Adenza

Calypso Commodities Analytics

Version 17.0

February 2022 – First Edition

Welcome to the Calypso Commodities Analytics guide which aims to provide an understanding of the analytics that underlie the Calypso Commodities pricing capability.

Revision Date	Comments
February 2022	First edition for version 17.0.

Contents

Section 1. Overview

1.1 Products and Pricers

The following pricers and models are available for pricing Commodities. Details of the models are described in subsequent chapters.

Pricer	Product Type supported	Valuation Model
PricerCommodityForward	CommodityForward	Deterministic
PricerCommoditySwap2	CommoditySwap2	Deterministic
PricerCommodity	Commodity	Deterministic
PricerCommodityCertificate	CommodityCertificate	Deterministic
PricerCommodityIndexSwap	CommodityIndexSwap	Deterministic
PricerFutureOptionCommodity	FutureOption (Commodity)	Black 76
PricerCommodityOTCOption2Clewlow	CommodityOTCOption2	Black 76 (Clewlow approximation for Asian)
PricerCommodityOTCOption2LTBlack	CommodityOTCOption2	Black 76 (Levy-Turnbull Asian Volatility Approximation)
PricerCommoditySwaption2	CommoditySwap	Black 76

1.2 Market Data Generators

The following market data item generators are available for pricing Commodities. Details of the generators are described in subsequent chapters.

MarketDataItem	Generator	Description
CurveCommodity	Commodity	Generates output in terms of implied future prices.
CurveCommodity	CommoditySwap	Generates output in terms of implied Monthly Asian swap prices.

MarketDataItem	Generator	Description
CurveCommoditySpread	CommoditySpreadAllInPoints	Generates output by adding a spread value to the output points of the base curve.
CurveCommoditySpread	MonthlyAsianSwapSpread	Generates output by first calculating implied Asian swap prices and then adding a spread value to the swap price.
ConvenienceYieldCurve	ConvenienceYield	Used to generate a 3-D pricing surface with axes of price, date and hour.
HyperSurface	CommodityElectricity	Generates a volatility surface using input points defined as the implied volatility of ATM future options and spreads to the ATM for the wings.
VolatilitySurface	CommodityDelta	Generates output by combining a commodity spot price, a commodity convenience yield curve and a zero curve in the quoting currency of the commodity.
VolatilitySurface	CommodityVolatilitySpread	Generates a volatility surface by adding a spread value to the output points of a commodity volatility surface.
VolatilitySurface	CommoditySimple	Generates a simple volatility surface using absolute or relative strikes and manually entered expiry tenors

Section 2. Commodity Curves

Calypso users are able to create forward curves using standard generation techniques, customise interpolation methods, create expiry calendars and are easily able to add new commodities and market conventions.

2.1 Forward Commodity Curve

The commodity forward curve analytics are accessed from *Main Entry > Market Data > Commodity Curves > Forward Curve.*

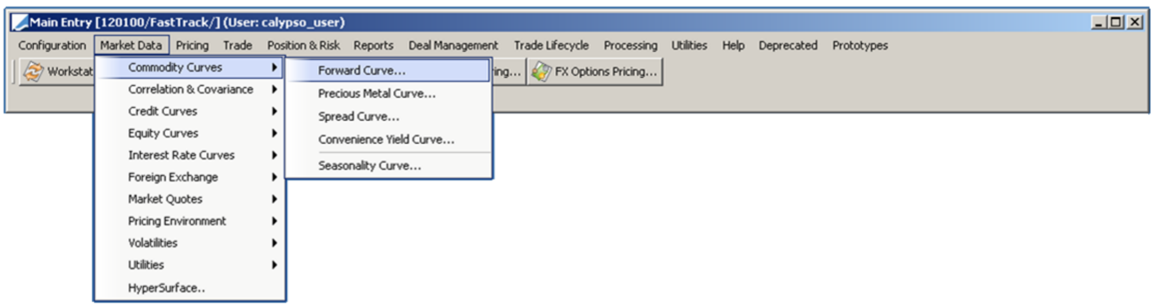


Figure 1 – Opening forward curve analytics

2.1.1 Selecting, Creating and Editing commodities

When the commodity forward curve window is opened the user is presented with the window shown in Figure 2. Clicking the ... button allows the user to select the commodity for which a forward curve is to be generated. Through FastTrack database setup there is a comprehensive list of commodity details pre-installed. All of the pre-loaded details can be edited and added to through the commodity editor shown in Figure 3 – Commodity editor.

Selecting *File > New* clears all fields so that a new commodity can be entered, to create a new commodity name click the ... button, enter the name in the left entry box, then pass it to the right by clicking >>, to save. Upon returning to the commodity selection box the created commodity is selectable from the drop down list.

The Settings section creates the attributes for the named commodity. Figure 3 – Commodity editor shows the details for WTI Crude Oil, which is stockpiled in Cushing, OK, measured in Barrels and sold through the NYMEX. For commodities that might be quoted in different units by either counterparty Calypso allows entry of conversions that when priced will display in your conventional unit. An example might be a trade where one counterparty trades Cotton in MTonnes while the other would prefer Pounds, to accommodate this the user can enter commodity conversions by going through *Main Entry > Configuration > Commodities > Commodity Conversion...*, see Figure 4 - Unit conversion. The effect of this would be that if we were to price our futures in $/US Gallon rather than $/Barrel, our forward curve points would be a factor of 42 smaller as shown in Figure 5 - Curve points shown in $/US Gallon.

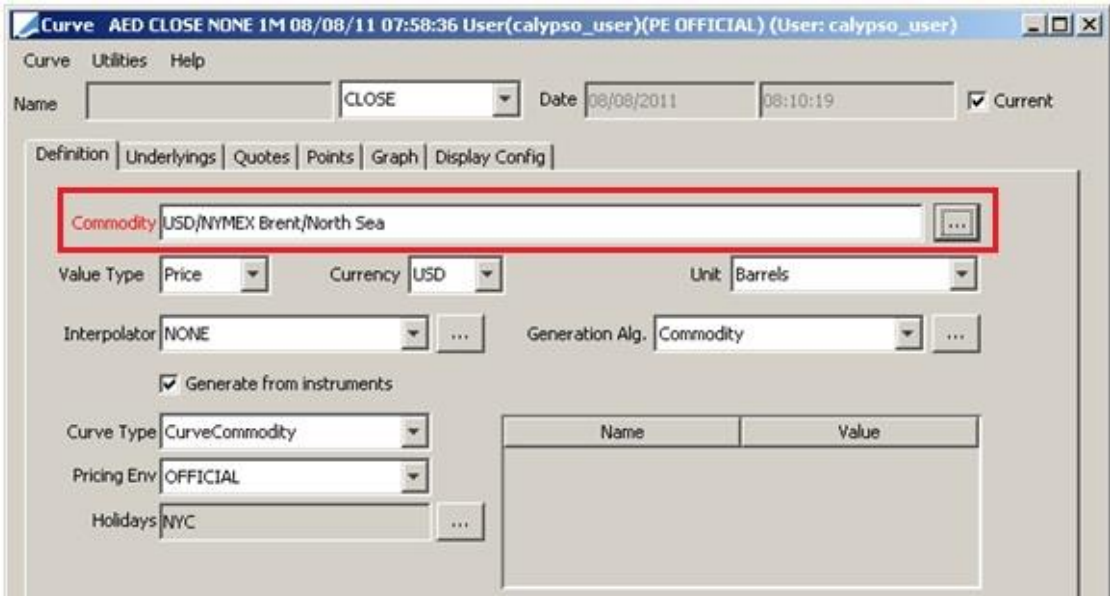


Figure 2 - Commodity forward curve definition tab

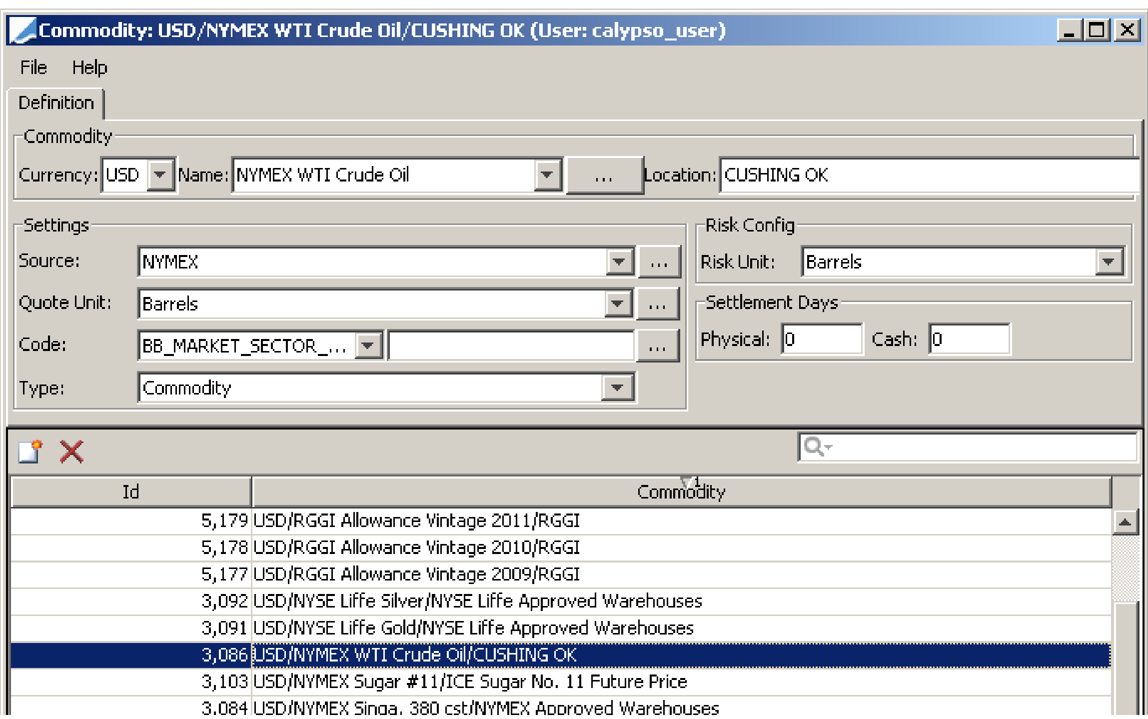


Figure 3 – Commodity editor

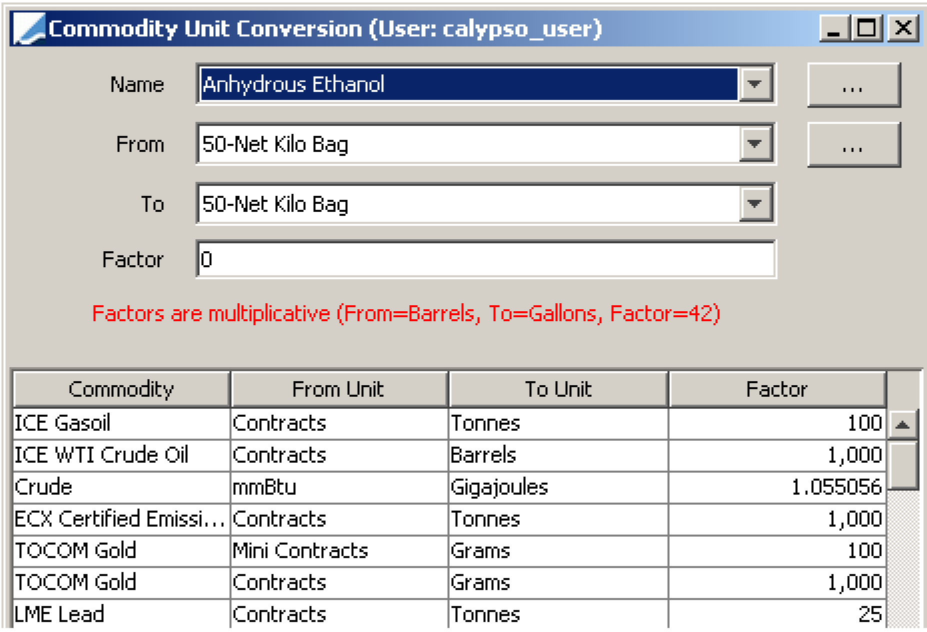


Figure 4 - Unit conversion

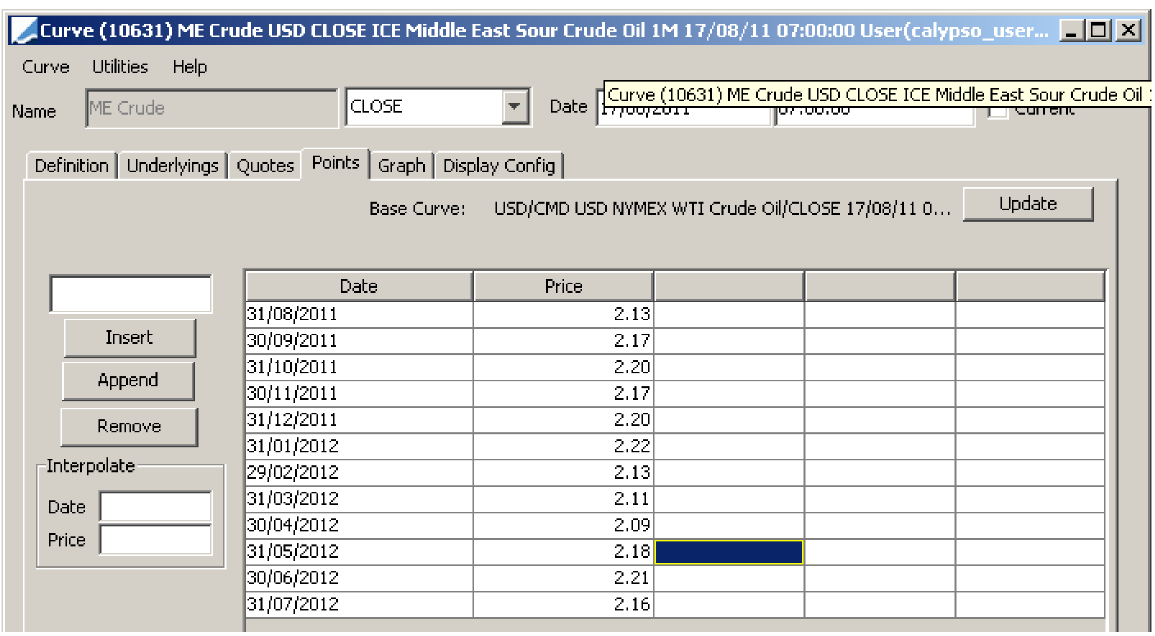


Figure 5 - Curve points shown in $/US Gallon

2.1.2 Commodity Generator

The "Commodity" forward curve generation algorithm is our most simple generator. The underlying's supported in this generator (in v11.1.04) are Futures, Commodity Forward Points, Monthly Asian Swaps, Future Calendar Spreads, Swap Roll Spreads and Spot. The generator places these underlying in three categories in order to generate output.

Some underlying's are used to generate output as a simple function of input date and price attributes. This function simply finds the curve date and the market quote of the underlying and generates an output point with that date and price. This logic applies to Future, Commodity Forward Point and Monthly Asian Swap underlying's.

Some underlying's represent the relative prices of two chronologically adjacent underlying's. These are referred to as Calendar Spreads or Swap Rolls, and represent a price spread between two future or swap underlying's.

2.1.3 CommoditySwap Generator

CommoditySwap generator calculates the average price ***P*** of a commodity over a period, ***T*** that is based upon underlying quotes $\boldsymbol{Q_i}$, which are valid for ***n*** sub period's of length $\boldsymbol{\tau_i}$. The generic equation for this is

$$P = \frac{\sum_{i=1}^{n} Q_i \tau_i}{T}$$

Equation 1

The period ***P*** might be a calendar month for commodity swaps that do not expire at the end of a calendar month and inside that calendar month if there is one expiry there are two distinct sub periods that are bounded by the expiry and the beginning and end of the month respectively. The quotes $\boldsymbol{Q_1}$ and $\boldsymbol{Q_2}$ are set based upon the forward price method selected (in this case WTI Nearby).

A simple example of this is the WTI Crude Oil curve. Using the Nearby forward price method the curve may be constructed from Futures quotes for monthly crude swap quotes on their expiry dates, which are the third business day prior to the last business day preceding the twenty-fifth calendar day (CME Group, 2011).

Taking the example of a WTI crude price calculation for September 2011, quotes are taken from September and October Futures, if these are $99 for September and $100 for October we would obtain an August 31st value of $99.3 (since we are averaging over the month of August) based upon formula.

An example of the Quotes and their calculated points, as displayed in Calypso, are shown in Figure 6 - Selected quotes for WTI Crude Oil Futures and Figure 7 - Points calculated from Equation 1.

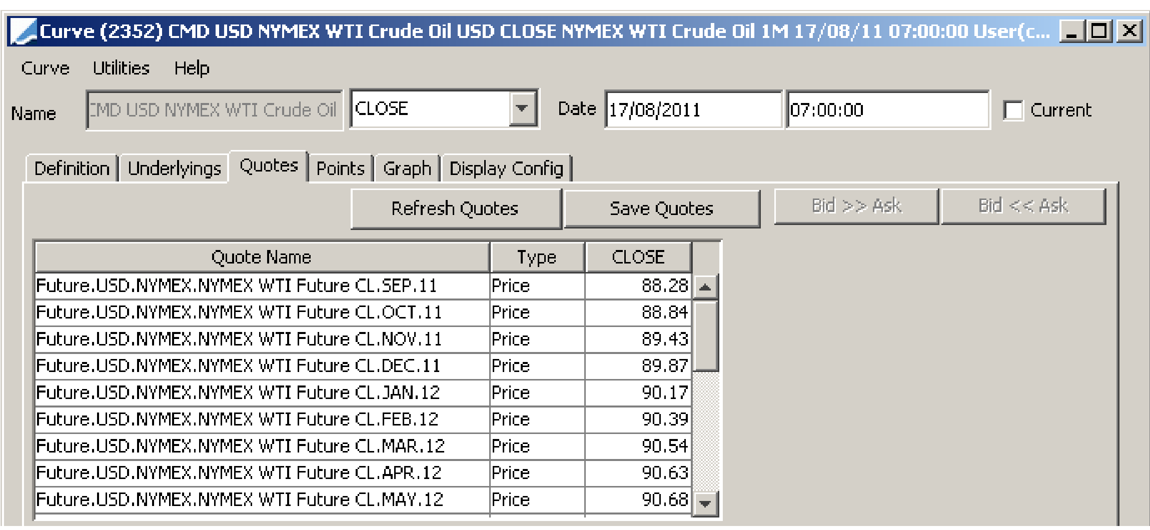


Figure 6 - Selected quotes for WTI Crude Oil Futures

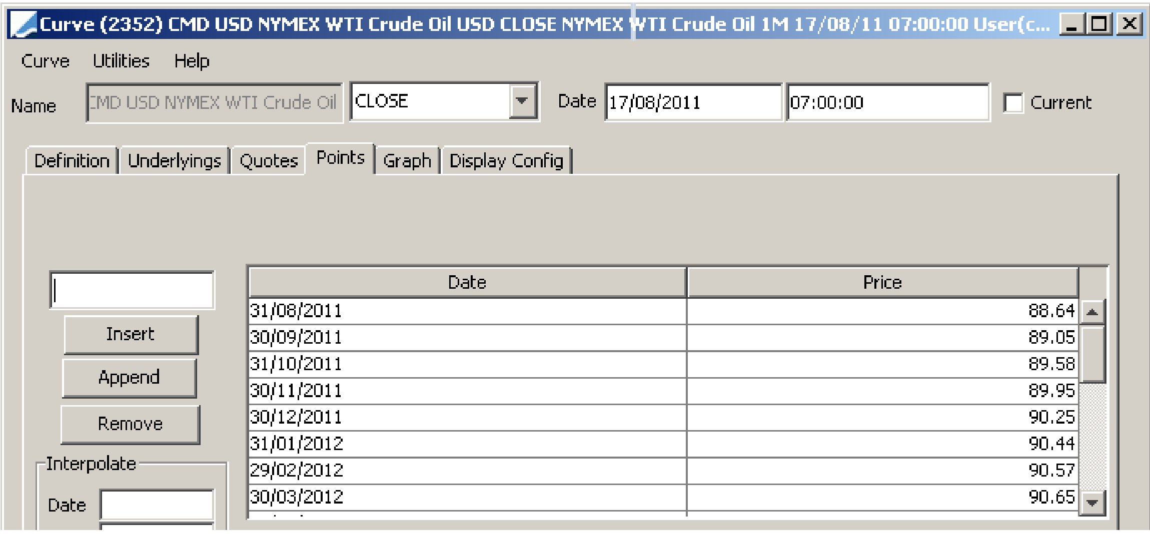


Figure 7 - Points calculated from Equation 1

2.2 Commodity Spread Curve

The commodity spread curve generation window is opened from *Main Entry > Market Data > Commodity Curves > Spread Curve.*

Spread curves are used to price illiquid commodities that are quoted in terms of spread to a more commonly traded commodity, or spreads, between the same commodity at different expiries.

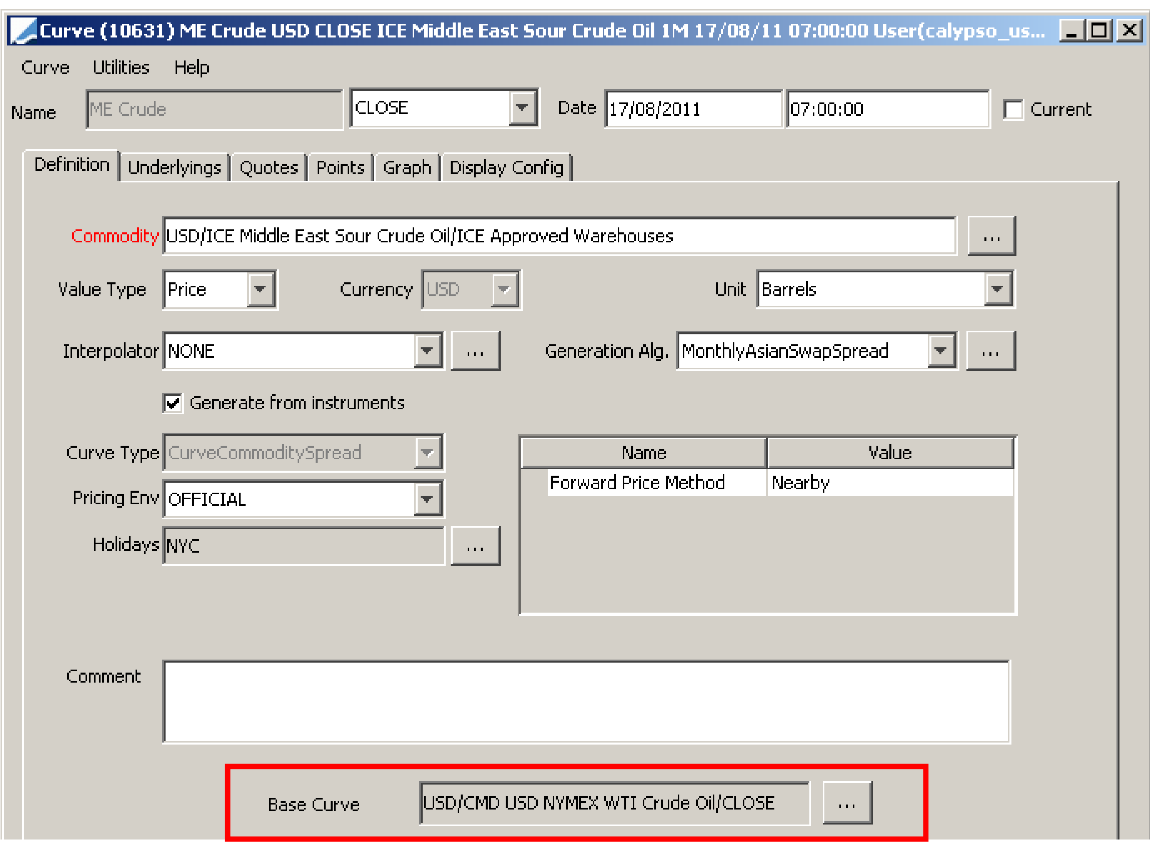


Figure 8 – Commodity spread curve definition tab

2.2.1 MonthlyAsianSwapSpread Generator

The Monthly Asian Swap Spread curve generator is available from the spread curve window when the Generate from instruments tick box is selected.

The underlying instruments are selected in the Underlying's tab and can be either Commodity Spreads or Commodity Forward Points. When creating the curve from commodity spreads the spreads are simply added onto the base commodity price that is calculated from Equation 1 in PricerCommoditySwap. So that the Spread points can be shown to be:

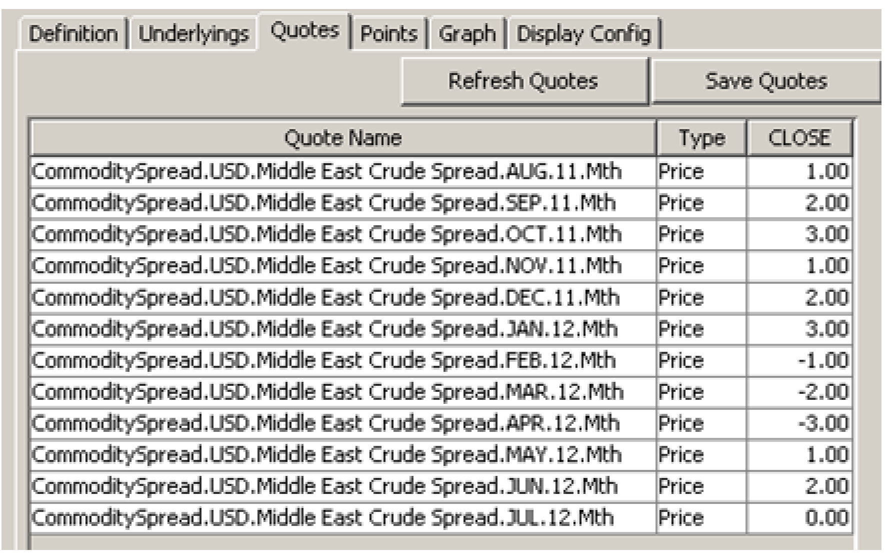
$$P_{Spread} = P + Spread$$

Equation 2

The price of Middle East Sour is often quoted as a spread against WTI, or Brent, which means that a Middle East Sour curve has to be created from the more liquid commodity curves. Calypso allows this through the introduction of a base curve on the Spread window. Figure 8 shows the Calypso definition tab for this example. The Generation Alg. dropdown allows for the selection of MonthlyAsianSwapSpread.

Clicking on the 'Underlyings' tab gives a choice CommoditySpread or CommodityFwdPoint, either of which can be used to create the spread curve. In this example 12 month of Middle East Crude Sour Oil spread underlying have been created and quotes have been entered in the usual way; either the quotes tab or through *Market Data > Market Quotes*.

The 'Points' tab shows the calculated prices; see Figure 10 – Spread curve calculated points. Note the relationship between the points calculated in Figure 7 - Points calculated from Equation 1, Figure 9 - Spread quotes and Figure 10 – Spread curve calculated points.



Quote Name	Type	CLOSE
CommoditySpread.USD.Middle East Crude Spread.AUG.11.Mth	Price	1.00
CommoditySpread.USD.Middle East Crude Spread.SEP.11.Mth	Price	2.00
CommoditySpread.USD.Middle East Crude Spread.OCT.11.Mth	Price	3.00
CommoditySpread.USD.Middle East Crude Spread.NOV.11.Mth	Price	1.00
CommoditySpread.USD.Middle East Crude Spread.DEC.11.Mth	Price	2.00
CommoditySpread.USD.Middle East Crude Spread.JAN.12.Mth	Price	3.00
CommoditySpread.USD.Middle East Crude Spread.FEB.12.Mth	Price	-1.00
CommoditySpread.USD.Middle East Crude Spread.MAR.12.Mth	Price	-2.00
CommoditySpread.USD.Middle East Crude Spread.APR.12.Mth	Price	-3.00
CommoditySpread.USD.Middle East Crude Spread.MAY.12.Mth	Price	1.00
CommoditySpread.USD.Middle East Crude Spread.JUN.12.Mth	Price	2.00
CommoditySpread.USD.Middle East Crude Spread.JUL.12.Mth	Price	0.00

Figure 9 - Spread quotes

Figure 10 – Spread curve calculated points

2.3 Bibliography

CME Group. (2011). *Light Sweet Crude contract specifications*. Retrieved from www.cmegroup.com: http://www.cmegroup.com/trading/energy/crude-oil/light-sweet-crude_contract_specifications.html.

Section 3. Pricers

3.1 PricerCommodityCertificate

Price (CertificateCommodity) = (CommodityMarketPrice – storageCost – gradeDiff – locationDiff - fees) * df –AccrualStorageCost.

3.2 PricerCommodityOTCOption2Clewlow

In this section we derive the equations in the paper ***A Model for Metal Exchange Average Price Option Contracts*** by Les Clewlow, August 1997. The paper specifies a pricing and risk management model for London Metal Exchange (LME) Average Price Option Contracts. These LME options are Asian options of European exercise type on the Monthly Average Settlement Price (MASP).

Introduction

We assume that he asset price at time t given by S_t follows a geometric Brownian motion with drift μ and volatility σ. The stochastic differential equation of S_t is

$$dS_t = \mu S_t dt + \sigma S_t d\bar{Z}_t \, , \tag{0.1}$$

where $d\bar{Z}_t$ is the increment in the Brownian motion. The value of $\ln S_T$ for some time $T > t$ has a normal distribution $N(.,.)$,

$$\ln S_T \sim N(a,b) \tag{0.2}$$

where the mean $\bar{a}$ and variance $\bar{b}$ are respectively given by

$$\begin{aligned} \bar{a} &= \ln S_t + (\mu - \tfrac{1}{2}\sigma^2)(T-t) \\ \bar{b} &= \sigma^2 (T-t) \end{aligned} \tag{0.3}$$

If the asset pays a continuous dividend yield d, the process of the asset Eqn. (1.1) becomes

$$dS_t = (\mu - d) S_t dt + \sigma S_t d\bar{Z}_t \, . \tag{0.4}$$

In the risk neutral world the process for S_t becomes

$$dS_t = (r - d) S_t dt + \sigma S_t dZ_t \tag{0.5}$$

where the real world drift μ is replaced by the risk-free rate r, and dZ_t is the Brownian increment in the risk-neutral world. The mean and variance in the risk-neutral world are

$$\begin{aligned} a &= \ln S_t + (r - \tfrac{1}{2}\sigma^2)(T-t) \\ b &= \sigma^2 (T-t) \end{aligned} \tag{0.6}$$

Note that the volatility remains unchanged so that the variance remains the same in the new measure.

Standard Black-Scholes theory gives us that the price of a European call option on the asset S_t with payoff function, $\max(S_T - K, 0)$ at expiry T is

$$C(S_t, t) = e^{-r(T-t)}\left(e^{a+\frac{1}{2}b}N(d_1) - KN(d_2)\right) \tag{0.7}$$

where K is the strike of the option and the arguments to the cumulative normal $N(.)$ are

$$\begin{aligned} d_1 &= \frac{a - \ln K + b}{\sqrt{b}}, \\ d_2 &= d_1 - \sqrt{b} \end{aligned} \tag{0.8}$$

Within the Black-Scholes framework the forward price at time t, $F_t(T)$ of the asset is simply related to the spot price S_t by

$$F_t(T) = E\left[S_T\right] = S_t e^{(r-d)(T-t)}, \tag{0.9}$$

where the expectation $E[.]$ is taken in the risk-neutral measure.

Let $S_1, S_2, \ldots, S_n$ be the price of the asset at different times up to the expiry time T. We define the arithmetic and geometric averages as follows,

Arithmetic Average:

$$A_T = \frac{1}{n}\sum_{k=1}^{n} S_k \tag{0.10}$$

Geometric Average:

$$G_T = \left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}}. \tag{0.11}$$

Using Jensen's inequality it can be shown that

$$G_T \le A_T \tag{0.12}$$

where the equality holds only when all the asset price observations assume the same values $S_1 = S_2 = \cdots = S_n$.

A nice property of the geometric average of asset prices is that the dynamics of this type of average is also log-normally distributed. It follows that we can evaluate a call option on the underlying G_T with payoff $\max(G_T - K, 0)$ within the standard Black-Scholes framework. On the other hand the arithmetic average A_T does not have a simple distribution – so in order to arrive at an analytical solution for the call option on A_T we have to resort to some approximation. The essence of the Clewlow model lies in approximating the distribution of A_T by the log-normal distribution. Simulation studies indicate that this is a reasonable approximation if the volatility σ is not too high (say $< 40\%$).

Put-Call Parity

Independent of the dynamics assumed for the underlying, no-arbitrage conditions require that put-call parity holds. Denote the price of a call and put option on the arithmetic average by $C_A(t)$ and $P_A(t)$ respectively. The strike and expiry are the same for the call and the put. Consider the portfolio where we go long a call option and short a put option. At expiry we have two states of the world; either $A_T \geq K$ or $A_T < K$. It is easy to see that the portfolio gives identical payoffs in either state of the world –

$$C_A(T) - P_A(t) = \begin{cases} (A_T - K) + 0, & A_T \geq K \\ 0 - (K - A_T), & A_T < K \end{cases} \qquad (0.13)$$

no matter what happens we have the *same* payoff:

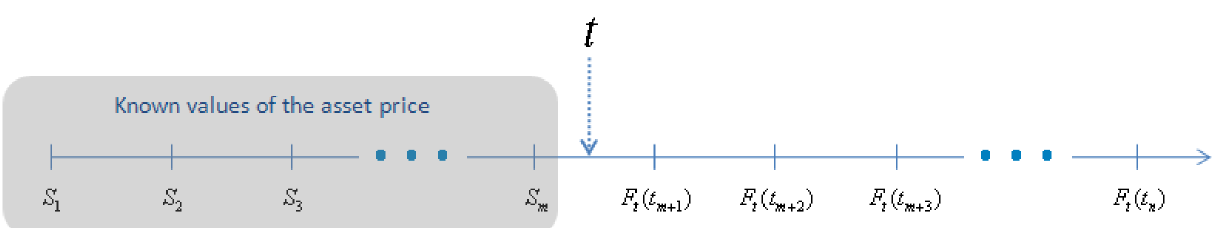
$$C_A(T) - P_A(t) = A_T - K \qquad (0.14)$$

What would one pay for a certain payoff at time T at an earlier time t? We would by the no-arbitrage principle simply pay its discounted value at the risk-free rate. It follows that:

$$C_A(t) - P_A(t) = \exp\left[-r(T-t)\right]\left(\frac{1}{n}\left[\sum_{k=1}^{m} S_i + \sum_{k=m+1}^{n} F_t(t_k)\right] - K\right), \qquad (0.15)$$

where the forward prices are given by Eqn. (1.9).

At the earlier time t, out of n observations only the first m are known.



Model in Detail

The payoff at expiry of an average price call option (APO call) is given by

$$\max(A_T - K, 0) \qquad (0.16)$$

In what follows it will be convenient to separate out at any given time t, the known part of the asset price average. If at time t we have that $S_1, S_2, \ldots, S_m$ are known then we let

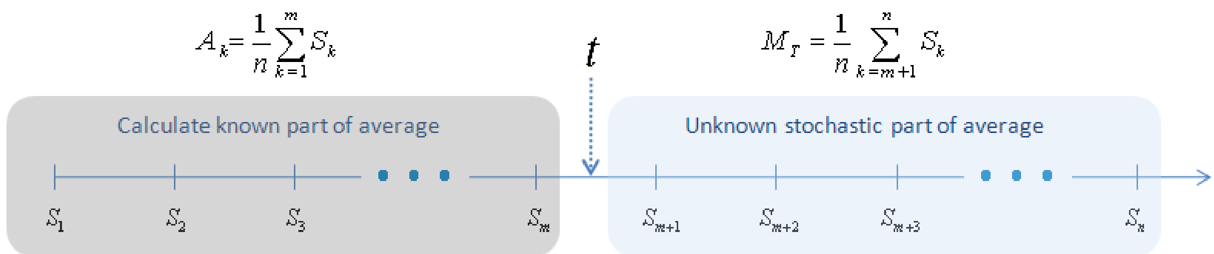
$$A_k = \frac{1}{n}\sum_{k=1}^{m} S_k$$
$$M_T = \frac{1}{n}\sum_{k=m+1}^{n} S_k \tag{0.17}$$

where the known part of the average is A_k and the unknown stochastic component of the average is M_T. We can express the payoff at expiry in terms of a strike that is adjusted by the known component of the average,

$$\max(M_T - \tilde{K}, 0) \tag{0.18}$$

where

$$\tilde{K} = K - A_k\,. \tag{0.19}$$



Now we apply the key assumption of the Clewlow model - we assume that M_T is log-normally distributed,

$$\ln M_T \sim N(a,b)\,. \tag{0.20}$$

A lognormal variable such as M_T has moments given by

$$E\left[M_T^k\right] = \exp\left(ka + k^2\frac{b}{2}\right) \tag{0.21}$$

A derivation of the lognormal moments is given in ***Appendix A***.

Using Eqn. (1.21) we have that,

$$\ln E\left[M_T\right] = a + \tfrac{1}{2}b \tag{0.22}$$

$$\ln E\left[M_T^2\right] = 2a + 2b \tag{0.23}$$

We have a system of two equations in two unknowns a and b. Solving for the unknowns we arrive at,

$$a = 2\ln E\left[M_T\right] - \tfrac{1}{2}\ln E\left[M_T^2\right] \tag{0.24}$$

$$b = -2\ln E\left[M_T\right] + \ln E\left[M_T^2\right] \tag{0.25}$$

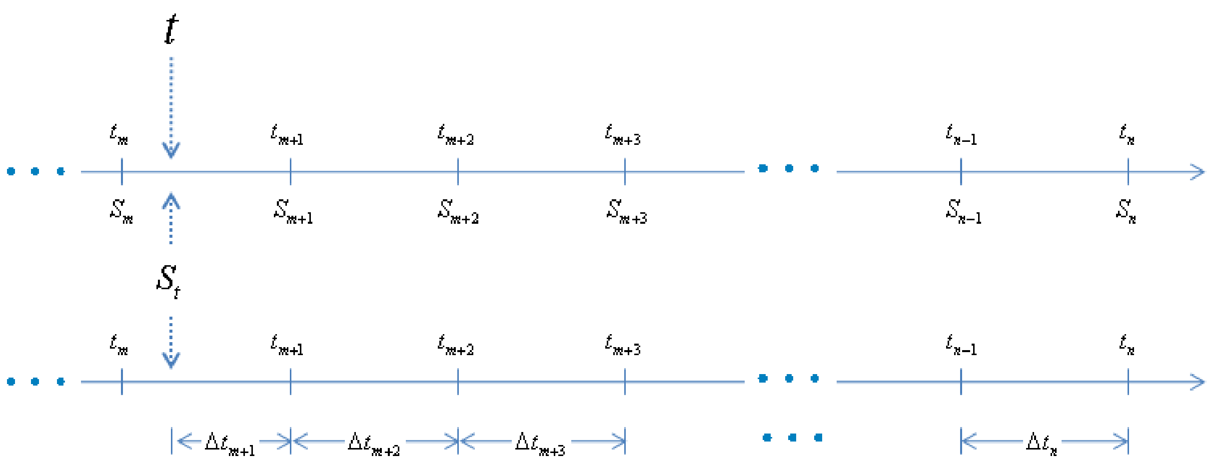
Now we proceed to calculate the first two moments of the stochastic component of the arithmetic average

$$M_T^2 = \frac{1}{n}\left(S_{m+1} + S_{m+2} + \cdots + S_n\right) \tag{0.26}$$

We can rewrite this as

$$M_T^2 = \frac{1}{n}\left(S_t x_{m+1} + S_t x_{m+1} x_{m+2} + \cdots + S_t x_{m+1} x_{m+2} x_{m+3} \cdots x_n\right) \tag{0.27}$$

where each x_i evolves the initial asset price forward over a time interval Δt_i.



$$x_i = \exp((r-d) - \tfrac{1}{2}\sigma^2)\Delta t_i + \sigma\sqrt{\Delta t_i}\, z_i \tag{0.28}$$

where z_i is a normal variate and z_i is independent of z_j if $i \neq j$. It follows that the x_i's are independent, log-normally distributed random variables. Using the results from ***Appendix A*** with $a = (r-d)\Delta t_i$ and $b = \sigma \Delta t_i$ we have that the first two moments of x_i are

$$E\left[x_i\right] = \exp\left((r-d)\Delta t_i\right) \tag{0.29}$$

and

$$E\left[x_i^2\right] = \exp\left((2(r-d)+\sigma^2)\Delta t_i\right). \tag{0.30}$$

Taking the expectation of Eqn. (1.27) we obtain

$$\begin{aligned} E\left[M_T\right] &= \frac{S_t}{n}\left(E\left[x_{m+1}\right] + E\left[x_{m+1}\right]E\left[x_{m+2}\right] + \cdots + E\left[x_{m+1}\right]E\left[x_{m+2}\right]\cdots E\left[x_n\right]\right) \\ &= \frac{S_t}{n}\sum_{k=m+1}^{n} E\left[x_{m+1}\right]\cdots E\left[x_k\right] \end{aligned} \tag{0.31}$$

Substituting for the expectation using Eqn. (1.29) we have that

$$
\begin{aligned}
E\left[M_T\right] &= \frac{S_t}{n}\Big(\exp\big((r-d)\Delta t_{m+1}\big)+\exp\big((r-d)\cdot(\Delta t_{m+1}+\Delta t_{m+2})\big)+\cdots \\
&\qquad +\exp\big((r-d)\cdot(\Delta t_{m+1}+\Delta t_{m+2}+\Delta t_{m+3}+\cdots+\Delta t_n)\big)\Big) \\
&= \frac{S_t}{n}\sum_{k=m+1}^{n}\exp\left((r-d)\cdot\sum_{j=m+1}^{k}\Delta t_j\right).
\end{aligned}
\tag{0.32}
$$

Taking the square of Eqn. (1.27) we have,

$$
\begin{aligned}
M_T^2 = \left(\frac{S_t}{n}\right)^2 x_{m+1}^2\big(1+x_{m+2}+x_{m+2}x_{m+3}+x_{m+2}x_{m+3}x_{m+4}+\cdots \\
+x_{m+2}x_{m+3}x_{m+4}x_{m+5}\cdots x_n\big)^2
\end{aligned}
\tag{0.33}
$$

The expectation is given by

$$
\begin{aligned}
E\left[M_T^2\right] = \left(\frac{S_t}{n}\right)^2 E\left[x_{m+1}^2\right]E\Big[\big(1+x_{m+2}+x_{m+2}x_{m+3}+x_{m+2}x_{m+3}x_{m+4}+\cdots \\
+x_{m+2}x_{m+3}x_{m+4}x_{m+5}\cdots x_n\big)^2\Big]
\end{aligned}
\tag{0.34}
$$

Substituting for $E\left[x_{m+1}^2\right]$ we obtain

$$
\begin{aligned}
E\left[M_T^2\right] = \left(\frac{S_t}{n}\right)^2 \exp\big((2(r-d)+\sigma^2)\Delta t_{m+1}\big)\times \\
E\Big[\big(1+x_{m+2}+x_{m+2}x_{m+3}+\cdots+x_{m+2}x_{m+3}x_{m+4}\cdots x_n\big)^2\Big]
\end{aligned}
\tag{0.35}
$$

To proceed further the Clewlow model makes another simplifying assumption. The interest rates, dividend yields, volatilities, and time intervals are taken to be the same except for the next nearest fixing at t_{m+1}. This allows the derivation of analytical expressions for $E\left[M_T\right]$ and $E\left[M_T^2\right]$.

In Eqn. (1.32) we separate out the factor depending on Δt_{m+1} and set all remaining time intervals other than Δt_{m+1} to the uniform value Δt. Eqn. (1.32) becomes

$$
\begin{aligned}
E\left[M_T\right] &= \frac{S_t}{n}\exp\big((r-d)\Delta t_{m+1}\big)\sum_{k=m+2}^{n}\exp\left((r-d)\cdot\Delta t\sum_{j=m+2}^{k}1\right) \\
&= \frac{S_t}{n}\exp\big((r-d)\Delta t_{m+1}\big)\times \\
&\qquad \Big\{\exp\big((r-d)\Delta t\big)+\exp\big(2(r-d)\Delta t\big)+\cdots+\exp\big((n-m-1)(r-d)\Delta t\big)\Big\}
\end{aligned}
\tag{0.36}
$$

Recognizing that the sum in brackets is a geometric sum of the form

$$c+cr+cr^2+\cdots+cr^n=c\frac{1-r^{n+1}}{1-r} \tag{0.37}$$

it follows that

$$E\left[M_T\right]=\frac{S_t}{n}\exp\left((r-d)\Delta t_{m+1}\right)\left(\frac{\exp\left((r-d)(n-m)\Delta t\right)-1}{\exp\left((r-d)\Delta t\right)-1}\right). \tag{0.38}$$

The evaluation of $E\left[M_T^2\right]$ is an extremely tedious exercise. We start by carrying out the multinomial expansion of the bracketed squared term in Eqn. (1.35) as follows:

$$\begin{aligned}(1+x_{m+2}+x_{m+2}x_{m+3}+\cdots+x_{m+2}x_{m+3}\ldots x_n)^2 &= \\ 1+\sum_{i=m+2}^{n}x_{m+2}^2\cdots x_i^2+2\sum_{i=m+2}^{n}x_{m+2}\cdots x_i+2x_{m+2}^2\sum_{i=m+3}^{n}x_{m+3}\cdots x_i+\cdots \\ \cdots+2x_{m+2}^2\cdots x_{n-1}^2\sum_{i=n}^{n}x_i\end{aligned} \tag{0.39}$$

The expansion can be "visualized" as follows. The first two terms in the expansion in Eqn. (1.39) $1+\sum_{i=m+2}^{n}x_{m+2}^2\cdots x_i^2$ arise from the products between the terms illustrated below:

$$1+x_{m+2}+x_{m+2}x_{m+3}+x_{m+2}x_{m+3}x_{m+4}+\cdots+x_{m+2}x_{m+3}x_{m+4}\cdots x_{n-1}+x_{m+2}x_{m+3}x_{m+4}\cdots x_n$$

$$1+x_{m+2}+x_{m+2}x_{m+3}+x_{m+2}x_{m+3}x_{m+4}+\cdots+x_{m+2}x_{m+3}x_{m+4}\cdots x_{n-1}+x_{m+2}x_{m+3}x_{m+4}\cdots x_n$$

The next term $2\sum_{i=m+2}^{n}x_{m+2}\cdots x_i$ arises from products of the form:

$$1+x_{m+2}+x_{m+2}x_{m+3}+x_{m+2}x_{m+3}x_{m+4}+\cdots+x_{m+2}x_{m+3}x_{m+4}\cdots x_{n-1}+x_{m+2}x_{m+3}x_{m+4}\cdots x_n$$

$$1+x_{m+2}+x_{m+2}x_{m+3}+x_{m+2}x_{m+3}x_{m+4}+\cdots+x_{m+2}x_{m+3}x_{m+4}\cdots x_{n-1}+x_{m+2}x_{m+3}x_{m+4}\cdots x_n$$

There is a degeneracy of two since products of the form below contribute an identical sum:

$$1+x_{m+2}+x_{m+2}x_{m+3}+x_{m+2}x_{m+3}x_{m+4}\cdots+x_{m+2}x_{m+3}x_{m+4}\cdots x_{n-1}+x_{m+2}x_{m+3}x_{m+4}\cdots x_n$$

$$1+x_{m+2}+x_{m+2}x_{m+3}+x_{m+2}x_{m+3}x_{m+4}+\cdots+x_{m+2}x_{m+3}x_{m+4}\cdots x_{n-1}+x_{m+2}x_{m+3}x_{m+4}\cdots x_n$$

The third term $2x_{m+2}^2 \sum_{i=m+3}^{n} x_{m+3} \cdots x_i$ arises from products of the form:

$$1 + x_{m+2} + x_{m+2}x_{m+3} + x_{m+2}x_{m+3}x_{m+4} + \cdots + x_{m+2}x_{m+3}x_{m+4} \cdots x_{n-1} + x_{m+2}x_{m+3}x_{m+4} \cdots x_n$$

$$1 + x_{m+2} + x_{m+2}x_{m+3} + x_{m+2}x_{m+3}x_{m+4} + \cdots + x_{m+2}x_{m+3}x_{m+4} \cdots x_{n-1} + x_{m+2}x_{m+3}x_{m+4} \cdots x_n$$

and,

$$1 + x_{m+2} + x_{m+2}x_{m+3} + x_{m+2}x_{m+3}x_{m+4} + \cdots + x_{m+2}x_{m+3}x_{m+4} \cdots x_{n-1} + x_{m+2}x_{m+3}x_{m+4} \cdots x_n$$

$$1 + x_{m+2} + x_{m+2}x_{m+3} + x_{m+2}x_{m+3}x_{m+4} + \cdots + x_{m+2}x_{m+3}x_{m+4} \cdots x_{n-1} + x_{m+2}x_{m+3}x_{m+4} \cdots x_n$$

Finally, the last term $2x_{m+2}^2 \cdots x_{n-1}^2 \sum_{i=n}^{n} x_i$ comes from products of the form:

$$1 + x_{m+2} + x_{m+2}x_{m+3} + x_{m+2}x_{m+3}x_{m+4} + \cdots + x_{m+2}x_{m+3}x_{m+4} \cdots x_{n-1} + x_{m+2}x_{m+3}x_{m+4} \cdots x_n$$

$$1 + x_{m+2} + x_{m+2}x_{m+3} + x_{m+2}x_{m+3}x_{m+4} + \cdots + x_{m+2}x_{m+3}x_{m+4} \cdots x_{n-1} + x_{m+2}x_{m+3}x_{m+4} \cdots x_n$$

The expectation Eqn. (1.39) is given by,

$$\begin{aligned} & E\left[(1 + x_{m+2} + x_{m+2}x_{m+3} + \cdots + x_{m+2}x_{m+3} \ldots x_n)^2\right] = \\ & 1 + \sum_{i=m+2}^{n} E\left[x_{m+2}^2\right] \cdots E\left[x_i^2\right] + 2 \sum_{i=m+2}^{n} E\left[x_{m+2}\right] \cdots E\left[x_i\right] + \\ & 2E\left[x_{m+2}^2\right] \sum_{i=m+3}^{n} E\left[x_{m+3}\right] \cdots E\left[x_i\right] + \cdots \\ & \cdots + 2E\left[x_{m+2}^2\right] \cdots E\left[x_{n-1}^2\right] \sum_{i=n}^{n} E\left[x_i\right] \end{aligned} \tag{0.40}$$

This cumbersome expression can be simplified using the fact that,

$$E\left[x_i\right] = \exp(\mu_1), \qquad \text{for all } i = m+2, \ldots, n \tag{0.41}$$

and

$$E\left[x_i^2\right] = \exp(\mu_2), \qquad \text{for all } i = m+2, \ldots, n \tag{0.42}$$

where

$$\mu_1 = (r-d)\Delta t, \qquad \mu_2 = \left(2(r-d)+\sigma^2\right)\Delta t \tag{0.43}$$

The trick to simplifying the expression in Eqn. (1.40) is to repeatedly factor out a geometric summation using the formula in Eqn. (1.37).

We evaluate the second term $\sum_{i=m+2}^{n} E\left[x_{m+2}^2\right]\cdots E\left[x_i^2\right]$ in Eqn. (1.40):

$$\begin{aligned}\sum_{i=m+2}^{n} E\left[x_{m+2}^2\right]\cdots E\left[x_i^2\right] = E\left[x_{m+2}^2\right] + E\left[x_{m+2}^2\right]E\left[x_{m+3}^2\right] + \cdots \\ \cdots + E\left[x_{m+2}^2\right]E\left[x_{m+3}^2\right]\cdots E\left[x_n^2\right]\end{aligned} \tag{0.44}$$

There are $(n-(m+2)+1)=(n-m-1)$ terms in the right-hand side of the above equation. Replacing each expectation by Eqn. (1.42) we obtain:

$$\sum_{i=m+2}^{n} E\left[x_{m+2}^2\right]\cdots E\left[x_i^2\right] = e^{\mu_2} + e^{2\mu_2} + \cdots + e^{(n-m-1)\mu_2} \tag{0.45}$$

This is a geometric summation so we can use Eqn. (1.37)

$$\sum_{i=m+2}^{n} E\left[x_{m+2}^2\right]\cdots E\left[x_i^2\right] = \frac{e^{(n-m)\mu_2} - e^{\mu_2}}{e^{\mu_2} - 1} \tag{0.46}$$

Now we try and rearrange the remaining terms in Eqn. (1.42) into a geometric series,

$$\begin{aligned}&2\sum_{i=m+2}^{n} E\left[x_{m+2}\right]\cdots E\left[x_i\right] + 2E\left[x_{m+2}^2\right]\sum_{i=m+3}^{n} E\left[x_{m+3}\right]\cdots E\left[x_i\right] + \cdots \\ &\qquad \cdots + 2E\left[x_{m+2}^2\right]\cdots E\left[x_{n-1}^2\right]\sum_{i=n}^{n} E\left[x_i\right] = \\ &2\left(E\left[x_{m+2}\right] + E\left[x_{m+2}\right]E\left[x_{m+3}\right] + \cdots + E\left[x_{m+2}\right]\cdots E\left[x_n\right]\right) + \\ &2E\left[x_{m+2}^2\right]\left(E\left[x_{m+3}\right] + E\left[x_{m+3}\right]E\left[x_{m+4}\right] + \cdots + E\left[x_{m+3}\right]\cdots E\left[x_n\right]\right) + \\ &2E\left[x_{m+2}^2\right]E\left[x_{m+3}^2\right]\end{aligned} \tag{0.47}$$

Using Eqns. (1.41) and (1.42) we have that,

$$2\sum_{i=m+2}^{n} E\left[x_{m+2}\right]\cdots E\left[x_i\right] + 2E\left[x_{m+2}^2\right]\sum_{i=m+3}^{n} E\left[x_{m+3}\right]\cdots E\left[x_i\right] + \cdots$$

$$\cdots + 2E\left[x_{m+2}^2\right]\cdots E\left[x_{n-1}^2\right]\sum_{i=n}^{n} E\left[x_i\right] =$$

$$= 2\left(\frac{e^{(n-m)\mu_1} - e^{\mu_1}}{e^{\mu_1} - 1}\right) + 2e^{\mu_2}\left(\frac{e^{(n-m-1)\mu_1} - e^{\mu_1}}{e^{\mu_1} - 1}\right) + 2e^{2\mu_2}\left(\frac{e^{(n-m-2)\mu_1} - e^{\mu_1}}{e^{\mu_1} - 1}\right) + \cdots \tag{0.48}$$

$$\cdots + 2e^{(n-m-2)\mu_2}\left(\frac{e^{2\mu_1} - e^{\mu_1}}{e^{\mu_1} - 1}\right)$$

We split the sum into two terms as follows,

$$2\left(\frac{e^{(n-m)\mu_1} - e^{\mu_1}}{e^{\mu_1} - 1}\right) + 2e^{\mu_2}\left(\frac{e^{(n-m-1)\mu_1} - e^{\mu_1}}{e^{\mu_1} - 1}\right) + 2e^{2\mu_2}\left(\frac{e^{(n-m-2)\mu_1} - e^{\mu_1}}{e^{\mu_1} - 1}\right) + \cdots$$

$$\cdots + 2e^{(n-m-2)\mu_2}\left(\frac{e^{2\mu_1} - e^{\mu_1}}{e^{\mu_1} - 1}\right) = S_1 + S_2 \tag{0.49}$$

where,

$$S_1 = \frac{2}{e^{\mu_1} - 1}\left(e^{(n-m)\mu_1} + e^{\mu_2}e^{(n-m-1)\mu_1} + e^{2\mu_2}e^{(n-m-2)\mu_1} + \cdots + e^{(n-m-2)\mu_2}e^{2\mu_1}\right) \tag{0.50}$$

and

$$S_2 = \frac{-2e^{\mu_1}}{e^{\mu_1} - 1}\left(1 + e^{\mu_2} + e^{2\mu_2} + \cdots + e^{(n-m-2)\mu_2}\right) \tag{0.51}$$

The sum S_2 comprises a geometric sum so we can directly apply Eqn. (1.37) to directly arrive at,

$$S_2 = \frac{2e^{\mu_1}\left(1 - e^{(n-m-1)\mu_2}\right)}{\left(e^{\mu_1} - 1\right)\left(e^{\mu_2} - 1\right)}. \tag{0.52}$$

Evaluation of the sum S_1 in Eqn. (1.51) requires a bit of manipulation to squeeze out a geometric sum as a factor. We first factor out a $e^{2\mu_1}$ term,

$$S_1 = \frac{2e^{2\mu_1}}{e^{\mu_1} - 1}\left(e^{(n-m-2)\mu_1} + e^{\mu_2}e^{(n-m-3)\mu_1} + e^{2\mu_2}e^{(n-m-4)\mu_1} + \cdots + e^{(n-m-2)\mu_2}\right) \tag{0.53}$$

Next we factor out the first term in the bracket,

$$\begin{aligned}
S_1 &= \frac{2e^{2\mu_1}e^{(n-m-2)\mu_1}}{e^{\mu_1}-1}\left(1+e^{\mu_2}e^{-\mu_1}+e^{2\mu_2}e^{-2\mu_1}+\cdots+e^{(n-m-2)\mu_2}e^{(n-m-2)\mu_1}\right) \\
&= \frac{2e^{2\mu_1}e^{(n-m-2)\mu_1}}{e^{\mu_1}-1}\left(1+e^{(\mu_2-\mu_1)}+e^{2(\mu_2-\mu_1)}+\cdots+e^{(n-m-2)(\mu_2-\mu_1)}\right) \\
&= \frac{2e^{2\mu_1}e^{(n-m-2)\mu_1}}{e^{\mu_1}-1}\left(\frac{e^{(n-m-1)(\mu_2-\mu_1)}-1}{e^{(\mu_2-\mu_1)}-1}\right) \\
&= \frac{2e^{2\mu_1}e^{(n-m-1)\mu_1}}{\left(e^{\mu_1}-1\right)e^{\mu_1}}\left(\frac{e^{(n-m-1)(\mu_2-\mu_1)}-1}{e^{(\mu_2-\mu_1)}-1}\right) \\
&= \frac{2e^{2\mu_1}}{\left(e^{\mu_1}-1\right)}\left(\frac{e^{(n-m-1)\mu_2}-e^{(n-m-1)\mu_1}}{e^{\mu_2}-e^{\mu_1}}\right).
\end{aligned} \tag{0.54}$$

Putting together the results in Eqn. (1.47), (1.53), and (1.55) we have that Eqn. (1.40) is given by,

$$\begin{aligned}
&E\left[(1+x_{m+2}+x_{m+2}x_{m+3}+\cdots+x_{m+2}x_{m+3}\ldots x_n)^2\right]= \\
&1+\frac{e^{(n-m)\mu_2}-e^{\mu_2}}{e^{\mu_2}-1}+\frac{2e^{2\mu_1}}{\left(e^{\mu_1}-1\right)}\left(\frac{e^{(n-m-1)\mu_2}-e^{(n-m-1)\mu_1}}{e^{\mu_2}-e^{\mu_1}}\right)+\frac{2e^{\mu_1}\left(1-e^{(n-m-1)\mu_2}\right)}{\left(e^{\mu_1}-1\right)\left(e^{\mu_2}-1\right)}.
\end{aligned} \tag{0.55}$$

It follows that from Eqn. (1.35) that,

$$\begin{aligned}
&E\left[M_T^2\right]=\left(\frac{S_t}{n}\right)^2\exp\left((2(r-d)+\sigma^2)\Delta t_{m+1}\right)\times \\
&\left(1+\frac{e^{(n-m)\mu_2}-e^{\mu_2}}{e^{\mu_2}-1}+\frac{2e^{2\mu_1}}{\left(e^{\mu_1}-1\right)}\left(\frac{e^{(n-m-1)\mu_2}-e^{(n-m-1)\mu_1}}{e^{\mu_2}-e^{\mu_1}}\right)+\frac{2e^{\mu_1}\left(1-e^{(n-m-1)\mu_2}\right)}{\left(e^{\mu_1}-1\right)\left(e^{\mu_2}-1\right)}\right).
\end{aligned} \tag{0.56}$$

If the average price call option (APO) depends on the forward price we need to set $d=r$ in the formulas for $E\left[M_T\right]$ and $E\left[M_T^2\right]$ in Eqn. (1.38) and (1.57). Unfortunately, the formulas diverge for $d=r$ so we need to derive the limits as $d\to r$.

Using L'Hopital's rule we have that,

$$\lim_{d\to r}E\left[M_T\right]=\frac{S_t(n-m)}{n} \tag{0.57}$$

The limiting expression for $E\left[M_T^2\right]$ is given by,

$$\lim_{d\to r} E\left[M_T^2\right] = \left(\frac{S_t}{n}\right)^2 exp(\sigma^2 \Delta t_{m+1}) \left(\frac{\left(1+e^{\sigma^2\Delta t}\right)\left(e^{\left((n-m)\sigma^2\Delta t\right)}-1\right)+2(n-m)\left(1-e^{\sigma^2\Delta t}\right)}{\left(e^{\sigma^2\Delta t}-1\right)^2} \right) \quad (0.58)$$

Pricing Formulas for London Metal Exchange APO Contracts

Let n be the number of asset price fixings which corresponds to the number of business days in the contract month. The time to the next fixing is Δt_{m+1} whereas the uniform time interval for all subsequent fixings is given by Δt. This uniform time interval should be set to,

$$\Delta t = \frac{t_n - t_{m+1}}{n}. \quad (0.59)$$

Let A_T denote the sum of the known part A_k and stochastic part M_T of the average of asset prices given in Eqn. (1.17),

$$A_T = M_T + A_k \quad (0.60)$$

The expectations $E\left[A_T\right]$ and $E\left[A_T^2\right]$ are given by

$$E\left[A_T\right] = E\left[M_T\right] + A_k \quad (0.61)$$

and

$$E\left[A_T^2\right] = E\left[M_T^2\right] + 2A_k E\left[M_T\right] + A_k^2 \quad (0.62)$$

Assuming A_T is log-normally distributed the formulas for a and b in Eqn. (1.24) and (1.25) can be written in terms of the above expectations,

$$a = 2\ln E\left[A_T\right] - \tfrac{1}{2}\ln E\left[A_T^2\right] \quad (0.63)$$

$$b = -2\ln E\left[A_T\right] + \ln E\left[A_T^2\right] \quad (0.64)$$

The pricing model for calls and puts in the Black-Scholes framework is:

$$\begin{aligned} C &= e^{-r(T-t)}\left(E\left[A_T\right]N(d_1) - KN(d_2)\right) \\ P &= e^{-r(T-t)}\left(KN(-d_2) - E\left[A_T\right]N(-d_1)\right) \end{aligned} \quad (0.65)$$

$$d_1 = \frac{\ln\left(\frac{E\left[A_T\right]}{K}\right) + \frac{1}{2}b}{\sqrt{b}} \qquad (0.66)$$
$$d_2 = d_1 - \sqrt{b}$$

$$b = \ln E_2 - 2\ln E_1 \qquad (0.67)$$

$$E_1 = \frac{\bar{F}(n-m)}{n} + A_k, \quad \bar{F} = \frac{1}{n-m}\sum_{k=m+1}^{n} F_t(t_k) \qquad (0.68)$$

$$E_2 = E_3 E_4 + E_5 \qquad (0.69)$$

$$E_3 = \left(\frac{\bar{F}}{n}\right)^2 \exp(\sigma^2 \Delta t_{m+1}) \qquad (0.70)$$

$$E_4 = \begin{cases} \frac{(1+e^{\sigma^2\Delta t})(e^{((n-m)\sigma^2\Delta t)}-1)+2(n-m)(1+e^{\sigma^2\Delta t})}{\left(e^{\sigma^2\Delta t}-1\right)}, & \text{for } m < n-1 \\ 1, & \text{for } m = n-1 \end{cases} \qquad (0.71)$$

$$E_5 = 2A_k\frac{\bar{F}(n-m)}{n} + A_k^2 \qquad (0.72)$$

Appendix A: Moments of a lognormal variable

Let x be a lognormal variable so that,

$$\ln x \sim N(a,b) \tag{0.73}$$

where a is the mean and b is the variance. The probability density function of x is given by

$$P(x) = \frac{1}{x\sqrt{2\pi b}} e^{-(\ln x - a)^2/(2b)} \tag{0.74}$$

The raw moments of x are given by:

$$E\left[x^k\right] = \frac{1}{\sqrt{2\pi b}} \int_0^\infty x^{k-1} e^{-(\ln x - a)^2/(2b)} dx \tag{0.75}$$

$b = -2\ln E\left[M_T\right] + \ln E\left[M_T^2\right]$ Making the change of variables $y = \ln x$ we obtain

$$\begin{aligned}
E\left[x^k\right] &= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} e^{y(k-1)} e^{-(\ln x - a)^2/(2b)} e^y dy \\
&= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \exp\left[-\frac{(y-a)^2}{2b} + ky\right] dy \\
&= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \exp\left[\frac{-(y-a)^2 + 2bky}{2b}\right] dy \\
&= \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \exp\left[\frac{-y^2 + 2ay - a^2 + 2bky}{2b}\right] dy \\
&= \exp\left[-\frac{a^2}{2b}\right] \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \exp\left[\frac{-y^2 + (2a + 2bk)y}{2b}\right] dy \\
&= \exp\left[-\frac{a^2}{2b}\right] \exp\left[\frac{(a+bk)^2}{2b}\right] \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} \exp\left[\frac{-(y-(a+bk))^2}{2b}\right] dy \\
&= \exp\left[-\frac{a^2}{2b}\right] \exp\left[\frac{a^2 + 2abk + b^2k^2}{2b}\right] \\
&= \exp\left[ka + k^2\frac{b}{2}\right]
\end{aligned} \tag{0.76}$$

3.3 PricerCommodityOTCOption2LTBlack

This pricer covers OTCOption2LTBlack valuating it by application of the lognormal Black & Scholes model.

It uses known analytical formulas to cover pricing of the different subtypes available within the CommodityOTCOption2.

3.3.1 References

The following references are provided as a contrast reference. Actual implementation can vary slightly in order to obtain improved performance or accuracy.

European Vanilla

Haug, E.G. The Complete Guide to Option Pricing Formulas 2nd edition. New York: Mc-Graw Hill. 2006. p 7-9

European Binary

Haug, E.G. The Complete Guide to Option Pricing Formulas 2nd edition. New York: Mc-Graw Hill. 2006. p 174, 175

Full Time Simple Barrier [with associated Rebate]

Haug, E.G. The Complete Guide to Option Pricing Formulas 2nd edition. New York: Mc-Graw Hill. 2006. p 152-155

Asian

Turnbull, S. L. & Wakeman, L. M., A Quick Algorithm for Pricing European Average Options, Journal of Financial and Quantitative Finance, 26, 377-389. (1991).

Levy, E., Pricing European Average Rate Currency Options, Journal of International Money and Finance, 14, 474-491. (1992)

3.3.2 Implementation details

Unlike other exotic payouts, the case of Average Rate options requires an approximation, since exact analytical formulas are not available.

The approach chosen in this case is the lognormal approximation suggested by Levy and Turnbull. Turnbull presented the possibility of approximating the distribution of the average by matching its known moments, while Levy refined the idea and suggested that just matching the first two moments with a lognormal distribution is sufficiently good in the majority of cases. This reduces the pricing to a direct application of the Black formula on the forward price, only that using an "adjusted" Asian volatility.

A brief summary of the calculations is presented.

Let

$$
\begin{aligned}
n &= \text{number of fixings} \\
m &= \text{number of realized fixings} \\
F_t(t_k) &= \text{forward price at time } t \text{ for fixing date } t_k
\end{aligned}
$$

We define the realized average by

$$A_{\mathrm{REL}} = \frac{1}{m}\sum_{k=1}^{m} S_k \tag{0.77}$$

and the float average by

$$A_{\mathrm{FLT}} = \frac{1}{n-m}\sum_{k=m+1}^{n} F_t(t_k) \tag{0.78}$$

Define the float variance by

$$\mathrm{var}_{\mathrm{FLT}} = \frac{1}{n^2} \times \sum_{j=m+1}^{n}\sum_{j'=m+1}^{n} F_t(t_j)F_t(t_{j'})\left[\exp\left(\sigma_j\sigma_j \times \min(t_j, t_{j'})\right) - 1\right] \tag{0.79}$$

The adjusted "Asian" volatility is given by

$$\sigma_A = \sqrt{\frac{\ln\left(\frac{\mathrm{var}_{\mathrm{FLT}}}{\tilde{A}^2} + 1\right)}{t_n - t}} \tag{0.80}$$

where

$$\tilde{A} = \tfrac{1}{n}(mA_{\mathrm{REL}} + (n-m)A_{\mathrm{FLT}}) \tag{0.81}$$

Now we can use the Black formula to calculate call option prices where the arguments to the normal cumulative functions are.

$$
\begin{aligned}
d_1 &= \frac{\ln(\tilde{A}/K)}{\sigma_A(t_n - t)} + \frac{1}{2}\sigma_A(t_n - t) \\
d_2 &= d_1 - \sigma_A(t_n - t)
\end{aligned} \tag{0.82}
$$