OXFORD INTERNET INSTITUTE UNIVERSITY OF OXFORD

Lecture 2: Statistical Inference & Uncertainty

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Applied Analytical Statistics

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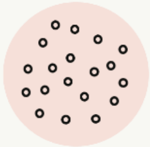
Plan for today | Statistical inference and uncertainty

Today we move **from description to inference**
We go beyond describing our sample towards learning about populations.



1. **Statistics as random variables**: the i.i.d. assumption and sampling distributions
2. **Large sample theory**: the Law of Large Numbers and the Central Limit Theorem
3. **Quantifying uncertainty:** standard errors and probability statements about statistics.
4. **Analytical inference**: construction and interpretation of confidence intervals
5. **Computational inference:** bootstrap confidence intervals, pros and cons

We will finish with a **class activity** to help you prepare for your summative.

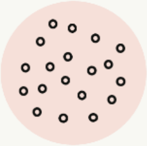
Inductive inference | Parameters and statistics

Goal of inductive inference: use data from a **sample** to learn about an unknown **population**.

RQ: What proportion of UK adults uses AI chat assistants at least once a week?

Data: Self-reported AI usage data from a representative survey of 1,000 UK adults.

The population is described by fixed but unknown **parameters**.
Examples: population mean, population proportion, population variance
μ p σ^2



Based on our sample, we compute **statistics** to estimate the population parameters.
Examples: sample mean, sample proportion, sample variance
$\bar{x}$ $\hat{p}$ s^2

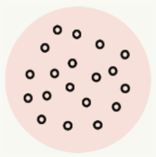
Inductive inference | Statistics as random variables

Last week, we learned about **random variables**, which take a numerical value for each possible outcome in the sample space (e.g. a coin, with heads/tails mapped to 1/0).

A **sample** (i.e. a specific set of observations) is the result of a random sampling process. Different samples result in different values of our statistic (e.g. the sample mean).

Data: Self-reported AI usage data from a representative survey of 1,000 UK adults.

Therefore, **a statistic is itself a random variable**.



The sample proportion $\hat{p}$ varies across samples.
The population proportion p is fixed but unknown.

Inductive inference | The i.i.d. assumption

To reason about samples and populations, we typically have to assume that our data is **independent and identically distributed → i.i.d.**

Independent: Each observation does not influence or inform any other observation.

Identically distributed: All observations are drawn from the same population distribution.

Formally, when $X_1, X_2, \ldots, X_n$ are i.i.d. draws from the same population, then:

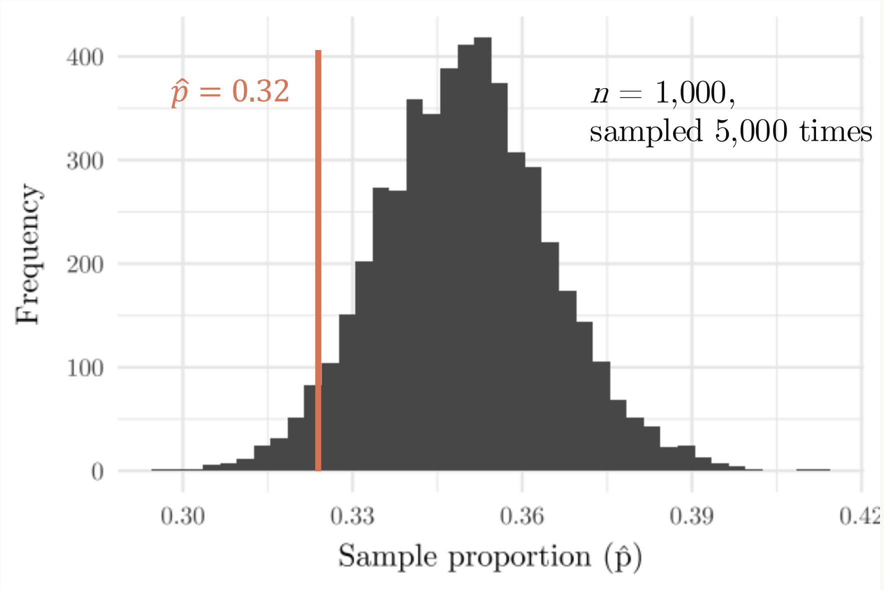
$\mathrm{P}(X_1 \in \mathrm{A}_1, \ldots, X_\mathrm{n} \in \mathrm{A}_\mathrm{n}) = \prod_{i=1}^{n} P\,(X_i \in A_i)$

and $X_i \sim F$ for all i

What kinds of data would likely violate these assumptions?

Without (approximate) i.i.d. assumptions, <u>standard</u> statistical inference becomes unreliable!

Sampling distributions | Linking samples to populations

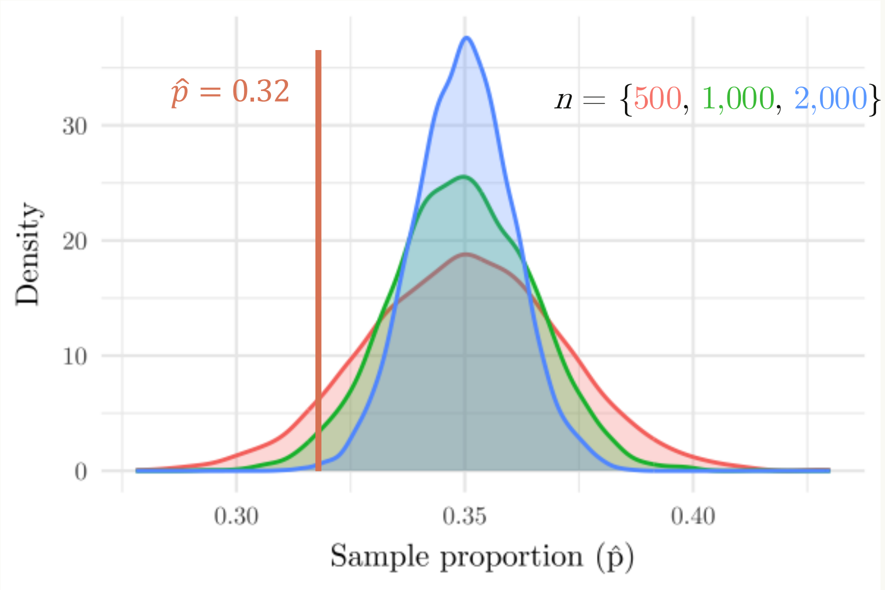


The **sampling distribution** is the distribution of a **statistic** across repeated i.i.d. samples from the same population.

This is a **hypothetical construct**. We do not usually observe this distribution, just one sample = one statistic.

How do we expect the shape of this distribution to change depending on sample size?

Sampling distributions | Variability and sample size



The **larger our sample**, the **less variability** there is in our sampling distribution.

How does the probability of observing $\hat{p} = 0.32$ change as we increase sample size?

Larger samples make it more likely that the statistic we observe is **closer to the centre of the sampling distribution**.

Large sample theory | Two key results for enabling inference

Random sampling introduces variability in our data. For sampling distributions, we saw that different samples from the same population produce different statistics.

The **Law of Large Numbers** (LLN) and **Central Limit Theorem** (CLT) describe what happens to this variability as sample size grows large, assuming i.i.d. samples.

→ asymptotic behaviour, i.e. what happens as $n \to \infty$

Most of classical statistical inference relies on these two results!

They help us today because they allow us to **approximate sampling distributions**.

Law of Large Numbers | Formula and intuition

Let $X_1, X_2, \dots, X_n$ be i.i.d. draws from the same population,

where the population mean $\mu = \mathbb{E}[X]$

and the sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, meaning that $\bar{X}_n$ is a random variable.

Then the **Law of Large Numbers** (LLN) states that $\bar{X}_n \to \mu$ when $n \to \infty$

Intuition:
- As sample size increases, the sample mean converges to the population mean.
- Random sampling variability averages out in large samples.

Now we are suddenly talking about means. Does the LLN apply to sample proportions?

Law of Large Numbers | Proportions as means

A **sample proportion** is just a sample mean of a Bernoulli random variable.

Let the random variable $X_i = \begin{cases} 1 & \text{if some specified event occurs} \\ 0 & \text{otherwise} \end{cases}$

Then $X_i \sim \text{Bernoulli}(p)$ and $\mathbb{E}[X_i] = p$, where p is the probability of the event.

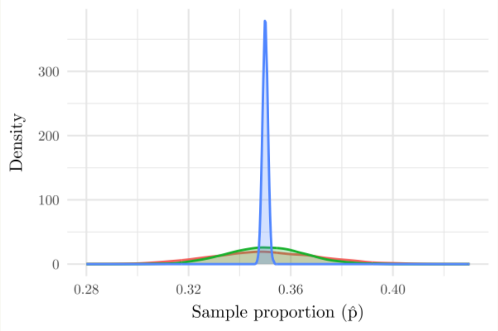
Then the sample proportion $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is exactly the sample mean of the observed X_i's.

Therefore, the LLN applies to sample proportions!

Law of Large Numbers | Convergence is key

The LLN tells us that **sample means converge to the true population mean**.

Estimates become **stable** as sample size increases, meaning that $P(|\bar{X}_n - \mu|) \to 0$ as $n \to \infty$. Large deviations between our statistic and the true parameter become increasingly unlikely.



For large n, our sampling distribution is extremely narrow, centered around the true population mean (in this case p).

However, the LLN **does not tell us how much variability remains for finite samples**, i.e. the data we are working with.

We need to describe the **shape of the sampling distribution** to quantify uncertainty.

Central Limit Theorem | Formula and intuition

Let $X_1, X_2, \ldots, X_n$ be i.i.d. draws from the same population,
where the population mean $\mu = \mathbb{E}[X]$ and population variance $\sigma^2 = \mathrm{Var}(X)$.

As before, the sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a random variable.

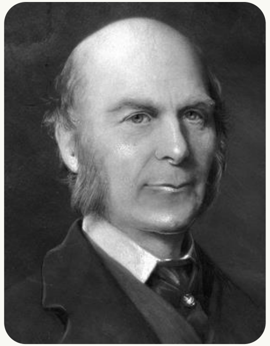
Then the **Central Limit Theorem** (CLT) states that $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$ as $n \to \infty$

Equivalently, for large n, we can say that $\bar{X}_n \approx N(\mu, \frac{\sigma^2}{n})$

Intuition:

- For large *n*, the sampling distribution of the sample mean is approximately normal.
- The sampling distribution is centered at the true population mean μ.
- The sampling distribution is narrower for larger samples, with a spread of σ^2/n.

Central Limit Theorem | An amazing law of nature



Francis Galton

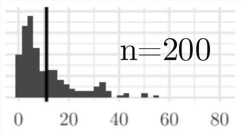
I know of scarcely anything so apt to impress the imagination as the **wonderful form of cosmic order expressed by the Central Limit Theorem**. The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement, amidst the wildest confusion.
The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the **supreme law of Unreason**.

Godfather of modern mathematical statistics
but also originator of eugenics (further reading here).

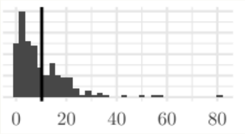
Central Limit Theorem | The great normaliser

Notice that we did not make many assumptions for the CLT. Most importantly:
The CLT does not require the sample itself to be normally distributed.

Sample 1

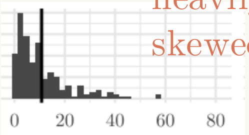


Sample 2



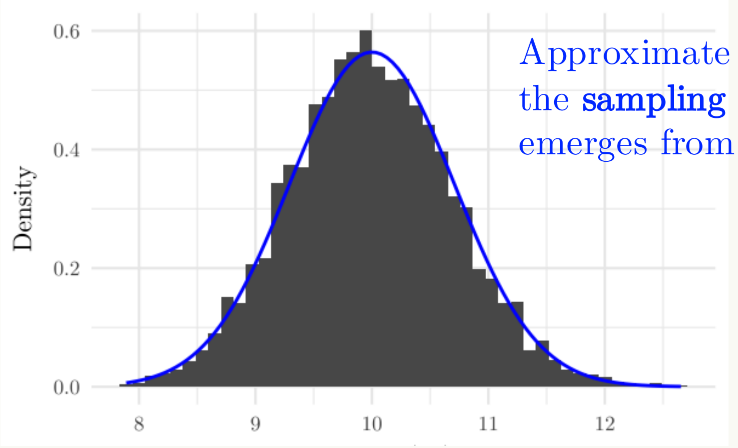
...

Sample n



heavily skewed

X



Approximate normality of the **sampling distribution** emerges from aggregation.

Sample mean $\overline{X}_n$

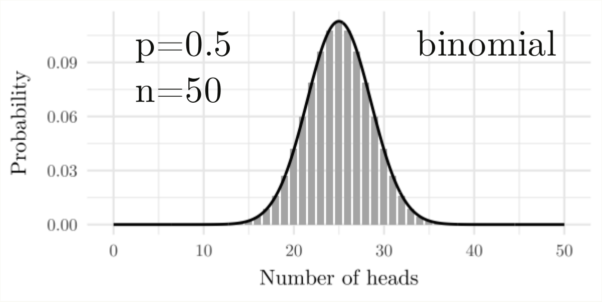
Central Limit Theorem | Approximate normality

In some cases, **we know the exact sampling distribution of a statistic.**

Example: Number of heads in a sequence of n independent coin tosses.

What distribution does this statistic follow?

The larger n, the better the normal distribution **approximates** the sampling distribution.



There is **no universal rule for what n is large enough** so that we can use the CLT approximation.

For sample means, even $n \geq 40$ can be large enough.
For sample proportions, a rule of thumb is $np \geq 10$.

Standard errors | Quantifying sampling uncertainty

The standard error **SE** is the **standard deviation of the sampling distribution** of a statistic.

specific to the sampling distribution

general measure of variability

Standard errors are defined by population parameters and sample size:

By CLT: $\bar{X}_n \approx N(\mu, \frac{\sigma^2}{n})$

For the sampling distribution of the **sample mean**, we have $SE(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}$

For the **sample proportion**, we have $SE(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$

Why is this the case?

In practice, we **replace unknown population parameters by their** sample estimates (e.g. $\hat{p}$).
As we will see, this is **justified for large samples**.

Standard errors | Exercise: SE of sample proportion

For a Bernoulli random variable: $\mathrm{Var}(X_i) = p(1 - p)$

For independent random variables: $\mathrm{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{Var}\,(X_i)$

For any constant a: $\mathrm{Var}(aY) = a^2\mathrm{Var}(Y)$

Use the above to show that $SE(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$

$$\mathrm{SE}(\hat{\mathrm{p}}) = \sqrt{\mathrm{Var}(\hat{\mathrm{p}})} = \sqrt{\mathrm{Var}\left(\frac{1}{\mathrm{n}}\sum_{i=1}^{n} X_i\right)} = \sqrt{\frac{1}{\mathrm{n}^2}\sum_{i=1}^{n} \mathrm{Var}(X_i)} = \sqrt{\frac{1}{\mathrm{n}^2}\mathrm{n}p(1-p)} = \sqrt{\frac{p(1-p)}{n}}$$

Standard errors | Estimating population standard deviation

To compute standard errors, we need the population standard deviation σ.
In practice, σ is unknown and must be estimated from the sample.

$$\mathrm{SE}(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}$$

The **sample standard deviation** is defined as $s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2}$

Compare this with the **population standard deviation** $\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2}$

The key difference is the **bias correction term**, dividing by $n - 1$ instead of n.
We make this correction because the sample mean $\bar{X}_n$ is itself an estimate for μ.
The naïve variance estimate (dividing by n) would be too small.

However, for large samples, $s \to \sigma$, and the correction becomes negligible. 🤔

See: consistency of sample variance

Sampling distributions | Probability statements about statistics

We now know, for any statistic, for large n and i.i.d. samples:

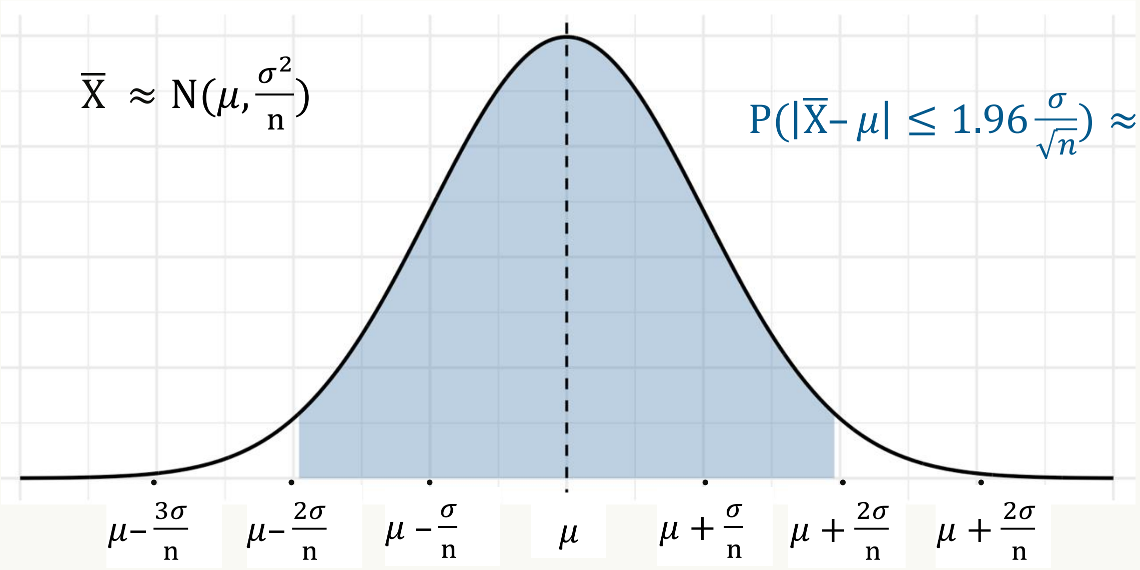
- The **shape** of the sampling distribution is approximately normal, by CLT
- The **center** of the sampling distribution is the true parameter, by LLN
- The **spread** of the sampling distribution is the standard error

We can use this knowledge to make **probability statements about statistics**.

In other words: The sampling distribution tells us how a statistic varies across samples.
We can approximate the sampling distribution based on the assumptions above.
This allows us to **quantify how unusual any observed value of the statistic would be**.

Sampling distributions | Probability of a sample mean

In 95% of repeated samples, $\bar{x}$ falls within ±1.96 standard errors of the true μ.



$P(|\bar{X} - \mu| \leq 1.96 \frac{\sigma}{\sqrt{n}}) \approx 0.95$

Where do we get the 1.96 from?

We can standardise the normal distribution

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \approx N(0,1)$$

In a standard normal:
$P(|Z| \leq 1.96) \approx 0.95$

Confidence intervals | Inverting the sampling distribution

Problem: We **cannot observe** the sampling distribution across all $\bar{X}$. We **do not know** μ.
Solution: We **can observe** one realisation of the statistic $\bar{x}$.

By the CLT, we can make **claims about the probability of statistics** $\bar{X}$:
$P(|\bar{X} - \mu| \leq 1.96 \cdot SE(\bar{X})) \approx 0.95$ where $SE(\bar{X}) = \frac{\sigma}{\sqrt{n}}$

We can now invert this statement to make **claims about the population mean $\boldsymbol{\mu}$**:
$P(\mu \in [\bar{X} \pm 1.96 \cdot SE(\bar{X})]) \approx 0.95$

What is the verbal interpretation of this equation?

Confidence intervals | Definition

Definition: A 95% confidence interval (**CI**) is a **procedure** that, in repeated sampling, produces intervals that contain the true population parameter 95% of the time.

CIs are random because they depend on the observed statistic.
The population parameter is fixed but unknown.

In general, for large n: $[\text{estimate} \pm (\text{critical value}) \cdot (\text{standard error})]$

For sample means: $\left[\bar{x} \pm 1.96 \cdot \frac{s}{\sqrt{n}}\right]$ where $1.96 = z_{0.975}$

For sample proportions: $\left[\hat{p} \pm 1.96 \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$ where $1.96 = z_{0.975}$

Confidence intervals | Interpretation

Definition: A 95% confidence interval (**CI**) is a **procedure** that, in repeated sampling, produces intervals that contain the true population parameter 95% of the time.

A CI that we construct based on a specific sample either contains the parameter or not.

✗ ~~"There is a 95% probability the true parameter lies in this interval."~~
✓ "If we repeated the study many times, 95% of the CIs would contain the true parameter."

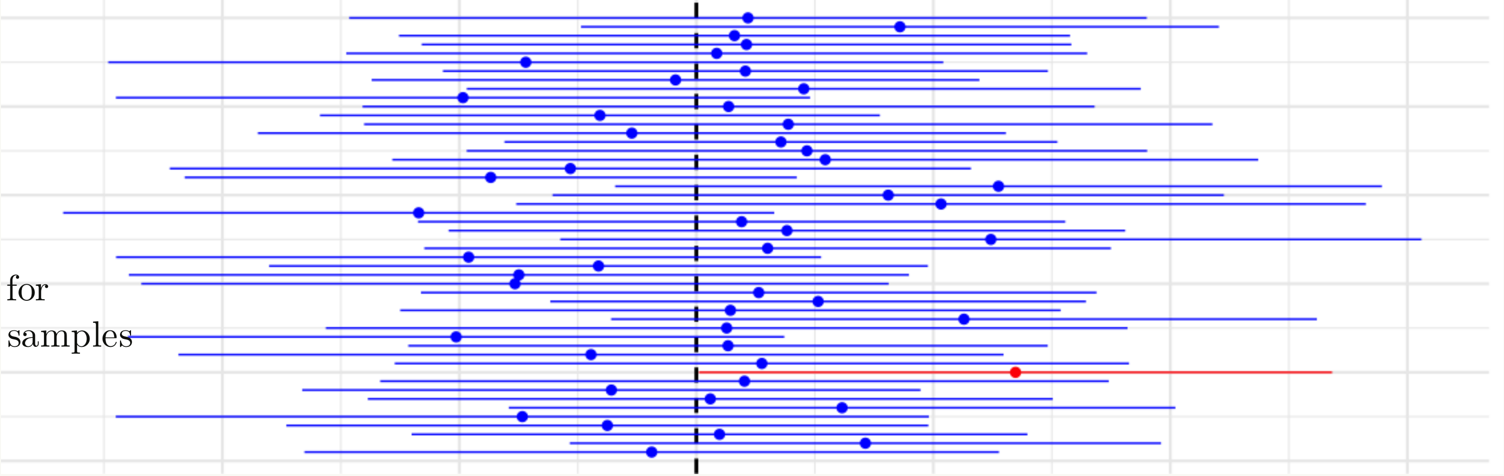
✗ ~~"95% of the data lie in this interval."~~
✓ "CIs describe uncertainty about point estimates, not dispersion of observations."

Confidence intervals | Coverage

The **coverage** of a confidence interval describes the long-run proportion of times that the **confidence interval procedure** contains the true population parameter under repeated sampling.

→ **A 95% CI has 95% coverage**

95% CIs for different samples



True population parameter

Confidence intervals | Scaling

Let's say we construct a 95% CI for a sample mean $\left[\bar{x} \pm 1.96 \cdot \frac{s}{\sqrt{n}}\right]$

🤔 How does CI width scale with sample size n?

CI width $\propto \frac{1}{\sqrt{n}}$. Diminishing returns! 4x the sample size = 0.5x CI width

🤔 How does CI width scale with sample standard deviation s?

CI width $\propto s$. Linear returns! 2x the variability = 2x uncertainty about your estimate.

Transition | From analytical to computational inference

So far, we relied on **analytical inference**, grounded in probability theory, LLN and CLT.

This **works well** when:
- Sample sizes are large
- Sampling distributions are approximately normal
- Standard errors have simple, known formulas

However, **analytical SEs and CIs become difficult** when:
- Sample sizes are small or moderate
- Sampling distributions are skewed or heavy-tailed
- Statistics are complex (e.g. medians, quantiles, complex estimators)

In these cases, theoretical approximations may be unreliable.
Therefore, we now introduce a **computational alternative** that requires **fewer assumptions**.

Bootstrap | Intuition

Goal: **Approximate the sampling distribution** of a statistic (e.g. sample mean)
Challenge: We **cannot repeatedly sample** from the population.

→ Same goal and challenge as before!

- We treat our sample as a **stand-in for the population**.
- We repeatedly draw samples **with replacement**.
- We **recompute the statistic** for each sample.
- We use the resulting "**bootstrap distribution" of the statistic** as a stand-in for the unobserved sampling distribution.

We no longer assume a theoretical distribution but estimate it directly from the data!

Bootstrap | Algorithm

Let $X_1, X_2, \ldots, X_n$ be an observed random sample from an unknown population,
and let $\hat{\theta} = T(X_1, \ldots, X_n)$ be a statistic estimating a population parameter θ.

Define a **bootstrap sample** $X_1^*, \ldots, X_n^*$ as a sample drawn **with replacement** from $\{X_1, \ldots, X_\mathrm{n}\}$

Let $\hat{\theta}^* = T(X_1^*, \ldots, X_n^*)$ be the estimated statistic for a given bootstrap sample.

Repeat the sampling procedure B times to obtain $\hat{\theta}_1^*, \ldots, \hat{\theta}_B^*$.

Then the empirical "bootstrap distribution" of $\hat{\theta}^*$ approximates the sampling distribution of $\hat{\theta}$.

Bootstrap | Confidence intervals

Once we have the bootstrap distribution, we can use it to construct confidence intervals.

The most common approach is the **percentile bootstrap CI**:

> For a given sample, the 95% bootstrap confidence interval is the interval between the 2.5^{th} and 97.5^{th} percentile of the bootstrap distribution.

The **interpretation** of bootstrap CIs is the **same as for analytical CIs**:
A 95% bootstrap CI is a procedure that, in repeated sampling, produces intervals that contain the true parameter 95% of the time.

Are bootstrap CIs symmetric?

Bootstrap | Strength = flexibility

Minimal distributional assumptions.
The bootstrap does not require normality of the sampling distribution, closed-form variance formulas, or large-sample approximations (e.g. CLT).

Applicable to many statistics.
The bootstrap works even for complex or nonlinear statistics (e.g. medians, quantiles), where analytical approximations of the sampling distribution are difficult to come by.

Naturally capturing skewness and asymmetry.
Bootstrap distributions can be asymmetric, skewed, reflecting whatever shape is implied by the sample data, whereas analytical CIs often assume symmetry around the point estimate.

Bootstrap | Limitation = sample quality

Observations must be independent from each other.
This assumption fails for time series, network data, clustered or panel data...
Regular bootstrappping would severely underestimate uncertainty for dependent data.

The sample must be sufficiently informative.
Since resampling produces many duplicate observations, the bootstrap distribution can be unstable with very small or imbalanced samples, especially if there are outliers.

The sample must be representative.
The observed sample is a stand-in for the population. If the sample is biased, unrepresentative, or systematically distorted, **the bootstrap will reproduce this bias**.

(+ practical limitation: computational costs can be high for complex statistics)

Comparison | Analytical vs. bootstrap inference

	Analytical inference	**Bootstrap inference**
Sampling assumption	i.i.d. sample	i.i.d. sample
Distributional assumption	Sampling distribution ≈ normal distribution (CLT)	Sample distribution ≈ population distribution
Required sample size	Large	Moderate
Sensitivity to skewness	High	Lower
Sensitivity to bias	High	High
Computational cost	Lower	Higher

Analytical inference is fast, elegant, theory-grounded. Best when assumptions clearly hold.
Bootstrap inference is robust and flexible. Best when sampling distributions are complex.

Neither method fixes bad data or bad research design!

Recap | Key takeaways from week 2

Statistical inference = learning from samples under uncertainty.
Statistics are random variables and estimates vary across samples from the same population.

The sampling distribution links samples and populations.
We approximate its shape to quantify uncertainty in our sample-specific estimates.

Large-sample theory lets us quantify uncertainty analytically.
For large n, LLN explains convergence. CLT states sampling distributions are ≈ normal.

When analytical assumptions are strained, computation offers an alternative.
Bootstraps approximate the sampling distribution directly from a single sample of data.

Next week | Hypothesis testing

So far, we used sampling distributions to **quantify uncertainty around estimates**.

Next week, we will take the next steps and start **drawing conclusions**:

Are our observed results compatible with a specific hypothesis about the population?

We will learn how to:
- Formulate **null and alternative hypotheses**
- Use **test statistics** and **reference distributions**
- Interpret **p-values** correctly (and avoid common mistakes)
- Connect **hypothesis tests and confidence intervals**

Class activity | Group assignment based on your RQs

G1: **Language, Communication, and Bias in AI & Media** – Caleb Agoha, Noha Mahgoub, Yunjia Qi

G2: **AI, Generative Models, and Evaluation** – Max Davy, Howard Leong, Audrey Yip

G3: **Media, Platforms, and Audience Response** – Sophie Bair, Charlotte Peart, Michi Wong

G4: **Political Economy, Policy, and Institutions** – Celikhan Baylan, Grahm Gaydos, Caleb Tan

G5: **Social Behaviour, Trust, and Adoption** – Min Jung, Mia Kussman, Isaac Backer

G6: **Health, Medicine, and Neuroscience** – Amelia Mercado, Laura Wegner, Ines Trichard

G7: **Education, Labor, and Socioeconomic Outcomes** – Rehmat Arora, Yilin Qian, Yue Zhang

G8: **Culture, Mobility, Lifestyle** – Teo Canmetin, Alena Tsvetkova, Fucheng Wang, Nesma Hammouda

Everyone not named: please get together in groups of 3.

Class activity | Overview

Please access the Week 2 Class Activity Google Doc on Canvas.

We will do this activity in two Phases.
In each Phase, you will first work individually and then discuss in your assigned group.

In **Phase 1** you will:
- Write down some basic facts about your RQ.
- Introduce your RQ to the rest of your group.

In **Phase 2** you will:
- Think through potential challenges in your own project and your group's projects.
- Discuss these challenges and, collectively, work towards solutions.

Goal: Help you prepare for the summative (W4 proposal deadline) and your own research.

Class activity | Anticipated challenges in answering your RQ

What are the **key challenges** you anticipate in **your own project**, and those in **your group?**

Data access

Sample size

Representativeness

Construct validity

Sampling bias

Ecological validity

Measurement reliability

Missing data

Dependence of observations

Temporal variation

Causality

... or any other challenge you think is relevant!