Family Name More

First Name Natalie

Student Number |__|__|__|__|__|__|__|__|

School Samuel Marsden Collegiate

UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

**No electronic/communication devices are permitted.**

**No exam materials may be removed from the exam room.**

## Mathematics and Statistics

## SEMESTER ONE FINAL TEST

Mid-year Examinations, 2022

# MATH199-22W (D) AIMS - Advancing in Mathematical Sciences

**For Examiner Use Only**

| Question | Mark |
|---|---|
| Q1 | |
| Q2 | |
| Q3 | |
| Q4 | |
| Q5 | |

Total ______

**Examination Duration:** 120 minutes

**Exam Conditions:**

Restricted Book exam: Approved materials only.

A non-graphics calculator is permitted.

**Materials Permitted in the Exam Venue:**

Restricted Book exam materials.

Students may bring into the exam: One A4, double sided, hand written page of notes.

**Materials to be Supplied to Students:**

1 x Write-on question paper/answer book

**Instructions to Students:**

Answer all FIVE questions.

There is a total of 76 marks.

Show all working.

There is no formula sheet.

1. **[16 marks]**

(a) Consider the following limit $L = \lim_{x \to -2} \frac{x^2 - 4}{x^2 - x - 6}$.

(i) Calculate L without using L'Hôpital's rule.

$$L = \lim_{x \to -2} \frac{x^2-4}{x^2-x-6} = \lim_{x \to -2} \frac{(x+2)(x-2)}{(x+2)(x-3)} = \lim_{x \to -2} \frac{x-2}{x-3} = \frac{-4}{-5} = \frac{4}{5}$$

(ii) Calculate L using L'Hôpital's rule.

$$L = \lim_{x \to -2} \frac{x^2-4}{x^2-x-6} = \frac{4-4}{4+2-6} = \frac{0}{0}$$ L'Hôpitals rule indeterminant form $\frac{0}{0}$

$$= \lim_{x \to -2} \frac{2x}{2x-1} = \frac{-4}{-4-1} = \frac{-4}{-5} = \frac{4}{5}$$

(b) Calculate the following limits using L'Hôpital's rule.

(i) $\lim_{x \to \infty} \frac{x^2 + 1}{e^x}$

$$\lim_{x \to \infty} \frac{x^2+1 \nearrow \infty}{e^x \nearrow \infty} = \frac{\infty}{\infty}$$ L'Hôpitals rule indeterminant form $\frac{\infty}{\infty}$

$$= \lim_{x \to \infty} \frac{2x \nearrow \infty}{e^x \nearrow \infty} = \frac{\infty}{\infty}$$ L'Hôpital's rule indeterminant form $\frac{\infty}{\infty}$

$$= \lim_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$

(ii) $\lim_{x \to 1} \ln(x) \csc(1-x)$

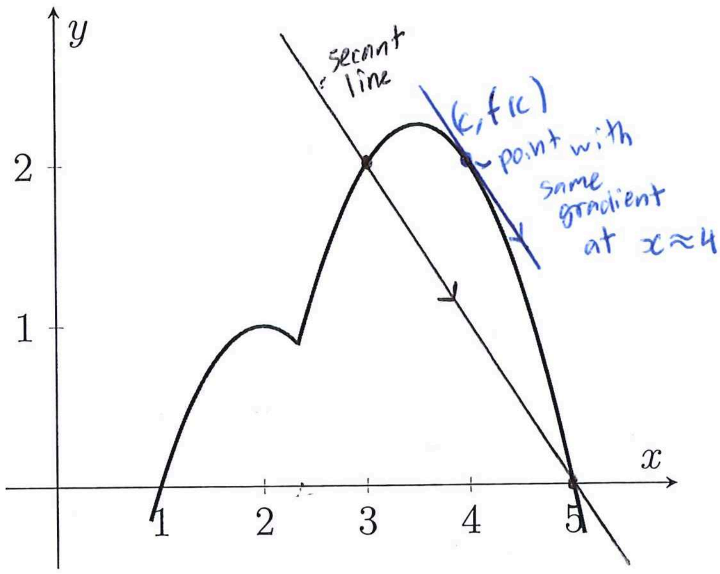
$\lim_{x\to 1} \ln|x| \csc(1-x)$ Indeterminant form $0\times\infty$ Can't use L'Hôpital's rule

$\lim_{x\to 1} \frac{\ln|x|}{\sin(1-x)} = \frac{0}{0}$ L'Hôpitals rule indeterminant form $\frac{0}{0}$

$\lim_{x\to 1} \frac{1/x}{\cos(1-x)} = 1$

(c) Recall that the Mean Value Theorem states that if $f$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$, then there is at least one point $c$ in $(a, b)$ where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Consider the graph of $y = f(x)$ below.

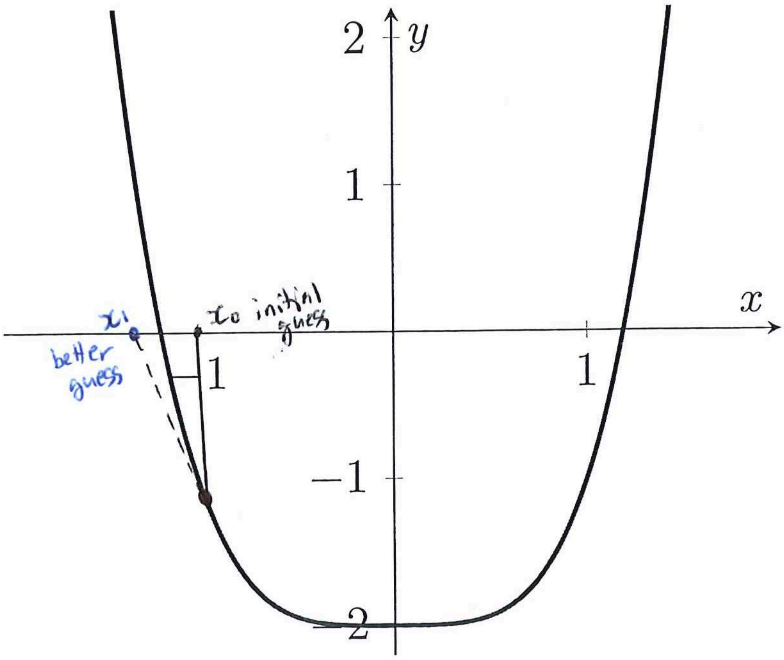



(i) Explain why the mean value theorem does not apply to the function $f(x)$ on the interval $[1, 5]$.

There is a sharp corner at around $x = 2.3$. This isn't differentiable so the function isn't differentiable on the open interval $(1, 5)$ and the mean value theorem doesn't apply

(ii) On the restricted domain $[3, 5]$ the mean value theorem does apply. Add details to the graph above to illustrate the mean value theorem on the interval $[3, 5]$.

(d) We can use Newton's method to estimate the roots of $f(x) = x^4 - 2$, the graph of which is provided below.




(i) State an appropriate initial approximation, $x_0$, for which Newton's method will converge to $\alpha = -\sqrt[4]{2}$.

$x = -1$

(ii) Illustrate the concepts in Newton's method by adding details to the given graph to show how the next estimate, $x_1$, is obtained from your initial approximation, $x_0$, given in (i).

(iii) Write down the recurrence relation for $x_{n+1}$ in terms of $x_n$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = x^4 - 2$$
$$f'(x) = 4x^3$$

$$= x_n - \frac{(x_n)^4 - 2}{4(x_n)^3}$$

$$= \frac{4(x_n)^4 - (x_n)^4 + 2}{4x_n^3}$$

$$= \frac{3(x_n)^4 + 2}{4(x_n)^3}$$

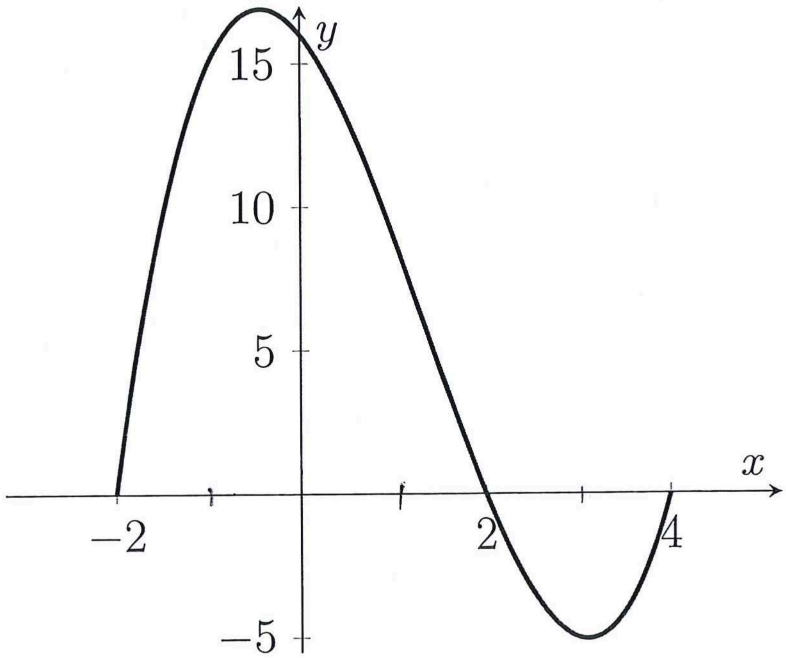
(iv) State an initial approximation, $x'_0$, for which Newton's method will *never* converge to $\alpha = -\sqrt[4]{2}$. Explain why it will not converge to $\alpha$.

$x = 0$

This point on the graph is a turning point and so has a gradient of 0. This means that it's tangent line is horizontal and parallell with the x axis so it will never touch the x-axis to give us the next approximation closer to $\alpha$

2. **[16 marks]**

(a) Consider the graph of $y = f(x)$ below, sketched for $-2 \leqslant x \leqslant 4$.




Use the graph to estimate:

(i) the open interval(s) on which $f$ is increasing.

$(-2, -\frac{1}{2})$ and $(3, 4)$

(ii) the open interval(s) on which $f$ is concave up.

$(1, 4)$

(iii) the $x$-value where the inflection point occurs. Justify your answer.

$x = 1$ as it looks like it goes from concave down to concave up at that point

(b) Consider the following function and its derivatives:

$$f(x) = \frac{x^2-3}{x-2} = x+2+\frac{1}{x-2}, \qquad f'(x) = \frac{(x-3)(x-1)}{(x-2)^2}, \qquad f''(x) = \frac{2}{(x-2)^3}.$$

(i) Find the $y$-intercept of the graph of $f(x)$.

y-intercept at ~~$f(x)=0$~~ $x=0$ $\qquad f(0) = \frac{0^2-3}{0-2} = \frac{3}{2}$

~~$0 = \frac{x^2-3}{x-2}$ $\quad 0 = x^2-3$ $\quad x^2=3$ $\quad x = \pm\sqrt{3}$~~

(ii) Find the $x$-intercept(s) of the graph of $f(x)$.

x-intercept at $f(x)=0$

$0 = \frac{x^2-3}{x-2} \qquad 0 = x^2-3 \qquad x^2 = 3 \qquad x = \pm\sqrt{3}$

(iii) Write down the equation of the vertical asymptote(s) of the graph of $f(x)$.

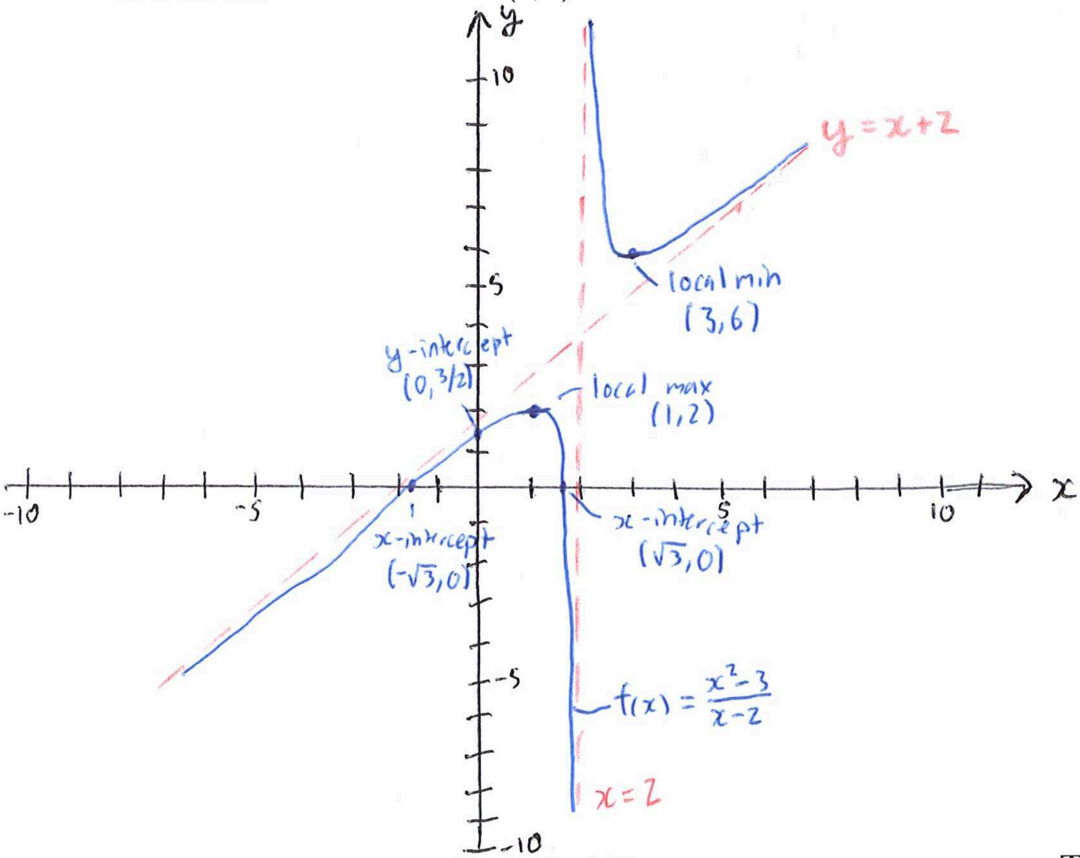
vertical asymdote at $x=2$

(iv) Write down the equation of the oblique asymptote of the graph of $f(x)$.

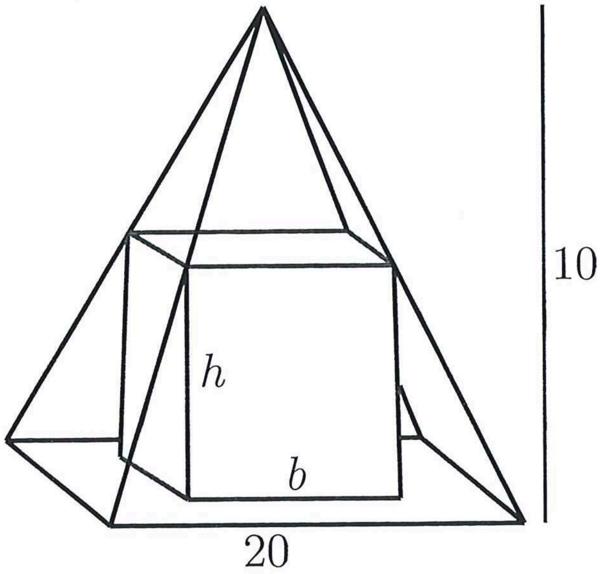
$y = x+2$

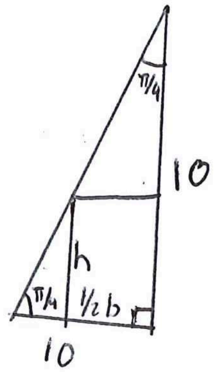
(v) Sketch the graph of $f(x)$ given the following additional information:

- A local minimum occurs at $(3,6)$.
- A local maximum occurs at $(1,2)$.



(c) Consider a pyramid of height 10 with a square base of side length 20. Furthermore suppose a rectangular cuboid (box) with base length $b$ and height $h$ is inscribed inside the pyramid as sketched below.




$\tan = \frac{o}{a}$

$h = \tan(\pi/4) \times (10 - \frac{1}{2}b)$

(i) Write the volume of the box as a function of $b$ only.

$V = hb^2 = (10 - \frac{1}{2}b)\tan(\pi/4)\, b^2$

$= 10b^2 \tan(\pi/4) - \frac{1}{2}b^3 \tan(\pi/4)$

(ii) Calculate the maximum possible volume for such an inscribed box. Justify why it is a maximum.

$V = 10b^2 \tan(\pi/4) - \frac{1}{2} b^3 \tan(\pi/4)$

$\frac{dV}{db} = 20b \tan(\pi/4) - \frac{3}{2} b^2 \tan(\pi/4)$ max at $\frac{dV}{db} = 0$

$0 = 20b \tan(\pi/4) - \frac{3}{2} b^2 \tan(\pi/4)$
$= (b)(20 \tan(\pi/4) - \frac{3}{2} b \tan(\pi/4))$

$b = 0$ or $20 \tan(\pi/4) - \frac{3}{2} b \tan(\pi/4) = 0$

b can't be 0 as there will be no box

$20 = \frac{3}{2} b$

$b = \frac{40}{3}$

$\frac{d^2V}{db^2} = 20 \tan(\pi/4) - 3b \tan(\pi/4)$

if $\frac{d^2V}{db^2} < 0$ at $b = \frac{40}{3}$ then $b = \frac{40}{3}$ is a maximum

$= 20 \tan(\pi/4) - 40 \tan(\pi/4)$
$= (20 \tan(\pi/4))(1 - 2)$
+ve × -ve = -ve

$\therefore 20 \tan(\pi/4) - 40 \tan(\pi/4) < 0$

and $b = \frac{40}{3}$ is a maximum

$V = 10b^2 \tan(\pi/4) - \frac{1}{2} b^3 \tan(\pi/4)$
$= 23.152$ units$^3$ (3DP)

~~or $(\tan(\pi/4))(\frac{400}{}$~~

3. **[15 marks]**

(a) Find the following indefinite integrals.

(i) $\int \sqrt{x} - \frac{5}{x}\,dx$

$$\int \left(x^{1/2} - \frac{5}{x}\right) dx$$

$$= \frac{2x^{3/2}}{3} - 5\ln|x| + c$$

(ii) $\int \sin^2(3\theta)\,d\theta$

$$\int \sin^2(3\theta)\,d\theta = \int \left(1/2 - 1/2\cos(6\theta)\right) d\theta$$

$$= 1/2\,\theta - 1/12\,\sin(6\theta) + c$$

(iii) $\int \frac{1}{1+4t^2}\,dt$

$$\int \frac{1}{1+4t^2}\,dt$$

$$= \frac{\tan^{-1}(2t)}{2} + c$$

(b) Consider the function $f(x) = (x-2)^2 - 1$.

(i) Evaluate the definite integral of $f$ from $x = 0$ to $x = 4$.

$$\int_0^4 ((x-2)^2 - 1)\,dx$$

$$= \int_0^4 (x^2 - 4x + 4 - 1)\,dx$$

$$= \int_0^4 (x^2 - 4x + 3)\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x + c\right]_0^4 \qquad = \frac{4^3}{3} - 2(4)^2 + 3(4) - \frac{0^3}{3} + 2(0)^2 - 3(0)$$

$$= 4/3 \text{ units}^2$$

(ii) What is the geometric area between the $x$-axis and the graph of $f$ from $x = 0$ to $x = 4$?

$$\int_0^1 f(x)\,dx - \int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_3^4$$

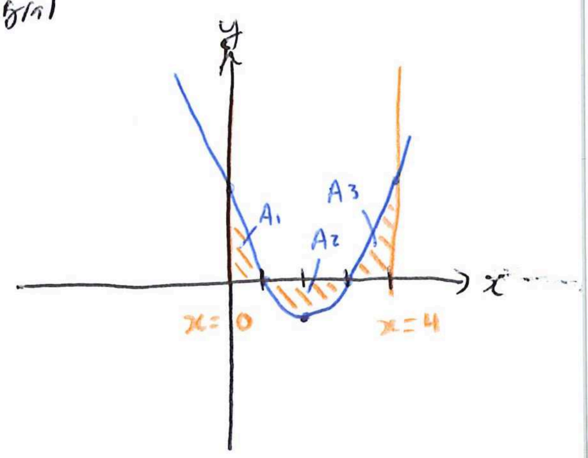
$$\cancel{\frac{1^3}{3} - 2(1)^2 + 3(1) = 4/3}$$

$$4/3 \quad - \quad -4/3 \quad + \quad 4/3$$

$$\cancel{= 10 \text{ units}^2} \quad = 4 \text{ units}^2$$

(iii) Why are the answers in (i) and (ii) different? Include a sketch in your answer.

The area given by the definite integral in i) is signed. This means that the area underneath the x-axis is taken away from the area above the x-axis so it doesn't show the ~~ah~~ actual area between the graph and the x-axis for this section.




in ii I split the areas up to give the unsigned area which is the total orange area shown in the graph

4. **[17 marks]**

(a) Use integration by parts to evaluate $\int x\cos(2x)\,dx$.

$\int u\,dv = uv - \int v\,du$

$u = x \qquad dv = \cos(2x)$

$du = 1 \qquad v = -\frac{\sin(2x)}{2}$

$$\int x\cos(2x)\,dx = -\frac{x\sin(2x)}{2} - \int -\frac{\sin(2x)}{2}\,dx$$

$$= -\frac{x\sin(2x)}{2} + -\frac{\cos(2x)}{4} + C$$

$$= -\frac{x\sin(2x)}{2} - \frac{\cos(2x)}{4} + C$$

(b) (i) Use a suitable substitution to evaluate $\int \frac{4x}{\sqrt{9-x^2}}\,dx$.

$$\int \frac{4x}{\sqrt{3^2-x^2}}\,dx = \int -2(u)^{-1/2}\,du$$

$u = 9 - x^2$

$\frac{du}{dx} = -2x \qquad dx = \frac{du}{-2x}$

$$= -(u)^{1/2} + C$$

$$= -(9-x^2)^{1/2} + C$$

$$= -\sqrt{9-x^2} + C$$

(ii) Explain why $\int_0^3 \frac{4x}{\sqrt{9-x^2}}\,dx$ is an improper integral.

There is a vertical asymptote at $x=3$ which is within the interval

(iii) Determine whether $\int_0^3 \frac{4x}{\sqrt{9-x^2}}\,dx$ converges or diverges. Evaluate the integral if it converges.

$$\lim_{l\to 3^-} \int_0^l \frac{4x}{\sqrt{9-x^2}}\,dx$$

$$= \lim_{l\to 3^-} \left[ -\sqrt{9-x^2} \right]_0^l$$

$$= \lim_{l\to 3^-} -\sqrt{9-l^2} + \sqrt{9-0^2}$$

$$= -\sqrt{0} + \sqrt{9}$$

$$= 3$$

the limit exists so the integral converges on 3

(c) Use partial fractions to find $I = \displaystyle\int \frac{x+9}{(x^2+4)(x-1)}\,dx$.

$$\int \frac{x+9}{(x^2+4)(x-1)}\,dx$$

$$\frac{x+9}{(x^2+4)(x-1)} = \frac{Ax+B}{(x^2+4)} + \frac{C}{(x+1)}$$

$$= \frac{(Ax+B)(x-1) + C(x^2+4)}{(x^2+4)(x-1)}$$

$$= \frac{Ax^2 - Ax + Bx - B + Cx^2 + 4C}{(x^2+4)(x-1)}$$

A

$$= \frac{(A+C)x^2 + (B-A)x + (4C-B)}{(x^2+4)(x-1)}$$

$(A+C) = 0$

$(B-A) = 1$

$(4C-B) = 9$

$$\begin{array}{cccc} A & B & C & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 4 & 9 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 4 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 5 & \cancel{9}10 \end{array}\right]$$

$C = 2 \quad B = -1 \quad A = -2$

$$\int \frac{x+9}{(x^2+4)(x-1)}\,dx$$

$$= \int \left( \frac{-2x-1}{(x^2+4)} + \frac{2}{(x-1)} \right) dx$$

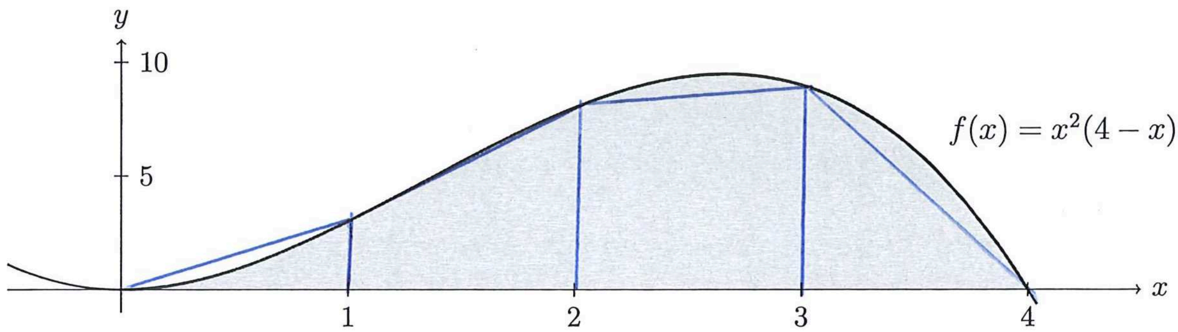
$$\cancel{= 2\ln|x+1|}$$

$$= \int \left( \frac{-2x}{x^2+4} - \frac{1}{x^2+4} + \frac{2}{x-1} \right) dx$$

$$= -\ln|x^2+4| - \frac{\tan^{-1}\left(\frac{x}{2}\right)}{2} + 2\ln|x-1| + c$$

5. **[12 marks]**

(a) Consider the region $R$ enclosed by the graph of the function $f(x) = x^2(4 - x)$ and the $x$-axis, from $x = 0$ to $x = 4$. $R$ is the shaded region sketched below.




(i) Use the Trapezium Rule with $n = 4$ to find an approximation to the area of $R$.

$$\int_0^4 x^2(4-x)\,dx \approx \frac{1}{2} h \left(y_0 + y_n + 2(y_1 + y_2 + \cdots y_{n-1})\right)$$

$$h = \frac{b-a}{n}$$

$$h = \frac{4-0}{4} = 1$$

$$\approx \frac{1}{2}\left(0 + 0 + 2(3 + 8 + 9)\right)$$

$$\approx 20 \text{ units}^2$$

(ii) Briefly explain whether this approximation will be an overestimate or an underestimate of the size of $R$.

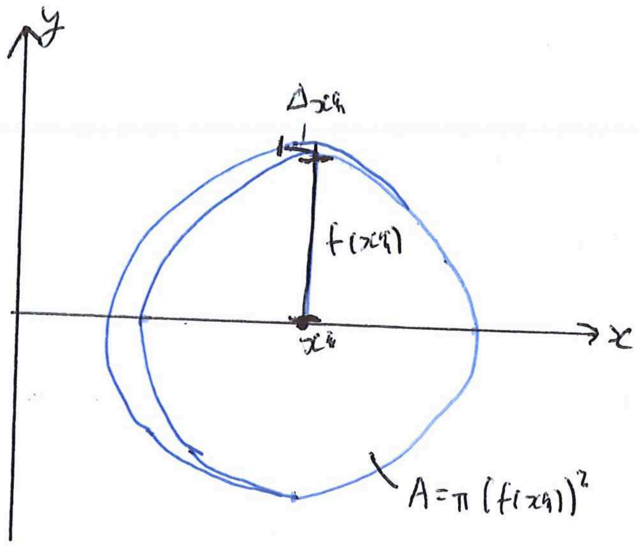
It will likely be an underestimate as most of the function in the interval is concave down

(iii) Briefly explain whether Simpson's Rule with $n = 4$ is likely to give a better or worse approximation than the Trapezium Rule with $n = 4$.

Simpson's Rule is likely to give a better approximation as it uses parabolas to approximate the curve of the graph so it is likely to be closer to the true area than the trapezium rule

(b) Consider the solid obtained when $R$ is revolved about the $x$-axis.

(i) A typical approximating disc at $x$ has volume $\pi f(x)^2 \Delta x$.
Sketch a typical disc, labelling $x$, $f(x)$, and $\Delta x$ (a 3-space diagram).




(ii) Calculate the volume of the solid.

$$V = \pi \int_0^4 (x^2(4-x))^2 \, dx$$

$$= \pi \int_0^4 (4x^2 - x^3)^2 \, dx$$

$$\cancel{= \pi [2(4x^2 - x^3)(8}$$

$$= \pi \int_0^4 (16x^4 - 8x^5 + x^6) \, dx$$

$$= \pi \left[ \cancel{4x^5} \frac{16x^5}{5} - \frac{8x^6}{6} + \frac{x^7}{7} \right]_0^4$$

$$= \pi \left[ \frac{16(4)^5}{5} - \frac{8(4)^6}{6} + \frac{(4)^7}{7} - \frac{16(0)^5}{5} + \frac{8(0)^6}{6} - \frac{0^7}{7} \right]$$

$$= \pi \frac{16384}{105}$$

$$= 490.2 \text{ units}^3 \text{ (2DP)}$$