Exercise 1.

(a) We have $P(\boldsymbol{x}) = P(x_1)P(x_2|x_1)P(x_3|x_2)P(x_4|x_3)$.

(b) Note that by the definition of the x_i, and letting $\omega_t \sim \mathcal{N}\left(0, \sigma^2\right)$, we have

$$\begin{aligned} X_1 &= \omega_1 \\ X_2 &= X_1 + \omega_2 \\ X_3 &= X_2 + \omega_3 \\ X_4 &= X_3 + \omega_4 \end{aligned}$$

from which we see that the X_i are linear combinations of the ω_i:

$$X_i = \sum_{1 \le j \le i} \omega_j$$

We can represent the X_i compactly as

$$\begin{aligned} X &= \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \\ &= \begin{pmatrix} \omega_1 \\ \omega_1 + \omega_2 \\ \omega_1 + \omega_2 + \omega_3 \\ \omega_1 + \omega_2 + \omega_3 + \omega_4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} \\ &=: A\boldsymbol{\omega} \end{aligned}$$

As a result, the covariance matrix is

$$\begin{aligned} A \cdot \operatorname{cov}(\boldsymbol{\omega}) \cdot A^\top &= A \cdot \sigma^2 I_4 \cdot A^\top \\ &= \sigma^2 A \cdot A^\top \\ &= \sigma^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \sigma^2 \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \end{aligned}$$

(c) Using standard techniques for matrix inversion, we compute the following inverse:

$$A^{-1} = \frac{1}{\sigma^2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

(d) We observe that $A^{-1}_{ij} = 0$ iff $i \neq j$ and x_i and x_j are not adjacent in (the undirected version of) graph.