Universität Rostock — Traditio et Innovatio
Institute of Communications Engineering

Prof. Dr.-Ing. Sascha Spors
Signal Theory and Digital Signal Processing
Institute of Communications Engineering (INT)
Faculty of Computer Science and Electrical Engineering (IEF)

Lab exercise IV in Digital Signal Processing, winter semester 2018/19 (course #24505)

Lecturer: Vera Erbes, email: vera.erbes@uni-rostock.de, room: W: 8.211
Authors: Frank Schultz, Vera Erbes

All 4 lab exercises are due 10th Feb. 2019.

Simple FIR and IIR filter design

Exercise 1: FIR filter design (10 points)

The ideal, discrete-time, linear-phase highpass filter with the cut-off frequency $\Omega_c > 0$ and the constant group delay of $\frac{N}{2}$ samples ($N > 0$) can be given as

$$H(\Omega) = \begin{cases} 0 \cdot \mathrm{e}^{-\mathrm{j}\Omega\frac{N}{2}} & , \quad |\Omega| < \Omega_c \\ 1 \cdot \mathrm{e}^{-\mathrm{j}\Omega\frac{N}{2}} & , \quad \Omega_c < |\Omega| \leq \pi. \end{cases} \tag{1}$$

Later a finite length impulse response (FIR) for $0 \leq k \leq N$ will be cut out, thus realising a FIR filter order of N, i.e. the FIR exhibits $N+1$ coefficients.

a) Show that the impulse response in general can be given as

$$h[k] = \frac{\sin\left(\pi\left(k-\frac{N}{2}\right)\right)}{\pi\left(k-\frac{N}{2}\right)} - \frac{\sin\left(\Omega_c\left(k-\frac{N}{2}\right)\right)}{\pi\left(k-\frac{N}{2}\right)} \tag{2}$$

for $-\infty \leq k \leq \infty$. **(2.5 points)**

b) Show that in the special case of even N

$$h[k] = \begin{cases} -\frac{\sin\left(\Omega_c\left(k-\frac{N}{2}\right)\right)}{\pi\left(k-\frac{N}{2}\right)} & , \quad k \neq \frac{N}{2} \\ 1 - \frac{\Omega_c}{\pi} & , \quad k = \frac{N}{2} \end{cases} \tag{3}$$

can be derived. **(2.5 points)**

The infinite impulse response shall now be limited to a finite length for $0 \le k \le N$, which inevitably degrades the transfer function characteristics of the ideal highpass. The rectangular window

$$w_{\mathrm{RECT}}[k] = \begin{cases} 1 & , \ 0 \le k \le N \\ 0 & , \ \text{else} \end{cases} \tag{4}$$

and the so-called Kaiser-Bessel window (in Matlab: `kaiser(N+1,beta)`)

$$w_{\mathrm{KB}}[k] = \begin{cases} \frac{I_0\left(\beta\left(1-\left(\frac{k-\alpha}{\alpha}\right)^2\right)^{\frac{1}{2}}\right)}{I_0(\beta)} & , \ 0 \le k \le N \\ 0 & , \ \text{else} \end{cases} \tag{5}$$

of length $N+1$ (i.e. the number of FIR coefficients) shall be used for the filter design. The Kaiser-Bessel function uses $\alpha = \frac{N}{2}$ and the modified Besselfunktion of 1st kind of zeroth order $I_0(\cdot)$ [Olv10, §10.25] (in Matlab: `besseli(0,arg)`). With the parameter β the attenuation A in dB of the highest side lobe in the filter's stop band can be adjusted. Kaiser empirically derived the following relation:

$$\beta = \begin{cases} 0 & , \ A < 21 \\ 0.5842 \cdot (A-21)^{0.4} + 0.07886 \cdot (A-21) & , \ 21 \le A \le 50 \\ 0.1102 \cdot (A-8.7) & , \ A > 50, \end{cases} \tag{6}$$

which is valid for large N. To demonstrate this, perform the following tasks:

c) Generate highpass FIR filters with different cut-off frequencies $f_c = 100, 1000, 10000$ Hz with adapted filter orders N by rectangular windowing ($h[k]$) and by Kaiser-Bessel windowing ($h_w[k] = h[k] \cdot w_{\mathrm{KB}}[k]$) for $0 \le k \le N$ and a sampling frequency of $f_s = 48000$ Hz. The stop band attenuation of the Kaisel-Bessel window shall be $A = 36$ dB. The specific filter order N is to be adapted to four cycle durations of the specific cut-off frequency. N should be even, i.e. an odd number $N+1$ of FIR coefficients results. This yields about the same slope steepnesses of the filter responses. Create the magnitude responses in fig. 1 on your own and check if the FIR filters are really linear-phase. **(2.5 points)**

d) Which window results when choosing $A < 21$ dB for the Kaiser-Bessel calculation method? Why is this the case? **(1 point)**

e) How is the slope steepness of the filter responses affected by choosing a constant, e.g. $N = 192$, filter order and using the different cut-off frequencies 100, 1000, 10000 Hz. Why is this the case? **(1.5 points)**

Task c), d) and e) should be provided as commented/explained and easy to read Matlab or Python code and further documentation when necessary. Recommended additional literature: [Opp10, Wer12, Ife02, Mey00, Lyo11].

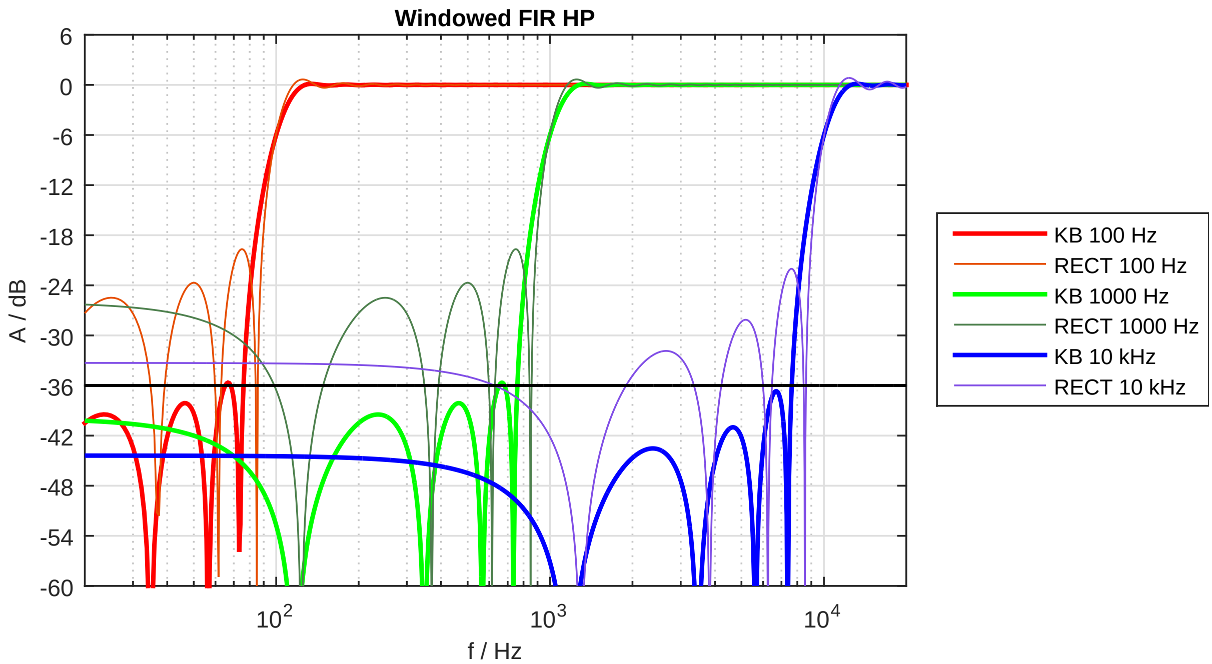

Figure 1: Highpass FIR filters for 100 Hz, 1 kHz, 10 kHz

Exercise 2: IIR filter design using the bilinear transform (10 points)

For a 1st order lowpass filter (simple RC circuit) a discrete-time IIR filter shall be designed using the so-called bilinear transform. The Laplace transfer function of this filter is

$$H(s) = \frac{1}{1 + \frac{s}{\omega_c}}, \tag{7}$$

where the -3 dB/$-45°$ cut-off frequency f_c is linked to $\omega_c = 2\pi f_c$. The Laplace variable s and the z-transform variable z are approximately linked as

$$s \approx 2 f_s \frac{z-1}{z+1}, \tag{8}$$

which is known as the bilinear transform (aka Tustin approximation) using the sampling frequency f_s ($[f_s] = \mathrm{Hz}$). This transform synthesises stable discrete-time filters from stable analogue filters. The left half plane of the Laplace domain is mapped into the unit circle in the z-domain. The infinite jω-axis of the Laplace domain is mapped to the unit circle's upper half in the z-domain and will thus be compressed which leads to frequency response distortion in the digital domain especially for frequencies near half the sampling frequency. However, for a single frequency – most often the cut-off frequency is chosen – an algorithm called pre-warping

$$\omega_c \quad \Leftarrow \quad 2 f_s \tan\left(\pi \frac{f_c}{f_s}\right) \tag{9}$$

allows for matching the analogue and digital transfer function at precisely this frequency. In order to demonstrate this the following tasks shall be performed:

a) Derive $H(z)$ in general form from $H(s)$ given in eq. 7 using $s = 2f_s\frac{z-1}{z+1}$ and formulate $H(z)$ in the so-called biquad structure, i.e. as an IIR filter, as follows

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}, \tag{10}$$

i.e. express the coefficients b_0, b_1, b_2 and a_1, a_2 in general form. **(3 points)**

b) Express the coefficients b_0, b_1, b_2 and a_1, a_2 using frequency pre-warping. **(2 points)**

c) State the numerical values of the resulting coefficients for a sampling frequency of f_s = 48 kHz and the analogue lowpass cut-off frequency f_c = 10 kHz with and without frequency pre-warping. **(1 point)**

d) Visualise the magnitude transfer function (logarithmic frequency axis up to 48 kHz, magnitude in dB, cf. fig. 2 top) for the three filters: analogue prototype, IIR filter without pre-warping and IIR filter with frequency pre-warping in one plot. Compare and discuss the results. What differences occur and why? **(2 points)**

e) Visualise the phase transfer function (logarithmic frequency axis up to 48 kHz, phase in degree, cf. fig. 2 bottom) for the three filters: analogue prototype, IIR filter without pre-warping and IIR filter with frequency pre-warping in one plot. Compare and discuss the results. What differences occur and why? **(2 points)**

Task c), d) and e) should be provided as commented/explained and easy to read Matlab or Python code and further documentation when necessary.

References

[Ife02] Ifeachor, E.C.; Jervis, B.W. (2002): *Digital Signal Processing.* Essex: Prentice Hall, 2. ed.

[Lyo11] Lyons, R.G. (2011): *Understanding Digital Signal Processing.* Upper Saddle River: Prentice Hall, 3. ed.

[Mey00] Meyer, M. (2000): *Signalverarbeitung.* Braunschweig, Wiesbaden: Vieweg, 2. ed.

[Olv10] Olver, F.W.J.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W. (2010): *NIST Handbook of Mathematical Functions.* Cambridge University Press, 1. ed.

[Opp10] Oppenheim, A.V.; Schafer, R.W. (2010): *Discrete-Time Signal Processing.* Upper Saddle River: Pearson, 3. ed.

[Wer12] Werner, M. (2012): *Digitale Signalverarbeitung mit MATLAB.* Wiesbaden: Vieweg+Teubner, 5. ed.

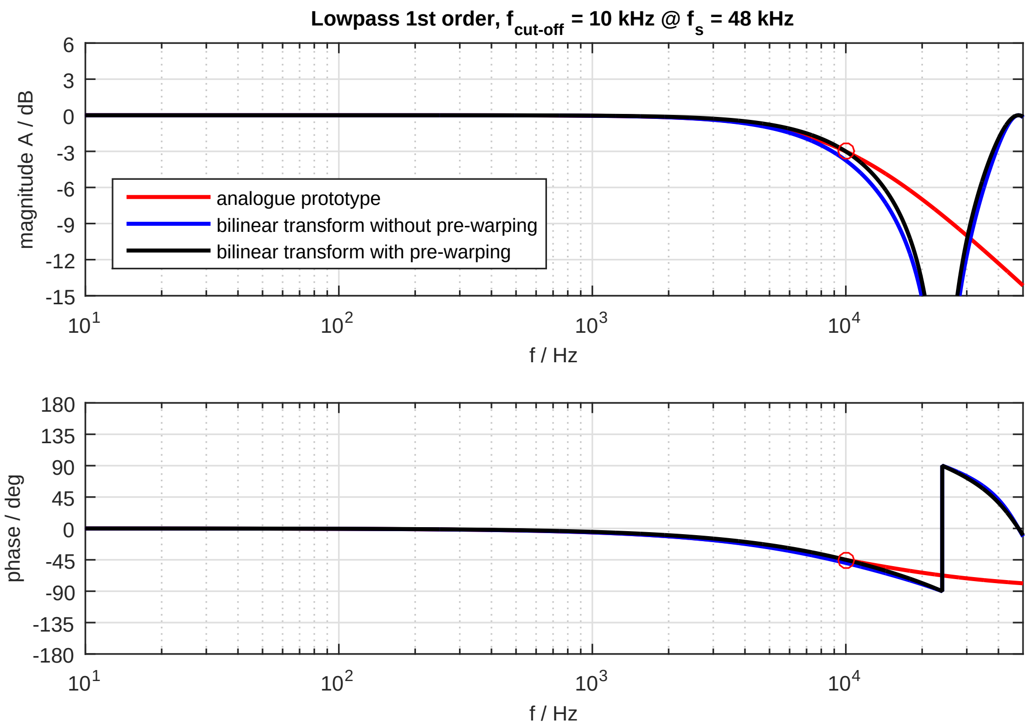

Figure 2: IIR lowpass filters of 1st order