per the appropriate case of Figs 5, 6, or 7. Like the initial data, the solution at time τ is still characterized by a sequence of traffic states $\mathbf{K}_j$ and the space coordinates at which they change, $x_j(\tau)$. It is also convenient to include in our description of the solution, $W_0(\tau)$ the velocity of these interfaces, $s_j = \mathrm{d}x_j/\mathrm{d}\tau$:

$$W_0(\tau) = \{\mathbf{K}_j, x_j(\tau), s_j; j = 1, 2, \ldots\}. \tag{9}$$

This solution will remain valid as long as the interfaces don't cross; i.e. $x_j(\tau) \leq x_{j+1}(\tau)$. The time at which this first happens, τ_1, is easily obtained because all the $x_j(\tau)$ are linear in τ. At that time one or more of the $\mathbf{K}_j$ ($j = j^*$) will be annihilated by the convergent interfaces, and the remaining elements of $W_0(\tau_1)$ can be combined into a data set similar to D_0 that can be viewed as a new set of initial conditions at $\tau = \tau_1$, D_1. These new data can be used to obtain a solution, $W_1(\tau)$, valid for $\tau_1 < t < \tau_2$. Iteration of this procedure should yield solutions for all t.† The procedure terminates after a specific step whenever we find $s_j \leq s_{j+1}$ for all j.

Figure 9 shows an example in which the solution for $t \in (0,\infty)$ only involves four iterations of the procedure. The example depicts the dissipation of a lump of semi-congested traffic in an uncrowded road until all the vehicles are traveling at the free-flow speed. It is assumed that the upstream traffic stream consists entirely of 1-vehicles. The figure, which has been drawn to scale using the above procedure, is self explanatory: part (a) displays the original and intermediate data in the (K,k)-plane; part (b) is the solution; and part (c) displays on a larger scale the type-1 and type-2 vehicle trajectories that intersect the semi-congested region 'C' of part (b). The result is easily interpreted. As one would hope the theory predicts that a queue of 1-vehicles spanning all the lanes will form upstream of the lump (state D), as 1-vehicles attempt to penetrate it, and that this penetration will increase the density of 1-vehicles at the back end of the lump (state C). The figure also shows (reasonably) that the lump will disentangle from the front by the acceleration of the 2-vehicles into the available road ahead (state B'), and that all of this happens without any improper overtakings; see part (c) of the figure.

Alternatively, one could display the solution to any problem in terms of cumulative count curves of vehicle number vs x (by vehicle type) at various times, and perhaps one

*All our interfaces satisfy the 'entropy condition' in which waves of the corresponding type run into (or parallel with) the discontinuity. The boundary satisfies a similar condition, where at most one wave can be emitted by it. The other three incident waves and eqn (5) define a system of five scalar equations that determine the velocity of the boundary, s, and the densities, $\mathbf{K}$, on both sides of it.

†For this statement to be rigorously true one would have to prove that the increasing sequence $\{\tau_i\}$ does not converge to a finite value. This however is of academic interest because any "stalling" problems would be revealed during the solution process. Furthermore, if $\tau_i \leq \tau_{max}$ for all i this would mean that the initial data would not have a solution beyond τ_{max}; i.e. the initial data didn't define a well-posed problem in the large, which is unlikely.

could even develop an algorithm that would operate on these curves instead of eqn (9); e.g. as in Newell's simplification of the original LWR theory (Newell, 1993b).

4.3. Approximate solution with piece-wise smooth data

Although in most applications traffic data are made available in piece-wise constant form (e.g. 30 s counts), the solution procedure presented in Section 4 can be used to solve, to an arbitrary level of accuracy, problems where the initial data are piece-wise continuous. One simply needs to approximate the initial data by a piece-wise constant data set (of the type used in Section 4) using as many discontinuities as necessary to achieve a desired level of approximation, and then apply the procedure of Section 4 to the modified data. The solution so developed should tend to the true solution as the modified data approaches the true data.

We note that the final solution will be piece-wise constant, and will not exhibit expansion waves where one (or both) of the densities decrease smoothly in the direction of increasing x. This property is what facilitates hand solutions. A more complicated model including

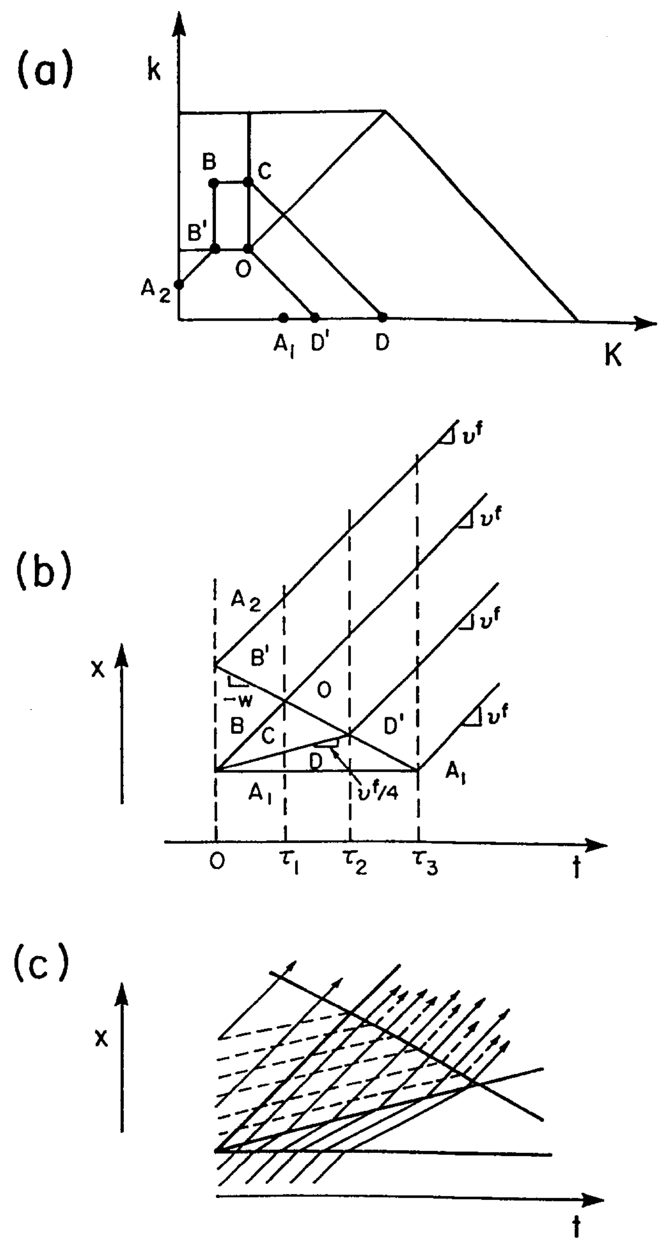


Fig. 9. Dissipation of a semi-congested region (with $w = v^f/2$).

expansion waves can be obtained by using eqn (8) in a more general form but it is not clear whether this would increase realism.*

5. THE FINITE HIGHWAY PROBLEM

The solution method of Section 4 can be extended to highways of finite length if one specifies suitable boundary conditions. Care must be exercised here to ensure that the resulting problem is not obviously ill-posed by defining conditions that are physically meaningful.† For example one may want to define the initial density distribution along the highway, the future flows entering the highway at its upstream end, assuming that queues do not grow all the way back, and any restrictions that may exist downstream. The latter may be accomplished by specifying the densities or the capacities that are allowed to prevail at the downstream end (possibly as a function of time), as per the existing restrictions. An understanding of these constraints, together with the diagrams of Figs 5–7 will determine what new interfaces (if any) are issued when a moving interface hits the downstream end of the freeway, or when traffic conditions change at the upstream end. These instants should be treated in the procedure in the same way as the τ_i of Section 4.

In order to illustrate these concepts, a detailed example depicting how a jam grows and dissipates upstream of a congested off-ramp is presented in Section 5.1. The same example is then used in Section 5.2 to introduce a streamlined procedure.

5.1. Example

As a form of illustration we use the above method to describe the phenomenon (well known to sports fans in many parts of the world) in which the stream of traffic along a freeway oversaturates an exit ramp, causing a semi-congested state to arise immediately upstream of the ramp. Under the 'right' conditions, and in agreement with the qualitative discussion at the end of Section 2, the theory predicts that a fully congested state will also develop further upstream, that will entrap even those drivers that are not taking the exit. This, of course, is a common occurrence (e.g. near Candlestick Park in San Francisco), but it only occurs if the congestion is severe. If the conditions are not 'right' (i.e. the traffic stream is 'poor' in through vehicles) then the fully congested state does not develop. This form of traffic jam is also fairly common; for example, it can be observed daily on the northbound Warren freeway upstream of its exit for freeway 24 in Oakland, California. Although not done in this paper, the reader can easily verify by repeating the example with a different set of traffic inputs that this is also a prediction of the theory.

We consider the freeway segment directly upstream of an exit ramp, as depicted in Fig. 10(a). We assume that the freeway is homogeneous along its length and that queued exiting traffic (if any) will occupy the right half of the freeway. This is shown on the left diagram, where it is noted that through traffic must travel at the free-flow speed (since no obstructions are assumed in our problem for these vehicles). If there is no exiting queue, then everyone is moving at the free-flow speed.

We assume that the exit ramp can only carry a fraction of the maximum flow of exiting vehicles that may approach it. [This may occur, e.g. because: (a) the ramp is too narrow, (b) an incident has occurred on it, or (c) there is a stop sign or some other restriction at the end of the ramp which prevents high flows from using it. Here we don't ask why; we simply demonstrate the consequences of such a situation on the freeway.]

For our purposes we will assume that the ramp can only carry 25% of the maximum possible approach flow of exiting vehicles, $\gamma_2 Q^o$, which in turn is 50% of the maximum freeway flow Q^o (i.e. $\gamma_2 = 0.50$). This means that the only semi-congested states that can

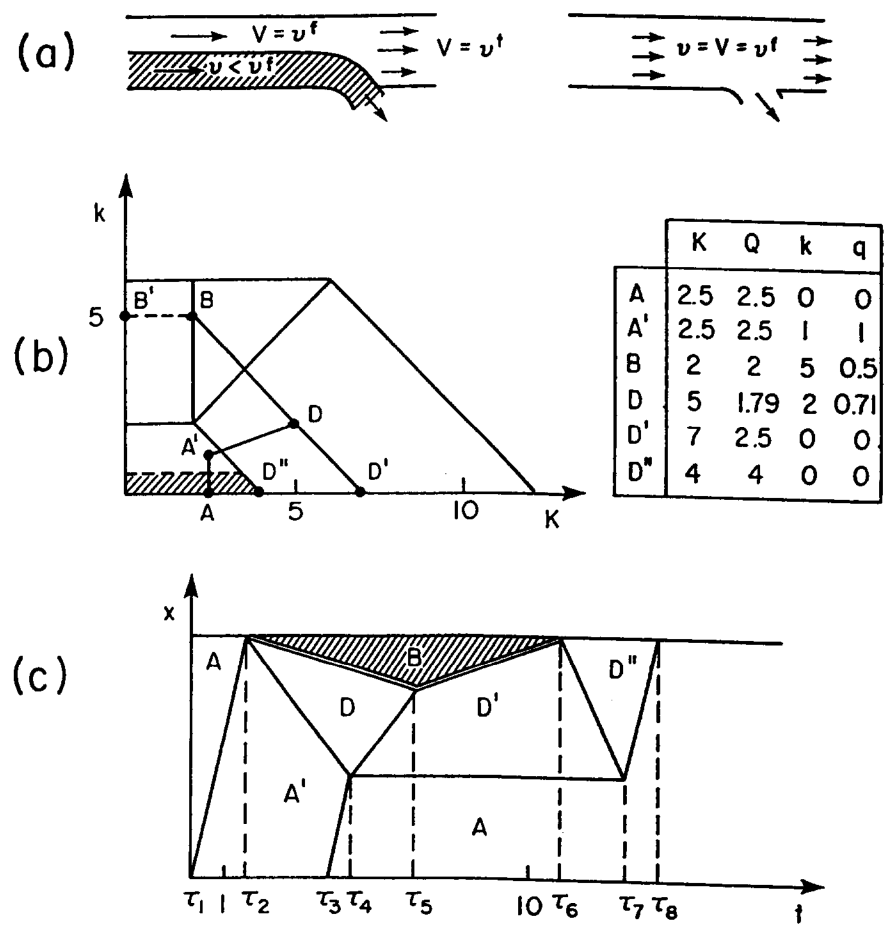
*We note that unless $v = v^f$ for densities below the 'optimum' density, the LWR model offers a poor explanation of certain aspects of the 'moving bottleneck' problem (see Gazis and Herman, 1992; Newell, 1993a) which arises here in the transitions from 1-pipe to 2-pipe regimes.

†If desired one can check that the waves emitted from the boundary point in the required direction (into or out of the time-space domain of the solution).

arise corresponding to the left portion of Fig. 10(a) are those on line B'B of Fig. 10(b), and that the only free-flowing states possible [corresponding to the right portion of Fig. 10(a)], are those in the shaded region next to the origin. (It is also possible for the system to be stable in certain congested i-pipe regimes, but those situations do not arise in our example and are not depicted in the figure).We assume that the freeway has been operating for some time in state 'A' (no exiting traffic) and that at $t = 0$ vehicles wishing to exit begin to arrive, defining the state denoted by point 'A'' in Fig. 10(b). We assume that this increased arrival rate lasts for 4 time units, after which the arrival flow returns to the prior level (A). The table shown in Fig. 10(b) depicts the flows and densities corresponding to points A and A', as well as other points which arise in the final solution. The units of measurement for time and space are unspecified as are those of flows and densities; they have been chosen to ensure that the solution can be clearly displayed.

The solution procedure can now be applied, and the result is given in Fig. 10(c). As explained in Section 4, a new τ_i is introduced each time that the traffic state changes at the upstream end of the freeway, and whenever an interface crosses either another interface or the downstream boundary. (Interfaces cannot cross the upstream boundary if our problem is well-posed; i.e. the queue cannot grow all the way up to the upstream freeway entrance if we have stated the entry flow.)

It should be clear that all changes in the upstream state propagate forward as a slip with speed v^f, and one such change generates the first interface at time $\tau_1 = 0$. The next iteration occurs at time τ_2 when the higher flow first reaches the off-ramp, and a queue begins to grow [Fig. 10(a), left]; the resulting semi-congested state must be on the



	K	Q	k	q
A	2.5	2.5	0	0
A'	2.5	2.5	1	1
B	2	2	5	0.5
D	5	1.79	2	0.71
D'	7	2.5	0	0
D"	4	4	0	0

Fig. 10. Growth and dissipation of a traffic jam caused by insufficient exit ramp capacity. The jam consists of a semi-congested portion (B) immediately upstream of the ramp and a fully congested portion (D) further upstream.

segment BB' of Fig. 10(b). For our data, the only stable state on this segment is point B. This corresponds to the Riemann case (A → B, subcase 1) of Fig. 7, with a backward-moving interregime boundary. The interfaces for this case must also appear in our solution now; as in Fig. 7, they generate an intermediate — fully congested — state D. The exact flows and densities of states B and D are obtained from the phase–space diagram. The velocities of the new interfaces are determined from the shock eqn (5) and the numerical data of Fig. 10(b).

The procedure is then iterated six more times to determine the complete solution. It should be noted that the same physical/stability considerations, together with Figs 5–7, uniquely determine the new states (D' and D" at times τ_4 and τ_6) as well as the set of interfaces issued from the boundary.

The final result, Fig. 10(c), has a reasonable interpretation. When the surge of exiting vehicles (state A') arrives at the exit ramp, a queue of these vehicles forms on the right side of the freeway (state B). As in the example of Section 4.2, this prevents the through vehicles from squeezing through so that a coalesced queue (state D) is also started directly upstream. When the surge of exiting vehicles terminates (state A) only the through vehicles continue to join the coalesced queue (state D') and this stops its growth. Later, when the last exiting vehicle reaches the back end of the downstream queue, that queue stops growing too and begins to recede. When it completely dissipates the freeway opens up and the coalesced queue, which then contains only through vehicles, begins to discharge at capacity (state D") until this queue finally dissipates.

5.2. A streamlined procedure

The solution to the above example can also be obtained without numerical computation of the interface velocities, using the graphical construction of Fig. 11.

This figure summarizes the information of the table in Fig. 10(b) in the (k,q)-plane, where each traffic state is identified by two points — one for 1-vehicles and another for 2-vehicles. Circles have been used to enclose the legend identifying the 1-vehicle points. The figure can be constructed easily because 2-pipe regime states must lie on the flow/density curves for each of the two pipes (which happen to be superimposed in

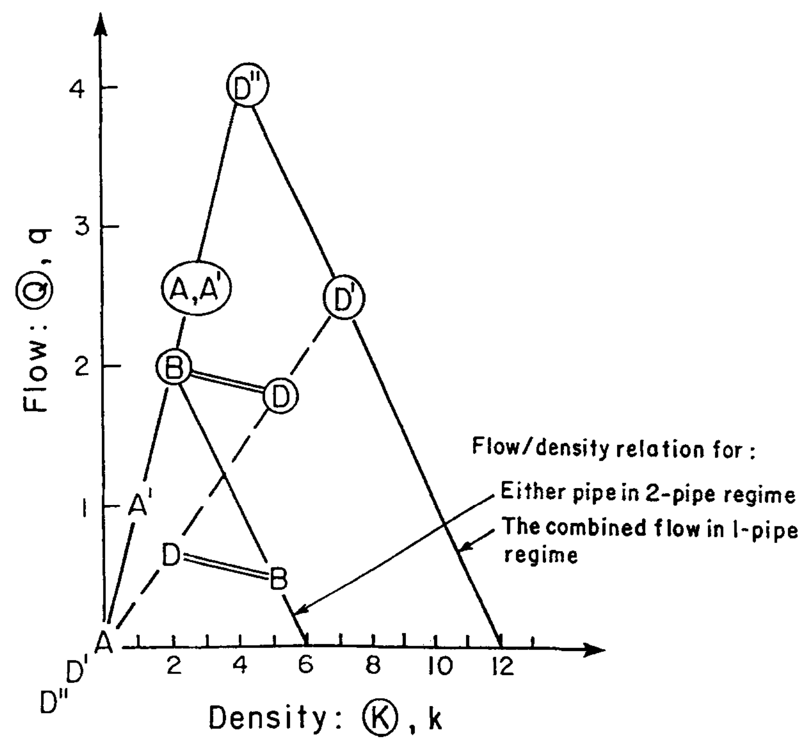


Fig. 11. Graphical determination of the interface velocities.

our example); thus, no side calculation is needed to determine the flows from the densities of Fig. 10(b). The 1-pipe states (e.g. 'D') must lie on the ray (dotted line) passing through the point on the combined flow-density curve with abscissa equal to the combined density of Fig. 10(b) (e.g. '7'); the specific location of the two points on the ray corresponds to the individual densities (e.g. '2' and '5').

Once the points identifying two neighboring states have been located in Fig. 11, the slopes of the lines connecting them represent the velocities of their interface. As in the scalar case, if the figures have been drawn with the proper scale, the trajectory of the interface in the (t,x)-plane should be parallel to the lines of Fig. 11. This figure displays by means of two double lines the two parallel segments connecting states D and B, which can be seen to be parallel to the corresponding interface of Fig. 10(c). The reader can easily verify that all the interfaces of Fig. 10(c) are parallel to their corresponding segment pairs in Fig. 11.*

The finite highway procedure just illustrated is general and can be applied to other geometries at the downstream end of the freeway, even if the downstream capacities are time-dependent. To do this one would have topp identify the stable states that can arise at the boundary when an interface reaches it, and then use this information in conjunction with Figs 5–7 to determine which (if any) new interfaces emanate from each point of contact. An exhaustive discussion of all possible cases and geometries is beyond the scope of this paper, however.

6. FINITE-DIFFERENCE APPROXIMATION

Although we have presented an 'exact' solution method for the initial value problem and a certain class of boundary value problems, the formulation in terms of a pair of conservation laws suggests that finite-difference approximation schemes may also be practical. They are appealing for their ease of implementation.

Although convergence proofs don't seem to exist for finite-difference schemes for general systems of conservation laws, experience with 'conservative' approximation schemes has been good (LeVeque, 1992). For our problem, Godunov's method (Godunov, 1961) seems appropriate because it has the feature that vehicles never move back.

In this method the freeway is discretized into (small) sections, m, of length Δx where the density vector is taken to be constant. These densities are revised every Δt time units† with a 'conservative' rule:

$$\mathbf{K}^{m}(t + \Delta t) = \mathbf{K}^{m}(t) + \left(\frac{\Delta t}{\Delta x}\right)[\mathbf{Q}^{m-1}(t) - \mathbf{Q}^{m}(t)], \tag{10}$$

where $\mathbf{Q}^{m}(t)$ is the average flow out of section m during $(t, t + \Delta t)$. In Godunov's method, $\mathbf{Q}^{m}(t)$ is taken to be the flow vector that would prevail at the interface of the two sections for the Riemann problem (given in Figs 5–7) with $\mathbf{K}^{u} = \mathbf{K}^{m}(t)$ and $\mathbf{K}^{d} = \mathbf{K}^{m+1}(t)$.‡ It is now shown that the flows corresponding to all the cases appearing in Figs 5, 6 and 7 can be summarized with just a few simple formulas.

In the expressions below $\mathbf{K}^{u} = (K^{u},k^{u})$, $\mathbf{K}^{d} = (K^{d}, k^{d})$ and $K_{T} = k + K$ represent the (constant) densities at time t, and $\mathbf{Q} = (Q,q)$ and $Q_{T} = q + Q$ represent the Godunov flows. The ratios $\alpha^{u} = K^{u}/K_{T}^{u}$ and $\alpha^{d} = K^{d}/K_{T}^{d}$ denote the proportion of 1-vehicles in the traffic streams in the upstream and downstream section. As in Lebacque (1993) and Daganzo (1993) it is also convenient to replace the flow–density curve for the 1-pipe regime (given in Fig. 2) by the lower envelope of a non-decreasing 'sending' curve S_{T} and a non-increasing 'receiving' curve R_{T}. These are plotted in Fig. 12. Likewise, sending

*Because the two segments corresponding to an interface must always be parallel, this geometrical property can be a useful check on the correctness of the construction process.

†The time increment should be small enough so that the fastest waves (of speed v^{f}) never travel more than one section in time Δt.

‡Application of the exact procedure of Section 4 reveals that the distribution of density along a section does not remain constant; and this is where the approximation is made when eqn (10) is iterated over time with the finite-difference scheme.

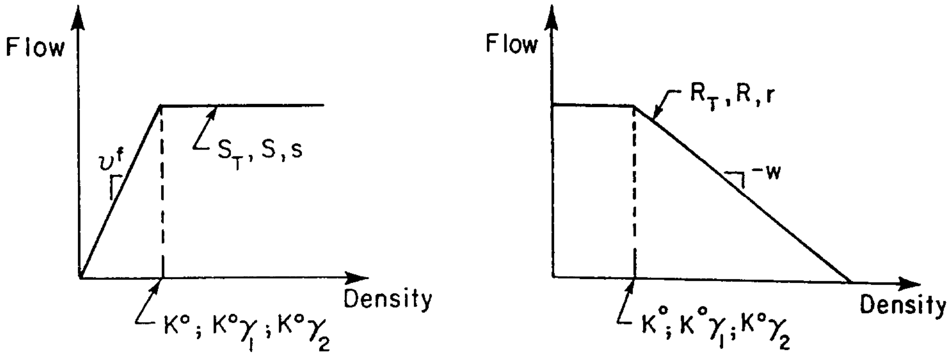


Fig. 12. 'Sending' and 'receiving' flow functions.

functions (S,s) and receiving functions (R,r) can be introduced in lieu of the two flow-density relations corresponding to the two sets of lanes in the 2-pipe regime. These are also shown in Fig. 12; the only difference between the three sets of curves is a scaling transformation.

Note now that all the cases in which transitions take place between 2-pipe regimes without boundary crossings (α^u, $\alpha^d \leq \gamma_1$) must obey:

$$q = \min\{s(k^u); r(k^d)\} \tag{11a}$$

and

$$Q = \min\{S(K^u); R(K^d)\} \tag{11b}$$

because each of these minimum relations describes the behavior of one of the pipes in the traditional LWR theory. For transitions between 1-pipe regimes without boundary crossings (α^u, $\alpha^d > \gamma_1$) the total flows should satisfy the LWR minimum relation without any discrimination across vehicle types; i.e

$$(Q;q) = (\alpha^u; 1 - \alpha^u) \min\{S_T(K_T^u), R_T(K_T^d)\}. \tag{12}$$

The reader can verify that the flows predicted by eqns (11) and (12) match in every case those in Fig. 5(a,b).

Recall now that all the cases in Fig. 6 are equivalent to a pair of lead vehicle problems (one on each set of lanes) in which both lead vehicles maintain the same speed. Therefore the resulting flow in each set of lanes should be that of a simple LWR Riemann problem with a downstream density that would cause traffic to move at the lead vehicles' speed. This means that all the cases of Fig. 6 reduce to eqn (11) if the following downstream densities are used:

$$\mathbf{K}'^d = (0,0) \qquad \text{if } \mathbf{K}^d \text{ is in region 'A' (i.e. } K_T^d \prec K^0)$$

$$= (\gamma_1, \gamma_2)K_T^d \qquad \text{if } \mathbf{K}^d \text{ is in region 'D' (i.e. } K_T^d \succ K^0)$$

This can be verified by direct inspection of the figure.

Similarly, direct inspection of the cases in Fig. 7 reveals that the following equations,

$$Q_T = r(k^d) + R(K^d) \tag{13a}$$

$$q = \min\{r(k^d); Q_T(1 - \alpha^u)\} \tag{13b}$$

$$Q = Q_T - q, \tag{13c}$$

apply if the upstream state is in region 'D'. The equations also apply to the two subcases '1', (where the upstream state is in region 'A') if the velocity of the trailing shock is negative (i.e. if $s_T(K_T^u) > R(K^d) + r(k^d)$). If this velocity is positive, then the flows are those of the upstream state; i.e.

$$\mathbf{Q} = \mathbf{Q}^u, \text{ if } S_T(K_T^u) \leq R(K^d) + r(k^d). \tag{14}$$

The remaining three cases of Fig. 7 obey eqn (11). The remaining two cases not displayed in Fig. 7, corresponding to transitions into region "A", obey eqn (12); see footnote on page 100.

7. CONCLUSION

This paper has presented a theory of traffic dynamics on freeway segments with special lanes. If it proves to be reasonable, it may eventually help predict more reliably the congestion build up and dissipation processes at major freeway weaves. This is just an example of the engineering designs involving special lanes that might be evaluated with the proposed model.

In order to be able to analyze more complicated problems (e.g. involving networks) it will be necessary to examine in more detail the boundary conditions that would properly represent various types of junctions and changes in freeway geometry (e.g. where a car-pool lane ends or begins), and to search for simplifications that may lead to a unified treatment of all the problems. The next step, however, should be testing the theory with field data.

Acknowledgement—Research supported by PATH MOU-165, University of California, Berkeley.

REFERENCES

Bick, J. H. and Newell, G. F. (1960) A continuum model for two-lane directional traffic flow. *O. App. Math* **18**, 191–204.

Daganzo, C. F. (1993) A finite difference approximation for the kinematic wave model. Inst. of Transp. Studies Research Report UCB-ITS-RR-93-11, Univ. of California, Berkeley. (Published in abbreviated form in *Transportation Research B* **29**, 261–276, 1995.)

Garabedian, P. R. (1986) *Partial Differential Equations.* Chelsea Publishing Co., New York.

Gazis, D. C. and Herman, R. (1992) The moving and 'phantom' bottlenecks. *Transportation Science* **26**, 223–229.

Gazis, D. C., Herman, R. and Weiss, G. H. (1962) Density oscillations between lanes on a multilane highway. *Operations Research* **10**, 658–667.

Godunov, S. K. (1961) Bounds on the discrepancy of approximate solutions constructed for the equations of gas dynamics. *Journal Computers Mathematics and Mathematical Physics* **1**, 623–637.

Lax, P. D. (1973) *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shockwaves.* Regional Conference Series in Applied Mathematics. SIAM, Philadelphia, PA.

Lebacque, J. P. (1993) Les modèles macroscopiques de Trafic. *Annales des Ponts* (3rd trim) **67**, 28–45.

LeVeque, R. J. (1992) *Numerical Methods for Conservation Laws.* Birkhäuser Verlag, Boston, MA.

Lighthill, M. J. and Whitham, J. B. (1955) On kinematic waves. I Flow movement in long rivers. II A theory of traffic flow on long crowded roads. *Proceedings Royal Society A*, **229**, 281–345.

Michalopoulos, P. G. and Beskos, D. E. (1984) Improved continuum models of traffic flow. *Proceedings 9th International Symposium on Transportation and Traffic Theory* (Vollmuller, J. and Hamerslag, R. Eds), pp. 89–111. VNU Science Press BV, Utrecht, The Netherlands.

Munjal, P. K. and Pipes, L. A. (1971) Propagation of on-ramp density perturbations on uni-directional two and three lane freeways. *Transportation Research* **5**, 257–266.

Newell, G. F. (1993a) A moving bottleneck. Institute of Transportation Studies Research Report UCB-ITS-RR-93-3, University of California, Berkeley.

Newell, G. F. (1993b) A simplified theory of kinematic waves in highway traffic, Part I: General theory; Part II: Queuing at freeway bottlenecks; Part III: Multi-destination flows. *Transportation Research B* **27**, 281–314.

Richards, P. I. (1956) Shockwaves on the highway. *Operations Research.* **4**, 42–51.

Whitham, G. B. (1974) *Linear and Nonlinear Waves.* Wiley, New York.