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Abstract

We develop and implement methods for determining whether introducing new securities or relaxing investment constraints improves the investment opportunity set for prospect investors. We formulate a new testing procedure for prospect spanning for two nested portfolio sets based on subsampling and Linear Programming. In an application, we use the prospect spanning framework to evaluate whether well-known anomalies are spanned by standard factors. We find that of the strategies considered, many expand the opportunity set of the prospect type investors, thus have real economic value for them. In-sample and out-of-sample results prove remarkably consistent in identifying genuine anomalies for prospect investors.

Keywords and phrases: Nonparametric test, prospect stochastic dominance efficiency, prospect spanning, market anomaly, Linear Programming.

JEL Classification: C12, C14, C44, C58, D81, G11, G40.

1 Introduction

Traditional models in economics and finance assume that investors evaluate portfolios according to the expected utility framework. The theoretical motivation for this goes back to

Von Neumann and Morgenstern (1944). Nevertheles, experimental and empirical work has shown that people systematically violate Expected Utility theory when choosing among risky assets. Prospect theory, first described by Kahneman and Tversky (1979) (see also Tversky and Kahneman (1992)), is widely viewed as a better description of how people evaluate risk in experimental settings. While the theory contains many remarkable insights, it has proven challenging to apply these insights in asset pricing, and it is only recently that there has been real progress in doing so (Barberis et al. (2019)). Barberis and Thaler (2003) and Barberis (2013) are excellent reviews on behavioral finance and prospect theory.

Stock market anomalies are key drivers of innovation in asset pricing. These are tradable portfolio strategies, usually constructed as long-short portfolios based on the top and bottom deciles of sorted stocks, according to specific characteristics (anomalies). Under the standard Mean-Variance (MV) paradigm, establishing a cross-sectional return pattern as an anomaly involves testing for pricing based on a factor model. If factors are traded, spanning regressions relate to MV criterion. Arbitrage pricing stipulates that a portfolio of factors is MV-efficient and no other portfolio can achieve a higher Sharpe Ratio (SR). In that sense, an anomaly is a strategy that exhibits higher SR and should be traded away. However, we can question MV spanning for portfolio selection if returns do not follow elliptical distributions, or investor preferences depend on more than the first two moments of the return distribution. Moreover, experimental evidence (Baucells and Heukamp (2006)) suggests that investors do not always act as risk averters. Instead, under certain circumstances, they behave in a much more complex fashion, exhibiting characteristics of both risk-loving and risk-averting. They behave differently on gains and losses, and they are more sensitive to losses than to gains (loss aversion). The relevant utility function could be concave for gains and convex for losses (S-Shaped).

The present study contributes to this literature by introducing, operationalizing and applying prospect spanning tests for portfolio analysis. The general research question is whether a given investment possibility set $\mathbb{K}$, namely the benchmark set, contains portfolios which prospect dominates all alternatives in an expanded investment possibility set $\mathbb{L}$.

Stochastic spanning (Arvanitis et al. (2019)) is a model-free alternative to MV spanning of Huberman and Kandel (1987) (see also Jobson and Korkie (1989), De Roon, Neyman, and Werker (2001)). Spanning occurs if introducing new securities or relaxing investment constraints does not improve the investment possibility set for a given class of investors. MV spanning checks if the mean-variance frontier of a set of assets is identical to the mean-variance frontier of a larger set made of those assets plus additional assets (Kan and Zhou (2012), Penaranda and Sentana (2012)). Here we investigate such a problem for investors with prospect type preferences which are interested in the whole return distributions generated by two sets of assets, namely stochastic dominance. First, we introduce the concept of prospect spanning, which is consistent with prospect type investors. We propose a theoretical measure for prospect spanning based on stochastic dominance and derive the exact limit distribution for the associated empirical test statistic for a general class of dynamic processes. To check prospect spanning on data, we develop consistent and feasible test procedures based on subsampling and Linear Programming (LP).

Similarly to Arvanitis et al. (2019), it is easy to see that if the prospect efficient set is non-empty, a prospect spanning set is essentially any superset of the former. As such, we can use a prospect spanning set to provide an outer approximation of the efficient set. This is useful in at least two ways. First, if the spanning set is small enough, the problem of optimal choice is reduced to a potentially simpler problem. Indeed, a spanning set is a reduction of the original portfolio set without loss of investment opportunities for any investor with S-shaped preferences. Secondly, if an algorithm for the choice of non-trivial canditate spanning sets is available, we can use this to construct decreasing sequences of prospect spanning sets that appropriately converge to the efficient set. Given the complexity of the prospect efficient set (see for example Ingersoll (2016)) such an approach can be useful for the determination of its properties.

The second contribution of the paper is to examine if we can explain well-known stock market anomalies by standard factor models for prospect investors. To do so, we test if trading strategies are genuine violations of standard factor models. More precisely, in the in-sample analysis, we use the prospect spanning test in order to check whether a portfolio set originating from a standard factor model, $\mathbb{K}$, spans the same set augmented with a market anomaly, $\mathbb{L}$. This check could be of significant relevance to the empirical analysis of financial markets. If the hypothesis of prospect spanning holds, the particular market anomaly can be explained by the factor model. Then the trading strategy that is identified in the literature as market anomaly may not be an attractive investment opportunity for prospect investors. On the contrary, if the hypothesis is not true, then the anomaly expands the opportunity set for prospect investors, and is useful to that extent. We also examine whether the cross-sectional patterns that found to expand the set of factors in-sample, maintain this abnormal return out-of-sample. Therefore, we use out-of-sample backtesting experiments as an independent criterion for robustness of in-sample test results (Harvey et al. (2016)). It turns out that prospect spanning tests produce remarkably consistent results both in- and out-of-sample in identifying trading strategies as genuine market anomalies for prospect investors. Thus, our framework helps validating stock market anomalies for prospect preferences.

Benartzi and Thaler (1995) utilize prospect theory to present an approach called myopic loss aversion which consists of two behavioural concepts, namely loss aversion and mental accounting. Barberis et al. (2001) study asset prices in an economy where investors derive direct utility not only from consumption but also from fluctuations in the value of their financial wealth. They are loss averse over these fluctuations and how loss averse they are depends on their prior investment performance. The design of their model is influenced by prospect theory. Barberis and Huang (2008) study the pricing of financial securities when investors make decisions according to cumulative prospect theory. Several other papers confirm that positively skewed stocks have lower average returns (Boyer, Mitton, and Vorkink (2010), Bali, Cakici, and Whitelaw (2011), Kumar (2009), Conrad, Dittmar, and Ghysels (2013)). Barberis and Xiong (2009, 2012) and Ingersoll and Jin (2013) show that theoretical investment models based on S-Shape utility maximisers help to understand the disposition effect found empirically in many studies (see e.g. Odean (1988), Grinblatt and Han (2005), Frazzini (2006), Calvet, Campbell, and Sodini (2009)). Kyle, Ou-Yang, and Xiong (2006) provide a formal framework to analyze the liquidation decisions of economic agents under prospect theory. He and Zhou (2011) study the impact of prospect theory on optimal risky exposures in portfolio choice through an analytical treatment. Ebert and Strack (2015) set up a general version of prospect theory and prove that probability weighting implies skewness preference in the small. Barberis et al. (2016) test the hypothesis that, when thinking about allocating money to a stock, investors mentally represent the stock by the distribution of its past returns and then evaluate this distribution in the way described by prospect theory. Moreover, Barberis et al. (2019) present a model of asset prices in which investors evaluate risk according to prospect theory and examine its ability to explain prominent stock market anomalies. In our paper, we test whether well-known factor models span the augmented universe with a prominent stock market anomaly, and if not, whether the result is supported out-of sample.

The paper is organised as follows. In Section 2, we review the definition of prospect stochastic dominance relation and we define the relevant concept of prospect spanning. We provide with a representation based on a class of S-shaped utility functions without assuming differentiability. Using an empirical approximation of the latter, we construct a test for the null hypothesis of spanning based on subsampling. The construction is based on the limiting null distribution of the test statistic which has the form of a saddle type point of a relevant Gaussian process. Under a weak condition on the structure of the parameter contact sets, we show that the test is asymptotically exact and consistent. This is weaker than the parameter extreme point comparisons used in Arvanitis, Scaillet and Topaloglou (2019) to obtain exactness in large samples.

In Section 3, we provide with a numerical approximation of the statistic that is based on the utility representation derived before. The utility functions are univariate, and normalized. We use a finite set of increasing piecewise-linear functions, restricted to the bounded empirical supports, that are constructed as convex mixtures of appropriate "ramp functions" (in the spirit of Russel and Seo (1989)) in our representation. For every such utility function, we solve two embedded linear maximization problems. This is an improvement over the implementation in Arvanitis and Topaloglou (2017) and Arvanitis, Scaillet and Topaloglou (2019) where they formulate tests in terms of Mixed-Integer Programming (MIP) problems. MIP problems are NP-complete, and far more difficult to solve. Our numerical approximations are simple and fast since they are based on standard LP. They suit better resampling methods, which otherwise become quickly computationally demanding in empirical work.

In Section 4, we perform an empirical application where we use the prospect spanning tests to evaluate stock market anomalies using standard factor models. We consider three such models that build on the pioneer three-factor model of Fama and French (1993): the four-factor model of Hou, Xue and Zhang (2015), the five-factor model of Fama and French (2015), and the four-factor model of Stambaugh and Yuan (2017). Given the extensive set of results produced under alternative spanning criteria, the analysis is confined to 11 well-known strategies used to construct Stambaugh-Yuan factors, along with 7 extra (18 overall) that attracted significant attention, namely Betting against Beta, Quality minus Junk, Size, Growth Option, Value (Book to Market), Idiosyncratic Volatility and Profitability. The 11 anomalies used in Stambaugh and Yuan (2017) are realigned appropriately to yield positive average returns. In particular, anomaly variables that relate to investment activity (Asset Growth, Investment to Assets, Net Stock Issues, Composite Equity Issue, Accruals) are defined low-minus-high decile portfolio returns, rather than high-minus-low. All the other anomalies are constructed as high-minus-low decile portfolio returns. These 18 trading strategies constitute our playing field for comparing spanning test results. Yet, we emphasize that this paper is not intended to compare factor models in terms of their ability to capture the cross-section of expected returns under prospect preferences. Instead, we use alternative factor models as a robustness check for testing the consistency of in- and out-of-sample results under the prospect spanning framework. Each factor model is our initial system of investment coordinates which we take as a granted opportunity set, without questioning its asset pricing validity. We view here the factors solely as investable assets (since they correspond to tradable strategies based on asset portfolios), and similarly for the anomalies. The anomalies might be labelled by other authors as factors if indeed priced in the cross-section, but we do not address such a research question in this paper.

Finally, Section 5 concludes the paper. In Appendix A, we provide a short description of the stock market anomalies used in the empirical application. In Appendix B, we also provide a short description of the performance measure used in the out-of-sample analysis. We give in a separate Online Appendix: i) the limiting properties of the testing procedures under sequences of local alternatives, ii) a Monte Carlo study of the finite sample properties of the test, iii) the proofs of the main results, as well as auxiliary lemmata and their proofs, iv) summary statistics of the factor and anomaly returns over our sample period from January 1974 to December 2016, and v) additional empirical results on out-of-sample analysis of market anomalies.

2 Prospect Stochastic Dominance and Stochastic Spanning

The theory of stochastic dominance (SD) gives a systematic framework for analyzing investor behavior under uncertainty (see Chapter 4 of Danthine and Donaldson (2014) for an introduction oriented towards finance). Stochastic dominance ranks portfolios based on general regularity conditions for decision making under risk (see Hadar and Russell (1969), Hanoch and Levy (1969), and Rothschild and Stiglitz (1970)). SD uses a distribution-free assumption framework which allows for nonparametric statistical estimation and inference methods. We can see SD as a flexible model-free alternative to mean-variance dominance of Modern Portfolio Theory (Markowitz (1952)). The mean-variance criterion is consistent with Expected Utility for elliptical distributions such as the normal distribution (Chamberlain (1983), Owen and Rabinovitch (1983), Berk (1997)) but has limited economic meaning when we cannot completely characterize the probability distribution by its location and scale. Simaan (1993), Athayde and Flores (2004), and Mencia and Sentana (2009) develop a mean-variance-skewness framework based on generalizations of elliptical distributions that are fully characterized by their first three moments. SD presents a further generalization that accounts for all moments of the return distributions without necessarily assuming a particular family of distributions.

Inspired by previous work, Levy and Levy (2002) formulate the notions of prospect stochastic dominance (PSD) (see also Levy and Wiener (1998), Levy and Levy (2004)) and Markowitz stochastic dominance (MSD). Those notions extend the well-know first degree stochastic dominance (FSD) and second degree stochastic dominance (SSD). PSD and MSD investigates choices by investors who have S-shaped utility functions and reverse S-shaped utility functions. Arvanitis and Topaloglou (2017) develop consistent tests for PSD and MSD efficiency which is an extension to the case where full diversication is allowed. Arvanitis, Scaillet and Topaloglou (2019) investigate MSD spanning. This paper extends those works to prospect spanning, which is consistent with prospect preferences.

2.1 Stochastic Spanning for Prospect Dominance and Analytical Representation

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, suppose that F denotes the cdf of some probability measure on $\mathbb{R}^n$. Let $G(z, \lambda, F)$ be $\int_{\mathbb{R}^n} 1_{\{\lambda^T u \leq z\}} dF(u)$, i.e., the cdf of the linear transformation $x \in \mathbb{R}^n \to \lambda^T x$ where λ assumes its values in $\mathbb{L}$, which denotes the portfolio space. We suppose that the portfolio space is a closed non-empty subset of $\mathbb{S} = \{\lambda \in \mathbb{R}^n_+ : \mathbf{1}^T \lambda = 1, \}$, possibly formulated by further economic, legal restrictions, etc. In many applications, we have that $\mathbb{L} = \mathbb{S}$. We denote by $\mathbb{K}$ a distinguished subcollection of $\mathbb{L}$ and generic elements of $\mathbb{L}$ by λ, κ, etc. In order to define the concepts of PSD and subsequently of stochastic spanning, we consider $\mathcal{J}(z_1, z_2, \lambda; F) := \int_{z_1}^{z_2} G(u, \lambda, F)\, du$.

Definition 1. κ *weakly Prospect-dominates* λ, written as $\kappa \succcurlyeq_P \lambda$, iff we have the inequalities $P_1(z, \lambda, \kappa, F) := \mathcal{J}(z, 0, \kappa, F) - \mathcal{J}(z, 0, \lambda, F) \leq 0$, $\forall z \in \mathbb{R}_-$ and $P_2(z, \lambda, \kappa, F) := \mathcal{J}(0, z, \kappa, F) - \mathcal{J}(0, z, \lambda, F) \leq 0$, $\forall z \in \mathbb{R}_{++}$.

Given the stochastic dominance relation above, stochastic spanning occurs when augmentation of the portfolio space does not enhance investment opportunities, or equivalently, investment opportunities are not lost when the portfolio space is reduced. The following definition clarifies the concept w.r.t. the Prospect dominance relation.

Definition 2. $\mathbb{K}$ Prospect-spans $\mathbb{L}$ ($\mathbb{K} \succcurlyeq_P \mathbb{L}$) iff for any $\lambda \in \mathbb{L}$, $\exists \kappa \in \mathbb{K} : \kappa \succcurlyeq_P \lambda$. If $\mathbb{K} = \{\kappa\}$, the element κ of the singleton $\mathbb{K}$ is termed as Prospect super-efficient.

The efficient set of the dominance relation is the subset of $\mathbb{L}$ that contains the maximal elements. The efficient set is a spanning subset of the portfolio space. Thereby, any superset of the efficient set is also a spanning subset of $\mathbb{L}$. We can consider a spanning set as an outer approximation of the efficient set. Given a candidate spanning set exists, the question is whether this actually spans the portfolio space. If a method for answering such a question also exists, we can accurately approximate the efficient set via the choice of finer spanning subsets of the portfolio space. This is important in the context of decision theory and investment choice.

Hence, the question we address here is: given a candidate $\mathbb{K}$, is $\mathbb{K} \succcurlyeq_P \mathbb{L}$? The following lemma provides an analytical characterization by means of nested optimizations, which is key for a numerical implementation on real data and statistical inference.

Lemma 3. *Suppose that* $\mathbb{K}$ *is closed. Then* $\mathbb{K} \succcurlyeq_P \mathbb{L}$ *iff we get the condition* $\rho(F) := \max_{i=1,2} \sup_{\lambda \in \mathbb{L}} \sup_{z \in A_i} \inf_{\kappa \in \mathbb{K}} P_i(z, \lambda, \kappa, F) = 0$, *where* $A_1 = \mathbb{R}_-$, $A_2 = \mathbb{R}_{++}$. *Moreover, we get that* κ *is Prospect super-efficient iff* $\sup_{\lambda \in \mathbb{L}} \max_{i=1,2} \sup_{z \in A_i} P_i(z, \lambda, \kappa, F) = 0$.

2.2 Representation By Utility Functions

We provide an expected utility characterization of spanning. Aside the economic interpretation, this is key to the numerical LP implementation of the inferential procedures that we construct in the next section. In doing so, we generalize the utility characterization of PSD in Levy and Levy (2002), in the sense that we do not require differentiability of the utilities. Our approach is in the spirit of the Russel and Seo (1989) representations for the second order stochastic dominance. We rely on utilities represented as unions of graphs of convex mixtures of appropriate "ramp functions" on each half-line.

To this end, we denote with $\mathcal{W}_-, \mathcal{W}_+$, the sets of Borel probability measures on the real line with supports that are closed subsets of $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively, with existing first moments and uniformly integrable. The latter requirement is convenient yet harmless since orderings are invariant to utility rescalings. Those sets are convex, and closed w.r.t. the topology of weak convergence and their union contains the set of degenerate measures. Define $V_- := \left\{ v_w : \mathbb{R}_- \to \mathbb{R},\ v_w(u) = \int_{\mathbb{R}_-} \left[z 1_{u \le z} + u 1_{z \le u \le 0} \right] dw(z),\ w \in \mathcal{W}_- \right\}$, and $V_+ := \left\{ v_w : \mathbb{R}_+ \to \mathbb{R},\ v_w(u) = \int_{\mathbb{R}_+} \left[u 1_{0 \le u \le z} + z 1_{z \le u < +\infty} \right] dw(z),\ w \in \mathcal{W}_+ \right\}$. Every element of V_+ is increasing and concave, and dually every element of V_- is increasing and convex. Furthermore, any function defined by the union of the graph of an arbitrary element of V_+ with the graph of an arbitrary element of V_- is the graph of an S-shaped utility function as defined by Levy and Levy (2002). Such a utility function is concave for gains and convex for losses. Denote the set of S-shaped utility functions obtained by such graph unions as V. Thereby,

$$V := \left\{ v : \mathbb{R} \to \mathbb{R},\ v(u) = \begin{cases} v_{w_1}(u), & u \le 0 \\ v_{w_2}(u), & u \ge 0 \end{cases},\ \text{where } v_{w_1} \in V_-, v_{w_2} \in V_+ \right\}.$$

Lemma 4. *We have* $\rho(F) = \max_{i=1,2} \sup_{v_w \in V_i} \left[\sup_{\lambda \in \mathbb{L}} \mathbb{E}_\lambda \left[1_{u \in A_i} v_w(u) \right] - \sup_{\kappa \in \mathbb{K}} \mathbb{E}_\kappa \left[1_{u \in A_i} v_w(u) \right] \right]$, *where* $\mathbb{E}_\lambda$ *denotes expectation w.r.t.* $G(z, \lambda, F)$*. If the hypotheses of Lemma 3 hold and* $\mathbb{K}$ *is convex, then* $\mathbb{K} \succcurlyeq_P \mathbb{L}$ *iff,* $\sup_{v \in V} [\sup_{\lambda \in \mathbb{L}} \mathbb{E}_\lambda [v] - \sup_{\kappa \in \mathbb{K}} \mathbb{E}_\kappa [v]] = 0$.

The fist part of the lemma connects the functional that represents spanning to the aforementioned classes of utilities. This is exploited below in order to obtain feasible numerical formulations based on LP. Those formulations are reminiscent of the LP programs developed in the early papers of testing for SSD efficiency of a given portfolio by Post (2003) and Kuosmanen (2004). The second part of Lemma 4 crystalizes the intuitive characterization of spanning w.r.t. investment opportunities. It states that spanning holds if and only if the reduction of investment opportunities from $\mathbb{L}$ to $\mathbb{K}$ does not reduce optimal choices uniformly w.r.t. this class of preferences.

2.3 An Asymptotically Exact and Consistent Test for Spanning

We cannot directly rely on Lemma 3 for empirical work if F is unknown and/or the optimizations are infeasible. We construct a feasible statistical test for the null hypothesis of $\mathbb{K} \succcurlyeq_P \mathbb{L}$ by utilizing an empirical approximation of F and by building feasible and fast optimisations with LP. The null and alternative hypotheses take the following forms: $\mathbf{H_0} : \rho(F) = 0$, and $\mathbf{H_a} : \rho(F) > 0$. In the special case of super-efficiency, the hypotheses write as in Arvanitis and Topaloglou (2017).

We consider a process $(Y_t)_{t \in \mathbb{Z}}$ taking values in $\mathbb{R}^n$. $Y_{i,t}$ denotes the i^{th} element of Y_t. The sample path of size T is the random element $(Y_t)_{t=1,\dots,T}$. In our empirical finance framework, it represents returns of n financial assets upon which we can construct portfolios via convex combinations. F is the cdf of Y_0 and F_T is the empirical cdf associated with the random element $(Y_t)_{t=1,\dots,T}$. Under our assumptions below, F_T is a consistent estimator of F, so we consider the following test statistic $\rho_T := \sqrt{T}\rho(F_T) = \sqrt{T} \max_{i=1,2} \sup_{\lambda \in \mathbb{L}} \sup_{z \in A_i} \inf_{\kappa \in \mathbb{K}} P_i(z, \lambda, \kappa, F_T)$, which is the scaled empirical analog of $\rho(F)$. As already mentioned, when $\mathbb{K}$ is a singleton, the test statistic coincides with the one used in Arvanitis and Topaloglou (2017). The following assumption enables the derivation of the limit distribution of ρ_T under $\mathbf{H_0}$ and is weaker than Assumption 2 in Arvanitis, Scaillet and Topaloglou (2019).

Assumption 5. *F is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^n$ with convex support that is bounded from below, and for some $0 < \delta$, $\mathbb{E}\left[\|Y_0\|^{2+\delta}\right] < +\infty$. $(Y_t)_{t\in\mathbb{Z}}$ is a-mixing with mixing coefficients $a_T = O(T^{-a})$ for some $a > 1 + \frac{2}{\eta}$, $0 < \eta < 2$, as $T \to \infty$.*

The lower bound hypothesis is harmless in our empirical finance framework since we are using financial returns. The mixing part is readily implied by concepts such as geometric ergodicity which holds for many stationary models used in the context of financial econometrics under parameter restrictions and restrictions on the properties of the underlying innovation processes. Examples are the strictly stationary versions of (possibly multivariate) ARMA or several GARCH and stochastic volatility type of models (see Francq and Zakoian (2011) for several examples). Counter-examples are models that exhibit long memory, etc. The moment condition is established in the aforementioned models via restrictions on the properties of building blocks and the parameters of the processes involved.

For the derivation of the limit theory of ρ_T under the null hypothesis, we consider the contact sets $\Gamma_i = \left\{\lambda \in \mathbb{L}, \kappa \in \mathbb{K}^{\succeq}_{\lambda}, z \in A_i : P_i\left(z, \lambda, \kappa, F\right) = 0\right\}$, where $\mathbb{K}^{\succeq}_{\lambda} := \{\kappa \in \mathbb{K} : \kappa \succcurlyeq_P \lambda\}$ which under the null contains elements different from λ for any element of $\mathbb{L} - \mathbb{K}$. For any i, the set Γ_i is non empty since $\Gamma^{\star}_i := \{(\kappa, \kappa, z), \kappa \in \mathbb{K}, z \in A_i\} \subseteq \Gamma_i$. Furthermore, $(\lambda, \kappa, 0) \in \Gamma_1$, $\forall \lambda, \kappa$. Since due to Assumption 5 $\underline{z} := \inf_{\lambda, Y_0} \lambda' Y_0$ exists, for all $z \leq \underline{z}$, $(\lambda, \kappa, z) \in \Gamma_i$, $\forall \lambda \in \mathbb{L}, \kappa \in \mathbb{K}^{\succeq}_{\lambda}$ for the i that corresponds to the sign of $\underline{z}$. In what follows, we denote convergence in distribution by $\rightsquigarrow$.

Proposition 6. *Suppose that $\mathbb{K}$ is closed, Assumption 5 holds and that $\mathbf{H_0}$ is true. Then as $T \to \infty$, $\rho_T \rightsquigarrow \rho_\infty$, where $\rho_\infty := \max_{i=1,2} \sup_\lambda \sup_z \inf_\kappa P_i\left(z, \lambda, \kappa, \mathcal{G}_F\right)$, $(\lambda, z, \kappa) \in \Gamma_i$, and $\mathcal{G}_F$ is a centered Gaussian process with covariance kernel given by $Cov(\mathcal{G}_F(x), \mathcal{G}_F(y)) = \sum_{t\in\mathbb{Z}} Cov\left(1_{\{Y_0 \leq x\}}, 1_{\{Y_t \leq y\}}\right)$ and $\mathbb{P}$ almost surely uniformly continuous sample paths defined on $\mathbb{R}^n$.*

The limiting random variables have the form of saddle points of Gaussian processes w.r.t. subsets of the relevant parameter spaces. This is well defined since $\mathrm{Var}\int_0^{+\infty}\mathcal{G}_{\lambda F}(u)\,du = \int_0^{+\infty}\sum_{t\in\mathbb{Z}}\mathrm{Cov}\left(1_{\{\lambda^T Y_0\leq u\}}, 1_{\{\lambda^{Tr} Y_t\leq u\}}\right)du \leq 2\sum_{t=0}^{\infty}\sqrt{a_T}\int_0^{+\infty}\sqrt{1-G(u,\lambda,F)}du < +\infty$, and $\mathrm{Var}\int_{-\infty}^{0}\mathcal{G}_{\lambda F}(u)\,du = \int_{-\infty}^{0}\sum_{t\in\mathbb{Z}}\mathrm{Cov}\left(1_{\{\lambda^T Y_0\leq u\}}, 1_{\{\lambda^{Tr} Y_t\leq u\}}\right)du \leq 2\sum_{t=0}^{\infty}\sqrt{a_T}\int_{-\infty}^{0}\sqrt{G(u,\lambda,F)}du$ $< +\infty$, where the first inequalities in each of the previous expressions follow from inequality 1.12b in Rio (2000), and the second ones follow from Assumption 5 (see also p. 196 of Horvath et al. (2006)).

Since F and Γ_i are unknown in practice, we use the results of the previous lemma to construct a decision procedure based on subsampling, in the spirit of Linton, Post and Whang (2014) (see also Linton, Maasoumi, and Whang (2005)).[1]

Algorithm 7. *This consists of the following steps:*

1. Evaluate ρ_T at the original sample value.

2. For $0 < b_T \leq T$, generate subsample values

   from the original observations $(Y_l)_{l=t,\dots t+b_T-1}$ for all $t = 1, 2, \dots, T - b_T + 1$.

3. Evaluate the test statistic on each subsample value

   thereby obtaining $\rho_{T,b_T,t}$ for all $t = 1, 2, \dots, T - b_T + 1$.

4. Approximate the cdf of the asymptotic distribution under the null of ρ_T

   by $s_{T,b}(y) = \frac{1}{T-b_T+1}\sum_{t=1}^{T-b_T+1} 1\left(\rho_{T,b_T,t}\leq y\right)$ and calculate its $1-\alpha$ quantile

   $q_{T,b_T}(1-\alpha) = \inf_y\left\{s_{T,b}(y)\geq 1-\alpha\right\}$, for the significance level $0 < \alpha < .5$.

5. Reject the null hypothesis $\mathbf{H_0}$ if $\rho_T > q_{T,b_T}(1-\alpha)$.

[1] The partitioning used to get the results in Proposition 6 directly leads to the consideration of subsampling as a resampling procedure. A testing procedure based on (block) bootstrap as in Scaillet and Topaloglou (2010), can, due to the form of the recentering, be consistent, but can be too conservative asymptotically, and thereby suffer from a lack of power compared to the subsampling under particular local alternatives (see also the relevant discussion in Arvanitis et al. (2019)). The potential of asymptotic exactness for the subsampling test justifies the particular resampling choice for inference.

In order to derive the limit theory for the testing procedure, namely its asymptotic exactness and consistency stated in the next theorem, we first use the following standard assumption that restricts the asymptotic behaviour of b_T governing the size $b_T + 1$ of each subsample.

Assumption 8. *Suppose that (b_T), possibly depending on $(Y_t)_{t=1,\ldots,T}$,* satisfies the condition $\mathbb{P}(l_T \leq b_T \leq u_T) \to 1$, where (l_T) and (u_T) are real sequences such that $1 \leq l_T \leq u_T$ for all T, $l_T \to \infty$ and $\frac{u_T}{T} \to 0$ as $T \to \infty$.

Theorem 9. *Suppose Assumptions 5 and 8 hold. For the testing procedure described in Algorithm 7, we have that*

1. *If $\mathbf{H_0}$ is true, and for $\lambda \in \mathbb{L} - \mathbb{K}$, $\inf_{Y_0} \lambda^{Tr} Y_0 \leq 0$ there exists $(\kappa, z) \in \mathbb{K}^{\succeq}_{\lambda} \times \mathbb{R}_{++}$ with $(\lambda, \kappa, z) \in \Gamma_2$ and that if $(\lambda, \kappa^{\star}, z^{\star}) \in \Gamma_2$ for $\kappa^{\star} \neq \kappa$ then $z^{\star} \neq z$, then for all $\alpha \in (0, .5)$ $\lim_{T\to\infty} \mathbb{P}(\rho_T > q_{T,b_T}(1-\alpha)) = \alpha$.*

2. *If $\mathbf{H_a}$ is true then $\lim_{T\to\infty} \mathbb{P}(\rho_T > q_{T,b_T}(1-\alpha)) = 1$.*

When for $\lambda \in \mathbb{L} - \mathbb{K}$, $\inf_{Y_0} \lambda^{Tr} Y_0 \leq 0$ then due to Assumption 5 for any contact triple $(\lambda, \kappa, z) \in \Gamma_2$ we have that $P_2(z, \lambda, \kappa, \mathcal{G}_F)$ must be non-degenerate. Whenever z corresponds solely to the particular κ, we obtain that ρ_∞ is non-degenerate and if its cdf jumps at the infimum of its support, then the jump magnitude is bounded above by .5. Hence in this case the test is asymptotically exact for all the usual choices of the significance level since the probability of rejection under the null hypothesis, i.e., the size of the test, reaches α in large samples. We combine Proposition 6 above and Theorem 3.5.1 of Politis, Romano and Wolf (1999) in the proof of the exactness statement, namely point 1 of Theorem 9. To get exactness, the condition imposed on $\mathbb{L} - \mathbb{K}$ is significantly weaker than the assumption on the relation between the extreme points of $\mathbb{L}$ and $\mathbb{K}$ adopted by Arvanitis, Scaillet and Topaloglou (2019). It amounts to the existence of a spanned portfolio whose support is not strictly positive and so that, in the event of positive returns, there exists an elementary increasing and concave utility for positive returns and a unique portfolio such that the latter dominates the former and we are indifferent between the two portfolios with this particular utility. Besides, the test is also consistent since the probability of rejection under the alternative hypothesis, i.e., the power of the test, reaches 1 in large samples. We show in the proof of the consistency statement, namely point 2 of Theorem 9, that the test statistic diverges to $+\infty$ under the alternative hypothesis when T goes to $+\infty$.

We opt for the "bias correction" regression analysis of Arvanitis et al. (2019) to reduce the sensitivity of the quantile estimates $q_{T,b_T}(1-\alpha)$ on the choice of b_T in empirically realistic dimensions for n and T (see also Arvanitis, Scaillet and Topaloglou (2019) for further evidence on its better finite sample properties). Specifically, given α, we compute the quantiles $q_{T,b_T}(1-\alpha)$ for a "reasonable" range of b_T. Next, we estimate the intercept and slope of the following regression line by OLS: $q_{T,b_T}(1-\alpha) = \gamma_{0;T,1-\alpha} + \gamma_{1;T,1-\alpha}(b_T)^{-1} + \nu_{T;1-\alpha,b_T}$. Finally, we estimate the bias-corrected $(1-\alpha)$-quantile as the OLS predicted value for $b_T = T$: $q_T^{BC}(1-\alpha) := \hat{\gamma}_{0;T,1-\alpha} + \hat{\gamma}_{1;T,1-\alpha}(T)^{-1}$. Since $q_{T,b_T}(1-\alpha)$ converges in probability to $q(\rho_\infty, 1-\alpha)$ and $(b_T)^{-1}$ converges to zero as $T \to 0$, $\hat{\gamma}_{0;T,1-\alpha}$ converges in probability to $q(\rho_\infty, 1-\alpha)$ and the asymptotic properties are not affected.

In the Online Appendix, we also show that under further assumptions, the test is asymptotically locally unbiased under given sequences of local alternatives. Besides, the Monte Carlo analysis reported in the Online Appendix shows that the test performs well with an empirical size close to 5% and an empirical power above 90% for a significance level $\alpha = 5\%$.

3 Numerical Implementation

In this section, we exploit the results of Lemma 4 in order to provide with a finitary approximation of the test statistic. We rely on this to provide with a numerical implementation based on LP below. We denote expectation w.r.t. the empirical measure by $\mathbb{E}_{F_T}$. Let $\mathcal{R}^-$ denote $\max_{i=1,\ldots,n} \text{Range}\left(Y_{i,t} 1_{Y_{i,t} \leq 0}\right)_{t=1,\ldots,T} = [\underline{x}, 0]$. Partition $\mathcal{R}^-$ into n_1 equally spaced values as $\underline{x} = z_1 < \cdots < z_{n_1} = 0$, where $z_n := \underline{x} - \frac{n-1}{n_1-1}\underline{x}$, $n = 1, \cdots, n_1$; $n_1 \geq 2$. Furthermore, partition the interval $[0,1]$, as $0 < \frac{1}{n_2-1} < \cdots < \frac{n_2-2}{n_2-1} < 1$, $n_2 \geq 2$. Similarly, $\mathcal{R}^+ := \max_{i=1,\ldots,n} \text{Range}\left(Y_{i,t}1_{Y_{i,t}\geq 0}\right)_{t=1,\ldots,T} = [0,\overline{x}]$. Partition $\mathcal{R}^+$ into p_1 equally spaced values as $0 = z_1 < \cdots < z_{p_1} = \overline{x}$, where $z_p := \frac{p-1}{p_1-1}\overline{x}$, $n = 1, \cdots, p_1$; $p_1 \geq 2$, and again partition the interval $[0,1]$, as $0 < \frac{1}{p_2-1} < \cdots < \frac{p_2-2}{p_2-1} < 1$, $p_2 \geq 2$. Using the above, we consider the test statistic:

$$\rho_T^\star := \sqrt{T} \max_{i=1,2} \sup_{v \in V_i^\star} \left[\sup_{\lambda \in \mathbb{L}} \mathbb{E}_{F_T}\left[v\left(\lambda^T Y\right)\right] - \sup_{\kappa \in \mathbb{K}} \mathbb{E}_{F_T}\left[v\left(\kappa^T Y\right)\right]\right], \tag{1}$$

where the set of utility functions for negative returns is:

$$V_-^\star := \left\{ v : v(u) = \sum_{n=1}^{n_1} w_n \left[z_n 1_{\underline{x} \leq u \leq z_n} + u 1_{z_n \leq u \leq 0}\right],\ (w_1, \ldots, w_{n_1}) \in \mathrm{W}^- \right\},$$

$$\mathrm{W}^- := \left\{ (\mathrm{w}_1, \ldots, \mathrm{w}_{\mathrm{n}_1}) \in \left\{0, \frac{1}{\mathrm{n}_2 - 1}, \cdots, \frac{\mathrm{n}_2 - 2}{\mathrm{n}_2 - 1}, 1\right\}^{\mathrm{n}_1} : \sum_{\mathrm{n}=1}^{\mathrm{n}_1} \mathrm{w}_\mathrm{n} = 1 \right\},$$

and the set of utility functions for positive returns is:

$$V_+^\star := \left\{ v : v(u) = \sum_{p=1}^{p_1} w_p \left[u 1_{0 \leq u \leq z_p} + z_p 1_{z_p \leq u \leq \overline{x}}\right],\ (w_1, \ldots, w_{p_1}) \in \mathrm{W}^+ \right\},$$

$$\mathrm{W}^+ := \left\{ (\mathrm{w}_1, \ldots, \mathrm{w}_{\mathrm{p}_1}) \in \left\{0, \frac{1}{\mathrm{p}_2 - 1}, \cdots, \frac{\mathrm{p}_2 - 2}{\mathrm{p}_2 - 1}, 1\right\}^{\mathrm{p}_1} : \sum_{\mathrm{p}=1}^{\mathrm{p}_1} \mathrm{w}_\mathrm{p} = 1 \right\}.$$

We obtain the following result on the approximation of ρ_T by $\rho_T^\star$.

Proposition 10. *When the support of F is also bounded from above, as $n_1, n_2, p_1, p_2 \to \infty$, we have $\rho_T^\star \to \rho_T$, $\mathbb{P}$ a.s.*

Our feasible computational strategy builds on LP formulations for the numerical evaluation using the previous finitary approximation of the test statistic.

We have a set of convex utility functions of the form: $v(u) = \sum_{n=1}^{n_1} w_n \max(u, z_n)$ for the negative part. For every $v \in V_-^\star$, we have at most n_2 line segments with knots at n_1 possible outcome levels. Then, we can enumerate all $n_3 = \frac{1}{(n_1-1)!} \prod_{i=1}^{n_1-1}(n_2 + i - 1)$ elements of $V_{-}^{\star}$. Our application in Section 4 uses $n_1 = 10$, and $n_2 = 5$, which gives $n_3 = 715$ distinct utility functions, and a total of 1430 small LP problems for the two embedded maximisation problems in (1). Solving (1) yields simultaneously the optimal factor portfolio κ, and the optimal augmented portfolio λ that maximize the expected utility. Below, we give the mathematical formulation for the first optimization problem $\sup_{\lambda\in\Lambda}\mathbb{E}_{F_N}\left[u\left(\lambda^T Y\right)\right]$, that yields the optimal augmented portfolio λ. The same formulation is used for the second optimization $\sup_{\kappa\in\kappa}\mathbb{E}_{F_N}\left[u\left(\kappa^T Y\right)\right]$.

Let us define: $c_{0,n} := \sum_{m=n}^{n_1}\left(c_{1,m} - c_{1,m+1}\right) z_m$, $c_{1,n} := \sum_{m=n}^{n_1} w_m$, and $\mathcal{N} := \{n = 1, \cdots, n_1 : w_n > 0\} \bigcup \{n_1\}$. For any given $u \in V_{-}$, $\sup_{\lambda\in\Lambda}\mathbb{E}_{F_N}\left[u\left(\lambda^T Y\right)\right]$ is the optimal value of the objective function of the following LP problem in canonical form:

$$\max T^{-1} \sum_{t=1}^{T} y_t \tag{2}$$

$$\text{s.t., for } t = 1, \cdots, T, \quad n \in \mathcal{N}, \quad i = 1, \cdots, M,$$

$$y_t \leq \lambda^T Y_t c_{1,n} + Q_t^- + Q_t^+, \qquad y_t \leq c_{0,n} + Q_t^- + Q_t^+,$$

$$Q_t^- \geq c_{0,n} - \lambda^T Y_t c_{1,n}, \qquad Q_t^+ \geq \lambda^T Y_t c_{1,n} - c_{0,n}, \qquad Q_t^- \geq 0, \qquad Q_t^+, \geq 0,$$

$$\sum_{i=1}^{M} \lambda_i = 1, \qquad \lambda_i \geq 0, \qquad \text{and} \qquad y_t \text{ being free.}$$

We have a set of concave utility functions of the form: $v(u) = \sum_{p=1}^{p_1} w_p \min(u, z_p)$, for the positive part. Again, for every $v \in V_{+}^{\star}$, we have at most p_2 line segments with knots at p_1 possible outcome levels. As before, the number of elements of $V_{+}^{\star}$ is $p_3 = \frac{1}{(p_1-1)!}\prod_{i=1}^{p_1-1}(p_2 + i - 1) = 1430$, for $p_1 = 10$ and $p_2 = 5$.

Let us define: $c_{0,p} := \sum_{m=p}^{p_1}\left(c_{1,m} - c_{1,m+1}\right) z_m$, $c_{1,p} := \sum_{m=p}^{p_1} w_m$, and $\mathcal{P} := \{p = 1, \cdots, p_1 : w_p > 0\} \bigcup \{p_1\}$. For any given $u \in V_{+}$, $\sup_{\lambda\in\Lambda}\mathbb{E}_{F_N}\left[u\left(\lambda^T Y\right)\right]$ is the optimal value of the objective function of the following LP problem in canonical form:

$$\max T^{-1} \sum_{t=1}^{T} y_t \tag{3}$$

$$\text{s.t., for } t = 1, \cdots, T, \quad n \in \mathcal{P}, \quad i = 1, \cdots, M,$$

$$y_t \leq \lambda^T Y_t c_{1,p}, \qquad y_t \leq c_{0,p}, \qquad \sum_{i=1}^{M} \lambda_i = 1 \qquad \lambda_i \geq 0, \qquad \text{and} \qquad y_t \text{ being free.}$$

The total run time for each computation does not exceed one minute when we use a desktop PC with a 3.6 GHz, 6-core Intel i7 processor, with 16 GB of RAM, using MATLAB and GAMS with the Gurobi optimization solver.

4 Empirical Application

In the empirical application, we examine if we can explain well-known stock market anomalies by standard factors within a new breed of asset pricing models, for prospect type investor preferences. For this purpose, we use the prospect spanning tests, both in- and out-of-sample.

4.1 Factor Models and Anomalies

We start with a benchmark factor model from a set of models that have generated support in the recent literature, and we ask whether a characteristic identified in the literature as stock market anomaly, is a market anomaly for prospect investors. To answer this question, we consider three models that build on the pioneer three-factor model of Fama and French (1993): the four-factor model of Hou, Xue and Zhang (2015), the five-factor model of Fama and French (2015), and the four-factor model of Stambaugh and Yuan (2017). Fama and French (1993) aim to capture the part of average stock returns left unexplained in CAPM of Sharpe (1964) and Lintner (1965) by including, in addition to the market factor, two extra risk factors relating to size (measured by market equity) and the ratio of book-to-market equity. In addition to the market excess return, the influential three-factor model of Fama and French (1993) includes a book-to-market or "value" factor, HML, and a size factor, SMB, based on market capitalization. Motivated by Miller and Modigliani (1961), Fama and French (2015) five-factor model (henceforth, FF-5) augments the original Fama-French three-factor model by two extra factors, one for profitability and another for investment. Hou, Xue and Zhang (2015) consider a four-factor model (dubbed the q-factor model) that includes the original market and size factors of Fama and French (1993) augmented by a profitability and investment factor. Stambaugh and Yuan (2017) consider a four-factor model (henceforth, M-4) including the standard market and size factors along with two composite factors for investment and profitability. To construct the composite factors, they combine information from 11 market anomalies relating to investment and profitability measures. We use alternative factor models as a robustness check, namely for testing the consistency of in- and out-of-sample results under the prospect preferences, and not for a horse race in cross-sectional asset pricing.

The stock market anomalies we examine in this paper have a long history in the relevant literature. A common theme in the original papers that first highlighted these patterns, is that they all challenge the rational asset pricing paradigm as they exhibit returns that are not in line with the risks taken. However, notwithstanding whether they are caused by sentiment (a catch-all term that stand for all kinds of irrational decision-making) or by market frictions (e.g. margin requirements), it is also acknowledged that most of them persist because they cannot be “arbitraged” away. From the perspective of the Arbitrage Pricing Theory this implies that arbitrageurs cannot trade against them without exposing themselves to significant risks. In this paper, we test the 11 strategies used to construct Stambaugh-Yuan factors, along with Betting against Beta, Quality minus Junk, Size, Growth Option, Value (Book to Market), Idiosyncratic volatility and Profitability. The 11 anomalies used in Stambaugh and Yuan (2017) are Accruals, Asset Growth, Composite Equity Issue, Distress, Growth Profitability Premium, Investment to Assets, Momentum, Net Operating Assets, Net Stock Issues, O-Score, and Return on Assets. They are realigned appropriately to yield positive average returns. In particular, anomaly variables that relate to investment activity (Asset Growth, Investment to Assets, Net Stock Issues, Composite Equity Isues, Aaccruals) are defined low-minus-high decile portfolio returns, rather than high-minus-low, as in Hou et al. (2015). All the other anomalies are constructed as high-minus-low decile portfolio returns. A short description of the 18 market anomalies that we study in the paper is given in Appendix A (see Stambaugh and Yuan (2017) for further details). Returns of the Fama and French 5 factors were downloaded from Kenneth French's site. The dataset consists of all monthly observations from January 1974 until December 2016. M-4 factor returns and anomaly spread return series were downloaded from the websites of Robert Stambaugh and AQR. In the Online Appendix, we report summary statistics of the factor and anomaly returns over our sample period.

4.2 In-Sample Analysis

In this section, we test in-sample the null hypothesis that the set of standard factors prospect spans the set enlarged with a particular market anomaly. We test separately for the Fama and French 5 factors, the Stambaugh-Yuan 4 factors as well as Hou-Xue-Zhang 4 factors, with respect to each one of the 18 additional anomalies. We get the subsampling distribution of the test statistic for subsample size $b_T \in \{T^{0.6}, T^{0.7}, T^{0.8}, T^{0.9}\}$. Using OLS regression on the empirical quantiles $q_{T,b_T}(1-\alpha)$ for a significance level $\alpha = 5\%$, we get the estimate q_T^{BC} for the bias-corrected critical value. We reject spanning if the test statistic $\rho_T^{\star}$ is higher than the regression estimate q_T^{BC}.

Tables 1-3 report the test statistics $\rho_T^{\star}$ as well as the regression estimates q_T^{BC} when we test for spanning of the alternative factor models w.r.t. each one of the 18 market anomalies.

Table 1: Test statistics: Fama and French (FF-5) Factors

Variable	Test statistic $\rho_T^\star$	Regression estimates q_T^{BC}	Result
Accruals	0.0016	0.0025	Spanning
Asset Growth	0.0	0.0	Spanning
Composite Equity Issue	0.0015	0.0003	Reject Spanning
Distress	0.0045	0.0005	Reject Spanning
Growth Profitability Premium	0.0015	0.0012	Reject Spanning
Investment to Assets	0.0014	0.0001	Reject Spanning
Momentum	0.0696	0.0204	Reject Spanning
Net Operating Assets	0.0268	0.0009	Reject Spanning
Net Stock Issues	0.0011	0.0003	Reject Spanning
O-Score	0.0129	0.0092	Reject Spanning
Return on Assets	0.0024	0.0047	Spanning
Betting against Beta	0.0235	0.0176	Reject Spanning
Quality minus Junk	0.0088	0.0061	Reject Spanning
Size	0.0	0.0	Spanning
Growth Option	0.0	0.0	Spanning
Value (Book to Market)	0.1921	0.1878	Reject Spanning
Idiosyncratic Volatility	01959	0.0100	Reject Spanning
Profitability	0.0	0.0	Spanning

Entries report the test statistics $\rho_T^\star$ and the regression estimates q_T^{BC} for spanning of the Fama and French (FF-5) model with respect to each one of the 18 market anomalies. We reject spanning at significance level $\alpha = 5\%$ if $\rho_T^\star > q_T^{BC}$. The dataset spans the period from January, 1974 to December, 2016.

Table 2: Test statistics: Stambaugh-Yuan (M-4) Factors

Variable	Test statistic $\rho_T^\star$	Regression estimates q_T^{BC}	Result
Accruals	0.0081	0.0083	Spanning
Asset Growth	0.0057	0.0069	Spanning
Composite Equity Issue	0.0143	0.078	Reject Spanning
Distress	0.0533	0.0020	Reject Spanning
Growth Profitability Premium	0.0113	0.0049	Reject Spanning
Investment to Assets	0.0116	0.0164	Reject Spanning
Momentum	0.1189	0.1143	Reject Spanning
Net Operating Assets	0.0653	0.0071	Reject Spanning
Net Stock Issues	0.0145	0.0073	Reject Spanning
O-Score	0.0133	0.0122	Reject Spanning
Return on Assets	0.0012	0.0015	Spanning
Betting against Beta	0.0755	0.0703	Reject Spanning
Quality minus Junk	0.0374	0.0099	Reject Spanning
Size	0.0	0.0	Spanning
Growth Option	0.0	0.0	Spanning
Value (Book to Market)	0.2939	0.2817	Reject Spanning
Idiosyncratic Volatility	0.2593	0.1039	Reject Spanning
Profitability	0.0	0.0	Spanning

Entries report the test statistics $\rho_T^\star$ and the regression estimates q_T^{BC} for spanning of the Stambaugh-Yuan (M-4) model with respect to each one of the 18 market anomalies. We reject spanning at significance level $\alpha = 5\%$ if $\rho_T^\star > q_T^{BC}$. The dataset spans the period from January, 1974 to December, 2016.

Table 3: Test statistics: Hou-Xue-Zhang (q) Factors

Variable	Test statistic $\rho^{\star}_T$	Regression estimates q^{BC}_T	Result
Accruals	0.0106	0.0039	Reject Spannin
Asset Growth	0.0176	0.0101	Reject Spanning
Composite Equity Issue	0.0163	0.0159	Reject Spanning
Distress	0.0386	0.0133	Reject Spanning
Growth Profitability Premium	0.0084	0.0038	Reject Spanning
Investment to Assets	0.0157	0.0123	Reject Spanning
Momentum	0.0835	0.0305	Reject Spanning
Net Operating Assets	0.0449	0.0059	Reject Spanning
Net Stock Issues	0.0178	0.0170	Reject Spanning
O-Score	0.0140	0.0109	Reject Spanning
Return on Assets	0.0235	0.0321	Spanning
Betting against Beta	0.0404	0.0424	Spanning
Quality minus Junk	0.0304	0.0177	Reject Spanning
Size	0.0	0.0	Spanning
Growth Option	0.0029	0.0	Reject Spanning
Value (Book to Market)	0.2045	0.1878	Reject Spanning
Idiosyncratic Volatility	0.2386	0.0101	Reject Spanning
Profitability	0.0	0.0	Spanning

Entries report the test statistics $\rho^{\star}_T$ and the regression estimates q^{BC}_T for spanning of the Hou-Xue-Zhang (q) model with respect to each one of the 18 market anomalies. We reject spanning at significance level $\alpha = 5\%$ if $\rho^{\star}_T > q^{BC}_T$. The dataset spans the period from January, 1974 to December, 2016.

We observe that the FF-5 model spans 6 out of 18 market anomalies, that is, Accruals, Asset Growth, Return on Assets, Size, Growth Option, and Profitability. The M-4 model spans the same 6 market anomalies, while the q model spans Return on Assets, Betting against Beta, Size, and Profitability. Thus, in most cases, optimal portfolios based on the investment opportunity set that includes a market anomaly is not spanned by the corresponding optimal portfolio strategies based on the original factors. We also observe that Return on Assets, Size, and Profitability are spanned by all the factor models, indicating the robustness of these characteristics being not considered as genuine market anomalies by prospect investors.

4.3 Out-of-Sample Analysis

In this section, we examine whether the inclusion of a market anomaly in the investment opportunity set benefits to prospect investors out-of-sample. Although we reject the null hypothesis of prospect spanning in most cases for the in-sample tests, it is not known a priori whether an optimal augmented portfolio also outperforms an optimal portfolio made of factors only in an out-of-sample analysis. This is because by construction we form these portfolios at time t, based on the information prevailing at time t, while we reap the portfolio returns over $[t, t+1]$ (next month). The out-of-sample test is a real-time exercise mimicking the way that a real-time investor acts.

Each time the hypothesized portfolio manager with prospect preferences forms optimal portfolios from two separate asset universes: the first universe consists only of factors from a factor model (FF-5, M-4, q), the set $\mathbb{K}$. The second universe is the respective set of factors augmented by a single trading (spread) strategy, the set $\mathbb{L}$. Portfolio managers are assumed to solve portfolio optimization problems, motivated by the prospect spanning framework, effectively looking for a portfolio picked from the augmented universe $\mathbb{L}$ that prospect stochastically dominates all portfolios of the respective factor universe $\mathbb{K}$..

The rejection of the prospect spanning hypothesis implies that there exists at least one portfolio in $\mathbb{L}$ build from the factors (of each particular factor model) and one market anomaly, which is weakly prefered to every factor portfolio in $\mathbb{K}$ by at least one S-shaped utility function (see Definition 2). Such a portfolio is by construction efficient w.r.t. $\mathbb{K}$ (see Definition 2.1 in Linton et al. (2014) for the SSD case which we can easily generalize to our PSD case). The empirical version of such a portfolio is the optimal portfolio λ that maximizes ρ_T for the particular sample value. In what follows, and given this characterization, we analyze the performance of such empirically optimal PSD portfolios through time, compared to the performance of the optimal factor portfolios solely derived from $\mathbb{K}$ by prospect investors.

We resort to backtesting experiments on a rolling horizon basis. The rolling windows cover the 516 months period from 01/1974 to 12/2016. At each month, we use the data from the previous 25 years (300 monthly observations) to calibrate the procedure. We solve the resulting optimization problem for the prospect stochastic spanning test and record the optimal portfolios. The clock is advanced and we determine the realized returns of the optimal portfolios from the actual returns of the various assets. Then we repeat the same procedure for the next time period and we compute the ex post realized returns over the period from 01/1999 to 12/2016 (216 months) for both portfolios.

We compute a number of commonly used performance measures: the average return (Mean), the standard deviation (SD) of returns, the Sharpe ratio, the downside Sharpe ratio (D. Sharpe ratio) of Ziemba (2005), the upside potential and downside risk (UP) ratio of Sortino and van der Meer (1991), the opportunity cost of Simaan (2013), and a measure of the portfolio risk-adjusted returns net of transaction costs (Return Loss) of DeMiguel et al. (2009). The downside Sharpe and UP ratios are considered to be more appropriate measures of performance than the typical Sharpe ratio given the asymmetric return distribution of the anomalies. For the calculation of the opportunity cost, we use the following utility function which satisfies the curvature of prospect theory (S-shaped): $U(R) = R^{\alpha}$ if $R \geq 0$ or $-\gamma(-R)^{\beta}$ if $R < 0$, where γ is the coefficient of loss aversion (usually $\gamma = 2.25$) and $\alpha, \beta < 1$. We provide a short description of those performance measures in Appendix B. In the next lines, we only detail the results of the out-of-sample tests for the Momentum market anomaly. The latter is well documented on diverse markets and asset classes (Asness, Moskowitz, and Pedersen (2013)). In the Online Appendix, we report the performance measures for the 5 Fama and French, the 4 Stambaugh and Yuan and the 4 Hou-Xue-Zhang optimal factor portfolios, and the optimal augmented portfolios for all the other market anomalies that we test.

Table 4 reports the performance measures for the Momentum anomaly under each factor model (Panels A, B and C, respectively). These performance measures supplement the evidence obtained from the in-sample analysis. We observe that the Mean, the Sharpe ratio, downside Sharpe ratio and UP ratio of the optimal augmented portfolio are improved with respect to the optimal factor portfolio. Although these measures are based on the first two moments, they support the in-sample result that the set enlarged with the momentum anomaly is not spanned by any factor model. The same is true when we take into account transaction costs. The Return Loss is always positive. The opportunity cost measure takes into account the entire distribution of returns under a given characterization of preferences. We observe that augmenting the factors by Momentum increases the performance of the optimal portfolio with respect to each factor model. The optimal weight of Momentum varies from 40% to 99%, indicating the superior performance of this characteristic.

In the Online Appendix, we present analogous Tables for the other market anomalies. Interestingly, based on the opportunity cost, enlarging the factor set by a market anomaly increases the performance of an optimal portfolio in 12 out of the 18 cases with respect to FF-5 factors (Composite Equity Issue, Distress, Growth Profitability Premium, Investment to Assets, Momentum, Net Operating Assets, O-Score, Net Stock Issues, Betting against Beta, Quality minus Junk, Value, and Idiosyncratic Volatility), in 10 cases with respect to M-4 factors (Composite Equity Issue, Distress, Investment to Assets, Momentum, Net Operating Assets, Net Stock Issues, Betting against Beta, Quality minus Junk, Value, and Idiosyncratic Volatility) and in 14 cases with respect to q factors (Accruals, Asset Growth, Composite Equity Issue, Distress, Growth Profitability Premium, Investment to Assets, Momentum, Net Operating Assets, O-Score, Net Stock Issues, Betting against Beta, Quality minus Junk, Size, Value, and Idiosyncratic Volatility). For all these additional market anomalies, we find a positive opportunity cost θ. One needs to give a positive return equal to θ to an investor who does not include the anomalies in her portfolio so that she becomes as happy as an investor who includes them. The computation of the opportunity cost requires the computation of the expected utility and hence the use of the probability density function of portfolio returns. Thus, the calculated opportunity cost has taken into account the higher order moments in contrast to the Sharpe ratios. Therefore, the opportunity cost estimates provide further convincing evidence for the diversification benefits of the inclusion of the market anomalies given their deviation from normality.

Additionally, although the rest of the performance measures depend mostly on the first two moments of the return distribution, they give consistent results. The Return Loss measure that takes into account transaction costs, is positive in all the above cases. This reflects an increase in risk-adjusted performance (i.e., an increase in expected return per unit of risk) and hence expands the investment opportunities of prospect investors. The same is true for the UP ratio. Finally, the Sharpe ratio and the downside Sharpe ratio agree that the performance of the optimal portfolios augmented with the above market anomalies is improved, although the differences are small in some cases.

The analysis indicates that the Composite Equity Issue, Distress, Investment to Assets, Momentum, Net Operating Assets, Net Stock Issues, Quality minus Junk, Value, and Idiosyncratic Volatility emerge as unambiguously genuine market anomalies under all factor sets, both in- and out-of-sample. Prospect investors would benefit from including these characteristics in their portfolios, expanding the investment opportunity set offered by factor portfolios. We stress that the prospect spanning approach is particularly robust in-sample and out-of-sample. The remarkable consistency of in-sample and out-of-sample results offers good incentives for adopting such an approach when exploring instances of apparent market inefficiency.

To sum up, the in-sample spanning tests, as well as the out-of-sample analysis given by the performance measures, indicate that in most cases (depending on the factor model used) the investment universe augmented with a market anomaly dominates the 5 Fama and French, the 4 Stambaugh and Yuan, and the 4 Hou-Xue-Zhang factors, yielding diversification benefits and providing better investment opportunities for investors with prospect type preferences towards risk.

Table 4: Performance measures. The case of the Momentum anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0056	0.0062	0.0044	0.0048	0.0073	0.0072
SD	0.0358	0.0370	0.0388	0.0409	0.0808	0.0385
Sharpe ratio	0.1507	0.1604	0.1063	0.1117	0.0879	0.1814
D. Sharpe ratio	0.1622	0.1706	0.1078	0.1108	0.0868	0.1995
UP ratio	0.6401	0.6693	0.5646	0.5853	0.5348	0.6769
Return Loss		0.0351%		0.0205%		0.3723%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.0416%		0.1446%		0.4338%
$\alpha = \beta = 0.4$		0.0210%		0.0129%		0.4093%
$\alpha = \beta = 0.6$		0.0129%		0.0152%		0.3229 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.5955	0.1507	-3.3717	10.9074
	SMB	0.0	0.0	-	-
	HML	0.0	0.0	-	-
	RMW	0.0	0.0	-	-
	CMA	0.0	0.0	-	-
	Momentum	0.4045	0.1507	3.3717	10.9074
M-4 Factors	Market	0.5331	0.2255	-1.6812	1.5383
	SMB	0.0	0.0	-	-
	MGMT1	0.0020	0.0113	7.4184	59.9621
	PERf1	0.0	0.0	-	-
	Momentum	0.4648	0.2273	1.6464	1.4817
q Factors	Market	0.0028	0.0411	14.6969	216.000
	ME	0.0	0.0	-	-
	IA	0.0	0.0	-	-
	ROE	0.0	0.0	-	-
	Momentum	0.9972	0.0411	-14.6969	216

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Momentum optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

5 Conclusions

In this paper, we develop and implement methods for determining whether introducing new securities or relaxing investment constraints improves the investment opportunity set for prospect investors. We develop a testing procedure for prospect spanning for two nested portfolio sets based on subsampling and standard LP.

In the empirics, we apply the prospect spanning framework to asset prices in which investors evaluate risk according to prospect theory and examine its ability to explain 18 well-known stock market anomalies. The setting deploys prospect theory in a fully nonparametric way. We find that of the strategies considered, many expand the opportunity set of the prospect investors, thus have real economic value for them.

Most importantly, we show that the prospect spanning approach is particularly robust between in-sample and out-of-sample applications. The paper contributes to a current strand of literature aiming to reevaluate published anomalies and discern those with real economic content for prospect investors. From a practitioner perspective, this robust framework for establishing investment opportunities for prospect investors can be of real value, especially in the case of quantitative investment funds that combine talent, capital and computational power to the purpose of exploiting the existing anomalies and discovering new ones.

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APPENDIX A: Description of Stock Market Anomalies

Below we provide the origin and a short description of the 18 market anomalies used in the empirical application.

1. Accruals: Sloan (1996) argues that investors tend to overestimate in their earnings expectations the persistence of the earnings' component that is due to accruals. As a result, firms with low accruals earn on average abnormally higher returns than firms with high accruals.

2. Asset Growth: Cooper, Gulen, and Schill (2008) maintain that investors tend to overreact positively right after asset expansions. According to the authors, this behavior causes firms with high growth in their total assets to exhibit relatively lower returns over the subsequent fiscal years.

3. Composite Equity Issues: Daniel and Titman (2006) base their analysis on a measure of equity issuance that they devised finding that equity issuers tend to underperform non-issuer firms.

4. Distress: Campbell, Hilscher, and Szilagyi (2008) find that firms with high default probability tend to exhibit lower subsequent returns. This pattern is counter-intuitive in the context of rational asset pricing, given that according to the standard models high risk entails high expected return and vice versa.

5. Gross Profitability Premium: Novy-Marx (2013) argues that gross profit is the most objective profitability metric. As a result, firms with the strongest gross profit have on average higher returns than the less profitable ones.

6. Investment to Assets: Titman, Wei, and Xie (2004) argue that investors are put off by empire-building managers who over-invest. For this reason, firms showing a significant increase in gross property, plant, equipment or inventories tend to underperform the market.

7. Momentum: Momentum (Jegadeesh and Titman (1993)) is perhaps the most cited

anomaly in asset pricing. Since Carhart factor model (1997), it has been included in various reduced-form models of the SDF as a factor. The momentum effect is attributed to sentiment and describes the pattern of "winner" stocks gaining higher subsequent returns and "loser" stocks relatively lower.

8. Net Operating Assets: Hirshleifer et al. (2004) suggest that investors often neglect information about cash profitability and focus instead on accounting profitability. Because of this bias, firms with high net operating assets (measured as the cumulative difference between operating income and free cash flow) get to have negative long-run stock returns.

9. Net Stock Issues: Ritter (1991) and Loughran and Ritter (1995) indicate that equity issuers underperform non-issuers with similar characteristics. Fama and French (2008) demonstrate that net stock issues are negatively correlated with subsequent returns.

10. O-Score: This anomaly coincides with the distress anomaly we mentioned earlier. In this case, the spread portfolios are constructed from stock ranking based on the O-score (Ohlson (1980)) to measure distress likelihood.

11. Return on Assets: Chen, Novy-Marx, and Zhang (2010) associate high past return on assets with abnormally high subsequent returns. Return on assets is measured as the ratio of quarterly earnings to last quarter's assets.

12. Betting against Beta: Black, Jensen and Scholes (1972) showed that low (high) beta stocks have consistently positive (negative) risk-adjusted returns. Frazzini and Pedersen (2014) propose an investment strategy ("betting-against-beta" (BAB)) that exploits this anomaly by buying low-beta stocks and shorting high-beta stocks. Because of its robustness, this anomaly is currently one of the most widely examined APT violations.

13. Quality minus Junk: Asness, Frazzini and Pedersen (2013) show that high-quality stocks (safe, profitable, growing, and well managed) exhibit high risk-adjusted returns. The authors attribute this pattern to mispricing.

14. Size: The market capitalization. is computed as the log of the product of price per share and number of shares outstanding, computed at the end of the previous month.

15. Growth Option: Growth Option measure represents the residual future-oriented firm growth potential. This future (yet-to-be exercised) growth option measure is calculated as the % of a firm's market value (V) arising from future-oriented growth opportunities (PVGO/V). It is inferred by subtracting from the current market value of the firm (V) the perpetual discounted stream of expected operating cash flows under a no-further growth policy (see, e.g., Kester (1984), Anderson and Garcia-Feijoo (2006), Berk, Green, and Naik (1999)).

16. Value (Book to market): The log of book value of equity scaled by market value of equity, computed following Fama and French (1992) and Fama and French (2008); firms with negative book value are excluded from the analysis.

17. Idiosyncratic Volatility: Standard deviation of the residuals from a firm-level regression of daily stock returns on the daily Fama-French three factors using data from the past month. See Ang et al. (2006).

18. Profitability.: It is measured as revenue minus cost of goods sold at time t, divided by assets at time t-1. Stocks with high profitability ratios tend to outperform on a risk-adjusted basis (Novy-Marx (2013), Novy-Marx and Velikov (2015)). Recent research suggests that profitability is one of the stock return anomalies that has the largest economic significance (see Novy-Marx (2013)).

APPENDIX B: Description of Performance Measures

For the downside Sharpe ratio, first we need to calculate the downside variance (or more precisely the downside risk), $\sigma^2_{P_-} = \frac{\sum_{t=1}^{T}(x_t-\bar{x})^2_-}{T-1}$, where the benchmark $\bar{x}$ is zero, and the x_t taken are those returns of portfolio P at month t below $\bar{x}$, i.e., those t of the T months with losses. To get the total variance, we use twice the downside variance namely $2\sigma^2_{P_-}$ so that the downside Sharpe ratio is, $S_P = \frac{\bar{R}_p-\bar{R}_f}{\sqrt{2}\sigma_{P_-}}$, where $\bar{R}_p$ is the average period return of portfolio P and $\bar{R_f}$ is the average risk free rate. The UP ratio compares the upside potential to the shortfall risk over a specific target (benchmark) and is computed as follows. Let R_t be the realized monthly return of portfolio P for $t = 1, ..., T$ of the backtesting period, where $T = 216$ is the number of experiments performed and let ρ_t be respectively the return of the benchmark (risk free rate) for the same period. Then, we have, UP ratio $= \frac{\frac{1}{K}\sum_{t=1}^{K}\max[0, R_t - \rho_t]}{\sqrt{\frac{1}{K}\sum_{t=1}^{K}(\max[0, \rho_t - R_t])^2}}$. It is obvious that the numerator of the above ratio is the average excess return over the benchmark and so reflects upside potential. In the same way, the denominator measures downside risk, i.e. shortfall risk over the benchmark.

Next, we use the concept of opportunity cost presented in Simaan (2013) to analyse the economic significance of the performance difference of the two optimal portfolios. Let R_{Aug} and R_F be the realized returns of the optimal augmented and the optimal factors portfolios, respectively. Then, the opportunity cost θ is defined as the return that needs to be added to (or subtracted from) the optimal factors portfolio return R_F, so that the investor is indifferent (in utility terms) between the strategies imposed by the two different investment opportunity sets, i.e., $E[U(1 + R_F + \theta)] = E[U(1 + R_{Aug})]$.

A positive (negative) opportunity cost implies that the investor is better (worse) off if the investment opportunity set allows for the market anomaly factor prospect type investing. The opportunity cost takes into account the entire probability density function of asset returns and hence it is suitable to evaluate strategies even when the asset return distribution is not normal. For the calculation of the opportunity cost, we use the following utility function which satisfies the curvature of prospect theory (S-shaped): $U(R) = R^\alpha$ if $R \geq 0$ or $-\gamma(-R)^\beta$ if $R < 0$, where γ is the coefficient of loss aversion (usually $\gamma = 2.25$) and $\alpha, \beta < 1$.

Finally, we evaluate the performance of the two portfolios under the risk-adjusted (net of transaction costs) returns measure, proposed by DeMiguel et al. (2009) which indicates the way that the proportional transaction cost, generated by the portfolio turnover, affects the portfolio returns. Let trc be the proportional transaction cost, and $R_{P,t+1}$ the realized return of portfolio P at time $t+1$. The change in the net of transaction cost wealth NW_P of portfolio P through time is, $NW_{P,t+1} = NW_{P,t}(1 + R_{P,t+1})[1 - trc \times \sum_{i=1}^{N}(|w_{P,i,t+1} - w_{P,i,t}|)$.

The portfolio return, net of transaction costs is defined as $RTC_{P,t+1} = \frac{NW_{P,t+1}}{NW_{P,t}} - 1$. Let μ_F and μ_{Aug} be the out-of-sample mean of monthly RTC factros and the Augmented optimal portfolio, respectively, and σ_F and σ_{Aug} be the corresponding standard deviations. Then, the return-loss measure is, $R_{Loss} = \frac{\mu_{Aug}}{\sigma_{Aug}} \times \sigma_F - \mu_F$, i.e., the additional return needed so that the factors performs equally well with the optimal augmented with the market anomaly portfolio. We follow the literature and use 35 bps for the transaction cost.

Spanning analysis of stock market anomalies under Prospect Stochastic Dominance

Stelios Arvanitis, Olivier Scaillet, Nikolas Topaloglou

Online Appendix

This online Appendix contains: i) the limiting properties of the testing procedures under sequences of local alternatives, ii) a Monte Carlo study of the finite sample properties of the test, iii) the proofs of the main results, as well as auxiliary lemmata and their proofs, iv) summary statistics of the factor and anomaly returns over our sample period from January 1974 to December 2016 and v) additional empirical results on out-of-sample analysis of market anomalies. We keep the numbering of assumptions and results as in the main text. We introduce a local numbering for assumptions and results that only appear here.

1 Local Alternatives

We enhance the consistency results of Theorem 9 by considering the limiting behavior of the testing procedure under a sequence of local to spanning alternatives. In this respect, $\underset{G}{\rightsquigarrow}$ denotes weak convergence under the measure with cdf G. Furthermore, $L_2^0(F)$ denotes the space of random variables with zero mean and finite second moment.

Assumption 1 (LOCAL)**.** *There exists a sequence of cdf* $(F_t^{\star})_{t\in\mathbb{N}}$ *such that for some* $h \in$

$L_2^0(F)$

$$\int_{\mathbb{R}^n}\left[\sqrt{T}\left[\sqrt{dF_T^\star}-\sqrt{dF}\right]-h\sqrt{dF}\right]\to 0,\;\; as\, T\to\infty.$$

Assumptions 5 and LOCAL along with Theorem 7.3 of Rio (2000) and the analogous extension of Theorem 1 of Wellner (1992) imply that $\sqrt{T}\,(F_T-F)\underset{F_t^\star}{\rightsquigarrow}\mathcal{G}_F+\delta_h$ where $\delta_h(x):=\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\dots\int_{-\infty}^{x_n}hdF$ for any $x\in\mathbb{R}^n$.

Proposition 2 (LOCLIM)**.** *Under Assumptions 5 and LOCAL, as $T\to\infty$,*

$$\rho_T\underset{F_t^\star}{\rightsquigarrow}\rho_\infty^\star:=\max_{i=+,-}\sup_\lambda\sup_z\inf_\kappa\left[P_i\left(z_i,\lambda,\kappa,\mathcal{G}_F\right)+P_{ih}\left(z,\lambda,\kappa,F\right)\right],\;(\lambda,z,\kappa)\in\Gamma_i,$$

where

$$P_{1h}(z,\lambda,\kappa,F):=\mathcal{J}_h(z,0,\kappa,F)-\mathcal{J}_h(z,0,\lambda,F)\,,\;z\in\mathbb{R}_-,$$

$$P_{2h}(z,\lambda,\kappa,F):=\mathcal{J}_h(0,z,\kappa,F)-\mathcal{J}_h(0,z,\lambda,F)\,,\;z\in\mathbb{R}_{++},$$

$$\mathcal{J}_h(z_1,z_2,\lambda;F):=\int_{z_1}^{z_2}G_h(u,\lambda,F)\,du,$$

and

$$G_h(z,\lambda,F):=\int_{\mathbb{R}^n}\mathbb{I}\{\lambda^{Tr}u\le z\}h(u)\,dF(u).$$

Using the premises of the previous proposition and if $P_{ih}>C>0$ for all i then a result such as the C2 one in Theorem 1 of Wellner (1992), and arguments analogous to the proof of Proposition 6 imply that

$$\rho\left(F_T^\star\right)=\frac{1}{\sqrt{T}}\max_{i=+,-}\sup_\lambda\sup_z\inf_\kappa\left[P_{ih}\left(z,\lambda,\kappa,F\right)\right],\;(\lambda,z,\kappa)\in\Gamma_i.$$

Hence, in such a case, we have that $\mathbb{K}\not\geq_P\mathbb{L}$, and we can construct the following sequence of local alternative hypotheses:

$$\mathbf{H}_T^{\star}:\ \rho\left(F_T^{\star}\right)=\frac{c}{\sqrt{T}},$$

where $c:=\max_{i=+,-}\sup_{\lambda}\sup_{z}\inf_{\kappa}\left[P_{ih}\left(z,\lambda,\kappa,F\right)\right]>0$ and $(\lambda,z,\kappa)\in\Gamma_i$. Obviously, $\mathbf{H}_T^{\star}$ approximates the null hypothesis as $T\to\infty$.

Proposition 3 (LOCUN)**.** *Suppose that Assumptions 5 and LOCAL hold, $P_{ih}>C>0$, while, for some $\lambda\in\mathbb{L}-\mathbb{K}$, $\inf_{Y_0}\lambda^{Tr}Y_0\leq 0$, there exists $(\kappa,z)\in\mathbb{K}_{\lambda}^{\succeq}\times\mathbb{R}_{++}$ with $(\lambda,\kappa,z)\in\Gamma_2$, and that if $(\lambda,\kappa^{\star},z^{\star})\in\Gamma_2$ for some $\kappa^{\star}\neq\kappa$ then $z^{\star}\neq z$. Under $\mathbf{H}_T^{\star}$* and as $T\to\infty$,

$$\lim_{T\to\infty}\mathbb{P}\left(\rho_T>q_{T,b_T}\left(1-\alpha\right)\right)=\mathbb{P}\left(\rho_{\infty}^{\star}>q\left(\rho_{\infty},\alpha\right)\right)>\alpha.$$

Hence the test is asymptotically locally unbiased under the chosen sequence of local alternatives.

2 Monte Carlo Study

We now design and perform a set of Monte Carlo experiments to evaluate the size and power of the proposed tests in finite samples. We do so in a framework of conditional heteroskedasticity that is partially consistent with empirical findings on returns of financial data and relevant to the empirical application that we develop in the main text. We construct $(Y_t)_{t\in\mathbb{Z}}$ as a vector GARCH(1,1) process that also contains an appropriately transformed element. Under the relevant restrictions, this allows for both temporal as well as cross sectional dependence between the random variables that constitute the vector process.

Suppose that $z_t\overset{\text{iid}}{\sim}N\left(0,1\right),\,t\in\mathbb{Z}$. Furthermore for all $t\in\mathbb{Z}$, for $i=1,2,3$, $\omega_i,\alpha_i,\beta_i\in\mathbb{R}_{++}$, $\mu_i\in\mathbb{R}_{+}$ define $y_{i_t}=\mu_i+z_t h_{i_t}^{1/2}$, with $h_{i_t}=\omega_i+\left(\alpha_i z_{t-1}^2+\beta_i\right)h_{i_{t-1}}$, such that $\mathbb{E}\left(\alpha_i z_0^2+\beta_i\right)^{1+\epsilon}<1$, for some $\epsilon>0$, while, for $i=4$ and $v_1,v_2\in\mathbb{R}$, define $y_{4_t}=v_1\left(z_t h_{3_t}^{1/2}\right)_{+}+v_2\left(z_t h_{3_t}^{1/2}\right)_{-}$. Suppose that $\mathbf{Y}_t=\left(y_{1_t},y_{2_t},y_{3_t},y_{4t}\right)'$. Arvanitis and Topaloglou

(2017) establish that the vector process above satisfies our assumption framework. Let $\tau = (0,0,1,0)$, $\tau^{\star} = (0,0,0,1)$ and $\mathbb{L} = \{(\lambda, 1-\lambda, 0, 0)\,,\ \lambda \in [0,1]\,, \tau, \tau^{\star}\}$. Using this portfolio space we obtain the following result on Prospect-spanning. Its proof follows directly from Proposition 4 of Arvanitis and Topaloglou (2017) and it essentially depends on the fact that $\tau^{\star}$ is a Prospect super-efficient portfolio w.r.t. the portfolio space.

Proposition 4 (MC)**.** *If* $\mu_i = 0$ *for* $i = 1,2,3$, $|v_1| > \sqrt{\frac{\max\{\omega_i,\alpha_i,\beta_i,\ i=1,2,3\}}{\min\{\omega_i,\alpha_i,\beta_i,\ i=1,2,3\}}}$ *and* $|v_2| < \sqrt{\frac{\min\{\omega_i,\alpha_i,\beta_i,\ i=1,2,3\}}{\max\{\omega_i,\alpha_i,\beta_i,\ i=1,2,3\}}}$ *then* $\mathbb{K} := \{(\lambda, 1-\lambda, 0, 0)\,,\ \lambda \in [0,1]\} \cup \{\tau^{\star}\}$ *Prospect-spans* $\mathbb{L}$*, while* $\mathbb{K} - \{\tau^{\star}\}$ *does not Prospect-span* $\mathbb{L}$*.*

2.1 Scenarios and Results

Scenarios We use as DGPs instances of the GARCH processes conforming to the previous Proposition 4 in order to evaluate the size and power under a fixed T .

Size Evaluation Scenario-Parameters Selection: To approximate the fixed T size, we test for PSD spanning by setting $\mu_i = 0$ for $i = 1,2,3$, $\omega_1 = 0.5$, $\omega_2 = 0.5$, and $\omega_3 = 0.5$, $a_1 = 0.4$, $a_2 = 0.45$, and $a_3 = 0.5$ and $\beta_1 = 0.5$, $\beta_2 = 0.45$, $\beta_3 = 0.4$, $v_1 = 1.5$ and $v_2 = 0.5$. In this case, we have that $|v_1| > \sqrt{\frac{\max\{\omega_i,a_i,\beta_i,\ i=1,2,3\}}{\min\{\omega_i,a_i,\beta_i,\ i=1,2,3\}}}$ and $|v_2| < \sqrt{\frac{\min\{\omega_i,a_i,\beta_i,\ i=1,2,3\}}{\max\{\omega_i,a_i,\beta_i,\ i=1,2,3\}}}$.

Power evaluation Scenario-Parameters Selection: To approximate the fixed T power, we test for PSD spanning by setting $\mu_i = 0$ for $i = 1,2,3$, $\omega_1 = 0.5$, $\omega_2 = 0.5$, and $\omega_3 = 0.8$, $a_1 = 0.3$, $a_2 = 0.4$, and $a_3 = 0.45$ and $\beta_1 = 0.3$, $\beta_2 = 0.4$, $\beta_3 = 0.45$, $v_1 = 2$ and $v_2 = 0.2$. In this case, we have that $\omega_1 < \omega_3$, $a_1 < a_3$ and $\beta_1 < \beta_3$.

Results We present our Monte Carlo results in Table 1. We use three cases. In the first case, $T = 300$ and we get the subsampling distribution of the test statistic for subsample size $b_T \in \{50, 100, 150, 200\}$. In the second case, $T = 500$ and $b_T \in \{100, 200, 300, 400\}$. Finally, in the third case, $T = 1000$ and $b_T \in \{120, 240, 360, 480\}$. We present the results using the original subsampling critical values (without bias correction) as well as the ones obtained using the bias correction method. We observe that for small samples (T=300 and T=500) the bias correction method is more efficient and more powerful. The test with the bias correction method seems to perform well in all cases with an empirical size close to 5% and an empirical power above 90% for a nominal size $\alpha = 5\%$.

We observe that the computational time is not increasing with the number of assets, it is only increasing with the number of observations.

Monte Carlo Results			
Without bias correction			
Cases	T=300	T=500	T=1000
Empirical size	12.9%	11.8%	9.5%
Empirical power	82.6%	85.8%	90.2%
With bias correction			
Cases	T=300	T=500	T=1000
Empirical size	3.9%	4.6%	4.1%
Empirical power	91.5%	92.3%	94.3%

Table 1: Monte Carlo Results. Entries report the empirical size and empirical power based on 1000 replications and a nominal size $\alpha = 5\%$. The rejection probabilities are calculated both without and with the bias correction method.

3 Proofs

3.1 Proofs of Main Results

Proof of Lemma 3. i. ($\Leftarrow$) If $\mathbb{K} \succcurlyeq_P \mathbb{L}$, we have from Definition 1 that, for any λ, there exists some κ such that $\sup_{z\in A_1} P_1(z,\lambda,\kappa,F) \leq 0$ and $\sup_{z\in A_2} P_2(z,\lambda,\kappa,F) \leq 0$. This implies that $\max_{i=1,2}\sup_{z\in A_i}\inf_{\kappa\in\mathbb{K}} P_i(z,\lambda,\kappa,F) \leq 0$, which in turn implies that $\rho(F) \leq 0$. Since $\mathbb{K}$ is closed and thereby compact, the Dominated Convergence Theorem implies that $\mathcal{J}(z,0,\kappa,F)$ is continuous w.r.t. κ. This along with the compactness of $\mathbb{K}$ implies that $\arg\min_{\kappa\in\mathbb{K}}\mathcal{J}(z,0,\kappa,F)$ is non empty. Let $\kappa^\star$ be an element of the latter. Then, the first equality follows from $\rho(F) \geq \inf_{\kappa\in\mathbb{K}}\mathcal{J}(z,0,\kappa,F) - \mathcal{J}(z,0,\kappa^\star,F) = 0$. If $\mathbb{K} \not\succcurlyeq_P \mathbb{L}$ then for some $\lambda^\star \in \mathbb{L}$, and any $\kappa \in \mathbb{K}$, there exists some $i^\star$ and $z^\star \in A_{i^\star}$ such that $P_{i^\star}(z^\star, \lambda^\star, \kappa, F) > 0$. Then the continuity of $\mathcal{J}(z, 0, \kappa, F)$ and $\mathcal{J}(0, \lambda, \kappa, F)$ w.r.t. κ and the compactness of $\mathbb{K}$, implies that for any $z \in A_i, \exists \kappa_{\lambda,z,i} \in \mathbb{K}$ such that $\inf_{\kappa \in \mathbb{K}} P_i(z, \lambda, \kappa, F) = P_i(z, \lambda, \kappa_{\lambda,z,i}, F)$, and thereby $\rho(F) \geq P_{i^\star}(z^\star, \lambda^\star, \kappa_{\lambda^\star,z^\star,i^\star}, F) > 0$.

ii. $(\Rightarrow)$ If $\rho(F) = 0$, then for any $\lambda \in \mathbb{L}$ we get that $\max_{i=1,2}\sup_{z\in A_i}\inf_{\kappa\in\mathbb{K}} P_i(z, \lambda, \kappa, F) \leq 0$. Hence, there exists an element of $\mathbb{K}$ for which $P_i(z, \lambda, \kappa, F) \leq 0$, for every $z \in A_i$, $i = 1, 2$. □

Proof of Lemma 4. Integrating by parts, for any $-\infty < \alpha < \beta < +\infty$, we have that

$$\int_\alpha^\beta G(u, \lambda, F)\, du = (u - \beta) G(u, \lambda, F)|_\alpha^\beta - \int_\alpha^\beta (u - \beta)\, dG(u, \lambda, F)$$

$$= (\beta - \alpha) G(\alpha, \lambda, F) - \int_{\mathbb{R}} (u - \beta) 1_{\alpha \leq u \leq \beta} dG(u, \lambda, F)$$

$$= \int_{\mathbb{R}} [(\beta - \alpha) 1_{u \leq \alpha} + (\beta - u) 1_{\alpha \leq u \leq \beta}]\, dG(u, \lambda, F).$$

Hence, for $\alpha = z \in \mathbb{R}_-$, $\beta = 0$, we obtain from Definition 1 that

$$P_1(z, \lambda, \kappa, F) = \int_{\mathbb{R}} [-z 1_{u \leq z} - u 1_{z \leq u \leq 0}]\, d[G(u, \kappa, F) - G(u, \lambda, F)]$$

$$= \int_{\mathbb{R}} [z 1_{u \leq z} + u 1_{z \leq u \leq 0}]\, d[G(u, \lambda, F) - G(u, \kappa, F)].$$

Analogously, for $\alpha = 0$, $\beta = z \in \mathbb{R}_{++}$, we obtain from Definition 1 that

$$P_2(z, \lambda, \kappa, F) = \int_{\mathbb{R}} [z 1_{u \leq 0} + (z - u) 1_{0 \leq u \leq z}]\, d[G(u, \kappa, F) - G(u, \lambda, F)]$$

$$= \int_{\mathbb{R}} [-u 1_{0 \leq u \leq z} + z 1_{-\infty < u \leq z}]\, d[G(u, \kappa, F) - G(u, \lambda, F)]$$

$$= \int_{\mathbb{R}} [u 1_{0 \leq u \leq z} - z(1 - 1_{z \leq u < +\infty})]\, d[G(u, \lambda, F) - G(u, \kappa, F)]$$

$$= \int_{\mathbb{R}} \left[u 1_{0\leq u\leq z} + z 1_{z\leq u<+\infty}\right] d\left[G\left(u,\lambda,F\right) - G\left(u,\kappa,F\right)\right].$$

The previous along with Fubini Theorem, enabled by the existence of the first moment for the elements of $\mathcal{W}_{-},\mathcal{W}_{+}$, and supposing that the analogous suprema and infima exist, imply that for any λ,κ,

$$\sup_{z\leq 0}\inf_{\kappa\in\mathbb{K}} P_1\left(z,\lambda,\kappa,F\right) = \sup_{w\in\mathcal{W}_{-}}\inf_{\kappa\in\mathbb{K}}\int_{\mathbb{R}_{-}} P_1\left(z,\lambda,\kappa,F\right) dw\left(z\right)$$

$$= \sup_{w\in\mathcal{W}_{-}}\inf_{\kappa\in\mathbb{K}}\int_{\mathbb{R}_{-}}\int_{\mathbb{R}} \left[z 1_{u\leq z} + u 1_{z\leq u\leq 0}\right] d\left[G\left(u,\lambda,F\right) - G\left(u,\kappa,F\right)\right] dw\left(z\right)$$

$$= \sup_{w\in\mathcal{W}_{-}}\inf_{\kappa\in\mathbb{K}}\int_{\mathbb{R}_{-}}\int_{\mathbb{R}_{-}} \left[z 1_{u\leq z} + u 1_{z\leq u\leq 0}\right] dw\left(z\right) d\left[G\left(u,\lambda,F\right) - G\left(u,\kappa,F\right)\right]$$

$$= \sup_{v_w\in V_{-}}\left[\mathbb{E}_{\lambda}\left[1_{u\leq 0} v_w\left(u\right)\right] - \sup_{\kappa\in\mathbb{K}}\mathbb{E}_{\kappa}\left[1_{u\leq 0} v_w\left(u\right)\right]\right],$$

and analogously,

$$\sup_{z\geq 0}\inf_{\kappa\in\mathbb{K}} P_2\left(z,\lambda,\kappa,F\right) = \sup_{w\in\mathcal{W}_{+}}\inf_{\kappa\in\mathbb{K}}\int_{\mathbb{R}_{+}} P_2\left(z,\lambda,\kappa,F\right) dw\left(x\right)$$

$$= \sup_{v_w\in V_{+}}\left[\mathbb{E}_{\lambda}\left[1_{u\geq 0} v_w\left(u\right)\right] - \sup_{\kappa\in\mathbb{K}}\mathbb{E}_{\kappa}\left[1_{u\geq 0} v_w\left(u\right)\right]\right].$$

This and the commutativity of suprema imply that

$$\rho\left(F\right) = \max_{i=1,2}\sup_{v_w\in V_i}\left[\sup_{\lambda\in\mathbb{L}}\mathbb{E}_{\lambda}\left[1_{u\in\mathbb{A}_i} v_w\left(u\right)\right] - \sup_{\kappa\in\mathbb{K}}\mathbb{E}_{\kappa}\left[1_{u\in\mathbb{A}_i} v_w\left(u\right)\right]\right],$$

and the first result follows. For the second one, due to Lemma 3, the previous implies that $\mathbb{K} \succcurlyeq_P \mathbb{L}$ iff, $\max_{i=1,2}\sup_{v_w\in V_i}\left[\sup_{\lambda\in\mathbb{L}}\mathbb{E}_{\lambda}\left[1_{u\in\mathbb{A}_i} v_w\left(u\right)\right] - \sup_{\kappa\in\mathbb{K}}\mathbb{E}_{\kappa}\left[1_{u\in\mathbb{A}_i} v_w\left(u\right)\right]\right] = 0$. If the latter holds then $\sup_{v_w\in V_i}\left[\sup_{\lambda\in\mathbb{L}}\mathbb{E}_{\lambda}\left[1_{u\in\mathbb{A}_i} v_w\left(u\right)\right] - \sup_{\kappa\in\mathbb{K}}\mathbb{E}_{\kappa}\left[1_{u\in\mathbb{A}_i} v_w\left(u\right)\right]\right] \leq 0$, $\forall i = 1,2$, which due to the convexity of $\mathbb{K}$ and a double application of the Sion (1958) Minimax Theorem implies that $\sup_{v\in V}\left[\sup_{\lambda\in\mathbb{L}}\mathbb{E}_{\lambda}\left[v\right] - \sup_{\kappa\in\mathbb{K}}\mathbb{E}_{\kappa}\left[v\right]\right] \leq 0$, and the result follows from $\mathbb{K}\subseteq\mathbb{L}$.

Now suppose that $\sup_{v\in V}\left[\sup_{\lambda\in\mathbb{L}}\mathbb{E}_\lambda[v]-\sup_{\kappa\in\mathbb{K}}\mathbb{E}_\kappa[v]\right]=0$, which implies from $\mathbb{K}\subseteq\mathbb{L}$ that $\sup_{v\in V_\star}\sup_{\lambda\in\mathbb{L}}\inf_{\kappa\in\mathbb{K}}\left[\mathbb{E}_\lambda[v]-\mathbb{E}_\kappa[v]\right]\leq 0$, $\sup_{v\in V_\star}\sup_{\lambda\in\mathbb{L}}\inf_{\kappa\in\mathbb{K}}\left[\mathbb{E}_\lambda[v]-\mathbb{E}_\kappa[v]\right]\leq 0$, where $V_\star = \left\{ v:\mathbb{R}\to\mathbb{R},\ v(u)=\begin{cases} v_w(u), & u\leq 0 \\ 0, & u\geq 0\end{cases}, \text{ where } v_w\in V_-\right\}$ and $V^\star=\left\{ v:\mathbb{R}\to\mathbb{R},\ v(u)=\begin{cases} 0, & u\leq 0 \\ v_w(u), & u\geq 0\end{cases}, \text{ where } v_w\in V_+\right\}$. Using the obvious identification of $V_\star, V^\star$ with V_-, V_+, the latter display implies that

$$\sup_{v_w\in V_i}\left[\sup_{\lambda\in\mathbb{L}}\mathbb{E}_\lambda\left[1_{u\in\mathbb{A}_i}v_w(u)\right]-\sup_{\kappa\in\mathbb{K}}\mathbb{E}_\kappa\left[1_{u\in\mathbb{A}_i}v_w(u)\right]\right]\leq 0,\ \forall i=1,2,$$

and the result follows from $\mathbb{K}\subseteq\mathbb{L}$. □

Proof of Proposition 6. The results in the auxiliary Lemma 15 imply that $\begin{pmatrix} P_1\left(z_1,\lambda,\kappa,\sqrt{T}(F_T-F)\right) \\ P_2\left(z_2,\lambda,\kappa,\sqrt{T}(F_T-F)\right)\end{pmatrix}$ weakly converges to $\begin{pmatrix} P_1(z_1,\lambda,\kappa,\mathcal{G}_F) \\ P_2(z_2,\lambda,\kappa,\mathcal{G}_F)\end{pmatrix}$ w.r.t. to the product topology of continuous (w.r.t. (z_1,z_2,λ)) epi-convergence (w.r.t. κ) on the product of the relevant spaces of lower semi-continuous lsc real valued functions (see e.g. Knight (1999) for the dual notion of epi-convergence). This product space is metrizable as complete and separable (see again Knight (1999)). Hence, Skorokhod representations are applicable (as above, see for example Theorem 1 in Cortissoz (2007)) and thereby for any (z_1,z_2,λ) and any sequence $(z_{1,T},z_{2,T},\lambda_T)\to(z_1,z_2,\lambda)$, there exist an enhanced probability space and processes $\begin{pmatrix} P_{1,T}(z_1,\lambda,\kappa) \\ P_{2,T}(z_2,\lambda,\kappa)\end{pmatrix}\overset{d}{=}\begin{pmatrix} P_1\left(z_{1,T},\lambda_T,\kappa,\sqrt{T}(F_T-F)\right) \\ P_2\left(z_{2,T},\lambda_T,\kappa,\sqrt{T}(F_T-F)\right)\end{pmatrix}$, $\begin{pmatrix} P_1^\star(z_1,\lambda,\kappa) \\ P_2^\star(z_2,\lambda,\kappa)\end{pmatrix}\overset{d}{=}\begin{pmatrix} P_1(z_1,\lambda,\kappa,\mathcal{G}_F) \\ P_2(z_2,\lambda,\kappa,\mathcal{G}_F)\end{pmatrix}$, defined on it such that $\begin{pmatrix} P_{1,T} \\ P_{2,T}\end{pmatrix}\to\begin{pmatrix} P_1^\star \\ P_2^\star\end{pmatrix}$ almost surely, w.r.t. to the product topology of epi-convergence, where $\overset{d}{=}$ denotes equality in distribution. Notice that,

$$\begin{pmatrix} P_1\left(z_{1,T},\lambda_T,\kappa,\sqrt{T}F_T\right) \\ P_2\left(z_{2,T},\lambda_T,\kappa,\sqrt{T}F_T\right) \end{pmatrix} \overset{d}{=} \begin{pmatrix} P_{1,T} \\ P_{2,T} \end{pmatrix} + \sqrt{T}\begin{pmatrix} P_1\left(z_{1,T},\lambda_T,\kappa,F\right) \\ P_2\left(z_{2,T},\lambda_T,\kappa,F\right) \end{pmatrix},$$

and for each i consider the function

$$P_i^{\infty}\left(z_i,\lambda,\kappa\right) := \begin{cases} P_i^{\star}\left(z_i,\lambda,\kappa\right), & P_i\left(z_i,\lambda,\kappa,F\right)=0 \\ +\infty, & P_i\left(z_i,\lambda,\kappa,F\right)>0 \\ -\infty, & P_i\left(z_i,\lambda,\kappa,F\right)<0 \end{cases} \cdot$$

Notice that for $P_i^{\infty}\left(z_i,\lambda,\kappa\right)=P_i^{\star}\left(z_i,\lambda,\kappa\right)$ for each $\left(z_i,\lambda,\kappa\right)\in\Gamma_i$. Suppose that, $\mathbf{H_0}$ holds. Then, for each i and for any compact $\mathcal{K}$ that contains $\kappa\in\mathrm{K}$ such that $\left(z_{i,T},\lambda_T,\kappa\right)$ converges on the boundary of Γ_i we have that almost surely,

$$\begin{aligned} &\liminf_{T\to\infty}\inf_{\mathcal{K}} P_i\left(z_{i,T},\lambda_T,\kappa,\sqrt{T}F_T\right) \\ &\geq \inf_{\mathcal{K}} P_i\left(z_i,\lambda,\kappa,\mathcal{G}_F\right)+\liminf_{T\to\infty}\sqrt{T}\inf_{\mathcal{K}} P_i\left(z_{i,T},\lambda_T,\kappa,F\right) \\ &\geq \inf_{\mathcal{K}} P_i^{\infty}\left(z_i,\lambda,\kappa\right). \end{aligned}$$

Hence, due to Proposition 3.2.(ii)-(iii) (ch. 5, p. 337) of Molchanov (2006), $P_i\left(z_i,\lambda,\kappa,\sqrt{T}F_T\right)$ almost surely epi-converges w.r.t. κ, continuously w.r.t. $\left(z_i,\lambda\right)$ to $P_i^{\infty}\left(z_i,\lambda,\kappa\right)$. The compactness of $\mathbb{K}$ and Theorem 3.4 (ch. 5, p. 338) of Molchanov (2006) imply that, almost surely

$$\inf_{\kappa} P_i\left(z_{i,T},\lambda_T,\kappa,\sqrt{T}F_T\right)\to\inf_{\kappa} P_i^{\infty}\left(z_i,\lambda,\kappa\right)=\begin{cases} \inf_{\kappa} P_i^{\star}\left(z_i,\lambda,\kappa\right), & \forall\kappa\in\mathbb{K}_{\lambda}^{\succeq},\left(z_i,\lambda,\kappa\right)\in\Gamma_i \\ -\infty & \exists\kappa\in\mathbb{K}_{\lambda}^{\succeq},\left(z_i,\lambda,\kappa\right)\notin\Gamma_i \end{cases}. \tag{1}$$

The existence of $\underline{z}$, the compactness of $\mathbb{L}$, the fact that $P_i\left(z_i,\lambda,\kappa,\sqrt{T}F_T\right)$ is a monotone transformation of $P_i\left(z_i, \lambda, \kappa, F_T\right)$, the fact that continuous convergence implies hypo-convergence by Theorem 7.11 of Rockafellar and Wets (2009), the dual version of Theorem 3.4 of Molchanov (2006), the fact that $\Gamma_i \neq \emptyset$ for all i, imply the result by reverting to the original probability space. $\square$

Proof of Theorem 9. The first result follows by a direct application of Theorem 3.5.1.i of Politis et al. (1999) due to the results of Proposition 6, since the limiting cdf is continuous at any $q_{1-\alpha}$ for all $\alpha \in \left(0, \frac{1}{2}\right)$ due to the auxiliary Lemma 16. Notice that if $\mathbf{H_a}$ is true then for $\lambda^\star \in \mathbb{L} - \mathbb{K}$, and any $\kappa \in \mathbb{K}$ there exists $i^\star, z^\star \in A_{i^\star}$ such that $P_{i^\star}\left(z^\star, \lambda^\star, \kappa, F\right) > 0$. Then we have that $\rho_T \geq \inf_{\kappa \in \mathbb{K}} P_{i^\star}\left(z^\star, \lambda^\star, \kappa, \sqrt{T}\left(F_T - F\right)\right) + \sqrt{T} \inf_{\kappa \in \mathbb{K}} P_{i^\star}\left(z^\star, \lambda^\star, \kappa, F\right)$, and due to arguments analogous to the ones used in the proof of Proposition 6, we have that the first term in the rhs of the last display is asymptotically tight, while due to the arguments used in the proof of Proposition 3, the second term in the rhs of the last display diverges to $+\infty$. The result follows from the properties of b_T. $\square$

Proof of Proposition LOCLIM. Analogous to the proof of Proposition 6. $\square$

Proof of Proposition LOCUN. Follows directly by Proposition LOCLIM and Theorem 3.5.1.iii of Politis et al. (1999). $\square$

Proof of Proposition 10. First notice that the integration by parts formula and the proof of Lemma 4 imply that

$$\rho_T = \sqrt{T} \max_{i=1,2} \sup_{v_w \in V_i} \left[\sup_{\lambda \in \mathbb{L}} \mathbb{E}_{F_T}\left[1_{\lambda^T Y \in A_i} v_w\left(\lambda^T Y\right)\right] - \sup_{\kappa \in \mathbb{K}} \mathbb{E}_{F_T}\left[1_{\kappa^T Y \in A_i} v_w\left(k^T Y\right)\right]\right].$$

Since $V_i^\star \subset V_i$, $i = -, +$ (as a matter of fact, we have that $\mathbb{P}$ a.s. $\rho_T \geq \rho_T^\star$), the result obtains if $\mathbb{E}_{F_T}\left[1_{\lambda^T Y \in A_i} v\left(\lambda^T Y\right)\right]$ converges uniformly $\mathbb{P}$ a.s. to $\mathbb{E}_{F_T}\left[1_{\lambda^T Y \in A_i} v_w\left(\lambda^T Y\right)\right]$ on $V_i \times \mathbb{L} \times \mathcal{R}^i$, $i = -, +$. This follows from the compactness of $\mathcal{R}^i$, the Density Theorem (see Theorem 15.10 of Aliprantis and Border (2006)), and Theorem 15.11 of Aliprantis and Border (2006). $\square$

Proof of Proposition MC. From Proposition 4 of Arvanitis and Topaloglou (2017), we have that $\tau^\star$ strictly Prospect dominates every portfolio in $\mathbb{L}$. Hence $\mathbb{K} \succcurlyeq_P \mathbb{L}$ and thereby $\mathbb{K} \supset \{\tau^\star\} \succcurlyeq_P \mathbb{L}$. Due to the same reasoning there is no element in $\mathbb{K} - \{\tau^\star\}$ that Prospect dominates $\tau^\star$. Hence $\mathbb{K} - \{\tau^\star\}$ cannot Prospect-span $\mathbb{L}$. □

3.2 Auxiliary Lemmata

The following are auxiliary results used in the proofs above.

Lemma 5. *Under Assumption 5*

$$\begin{pmatrix} P_1\left(z_1, \lambda, \kappa, \sqrt{T}\left(F_T - F\right)\right) \\ P_2\left(z_2, \lambda, \kappa, \sqrt{T}\left(F_T - F\right)\right) \end{pmatrix} \rightsquigarrow \begin{pmatrix} P_1\left(z_1, \lambda, \kappa, \mathcal{G}_F\right) \\ P_2\left(z_2, \lambda, \kappa, \mathcal{G}_F\right) \end{pmatrix}$$

as random elements with values on the space of $\mathbb{R}^2$-valued bounded functions on $\mathbb{L} \times \mathbb{K} \times \mathbb{R}_- \times \mathbb{R}_{++}$ equiped with the sup-norm. The limiting process has continuous sample paths.

Proof. See Lemma 2 in Arvanitis and Topaloglou (2017). □

Lemma 6. *Under Assumptions 5, $\mathbb{P}\left(\rho_\infty \geq 0\right) = 1$, its cdf is absolutely continuous on $(0, +\infty)$ and it may have a jump discontinuity at zero. Suppose moreover that for $\lambda \in \mathbb{L} - \mathbb{K}$, $\inf_{Y_0} \lambda^{Tr} Y_0 \leq 0$ and there exists $(\kappa, z) \in \mathbb{K}_\lambda^{\succeq} \times \mathbb{R}_{++}$ with $(\lambda, \kappa, z) \in \Gamma_2$ and that if $(\lambda, \kappa^\star, z^\star) \in \Gamma_2$ for $\kappa^\star \neq \kappa$ then $z^\star \neq z$. Then $\mathbb{P}\left(\rho_\infty > 0\right) \geq \frac{1}{2}$.*

Proof. Notice first that for all (λ, κ), $P_i\left(\lambda, \kappa, \min_{i \leq N, t \leq T}\left(Y_t\right), \sqrt{T} F_T\right) = 0$, for the i that corresponds to the sign of $\underline{z}$ which then implies that $P_i\left(z, \lambda, \kappa, \sqrt{T} F_T\right) \geq 0$ a.s., and then due to the Portmanteau Theorem and Proposition 6, we get $0 = \liminf_{T \to \infty} \mathbb{P}\left(\rho_T < 0\right) \geq \mathbb{P}\left(\rho_\infty < 0\right)$. Now, for $\Lambda = \mathbb{L} \times \mathbb{K} \times \{1, 2\} \times \mathbb{R}_- \times \mathbb{R}_{++}$ where $\{1, 2\}$ is equipped with the discrete metric, consider $X_\mu := 1_{i=1} P_1\left(z_1, \lambda, \kappa, \mathcal{G}_F\right) + 1_{i=2} P_2\left(z_2, \lambda, \kappa, \mathcal{G}_F\right)$, for $\mu = (\lambda, \kappa, i, z_1, z_2)$, and 1_j is the indicator of $\{j\}$. X_μ is zero mean Gaussian and has continuous sample paths due to the final assertion of Lemma 5. Since $P_i(z,\kappa,\kappa,\mathcal{G}_F)=0$ almost surely for all z and i, and due Lemma 18.15 of van der Vaart (2000), we have that for $\mu^\star=(\kappa,\kappa,i,z_1,z_2)$

$$0\leq\sigma^2:=\sup_\Lambda\mathbb{E}\left(X_\mu^2\right)=\sup_\Lambda\mathbb{E}\left((X_\mu-X_{\mu^\star})^2\right)\leq\sup_{\mu,v\in\Lambda}\mathbb{E}\left((X_\mu-X_v)^2\right)<+\infty.$$

Hence due to the zero mean function of X_μ, and Furnique inequality (see Relation (1,1) in Samorodnitsky (1991)), we have that for $0<\varepsilon<1$, there exists $\kappa(\varepsilon)$, such that

$$\mathbb{E}\left(\sup_\Lambda X_\mu^2\right)=\int_0^{+\infty}\mathbb{P}\left(\sup_\Lambda|X_\mu|>\sqrt{y}\right)dy\leq 2\kappa(\varepsilon)\int_0^{+\infty}\exp\left(\frac{-(1-\varepsilon)}{2\sigma^2}y\right)dy<+\infty.$$

Then Ch. 2 of Nualart (2006), (see the remark after the proof of Proposition 2.1.11 (p. 109)) implies the existence of the square integrable Malliavin derivative for X_μ. The zero mean Gaussianity, via the exclusion of $\mathbb{P}$-negligible events, implies that X_μ is zero only when $\kappa=\lambda$ or $\kappa\neq\lambda$ and $z\leq\inf Y_0$, and at most only then that X_μ has degenerate variance. Hence, Nualart (2006) implies then that the Malliavin derivative of X_μ equals zero only then. The previous lines imply the validity of Assumption 1 of Arvanitis, Scaillet and Topaloglou (2019) for $\mathcal{T}=\{0\}$ in their notation, and the second assertion follows by Theorem 1 there. For the final assertion notice that since $\inf_{Y_0}\lambda^{Tr}Y_0\leq 0$ and $\lambda\in\Lambda-K$, then $\operatorname{Var}(P_2(z,\lambda,\kappa,\mathcal{G}_F))>0$ due to Assumption 5. Then since for any $(\lambda,\kappa^\star,z^\star)\in\Gamma_2$ with $\kappa^\star\neq\kappa$ then $z^\star\neq z$, we have that $\mathbb{P}(\sup_z\inf_\kappa P_2(z,\lambda,\kappa,\mathcal{G}_F)>0,(\lambda,\kappa,z)\in\Gamma_2)=\mathbb{P}(\sup_z P_2(z,\lambda,\kappa,\mathcal{G}_F)>0,(\lambda,\kappa,z)\in\Gamma_2)\geq\mathbb{P}(P_2(z,\lambda,\kappa,\mathcal{G}_F)>0,(\lambda,\kappa,z)\in\Gamma_2)=\frac{1}{2}$, due to non-degeneracy and zero mean Gaussianity. We deduce the result since $\mathbb{P}(\rho_\infty>0)$ is greater than or equal from the probability in the lhs of the inequality. $\square$

4 Summary Statistics of the Factor and Anomaly Returns

Table 2 reports summary statistics of the factor and anomaly returns over our sample period.

Table 2: Descriptive Statistics of monthly returns

Panel Composite EquityA: Factor Models	Mean	SD	Skewness	Kurtosis	Sharpe ratio
Market	0.0098	0.0454	-0.5234	2.0989	0.1306
FF-5 model					
SMB	0.0028	0.0299	0.3853	4.2534	-0.0363
HML	0.0036	0.0293	0.0777	2.1566	-0.0108
RMW	0.0030	0.0235	-0.3615	12.444	-0.0387
CMA	0.0034	0.0198	0.3913	1.9190	-0.0251
M-4 model					
SMB	0.0045	0.0280	0.2565	2.0487	0.0198
MGMT1	0.0061	0.0283	0.1510	1.8210	0.0789
PERF1	0.0065	0.0393	-0.0486	3.8711	0.0650
q model					
ME	0.0034	0.0305	0.6317	6.3636	-0.0150
IA	0.0040	0.0184	0.2052	1.8422	0.0041
ROE	0.0055	0.0261	-0.7203	4.8811	0.0624
Panel B: Anomalies	**Mean**	**SD**	**Skewness**	**Kurtosis**	**Sharpe ratio**
Accruals	0.0031	0.0310	0.0066	1.0947	-0.0272
Asset Growth	0.0052	0.0328	0.5986	3.6047	0.0390
Composite Equity Issues	0.0049	0.0337	0.0480	2.3218	0.0301
Distress	0.0045	0.0638	0.0830	3.6412	0.0102
Growth Profitability Premium	0.0021	0.0377	0.2643	1.2312	-0.0480
Investment to Assets	0.0053	0.0291	0.0804	0.1216	0.0478
Momentum	0.0107	0.0652	-0.8537	5.5629	0.1046
Net Operating Assets	0.0056	0.0291	0.1552	1.0253	0.0595
O-Score	0.000	0.0362	0.2574	1.1787	-0.1070
Return on Assets	0.0057	0.0417	0.3637	2.6874	0.0425
Net Stock Issues	0.0051	0.0271	0.1013	2.5132	0.0433
Betting against Beta	0.0088	0.0340	-0.6509	3.3393	0.1431
Quality minus Junk	0.0051	0.0454	0.0979	1.5959	0.0269
Size	-0.0186	0.0637	-1.8285	8.4206	-0.3533
Growth Option	-0.0218	0.0535	2.0057	10.1429	-0.4819
Value (Book to Market)	0.0181	0.0596	0.09919	13.4067	0.2384
Idiosyncratic Volatility	0.0103	0.0857	1.8921	8.6201	0.0743
Profitability	-0.0037	0.0515	-1.9601	8.9951	-0.1481

Entries report the descriptive statistics of the factor and anomaly returns. The dataset spans the period from January, 1974 to December, 2016.

5 Out-of-Sample Analysis: Tables

Tables 3-19 reports the performance measures for the 5 Fama and French, the 4 Stambaugh and Yuan and the 4 Hou-Xue-Zhang optimal factor portfolios, and the augmented portfolios with each of the market anomaly (Panels A, B and C respectively).

Table 3: Performance measures. The case of the Accruals anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0059	0.0043	0.0068	0.0062	0.0033	0.0028
SD	0.0268	0.0278	0.0183	0.0182	0.0170	0.0137
Sharpe ratio	0.2113	0.1477	0.3606	0.3274	0.1778	0.1862
D. Sharpe ratio	0.2344	0.1645	0.5318	0.4538	0.1912	0.2087
UP ratio	0.7288	0.6683	1.0322	0.9529	0.5671	0.5859
Return Loss		-0.1715%		-0.0607%		0.0202%
Opportunity Cost						
$\alpha = \beta = 0.2$		-0.7316%		-0.4870%		0.0201%
$\alpha = \beta = 0.4$		-0.3076%		-0.1571%		0.0079%
$\alpha = \beta = 0.6$		-0.1321%		-0.0567%		0.0074 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.5707	0.0548	0.0215	-0.0380
	SMB	0.0	0.0	-	-
	HML	0.0842	0.0952	0.5712	-1.1834
	RMW	0.0050	0.0127	2.4771	5.0953
	CMA	0.0	0.0	-	-
	Accruals	0.3402	0.0695	0.0788	-1.2977
M-4 Factors	Market	0.3510	0.2023	0.7799	-1.2026
	SMB	0.1108	0.0850	-0.2305	-1.4947
	MGMT1	0.2939	0.1477	-1.0372	-0.6208
	PERf1	0.1238	0.0835	-0.6683	-1.4185
	Accruals	0.1207	0.1052	0.8907	-1.0240
q Factors	Market	0.1290	0.0397	1.4982	3.5138
	ME	0.0388	0.0561	0.8400	-1.1400
	IA	0.1237	0.1683	0.7041	-1.4664
	ROE	0.5109	0.1550	-0.4084	-1.4611
	Accruals	0.1975	0.0637	-0.0610	-1.0143

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Accruals optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 4: Performance measures. The case of the Asset Growth anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0056	0.0056	0.0047	0.0048	0.0028	0.0033
SD	0.0213	0.0215	0.0176	0.0176	0.0201	0.0219
Sharpe ratio	0.2511	0.2503	0.2563	0.2560	0.1295	0.1404
D. Sharpe ratio	0.2870	0.2894	0.2847	0.2836	0.1400	0.1590
UP ratio	0.7416	0.7406	0.6359	0.6359	0.5316	0.5559
Return Loss		-0.0019%		-0.0005%		0.0199%
Opportunity Cost						
$\alpha = \beta = 0.2$		-0.1492%		-0.0007%		0.1506%
$\alpha = \beta = 0.4$		-0.0358%		-0.0005%		0.0400%
$\alpha = \beta = 0.6$		-0.0064%		-0.0003%		0.0120 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.4177	0.1572	1.0019	0.1074
	SMB	0.0056	0.0188	3.8407	14.8344
	HML	0.1797	0.0713	-1.2210	0.7154
	RMW	0.2346	0.1295	-0.7992	-0.8630
	CMA	0.0381	0.0791	2.3286	4.6684
	Asset Growth	0.1243	0.0667	0.5853	1.6736
M-4 Factors	Market	0.2406	0.1154	3.7302	18.5626
	SMB	0.1508	0.0808	-1.1610	-0.4733
	MGMT1	0.4274	0.0533	-4.9232	37.9211
	PERf1	0.1812	0.0488	-1.4616	4.6238
	Asset Growth	0.0	0.0	-	-
q Factors	Market	0.1084	0.0373	0.8722	0.8554
	ME	0.0049	0.0171	3.4229	10.5101
	IA	0.1127	0.1367	0.6434	-1.2765
	ROE	0.5883	0.1091	-0.2565	-1.0208
	Asset Growth	0.1857	0.0790	-0.2119	-1.0005

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Asset Growth optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 5: Performance measures. The case of the Composite Equity Issues anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0045	0.0050	0.0048	0.0048	0.0033	0.0036
SD	0.0235	0.0228	0.0176	0.0177	0.0144	0.0138
Sharpe ratio	0.1832	0.2107	0.2574	0.2582	0.2133	0.2417
D. Sharpe ratio	0.1944	0.2255	0.2869	0.2870	0.2658	0.3199
UP ratio	0.6421	0.6612	0.6413	0.6404	0.6568	0.7168
Return Loss		0.0655%		0.0013%		0.0419%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.0918%		0.0203%		0.2121%
$\alpha = \beta = 0.4$		0.0807%		0.0045%		0.0836%
$\alpha = \beta = 0.6$		0.0469%		0.0051%		0.0350 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.5150	0.1773	-1.1062	0.2813
	SMB	0.0216	0.0453	1.6409	0.7124
	HML	0.0910	0.0904	0.3713	-1.4702
	RMW	0.0707	0.1280	1.4095	0.1273
	CMA	0.0550	0.1137	1.6936	0.9982
	Composite Equity Issues	0.2467	0.1028	-0.2571	-0.8688
M-4 Factors	Market	0.2505	0.1414	3.3818	12.5893
	SMB	0.1506	0.0807	-1.1619	-0.4676
	MGMT1	0.4185	0.0636	-4.4121	24.3000
	PERf1	0.1792	0.0514	-1.4698	4.4559
	Composite Equity Issues	0.0013	0.0030	2.1122	2.6807
q Factors	Market	0.1121	0.0170	0.4119	0.0547
	ME	0.1321	0.0276	-2.5370	9.9149
	IA	0.2602	0.0472	-3.6698	18.8782
	ROE	0.3462	0.0439	5.5208	32.2358
	Composite Equity Issues	0.1494	0.0179	1.9425	9.2660

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Composite Equity Issues optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 6: Performance measures. The case of the Distress anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0054	0.0060	0.0058	0.0059	0.0032	0.0032
SD	0.0258	0.0192	0.0194	0.0192	0.0155	0.0156
Sharpe ratio	0.2011	0.3017	0.2873	0.2960	0.1923	0.1928
D. Sharpe ratio	0.2163	0.3818	0.3520	0.3697	0.2241	0.2248
UP ratio	0.6629	0.8655	0.8084	0.8306	0.6155	0.6161
Return Loss		0.2673%		0.0171%		0.0007%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.3196%		0.0536%		0.0206%
$\alpha = \beta = 0.4$		0.1777%		0.0202%		0.0022%
$\alpha = \beta = 0.6$		0.1016%		0.0087%		-0.0001 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.4809	0.1439	-1.0704	0.1330
	SMB	0.0230	0.0490	1.7887	1.3365
	HML	0.2374	0.0949	-0.6449	-0.1668
	RMW	0.0565	0.0971	1.5373	1.0492
	CMA	0.0283	0.0603	1.9261	2.3037
	Distress	0.1739	0.0694	0.3919	-0.4916
M-4 Factors	Market	0.3702	0.1819	0.6927	-0.5093
	SMB	0.0812	0.0963	0.3925	-1.8170
	MGMT1	0.4138	0.0848	-1.9998	4.1059
	PERf1	0.1176	0.0777	-0.2801	-1.1894
	Distress	0.0171	0.0323	3.1190	9.9059
q Factors	Market	0.1201	0.1239	5.8609	36.9753
	ME	0.0985	0.0558	-0.9635	-0.7772
	IA	0.3880	0.1316	-2.2555	3.8149
	ROE	0.3926	0.1556	2.0878	5.4582
	Distress	0.0007	0.0017	3.5167	13.3486

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Distress optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 7: Performance measures. The case of the Growth Profitability Premium anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0039	0.0038	0.0054	0.0051	0.0033	0.0033
SD	0.0159	0.0144	0.0147	0.0140	0.0142	0.0130
Sharpe ratio	0.2303	0.2452	0.3520	0.3484	0.2175	0.2397
D. Sharpe ratio	0.2952	0.3320	0.5170	0.5188	0.2720	0.3098
UP ratio	0.6675	0.7192	0.9510	0.9542	0.6621	0.7014
Return Loss		0.0262%		-0.0042%		0.0336%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.0179%		-0.0822%		0.0283%
$\alpha = \beta = 0.4$		0.0089%		-0.0410%		0.0239%
$\alpha = \beta = 0.6$		0.0079%		-0.0159%		0.0154 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.1659	0.0172	0.5357	0.6656
	SMB	0.0856	0.0169	-0.6251	-0.4621
	HML	0.2252	0.0218	0.1375	0.4455
	RMW	0.2209	0.0330	0.1250	-0.0319
	CMA	0.1768	0.0266	-0.2260	-0.6922
	Growth Profitability Premium	0.1255	0.0120	-0.5937	-0.8303
M-4 Factors	Market	0.2005	0.0125	0.5748	0.4893
	SMB	0.1846	0.0119	-0.3763	-1.0212
	MGMT1	0.4162	0.0184	1.1284	-1.3355
	PERf1	0.1262	0.0218	-0.1952	-0.0222
	Growth Profitability Premium	0.0725	0.0299	0.9256	-0.7726
q Factors	Market	0.1235	0.0246	0.4001	-1.0900
	ME	0.0991	0.0218	-0.0259	0.5554
	IA	0.4562	0.0233	-1.6101	2.1101
	ROE	0.2687	0.0192	0.2681	-0.1388
	Growth Profitability Premium	0.0525	0.0087	-0.8981	4.1834

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Growth Profitability Premium optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 8: Performance measures. The case of the Investment to Assets anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0052	0.0059	0.0053	0.0052	0.0026	0.0030
SD	0.0236	0.0224	0.0136	0.0133	0.0215	0.0191
Sharpe ratio	0.2092	0.2543	0.3694	0.3750	0.1111	0.1440
D. Sharpe ratio	0.2357	0.3074	0.5410	0.5534	0.1144	0.1496
UP ratio	0.7136	0.7992	0.9760	0.9930	0.4993	0.5480
Return Loss		0.1077%		0.0081%		0.0737%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.0776%		0.0123%		0.1252%
$\alpha = \beta = 0.4$		0.0457%		0.0047%		0.0740%
$\alpha = \beta = 0.6$		0.0451%		0.0010%		0.0406 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.5120	0.1337	0.2717	0.2998
	SMB	0.0	0.0	-	-
	HML	0.1287	0.0545	-0.8572	0.4498
	RMW	0.1094	0.1295	0.9001	-0.3029
	CMA	0.0062	0.0190	3.6493	13.3856
	Investment to Assets	0.2446	0.0919	0.8896	0.2064
M-4 Factors	Market	0.2794	0.1326	2.2679	8.0248
	SMB	0.1143	0.0980	-0.2727	-1.9057
	MGMT1	0.4213	0.0529	-5.1549	38.3569
	PERf1	0.1710	0.0561	-0.9983	2.2962
	Investment to Assets	0.0141	0.0295	2.3963	5.6739
q Factors	Market	0.1239	0.0421	1.6964	3.9373
	ME	0.0	0.0	-	-
	IA	0.0368	0.0716	2.7607	8.3445
	ROE	0.6357	0.0706	-0.1423	1.5461
	Investment to Assets	0.2036	0.0709	0.0894	-1.0312

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Investment to Assets optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 9: Performance measures. The case of the Net Operating Assets anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0060	0.0061	0.0059	0.0061	0.0042	0.0047
SD	0.0205	0.0188	0.0160	0.0160	0.0154	0.0152
Sharpe ratio	0.2806	0.3093	0.3563	0.3658	0.2537	0.2917
D. Sharpe ratio	0.3716	0.4014	0.5355	0.5423	0.3109	0.3878
UP ratio	0.8375	0.8611	0.9866	0.9982	0.7113	0.8193
Return Loss		0.0610%		0.0151%		0.0591%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.1756%		0.1628%		0.1106%
$\alpha = \beta = 0.4$		0.0830%		0.0309%		0.0079%
$\alpha = \beta = 0.6$		0.0364%		0.0035%		0.0199 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.2921	0.2118	0.8839	-0.6001
	SMB	0.0567	0.0634	0.2552	-1.9347
	HML	0.0568	0.0603	1.4862	4.4021
	RMW	0.2455	0.1540	-0.9128	-1.0649
	CMA	0.1765	0.1213	-0.4828	-1.3261
	Net Operating Assets	0.1725	0.1334	1.2569	0.2194
M-4 Factors	Market	0.2276	0.0887	2.2317	5.3869
	SMB	0.1343	0.0752	-1.1034	-0.5756
	MGMT1	0.3563	0.0503	-3.0522	13.5934
	PERf1	0.2036	0.0496	-2.0178	6.6946
	Net Operating Assets	0.0782	0.0521	2.2033	5.9295
q Factors	Market	0.0941	0.0471	3.9241	25.8403
	ME	0.0522	0.0530	0.2838	-1.3734
	IA	0.2319	0.1980	-0.1403	-1.9270
	ROE	0.4707	0.1970	0.3297	-1.0640
	Net Operating Assets	0.1511	0.1005	2.2749	8.2326

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Net Operating Assets optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 10: Performance measures. The case of the O-Score anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0040	0.0034	0.0053	0.0043	0.0033	0.0031
SD	0.0153	0.0117	0.0146	0.0119	0.0138	0.0109
Sharpe ratio	0.2448	0.2738	0.3428	0.3441	0.2220	0.2592
D. Sharpe ratio	0.3256	0.3793	0.5000	0.4989	0.2789	0.3606
UP ratio	0.7155	0.7523	0.9321	0.9137	0.6647	0.7573
Return Loss		0.0518%		-0.0074%		0.0581%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.0941%		-0.0052%		0.0236%
$\alpha = \beta = 0.4$		0.0335%		-0.0033%		0.0096%
$\alpha = \beta = 0.6$		0.0296%		-0.0013%		0.0027 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.1586	0.0887	4.2729	-17.2761
	SMB	0.1505	0.0364	-3.2311	10.9244
	HML	0.1055	0.0262	-0.2659	-0.9961
	RMW	0.1656	0.0478	-1.7096	4.1146
	CMA	0.2679	0.0671	-2.7919	9.2713
	O-Score	0.518	0.0427	2.9879	10.2264
M-4 Factors	Market	0.1808	0.0104	0.0090	-0.9814
	SMB	0.2397	0.0097	-0.0779	-0.5587
	MGMT1	0.3633	0.0082	-1.1236	1.3092
	PERf1	0.0591	0.0223	0.2919	-0.5503
	O-Score	0.1571	0.0224	-0.2357	-1.0662
q Factors	Market	0.1052	0.0197	0.5103	0.4478
	ME	0.1844	0.0301	-1.1632	0.2721
	IA	0.4265	0.0176	-0.6765	-0.7199
	ROE	0.1539	0.0415	0.0814	-1.0066
	O-Score	0.1300	0.0182	-1.2697	1.1189

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios,, as well as the augmented with the O-Score optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 11: Performance measures. The case of the Return on Assets anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0058	0.0048	0.0064	0.0055	0.0050	0.0054
SD	0.0215	0.0186	0.0173	0.0163	0.0226	0.0285
Sharpe ratio	0.2592	0.2434	0.3577	0.3226	0.212	0.1801
D. Sharpe ratio	0.3054	0.2967	0.4874	0.4360	0.2340	0.2291
UP ratio	0.7749	0.7550	0.9438	0.8897	0.6392	0.6968
Return Loss		-0.0302%		-0.0593%		-0.0774%
Opportunity Cost						
$\alpha = \beta = 0.2$		-0.0690%		-0.0281%		-0.0419%
$\alpha = \beta = 0.4$		-0.0301%		-0.0118%		-0.0216%
$\alpha = \beta = 0.6$		-0.0124%		-0.0486%		-0.0096 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.3706	0.0600	0.4847	-0.1630
	SMB	0.0049	0.0139	3.6965	16.9345
	HML	0.2500	0.1110	-1.2613	0.6127
	RMW	0.0	0.0	NA	NA
	CMA	0.0454	0.0624	0.9837	-0.5813
	Return on Assets	0.3290	0.0969	01.1293	0.3935
M-4 Factors	Market	0.2923	0.1026	0.2693	-1.4140
	SMB	0.0956	0.0940	0.0549	-1.9457
	MGMT1	0.3641	0.0814	-2.6871	8.0730
	PERf1	0.0392	0.0421	0.3992	-1.2842
	Return on Assets	0.2088	0.0886	0.9793	0.5493
q Factors	Market	0.1399	0.1881	2.4086	5..7135
	ME	0.0739	0.0662	-0.1287	-1.7956
	IA	0.2614	0.2284	-0.2561	-1.9239
	ROE	0.1370	0.1266	0.5857	0.9076
	Return on Assets	0.3877	0.4080	0.6962	-1.3660

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Return on Assets optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 12: Performance measures. The case of the Net Stock Issues anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0056	0.0055	0.0053	0.0054	0.0033	0.0037
SD	0.0210	0.0193	0.0146	0.0146	0.0142	0.0134
Sharpe ratio	0.2569	0.2741	0.3431	0.3503	0.2124	0.2604
D. Sharpe ratio	0.2991	0.3102	0.5002	0.5156	0.02641	0.3523
UP ratio	0.7848	0.7697	0.9326	0.9443	0.6536	0.7432
Return Loss		0.0382%		0.0107%		0.0696%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.0542%		0.0932%		0.1313%
$\alpha = \beta = 0.4$		0.0617%		0.0275%		0.0893%
$\alpha = \beta = 0.6$		0.0341%		0.0100%		0.0479 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.4957	0.1309	-0.6432	1.4947
	SMB	0.0105	0.0346	3.1457	8.3998
	HML	0.1128	0.0936	0.7579	-0.2461
	RMW	0.0433	0.1105	2.3566	3.8523
	CMA	0.0269	0.0693	2.4315	4.5644
	Net Stock Issues	0.3108	0.1239	-0.9356	-0.1402
M-4 Factors	Market	0.1807	0.0138	0.8270	-0.0001
	SMB	0.2087	0.0165	-0.5718	-0.9720
	MGMT1	0.3833	0.0274	0.2426	-0.2977
	PERf1	0.1758	0.0191	1.2487	0.3269
	Net Stock Issues	0.0515	0.0214	-0.1445	0.8229
q Factors	Market	0.1065	0.0165	0.1272	-0.4359
	ME	0.1424	0.0288	-0.6258	0.2834
	IA	0.3063	0.0159	0.6399	-0.7070
	ROE	0.2878	0.0316	0.6894	0.0121
	Net Stock Issuese	0.1569	0.0253	-0.5046	-1.3702

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Net Stock Issues optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 13: Performance measures. The case of the Betting against Beta anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0048	0.0061	0.0056	0.0060	0.0042	0.0032
SD	0.0315	0.0257	0.0293	0.0271	0.0294	0.0395
Sharpe ratio	0.1451	0.2293	0.1844	0.2126	0.1353	0.0757
D. Sharpe ratio	0.1494	0.2354	0.1980	0.2193	0.1440	0.0742
UP ratio	0.6249	0.6778	0.6715	0.6655	0.5516	0.4792
Return Loss		0.2709%		0.0845%		-0.1813%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.8951%		0.3222%		-0.0184%
$\alpha = \beta = 0.4$		0.4312%		0.1624%		-0.0873%
$\alpha = \beta = 0.6$		0.2052%		0.0774%		-0.0994 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.4927	0.0780	0.4549	-0.5794
	SMB	0.0	0.0	-	-
	HML	0.0	0.0	-	-
	RMW	0.0	0.0	-	-
	CMA	0.0	0.0	-	-
	Betting against Beta	0.5073	0.0780	-0.4549	-0.5794
M-4 Factors	Market	0.5757	0.0963	1.2783	1.3169
	SMB	0.0	0.0	-	-
	MGMT1	0.0005	0.0043	10.5695	117.9659
	PERf1	0.0	0.0	-	-
	Betting against Beta	0.4238	0.0969	-1.2686	1.2466
q Factors	Market	0.0238	0.0521	4.4753	25.9427
	ME	0.0	0.0	-	-
	IA	0.0	0.0	-	-
	ROE	0.2347	0.2990	0.5188	-1.7113
	Betting against Beta	0.7415	0.3211	-0.4558	-1.7923

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Betting against Beta optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 14: Performance measures. The case of the Quality minus Junk anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0065	0.0063	0.0068	0.0070	0.0040	0.0079
SD	0.0228	0.0164	0.0191	0.0167	0.0202	0.0262
Sharpe ratio	0.2750	0.3704	0.3455	0.4087	0.1869	0.2936
D. Sharpe ratio	0.3299	0.4733	0.4712	0.5982	0.1988	0.4115
UP ratio	0.8220	0.9440	0.9099	1.0587	0.5580	0.7862
Return Loss		0.2269%		0.1242%		0.2103%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.6876%		0.1349%		0.8794%
$\alpha = \beta = 0.4$		0.2887%		0.0773%		0.4153%
$\alpha = \beta = 0.6$		0.1242%		0.0397%		0.1996 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.4495	0.0868	-0.4931	0.8726
	SMB	0.0212	0.0549	2.3423	3.76587
	HML	0.1986	0.1008	-0.5960	-0.6281
	RMW	0.0022	0.0093	4.9980	28.6591
	CMA	0.0066	0.0297	4.6831	20.5962
	Quality minus Junk	0.3219	0.0693	-0.0925	-0.8967
M-4 Factors	Market	0.2821	0.1704	1.4413	0.3443
	SMB	0.1818	0.1001	-1.2589	-0.3806
	MGMT1	0.2982	0.1064	-1.8810	2.0838
	PERf1	0.0737	0.0413	-1.1342	-0.5347
	Quality minus Junk	0.1634	0.0734	1.6537	1.4978
q Factors	Market	0.1440	0.1224	2.6661	7.7854
	ME	0.1363	0.0757	-0.9990	-0.6671
	IA	0.3172	0.1598	-1.4410	0.1860
	ROE	0.1558	0.0914	0.0235	1.9922
	Quality minus Junk	0.2467	0.2893	1.9998	2.3764

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Quality minus Junk optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 15: Performance measures. The case of the Size anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0020	0.0020	0.0056	0.0056	0.0025	0.0025
SD	0.0226	0.0226	0.0162	0.0162	0.0168	0.0167
Sharpe ratio	0.0788	0.0785	0.3289	0.3292	0.1327	0.1354
D. Sharpe ratio	0.0716	0.0713	0.4955	0.4963	0.1439	0.1472
UP ratio	0.3715	0.3711	0.9535	0.9545	0.5223	0.5245
Return Loss		-0.0006%		-0.0004%		0.0047%
Opportunity Cost						
$\alpha = \beta = 0.2$		-0.0006%		-0.0002%		0.0059%
$\alpha = \beta = 0.4$		-0.0001%		-0.0002%		0.0050%
$\alpha = \beta = 0.6$		-0.0001%		-0.0001%		0.0032 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.2277	0.2259	2.7209	6.0253
	SMB	0.0842	0.0496	-0.9628	-0.7832
	HML	0.0835	0.0661	2.1073	8.8455
	RMW	0.3452	0.1127	-2.6311	5.2458
	CMA	0.2594	0.0997	-1.7785	2.2821
	Size	0.0	0.0	-	-
M-4 Factors	Market	0.2579	0.1491	3.3910	13.0018
	SMB	0.1403	0.0866	-0.9070	-1.0637
	MGMT1	0.4206	0.0727	-4.6324	24.3892
	PERf1	0.1812	0.0602	-1.4963	3.3174
	Size	0.0	0.0	-	-
q Factors	Market	0.1103	0.0950	7.8161	70.1779
	ME	0.0977	0.0579	-0.9870	-0.8295
	IA	0.3843	0.1322	-2.0914	3.2149
	ROE	0.4077	0.1633	1.8647	3.6192
	Size	0.0	0.0	-	-

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Size optimal portfolio.The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 16: Performance measures. The case of the Growth Option anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0025	0.0025	0.052	0.0052	0.0028	0.0028
SD	0.0220	0.0220	0.0180	0.0181	0.0163	0.0163
Sharpe ratio	0.1018	0.1015	0.2739	0.2736	0.1580	0.1574
D. Sharpe ratio	0.0931	0.0928	0.3128	0.3115	0.1776	0.1767
UP ratio	0.3876	0.3870	0.6702	0.6677	0.5396	0.5385
Return Loss		-0.0007%		-0.0006%		-0.0010%
Opportunity Cost						
$\alpha = \beta = 0.2$		-0.0009%		-0.0010%		-0.0067%
$\alpha = \beta = 0.4$		-0.0007%		-0.0007%		-0.0031%
$\alpha = \beta = 0.6$		-0.0001%		-0.0004%		-0.0015 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.2176	0.1899	2.8779	8.0596
	SMB	0.0853	0.0484	-0.9662	-0.6778
	HML	0.0956	0.0783	1.6905	3.9285
	RMW	0.3445	0.1080	-2.5427	4.9875
	CMA	0.2569	0.0945	-1.6888	2.1760
	Growth Option	0.0	0.0	-	-
M-4 Factors	Market	0.2568	0.1472	3.1545	10.9871
	SMB	0.1451	0.0847	-0.9958	-0.8732
	MGMT1	0.4211	0.0679	-3.9884	19.9554
	PERf1	0.1770	0.0555	-1.5982	3.5359
	Growth Option	0.0	0.0	-	-
q Factors	Market	0.1218	0.1258	6.3636	41.9992
	ME	0.0885	0.0591	-0.6129	-1.4018
	IA	0.3866	0.1326	-2.0673	3.2401
	ROE	0.4031	0.1539	1.4330	3.1234
	Growth Option	0.0	0.0	-	-

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Growth Option optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 17: Performance measures. The case of the Value (Book to Market) anomaly.

	Panel A FF-5	+ anom.	Panel B M-4	+ anom.	Panel C q	+ anom.
Mean	0.0050	0.0129	0.0048	0.0132	0.0072	0.0174
SD	0.0323	0.0454	0.0333	0.0494	0.0385	0.0697
Sharpe ratio	0.1484	0.2780	0.1356	0.2617	0.1817	0.2463
D. Sharpe ratio	0.1517	0.3820	0.1348	0.3300	0.1999	0.3140
UP ratio	0.6275	0.8444	0.5940	0.7677	0.6803	0.7438
Return Loss		0.4114%		0.4127%		0.2379%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.8846%		0.4433%		0.1074%
$\alpha = \beta = 0.4$		0.4954%		0.3685%		0.1854%
$\alpha = \beta = 0.6$		0.2853%		0.2500%		0.1573 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.4059	0.0627	-2.7731	15.2183
	SMB	0.0	0.0	-	-
	HML	0.0	0.0	-	-
	RMW	0.0	0.0	-	-
	CMA	0.0	0.0	-	-
	Value (Book to Market)	0.5941	0.0627	2.7731	15.2183
M-4 Factors	Market	0.3299	0.1740	-0.9762	-0.2593
	SMB	0.0	0.0	-	-
	MGMT1	0.0	0.0	-	-
	PERf1	0.0	0.0	-	-
	Value (Book to Market)	0.6701	0.1740	0.9762	-0.2593
q Factors	Market	0.0	0.0	-	-
	ME	0.0	0.0	-	-
	IA	0.0	0.0	-	-
	ROE	0.0	0.0	-	-
	Value (Book to Market)	1.0	0.0	-	-

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Value (Book to Market) optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 18: Performance measures. The case of the Idiosyncratic Volatility anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0057	0.0057	0.0058	0.0061	0.0031	0.0052
SD	0.0235	0.0230	0.0161	0.0156	0.0205	0.0281
Sharpe ratio	0.2337	0.2365	0.3445	0.3746	0.1379	0.1749
D. Sharpe ratio	0.2753	0.2786	0.4957	0.5692	0.1548	0.1893
UP ratio	0.7260	0.7300	0.9531	1.0274	0.5647	0.6616
Return Loss		0.0072%		0.0492%		0.0695%
Opportunity Cost						
$\alpha = \beta = 0.2$		0.0420%		0.0501%		0.0822%
$\alpha = \beta = 0.4$		0.0093%		0.0387%		0.0631%
$\alpha = \beta = 0.6$		0.0006%		0.0227%		0.0411 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.3635	0.3229	0.8736	-1.0761
	SMB	0.0560	0.0514	-0.1043	-1.8739
	HML	0.0928	0.0586	-0.5338	-0.9200
	RMW	0.2705	0.1686	-0.9693	-1.2245
	CMA	0.1714	0.1139	-0.6073	-1.2245
	Idiosyncratic Volatility	0.0458	0.0434	1.2932	0.1275
M-4 Factors	Market	0.3078	0.0804	3.4567	26.7380
	SMB	0.0238	0.0605	2.4099	4.1298
	MGMT1	0.4340	0.0514	-2.9292	21.4974
	PERf1	0.2011	0.0380	-2.3557	9.2464
	Idiosyncratic Volatility	0.0333	0.0220	0.1900	-1.3584
q Factors	Market	0.2753	0.1411	1.7242	7.850
	ME	0.0049	0.0148	4.4966	25.9102
	IA	0.2262	0.1673	-0,5589	-1.5864
	ROE	0.2005	0.1486	1.3694	1.6745
	Idiosyncratic Volatility	0.2930	0.2479	2.1754	3.3433

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Idiosyncratic Volatility optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

Table 19: Performance measures. The case of the Profitability anomaly.

	Panel A		Panel B		Panel C	
	FF-5	+ anom.	M-4	+ anom.	q	+ anom.
Mean	0.0042	0.0040	0.0054	0.0047	0.0030	0.0027
SD	0.0154	0.0156	0.0147	0.0148	0.0151	0.0157
Sharpe ratio	0.2581	0.2408	0.3506	0.3036	0.1848	0.1550
D. Sharpe ratio	0.3511	0.3132	0.5152	0.4193	0.2221	0.1791
UP ratio	0.7410	0.6963	0.9476	0.8310	0.6108	0.5611
Return Loss		-0.0027%		-0.0692%		-0.2254%
Opportunity Cost						
$\alpha = \beta = 0.2$		-0.0095%		-0.0943%		-0.2550%
$\alpha = \beta = 0.4$		-0.0034%		-0.0412%		-0.1119%
$\alpha = \beta = 0.6$		-0.0016%		-0.0230%		-0.0865 %

Descriptive statistics of the weight allocation of the optimal portfolios

		Mean	Std. Dev.	Skewness	Kurtosis
FF-5 Factors	Market	0.1657	0.0955	7.3826	58.3022
	SMB	0.0976	0.0220	-3.0034	10.3874
	HML	0.1029	0.0642	0.0701	-0.7638
	RMW	0.3614	0.0518	-5.4348	34.7300
	CMA	0.2641	0.0620	-1.2678	4.2188
	Profitability	0.0083	0.0058	-0.1242	-1.1423
M-4 Factors	Market	0.1851	0.0143	0.7369	-0.8036
	SMB	0.1957	0.0104	-0.8692	0.4101
	MGMT1	0.4073	0.0113	-1.3624	0.7675
	PERf1	0.1792	0.0158	1.2205	1.7524
	Profitability	0.0327	0.0144	0.0186	-1.4605
q Factors	Market	0.1028	0.1107	7.4578	58.2318
	ME	0.1281	0.0404	-2.4151	5.2278
	IA	0.4141	0.0968	-3.6550	12.9320
	ROE	0.3327	0.1116	2.8444	13.5653
	Profitability	0.0222	0.0183	0.2704	-1.5003

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio, Returns Loss and Opportunity Cost) for the factor optimal portfolios, as well as the augmented with the Profitability optimal portfolio. The dataset spans the period from January, 1999 to December, 2016. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. In the second half, the Table exhibits the descriptive statistics of the weight allocation of the optimal augmented portfolios.

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