Lecture 6: Logistic Regression

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Applied Analytical Statistics

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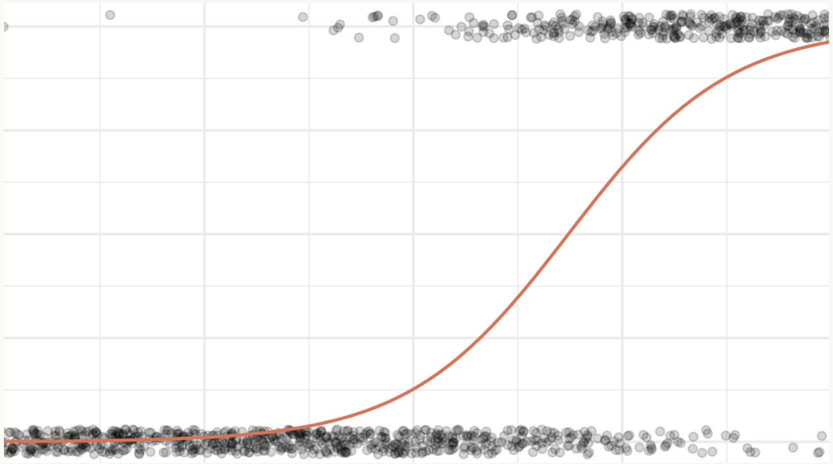
Plan for today | Logistic regression and GLMs



Today we **keep expanding regression** to other types of outcome variables.

We focus on **logistic regression** for binary outcome variables.

We introduce **generalised linear models** (GLMs) as a more general framework.

We also cover different approaches for **evaluating and comparing model fit**.

Recap | Bernoulli distribution

The **Bernoulli distribution** is the discrete probability distribution of a random variable Y which takes the value 1 with probability p and the value 0 with probability $1 - p$.

$$f_Y(p) = \begin{cases} p \text{ if } Y = 1 \\ 1 - p \text{ if } Y = 0 \end{cases}$$

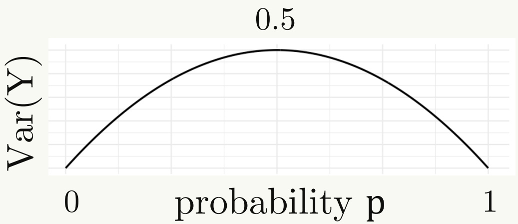
: generalised version of a single coin toss

$$E(Y) = p$$

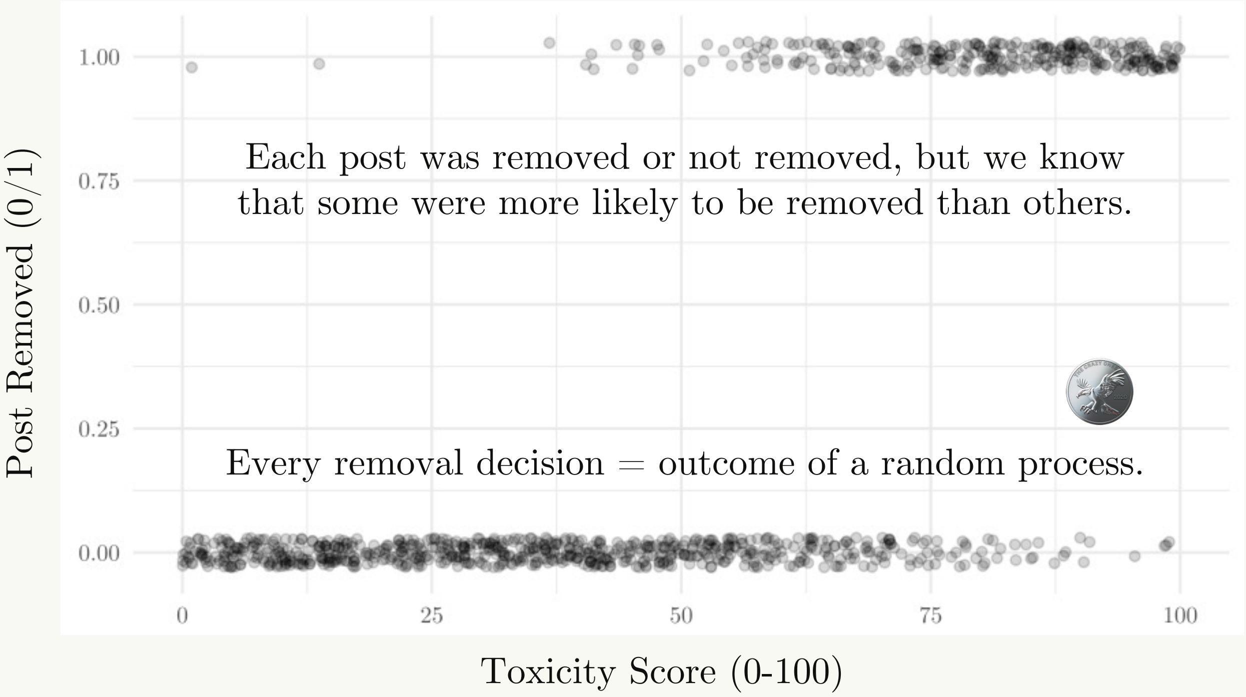
: expected value, i.e. mean = probability p

$$Var(Y) = p(1 - p)$$

: variance depends on probability p



Binary outcomes | Working example



Data: 1,000 content moderation decisions

(simulated)

Outcome: post removed?

Regressors:

- Toxicity score (0-100)
- Verified author (0/1)
- Number of followers

Binary outcomes | Conditional probability

Linear regression models the **conditional mean** of an outcome:

$$E(Y \mid \mathbf{X}) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k$$

where $E(\boldsymbol{\varepsilon} \mid \mathbf{X}) = 0$

The conditional mean of a binary outcome is the conditional probability of $Y = 1$:

$$E(Y \mid \mathbf{X}) = P(Y = 1 \mid \mathbf{X})$$

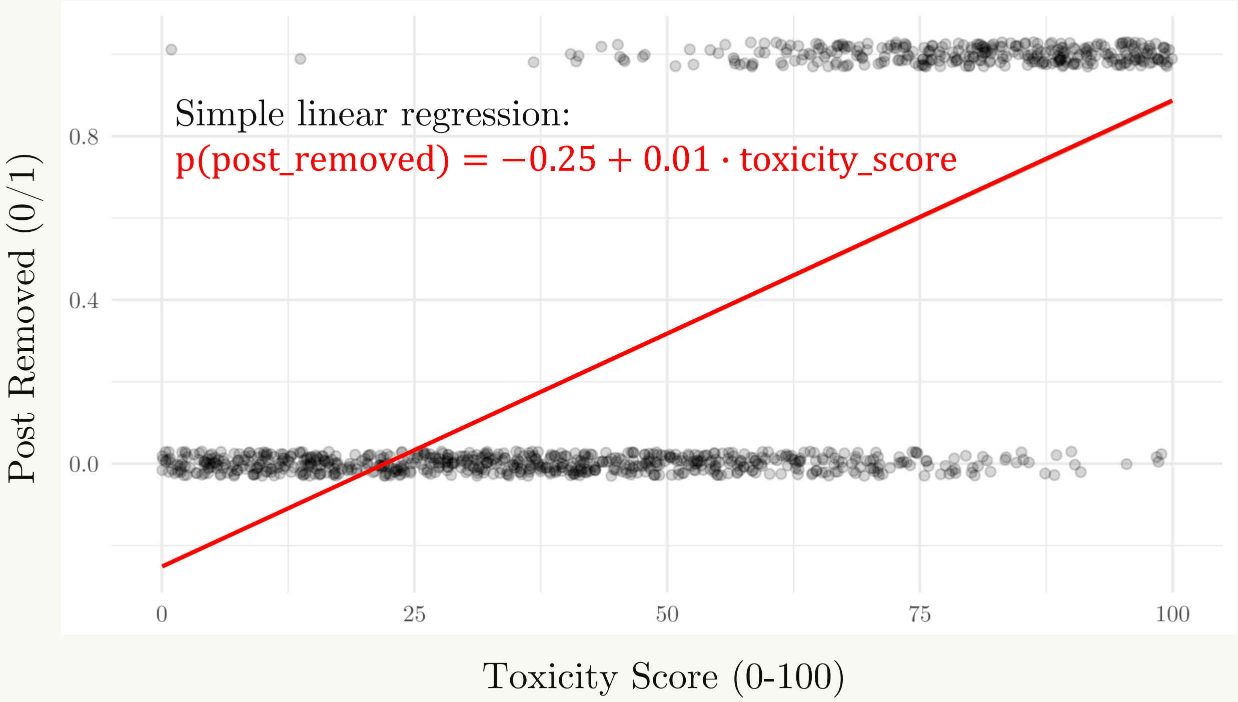
where Y is a **Bernoulli random variable** with $Y \in \{0,1\}$.

In principle, we could still fit linear regression to binary outcome data using OLS:

$$P(Y = 1 \mid \mathbf{X}) = \mathbf{X}\boldsymbol{\beta}$$

where β_1 = change in probability of Y for a one-unit increase in X_1

Binary outcomes | Linear regression



$\hat{\beta}_1 = 0.01$: Each 1-point increase in toxicity score is associated with a 1pp increase in the probability of a post being removed.

$\hat{\beta}_0 = -0.25$: On average, a post with a toxicity score of 0 has a **-25% chance of being removed.**

Binary outcomes | Why linear regression fails

Problem #1: Probabilities are bounded:

$0 \leq P(Y = 1 \mid \mathbf{X}) \leq 1$ for all $\mathbf{X}$ whereas linear functions are **unbounded**.

Linear regression for binary outcomes can predict probabilities < 0 and > 1.

Problem #2: The relationship between $\mathbf{X}$ and $P(Y = 1 \mid \mathbf{X})$ must be non-linear.

In linear regression $\partial P(Y = 1 \mid \mathbf{X}) / \partial X_1 = \beta_1$ is constant for all $\mathbf{X}$.

BUT a constant rate of change is not compatible with hard limits at $Y = 0$ and $Y = 1$.

Therefore, the linear model is misspecified for binary outcomes.

Logistic regression | Key ingredients

Our goal: **ensure predicted probabilities lie in the probability space (0,1)**.

Let $Y \in \{0,1\}$ be a Bernoulli random variable with $p = P(Y = 1) = E(Y)$. Then:

$$\text{odds} = \frac{p}{1-p}$$

map p from $(0,1) \rightarrow (0,\infty)$

$p = 0.8 \leftrightarrow$ odds of 4:1

$p = 0.5 \leftrightarrow$ odds of 1:1

$p = 0.01 \leftrightarrow$ odds of 1:99

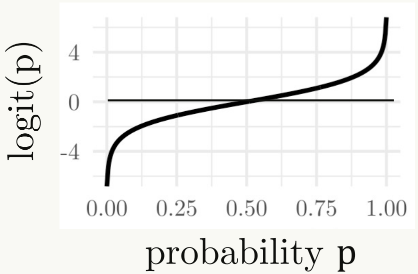
Which transformation can remove the boundary at 0?

By taking the natural logarithm, we arrive at the **logit of p**:

$$\text{logit}(p) = \log\left(\frac{p}{1-p}\right)$$

maps p from $(0,1) \rightarrow (-\infty,\infty)$

"log-odds"



Logistic regression | The logistic model

For **univariate logistic regression** we model:

$$\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 X_i$$

which we denote as

$$\text{logit}(p_i) = \beta_0 + \beta_1 X_i$$

$$p_i = E(Y|X_i)$$

→ Both sides of the equation take values in (-∞,∞).

By exponentiating we can show this is equivalent to:

$$p_i = \frac{1}{1+e^{-(\beta_0+\beta_1 X_i)}}$$

which we denote as

linear model still in here!

$$p_i = \text{logit}^{-1}(\beta_0 + \beta_1 X)$$

"linked" to conditional mean of outcome by logit

→ Both sides of the equation take values in (0,1).

Interpretation | Predicted probabilities

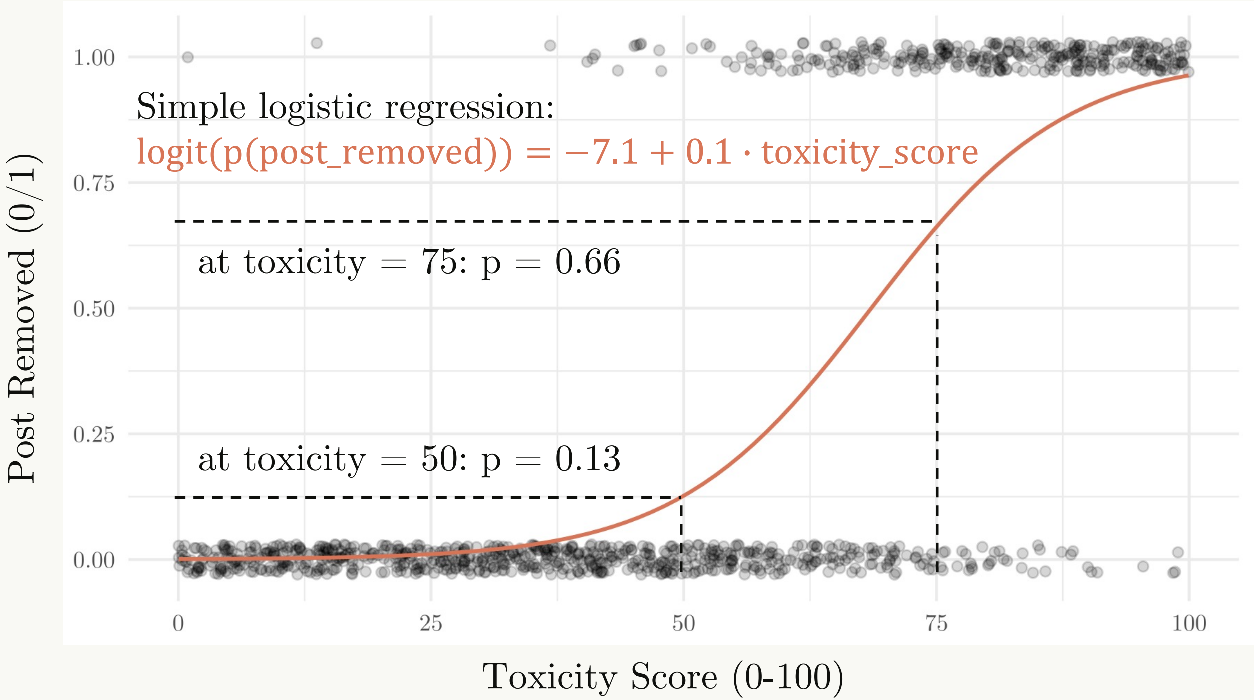
We now have a **well-specified logistic model** that we can fit for binary outcomes.
For every observation X_i , the model specifies the conditional mean $p_i = E(Y \mid X_i)$

$$\log\left(\frac{p_i}{1 - p_i}\right) = \beta_0 + \beta_1 X_i \quad \text{which is equivalent to} \quad p_i = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X_i)}}$$

β_1 = change in **log-odds** for a one-unit increase in X → **unintuitive!**

More intuitive: how does **predicted probability change as X changes**?

Interpretation | Predicted probabilities (cont'd)



We can obtain and interpret predictions at specified values of **X**:

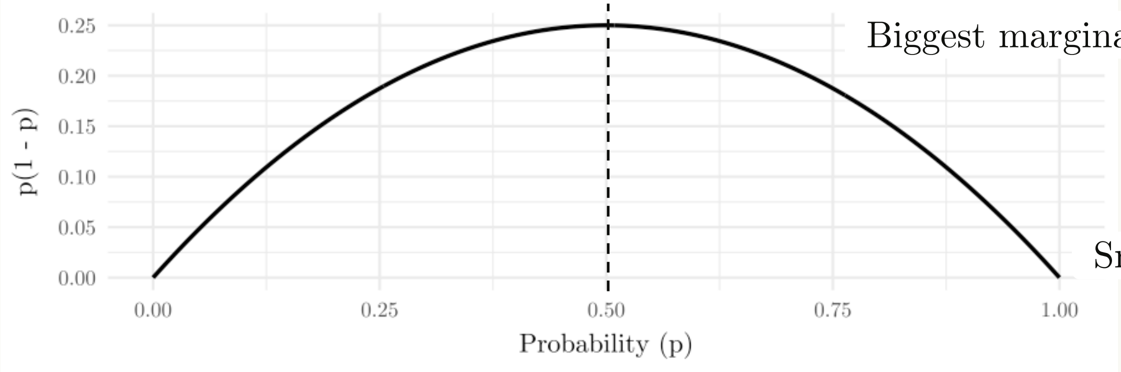
An increase in toxicity score from 50 to 75 is associated with an increase in probability of a post being removed by 53pp.

Interpretation | Marginal effects

More generally, we can quantify the **slope of the fitted logistic regression curve**:

$$p_i = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X)}}$$ taking derivative: $$\frac{\partial p}{\partial X_i} = \beta_1 p_i (1 - p_i)$$ → "marginal effect" of X (not causal)

Slope is not constant, depends on predicted probability level p_i multiplied by constant β_1.

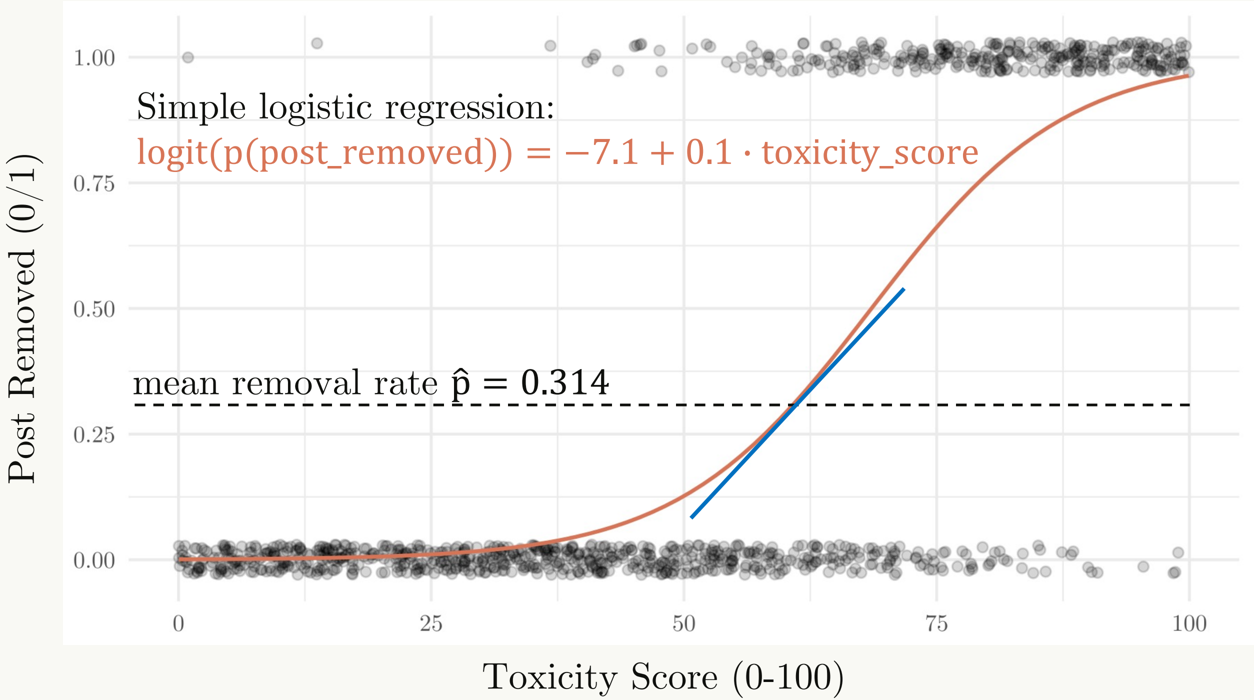


Biggest marginal effect of X at $p_i = 0.5$

→ S-shape of logistic curve

Small marginal effect of X at extremes

Interpretation | Marginal effects at the mean



How do we interpret $\hat{\beta}_1$?

$\partial p/\partial X = \hat{\beta}_1 \hat{p}(1-\hat{p})$
$= 0.1 \cdot 0.31 \cdot (1-0.31)$
$= 0.02$

At mean removal rate $\hat{p}$, a 1-point increase in toxicity score is associated with a 2pp increase in probability of a post being removed.

Interpretation | Odds ratios

Finally, we can interpret coefficients in terms of **odds ratios** (OR).

$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 X$$ is equivalent to $$\text{odds} = \frac{p}{1-p} = e^{\beta_0 + \beta_1 X}$$

As we increase regressor X by one unit:

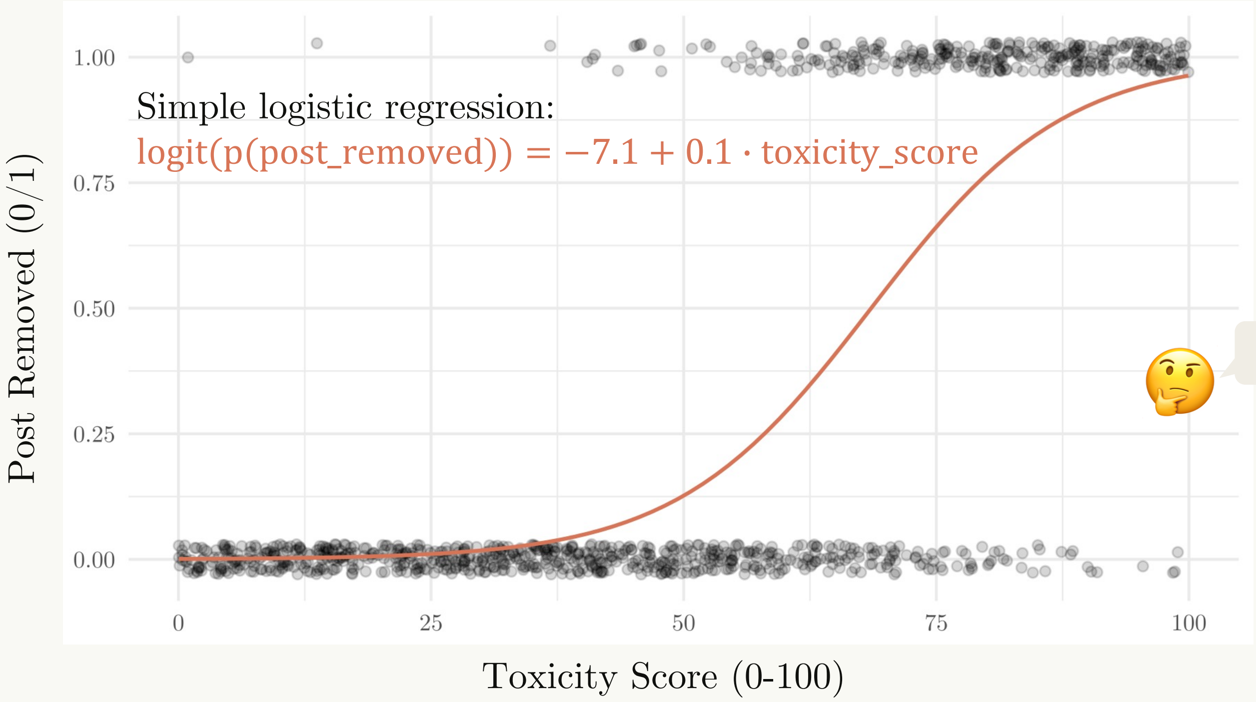
$$\text{OR} = \frac{\text{odds}(X+1)}{\text{odds}(X)} = \frac{e^{\beta_0 + \beta_1 (X+1)}}{e^{\beta_0 + \beta_1 X}} = e^{\beta_1}$$ which is constant across X!

For every one-unit increase in X, the odds in favour of Y=1 multiply by e^{β_1}.

→ **OR>1**: on average, Y=1 becomes more likely as X grows

→ **OR<1**: on average, Y=1 becomes less likely as X grows

Interpretation | Odds ratios (cont'd)



$\hat{\beta}_1 = 0.1 \rightarrow \mathrm{OR} = e^{0.1} = 1.1$:

Each one-point increase in toxicity score is associated with a 10% increase in the odds of a post being removed.

What about a 10-point increase?

$\mathrm{OR} = e^{0.1*10} = 2.7$:

Each 10-point increase in toxicity score is associated with a 170% increase in the odds of a post being removed.

Logistic regression | Multiple regressors

Multivariate logistic regression is the direct analogue of multivariate linear regression:

$$\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 X_{1i} + \cdots + \beta_k X_{ki}$$

where

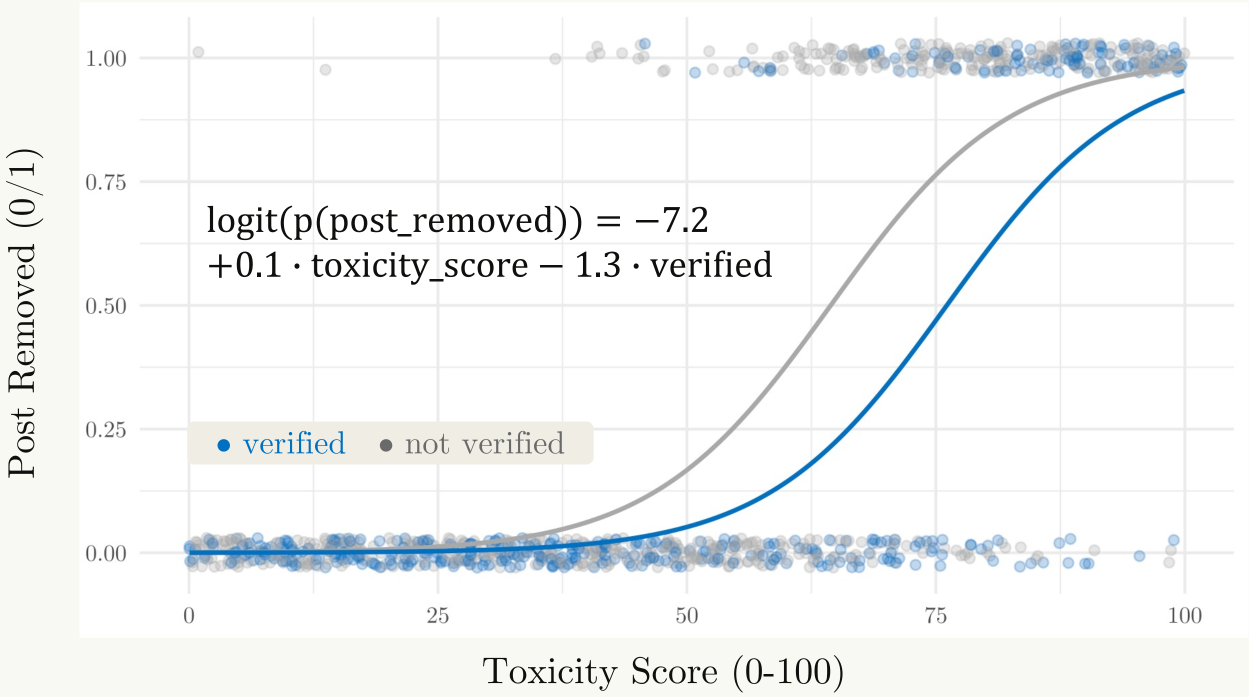
$$p_i = P(Y_i = 1 \mid X_i)$$
$$Y_i \sim \text{Bernoulli}(p_i)$$

We now interpret coefficients **ceteris paribus**, i.e. holding other regressors constant.

β_j = change in log-odds of $Y = 1$ for a one-unit increase in X_j, ceteris paribus

e^{β_j} = multiplicative change in odds of $Y = 1$ for a one-unit increase in X_j, ceteris paribus

Logistic regression | Multiple regressors (cont'd)



$\hat{\beta}_1 = 0.1 \rightarrow \text{OR} = 1.1$:
Each 1-point increase in toxicity score is associated with an increase in the odds of a post being removed by 10%, **ceteris paribus**.

$\hat{\beta}_2 = -1.3 \rightarrow \text{OR} = 0.27$:
Posts from verified accounts have 73% lower odds of being removed than from non-verified accounts, **ceteris paribus.**

Logistic regression | Estimating coefficients

In linear regression, we minimised squared residuals:

$\hat{\beta} = \arg\min \sum (Y_i - \hat{Y}_i)^2$ by OLS, producing closed-form solution $\hat{\beta} = (X'X)^{-1}X'Y$

This breaks down for logistic regression (and other non-linear models).

Instead, we estimate parameters using **Maximum Likelihood Estimation** (**MLE**):

$L(\beta) = \prod_i p_i^{Y_i} (1 - p_i)^{1-Y_i}$ under a Bernoulli model, where $p_i = \dfrac{1}{1 + e^{-X_i\beta}}$

MLE chooses β that maximises $L(\beta)$ via p_i, by numerical optimisation.
We **find the coefficients that make the observed outcomes most likely**.

Logistic regression | Assumptions

Assumptions for consistent estimates, i.e. unbiased in large samples:

Correct functional form: The log–odds are a linear function of parameters $\boldsymbol{\beta}$.

Exogeneity: Regressors X are uncorrelated with unobserved determinants of Y.

Assumptions for MLE to function:

No perfect multicollinearity: Regressors X are not perfectly correlated with each other.

No complete separation: Outcome Y is not perfectly predicted by X.

Specific to logistic regression

Assumptions for correct standard errors and inference:

Independence: Observations are not correlated with each other.

$\approx$ homoskedasticity in OLS

Correct variance specification: Conditional variance depends only on mean: $\text{Var}(Y_i \mid X_i) = p_i(1 - p_i)$

Logistic regression | Uncertainty

In linear regression, we included an error term with specified variance:

$Y_i = \mathbf{X}_i\beta + \varepsilon_i$ where $\varepsilon_i \sim N(0, \sigma^2)$

$Y \in \{0,1\}$

In logistic regression, randomness comes from the Bernoulli outcome itself:

$Y_i \sim \text{Bernoulli}(p_i)$ where $p_i = \text{logit}^{-1}(\boldsymbol{X_i}\beta)$ and $\text{Var}(Y_i \mid X_i) = p_i(1 - p_i)$

heteroskedastic by design

For large n, the MLE of β has an approximately normal sampling distribution:

$\hat{\boldsymbol{\beta}} \sim N(\boldsymbol{\beta}, [\mathbf{X}'\mathbf{W}\mathbf{X}]^{-1})$ where $\mathbf{W} = \text{diag}\big(p_i(1 - p_i)\big)$ variance matrix of Y

Coefficient standard errors depend on the curvature of the likelihood.

→ how much does likelihood vary as we move around $\boldsymbol{\beta}$ flat curve = many plausible $\boldsymbol{\beta}$

Logistic regression | Inference

To test for significance of coefficients in logistic regression, we use a **Wald test**:

$$z = \frac{\hat{\beta}_j - \beta_{j,0}}{SE(\hat{\beta}_j)}$$

The test statistic measures the distance between our sample coefficient and the coefficient value under the null in SE units (see Week 3).

For large n, under H_0, z approximately follows a standard normal distribution: $z \sim N(0,1)$.

🤔 Why can we use standard normal rather than t-distribution?

In logistic regression, there is no constant variance σ^2 that requires separate estimation.
→ **no df correction required**, standard errors follow directly from data and MLE

Generalised linear models | Motivation

We now covered two regression models for two types of outcome variables:
Linear regression for unbounded continuous outcome variables
Logistic regression for binary outcome variables

These models share a common structure:

$$\mathrm{E}(Y_i|\mathbf{X}_i) = \mathbf{X}_i\boldsymbol{\beta} \quad \text{and} \quad p_i = \mathrm{logit}^{-1}(\mathbf{X}_i\boldsymbol{\beta}) \;\; \text{where} \;\; p_i = \mathrm{E}(Y_i|\mathbf{X}_i)$$

Both model **conditional means** and include a **linear component**.

Generalised linear modelling (GLM) extends this structure to other outcomes.
→ all regression is about describing how the expected value of Y changes with X.

Generalised linear models | Three components

Generalised linear modelling (GLM) is a framework for statistical analysis that includes linear regression and logistic regression as special cases. GLMs have three components:

The **systematic component** is the linear predictor $X\beta$.

→ same in all GLMs

The **random component** specifies the distribution of the outcome variable Y.

→ modelling assumption based on type of outcome variable, determines variance structure

$Y_i \sim N(\mu_i, \sigma^2)$ where $\mu_i = E(Y_i|X)$

$Y_i \sim \text{Bernoulli}(p_i)$ where $p_i = E(Y_i|X)$

The **link function** connects the expected value of Y to the linear predictor: $g(E(Y|X)) = X\beta$

→ ensures that transformed outcome is linearly related to predictors

Identity link: $g(\mu_i) = \mu_i$

Logit link: $g(p_i) = \log\left(\frac{p_i}{1-p_i}\right)$

Generalised linear models | Estimating coefficients

We fit all GLMs, like logistic regression, using **maximum likelihood estimation** (MLE).
→ finding the parameters that make the observed data most likely

The likelihood function depends on the random component and link function.

```
glm(
  post_removed ~ toxicity_score,
  data = df,
  family = binomial(link = "logit")
  )
```

Logistic regression in R

(Bernoulli is special case of binomial)

```
glm(
  post_removed ~ toxicity_score,
  data = df,
  family = gaussian(link = ”identity")
  )
```

Linear regression in R

Poisson regression | GLM version

RQ: Is higher ad spend associated with higher engagement on social media ads?

Data: Number of likes for 1,000 Instagram ads.

The outcome is a non-negative integer with no upper boundary $Y \in \{0,1,2,\ldots\}$

The corresponding **random component** is a Poisson distribution:

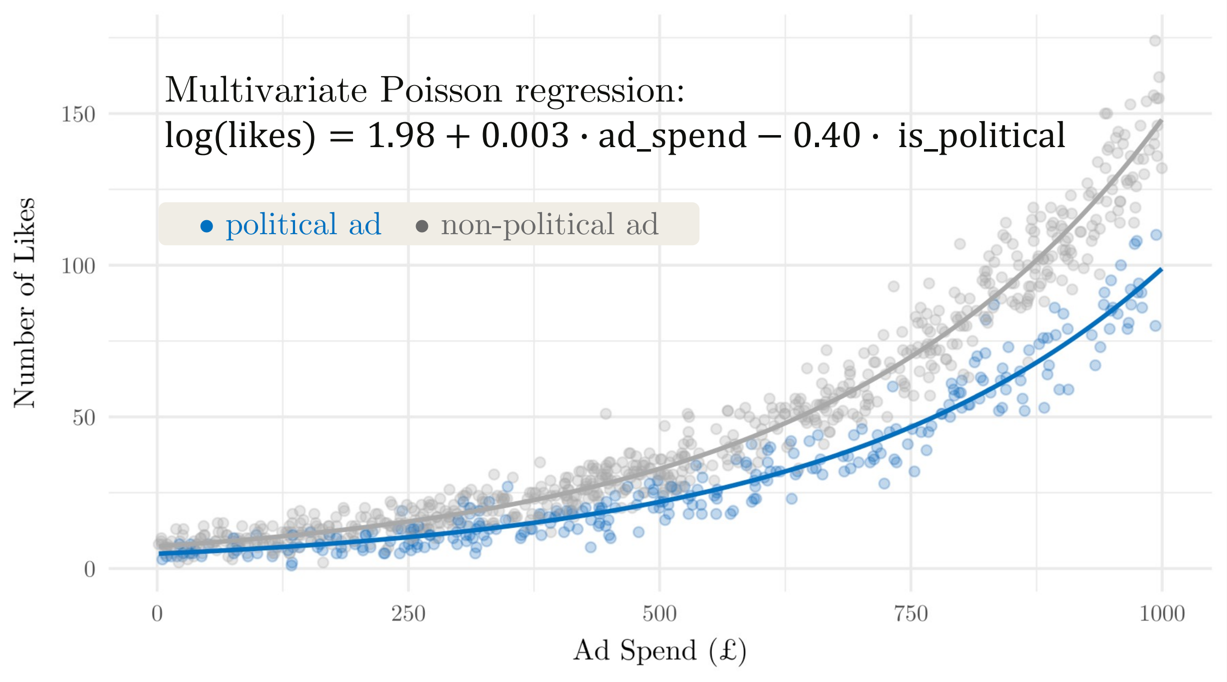
$Y_i \sim \text{Poisson}(\lambda_i)$ where $\lambda_i = E(Y_i|X)$ and $\underline{Var(Y_i|X) = \lambda_i}$

Strong assumption: variance = mean.
Fit negative binomial if violated.

The **link function** needs to map $(0,\infty) \rightarrow (-\infty,\infty)$:

$g(\lambda_i) = \log(\lambda_i)$ from which follows the **Poisson GLM** $\log(\lambda_i) = \mathbf{X}_i\beta$

Poisson regression | Coefficient interpretation



$\hat{\beta}_1 = 0.003$, $e^{0.003*100} = 1.35$: Each 100£ increase in ad spend is associated with a 35% increase in expected likes, **ceteris paribus**.

$\hat{\beta}_2 = -1.3$, $e^{-0.4} = 0.67$: Political ads, on average, receive 33% fewer likes than non-political ads, **ceteris paribus.**

Model comparison | Nested models

Model A is **nested** in model B if B contains all regressors in A plus additional regressors:

A: `likes ~ ad_spend` is **nested** in B: `likes ~ ad_spend + is_political`

Nested models allow for **stepwise theoretical expansion**:
$\rightarrow$ fit baseline THEN add controls THEN add interactions etc.

Nested models allow us to **understand omitted variable bias**:
$\rightarrow$ how much of **ad_spend** coefficient in A was indirect association via **is_political**?

Nested models enable **joint hypothesis testing:**
$\rightarrow$ do **is_political** and other controls **jointly** improve our model?

Model comparison | The Likelihood Ratio (LR) test

A **likelihood ratio (LR) test** compares how well **two <u>nested</u> models** explain observed data:

$$\mathrm{LR} = -2(\log \mathrm{L}_{\mathrm{restricted}} - \log \mathrm{L}_{\mathrm{full}})$$

where **logL** is the fitted model log-likelihood.

Under H_0 of no difference, the test statistic follows a chi-squared distribution: $\mathrm{LR} \sim \chi^2_{\mathrm{df}}$
where df = number of additional parameters (coefficients) in the full model.

This is a very flexible test for significance of one or multiple coefficients:
Does adding these regressors significantly improve model fit?
(full vs. restricted)

For adding a single predictor, for large n, LR test ≈ Wald test.

Model comparison | AIC

We may also want to compare **non-nested models with different functional forms**.

For this, we can use the Akaike Information Criterion (AIC):

$\mathrm{AIC} = -2\ \mathrm{LogL} + 2\mathrm{k}$ where k is the number of parameters

When comparing models, a **lower AIC indicates better model fit**.

When comparing two models A and B, where AIC of A is smaller than AIC of B:

$\exp((\mathrm{AIC_A} - \mathrm{AIC_B})/2)$ is the relative likelihood of B with respect to A

Example: value of 0.5 → B is 50% as likely as A to minimise expected information loss.

Recap | Key takeaways from Week 6

Logistic regression models conditional probabilities.
We model the log-odds of a binary outcome as a linear function of predictors.

Logistic coefficients are interpreted in terms of log-odds and odds ratios.
Coefficients multiply the odds by e^{β}, while probability effects are nonlinear.

Generalised linear models (GLMs) provide a unified regression framework.
We extend regression to different outcome types using distributions and link functions.

Estimation and inference in GLMs rely on maximum likelihood.
We maximise the likelihood implied by the assumed distribution, to enable SEs, tests