Ex4_2

February 10, 2019

1 Exercise 2

1.1 a)

Analoger Filter:

$$H(s) = \frac{1}{1 + \frac{s}{\omega_c}} \tag{1}$$

Bilineare transformation mit:

$$s = 2f_s \frac{z-1}{z+1} \tag{2}$$

Einsetzen ergibt die generelle Form:

$$H_d(z) = \frac{1}{1 + \frac{2f_s}{\omega_c}\frac{z-1}{z+1}} = \frac{1 + z^{-1}}{(1 + \frac{2f_s}{\omega_c}) + (1 - \frac{2f_s}{\omega_c})z^{-1}} \tag{3}$$

Analoger Tiefpassfilter kann geschrieben werden als:

$$H(s) = \frac{1}{1 + a_1 \frac{s}{\omega_c} + b_1 (\frac{s}{\omega_c})^2} \tag{4}$$

Unser gegebener Tiefpassfilter erster Ordung:

$$H(s) = \frac{1}{1 + \frac{s}{\omega_c}} \tag{5}$$

Normale Notation für die Transferfunktion von Analogen Filtern zweiter Ordnung:

$$H(s) = \frac{B_0 s^2 + B_1 s + B_2}{A_0 s^2 + A_1 s + A_2} \tag{6}$$

Durch den Vergleich der normalen Notation und unserem gegeben Filter ergibt sich:

$$B_0 = 0, B_1 = 0, B_2 = 1, A_0 = \frac{b_1}{\omega_c^2} = 0, A1 = \frac{a_1}{\omega_c} = \frac{1}{\omega_c}, A2 = 1 \tag{7}$$

Nach der Substitution von $s = 2f_s \frac{z-1}{z+1}$ wollen wir folgende Form erhalten:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{8}$$

Rechenweg:

$$\frac{B_0(2f_s\frac{z-1}{z+1})^2 + B_1(2f_s\frac{z-1}{z+1}) + B_2}{A_0(2f_s\frac{z-1}{z+1})^2 + A_1(2f_s\frac{z-1}{z+1}) + A_2} = \dots = \frac{\frac{B_0 4f_s^2 + B_1 2f_s + B_2}{A_0 4f_s^2 + A_1 2f_s + A_2} + \frac{-B_0 8f_s^2 + 2B_2}{A_0 4f_s^2 + A_1 2f_s + A_2} z^{-1} + \frac{B_0 4f_s^2 - B_1 2f_s + B_2}{A_0 4f_s^2 + A_1 2f_s + A_2} z^{-2}}{1 + \frac{-A_0 8f_s^2 + 2A_2}{A_0 4f_s^2 + A_1 2f_s + A_2} z^{-1} \frac{A_0 4f_s^2 - A_1 2f_s + A_2}{A_0 4f_s^2 + A_1 2f_s + A_2} z^{-2}} \tag{9}$$

Die Koeffizienten des digitalen Filters sind daher:

$$b_0 = \frac{B2 + 2B_1 f_s + 4B_0 f_s^2}{A_2 + 2A_1 f_s + 4A_0 f_s^2} = \frac{1}{1 + \frac{2f_s}{\omega_c}} b_1 = \frac{2B2 - 8B_0 f_s^2}{A_2 + 2A_1 f_s + 4A_0 f_s^2} = \frac{2}{1 + \frac{2f_s}{\omega_c}} b_2 = \frac{B2 + 2B_1 f_s - 4B_0 f_s^2}{A_2 + 2A_1 f_s + 4A_0 f_s^2} = \frac{1}{1 + \frac{2f_s}{\omega_c}} a_1 = \tag{10}$$

1.2 b)

$$b_0 = \frac{B2 + 2B_1 f_s + 4B_0 f_s^2}{A_2 + 2A_1 f_s + 4A_0 f_s^2} = \frac{1}{1 + \frac{1}{tan(\pi\frac{f_c}{f_s})}} b_1 = \frac{2B2 - 8B_0 f_s^2}{A_2 + 2A_1 f_s + 4A_0 f_s^2} = \frac{2}{1 + \frac{1}{tan(\pi\frac{f_c}{f_s})}} b_2 = \frac{B2 + 2B_1 f_s - 4B_0 f_s^2}{A_2 + 2A_1 f_s + 4A_0 f_s^2} = \frac{}{1 -} \tag{11}$$

1.3 c), d), e)

c) calculate the coefficients

In [1]:
```python
import numpy as np
import matplotlib.pyplot as plt
import scipy.signal as sig

fs = 48000 #Hz
fc = 10000 #Hz

omc = 2*np.pi*fc
omc_wrapped = 2*fs *np.tan(np.pi * fc / fs)

#Coefficients without pre-wrapping:
b0 = 1 / ( 1 + 2 *fs / omc)
b1 = 2 / ( 1 + 2 *fs / omc)
b2 = 1 / ( 1 + 2 *fs / omc)
b = [b0, b1, b2]

a1 = 2 / ( 1 + 2 *fs / omc)
#a2 = (1 - 2 * fs / omc) / ( 1 + 2 *fs / omc)
a2 = (omc - 2 * fs) / (omc + 2 *fs)
a = ([1, a1, a2])

#Coefficients with pre-wrapping
b0_w = 1 / ( 1 + 2 *fs / omc_wrapped)
b1_w = 2 / ( 1 + 2 *fs / omc_wrapped)
```

```
b2_w = 1 / ( 1 + 2 *fs / omc_wrapped)
b_wrapped = [b0_w, b1_w, b2_w]

a1_w = 2 / ( 1 + 2 *fs / omc_wrapped)
a2_w = (1 - 2 * fs / omc_wrapped) / ( 1 + 2 *fs / omc_wrapped)
a_wrapped = [1, a1_w, a2_w]

print("Coefficients of the digital filter in biquad structure:\n")
print("a = ", a)
print("b = ",b)


print("\nCoefficients of the digital filter in biquad structure with pre-wrapping:\n")
print("a = ",a_wrapped)
print("b = ",b_wrapped)
```

```
Coefficients of the digital filter in biquad structure:

a =  [1, 0.7911744635176464, -0.2088255364823536]
b =  [0.3955872317588232, 0.7911744635176464, 0.3955872317588232]

Coefficients of the digital filter in biquad structure with pre-wrapping:

a =  [1, 0.8683475024126042, -0.13165249758739586]
b =  [0.4341737512063021, 0.8683475024126042, 0.4341737512063021]
```

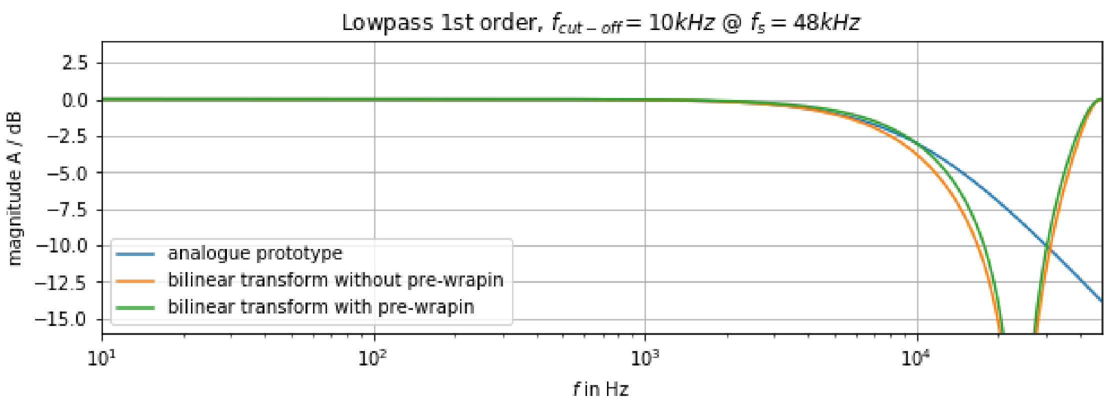
d) visualise magnitude transfer function

In [18]:
```
# Analog Filter
A = [0, 1/omc, 1]
B = [0, 0, 1]

#bilinear transform without pre-wraping
Om, Hd = sig.freqz(b, a, whole=True)
temp, Ha = sig.freqs(B, A, worN=fs*Om)
#Compute the frequency response of a digital filter and Analog Filter

#bilinear transform with pre-wraping
Om_w, Hd_wrapped = sig.freqz(b_wrapped, a_wrapped, whole=True)

#freuquency vector
#f = np.arange(0,fs,1) #frquency range
f = Om * fs / (2*np.pi)
#Plot Magnitude
plt.figure(figsize=(10,3))
plt.semilogx(f, 20*np.log10(np.abs(Ha)), label=r'analogue prototype')
plt.semilogx(f, 20*np.log10(np.abs(Hd)), label=r'bilinear transform without pre-wrapi
```

```
    plt.semilogx(f, 20*np.log10(np.abs(Hd_wrapped)), label=r'bilinear transform with pre-w
    plt.xlabel(r'$f$ in Hz')
    plt.ylabel(r'magnitude A / dB')
    plt.grid()
    plt.title(r'Lowpass 1st order, $f_{cut-off} = 10 kHz$ @ $f_s = 48 kHz$')
    plt.legend()
    plt.axis([10, fs, -16, 4])
```

```
/home/max/anaconda3/lib/python3.7/site-packages/ipykernel_launcher.py:19: RuntimeWarning: divic
/home/max/anaconda3/lib/python3.7/site-packages/ipykernel_launcher.py:20: RuntimeWarning: divic
```

```
Out[18]: [10, 48000, -16, 4]
```

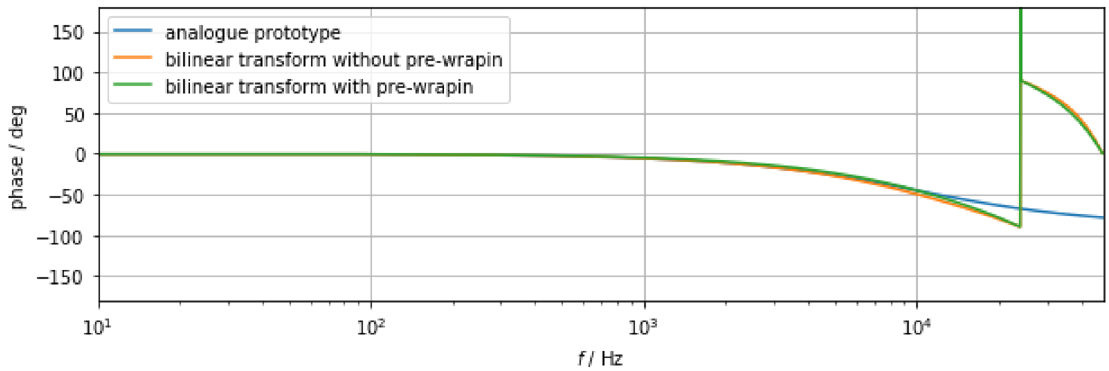


Die billineraren transformation der Frequenzantwort ist verschieden zu der des analogen Filters, durch die first-order approximation während des Mappings von der s-plane zur z-plane. Unter anderem verschiebt sich die corner frequency. Dieser Effekt wird als frequenzy wraping bezeichnet. Durch das pre-wraping, bei der die corner frequency ersetzt, sodass die Frequenz an dieser Stelle korregiert wird.

e)

```
In [14]: #Plot Angle
    plt.figure(figsize=(10,3))
    plt.semilogx(f, (180 / np.pi) * np.angle(Ha), label=r'analogue prototype') # transfor
    plt.semilogx(f, (180 / np.pi) * np.angle(Hd), label=r'bilinear transform without pre-w
    plt.semilogx(f, (180 / np.pi) * np.angle(Hd_wrapped), label=r'bilinear transform with
    plt.xlabel(r'$f$ / Hz')
    plt.ylabel(r'phase / deg')
    plt.grid()
    plt.legend()
    #plt.title(r'Lowpass 1st order, $f_{cut-off} = 10 kHz$ @ $f_s = 48 kHz$')
    plt.axis([10, fs, -180, 180])
```

```
Out[14]: [10, 48000, -180, 180]
```



Die Stabilität und minimale phase des Systems sind durch die billineare Transformation erhalten geblieben.