MA4090 Final Report

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Exploring Bell's Theorem through a Card Game

Contents

1 Introduction

For the last 100 years quantum theory has been a hot topic, dividing opinion among academics and causing debate about our understanding of reality itself.
Classical mechanics provides us with a description of the reality we experience in our daily lives, all things have their definite properties. However, quantum theory seems to go against this idea of certainty. The fundamental particles that make up the world appear to behave in a probabilistic manner.
This strangeness didn't sit well with some of the world's brightest minds, most notably Albert Einstein, who famously called quantum entanglement "spooky action at a distance" [1].
In a paper published in 1935, Einstein, along with Boris Podolsky and Nathan Rosen, set out to highlight that the physical description of the behaviour of fundamental particles proposed by Neils Bohr and Werner Heisenberg was an incomplete description [2].
It was believed that there was no experiment that could be conducted to confirm or deny whether Einstein, Podolsky and Rosen's "hidden variable theory" could provide a more complete description of the behaviour of fundamental particles. However, in 1964 Irish physicist John Stewart Bell published a paper that reignited the scientific discussion on the topic [3]. In this paper, he derived his inequalities and formulated the idea for an experiment that would finally settle the debate.
We will begin the paper by describing the mathematical framework of quantum probability, introducing the most useful mathematical definitions along the way. The reader will then learn about some of the practical experiments which confirm Bell's inequalities are violated by quantum systems. We will then conclude by exploring Bell's findings through a card game.

2 Introduction to Quantum Probability

Quantum probability theory is a generalisation of Kolmogorov's classical probability theory. It uses the existing model to build a framework that can describe quantum mechanical systems. Quantum systems usually contain observables that do not commute, therefore, quantum probability falls under the category of a non-commuting probability theory.
So, what we would like to do is to firstly introduce this classical probability theory and the laws that govern it, then find a way that we can make the mathematical structure of the classical probability space non-commutative so it can successfully describe a quantum system as desired.
We will also introduce some quantum mechanics throughout this section as we formulate this new probability theory.

2.1 Classical Probability

2.1.1 Kolgomorov's Model

Modern probability is based on Andrey Kolmogorov's axioms. Events, E, are represented in the form of sets, subsets of some Ω called the sample space. The $\sigma - algebra$ of sets is the system of sets representing events. This $\sigma - algebra$ is closed with respect to the following operations:
Countable union, intersection and complement.
Kolmogorov formed the basis for his model of probability with the following two natural axioms:
1: Events are represented as elements of a $\sigma - algebra$ and operations of events are described by boolean logic.
2: Probability is represented as a probabilistic measure (countably additive function on a $\sigma - algebra$) [4].
A probability space in the Kolmogorov model is defined as a triple (Ω, F, P), with Ω being the sample space, F being the event space and $P(E), E \in F$ being the probability of the event E happening. Kolmogorov's famous axioms of probability that govern today's classical probability are as follows:
First Axiom: The probability of an event happening is a non-negative real number. This gives us the following condition:

$$P(E) \geq 0, \ P(E) \in \mathbb{R}, \ \forall E \in F.$$

Second Axiom: The probability that at least one of the events in the sample space will occur is 1:

$$P(\Omega) = 1.$$

Third Axiom: The assumption of $\sigma-$additivity. Any countable sequence of disjoint sets satisfies the following:

$$P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i).$$

Random variables are measurable functions $a : \Omega \to \mathbb{R}$, these would represent observables in quantum mechanics. The expectation value for a random variable, a, is given by the Lebesgue integral:

$$\int_{\Omega} a(\omega)\, dP(\omega).$$

Conditional probability in the Kolmogorov model is given by Bayes formula:

$$P(B|C) = \frac{P(B \cap C)}{P(C)}.$$

A family $(A_i)_{i \in I}$ of events is called **independent** if for all finite $J \subset I$ we have the following:

$$\mathbb{P}(\cap_{i \in J} A_i) = \prod_{i \in J} \mathbb{P}(A_i).$$

A theorem was needed to enable us to measure probabilities, albeit with some inaccuracy and uncertainty. When dealing with a large number of trials, the 'success' rate (i.e. the percentage of times that the event A_j occurs) will be close to the assigned probability p after sufficiently many of the trials. The following theorem provided the desired result:

Theorem (**Weak law of large numbers**).

If $A_1, A_2, A_3 ...$ are independent events, all with the same probability of success p, then we have for all $\epsilon \geq 0$:

$$\lim_{n \to \infty} \mathbb{P}[|\frac{1}{n} K_n - p| \leq \epsilon] = 1.$$

where, $K_n(\omega) :=$ number of elements in (or cardinality of) $\{j \in (1, 2, ..., n) \Big| \omega \in A_j\}$.

John Bell added his knowledge to the field of probability theory in his groundbreaking paper in 1964, when he set out to formulate an experiment that would confirm or deny the possibility of hidden variables governing the behaviour of quantum particles. Bell's inequalities, which I will introduce in the next short section and work closely with throughout this paper, are statements that always hold in classical probability (i.e. in a Kolmogorov probability space as described above) but are violated by the behaviour of particles in quantum mechanical systems. Since the quantum system describes the behaviour of the tiny quantum particles that make up the world around us, the violation of Bell's inequalities have lead many leading to question the true nature of reality as we know it.

2.1.2 Bell's Inequalities

In this brief section, I will introduce the inequalities proposed by John Bell for both the three and four random variable cases and outline the short proofs as seen in Hans Maassen's paper [5].

Bell's Three Variable Inequality: If we have any triple of 0-1 valued random variables, P_1, P_2, P_3, on a probability space $(\Omega, \mathbb{P})$ then the following inequality holds:

$$\mathbb{P}[P_1 = 1, P_3 = 0] \leq \mathbb{P}[P_1 = 1, P_2 = 0] + \mathbb{P}[P_2 = 1, P_3 = 0].$$

Proof: The proof for this is trivial:

$$\mathbb{P}[P_1 = 1, P_3 = 0] = \mathbb{P}[P_1 = 1, P_2 = 0, P_3 = 0] + \mathbb{P}[P_1 = 1, P_2 = 1, P_3 = 0] \leq$$

$$\mathbb{P}[P_1 = 1, P_2 = 0] + \mathbb{P}[P_2 = 1, P_3 = 0]$$

□

Bell's Four Variable Inequality: If we have any quadruple 0-1 valued random variables, P_1, P_2, P_3, P_4, on a probability space $(\Omega, \mathbb{P})$ then the following inequality holds:

$$\mathbb{P}[P_1 = P_3] \leq \mathbb{P}[P_1 = P_4] + \mathbb{P}[P_2 = P_3] + \mathbb{P}[P_2 = P_4].$$

(By symmetry, none of these four probabilities can be larger than the sum of the other three!).

Proof: The proof for this has a little more to it, but again it is very simple:
Let A_{jk}, $(j = 1, 2,\ k = 3, 4)$, be the random variable which is 1 if $P_j = P_k$ and 0 otherwise.
Let us assume that the following inequality does not hold: $A_{13} \leq A_{14} + A_{23} + A_{24}$. Note that taking expectations of each of the terms in the above inequality gives us Bell's four variable inequality, because $\mathbb{P}[P_j = P_k] = \mathbb{E}(A_{jk})$. To violate the above inequality, there would need to be at least one $\omega \in \Omega$ such that $A_{13}(\omega) = 1$ and $A_{14}(\omega) = A_{23}(\omega) = A_{24}(\omega) = 0$. So, this would mean that $P_1(\omega) \neq P_4(\omega) \neq P_2(\omega) \neq P_3(\omega)$. Clearly, this is a contradiction, because there

are an odd number of inequality signs. We have proved by contradiction that Bell's four variable inequality does hold!

□

<u>Bell's Theorem:</u> Bell's theorem states that if certain predictions of quantum theory are correct then our world is non-local. "Non-local" here means that there exist interactions between events that are too far apart in space and too close together in time for the events to be connected even by signals moving at the speed of light [6]. i.e based on Bell's inequalities not being satisfied by quantum systems, fundamental particles can not be described by a local hidden variable theory.

2.2 Non-Commutative Probability

As I have stated previously, in order to provide a framework for quantum probability, we must somehow make the mathematical structure of a classical probability space non-commutative. Strategies to make a commutative mathematical structure into a non-commutative one have been formulated since as long ago as the 1930s, for example the basis for non-commutative integration theory was formulated by John von Neumann in 1932 and brought together by Irving Segal in 1953 [7]. More recently, the example of Alain Connes' non-commutative geometry theory, which was formulated by him in his book in 1994, is a very topical theory today [8].

All of these examples employ a similar strategy in making some commutative classical mathematical structure into a non-commutative one, generally consisting of the following three steps:

· Firstly, we put the information found in this classical structure into an algebra of functions on this classical structure.

· Secondly, we identify axioms that govern this algebra.

· Thirdly, we drop the commutativity axiom, (this would be one of the axioms from the second step).

So, as outlined earlier, to formulate the framework for quantum probability the above strategy was applied to the structure of a classical probability space and a non-commutative probability space was formulated as desired.

In the case of quantum probability theory a non-commutative probability space can be defined as follows:
A **non − commutative probability space** is defined as a pair (A, ϕ), where A is a $*$-algebra (or a von Neumann algebra) of operators on some Hilbert space H, and ϕ is a state on A [9].

2.2.1 Some Definitions

We will now provide the main definitions and explanations that allow one to arrive at the above definition and to formulate the framework for this new probability theory:

A **Hilbert space** is a complex linear space, H, with a sesquilinear form:

$$H \times H \to \mathbb{C} : (\psi, \chi) \mapsto \langle \psi, \chi \rangle,$$

where $\langle \psi, \chi \rangle$ is the **inner product** of ψ and χ.

Please note that H can be assumed to be finite dimensional for the purpose of our description of quantum probability.

In this case we will define an **operator** on H to be a linear map:

$$A : H \to H$$

These operators can be added together or multiplied in the usual manner.

We will also define the **adjoint** of an operator, A, to be the unique operator A^* on H that satisfies the following:

$$\forall \psi, \theta \in H : \ \langle A^* \psi, \theta \rangle = \langle \psi, A\theta \rangle$$

The **norm** of an operator A is defined in the following way:

$$||A|| := \sup\{||A\psi|| \;\Big|\; \psi \in H, ||\psi|| = 1\}.$$

The adjoint and norm also give us the following property:

$$||A^* A|| = ||A||^2 \ .$$

Now, we will introduce a simple definition of a (unital) $*-$ **algebra** of operators on H. In this case, we mean a subspace $\mathbb{A}$ of $A : H \to H$ such that $\mathbb{I} \in \mathbb{A}$ and

$$A, B \in \mathbb{A} \ \Rightarrow \ \lambda A, \ A + B, \ A \cdot B, \ A^* \in \mathbb{A}.$$

We will define a **state** ϕ on $\mathbb{A}$ to be a linear functional $\phi : \mathbb{A} \to \mathbb{C}$ that satisfies the following two requirements:

$$1.\ \forall A \in \mathbb{A} :\ \phi(A^* A) \geq 0,$$

$$2.\ \phi(\mathbb{I}) = 1.$$

Now, after introducing the above definitions, a quantum probability space can be called a pair $(\mathbb{A}, \phi)$ using the above definitions.

2.3 At the Example of $\mathbb{M}_2$

The simplest example of a non-commutative $*-$algebra is $\mathbb{M}_2(\mathbb{C})$, this is the algebra of all 2×2 matrices with complex valued entries.
So, we now have the non-commutative probability space given by the pair: $(\mathbb{M}_2(\mathbb{C}), \phi)$.
A state in this example is now given by the linear functional $\phi : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$. This linear functional satisfies the following:

$$\phi(\mathbb{I}_2) = 1,\ \phi(x^* x) \geq 0\ \ \forall\ x \in \mathbb{M}_2(\mathbb{C})$$

From the state ϕ, we can define the inner product of x, y on $\mathbb{M}_2(\mathbb{C})$ by the following:

$$\langle x, y \rangle_\phi := \phi(x^* y).$$

This, $(\mathbb{M}_2(\mathbb{C}), \langle \cdot, \cdot \rangle)$, will now give us a Hilbert space as we have stated earlier.
To test out the example of $\mathbb{M}_2$ further, we can try taking the trace of the matrices to be the linear functional representing a state. We will now check if taking the trace as a state satisfies the requirements we have outlined previously in this paper.

$$Tr_2 :\ \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$$

$$x \mapsto x_{11} + x_{22}$$

The trace always satisfies the inequality:

$$Tr_2(x^* x) \geq 0$$

We can now take the trace of the identity matrix, we get the following:

$$Tr_2(\mathbb{I}_2) = 2$$

Clearly, this violates the desired equality that we outlined earlier:

$$\phi(\mathbb{I}) = 1$$

So, we clearly must alter this linear functional in a way to allow it to satisfy the second requirement in the definition of a state as we have outlined previously. we can instead try the following linear functional:

$$tr_2 : \ \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$$

$$x \mapsto \frac{1}{2}(x_{11} + x_{22})$$

This still satisfies the following inequality, similar to before:

$$tr_2(x^*x) \geq 0$$

Now, it is also easy to see that applying this linear functional to the identity matrix will satisfy the desired equality:

$$tr_2(\mathbb{I}_2) = 1$$

Now, it can also be noted that we have a inner product here, as:

$$tr_2(x^*x) = 0$$

when,

$$x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, clearly the following defines an inner product on $\mathbb{M}_2$:

$$\langle x, y \rangle_{tr_2} \ := \ tr_2(x^*y)$$

2.3.1 Riesz's Representation Theorem/Riesz's Lemma

An important theorem that we will outline here is Riesz's representation theorem, which is a theorem in functional analysis that is defined as follows:

$$\textbf{Riesz's Representation Theorem} :$$

Let H be a Hilbert space whose inner product is linear in it's first argument and antilinear in it's second argument. For every continuous linear functional $\varphi \in H^*$, there exists a unique $f_\varphi \in H$ such that the following equality holds:

$$\varphi(x) = \langle x, f_\varphi \rangle, \ \forall \ x \in H$$

H^* here is the dual space of the Hilbert space H. The dual space of H is a set consisting of all linear forms on H, in our case this means the dual is the set of all linear functionals that map from H to $\mathbb{C}$.

So, if we take $(V, \langle , \rangle)$ to be a finite dimensional Hilbert space, then any linear functional (bounded/continuous) $L : V \to \mathbb{C}$ is of the form

$$L(\underline{a}) = \langle \underline{a}, \underline{x} \rangle$$

for some vector $\underline{a} \in V$.

So, bringing Riesz's representation theorem into the example of $\mathbb{M}_2(\mathbb{C})$, we have the following:

We will let $\phi : \mathbb{M}_2 \to \mathbb{C}$ be a linear functional, then by applying the theorem $\exists$ a unique $D \in \mathbb{M}_2(\mathbb{C})$ such that the following equality holds:

$$\phi(x) = \langle D, x \rangle_{tr_2} = tr_2(D^* x).$$

This theorem is very useful in the context of quantum mechanics because given any state (linear functional) on the finite dimensional Hilbert space, we can find a matrix/vector that satisfies the theorem. We will now demonstrate the importance of this theorem with the help of an example:

If we take $A := \mathbb{M}_n(\mathbb{C})$ as our unital $*-$ algebra and take $\varphi : A \to \mathbb{C}$ to be a state on this algebra. This state has the following properties:

$$\varphi : A \to \mathbb{C}$$

$$\varphi(\mathbb{I}) = 1$$

$$\varphi(a^* a) \geq 0 \;\; \forall a \in A$$

We define the trace operator to be the following:

$$Tr : A \to \mathbb{C}, \; a \mapsto Tr(a) = \sum_1^n a_{ii}$$

and the normalised trace operator to be the following:

$$tr : A \to \mathbb{C}, \; a \mapsto tr(a) = \frac{1}{n} \sum_1^n a_{ii}$$

So, we have the following equalities and inequalities:

$$Tr(\mathbb{I}) = n$$

$$tr(\mathbb{I}) = 1$$

$$Tr(a^*a) \geq 0$$

$$tr(a^*a) \geq 0$$

The above equalities and inequalities all hold $\forall\ a \in \mathbb{M}_n(\mathbb{C})$.
Equipping A with an inner product defined by one of the above trace operators, we can say that $(A, \langle,\rangle)$ is a Hilbert space. The inner product we will take here is the normalised trace operator, i.e $\langle b, a\rangle := tr(b^*a)$. Using Riesz' theorem, we can say that any linear functional $\varphi : A \to \mathbb{C}$ is of the following form:

$$\varphi(a) = \langle D, a\rangle_{tr} = tr(D^*a)$$

The above holds for some $D \in A$.

Now, let us go one step further and say that this linear functional is a state on A. As we have previously outlined, this implies some strict properties for the linear functional and subsequently for D. D Is defined as a density operator, which means that the following two properties always hold:

$$1 : D \geq 0$$

$$2 : tr(D) = 1$$

Short proof of the above :
By definition of the state and of the adjoint operator we have the following for the proof of the second property:

$$1 = \varphi(\mathbb{I}) = tr(D^*\mathbb{I}) = tr(D^*) = \overline{tr(D)} = tr(D) = 1$$

For the first property we will use that the state squared must be greater than or equal to zero, again this comes from the definition of a state as we have introduced earlier:

$$0 \leq \varphi(a^*a) = tr(D^*a^*a) = tr((D^*a^*a)^*) = \overline{tr(a^*aD)} = tr(D(a^*a))\ \forall a \in A$$

Clearly, D must be greater than or equal to zero. So, the positivity of states proves the first property for us that $D \geq 0$.

$\square$

Relating this to the commutative setting, we can take D as a diagonal nxn matrix.

$$\begin{bmatrix} D_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & D_{nn} \end{bmatrix} \in \mathbb{D}_n(\mathbb{C})$$

Taking some $f \in \mathbb{D}_n(\mathbb{C})$ and we have a state defined as the following

$$\varphi_D : \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$$

From Riesz's theorem/lemma we have the following:

$$\varphi_D(f) = tr(Df) = \sum_{x \in X} D_x f_x$$

where $X = \{1, 2, 3, .., n\}$.
Going one step further, we conclude that this gives precisely the expectation value of f. We can say this because using the definition of a state we can show that a probability measure μ is defined on X by the state.

Proof :

$$\varphi_D(\mathbb{I}) = tr(D) = \sum_{x \in X} D_x = 1$$

$$\varphi(f^* f) \geq 0 \ \forall f$$

$$\Rightarrow \varphi(\delta_x) \geq 0$$

$$\Rightarrow a_{x_0} \geq 0$$

where

$$\delta_x = \begin{cases} 1 & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$\Rightarrow a_x \geq 0 \ \forall x \in X$$

So, as a consequence of Riesz's representation theorem, any state on our Hilbert space defines a probability measure. Where the expectation value is given as follows:

$$\varphi(f) = \sum_x a_x f(x) = \sum_{x \in X} f(x)\mu(\{x\}) = \mathbb{E}_\mu(f)$$

By Riesz's findings, there is a one to one correspondence between defining a state on the Hilbert space and having a probability measure on X.

2.3.2 Pauli Spin Matrices

Let us now consider $\mathbb{M}_2(\mathbb{C})$ as a vector space.
Using Riesz's theorem as introduced above, we can say that any state $\varphi : \mathbb{M}_2\mathbb{C} \to \mathbb{C}$ can be written as $\varphi(A) = Tr(DA)$ where $D \geq 0, Tr(D) = 1$ for $D \in \mathbb{M}_2\mathbb{C}$, so we get an expression for D as the following:

$$\begin{bmatrix} 1-z & x-iy \\ x+iy & 1+z \end{bmatrix}$$

with

$$x^2 + y^2 + z^2 \geq 1,\ x, y, z \in \mathbb{R}$$

This will allow us to define Wolfgang Pauli's famous spin matrices using the above formulation, which are related to an angular momentum operator. Each of the matrices that Pauli brought to the table is a Hermitian matrix, meaning that it is equal to it's own conjugate transpose. Each matrix represents an observable corresponding to the spin along one of the three axes in $\mathbb{R}^3$, and the identity matrix is also included. So, the following are defined as the Pauli spin matrices:

$$\{\mathbb{I}_2,\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}$$

These matrices are often denoted by the following, $\{\sigma_0, \sigma_x, \sigma_y, \sigma_z\}$. These matrices are very important in quantum mechanics, they give you the components of the spin of a spin $\frac{1}{2}$ particle in the x,y and z directions of the three dimensional plane $\mathbb{R}^3$. Note, that these matrices do not commute, therefore if you measure in one direction it will disturb your measurement in another direction and vice versa.

2.3.3 The Bloch Sphere

In this short section we will introduce the Bloch sphere. The Bloch sphere gives us the full mathematical truth about the behaviour of a qubit, presented in a very nice way. The Bloch sphere, also known to many mathematicians as the Riemann sphere, is a unit 2 sphere in the three dimensional real plane $S^2 \in \mathbb{R}^3$. The points on the exterior of the sphere correspond to the pure states of the two level quantum mechanical system. These are any state that can be written as a superposition of the two basis states/vectors, $(1,0), (0,1) \in \mathbb{C}^2$ or written in bra ket notation these would be $|0\rangle, |1\rangle$, where the coefficients in front each of the basis vectors are complex numbers.
Note that the two basis vectors are also pure states themselves.

As we have mentioned above, this description leads us to the two level quantum mechanical system or the qubit:

States correspond to one dimensional subspaces of $\mathbb{C}^2$, described by unit vectors spanning the subspace.
The $0-1$ valued random variables correspond to orthogonal projections P onto a complex one dimensional subspace.
We can parametrize the set of all states by the unit vectors that are of the following form:

$$(\cos\theta, e^{i\varphi}\sin\theta) \in \mathbb{C}^2,\quad \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 0 \leq \varphi \leq \pi$$

This set of unit vectors can be identified by points on the unit sphere, $S^2 \in \mathbb{R}^3$, as stated above.

So, a representation of the Bloch sphere can be seen in the figure below with states $\psi, |0\rangle$ and $|1\rangle$ drawn as vectors in green, blue and red:

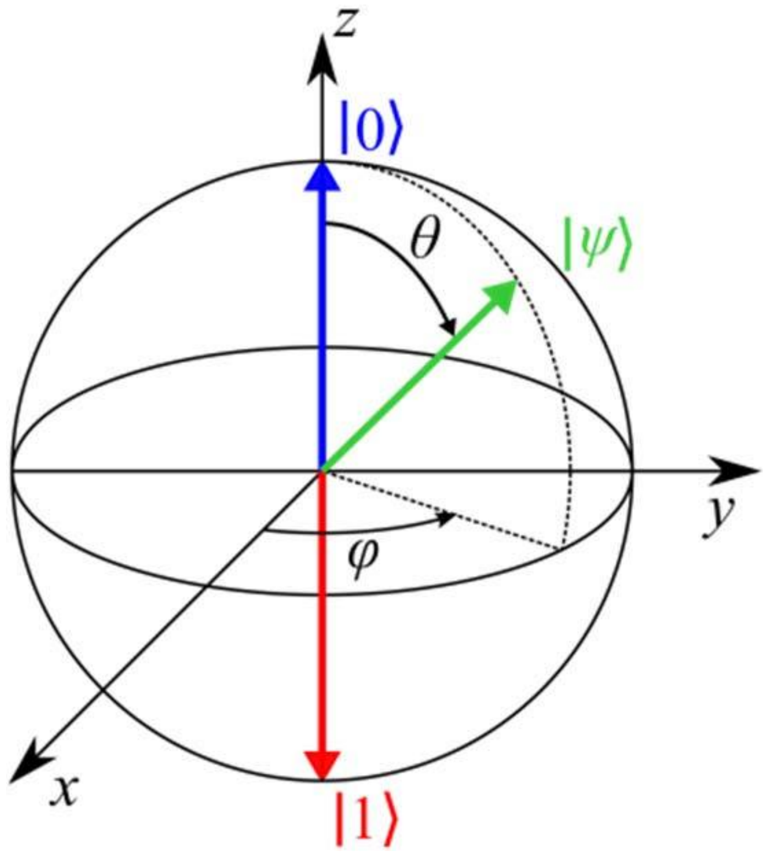

Figure 1

In figure 1 the basis vectors are represented in Dirac notation (bra ket notation) and $|\psi\rangle$ represents a pure state on the exterior of the Bloch sphere. The following would be the equation for the pure state $|\psi\rangle$ as a linear combination of the two base states:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

This is written again in Dirac notation for continuity with the notation in figure 1.

2.3.4 Tensor Products

The tensor product of two vector spaces V and W is denoted $V \otimes W$. It is a was of creating a new vector space from the combination of V and W, quite like the multiplication of integers. The dimensions of a tensor product come about as follows: $\mathbb{R}^n \otimes \mathbb{R}^k \cong \mathbb{R}^{nk}$. Tensor products can also operate on linear maps between vector spaces; given two linear maps $f : V \to X$ and $g : W \to Y$, the tensor product is defined as follows:

$$f \otimes g : V \otimes W \to X \otimes Y$$

$$(f \otimes g)(v \otimes w) = f(v) \otimes g(w)$$

By choosing basis vectors for all vector spaces in the description, we can represent the linear maps in the form of matrices. The matrix describing the tensor product of the two matrices (linear maps) is precisely the Kronecker product of the matrices. The definition of the Kronecker product is the following: If we have two matrices A, a $m \times n$ matrix, and B, a $p \times q$ matrix, then the Kronecker product, $A \otimes B$, is the $pm \times qn$ block matrix:

$$\begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{bmatrix}$$

Note that the trace operator gives us the following equality:

$$Tr(A \otimes B) = Tr(A) \times Tr(B)$$

Tensor products are very helpful in the setting of quantum mechanics. When looking at multiple particles, the calculations are much more manageable if we use tensor products. Later in this paper, we will use tensor products to describe a pair of quantum particles, (photons). So, sticking with the example of $\mathbb{M}(\mathbb{C}^2)$, the pair of particles would be described by a unit vector in the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$. The basis vectors for this tensor product would be defined as follows:

$$(1, 0, 0, 0) = e_1 \otimes e_1 =: e_{11}$$

$$(0, 1, 0, 0) = e_1 \otimes e_2 =: e_{12}$$

$$(0, 0, 1, 0) = e_2 \otimes e_1 =: e_{21}$$

$$(0, 0, 0, 1) = e_2 \otimes e_2 =: e_{22}$$

where e_1 and e_2 are the basis vectors in $\mathbb{C}^2$, i.e $(1, 0), (0, 1)$.

2.4 The Quantum Mechanical Coin Toss

At the example of $\mathbb{M}_2(\mathbb{C})$ we can explore the idea of a quantum mechanical fair coin toss. Taking $(\mathbb{M}_2, \frac{1}{2}tr)$ as our space, we will class events as orthogonal projections $P \in \mathbb{M}_2$. The matrices P will satisfy the following:

$$P^2 = P = P^*.$$

The right hand side of the above equality means that these matrices are self-adjoint (also known as Hermitian).

Proposition: Self-adjoint matrices in $\mathbb{M}_2$ will always have two real valued eigenvalues.

Proof :

Let λ be an eigenvalue of P and let x be the eigenvector corresponding to λ. From the definition of eigenvalues and eigenvectors, we have the following:

$$Px = \lambda x$$

Now let us bring the inner product in to the frame by computing $\lambda\langle x, x\rangle$:

$$= \langle \lambda x, x\rangle$$

$$\begin{aligned} &= \langle Px, x\rangle \\ &= \langle x, Px\rangle \\ &= \overline{\langle Px, x\rangle} \\ &= \overline{\langle \lambda x, x\rangle} \\ &= \overline{\lambda}\langle x, x\rangle \end{aligned}$$

An eigenvector cannot be 0, so $x \neq 0$, given that it's an eigenvector. By another property of inner products, (which we are assuming knowledge of), we conclude the following:

$$\Rightarrow \langle x, x\rangle \neq 0$$

So, we can divide across by $\langle x, x\rangle$:

$$\Rightarrow \lambda = \overline{\lambda}$$

This implies that λ (i.e any eigenvalue) is real. □

We know that given $P = P^2$, the eigenvalues can only be 0 or 1. Using this conclusion, we come to the following description for the possible situations that we could encounter:

$$\text{Case 1: } \lambda_1 = 0,\ \lambda_2 = 0,\ \text{So, P} = 0,$$

$$\text{Case 2: } \lambda_1 = 0,\ \lambda_2 = 1\ or\ \lambda_1 = 1,\ \lambda_2 = 0,$$

$$\text{Case 3: } \lambda_1 = 1,\ \lambda_2 = 1,\ \text{so this means } P = \mathbb{I}.$$

So, we have specified what P is equal to for case 1 and case 3. However, we need to find a matrix for P in case 2. P in this case is a one-dimensional projection satisfying the following :

$$trP = 0 + 1 = 1$$

$$DetP = 0 \cdot 1 = 0$$

If we use the fact that $P = P^*$ and that the trace is 1, we can write:

$$P = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

Now, using the fact that $DetP = 0$:

$$\frac{1}{4}((1-z^2) - (x^2+y^2)) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 = 1$$

This means that the set of all P projections in $\mathbb{M}_2$ are parametrized by the unit sphere S_2 in $\mathbb{R}^3$, similar to the Bloch sphere (but the Bloch sphere being the unit ball in $\mathbb{R}^3$). Recalling the Pauli spin matrices from section 2.3.2, we can now string together what we have described in this section with what a quantum mechanical die means and in turn what the spin of a particle means in probabilistic terms:

Taking a vector in $\mathbb{R}^3$, namely $a = (a_1, a_2, a_3)$, we have the following spin matrix:

$$\sigma(a) := \begin{bmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{bmatrix} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3.$$

Where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli spin matrices as defined previously. So, similar to the states the one dimensional projections can be parametrized nicely in $\mathbb{R}^3$. We will now look for a relation between these to calculate the probability of an event in a certain state. We first note that for any $a, b \in \mathbb{R}^3$ we have:

$$\sigma(a)\sigma(b) = \langle a, b\rangle \cdot 1 + i\sigma(a \times b)\ [9]$$

From before, we have the following when applying the projection operator to the vector a:

$$P(a) = \frac{1}{2}(\mathbb{I} + \sigma(a)), \ ||a|| = 1.$$

The probability of the event $P(a)$ in a state, namely $p(b)$, is given by:

$$tr(p(b)P(a)) = \frac{1}{2}(1 + \langle a, b \rangle).$$

Also, the events $P(a), P(b)$ are compatible iff $a = \pm b$. So, for all vectors a in the unit sphere, we have the following:

$$P(a) + P(-a) = 1, \ P(a)P(-a) = 0. \ [9]$$

Proof in [9]. Interpreting the above proposition, we have the probability distribution of the quantum mechanical coin toss to be a vector, b, in the unit sphere. As we have just stated if two events are compatible, then the probability of exactly one of the two of them occurring is equal to 1. The probability of a certain $P(a)$ occurring is precisely equal to $\frac{1}{2}(1 + \langle a, b \rangle)$, as we have stated above.
So, comparing this with the setting of a classical coin toss: Where we would simply have a probability of $\frac{1}{2}$ of our coin landing on heads in the classical setting, in the quantum coin toss we would have a probability of $\frac{1}{2}(1 + \langle a, b \rangle)$ of landing on heads and this would be the case for every direction of vector in $\mathbb{R}^3$. Quantum coin tosses in alternative directions are incompatible as we have stated above.
So, we can conclude that the notion of a quantum fair coin is modelled by the Hilbert space $(\mathbb{M}_2, \frac{1}{2}tr)$. [9]
The coin toss we have described in this section describes precisely the behaviour of spin $\frac{1}{2}$ particles, which are part of the makeup of the world around us.

3 Polarization Experiments

In this section we will introduce some basic polarization experiments which will help us to build a mathematical model that describes quantum probability.

3.1 Polarization Filters

We will consider a beam of (unpolarized) light with intensity I_0. We will then set up two polarization filters for the beam of light to pass through. We will fix the angle of the first polarization filter and rotate the second by an angle α. We will call the intensity of the light after passing through the first polarization filter I_1 and observe that $I_1 = \frac{I_0}{2}$. The $\cos^2$ law for photon absorption states that the intensity of light that passes through a second polarization filter is proportional to $\cos^2(\alpha)$ where α is the angle at which the second polarization filter is at, this is known as Malus' Law [10]. Calling the intensity of the light after passing through the second polarization filter I_2, we get $I_2 = I_1 \cos^2(\alpha)$.

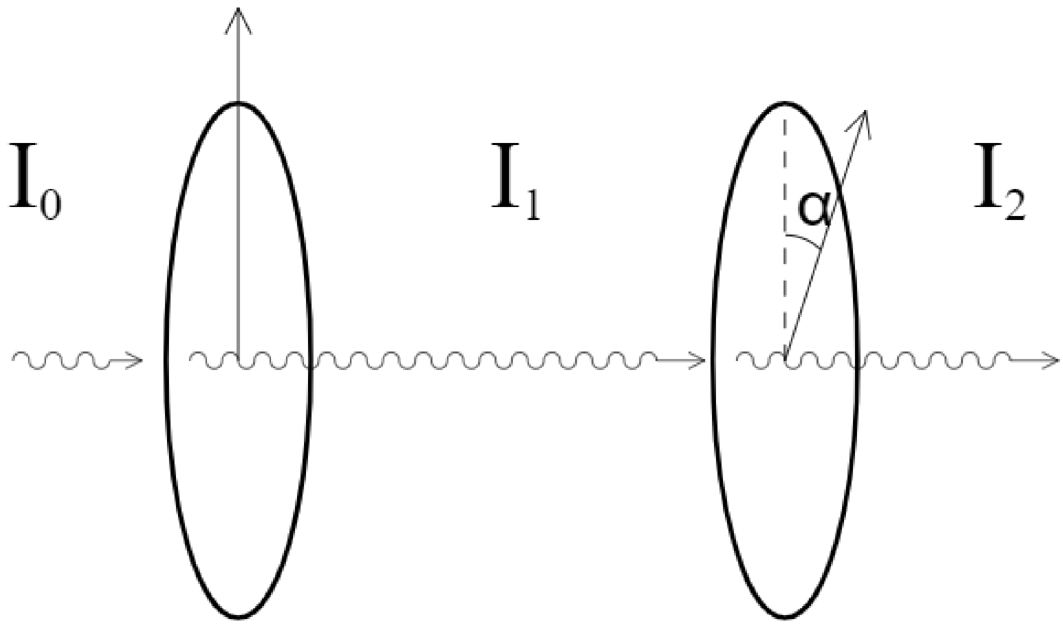

Figure 2

The intensity of light must be proportional to the number of photons.
We, therefore, conclude that the probability that a photon which passed through the first polarization filter will pass through the second is given by $\cos^2(\alpha)$.
So, $I_2 = I_1 \cos^2(\alpha)$ holds for large numbers of photons.

We are aware, (from the work of Max Planck and Albert Einstein), that light at extremely low intensities comes in small bundles of electromagnetic energy called photons, which cannot be broken down further and are independent of the total intensity of the light.
Each polarization filter reacts with each photon in such a way that depending on the angle of the polarization filter the photon may or may not pass through. (It must either fully pass through or be fully stopped given that the photon cannot be broken down further).

The real strangeness of the situation begins to appear once a third polarization filter is added. This can be demonstrated easily by you at home:

- Take three pairs of sunglasses, (or three polarization filters of some sort if you have them lying around the house).
- First, line two of the polarization filters/sun glass lenses up to a light and observe the change in intensity of the light that passes through the two filters as you rotate the angle of the second filter.
- Now, rotate the second filter until there is no longer any light passing through them, this should be at the angles such that the filters are perpendicular to each other.
- Finally, use the third filter by putting it in between the first two filters. Rotate this second filter to a 45 degree angle and notice that it will appear that a lot more light is created by doing so, demonstrating the strange nature of this quantum system.

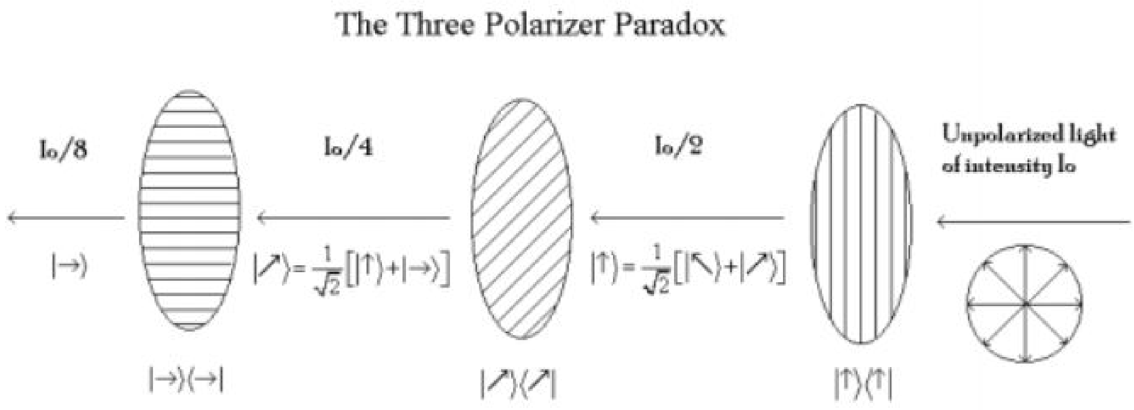

Figure 3

This simple home experiment can be summarised in figure 3. No light would pass through the filter on the left until the third filter is added. When we begin to measure the exact number of extra photons that are allowed through after the third filter is added, we discover just how strange the behaviour really is.

3.2 Classical Description of Polarization will not suffice

We will see through various experiments that classical probability will not suffice when trying to describe a quantum system, i.e. Bell's inequalities will not be satisfied when dealing with quantum behaviour. The experiments will be very similar to the one described above. However, physicists were forced to make the experiments more sophisticated over the years in attempts to answer any questions that advocates of classical probability theory were asking of them or to deal with any issues that were found to be problematic or incomplete in the simple experiments.

3.2.1 Experiments

The technology needed to produce the extremely low intensity light that we reffered to earlier was not available in the time of Einstein, however, it became readily available in the years since. This has made it possible to conduct comprehensive experiments to prove Bell's theorem. There are various experiments that have been conducted to demonstrate and prove Bell's findings. We will begin here by introducing the most simple experiment.
So, now considering an experiment with 3 polarization filters, similar to the three polarization filter experiment described above, but a bit more sophisticated, (using extremely low intensity light), we will see that Bell's three variable inequality is violated by this quantum system as we expected.
If we are to think of a classical probabilistic description of this experiment, we would assign a random variable to each polarization filter. This random variable would take the value one if the photon in question passed through the filter and would take the value zero if the photon was stopped by the filter. For a filter at an angle α, we would call this random variable P_α such that $P_\alpha(\omega) = 1$ if the photon ω passes through and $P_\alpha(\omega) = 0$ if the photon ω is absorbed. If we have two filters, the first at an angle α and the second at an angle β. Using Malus' $\cos^2$ law, as outlined previously, we can define the expectation value for the correlation the two random variables to be as follows:

$$\mathbb{E}(P_\alpha P_\beta) = \mathbb{P}[P_\alpha = 1 \textit{ and } P_\beta = 1] = \frac{1}{2}\cos^2(\alpha - \beta),$$

where $[P_\alpha = 1 \textit{ and } P_\beta = 1]$ is the set of all ω that pass through both of the filters.

We will now introduce a third filter and check if the mathematics we have outlined can hold up when testing it with Bell's three variable inequality. So, we now have three filters at angles a,b and c for simplicity. As we have seen above, the following correlation will hold between any pair of filters at angles i,j:

$$\mathbb{E}(P_iP_j) = \mathbb{P}[P_i = 1 \ and \ P_j = 1] = \frac{1}{2}\cos^2(i-j).$$

The experiment confirms that we have the following:

$$\mathbb{P}[P_i = 1, P_j = 0] = \mathbb{P}[P_i = 1] - \mathbb{P}[P_i = 1, P_j = 1] = \frac{1}{2} - \frac{1}{2}\cos^2(i-j) = \frac{1}{2}\sin^2(i-j).$$

Fitting this result into Bell's three variable inequality yields the following result:

$$\frac{1}{2}\sin^2(a-c) \leq \frac{1}{2}\sin^2(a-b) + \frac{1}{2}\sin^2(b-c).$$

So, now if we check this inequality with the following angles; $a = 0, b = \frac{1}{6}\pi, c = \frac{1}{3}\pi$, it will give us the following:

$$\frac{3}{8} \leq \frac{1}{8} + \frac{1}{8}.$$

Clearly, this violates the inequality, thus proving (in this simple example) that classical probability will not suffice when describing a quantum system!

3.3 Formulating the Mathematical Model for Polarization

Through studying the various polarization experiments discussed above and in [5], it becomes clear that classical probability will not suffice in attempting to describe these experiments mathematically. We can, however, use the information we have obtained from the experiments to formulate a model for an alternative probability theory, a quantum probability as introduced in section 1.

Polarization is completely characterized by a direction in the plane perpendicular to the light beam being polarized. We can, therefore, describe different directions of the polarization by unit vectors in a two-dimensional real plane $\mathbb{R}^2$, $(\psi \in \mathbb{R}^2, \|\psi\| = 1)$, pointing in this direction. Since two states that differ by a rotation of π only differ by a sign, we must describe states of polarization as one-dimensional subspaces of $\mathbb{R}^2$. Given $\psi, \theta \in \mathbb{R}^2$, two unit vectors, the transition probability can be expressed as $\cos^2(\alpha) = \langle\psi, \theta\rangle^2$.

Note: this expression does not depend on the sign of ψ, θ as $\cos^2(\alpha) = \cos^2(\pi - \alpha)$.

Using the orthogonal projection P onto the one-dimensional subspace, we can obtain the 0-1 valued random variable which tells us whether a single photon is polarized in a certain direction or not. This then allows us to write the transition probability as the following:

$$\cos^2(\alpha) = \langle \psi, \theta \rangle^2 = \langle \psi, P\psi \rangle.$$

Using the above findings, we can then formulate a mathematical model to describe polarization (the mathematical model used for quantum mechanics):

States of polarization of a photon correspond to one-dimensional subspaces of $\mathbb{R}^2$ described by unit vectors spanning the subspace.
Polarization Filters correspond to orthogonal projections P from $\mathbb{R}^2$ onto these one-dimensional subspaces.
Probability that a photon passes through is given by the following:

$$\cos^2(\alpha) = \langle \psi, \theta \rangle^2 = \langle \psi, P\psi \rangle = \mathbb{E}(P).$$

The above description is lacking, however.

We have disregarded circular polarization here. If we include this, we can obtain the full description of polarization (This leads to the 2-level system or Qubit). The model would then be:

States of polarization of a photon again correspond to one-dimensional subspaces of $\mathbb{C}^2$ described by unit vectors spanning the subspace.
Polarization Filters correspond to orthogonal projections P from $\mathbb{C}^2$ onto these one-dimensional subspaces.
Probability that a photon passes through is given by the following:

$$\cos^2(\alpha) = \langle \psi, \theta \rangle^2 = \langle \psi, P\psi \rangle = \mathbb{E}(P).$$

The set of all states can be parametrized by the unit vectors of the form:

$$(\cos\alpha, e^{i\phi}\sin\alpha) \in \mathbb{C}^2, \frac{-\pi}{2} \leq \alpha \leq \frac{\pi}{2}, 0 \leq \phi \leq \pi$$

The mathematical model that is used by quantum mechanics is a generalisation of the model from above. We keep things simple by working only in finite dimensions, i.e. we have only finitely many states.

The model is formulated as follows:

<u>States</u> of the system are given by one-dimensional subspaces of $\mathbb{C}^n$, with the state being described by some unit vector spanning the respective one-dimensional subspace.

0-1 valued random variables are given by Orthogonal projections onto a linear subspace of $\mathbb{C}^n$. Projections onto a subspace κ answer the questions whether the system is in any of the states represented by a unit vector in κ.

Probability is given by a measurement of a random variable P on a system in a state ψ gives the value 1 with probability still given by $\langle \psi, P\psi \rangle$.

We can attempt to embed this into a finite classical model as follows:
Taking a finite probability space $\Omega = \{\omega_1, ..., \omega_n\}$, with a probability distribution $(p_1, ..., p_n),\ 0 \leq p_i \leq 1,\ \sum_i p_i = 1$ where the probability for ω_i is p_i. For the 0-1 valued random variable, we define a function on Ω, i.e a characteristic function χ_A with $A \in \Omega$.

So, let us now construct a model for our situation: In our model, we can think of $\mathbb{C}^n$ as the space of complex valued functions on Ω, with $\delta_i,\ \delta_i(\omega_j) = \delta_{i,j}$ as basis. States here could be seen as the unit vectors δ_i.
Orthogonal projections P_A onto the linear span of these state vectors can be seen as the 0-1 valued random variable χ_A. In our basis χ_A is a diagonal matrix with 1 in the i^{th} diagonal entry is $\omega_i \in A$ and 0 otherwise. This would mean that $\omega_i \in A$ iff (if and only if) $\chi_A(\omega_i) = 1$ iff $\langle \delta_i, P_A \delta_i \rangle = 1$.
Any set of pairwise commuting projections on $\mathbb{C}^n$ can be simultaneously diagonalised, these can interpreted as a set of classical 0-1 valued random variables. So, therefore, classical probability corresponds to sets of pairwise commuting projections [5].

In our above description using classical probability theory, we neglect mixed states, i.e we didn't describe a probability distribution. We have not yet described the situation where the system is in a certain state ψ_1 with probability of q and in another state ψ_2 with probability of 1-q.
As we have seen from the Bloch sphere, the set of all states is a convex set. The pure states that we have described here are the points on the exterior of the sphere, whereas the mixed states that we have not yet described are interior points, as we have seen. So, we will fit them into our classical description as follows: Generally if we have P_i, a 0-1 valued quantum random variable and we take $\psi_1, ...\psi_k$ to be states with p_i being the probability that ψ_i occurs. Then the probability that a measurement of P gives the value 1 is given as follows:

$$\mathbb{P}(P = 1) = \sum_i p_i \langle \psi_i, P\psi_i \rangle.$$

So, if we now take $\psi \in \mathbb{C}^n$ to be a unit vector and take D_ψ to be the orthogonal projection onto the one-dimensional subspace generated by the unit vector ψ. We have the following:

$$\langle \psi, P\psi \rangle = tr(D_\psi \cdot P)$$

$$\sum_i p_i \langle \psi_i, P\psi_i \rangle = tr(\sum_i p_i D_{\psi_i} \cdot P) = tr(D \cdot P)$$

where $D := \sum_i p_i D_{\psi_i}$.
So, D is a convex combination of one-dimensional projections which can be seen as a $n \times n$ matrix that is Hermitian and positive semi-definite with trace equal to 1. The set of these matrices forms a closed convex set with extreme points being one-dimensional projections corresponding to pure states.

These matrices are called Density Matrices.

General mixed states are represented by these density matrices and the probability that a measurement of P gives 1 is given by:

$$\mathbb{P}(P = 1) = tr(D \cdot P).$$

Note that we have seen these matrices previously in the introduction to quantum probability, but we had not yet defined them.

4 Aspect's Experiment

In 1980, Alain Aspect set out to execute an experiment that even the most devoted classical probability theorists could not argue with the results of. The final experiment, conducted in Orsay in 1982, was immediately recognised by the scientific community and earned the title of "the decisive experiment". Although the experiment was not without it's flaws, it is credited with being the experiment that confirmed once and for all that quantum mechanics does in fact violate Bell's inequalities. It led to numerous other more sophisticated experiments, (known as Bell tests), over the years since 1982. Each one of these tests has confirmed the findings of John Bell, in this section we will stick with the example of describing Aspect's "decisive" experiment and introduce a card game that helps us delve deep into the essence of Bell's findings, Aspect's experiment and quantum probability itself.

4.1 Description of the Experiment

So, let's start by introducing the experiment and formulating a mathematical description of the system using the framework we have built throughout this project. As we have previously noted, the main criticism of our previous polarization experiments was that the filters didn't act on each of the photons without the influence of each filter on each other. So, the idea was now to somehow use entangled pairs of photons in the experiment and have them pass through two different filters at precisely the same time. It had become possible to build a device that produces pairs of entangled photons where each one moves in the opposite direction to it's paired photon and they behave oppositely to each other when polarized, i.e if one of the photons passes through it's filter, it implies that it's paired photon will be absorbed by it's filter (this is precisely what it means to be in entanglement in this case).

So, we will now introduce the exact framework for this experiment:
We have the photon pair producing device in the centre with a polarization filter on each side. The filters on the left and right can each be positioned in two different polarization measurements α_1 or α_2 on the left and β_1 or β_2 on the right. The directions of polarization for each trial of the experiment are chosen at random.

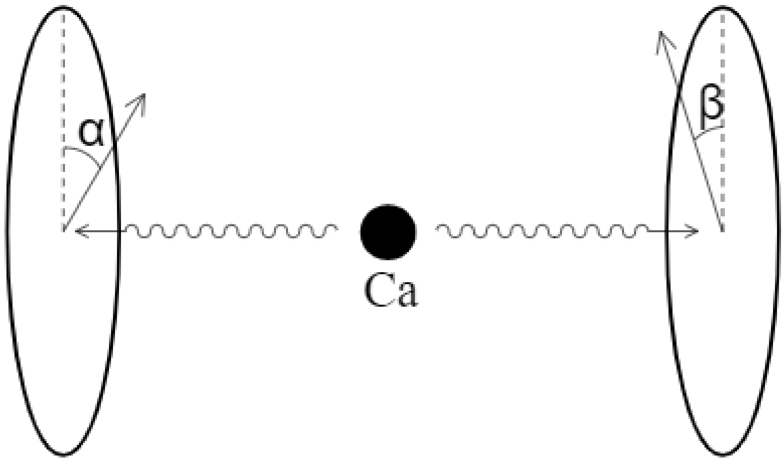

Figure 4

Figure 4 displays the simple set up of Aspect's experiment, with the calcium atom in the centre instead of the pair producing device.
The experiment gives us the following four random variables, (with two measured and compared at each trial):

$$P_1 := P(\alpha_1),\ P_2 := P(\alpha_2),\ Q_1 := Q(\beta_1),\ Q_2 := Q(\beta_2)$$

For checking the results of this experiment, we will need to recall Bell's four variable inequality previously outlined in this paper.
The quantum mechanical prediction for the probability of the two photons passing through or being absorbed is given by the following:

$$\mathbb{P}[P(\alpha) = Q(\beta) = 1] = 1 - \frac{1}{2}\cos^2(\alpha - \beta) = \frac{1}{2}\sin^2(\alpha - \beta)$$

$$\mathbb{P}[P(\alpha) = Q(\beta) = 0] = 1 - \frac{1}{2}\cos^2(\alpha - \beta) = \frac{1}{2}\sin^2(\alpha - \beta)$$

So, we have for the probability of both measurements giving the same value:

$$\mathbb{P}[P(\alpha) = Q(\beta)] = \sin^2(\alpha - \beta)$$

We can now plug this into Bell's four variable inequality and we are left with the following inequality, which needs to always be satisfied for classical probability theory to suffice:

$$\sin^2(\alpha_1 - \beta_1) \leq \sin^2(\alpha_1 - \beta_2) + \sin^2(\alpha_2 - \beta_1) + \sin^2(\alpha_2 - \beta_2) \tag{2}$$

Plugging in the following values; $\alpha_1 = 0,\ \alpha_2 = \frac{\pi}{3},\ \beta_1 = \frac{\pi}{2},\ \beta_2 = \frac{\pi}{6}$, we get:

$$1 \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \tag{3}$$

Clearly, the above is a contradiction and classical probability does not suffice in the description of this quantum system.

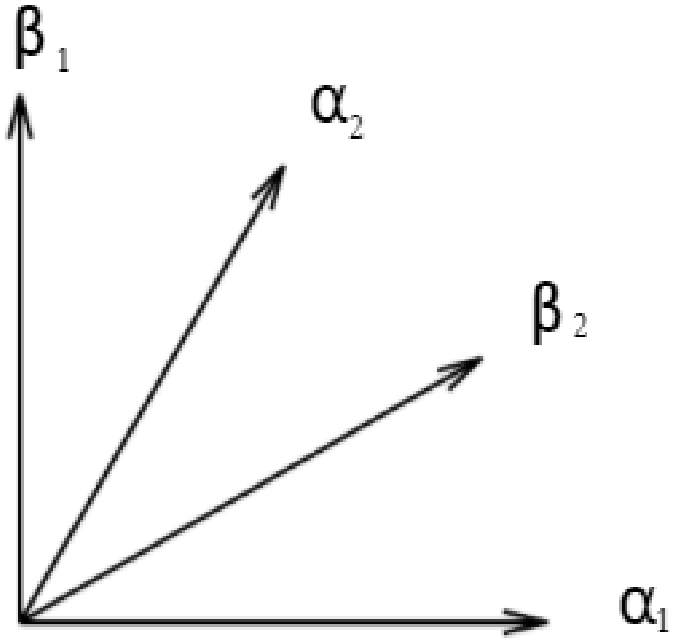

Figure 5

Clearly, there is no way to choose four angles (as shown above) that will satisfy our inequality.

4.1.1 Assumptions Made

1: For each $\omega \in \Omega$ the values of $P_j(\omega)$ and $Q_j(\omega)$ are well defined, i.e it is known how each photon will react to any filter, even the ones the photon doesn't react with. This is a standard assumption in classical probability theory, however it doesn't transfer over to the quantum mechanical system. These unmeasured quantities are called 'hidden variables'.

2: The outcome on the left should not depend on the angle of the polarization filter on the right and vice versa. Aspect ensured this by making the choice of what to measure on the left (α_1 or α_2) and the right (β_1 or β_2) during the flight of the photons. Therefore, any influence from left to right or right to left would have to travel faster than the speed of light. By the causality principle of relativity theory this is not possible. This is known as the "Locality" assumption [11].

Note: This description of quantum systems does not include the possibility of an explanation in classical terms if you are prepared to give up the causality principle.

4.2 Mathematical Model of Aspect's Experiment

Similar to the mathematical model that we formulated for the simple polarization experiment, we will now introduce the mathematical model for Aspect's experiment. Throughout this description, we will need to keep in mind the $\cos^2$ law for photon absorption that we introduced in section 3.1. We will need to change the situation from the previous description given that the filters are acting on two different photons, but the $\cos^2$ law will still be central in our mathematical model. This system is precisely the kind of two-level quantum system that we described previously, so we will use the tensor product of the matrices here to simplify our calculations. We will also revert back to the Bloch sphere and delve a little deeper in to what we can take from it in terms of polarization of a photon.

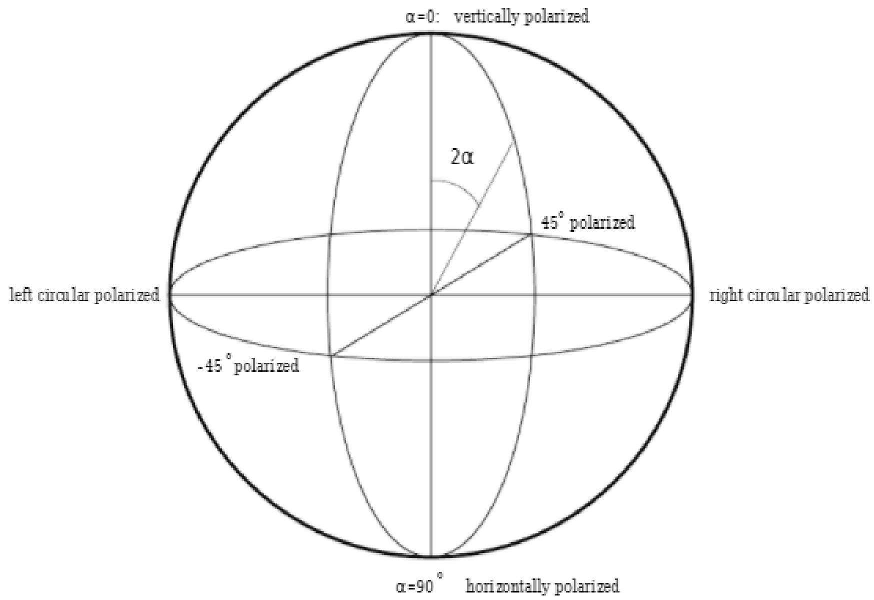

Figure 6

The above figure presents the Bloch sphere in terms of states of a photon depending on the angle of polarization [5]. With this representation, we can build a mathematical model of matrices that describe polarization filters:

$$\text{Polarized light at an angle } \alpha \cong (\cos\alpha, \sin\alpha) \in \mathbb{C}^2$$

$$\text{So, vertical and horizontal polarized light} \cong \text{unit vectors } (1,0),\ (0,1) \in \mathbb{C}^2$$

$$\text{Left or right circular polarized light} \cong \left(\frac{1}{\sqrt{2}}, \pm\frac{i}{\sqrt{2}}\right) \in \mathbb{C}^2$$

So, getting the tensor products of these matrices with their conjugate transposes, we get the following polarization matrices in our mathematical model:

$$\text{Polarization matrix for angle } \alpha \cong \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}$$

$$\text{Vertical polarization matrix} \cong \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Horizontal polarization matrix} \cong \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Horizontal polarization matrix} \cong \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Left or right circular polarization matrix} \cong \begin{bmatrix} \frac{1}{2} & \mp\frac{i}{2} \\ \pm\frac{i}{2} & \frac{1}{2} \end{bmatrix}$$

So, back to Aspect's experiment, using the bases $e_{11}, e_{12}, e_{21}, e_{22}$ from our tensor product space ($\mathbb{C}^4$) we have that the state of the emitted photons from the Calcium atom/paired photon producer has the following relation between the basis polarization vectors in $\mathbb{C}^4$, e_{12} (left vertical and right horizontal) and e_{21} (left horizontal and right vertical):

$$\psi = \frac{1}{\sqrt{2}}(e_{12} - e_{21})$$

So, now we can move forward and define the matrices for both the filter on the left $P(\alpha)$ and the filter on the right $Q(\beta)$ in Aspect's experiment using the framework we have introduced. Both are represented by projection operators in $\mathbb{C}^4$ using the angle alpha polarization matrix from the mathematical model of polarization. So, getting the tensor product of the matrix polarization matrix from the one level system with the identity matrix in $\mathbb{M}(\mathbb{C}^2)$ will give us:

$$P(\alpha) = \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha & 0 & \cos\alpha\sin\alpha & 0 \\ 0 & \cos^2\alpha & 0 & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & 0 & \sin^2\alpha & 0 \\ 0 & \cos\alpha\sin\alpha & 0 & \sin^2\alpha \end{bmatrix}$$

$$Q(\beta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta & 0 & 0 \\ \cos\beta\sin\beta & \sin^2\beta & 0 & 0 \\ 0 & 0 & \cos^2\beta & \cos\beta\sin\beta \\ 0 & 0 & \cos\beta\sin\beta & \sin^2\beta \end{bmatrix}$$

These projections commute, so we note that $1:\ P(\alpha)Q(\alpha)$ is a projection in it's self. To help us describe our probability in a classical probability framework, we will also note that the following operations are projections that also commute: $2:\ P(\alpha)(\mathbb{I}-Q(\beta)),\ 3:\ (\mathbb{I}-P(\alpha))Q(\beta),\ 4:\ (\mathbb{I}-P(\alpha))(\mathbb{I}-Q(\beta))$.
So, putting the above operations in to word form to understand their physical meaning in our context:

1: (Left passes through, right photon passes),

2: (Left passes through, right photon absorbed),

3: (Left passes absorbed, right photon passes),

4: (Left passes absorbed, right photon absorbed).

So, to find the probabilities of the above four events happening we need to operate on the original state ψ of the paired photons using each of the 4 projection operators that we have defined. Let us first note what ψ is equal to:

$$\psi = \frac{1}{\sqrt{2}}(e_{12} - e_{21}) = \frac{1}{\sqrt{2}}(0, 1, -1, 0)$$

So, using the fact that $\langle \psi, P\psi \rangle$ is the probability that a measurement of P gives the value 1 (as we have previously seen in section 3.3), the probability for event 1 occurring is given by the following:

$$\langle \psi, P(\alpha)Q(\beta)\psi \rangle = \frac{1}{2}(0, 1, -1, 0)\times$$

$$\begin{bmatrix} \cos^2\alpha\cos^2\beta & \cos^2\alpha\cos\beta\sin\beta & \cos\alpha\sin\alpha\cos^2\beta & \cos\alpha\sin\alpha\cos\beta\sin\beta \\ \cos^2\alpha\cos\beta\sin\beta & \cos^2\alpha\sin^2\beta & \cos\alpha\sin\alpha\cos\beta\sin\beta & \cos\alpha\sin\alpha\sin^2\beta \\ \cos\alpha\sin\alpha\cos^2\beta & \cos\alpha\sin\alpha\cos\beta\sin\beta & \sin^2\alpha\cos^2\beta & \sin^2\alpha\cos\beta\sin\beta \\ \cos\alpha\sin\alpha\cos\beta\sin\beta & \cos\alpha\sin\alpha\sin^2\beta & \sin^2\alpha\cos\beta\sin\beta & \sin^2\alpha\sin^2\beta \end{bmatrix}$$

$$
\begin{aligned}
&\times \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \\
&= \frac{1}{2}(\cos^2\alpha\sin^2\beta + \sin^2\alpha\cos^2\beta - 2\cos\alpha\sin\alpha\cos\beta\sin\beta) \\
&= \frac{1}{2}(\cos\alpha\sin\beta - \sin\alpha\cos\beta)^2 \\
&= \frac{1}{2}\sin^2(\alpha-\beta).
\end{aligned}
$$

Similarly for events 2, 3 and 4 we can use the following to calculate their respective probabilities:

$$
\begin{gathered}
\langle\psi, P(\alpha)(\mathbb{I} - Q(\beta))\psi\rangle, \\
\langle\psi, (\mathbb{I} - P(\alpha))Q(\beta)\psi\rangle, \\
\langle\psi, (\mathbb{I} - P(\alpha))(\mathbb{I} - Q(\beta))\psi\rangle.
\end{gathered}
$$

The calculations for the probabilities for these 3 events are a bit more arduous, so we will not include them.

5 Aspect's Experiment as a Card Game

A helpful way of exploring the process and results of Aspect's experiment and demonstrating the power of the quantum mechanical die would be the following card game proposed by Hans Maassen [5]. We will lay out the framework for this two player game and investigate whether there exists a classical strategy that the two players can use to win this game.

5.1 How to Play

Taking two players, P and Q, who will work together to try to win the game and a third person who plays the roll of recording the results and dealing the cards. There are dice and a well shuffled deck of cards (we will actually assume this is an infinite sequence of independent cards for complete randomness). The game plays out as follows:

1: The person dealing the cards (who we will refer to as the Z) hands out one card to each player. P and Q look at their own cards but not at each others cards. (The colour of the card is what is important for the game).

2: The dice are then thrown.

3: P and Q then simultaneously say 'yes' or 'no'. They can base this on any information they have, i.e this can be based on a combination of colour of their card, number on their card, numbers the dice land on, the time or anything else that might help them.

4: The cards are then shown and this determines which of the four squares the result of this trial will be recorded in. On the game board the squares are as follows: (red,red), (red,black), (black,red), (black,black).

5: Now Z will record this result in it's respective square. Z will put a '1' if the answers given by P and Q were the same or '1' if their answers were different.

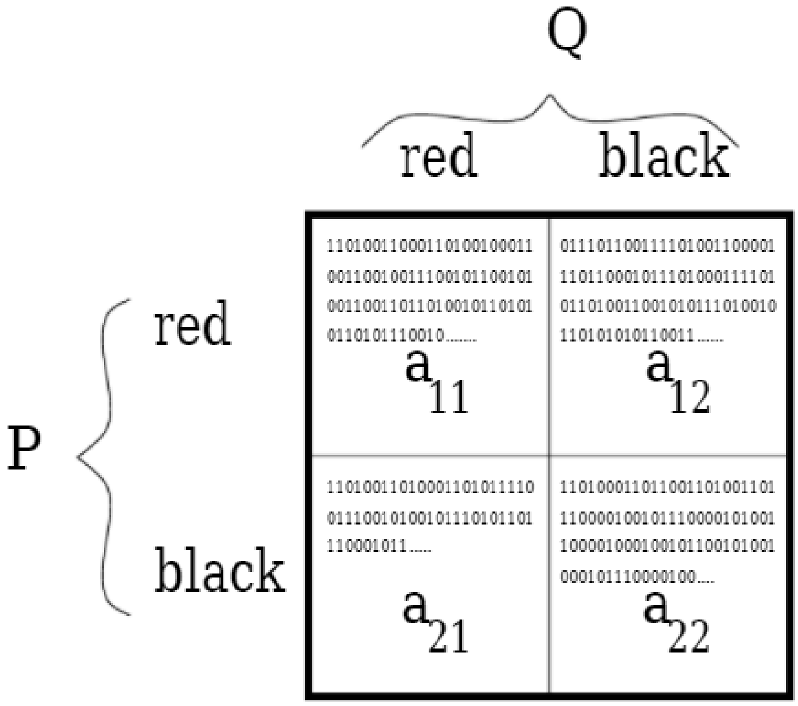

Figure 7: This figure shows the game board (the board of results recorded by Z).

Note that P and Q can not communicate at all during the game. However, they can agree on a strategy before the game starts. This strategy would target step 3 of the game and should influence whether they answer yes or no each time.

Z keeps track of the percentage of '1's in each square in proportion to the total number of digits in the square in question as the game proceeds. As we tend to infinity trials, the limits of these percentages can be called $a_{11}, a_{12}, a_{21}, a_{22}$ as we see in figure 7.

To win this game P and Q have a singular goal, this is to employ a strategy so that the resulting limiting percentages, $a_{11}, a_{12}, a_{21}, a_{22}$, satisfy the following requirement: a_{11} is larger than the sum of the other three, i.e P and Q try to give identical answers whenever their cards are both red, but different answers otherwise. So, the inequality for the game is as follows:

$$a_{11} \geq a_{12} + a_{21} + a_{22}.$$

5.2 Bell's Inequality for the Game

We propose that Bell's inequality for the game reads as follows:

$$a_{11} \leq a_{12} + a_{21} + a_{22}. \tag{4}$$

Proposition :

This inequality tells us that P and Q will be unable to win this game by classical means, i.e there's no strategy that they can come up with to result in them winning the game.

Proof :

Recall in section 2.1.2, we defined Bell's four variable inequality.
There are extremal strategies that P and Q can employ to try to win this game, these are ones that result in values of 1 in two of the squares and 0 in the other two. Examples of these strategies would be when one player says 'yes' all the time and the other answers based solely on whether his or her card is red, this would result in $a_{11}, a_{12} = 1$ and 0 in the other two squares. There are sixteen extremal strategies and many other random strategies that can be combinations of the extremal strategies or something more random. By Bell's inequality we see that none of these sixteen strategies can win the game. Bringing randomness into the strategy will not help the players to win i.e randomness from the dice or some other piece of random information that the players could look for during the game. These random strategies will not lead to results that satisfy Bell's linear inequality, so this acts to (informally) prove the above proposition [5].

5.3 Nature Can Win the Game

So, we have concluded that classical probability theory does not provide P and Q with any possible strategy that they could employ to win this game, but does quantum mechanics provide a solution?
The answer is yes. Using the mathematical framework we have built, we have been able to determine the probabilistic description of the quantum system in Aspect's experiment. By replacing the classical die with a photon-pair emitting device placed on the table between P and Q, who each have a polarization filter (we will call this the quantum mechanical die for simplicity) and using the probabilities that we have derived from the cos^2 law of photon absorbtion, P and Q can use their polarization filters to win the game:

When the cards are dealt, P and Q must position their polarizers at a certain angle. This will depend on what cards that each player is dealt. The players will say yes if the photon passes through their filter and no if not. They will choose their polarization angles based on the following strategy:

Player P chooses the angle $\alpha_1 = 0$ anytime his or her card is red.
Player P chooses the angle $\alpha_2 = \frac{\pi}{3}$ anytime his or her card is black.

Player Q chooses the angle $\beta_1 = \frac{\pi}{2}$ anytime his or her card is red.
Player Q chooses the angle $\beta_2 = \frac{\pi}{6}$ anytime his or her card is black.

We now recall the following conclusion based off our mathematical description of Aspect's experiment (in section 4.2):

$$\mathbb{P}(P(\alpha) = Q(\beta)) = \sin^2(\alpha - \beta) \tag{5}$$

Plugging the values for the four different angles that we have specified into the limiting percentages (based on the probability equation we have defined):

$$a_{11} = \sin^2(0 - \frac{\pi}{2}) = \sin^2(\frac{\pi}{2}) = 1$$

$$a_{12} = \sin^2(0 - \frac{\pi}{6}) = \sin^2(\frac{\pi}{6}) = \frac{1}{4}$$

$$a_{21} = \sin^2(\frac{\pi}{3} - \frac{\pi}{2}) = \sin^2(\frac{\pi}{6}) = \frac{1}{4}$$

$$a_{22} = \sin^2(\frac{\pi}{3} - \frac{\pi}{6}) = \sin^2(\frac{\pi}{6}) = \frac{1}{4}$$

So, without any communication during the game, P and Q's answers will achieve the following results when plugged into the games inequality in the long run:

$$1 \geq \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

This clearly satisfies the inequality, P and Q can successfully win the game with the help of the quantum mechanical die!

5.4 Other Quantum Games

Although we have been unable to cover any further examples of quantum games in this paper, the reader would be strongly encouraged to give the following paper a read if they wish to delve deeper into the idea of quantum theory in the context of a game [12].

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