Lecture 4: Univariate Linear Regression

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Applied Analytical Statistics

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Correction | One-sided hypothesis testing

For a two-sided test: $H_0\colon \mu = \mu_0$ and $H_A\colon \mu \neq \mu_0$ → all possible outcomes

For a one-sided test: $H_A\colon \mu > \mu_0$ … **but what is H_0?**

In principle, we would want H_0 to be complementary: $H_0\colon \mu \leq \mu_0$

However, hypothesis testing requires us to **specify a single null distribution**.
This is why it is convenient to **use a simple** null $H_0\colon \mu = \mu_0$ even in the one-sided case.

In practice, $H_0\colon \mu \leq \mu_0$ and $H_0\colon \mu = \mu_0$ usually lead to equivalent results because the latter describes the "worst case". H_0 is easier to reject for values $\mu < \mu_0$ than for $\mu = \mu_0$. Therefore, only $\mu = \mu_0$ is relevant for controlling α and calculating p-values.

Details | Holm-Bonferroni correction for multiple comparisons

Family-wise error rate (FWER) is the probability of making one or more false discoveries, i.e. Type I errors, when performing multiple hypothesis tests.

Without correction, FWER increases as we perform more tests that each have fixed α.

Simple Bonferroni correction controls FWER by dividing per-test α by number of tests m.

Holm-Bonferroni correction is uniformly more powerful → less increase in Type II error rate

Sort p-values from smallest to largest: $p_{(1)}, \dots, p_{(m)}$.

Compare $p_{(k)} \leq \frac{\alpha}{m-k+1}$ **sequentially**, starting from $p_{(1)}$.

Stop once a test fails to reject null, do not reject null for any larger p-values.

$$p_{(1)} \leq \frac{\alpha}{m}$$

$$p_{(2)} \leq \frac{\alpha}{m-1}$$

$$\cdots$$

Plan for today | Univariate linear regression

[TO ADD]

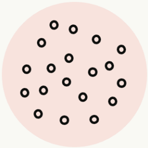
Regression | Motivation

Regression provides a **unified framework** for describing the relationship between variables.

1. Quantifying **strength and direction** of associations.
2. Expressing **uncertainty** about associations.
3. Drawing **conclusions** about statistical hypotheses.

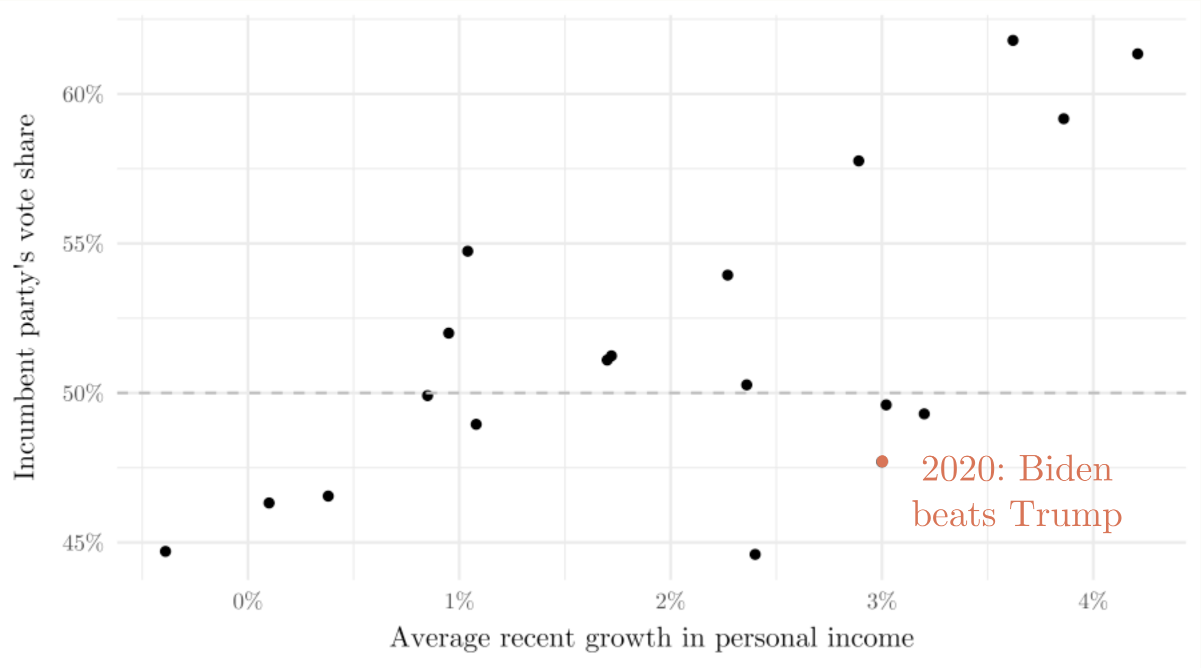
Familiar ideas:
correlations
confidence intervals
hypothesis tests

Regression can handle **more than just two variables**.
Regression can handle **different variable types**.



In this course, we mainly use regression for **analytical inference** but regression can also be a powerful tool for prediction.

Simple linear regression | Working example



Data: US election results vs. economic performance

Source: RegOS + updates

What kind of relationship can we **plausibly assume** here?

Simple linear regression | Population model

The **simple linear regression model** specifies the relationship between an outcome Y and a single predictor X **at the population level** based on coefficients β_0 and β_1:

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

where ε_i is the **error term**, capturing unobserved factors affecting Y_i.

We **assume** that the conditional mean of Y given X is linear in X :

$$E[Y \mid X] = \beta_0 + \beta_1 X$$

so that

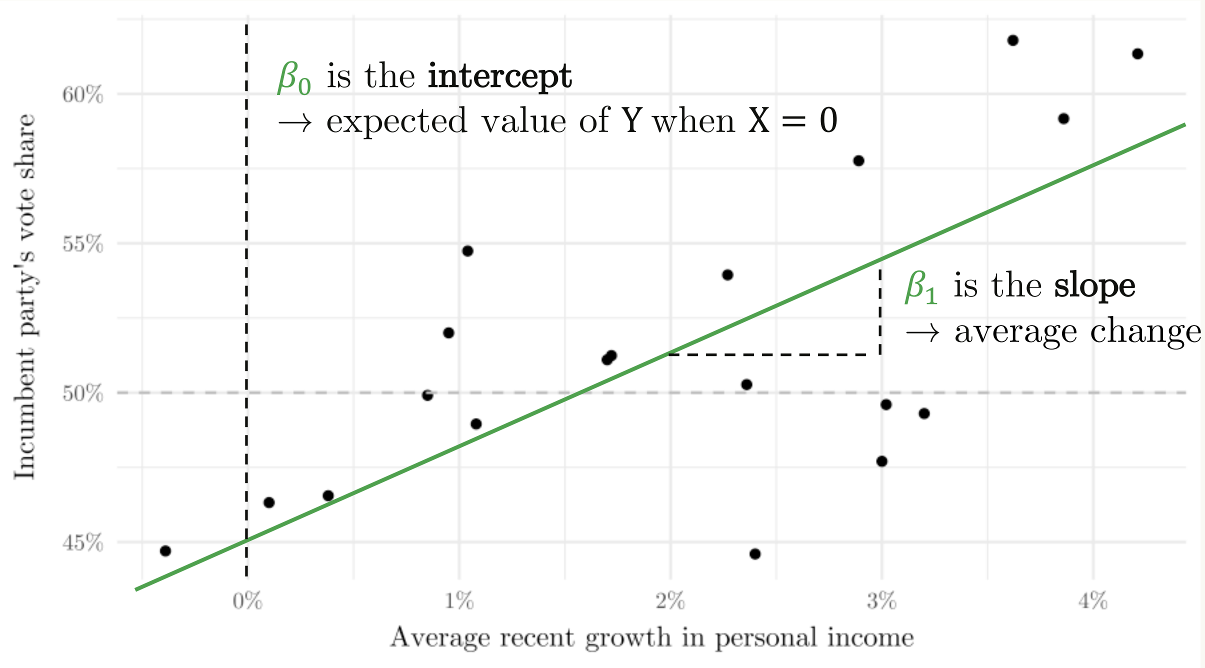
$$Y_i = E[Y \mid X_i] + \varepsilon_i$$

This is a **modelling assumption** about the average relationship between X and Y.

All models are wrong, but some are useful!

George Box (1919–2013)

Simple linear regression | Interpreting regression coefficients



β_0 is the **intercept**
→ expected value of Y when X = 0

β_1 is the **slope**
→ average change in Y for a one-unit increase in X

Example population model:
$\beta_0 = 45, \;\; \beta_1 = 2.5$
$\rightarrow E[Y \mid X_i] = 45 + 2.5X_i$

Simple linear regression | Estimated model

As always, we do not observe the **population** but one finite, noisy **sample**.
The **parameters** in our assumed population model are fixed but unknown: $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$

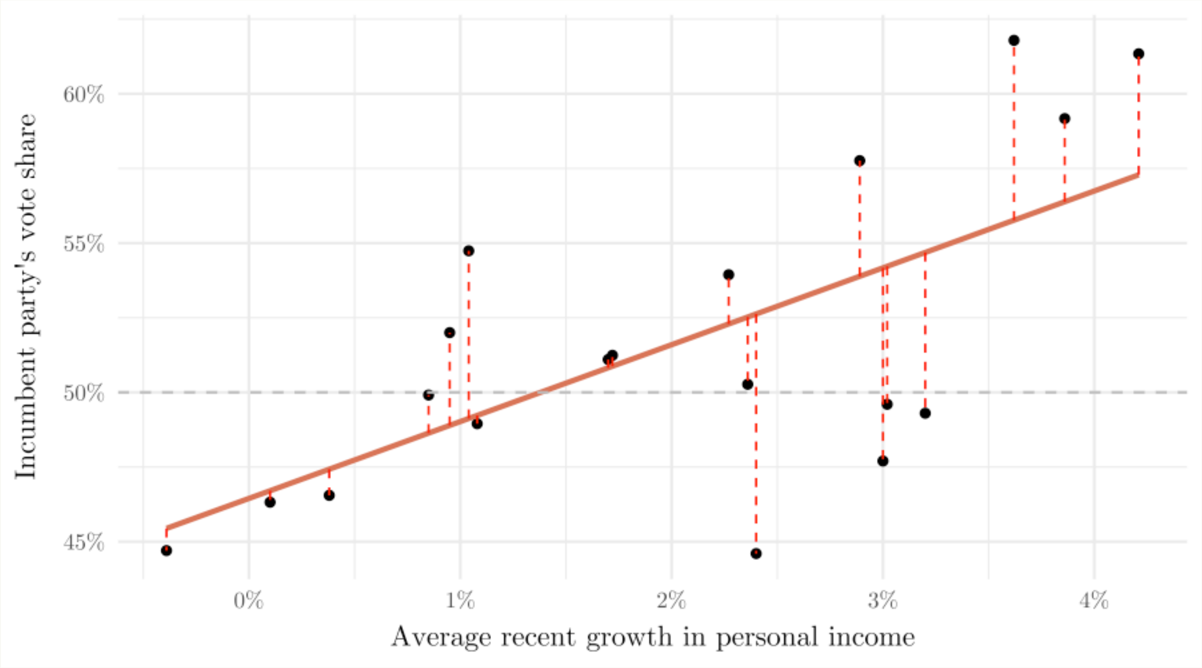
As always, we want to obtain **sample estimates** of population parameters: $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$
Here, our regression coefficients are **statistics** that vary across samples.

🤔 How do we choose $\hat{\beta}_0$ and $\hat{\beta}_1$? $\rightarrow$ find a "line of best fit"

The **residuals** $\hat{\varepsilon}_i = Y_i - \hat{Y}_i$ are the observed errors from our estimated model.

The most common method for estimating $\hat{\beta}_0$ and $\hat{\beta}_1$ is by **ordinary least squares (OLS)**:
We choose $\hat{\beta}_0$, $\hat{\beta}_1$ to **minimise the sum of the squared residuals** $\sum_i \hat{\varepsilon}_i^2$

Ordinary Least Squares | Residuals and OLS



$\hat{\varepsilon}_i$ is the vertical distance between the estimated ("fitted") regression line and observation i.

We select intercept $\hat{\beta}_0$ and slope $\hat{\beta}_1$ to minimise the sum of squared residuals $\hat{\varepsilon}_i$

🤔 Why do we **square** residuals before minimising their sum?

Ordinary Least Squares | Deriving OLS estimators

We choose $\hat{\beta}_0$, $\hat{\beta}_1$ to minimise the sum of squared residuals / residual sum of squares (RSS):

$$\min_{\hat{\beta}_0,\hat{\beta}_1} \sum_{i=1}^{n} \text{RSS} = \min_{\hat{\beta}_0,\hat{\beta}_1} \sum_{i=1}^{n} \hat{\varepsilon}_i^2 = \min_{\hat{\beta}_0,\hat{\beta}_1} \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \min_{\hat{\beta}_0,\hat{\beta}_1} \sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2$$

We solve this **minimisation problem** by taking partial derivatives and setting to zero:

$$\frac{\partial \text{RSS}}{\partial \hat{\beta}_0} = -2 \sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0 \quad \Rightarrow \sum_{i=1}^{n} \hat{\varepsilon}_i = 0.$$

→ residuals sum to zero by design!

geometrically: $\hat{\varepsilon} \perp \mathbf{1}$ (from intercept)

the "normal equations"

$$\frac{\partial \text{RSS}}{\partial \hat{\beta}_1} = -2 \sum_{i=1}^{n} X_i (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0 \quad \Rightarrow \sum_{i=1}^{n} X_i \hat{\varepsilon}_i = 0.$$

→ residuals are orthogonal to the predictor

geometrically: $\hat{\varepsilon} \perp \mathbf{X}$ (from slope)

Ordinary Least Squares | Deriving OLS estimators (cont'd)

By solving the system of equations on the previous slide we get to:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} \quad \text{and} \quad \hat{\beta}_0 = \overline{Y} - \hat{\beta}_1\overline{X}$$

$\hat{\beta}_0$ and $\hat{\beta}_1$ are the **OLS estimators** of population slope β_1 and population intercept β_0.

We know that $\mathrm{Var}_n(X) = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ and $\mathrm{Cov}_n(X,Y) = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\,(Y_i - \overline{Y})^2$

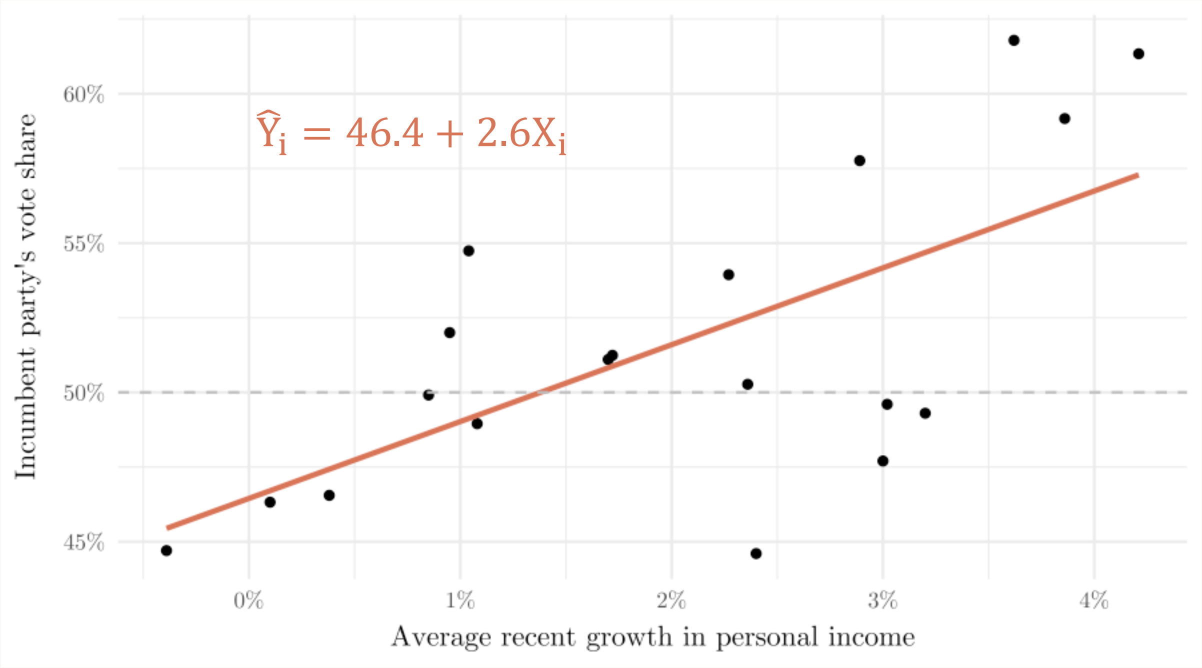
Therefore: $\hat{\beta}_1 = \frac{\mathrm{Cov}_n(X,Y)}{\mathrm{Var}_n(X)}$.

how strongly do X and Y move together?

→ enables per-unit interpretation of slope coefficient

how much does X itself vary?

Ordinary Least Squares | Fitted regression model

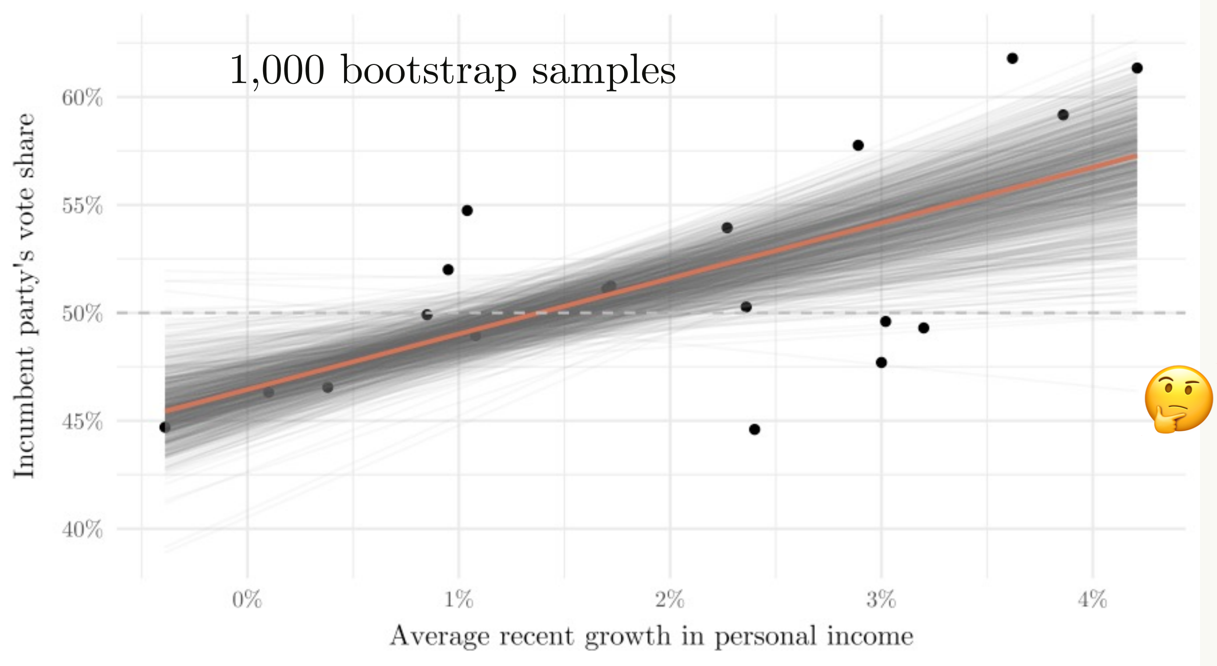


$\rightarrow$ expected change

$\widehat{\beta}_1 = 2.6$: a 1pp increase in "average recent growth in personal income" is associated with a 2.6pp increase in "incumbent party's vote share"

$\widehat{\beta}_0 = 46.4$: for 0% "average recent growth in personal income", the expected "incumbent party's vote share" is 46.4%.

Uncertainty in regression | Coefficients as random variables



We derived $\hat{\beta}_1$ and $\hat{\beta}_2$ as functions of our sample.

$\hat{\beta}_1$ and $\hat{\beta}_2$ will vary across repeated samples.

🤔 How do we quantify uncertainty?

As in previous weeks, we want to **approximate the sampling distribution** in order to perform inference.

Uncertainty in regression | Sampling distribution of OLS

Under **standard OLS assumptions**, the OLS slope has a well-defined sampling distribution:
→ will cover next week

$$\hat{\beta}_1 \sim N(\beta_1, \mathrm{Var}(\hat{\beta}_1)) \quad \text{where} \quad \mathrm{Var}(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2} \quad \text{and} \quad \mathrm{Var}(\varepsilon|X) = \sigma^2$$

→ "homoskedasticity"

What is the intuition behind the different terms in $\mathrm{Var}(\hat{\beta}_1)$?

σ^2 is the population variance of the error term, given X.
→ irreducible noise in outcome after controlling for predictor

$\sum_{i=1}^{n}(X_i - \bar{X})^2$ is the spread of our predictor.
→ horizontal information in predictor, growing with n

Familiar problem: σ^2 is a fixed but unknown population parameter.

Uncertainty in regression | Coefficient standard error

Our goal is to **characterise the sampling distribution of our regression coefficients**, so that we can quantify uncertainty around our estimated coefficients.

The **standard error of coefficient $\widehat{\beta}_1$** is the standard deviation of its sampling distribution:

$$\mathrm{SE}(\hat{\beta}_1) = \sqrt{\frac{\hat{\sigma}^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}} \qquad \text{where} \qquad \hat{\sigma}^2 = \frac{1}{n-2}\sum \hat{\varepsilon}_i^2$$

Why 1/(n-2) in $\hat{\sigma}^2$?

residual df = 2, one for each estimated parameter

Larger samples $n \rightarrow$ less uncertainty.
Larger error terms $\hat{\varepsilon}_i^2 \rightarrow$ more uncertainty.
More variation in X $\rightarrow$ less uncertainty.

Uncertainty in regression | Coefficient confidence intervals

Definition: A 95% confidence interval (**CI**) is a **procedure** that, in repeated sampling, produces intervals that contain the true population parameter 95% of the time.

Now that we know $\mathrm{SE}(\hat{\beta}_1)$, we can apply the same logic as for sample statistics (Week 2):

$$\mathrm{CI}_{1-\alpha} = \left[\hat{\beta}_1 \pm t_{\alpha/2,\, n-2} \times \mathrm{SE}(\hat{\beta}_1)\right]$$

🤔 Why the t distribution instead of the normal?

We use $\hat{\sigma}^2$ to estimate σ^2 when calculating $\mathrm{Var}(\hat{\beta}_1)$.
This introduces uncertainty that we need to adjust for in our sampling distribution.

→ same intuition as for z-test vs. t-test (Week 3)

We adjust by using the t distribution, which has heaver tails than the normal distribution.
As before, each estimated coefficient uses up one degree of freedom $df = n - 2$.

Uncertainty in regression | Hypothesis tests about coefficients

We test claims about population parameters: H_0: $\beta_1 = 0$ vs. H_A: $\beta_1 \neq 0$

Under the null, there is no linear association between X and Y in the population.
Knowing predictor X, on average, does not provide information about outcome Y.

$$t = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)}$$

The test statistic measures the distance between our sample coefficient and the coefficient value under the null in SE units (see Week 3).

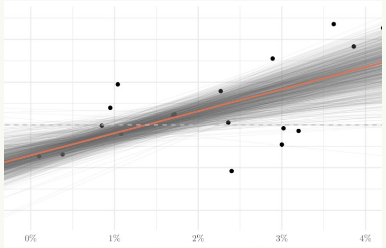
Under standard OLS assumptions and H_0: $\beta_1 = 0$, the t statistic is t-distributed: $T \sim t_{n-2}$

The p-value is the probability of observing a t-statistic at least this extreme under the null.

We reject H_0 if $p < \alpha$ and do not reject H_0 if $p \geq \alpha$, where $p = \Pr(|T_{n-2}| \geq |t| \mid H_0)$

Uncertainty in regression | Bootstrap for coefficients

Analytical SEs and CIs rely on theoretical assumptions about the sampling distribution.
Instead, we can estimate the sampling distribution of $\hat{\beta}_0$, $\hat{\beta}_1$ directly from the data.



- We treat our sample as a **stand-in for the population**.
- We repeatedly draw samples **with replacement**.
- We **recompute the** coefficient for each sample.
- We use the resulting "**bootstrap distribution" of the** coefficient as a stand-in for the unobserved sampling distribution.

We can **estimate coefficient SEs** based on the bootstrap distribution.
We can **construct coefficient CIs** from percentiles of the bootstrap distribution.
We can **reject $\mathbf{H_0}$** at α significance level if the null value lies outside the bootstrapped $\mathtt{CI}_{1-\alpha}$.

Goodness of fit | Decomposition of outcome variation

Goodness of fit describes how well our fitted regression model fits our data.

To quantify goodness of fit, we first **decompose the variation in our outcome Y**:

$$\sum (Y_i - \overline{Y})^2 = \sum (\hat{Y}_i - \overline{Y})^2 + \sum (Y_i - \hat{Y}_i)^2$$

Residual Sum of Squares (RSS):
Unexplained variation, in residuals

Total Sum of Squares (TSS):
Total variation in outcome Y

Explained Sum of Squares (ESS):
Variation explained by fitted regression

Total variation in the outcome is the sum of explained and unexplained variation.

$$TSS = ESS + RSS$$

These are all sample statistics that we can easily calculate.

Goodness of fit | R^2 coefficient of determination

In linear regression, the most common goodness-of-fit measure is R^2

$$R^2 = \frac{\text{Explained Sum of Squares (ESS)}}{\text{Residual Sum of Squares (RSS)}} = \text{\% of total variation explained by fitted model}$$

R^2 is a proportion, so measured on a 0-1 scale. In simple linear regression, $R^2 = \mathrm{Corr}(X, Y)^2$.

Small R^2 = limited practical significance, even if coefficients are statistically significant.

🤔 When is a large R^2 achievable? When is it not?

→ simple mechanical process vs human behaviour

🤔 When is a small R^2 acceptable?

→ okay for explanation, not prediction

Simple linear regression | Categorical predictors

Data: Human ratings (0-100) for answers from LLM A vs. LLM B for 50 questions.

We focused on scalar predictors, but regression can also handle categorical predictors:

Assumed population model: $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$ where $X \in \{0,1\}$ indicates LLM $\in \{A, B\}$.

Let's assume $\hat{\beta}_0 = 50$ and $\hat{\beta}_1 = 20$. 🤔 What is the interpretation of these coefficients?

The expected (mean) human rating for LLM A is 50.

LLM B is rated, on average, 20 points higher than LLM A.

In this setting, testing $H_0\colon \beta_1 = 0$ is the same as testing for equality of group means.
→ **Regression generalises the t-test framework!**

Recap | Key takeaways from Week 4

[TO ADD]

Class activity | Group assignment based on your RQs

G1: **Language, Communication, and Bias in AI & Media** – Caleb Agoha, Noha Mahgoub, Yunjia Qi

G2: **AI, Generative Models, and Evaluation** – Max Davy, Howard Leong, Audrey Yip

G3: **Media, Platforms, and Audience Response** – Sophie Bair, Charlotte Peart, Michi Wong

G4: **Po** **Institutions** – Celikhan Baylan, Grahm Gaydos, Caleb Tan

G5: **So** **Adoption** – Min Jung, Mia Kussman, Isaac Backer

G6: **Health, Medicine, and Neuroscience** – Amelia Mercado, Laura Wegner, Ines Trichard

G7: **Education, Labor, and Socioeconomic Outcomes** – Rehmat Arora, Yilin Qian, Yue Zhang

G8: **Culture, Mobility, Lifestyle** – Teo Canmetin, Alena Tsvetkova, Fucheng Wang, Nesma Hammouda

Same groups as in week 2!

Everyone not named: please get together in groups of 3.

Class activity | Proposal instructions

Dear students,

This formative assignment is for you to submit a **project proposal of ≤250 words** that outlines your summative plans. Please submit this in **pdf format**.

Please include:

1. A **title**.
2. Your **name**, underneath the title.
3. A clear **research question**.
4. A short description of **why it is interesting** to answer this question.
5. Which **dataset(s)** you are planning to use
6. Which **statistical method(s)** you are planning to use.

Remember:

- The main goal of the summative is for you to demonstrate that you can apply analytical statistics to answer a research questions
- You are strongly encouraged to use the statistical methods taught in this class.
- There will be no extra credit for collecting novel data.

Class activity | Peer review questions

Is there a clear **motivation** for this project?

Does this project seem **feasible** within the constraints of this course?

Does the **dataset** seem appropriate for answering the RQ?

Does the **statistical method** seem appropriate for answering the RQ?
  Does it seem **in scope** for this course?

[TO UPDATE]