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Exercise 1.

a)
The transfer function can also be written as:

$$\mathrm{H}(\Omega) = (\mathrm{rect}(\frac{\Omega}{2\pi}) - \mathrm{rect}(\frac{\Omega}{2\Omega_c})) \cdot \mathrm{e}^{-\mathrm{j}\Omega\frac{N}{2}} \tag{1}$$

Excluding the delay, applying DTFT using the continuous Fourier Transform,

$$\frac{W}{\pi}\ \mathrm{si}(Wt) \quad \circ\!\!-\!\!\bullet \quad \mathrm{rect}(\frac{\omega}{2W}) \tag{2}$$

and sampling,

$$t = kT \tag{3}$$

$$T = \frac{1}{f_s} = \frac{2\pi}{\omega_s} = \frac{\pi}{\omega_{max}} = 1 \tag{4}$$

yields:

$$\frac{\sin(\pi k)}{\pi k} - \frac{\sin(\Omega_c k)}{\pi k} \quad \circ\!\!-\!\!\bullet \quad \mathrm{rect}(\frac{\Omega}{2\pi}) - \mathrm{rect}(\frac{\Omega}{2\Omega_c}) \tag{5}$$

With the delay we get:

$$h[k] = \frac{\sin(\pi(k-\frac{N}{2}))}{\pi(k-\frac{N}{2})} - \frac{\sin(\Omega_c(k-\frac{N}{2}))}{\pi(k-\frac{N}{2})} \tag{6}$$

b)
In the case of $k = \frac{N}{2}$:

$$h[\frac{N}{2}] = \frac{\sin(0\pi)}{0\pi} - \frac{\sin(0\Omega_c)}{0\pi} = \frac{0}{0} = \text{undefined} \tag{7}$$

Using limits:

$$h[\frac{N}{2}] = \lim_{x\to 0}(\frac{\sin(\pi x)}{\pi x} - \frac{\sin(\Omega_c x)}{\pi x}) = \text{“}\frac{0}{0}\text{”} \tag{8}$$

Here we can apply l'Hôpital's rule:

$$\begin{aligned} h[\frac{N}{2}] &= \lim_{x\to 0}(\frac{\pi\cos(\pi x)}{\pi} - \frac{\Omega_c\cos(\Omega_c x)}{\pi}) && (9)\\ &= \lim_{x\to 0}(\cos(\pi x) - \frac{\Omega_c\cos(\Omega_c x)}{\pi}) && (10)\\ &= 1 - \frac{\Omega_c}{\pi} && (11)\end{aligned}$$

For $k \neq \frac{N}{2}$ and even N:

$$(k - \frac{N}{2}) \in \mathbb{Z} \setminus 0 \tag{12}$$

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$$\frac{\sin(\pi(k - \frac{N}{2}))}{\pi(k - \frac{N}{2})} = 0 \tag{13}$$

This yields the overall result for even N:

$$h[k] = \begin{cases} -\frac{\sin(\Omega_c(k-\frac{N}{2}))}{\pi(k-\frac{N}{2})} & k \neq \frac{N}{2} \\ 1 - \frac{\Omega_c}{\pi} & k = \frac{N}{2} \end{cases} \tag{14}$$

d)

For $A < 21$ the definition of the Kaiser-Bessel windows is identical to that of the rectangular window.

e)

The slope is a result of the leakage effect and is frequency independent. If the different filters have the same length the slope is also identical, but since we plot the spectrum with a logarithmic frequency axis, it seems that the slope for the lower frequency filter is lower. In the earlier tasks this was compensated by using more samples for the lower frequency filters.