Operations Research: Personal Notes

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1 Terms

algebraic method

- can be used for any number of decision variables unlike the *graphical method*

Say you have n constraints and m decision variables. You transform the inequalities from the constraints into equations by adding n slack variables. Now you have $n + m$ total variables and a system of n equations.

You can find the corner points by setting m of the variables to 0 and solving the system (some solutions will be unfeasible in which case they are not corner points). This means there are C_{n+m}^{m} combinations (some unfeasible).

(If you set the variables to any point other than 0 you might get inner points, not corner points.)

basic solutions
The solutions you get by using the *algebraic method*. If they are feasible they are called *basic feasible solutions*.

The variables which are set to 0 are the *non-basic variables* and the ones which are solved are called *basic variables*.

constraints ...

corner points ...

decision variable ...

feasible region
: All the feasible solutions. Is withing the corder points.

feasible solution
: Solutions which satisfy all the constraints and the non-negativity requirements.

graphical method

- for two decision variables

non negativity restrictions ...

objective function ...

optimal solution
: Something something.

slack variables
: The variables which are introduced to transform the inequalities in the constraints into equations. For example (1) into (2) by adding x_3.

$$2x_1 + 5x_2 \geq 3 \tag{1}$$
$$2x_1 + 5x_2 + x_3 = 3 \tag{2}$$

They are subject to the non-negativity restrictions but do not count to the objective function.

When $\geq$ inequalities are used, they are subtracted and are called *negative slack variables* or *surplus variables*. But still $x_i \geq 0$.

2 Simplex algorithm

2.1 Algebraic form

Example algorithm for:

$$\begin{aligned}
\max\ 6x_1 + 5x_2 \\
x_1 + x_2 \leq 5 \\
3x_1 + 2x_2 \leq 12 \\
x_1, x_2 \geq 0
\end{aligned}$$

It is transformed into:

$$\begin{aligned} \max\ 6x_1 + 5x_2 \\ x_1 + x_2 + x_3 = 5 \\ 3x_1 + 2x_2 + x_4 = 12 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The constraints should not have negative values so we write the equations by x_3 and x_4 and set the rest to 0 (thus having a basic feasible solution).

Iteration 1:

$$x_3 = 5 - x_1 - x_2 \tag{1}$$
$$x_4 = 12 - 3x_1 - 2x_2 \tag{2}$$
$$z = 6x_1 + 5x_2$$

The objective is to increase z so it appears that x_1 will increase it faster. (1) limits x_1 to 5 and (2) limits it to 4. So we rewrite (2) by x_1 and (1) with the new x_1. This is a new basic feasible solution.

Iteration 2:

$$3x_1 = 12 - 2x_2 - x_4 \Leftrightarrow x_1 = 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4 \tag{3}$$
$$x_3 = 5 - (4 - \frac{2}{3}x_2 - \frac{1}{3}x_4) - x2 \Leftrightarrow x_3 = 1 - \frac{1}{3}x_2 + \frac{1}{3}x_4 \tag{4}$$
$$z = 6(4 - \frac{2}{3}x_2 - \frac{1}{3}x_4) + 5x_2 \Leftrightarrow z = 24 + x_2 - 2x_4$$

Now z can be increased further by either increasing x_2 or decreasing x_4 (because of the negative coefficient -2). But decreasing x_4 would make it negative so we have to increase x_2 which is limited to 6 in (3) and 3 in (4).

Iteration 3:

$$\frac{1}{3}x_2 = 1 - x_3 + \frac{1}{3}x_4 \Leftrightarrow x_2 = 3 - 3x_3 + x_4 \tag{5}$$
$$x_1 = 4 - \frac{2}{3}(3 - 3x_3 + x_4) \Leftrightarrow x_1 = 2 + 2x_3 - x_4 \tag{6}$$
$$z = 24 + (3 - 3x_3 + x_4) - 2x_4 \Leftrightarrow z = 27 - 3x_3 - x_4$$

Since x_3 and x_4 cannot be increased, this is the final and optimal solution.

Advantages of the simplex method:

- Infeasible solutions aren't evaluated.
- Evaluates progressively better solutions.
- Stops at the optimal solution.

2.2 Tabular form

Stages:

- Initialization: Start with a basic feasible solution (the RHS variables are always non-negative).
- Iteration:
- Termination (when there is no entering column).

c_j
: The coefficient of x_j in the objective function.

θ
: The limiting value for the chosen row. The one which limits it the most (the smallest) is chosen as the *pivot row*.

  You don't compute θ for negative elements.

entering column
: ...

pivot element
: The intersection of the *pivot row* and *entering column*.

pivot row
: The leaving row.

RHS
: Right hand side.

⋆ ⋆ ⋆

- The values for the row need to contain the identity matrix for the variables on the left (in that order).
- The *non-basic variables* will have the values of $c_j - z_j$ equal to 0.

2.3 Minimization

For this example:

$$
\begin{aligned}
\min\ 3x_1 + 4x_2 \\
2x_1 + 3x_2 \geq 8 \\
5x_1 + 2x_2 \geq 12 \\
x_1, x_2 \geq 0
\end{aligned}
$$

we transform it into

$$
\begin{aligned}
\min\ 3x_1 + 4x_2 + 0x_3 + 0x_4 \\
2x_1 + 3x_2 - x_3 = 8 \\
5x_1 + 2x_2 - x_4 = 12 \\
x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

Unlike in the first example, here x_3 and x_4 have negative coefficients. This is done because $x_i \geq 0$ must hold for the minimum θ rule to work.

Simplex works for maximization problems. A minimization objective can be turned into a maximization objective by negating the coefficients:

$$\max\ -3x_1 - 4x_2 - 0x_3 - 0x_4$$

You can see that the slack variables do not qualify to be the initial basic variables since they turn out negative. So the initial basic feasible solution must be obtained by other means.

We rewrite it by adding two artificial variables with no meaning to the problem: $a_1 = 8$ and $a_2 = 12$ (the corresponding RHS number):

$$
\begin{aligned}
2x_1 + 3x_2 - x_3 + a_1 = 8 \\
5x_1 + 2x_2 - x_4 + a_2 = 12
\end{aligned}
$$

2.3.1 Big M method

We have to modify the objective function and give a_1 and a_2 very small contributions (thus preventing them from being chosen). So for maximization, we give a very small number $-M$. It becomes:

$$\max\ -3x_1 - 4x_2 - 0x_3 - 0x_4 - Ma_1 - Ma_2$$

M is a large, positive, tends to infinity and is called big M.

a_1 and a_2 are chosen as the basic variables.

Artificial variables will never try to enter so there's no point in computing $c_j - z_j$ for them.

2.3.2 Two phase method

This method eliminates big M which makes it better to be implemented on computers.

The first phase eliminates the artificial variables and generates a basic feasible solution.

The second phase generates the correct one.

With this method the objective function is changed so that all x_i have a 0 contribution and the artificial variables have a -1 contribution (for a maximization objective).

2.4 Iteration

degeneracy

Additional iteration done without increasing the objective function value.

Degeneracy always happens if there is a tie for the leaving variable (multiple equal θ values). This leads to one of the next θ values being equal to 0 which means that row will have to leave since it is the minimum. Since the increase depends on the θ, there will be no increase for that iteration.

There is no way to overcome degeneracy.

2.5 Termination

alternate optimum

Multiple optimal solutions.

Indicated by the current solution being optimal and a non-basic variable with $c_j - z_j = 0$ which will want to enter. Entering it will give another solution with the same objective function value.

When they exist, there are an infinite number of optimal solutions. But the simplex method only looks for the corners.

unboundedness

When there is an entering column, but no leaving row.

Unbounded region vs unbounded solution.

infeasibility

No solution is possible.

With the big M method, the artificial variable remains in the basis even after optimality has been reached.

cycling

Very rare in practice.