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Exercise 1.

a)
The transfer function can also be written as:

$$\mathrm{H}(\Omega) = (\mathrm{rect}(\frac{\Omega}{2\pi}) - \mathrm{rect}(\frac{\Omega}{2\Omega_c})) \cdot \mathrm{e}^{-\mathrm{j}\Omega\frac{N}{2}} \tag{1}$$

Excluding the delay, applying DTFT using the continuous Fourier Transform,

$$\frac{W}{\pi}\,\mathrm{si}(Wt) \quad \circ\!\!-\!\!\bullet \quad \mathrm{rect}(\frac{\omega}{2W}) \tag{2}$$

and sampling,

$$t = kT \tag{3}$$

$$T = \frac{1}{f_s} = \frac{2\pi}{\omega_s} = \frac{\pi}{\omega_{max}} = 1 \tag{4}$$

yields:

$$\frac{\sin(\pi k)}{\pi k} - \frac{\sin(\Omega_c k)}{\pi k} \quad \circ\!\!-\!\!\bullet \quad \mathrm{rect}(\frac{\Omega}{2\pi}) - \mathrm{rect}(\frac{\Omega}{2\Omega_c}) \tag{5}$$

With the delay we get:

$$h[k] = \frac{\sin(\pi(k-\frac{N}{2}))}{\pi(k-\frac{N}{2})} - \frac{\sin(\Omega_c(k-\frac{N}{2}))}{\pi(k-\frac{N}{2})} \tag{6}$$

b)
In the case of $k = \frac{N}{2}$:

$$h[\frac{N}{2}] = \frac{\sin(0\pi)}{0\pi} - \frac{\sin(0\Omega_c)}{0\pi} = \frac{0}{0} = \text{undefined} \tag{7}$$

Using limits:

$$h[\frac{N}{2}] = \lim_{x\to 0}(\frac{\sin(\pi x)}{\pi x} - \frac{\sin(\Omega_c x)}{\pi x}) = \text{“}\frac{0}{0}\text{”} \tag{8}$$

Here we can apply l’Hôpital’s rule:

$$\begin{aligned} h[\frac{N}{2}] &= \lim_{x\to 0}(\frac{\pi\cos(\pi x)}{\pi} - \frac{\Omega_c\cos(\Omega_c x)}{\pi}) & (9)\\ &= \lim_{x\to 0}(\cos(\pi x) - \frac{\Omega_c\cos(\Omega_c x)}{\pi}) & (10)\\ &= 1 - \frac{\Omega_c}{\pi} & (11)\end{aligned}$$

For $k \neq \frac{N}{2}$ and even N:

$$(k - \frac{N}{2}) \in \mathbb{Z} \setminus 0 \tag{12}$$

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$$\frac{\sin(\pi(k-\frac{N}{2}))}{\pi(k-\frac{N}{2})} = 0 \tag{13}$$

This yields the overall result for even N:

$$h[k] = \begin{cases} -\frac{\sin(\Omega_c(k-\frac{N}{2}))}{\pi(k-\frac{N}{2})} & k \neq \frac{N}{2} \\ 1-\frac{\Omega_c}{\pi} & k = \frac{N}{2} \end{cases} \tag{14}$$