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ABSTRACT

This article presents a framework for analyzing the dynamic effects of anticipated large demand pressures on asset risk premia. We show that large institutions who can time their entry into the market will trade either at the open, or during periods of unusual demand pressures. We show that if these institutions do enter later in the day, they trade in the same direction as institutions which provide liquidity continuously; institutions therefore appear to exhibit "herding" behavior. We also explore how changing the uncertainty of demand pressures late in the day affects trading costs throughout the day.

MOST FINANCIAL MARKET INVESTORS do not continuously monitor intraday fluctuations in stock prices. Other activities (e.g., outside employment) often take precedence over following the stock market, and there are limitations to processing information (for example, a reader of this page is likely to be unaware of current financial market prices). In fact, for many investors, entry into financial markets occurs once a day or even less frequently. Given these institutional realities, it is of interest to explore how asset markets react when it is known that a set of very large predictable demands will arrive during the afternoon. Does their arrival affect prices only when they arrive, or throughout the day? If the price impact is spread out, then how do trading costs behave during the day? If some traders have the flexibility to "time" their entry into the market, when will they arrive during the day, and how will their entry decisions affect asset risk premia? If market participants anticipate that demands in the afternoon will be unusually volatile, how does this affect the intertemporal behavior of prices and expected returns earlier in the day? The primary focus of this article is the development of tractable methodologies that can address these issues.

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Our basic model posits the existence of two classes of risk-averse, competitive traders. The first class of traders, termed "market makers," remain continuously in the market and attempt to profit from minute-by-minute price fluctuations. The second class of investors are labeled "outside investors," and they represent the remainder of the investors who find continuous monitoring of market movements prohibitively costly. The interaction between market makers and outside investors who participate sporadically has been analyzed previously in the inventory-theoretic models of Ho and Stoll (1983), O'Hara and Oldfield (1986), Zabel (1981), and Grossman and Miller (1988), among others. However, these articles focus on the effect of risky inventory on contemporaneous prices, whereas the contribution of our article derives (i) from demonstrating the dynamic impact of anticipated *future* trades on asset risk premia and expected returns throughout the day and (ii) from analyzing the optimal behavior of outsiders who can time their entry into the market.

Since a primary concern of the article is to explore how the market makers (and, consequently, asset risk premia) react to anticipated future demands, we initially develop a continuous-time model under the assumption that the market makers can perfectly predict the arrival rate for the outside investors during the day.[1] While this is admittedly a strong assumption, it adds considerable tractability. In fact, we find that closed-form solutions are possible for any arrival process, even for a *nonstationary* one. The article thus represents the first attempt to address such general processes without resorting to numerical methods. The analysis shows that during a single trading day the equilibrium risk premium can be described by a combination of modified Bessel functions, which have many properties that lend tractability to the model.

In the latter part of the article, we extend the basic model in two separate ways. The first extension introduces an additional class of outside investors (e.g., large financial institutions), who have the flexibility to select the time at which they wish to enter the market. We find that these traders enter at the commencement of trading if unusual demand pressures are not expected during the day. However, if they know that large order flows will be arriving later in the day, they delay their trades to earn additional expected returns and, in effect, help the market makers to cover the unusual demand. We show that if such traders do trade later in the day, they trade in the same direction as the contemporaneous trades of the institutions that remain in the market continuously. This provides an explanation for the "herding" behavior of institutions, which is based purely on optimal risk sharing.[2] The second extension considers a discrete-time environment where the future demand pressure from the outside investors contains an unpredictable component. This variant of the model provides additional insight by demonstrat-

[1] However, it is not necessary that the outside investors be able to predict the future arrivals.

[2] See Grinblatt, Titman, and Wermers (1993) and Lakonishok, Shleifer, and Vishny (1992) for evidence of institutional herding behavior.

ing the dynamic impact of anticipated highly volatile trades in the afternoon on expected returns and the trading costs of investors earlier in the day.

The article is organized as follows. Section I presents a continuous-time analysis, which assumes that market makers can perfectly anticipate outside investor demands. Section II considers the model's equilibrium when some outside investors can time their trades. Section III presents a model with random demands, and Section IV concludes.

I. The Basic Model

A. Securities and Trading

Trading takes place in the economy during $1/h$ periods, each of length h, beginning at time 0 and ending at time 1. For expositional purposes, time 0 is referred to as the beginning of a "trading day," and time 1 as the end of the day. During the day agents can trade in a riskless bond and a risky security termed the "market portfolio." Payoffs for both securities are realized at the end of the day. The bond's payoff always equals one dollar, while the market portfolio pays $\mu + \sum_{i=1}^{1/h} \varepsilon_{hi}$. We use the variable t to denote calendar time; thus, in the previous expression, $hi = t$. It is assumed that each ε_t is independently normally distributed with mean zero and variance v_e, except for ε_1, which has variance v_1. By distinguishing between the variance of the ε_t for $t < 1$ and ε_1 the model can allow for the possibility that the *interperiod* trading risk during the day (v_e) is different from the *interday* trading risk (v_1).

B. Agents and Equilibrium

All investors are expected utility maximizers. Since the analysis concerns changes in the *market* risk premium, it seems appropriate to assume that there is a continuum of traders, so that each behaves as a competitive price-taking individual. The traders are divided into two groups: "market makers" and "outside investors." We normalize the mass of each investor group to one. That is, if $f(i)$ represents the density of individuals of type i (where i represents either the market makers or the outside investors) then $\int f(i)\, di = 1$. Both groups have negative exponential utility functions over end-of-period wealth, with θ and ϕ representing the risk aversion parameter for each market maker and each outside investor, respectively. Since the mass of the market makers is normalized to one, the holdings of an individual market maker at any time equals the aggregate holdings of all the market makers at that time. It is also evident that one can represent a change in the size of an investor pool by an alteration in its risk aversion parameter.

The market makers represent professional investors who continuously monitor stock market prices for profitable opportunities. These investors therefore trade in every period. The outside investors represent agents who participate in the market on a more sporadic basis. Since the model is designed to analyze trading over a few hours, in this section and the next, we assume that each outside investor only enters the market at a single exogenously specified date. In the next section, we explore the equilibrium when some outside investors have the flexibility to time their trades.

We refer to an outside investor who trades at time t as the "period t outside investor." For expositional purposes, throughout the article "demand" refers to an agent or agents' posttrade holdings. An agent's net purchases therefore equal his demand minus his initial endowment. We denote B_t to be the market makers' aggregate time t demand for the bond, and B_0 to be their initial endowment. Next, let $\hat{b}_t$ and b_t respectively represent the demand and the endowments of each time t outside investor. Similarly, let S_t represent the market makers' time t demand of the risky security, and s_t the demand of the period t outside investors. Initially, the market makers are endowed with S_0 shares, while the period t outside investors enter the market with an endowment of $\hat{s}_t$ shares. For now, we assume that both S_0 and $\hat{s}_t$ are known to the market makers at time zero. In Section III, we develop a model with stochastic future demands and argue that many of the key insights obtained from the present analysis are robust to the assumption of perfect predictability.

Our ultimate goal is to determine the equilibrium price P_t for the risky security in every period. To accomplish this, we begin by conjecturing that in equilibrium the period t risk premium R_t does not depend upon the realizations of the ε_t's. (This conjecture is later confirmed in equilibrium.) One can then write P_t in the following form

$$P_t = \mu - R_t + \sum_{i=1}^{t/h-1} \varepsilon_{hi}. \tag{1}$$

Using equation (1), and letting EU_t represent the expected utility at date t, the market makers' problem can be written as

$$\max_{S_t} EU_t\left[S_1\left(\mu + \sum_{i=1}^{1/h} \varepsilon_{hi}\right) + B_1 \middle| \varepsilon_1, \ldots, \varepsilon_{t-h}\right], \tag{2}$$

given the budget constraint

$$P_t(S_t - S_{t-h}) + B_t - B_{t-h} = 0. \tag{3}$$

Repeated use of equation (3) then permits one to substitute out the B_1 and B_t terms successively from equation (2). Then, maximizing the mean-variance objective obtained from equation (2), it is easy to show that S_1 is given by

$$S_1 = R_1/\theta v_1. \tag{4}$$

Notice that the solution does not depend upon either any of the past values of ε or on any previous value of S_t. Therefore, when solving for S_{1-h}, one knows that the realization of ε_{1-h} will not alter the equilibrium value of S_1. Further, it is evident that S_{1-h} does not depend upon past values of ε either.

These observations imply that the market makers' period-by-period demand can be calculated ex ante at period zero. In fact, upon differentiating the mean-variance objective with respect to each of the $S_t' s$ we obtain

$$S_t = \frac{R_t - R_{t+h}}{\theta v_e}, \tag{5}$$

for $t = h$ to $1 - h$.

The problem of the outside investors is considerably simpler than that of the market makers. Given that all uncertainty occurs only through the normally distributed $\varepsilon_t's$, one is left with the classical one-period mean-variance problem, the solution of which yields the following expression for the *aggregate* demand (i.e., posttrade holdings as per our convention defined earlier) of the period t outside investors:[3]

$$s_t = hR_t/[\phi((1-t)v_e/h + v_1)]. \tag{6}$$

In equilibrium, markets must clear in every trading round; therefore, the sum of the aggregate holdings of the market makers and outside investors must equal zero. Thus, for all time periods during which trading occurs

$$\sum_{i=1}^{t/h} (\hat{s}_{hi} - s_{hi}) = S_t - S_0. \tag{7}$$

Using equations (5) and (6), the above relationship can be rewritten as

$$\sum_{i=1}^{t/h} \left[\hat{s}_{hi} - \frac{hR_{hi}}{\phi\left[(1-t)\frac{v_e}{h} + v_1\right]} \right] = \frac{R_t - R_{t+h}}{h} \frac{h}{\theta v_e} - S_0. \tag{8}$$

We now examine the continuous-time limit as the time between periods, h, goes to zero. Let v equal the limit of v_e/h as h goes to zero. One can think of v as representing the total daily variance of the ε. In addition, let

$$\operatorname*{Lim}_{h \to 0} \sum_{i=1}^{t/h} \hat{s}_{hi} \to \int_0^t \hat{s}(z)\, dz. \tag{9}$$

Then, in the limit, equation (8) becomes

$$\int_0^t \left[\hat{s}(z) - \frac{R(z)}{\phi(v_1 + (1-z)v)} \right] dz = -\frac{dR}{dt} \frac{1}{\theta v} - S_0. \tag{10}$$

Equation (10) is a mixed integral and differential equation that describes the equilibrium behavior of the risk premium. In many cases it is the negative of the risk premium's time derivative that is of interest, since this value

[3] Note that since the total mass of the outside investors is unity, the effective mass of outside investors trading in any given period is h and their aggregate effective risk aversion is therefore ϕ/h, given that trading takes place in the unit interval over $1/h$ periods, each of length h.

corresponds to the instantaneous expected rate at which prices appreciate. It is thus useful to define $r(t) = -dR/dt$.

To calculate a closed-form solution for $R(t)$, we need to eliminate the integral from equation (10). To do so, differentiate with respect to t in order to obtain the following second-order differential equation

$$-\frac{\phi}{\theta}\left[1 - t + \frac{v_1}{v}\right]\frac{d^2R}{dt^2} + R = \hat{s}\phi[v(1-t) + v_1]. \tag{11}$$

The general solution to this equation must involve two arbitrary constants that are fixed through the initial conditions. From equation (10), at time zero one obtains

$$r(0) = \theta v S(0), \tag{12}$$

which constitutes the first initial condition. The second initial condition is derived from the economics of the problem. Equation (4) gives the market maker's demand at the last trading instant, time 1, while equation (5) determines their demand at all other instants. In equilibrium, the demand obtained from equation (5) at $t = 1$ should equal the demand represented by equation (4). Now, as h goes to zero, equation (5) can be written as

$$S(t) = r(t)/\theta v. \tag{13}$$

Therefore, the second initial condition is that at $t = 1$,

$$R(1)v = r(1)v_1. \tag{14}$$

A pair of transformations on r and t show that equation (11) is a variant of a classical problem from physics and engineering, known as Bessel's equation. Let $\tau = \phi[1 - t + v_1/v]/\theta$, implying $dt/d\tau = -\theta/\phi$, $dR/d\tau = -(\theta/\phi)(dR/dt)$, and $d^2R/d\tau^2 = (\theta/\phi)^2 d^2R/dt^2$. Now rewrite equation (11) as

$$\tau\frac{d^2R}{d\tau^2} - \left(\frac{\theta}{\phi}\right)^2 R = -\tau\frac{\theta^3}{\phi^2}v\hat{s}. \tag{15}$$

While the homogenous part of equation (15) is a form of Bessel's equation, the general solution to equation (15) is considerably simplified by assuming that $\hat{s}(\tau)$ is analytic in τ. (Although, later on, much of the analysis will dispense with this requirement.) Thus, for now let

$$\hat{s}(\tau) = \sum_{i=0}^{\infty} a_i\tau^i. \tag{16}$$

For the purposes of application, equation (16) is general enough that it should not pose any difficulties.

The solution to equation (15) involves a linear combination of modified Bessel functions of the first and second kinds. These functions are usually denoted by $I_p(x)$ and $K_p(x)$ where x (strictly positive) is an independent variable, and p is the "order." The functions I_p and K_p satisfy numerous recursive relationships, and are well-behaved monotonic functions. For example, I_p and K_p are both strictly positive, with $dI_p/dx > 0$ and $d^2I_p/dx^2 < 0$, while $dK_p/dx < 0$ and $d^2K_p/dx^2 > 0$.

The theorem below states the solution to equation (15).

THEOREM 1: *Let C_1 and C_2 represent two arbitrary constants. Then, if the outsiders' endowment process satisfies equation* (16), *the solution to equation* (15) *is given by*

$$R = \left\{ \sum_{n=0}^{\infty} \left[\sum_{j=0}^{n} \left(\frac{2\theta}{\phi} \right)^{1-2j} \tau^{n+1-j} \prod_{k=0}^{j-1} 4(n+1-k)(n-k) \right] a_n \right\} \frac{v\phi}{2} + C_1 \tau^{1/2} I_1\left(2\frac{\theta}{\phi}\tau^{1/2} \right) + C_2 \tau^{1/2} K_1\left(2\frac{\theta}{\phi}\tau^{1/2} \right) \tag{17}$$

where, for notational convenience, we define

$$\prod_{k=0}^{-1} 4(n+1-k)(n-k) \equiv 1, \tag{18}$$

and where C_1 and C_2 are defined by equations (54) *and* (55) *in the Appendix.*

Proof: See the Appendix.

The solution given in Theorem 1 imposes some restrictions on price movements during the day. Notice that if the endowment of the outside investors follows a polynomial of order n, then the inhomogeneous part of the risk premium (i.e., R given $C_1 = C_2 = 0$) follows a polynomial of order $n + 1$. Given the properties of I_p and K_p, this significantly restricts R's path through time. For example, if $a_i = 0$ for $i > 0$ then the expected rate at which prices appreciate (r) has its derivative in time change sign at most once during the day, as the proposition below states.

PROPOSITION 1: *If $a_i = 0$ for all $i > 0$, then the expected rate of price appreciation changes direction ($dr/dt = 0$) at most once during the day.*

Proof: Using equation (17) and the fact that all the a_is except a_0 equal zero, we obtain

$$r = -\theta a_0 v - C_1 \frac{\theta}{\phi} I_0\left(2\frac{\theta}{\phi}\tau^{1/2} \right) + C_2 \frac{\theta}{\phi} K_0\left(2\frac{\theta}{\phi}\tau^{1/2} \right). \tag{19}$$

Next differentiate r with respect to $\tau^{1/2}$, yielding

$$\frac{dr}{d\left(2\frac{\theta}{\phi}\tau^{1/2} \right)} = -C_1 \frac{\theta}{\phi} I_1 - C_2 \frac{\theta}{\phi} K_1. \tag{20}$$

From the linear relationship between t and τ, the above equation implies that, if $dr/dt = 0$ at some point during that day, then C_1 and C_2 have opposite signs. However, the derivatives of I_1 and K_1 have opposite signs (see, e.g., Andrews (1985)), so if dr/dt ever equals zero, then r is either a strictly convex or concave function. Q.E.D.

The case considered in the above corollary is of interest since it corresponds to a day in which the outside investors arriving at one point in time are identical in terms of their risk aversion and endowments to those arriving at any other point in time. If this represents a typical trading day, the analysis predicts that "on average" the market's expected rate of price appreciation should change sign at most once during the day. In fact, Harris (1986) and Jain and Joh (1988) find that the average pattern involves a large positive return during the first and last 15 minutes of each day, with a small return in the middle of the day. Since this involves only one sign change on average, the empirical results are consistent with the corollary discussed above.

C. Comparative Statics

It is possible to obtain some sharp conclusions from the model when θ/ϕ, the risk aversion of market makers relative to that of outside investors, is small. The case for studying the solution for small values of θ/ϕ is compelling because market makers are typically large financial institutions, which can be expected to have a large risk-bearing capacity. From a technical standpoint, the analysis simplifies for small values of θ/ϕ because the integral differential equation (10) can be approximated by an ordinary differential equation. As the next theorem shows, when the market makers are far less risk averse than the outsiders, the expected rate of price appreciation is approximately proportional to the aggregate endowment brought to market.

Before proceeding, the analysis requires one last definition. We write $g(x) \sim h(x)$ as $x \to \underline{x}$ if $\lim_{x \to \underline{x}} g(x)/h(x) = 1$.

THEOREM 2: *As $\theta/\phi \to 0$, the solution to equation* (10) *approaches the solution to*

$$\theta \int_0^t \hat{s}(z)\,dz = -\frac{dR}{dt}\frac{1}{v} - \theta S_0. \tag{21}$$

Formally, if R^* solves the above equation, then as $\theta/\phi \to 0$, $R \sim R^*$.

Proof: Let R^* represent the solution to equation (21), and define $\delta(t) = R(t) - R^*(t)$. One can therefore rewrite equation (10) as

$$\int_0^t \left[\theta \hat{s}(z) - \frac{\theta}{\phi}\frac{R^*(z) + \delta(z)}{(v_1 + (1-z)v)}\right] dz = -\left[\frac{dR^*}{dt} + \frac{d\delta(t)}{dt}\right]\frac{1}{v} - \theta S_0. \tag{22}$$

From equation (21) R^* is finite for all t, and from Theorem 1, R is finite for all t. Thus, δ is also bounded for all t, and the terms multiplied by θ/ϕ in the integral become negligible as $\theta/\phi \to 0$. This, together with the observation that R^* must solve equation (21), reduces the above equation to

$$0 = -\frac{d\delta(t)}{dt}\frac{1}{v} \tag{23}$$

for all t, leading one to conclude $R^*(t) \to R(t)$ as $\theta/\phi \to 0$. Q.E.D.

Notice that Theorem 2 does not require $\hat{s}$ to be analytic, and so allows one to produce solutions under an even larger set of endowment processes. Note also that the second term on the left-hand side of equation (10) disappears in equation (21), so that the finite horizons of the outside investors cease to matter for small θ/ϕ.

For notational convenience, let $\hat{S}(t) = \int_0^t \hat{s}(z)\,dz$. Then equation (21) and the boundary condition given by equation (14) imply that

$$R = \theta v_1\left[S_0 + \hat{S}(1)\right] + \theta v S_0 + \theta v \int_0^1 \hat{S}(z)\,dz - \theta v S_0 t - \theta v \int_0^t \hat{S}(z)\,dz. \tag{24}$$

The simplicity of R's characterization for small θ/ϕ allows one to develop a wide range of comparative statics. We only develop a few of the many possible variations and leave the rest for the readers, depending on their interest. Consider first the case of two days that are identical in all ways except for the trade of a single large institution. Further, assume that this institution will enter the market in the late trading hours and has a large endowment of the security. This situation corresponds, for example, to sunshine trades or the anticipated trades of index funds designated for execution late in the day. The next proposition describes how the anticipated afternoon trade imposes trading costs on investors throughout the day.

PROPOSITION 3: *For small values of θ/ϕ, increasing $\hat{s}(1)$ has no impact on the expected rate of price appreciation. Instead, the market adjusts to the expected increase in supply by increasing the risk premium throughout the day. The expressions for the derivatives are given by*

$$\frac{dR}{d\hat{s}(1)} \sim \theta(v_1 + v) \quad \text{as } \theta/\phi \to 0^+ \tag{25}$$

$$\frac{dr}{d\hat{s}(1)} \sim 0 \quad \text{as } \theta/\phi \to 0^+. \tag{26}$$

Proof: These results follow directly from equations (21) and (24). Q.E.D.

At first glance the results in Proposition 3 may seem odd. One expects the market makers to keep their morning holdings low in order to accommodate the late day trades. If so, then Proposition 1 suggests that a late-day increase in selling pressure reduces the expected rate of price appreciation during the entire trading day. However, as Proposition 3 indicates, this is not the case. As θ/ϕ becomes small, any increase in the risk premium has only a second-order effect on the sales from the outside investors to the market makers. The larger market risk premium thus acts only to encourage the market makers to hold a larger end-of-day position.

The issue we consider next is the effect of an increase in the selling or buying pressure at an arbitrary point in the *middle* of the day on trading costs in the morning and afternoon. As with the previous result, one obtains the answer through an examination of equations (21) and (24). From equation (21), it follows that the expected rate of price appreciation is only altered following the time at which the trade arrives (i.e., the $\hat{s}$ changes). On the other hand, equation (24) shows that the actual risk premium increases throughout the day, beginning at $t = 0$. The analysis thus demonstrates how both morning and afternoon traders pay the cost of absorbing a large trade in the middle of the day.

II. Equilibrium When Some Outside Investors Can Time Their Entry

The basic model assumed that costs of monitoring the market preclude outside investors from timing their trades. The present section begins with the reasonable premise that some outside investors (e.g., large financial institutions) can time when they will enter the market. These agents (termed "discretionary outsiders") thus enter the market during the day at a time of their choosing.

Because a discretionary outsider only trades once, once he has entered the market his problem is exactly the same as that faced by the nondiscretionary outsiders. Therefore, the number of shares he holds at the end of trading must solve the mean-variance objective of outside investors. Addressing the question of when a discretionary outsider will enter the market requires one to calculate a discretionary outsider's welfare for each possible entry date. Let y_0 represent a discretionary outsider's initial holdings of the bond, x_0 his initial holdings of the stock, and γ his level of risk aversion. Then a discretionary outsider's utility ($U(t)$) if he enters at time t can be written as,

$$U(t) = \frac{R(t)^2}{2\gamma[(1-t)v + v_1]} + y_0 + x_0[\mu - R(t)] - \frac{\gamma}{2}x_0^2 tv. \tag{27}$$

If $x_0 \neq 0$, then a discretionary outsider has an incentive to enter the market as early as possible in order to avoid holding an unhedged position. It follows that if the absolute value of x_0 is sufficiently large, all discretionary outsiders will have an incentive to enter at time zero. However, as will be seen, if demand pressures later in the day are significant enough relative to x_0, then the discretionary outsiders will delay their trades to take advantage of higher expected returns later in the day. This effectively puts an upper bound on how quickly prices can change during the day.

The next proposition begins the process of characterizing the behavior of the discretionary outsiders by showing that, except possibly at time zero, they never concentrate their trading together. That is, outsiders never enter simultaneously to purchase or sell a strictly positive mass of the stock after the opening bell.

PROPOSITION 4: *If the integral of $\hat{s}$ is continuous, then with the exception of time $t = 0$, the discretionary outsiders never purchase or sell a strictly positive mass of the stock at any one point in time.*

Proof: The proof is by contradiction. Suppose the discretionary outsiders sell a strictly positive mass of the stock at time t^*. Since it is assumed that the integral of the endowment process for the nondiscretionary outsiders is continuous, the aggregate endowment brought to market is continuous throughout the day, and in particular at time t^*. This implies that if the discretionary outsiders sell a strictly positive mass at time t^*, the market makers must buy a strictly positive mass at time t^*. From equation (13), one must therefore have $\lim_{t \to t^*-} r(t) < r(t^*)$.

For a discretionary outsider to sell stock at time t^*, it must be that $U(t^*)$ is strictly greater than $U(t)$ for all t, and in particular for $t < t^*$. Next consider dU/dt, which equals

$$\frac{dU}{dt} = \frac{R^2 v}{2\gamma[(1-t)v + v_1]} - \frac{\gamma}{2}x_0^2 v + r\left[x_0 - \frac{R}{\gamma[(1-t)v + v_1]}\right]. \tag{28}$$

If the discretionary outsiders enter at time t^*, then the above expression must be at least weakly negative for $t < t^*$ and t close enough to t^*. Also note that the final expression in square brackets equals the discretionary trader's net sales. (Recall that equation (6) represents the demand of a discretionary outsider, with s_t replaced by y_t, h set equal to 1, and v_e replaced by v.) If dU/dt is weakly positive for $t < t^*$, then at t^* it must be strictly positive, since the term in square brackets above is positive, and r increases by a discrete amount at t^*. Therefore, the discretionary outsider cannot maximize his utility by entering at t^*, a contradiction. A similar argument provides a contradiction for the case where the discretionary outsiders are net buyers. Q.E.D.

Proposition 4 shows that if the discretionary outsiders enter during the day, they must spread out their entry. The question then arises as to the characterization of the risk premium during such periods. Intuitively, the discretionary outsiders will try to "help out" the market by entering during times of extreme demands. For example, suppose $x_0 = 0$. Then, in terms of risk, it costs a discretionary outsider nothing to delay his trades. His optimal entry time must therefore correspond to a period when the risk premium is either very large (if he will purchase shares) or very small (if he will sell shares). Such periods occur precisely when the nondiscretionary demands are unusually large or small. If x_0 is not zero, then the same intuition applies, except that now a discretionary outsider who delays his trade faces a cost in terms of unhedged risk. Thus, entry during the day will still occur only in times of unusual demand or supply by the nondiscretionary outsiders, with the additional stipulation that fluctuations in the order flow must be sufficiently large to warrant the delay in trade. Otherwise, discretionary outsiders with non-zero endowments will enter only at time zero. The proposition below formalizes this intuition by further characterizing the discretionary outsiders' optimal trading pattern. Note that given the trading pattern, the complete equilibrium is characterized by adding the discretionary outsiders' endowments to the first term on the left-hand side of equation (10) and solving the resulting differential equation; for small θ/ϕ, the solution is given by equation (24).

PROPOSITION 5: *Suppose the discretionary outsiders trade during some time interval $t \in [t_1, t_2]$, where t_1 and t_2 are not equal to either 0 or 1. Then if the discretionary outsiders are net buyers (sellers):*

a. The expected rate of price appreciation increases (decreases) during the interval, and
b. The market makers are also net buyers (sellers).

Proof: Let U_0 represent the expected utility of a discretionary outsider who retains his initial position throughout the trading day. Then,

$$U_0 = \mu x_0 + y_0 - 0.5\gamma x_0^2[v + v_1]. \tag{29}$$

Note from Proposition 4 that the discretionary outsiders spread themselves out between t_1 and t_2. This implies that the utility of the traders, $U(t)$, must be constant throughout this interval (else these traders would concentrate at the time points with the highest utility). We can therefore write $U(t) = k$ for $t \in [t_1, t_2]$, where k is a constant. Thus, setting $U(t) = k$ in equation (27) and performing some algebra yields

$$\begin{aligned} R^2 - 2x_0R\gamma[(1-t)v + v_1] + x_0^2\gamma^2[(1-t)v + v_1]^2 \\ = 2\gamma[k - U_0][(1-t)v + v_1]. \end{aligned} \tag{30}$$

Now differentiate with respect to t, and solve for r to obtain

$$r = x_0\gamma v + \frac{\gamma v[k - U_0]}{R - x_0\gamma[(1-t)v + v_1]}. \tag{31}$$

Since the terminal holdings of the discretionary outsiders equal $R/\gamma[(1-t)v + v_1]$, the denominator of the second term in equation (31) is positive if they are net buyers, and negative if they are net sellers. To prove the proposition's claim the remaining step is to show that the denominator of the second term decreases as t increases. Once again, if one performs some algebraic manipulations on equation (30), it can be rewritten as

$$(R - x_0\gamma[(1-t)v + v_1])^2 = 2\gamma[(1-t)v + v_1][k - U_0]. \tag{32}$$

Since entering the market and trading must provide a discretionary outsider a higher expected utility than holding his initial endowment, the last term in the equation must be positive. Therefore, the right-hand side of the equation decreases as t increases, and so the left-hand side must also be decreasing in t. Q.E.D.

Proposition 5 shows that when the market faces unusually large order fluctuations, the discretionary outsiders enter to offset the order flow. This creates a "herd" effect in the sense that when the financial institutions that remain continuously in the market are heavy buyers, a large number of other institutions enter and follow their lead.[4] Contrast this to the result in Scharfstein and Stein (1990). In their article, asymmetric information encourages institutional traders to mimick each other for reputational reasons. Here the herd effect obtains purely due to price pressures. When unusual order flows arrive, the market rewards those who take the other side with unusually high returns, which, in turn, encourages entry. Thus, we show that "lemming-like" behavior can be completely rational and stem purely from optimal risk sharing. This result is consistent with the herding behavior of financial institutions documented in Grinblatt, Titman, and Wermers (1993) and Lakonishok, Shleifer, and Vishny (1992).

III. A Model with Random Future Demands

The previous two sections developed a continuous-time model under the assumption that market makers are able to predict future demands by outside investors. As argued earlier, while this is a strong assumption, it does allow us to examine questions of interest within a tractable setting. We now develop an intertemporal model with uncertain future demands and examine the effect of anticipated, highly *volatile* trades (as opposed to trades of large magnitude) on risk premia and expected price changes throughout the day.

We use the techniques of Hirshleifer, Subrahmanyam, and Titman (1994) to develop an intertemporal discrete-time setting in which market makers face random demands in the future. Departing somewhat from the article's previous notation, assume there are $T - 1$ trading rounds. Period T represents the terminal date at which the security is liquidated at a value V_T. The market makers' wealth evolves according to

$$W_t = W_{t-1} + S_{t-1}(P_t - P_{t-1}) \tag{33}$$

[4] Note that discretionary outsiders trade at points other than the open only if they expect large *non-discretionary* demands to arrive during the day. Our analysis thus implies that some degree of non-discretionary trading is necessary to sustain intraday trade. In particular, it is easy to show that if all outside investors are discretionary, they will trade only at the open to share risk as quickly as possible with the market making institutions. Also note that unlike Admati and Pfleiderer (1988), our model quantifies exactly when the discretionary outsiders will concentrate their trading.

Because $T-1$ represents the final trading round, the market makers must solve the standard mean-variance problem in this period, which yields

$$S_{T-1} = \frac{E_{T-1}(V_T) - P_{T-1}}{\theta \operatorname{var}_{T-1}(V_T)}. \tag{34}$$

Using the methodology of Hirshleifer, Subrahmanyam, and Titman (1994), one can show that for $t = 1, \ldots, T-2$

$$S_t = \frac{E_t(P_{t+1}) - P_t}{\theta a_1^t} + E_t(S_{t+1}) \frac{a_1^t - a_2^t}{a_1^t} \tag{35}$$

where a_1^t and a_2^t are the elements in the first row of the matrix

$$\left[V_t^{-1} + \frac{1}{a_1^{t+1}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right]^{-1}$$

and V_t is the variance-covariance matrix of P_{t+1} and $E_{t+1}(P_{t+2})$ conditional on the information set at t, and $a_1^{T-1} = v$.[5] The details of this derivation are mechanical and are omitted for brevity (but are available from the authors).

We assume that, in each period t, there is a random supply shock of z_t. The mean supply at each date is denoted $\overline{Y}_t$, and this can vary over time. Thus, in each period the total supply in the market equals $\overline{Y}_t + z_t$.[6] As in the previous sections of the article, the value of the stock evolves stochastically over time, so that $V_T = \mu + \sum_{i=1}^{T} \varepsilon_t$, where each ε_t is revealed following trade at time t. To solve for the equilibrium, conjecture that the risky security's price $P_t = \mu + \sum_{i=1}^{t} \varepsilon_i - R_t$, where R_t is a risk premium and assume that $R_t = \kappa_t + k_t z_t$, where κ_t is the mean risk premium at time t and k_t is the response of the risk premium to the supply shock. Market clearing implies $S_t = \overline{Y}_t + z_t$. Define

$$M_t = \left[a_1^t - a_2^t \right] E_t(S_{t+1}). \tag{36}$$

Straightforward manipulations using the method of undetermined coefficients show that

$$\frac{k_t}{\theta a_1^t} = 1 \tag{37}$$

$$\frac{\kappa_t - \kappa_{t+1}}{\theta a_1^t} + \frac{M_t}{a_1^t} = \overline{Y}_t \tag{38}$$

$$M_t = \left(\frac{\kappa_{t+1} - \kappa_{t+2}}{\theta a_1^{t+1}} + \frac{M_{t+1}}{a_1^{t+1}} \right) (a_1^t - a_2^t) \tag{39}$$

[5] The expression for S_t we obtain is analogous to expression (6b) in the two-period model of Brown and Jennings (1989, p. 531).

[6] Note that here, for tractability, we have assumed price-insensitive outside investor demands, unlike in the earlier sections.

for $t = 1, 2, \ldots, T-2$. Further,

$$V_t = \begin{bmatrix} v + k_{t+1}^2 \operatorname{var}(z_{t+1}) & v \\ v & v \end{bmatrix}. \tag{40}$$

This relationship, together with equation (37), can be shown to imply that

$$a_1^t = \frac{(a_1^{t+1})^2 \theta^2 \operatorname{var}(z_{t+1})}{1 + \theta^2 a_1^{t+1} \operatorname{var}(z_{t+1})} + v \tag{41}$$

and that $a_2^t = v$ for all t. The values to begin the backward recursion are $k_{T-1} = \theta v$, $M_{T-1} = 0$, $\kappa_{T-1} = \theta v \overline{Y}_{T-1}$, $a_1^{T-1} = v$. The variable $M_{T-2} = (a_1^{T-2} - a_2^{T-2})(\kappa_{T-1}/\theta v)$, and κ_{T-2} then solves

$$\frac{\kappa_{T-2} - \kappa_{T-1}}{\theta a_1^{T-2}} + \frac{M_{T-2}}{a_1^{T-2}} = \overline{Y}_{T-2}. \tag{42}$$

The mechanics of the recursion are as follows. Obtain the $a_1^t s$ from equation (41) (using $a_1^{T-1} = v$) and note that $a_2^t = v$ for all $t = 1, \ldots, T-1$. Then obtain the k_t's from equation (37). We thus obtain the endogenous quantities a_1^t, a_2^t, and k_t. Knowing κ_{T-1}, κ_{T-2}, and M_{T-2} from above, one can then calculate κ_{T-3} from equation (38) and M_{T-3} from equation (39) and we have thus recursed the system back a step. Repeating the process, one can obtain all the remaining endogenous quantities κ_t and M_t.

The following simulations assume 20 trading rounds and consider the parameter values $\theta = 0.5$ and $v = 1$. Figure 1 examines the behavior of the mean risk premium κ_t in three different scenarios to the base case in which the mean supply and variance of the unanticipated supply shocks are set to unity in each period.

In the first case, there is a permanent change in the mean supply of 10 in the first trading round. Notice that the mean risk premium begins much higher and then eventually reaches a level much closer to the base case. In the second case, there is a change of supply by 10 in the 10th period, i.e., in the "middle of the day." Here the risk premium is much lower than the first case, until the extra supply enters the market. After period 10, both Cases 1 and 2 have identical expected supplies from the outsiders, and identical mean risk premia. This result also obtains in the basic model (see the discussion following Proposition 2 in Section I). Finally, in the third case, there is a change in the mean supply of 10 at the last trading round alone. As one can see, the mean risk premium remains above the base case and then eventually equals that of the first case in the last period. These simulations indicate that the more general results derived in the continuous-time case are not qualitatively altered in the case of stochastic demands.[7] In addition, the stochastic demand model yields insights into how *uncertainty* about late-day demands alters early morning prices and expected returns. Figure 2 considers the

[7] Further, since outside investor demands are inelastic in this section, the analysis also indicates that the elasticity of outside investor demand does not appear to affect our results.

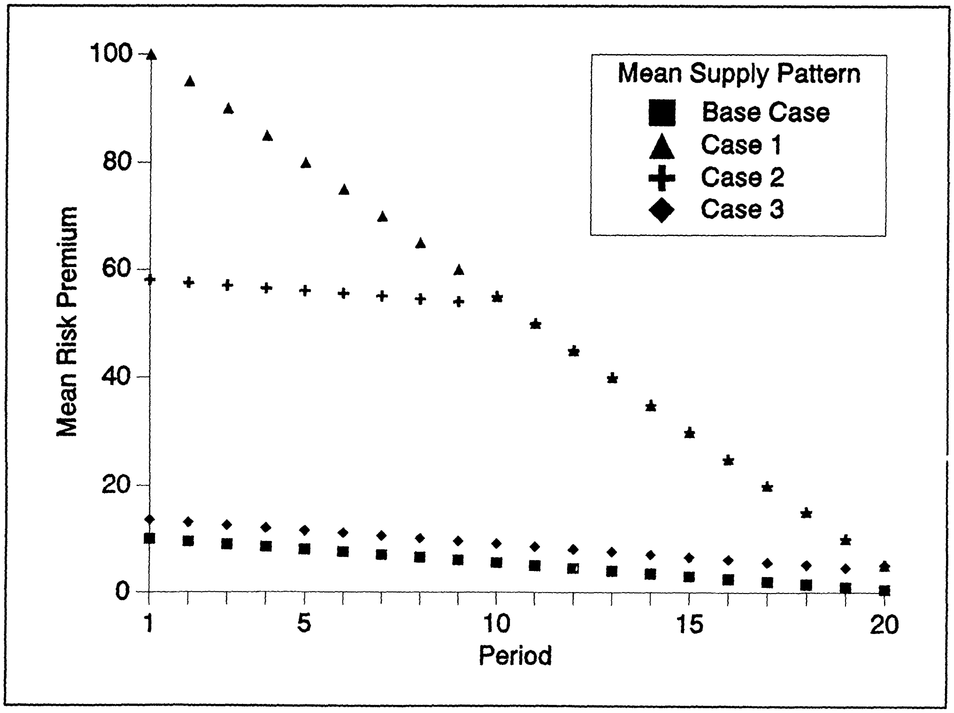

Figure 1. The mean risk premium for different patterns of mean supply of the risk asset. The mean risk premium is plotted for different patterns of mean supply. The risk aversion of market makers θ is 0.5 and there are 20 trading periods. The variance of the value innovation, v, and the variance of the supply shock, var(z_t), are each set to unity in all periods. The following values of the mean supply of the risky asset, $\bar{Y}_t$, are assumed: Base Case, 1 in all periods; Case 1, 10 in all periods; Case 2, 1 in Periods 1 to 9, and 10 in periods 10 to 20; and Case 3, 1 in Periods 1 to 19, and 10 in Period 20.

effect of a high-variance *random* supply shock with a variance of 10 on the responsiveness of the risk premium to the contemporaneous supply shock k_t.[8]

The figure compares two cases to the base case in which the variance of the random demand is unity throughout the day: in the first case there is a high variance shock at the end of the day (in the last trading round), and in the second case there is a high-variance shock in the *middle* of the day (in the 10th trading round). (The mean supply $\bar{Y}_t$ is set to unity in each period.) The figure clearly shows that the risk premia begin responding to the future high-variance shock several rounds in advance. This quantifies how anticipated high-variance trades towards the afternoon increase the trading costs of the outside investors who trade in the morning as well.

[8] It can easily be shown that in our setup, the mean risk premia κ_t are invariant to changes in the variance of the supply shock, var(z_t), provided the mean supply, $\bar{Y}_t$, is constant throughout the day. Thus, we focus on k_t, the responsiveness of the risk premium to the contemporaneous supply shock.

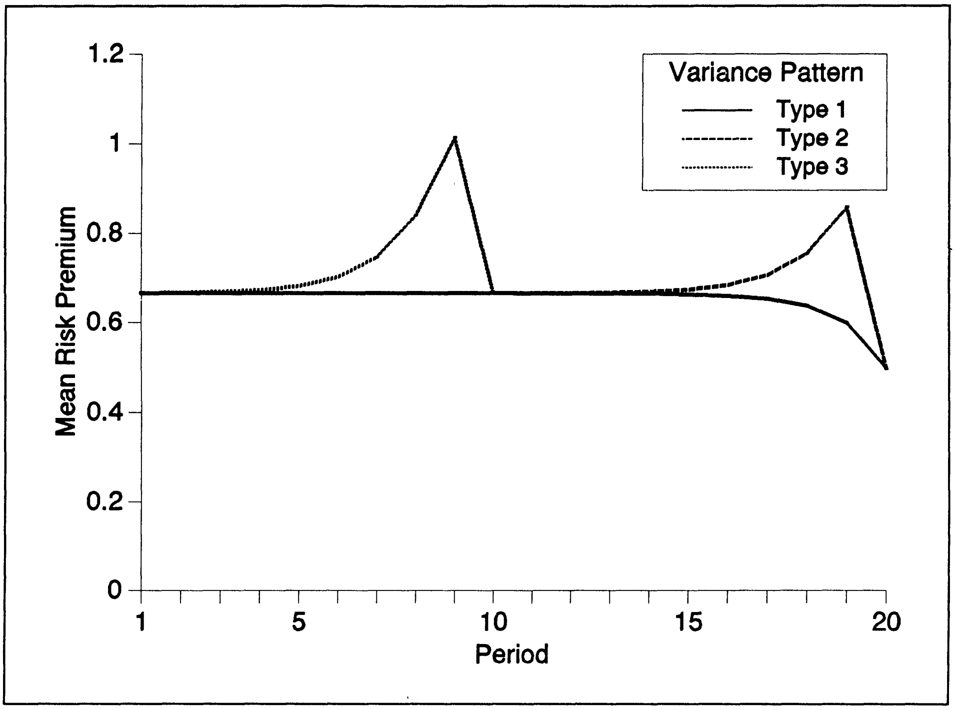

Figure 2. The responsiveness of the risk premium to the contemporaneous supply shock, for different patterns of supply volatility of the risky asset. The responsiveness of the risk premium to the contemporaneous supply shock (i.e., the coefficient of the contemporaneous supply in the expression for the risk premium) is plotted as a function of time, for different supply volatility patterns. The risk aversion of market makers θ is 0.5, and there are 20 trading periods. The variance of the value innovation, v, and the mean supply, $\bar{Y}_t$, are each set to unity in all periods. The following values for the supply variance, var(z_t), are assumed: Type 1, 1 in all periods; Type 2, 1 in Periods 1 to 19, and 10 in Period 20; and Type 3, 1 in Periods 1 to 9, 10 in Period 10, and 1 in Periods 11 to 20.

An interesting application of this analysis is to the case of "triple witching hour" Fridays (i.e., Fridays on which index futures, index options, and individual stock options all expire). It is often alleged that there is significant price pressure on the market during these days as institutions reverse arbitrage positions. In fact, Stoll and Whaley's (1988) study finds a significant price effect during the last hour before closing on such days. Our model implies that such effects should be spread out throughout the Friday, with the strongest effects occurring during the last hour of trading.

IV. Conclusion

For most individuals, the costs of continuously monitoring financial markets for trading opportunities cannot be ignored. Consequently, the general investor population must rely on "Wall Street institutions" and floor brokers to absorb their order flows and thereby provide them with intertemporal liquidity. Recognizing these institutional realities, we have developed a model that analyzes how anticipated large or highly volatile demand pressures late in the day affect asset risk premia and expected returns throughout the day.

Some conclusions that the analysis yields are as follows. First, if the risk aversion of market makers is small relative to that of the outside investors, increasing the selling pressue from traders late in the day only alters the risk premium during the day, and not the expected rate of price appreciation. In contrast, the same pressure at an arbitrary point during the day alters both. These results demonstrate how morning traders who sell securities bear part of the cost for providing the afternoon traders with liquidity. The analysis also indicates that large institutions who can time their trades will either concentrate their trading at the beginning of the day, or trade in the same direction as the market making institutions during periods of unusual buying and selling pressures. Institutions will therefore appear to exhibit "herding" behavior that stems purely from risk-sharing considerations.

Our model also adds perspective to the concept of market liquidity in an intertemporal setting. If one defines "liquidity" as the efficiency with which market makers can transfer demands intertemporally among outside investors, our analysis formalizes the notion that this measure depends not only on contemporaneous inventory and volume, but also on the distribution of volume that is expected to arrive in the future. Further, the model provides several empirically testable predictions. We show that the analysis has implications for predictable variations in intraday expected returns documented, for example, by Harris (1986) and Jain and Joh (1988), the behavior of returns during "triple witching hour" days, and also for days on which there are significant institutional trades, both predictable ones (e.g., sunshine trades) and unpredictable ones.

Appendix

The Derivation of R

Equation (15) is a special case of the class of equations that can be transformed into Bessel's equation. The discussion given below shows how to accomplish this transformation for the problem presented in this article; interested readers should consult Andrews (1985) for more general cases.

Let $x = 2\theta\tau^{1/2}/\phi$, and $y = r\tau^{-1/2}$ producing the following relationships:

$$dR/d\tau = (dR/dx)(dx/d\tau), \tag{43}$$

$$d^2R/d\tau = (d^2R/dx^2)(dx/d\tau)^2 + (dR/dx)(d^2x/d\tau^2), \tag{44}$$

$$R = \phi xy/2\theta, \tag{45}$$

$$dR/dx = \phi y/2\theta + (\phi x/2\theta)(dy/dx), \tag{46}$$

and

$$d^2R/dx^2 = (\phi/\theta)(dy/dx) + (\phi x/2\theta)(d^2y/dx^2). \tag{47}$$

Substituting for these derivatives into equation (15) and multiplying through by $2x\phi/\theta$ produces

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (1 + x^2)y = -\frac{1}{2}x^3 v\phi\hat{s}. \tag{48}$$

The homogenous part of the above equation is the "modified Bessel's equation," the solution of which (see Andrews (1985)) is

$$y = C_1 I_1(x) + C_2 K_1(x). \tag{49}$$

To get the particular solution, we use the method of variation of parameters. Divide equation (48) by x^2 to get it into standard form. The following fact holds for Bessel functions (see, e.g., Andrews (1985)):

FACT 1: *Recurrence relations and the Wronskian of* I_p *and* K_p.

$$\frac{d}{dx}\left[x^p I_p(x)\right] = x^p I_{p-1}(x), \tag{50}$$

and

$$\frac{d}{dx}\left[x^p K_p(x)\right] = -x^p K_{p-1}(x). \tag{51}$$

The Wronskian, $W(I_p, K_p) \equiv -I_p K_{p+1} - I_{p+1}K_p = -1/x$.

From this fact, the Wronskian of I_1 and K_1 equals $-1/x$, therefore the particular solution for the risk premium (R_p) satisfies

$$y_p = \int 0.5\alpha^2 v\phi[I_1(\alpha)K_1(x) - I_1(x)K_1(\alpha)] \sum_{n=0}^{\infty} a_n\left(\frac{\phi\alpha}{2\theta}\right)^{2n} d\alpha. \tag{52}$$

Integration by parts, and the use of Fact 1, provides a solution for y_p. After solving for a few terms a pattern emerges and one can conjecture that in general

$$\int \alpha^{2n+2}[I_1(\alpha)K_1(x) - I_1(x)K_1(\alpha)]\, d\alpha$$
$$= \sum_{j=0}^{n} x^{2n+1-2j} \prod_{k=0}^{j-1} 4(n + 1 - k)(n - k). \tag{53}$$

Combining the particular and homogenous solutions of the second order differential equation and using the boundary conditions, one obtains the result that $y = y_p + C_1 I_1(x) + C_2 K_1(x)$, where C_1 and C_2 are given by

$$\left\{\sum_{n=0}^{\infty}\left[\sum_{j=0}^{\infty}(n+1-j)\left(\frac{2\theta}{\phi}\right)^{1-2j}\tau_0^{n-j}\prod_{k=0}^{j-1}4(n+1-k)(n-k)\right]a_n\right\}\frac{\phi v}{2}$$

$$+ C_1\frac{\theta}{\phi}I_0\left(2\frac{\theta}{\phi}\tau_0^{1/2}\right) - C_2\frac{\theta}{\phi}K_0\left(2\frac{\theta}{\phi}\tau_0^{1/2}\right) = \frac{\theta^2}{\phi}vS_0 \tag{54}$$

and

$$\left\{\sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}\left(\frac{2\theta}{\phi}\right)^{1-2j}\left(\frac{\phi v_1}{\theta v}\right)^{n+1-j}\prod_{k=0}^{j-1}4(n+1-k)(n-k)\right]a_n\right\}\frac{v}{2v_1}$$

$$+ C_1(\phi\theta v_1 v)^{-1/2}I_1\left(2\sqrt{\frac{\theta v_1}{\phi v}}\right) + C_2(\phi\theta v_1 v)^{-1/2}K_1\left(2\sqrt{\frac{\theta v_1}{\phi v}}\right)$$

$$= \left\{\sum_{n=0}^{\infty}\left[\sum_{j=0}^{n}(n+1-j)\left(\frac{2\theta}{\phi}\right)^{1-2j}\left(\frac{\phi v_1}{\theta v}\right)^{n-j}\right.\right.$$

$$\left.\left.\times\prod_{k=0}^{j-1}4(n+1-k)(n-k)\right]a_n\right\}\frac{\phi}{2\theta}$$

$$+ C_1\frac{1}{\phi v}I_0\left(2\sqrt{\frac{\theta v_1}{\phi v}}\right) - C_2\frac{1}{\phi v}K_0\left(2\sqrt{\frac{\theta v_1}{\phi v}}\right) \tag{55}$$

where $\tau_0 \equiv (\phi/\theta)(1 + v_1/v)$.

Next, substitute out for x and y in terms of τ and R to obtain equation (17). One can then verify that this is indeed the solution by showing that equation (17) satisfies equation (15).

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