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Portfolio Inefficiency and the Cross-section of Expected Returns

SHMUEL KANDEL and ROBERT F. STAMBAUGH*

ABSTRACT

The Capital Asset Pricing Model implies that (i) the market portfolio is efficient and (ii) expected returns are linearly related to betas. Many do not view these implications as separate, since either implies the other, but we demonstrate that either can hold nearly perfectly while the other fails grossly. If the index portfolio is inefficient, then the coefficients and R^2 from an ordinary least squares regression of expected returns on betas can equal essentially any values and bear no relation to the index portfolio's mean-variance location. That location does determine the outcome of a mean-beta regression fitted by generalized least squares.

EXPECTED RETURNS ON A set of risky assets obey an exact linear relation to betas computed against an index portfolio that lies on the minimum-variance boundary of those assets. If the betas are computed instead against an index portfolio that lies inside the minimum-variance boundary, then expected returns must deviate to some degree from any fitted cross-sectional linear relation.[1] These properties are well known, but they leave open the question of whether, in the latter case, the extent to which expected returns are approximated by a linear function of beta is at all related to the mean-variance location of the index portfolio. For example, one might ask whether, with only negligible inefficiency in the index portfolio, a plot of expected returns versus betas would display a near perfect linear relation.

In fact, the mean-variance location of an inefficient index portfolio bears essentially no relation to the plot of expected returns versus betas. For example, expected returns can display essentially no correlation with betas computed against an index portfolio that has an expected return arbitrarily close to that of the efficient portfolio with the same variance. Alternatively, expected returns can display a nearly perfect linear relation to betas computed against an index portfolio that is grossly inefficient. Such plots of expected returns versus betas can be summarized by ordinary least squares (OLS) regression. We show that, if the index portfolio is inefficient, the OLS regression coefficients and R^2 can equal essentially any values desired. This

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[1] See Fama (1976), Roll (1977), and Ross (1977).

general result, as well as the two examples noted, can be demonstrated by repackaging a given set of risky assets into alternative sets that generate the same portfolio opportunities. Such repackagings change neither the index portfolio nor the minimum-variance boundary, but they can change the cross-sectional mean-beta relation in virtually any manner desired.

This study shows that generalized least squares (GLS) regression provides a framework wherein the *exact* linear mean-beta relation implied by strict efficiency of the index portfolio can be generalized to an *approximate* linear relation when the index is inefficient. The GLS regression uses the covariance matrix of the asset returns, and much of the information in that matrix is omitted in a plot of expected return versus beta. An index portfolio's location in mean-variance space is unaffected by repackaging the individual assets, and we define a measure of relative efficiency that is determined by a portfolio's mean-variance location. This relative efficiency measure approaches its maximum value of unity as the index portfolio moves closer to the upper portion of the minimum-variance boundary. We find that this measure provides a simple link between the index portfolio's mean-variance location and the properties of the fitted GLS mean-beta relation. As the index portfolio's relative efficiency moves closer to unity, the fitted GLS mean-beta relation moves closer to the exact linear relation corresponding to an efficient portfolio with the same variance as the index. A slope coefficient of zero occurs only when the mean return on the index is equal to that of the global minimum-variance portfolio. Moreover, the goodness-of-fit measure for the GLS cross-sectional regression is simply the squared relative efficiency of the index portfolio.

In the absence of an exact linear relation between expected returns and betas, a variety of criteria could be used to fit a line and judge its goodness-of-fit. Developing such criteria is difficult without an economic context in which to view a fitted linear mean-beta relation. We consider a context in which such a relation is judged by its ability to provide fitted expected returns that are useful substitutes for true expected returns as inputs to a standard one-period portfolio optimization. For a given set of cross-sectional independent variables, including but not limited to beta, using the expected returns fitted from a GLS regression produces a portfolio with a higher expected return than using any other linear function of the independent variables. The squared relative efficiency of that portfolio is simply the goodness-of-fit for the GLS regression.

The remainder of the article is organized as follows. Section I shows that the cross-sectional mean-beta relation fitted by OLS bears essentially no relation to the mean-variance location of an inefficient index portfolio. Section II defines a portfolio's relative efficiency, using a measure that can be stated in terms of either expected returns or variances. Section III provides simple relations between the index portfolio's relative efficiency and a GLS regression of expected returns on betas. Section IV offers a portfolio-optimization setting in which to compare GLS to other methods for fitting and judging cross-sectional relations for expected returns. Although this study deals almost exclusively with population moments, Section V presents a brief discussion of issues related to estimation and inference. Conclusions are then presented in Section VI. The Appendix contains proofs of all propositions.

I. Inefficiency and Deviations from Mean-Beta Linearity

For a universe of n risky assets, define

R: n-vector of returns realized in a given period,
E: n-vector of expected returns,
V: $n \times n$ covariance matrix of returns, assumed to be nonsingular.

For a given portfolio p, a combination of the n assets, define

w_p : n-vector of weights in portfolio p,
μ_p : mean return on portfolio p $(= w_p' E)$,
σ_p^2: variance of return on portfolio p $(= w_p' V w_p)$,
β : n-vector of betas with respect to p $[= (1/\sigma_p^2) V w_p]$.

Let ι denote an n-vector of ones, and define

$$X = [\iota \quad \beta]. \tag{1}$$

Assume that neither E nor β are proportional to ι.

The mean-variance location of portfolio p has virtually no bearing on the degree to which the elements of E and β conform to a linear relation, when goodness-of-fit is measured by the standard Euclidean norm. That is, portfolio p can lie arbitrarily close to the minimum-variance boundary and yet produce an OLS slope and R^2 that are arbitrarily close to zero. Similarly, portfolio p can lie far from the minimum-variance boundary (by whatever metric desired) and yet still produce an OLS fit between expected returns and betas that is arbitrarily close to exact linearity.

We verify the above statements by considering "repackagings" of assets. The portfolio opportunities generated by one set of n assets are identical to those generated by an alternative set of n assets that simply repackage the original set, provided that returns on the new assets also have a nonsingular covariance matrix. Such a repackaging does not change the minimum-variance boundary or the location of portfolio p in mean-variance space, but it can change the relation between the n assets' expected returns and their betas with respect to portfolio p.

A given repackaging of assets can be represented by a nonsingular $n \times n$ matrix A, where $A\iota = \iota$. The returns on the repackaged assets are constructed as $R^* = AR$, so the means and betas of the repackaged assets are given by $E^* = AE$ and $\beta^* = A\beta$. For a given repackaging of the n assets, let γ^* denote the vector of coefficients in an OLS regression of expected returns on betas with respect to portfolio p. That is,

$$\gamma^* = (X^{*\prime} X^*)^{-1} X^{*\prime} E^*, \tag{2}$$

where

$$X^* = [\iota \quad \beta^*] = AX. \tag{3}$$

The goodness-of-fit in this regression is given by

$$R^2_{OLS} = 1 - \frac{(E^* - X^*\gamma^*)'(E^* - X^*\gamma^*)}{\left(E^* - \dfrac{\iota' E^*}{n}\iota\right)'\left(E^* - \dfrac{\iota' E^*}{n}\iota\right)}. \tag{4}$$

If portfolio p is inefficient, the following proposition states that one can always find a repackaging such that expected returns on the new set of n assets obey essentially any desired OLS regression outcome.

PROPOSITION 1: *If portfolio p is inefficient, then for any* $\omega \in (0, 1)$, $\varepsilon > 0$, *and two-element vector* θ, *there exists a nonsingular* $n \times n$ *matrix* A, *with* $A\iota = \iota$, *such that*[2]

$$\|\gamma^* - \theta\| < \varepsilon, \quad and \tag{5}$$

$$R^2_{OLS} = \omega. \tag{6}$$

The results of an OLS regression correspond closely, of course, to what one would infer visually from a simple plot of expected returns versus betas. Proposition 1 implies that such a plot could in fact appear to contradict standard theory, since small degrees of portfolio inefficiency or deviations from perfect mean-beta linearity may not be visible in a plot. Two such examples are presented in Figure 1. The minimum-variance boundaries in Figures 1*a* and 1*c* are identical, and they are generated using sample means and covariances of monthly returns on ten portfolios of common stocks sorted by equity capitalization (firm size) for the period from 1926 to 1992.[3]

The ten points plotted in Figure 1*a* as solid dots represent means and variances on ten assets that simply repackage the ten size portfolios. Portfolio p, shown as a small circle, is inefficient, having a monthly expected return that is 88 basis points less than the expected return on the efficient portfolio with the same variance. Figure 1*b* plots the expected returns on the ten assets versus the assets' betas with respect to portfolio p. The mean-beta relation is not exactly linear, although the violations of exact linearity are too slight to be visible on the graph. The OLS regression line on which all of the points appear to lie has an intercept of 30 basis points, close to the average monthly interest rate for the 1926 to 1992 period, and the slope of the line is 76 basis points, the average excess return on portfolio p. In other words, shown only Figure 1*b*, one would be inclined to conclude that portfolio p is the Sharpe-Lintner tangent portfolio of the ten assets.

[2] The Euclidean norm of an n-vector v is defined as $\|v\| = (v'v)^{1/2}$.

[3] The portfolios include all stocks on the New York Stock Exchange, and the returns within a portfolio are value-weighted. Portfolio returns were obtained from the Index File supplied by the Center for Research in Security Prices.

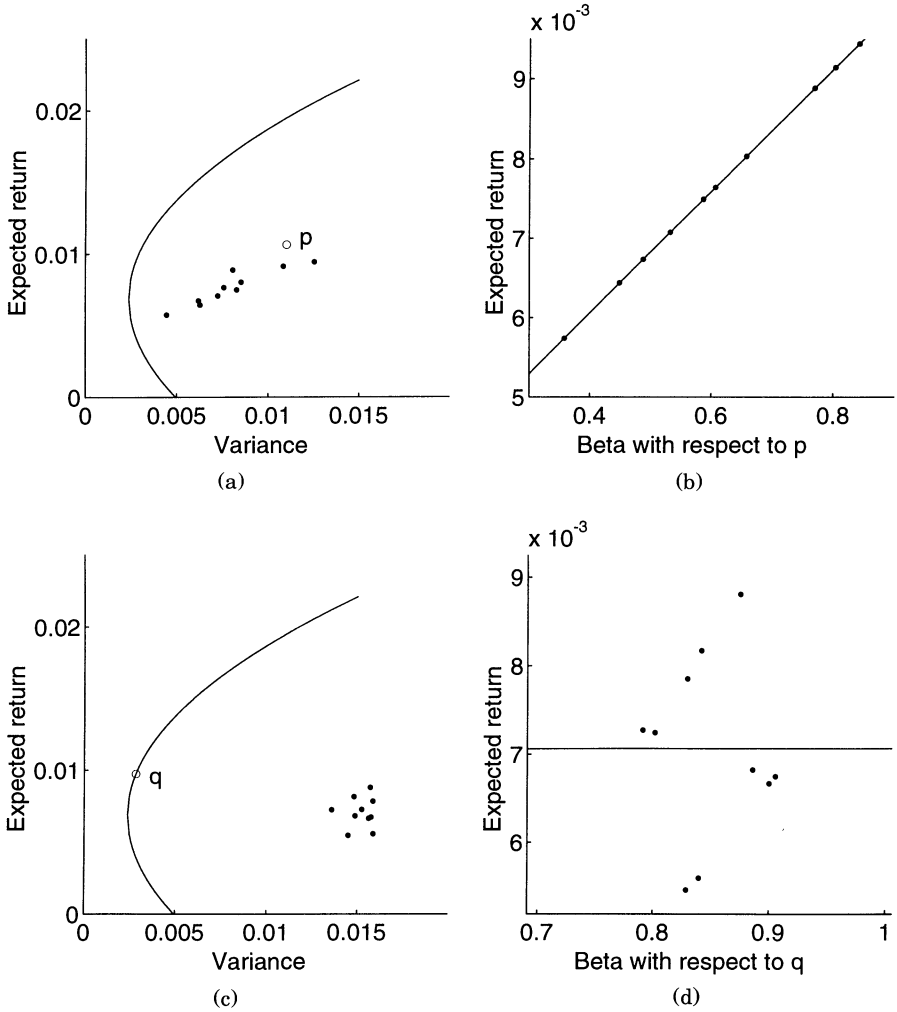

Figure 1. Examples of mean-beta relations for different mean-variance locations of the index portfolio. Figure 1*a* plots ten assets (*solid dots*), their minimum-variance boundary, and a portfolio p of those ten assets (*circle*). Figure 1*b* plots the expected returns and betas of those assets with respect to portfolio p as well as the OLS regression line through those points. Figures 1*c* and 1*d* display a similar case, except that the ten assets are a "repackaging" of those in the first case. The points in Figure 1*b* do not lie *exactly* on the regression line, and portfolio q in Figure 1*c* does not lie *exactly* on the minimum-variance boundary.

The ten assets whose means and variances are plotted in Figure 1*c* are obtained as a different repackaging of the ten size portfolios. Portfolio q is inefficient, although it lies too close to the minimum-variance boundary for the inefficiency to be visible on the graph. For the ten assets, a least squares regression of expected returns on betas with respect to portfolio q produces an R^2_{OLS} less than 0.0001, and the corresponding plot is shown in Figure 1*d*.

Shown only that plot, one would be inclined to conclude that portfolio q is inefficient. Such a conclusion must be correct, of course, but the degree of inefficiency can be of no economic significance.

Although we focus in this study on the cross-sectional relation between expected returns and betas with respect to a single portfolio, we should note that an extension of Proposition 1 to multifactor models is straightforward. That is, suppose X has instead $k + 1$ columns, where columns 2 through $k + 1$ contain the assets' sensitivities to a set of k factors—or betas with respect to k portfolios that mimic the factors. Then, unless expected returns conform exactly to some linear combination of the columns of X, the n assets can again be repackaged to produce essentially any desired OLS coefficients and R^2_{OLS}. In other words, if no portfolio of the mimicking portfolios is *exactly* mean-variance efficient, then a multifactor model faces the same problems associated with the single-beta model.[4]

Issues related to OLS regressions of expected returns on betas are discussed in several recent studies. Roll and Ross (1994) derive the region in mean-variance space containing the portfolios that produce a mean-beta relation whose OLS slope is exactly zero (and thus R^2_{OLS} is zero). Our observation that inefficient portfolios can also give arbitrarily good fits to any given linear mean-beta relation goes beyond the analysis of Roll and Ross, who do not consider goodness-of-fit measures.[5] Jagannathan and Wang (1994) construct a four-asset example in which repackaging changes R^2_{OLS} from 0.95 to 0.0, although those authors do not address the generality of the example or its relation to the mean-variance location of the index portfolio. Grauer (1994) constructs a number of examples illustrating that the difference between the OLS intercept and the riskless rate does not correspond to the proximity of the index portfolio to the Sharpe-Linter tangent portfolio.

Roll and Ross (1994) show that the distance between the minimum-variance boundary and the zero-slope-producing region is increasing in the cross-sectional variance of expected returns on the n assets, holding constant the product of the cross-sectional mean of the expected returns and the variance of the payoff on a specific zero-investment position.[6] As shown here, repackaging the n assets allows the goodness-of-fit to become arbitrarily close to zero for *any* inefficient index portfolio. Moreover, the following proposition states that such a repackaging can be constructed to produce expected returns and betas exhibiting any desired cross-sectional mean and variance.[7]

[4] Huberman, Kandel, and Stambaugh (1987) define mimicking portfolios and analyze their relation to the minimum-variance boundary under exact k-factor pricing.

[5] Roll and Ross derive mean-variance regions for portfolios that produce a given positive OLS slope for the mean-beta relation, but, other than in the case where the slope is zero, the value for the slope does not provide information about the goodness-of-fit.

[6] The zero-investment position goes long $1 in the portfolio with weights $(1/\iota' E)E$ and short $1 in the equally weighted portfolio of the n assets.

[7] This repackaging would not necessarily satisfy additional constraints involving the covariance matrix of returns, such as an upper bound on the variance of the zero-investment position defined by Roll and Ross (1994).

PROPOSITION 2: *If portfolio p is inefficient, then for any scalars* $\bar{\beta}^*$, $\bar{E}^*$, $\sigma_{\beta^*} > 0$, $\sigma_{E^*} > 0$, *and* $\omega \in (0,1)$, *there exists a nonsingular* $n \times n$ *matrix* A, *with* $A\iota = \iota$, *such that*

$$\frac{1}{n}\iota' A\beta = \bar{\beta}^*, \tag{7}$$

$$\frac{1}{n}\iota' AE = \bar{E}^*, \tag{8}$$

$$\frac{1}{n}(A\beta - \bar{\beta}^*\iota)'(A\beta - \bar{\beta}^*\iota) = \sigma_{\beta^*}^2, \tag{9}$$

$$\frac{1}{n}(AE - \bar{E}^*\iota)'(AE - \bar{E}^*\iota) = \sigma_{E^*}^2, \quad \textit{and} \tag{10}$$

$$R_{OLS}^2 = \omega. \tag{11}$$

Numerous empirical investigations of asset pricing have tested the hypothesis that the OLS slope in the cross-sectional mean-beta relation is equal to zero. (A recent example can be found in Fama and French (1992).) Failure to reject this null hypothesis with a finite number of time-series observations does not, of course, translate into a rejection of the hypothesis of mean-variance efficiency. In an infinite sample, failure to reject a zero OLS slope must reject *exact* efficiency of the index. Roll and Ross show that such a result could occur with an index portfolio that is close to efficient, so, following Roll (1977), inferences about the pricing theory could be sensitive to construction of the index.

Some readers might interpret the Roll-Ross analysis as implying that the outcome of a zero slope with a near-efficient index portfolio requires low dispersion in expected returns. If that were indeed the case, then the above criticism might not be very relevant to many empirical studies. That is, such investigations often select assets so as to create substantial dispersion in expected returns. Given Proposition 2, however, an outcome of a near-zero slope with a near-efficient index portfolio can occur with large dispersions in both expected returns and betas. In other words, it seems difficult to argue that simply selecting assets with disperse expected returns or betas necessarily endows the zero-slope test with power against an alternative hypothesis of near efficiency in the index portfolio.

A reasonable reaction to the examples in Figure 1 could be that the sets of ten assets are unusual, so that, although these special cases illustrate theoretical possibilities, one could simply avoid using such assets in empirical investigations. Although the assets selected in the examples are no doubt unusual by some criteria, the relevant question is how one would develop such criteria. We employ repackaging as an expositional and analytical device. Our use of this device might lead one to suggest that assets constructed using extreme values in the matrix A could be ruled out, but such a suggestion misses the point. Any given set of assets can be viewed as an extreme repackaging of one set but a modest repackaging of another. In other words, the assets selected by an empirical researcher do not come with a well-defined A matrix. So if one seeks to admit only modest repackagings, or even no repackagings, the question arises, "Repackagings of what?".

One source of information about how "unusual" a set of assets might be is the covariance matrix of their returns. The plots in Figure 1 omit much information about the covariance matrix. For example, the covariance matrix of the assets plotted in Figure 1a, although nonsingular, has one very small eigenvalue. We consider below a framework that uses this additional information to measure the relation between expected returns and betas with quantities that correspond directly to portfolio p's position in mean-variance space. A portfolio's position in mean-variance space will be characterized by a simple measure of relative mean-variance efficiency.

II. A Measure of Relative Portfolio Efficiency

For a given portfolio p, let x denote the efficient portfolio with the same variance as p, and let y denote the minimum-variance portfolio with the same mean as p. Define

μ_x : mean return on portfolio x,
μ_{x0}: mean return on portfolios uncorrelated with portfolio x,
σ_y^2 : variance of portfolio y,
μ_g : mean of the global minimum-variance portfolio,
σ_g^2 : global minimum variance.

The *relative efficiency* of portfolio p is defined as

$$\psi_p \equiv \frac{\mu_p - \mu_g}{\mu_x - \mu_g}. \tag{12}$$

The relative efficiency measure defined in equation (12) has a range from -1 to 1, with the latter value corresponding to exact efficiency. Relative efficiency is undefined for the global minimum-variance portfolio. When portfolio p lies on the minimum-variance boundary but has the *lowest* expected return for its variance, then $\psi_p = -1$. The square of this efficiency measure can also be expressed in terms of variances, as given by the following proposition.

PROPOSITION 3:

$$\psi_p^2 = \frac{\sigma_y^2 - \sigma_g^2}{\sigma_p^2 - \sigma_g^2}. \tag{13}$$

Both equations (12) and (13) are represented graphically in Figure 2. Figure 3 displays the locations in mean-variance space of portfolios with given values of ψ_p. The minimum-variance boundary is the same as that constructed in Figures 1a and 1c.

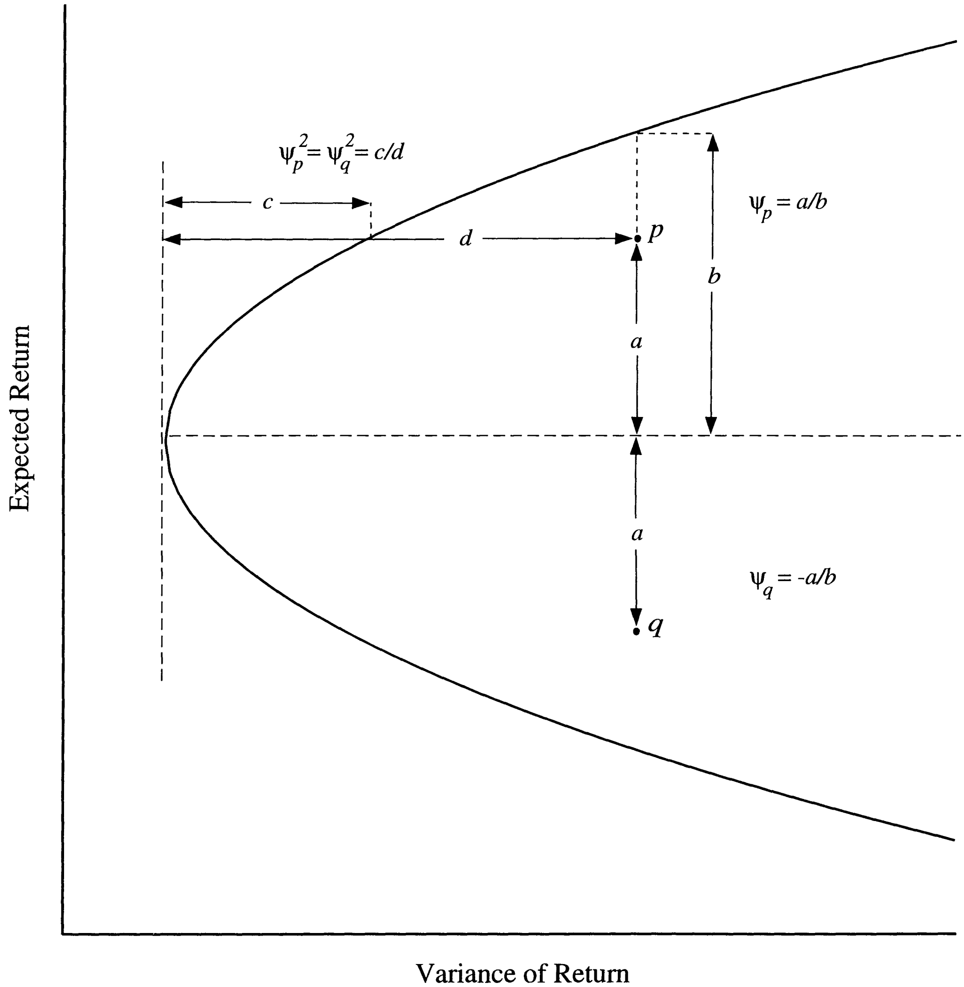

Figure 2. The relative efficiency measure ψ in mean-variance space. The *solid curve* represents the minimum-variance boundary of portfolio opportunities. The relative efficiencies of portfolios p and q are denoted by ψ_p and ψ_q.

A portfolio's inefficiency can also be characterized in terms of correlation. Kandel and Stambaugh (1987) and Shanken (1987) show that ρ_p, the maximum correlation between the return on portfolio p and the return on any minimum-variance portfolio, is given by

$$\rho_p = \frac{\sigma_y}{\sigma_p}. \tag{14}$$

This measure, like ψ_p, approaches unity as portfolio p approaches the minimum-variance boundary, but it is bounded below by zero. Combining

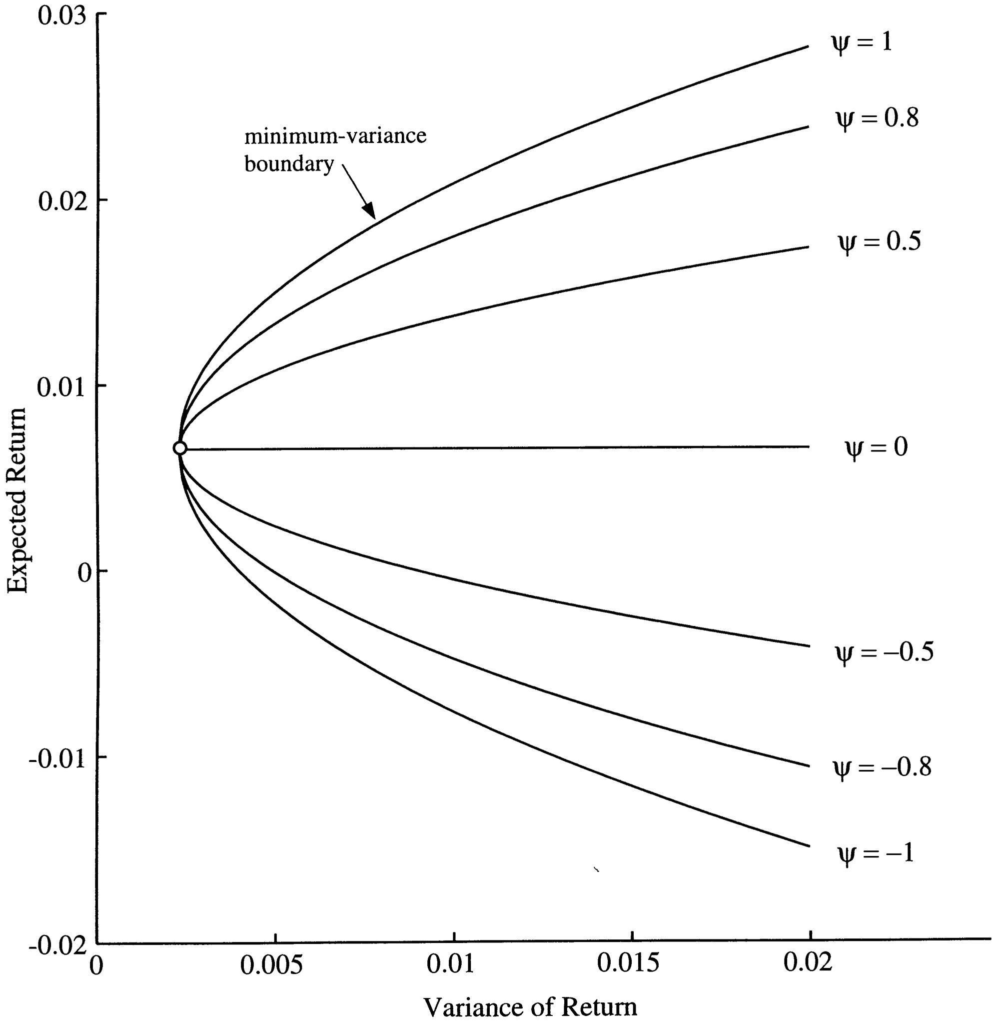

Figure 3. Mean-variance locations of portfolios with various levels of relative efficiency. Each *curve* displays the locus of portfolios with a given measure of relative efficiency, ψ. Relative efficiency is undefined for the global minimum-variance portfolio (denoted by the *small circle*).

equations (13) and (14) gives

$$1 - \rho_p^2 = \left(1 - \frac{\sigma_g^2}{\sigma_p^2}\right)\left(1 - \psi_p^2\right), \tag{15}$$

which implies that

$$\psi_p < \rho_p \tag{16}$$

if portfolio p is inefficient.

III. The Mean-Beta Relation and the Covariance Matrix

Consider a cross-sectional regression of E on β, where the covariance matrix V is used to perform GLS. That is, the coefficient vector in the regression is given by

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = (X'V^{-1}X)^{-1}X'V^{-1}E. \tag{17}$$

PROPOSITION 4: *The slope coefficient ϕ_2 is given by*

$$\phi_2 = \psi_p(\mu_x - \mu_{x0}) \tag{18}$$

and the intercept ϕ_1 is given by $\phi_1 = \mu_p - \phi_2$ or

$$\phi_1 = \mu_{x0} + (1 - \psi_p)(\mu_g - \mu_{x0}). \tag{19}$$

If p is efficient, so $\psi_p = 1$, then ϕ_1 must equal μ_{x0} and ϕ_2 must be the portfolio's premium over that zero-beta rate, $\mu_x - \mu_{x0}$. The above proposition reveals that, if p is inefficient, then $\phi_1 > \mu_{x0}$ and $\phi_2 < \mu_x - \mu_{x0}$. As ψ_p gets closer to 1, ϕ_1 approaches μ_{x0} and ϕ_2 approaches its maximum value, $\mu_x - \mu_{x0}$. A negative slope occurs for $\mu_p < \mu_g$, and a zero slope occurs if and only if $\psi_p = 0$, or when $\mu_p = \mu_g$.[8]

The standard measure for the GLS regression's goodness-of-fit is

$$R^2_{GLS} = 1 - \frac{(E - X\phi)'V^{-1}(E - X\phi)}{(E - \iota\bar{\mu})'V^{-1}(E - \iota\bar{\mu})}, \tag{20}$$

where

$$\bar{\mu} = (E'V^{-1}\iota)/(\iota'V^{-1}\iota), \tag{21}$$

which is the coefficient in a GLS regression of E on ι. Note that exact linearity gives $R^2_{GLS} = 1$, a slope of zero gives $R^2_{GLS} = 0$, and $0 \leq R^2_{GLS} \leq 1$.

PROPOSITION 5:

$$R^2_{GLS} = \psi_p^2. \tag{22}$$

We see that, unlike the OLS regression, the outcome of a GLS regression of expected returns on betas is determined completely by portfolio p's location in mean-variance space, as summarized by ψ_p. In Figure 1*a*, $\psi_p = 0.3$, so the goodness-of-fit in a GLS regression of means on betas with respect to portfolio p is 0.09. In Figure 1*c*, ψ_q is nearly 1, and so is the goodness-of-fit in the GLS mean-beta regression. Although it can be shown algebraically that the coefficient vector ϕ and the goodness-of-fit measure R^2_{GLS} are invariant to repackaging the n assets, this result follows immediately from the fact that portfolio p's location in mean-variance space is unaffected by repackaging the assets used to generate the set of portfolio opportunities.

[8] This last point is made independently by Roll and Ross (1994), who attribute private correspondence with Simon Wheatley.

The GLS regression constructs a least squares fit between means and betas that are transformed using the factored inverse of the covariance matrix, and, as is obvious from Figure 1, the outcome of that regression need bear no resemblance to a plot of the "raw" expected returns versus betas. To decide whether fitted lines and goodness-of-fit measures are more relevant when computed with the raw means and betas than with their transformed counterparts, it may be useful to have a context in which fitted cross-sectional relations for expected returns would be used. The next section considers the use of such relations in providing expected returns as inputs to portfolio optimization. It is shown that the fitted GLS regression provides the optimal inputs for the optimization, and the regression's goodness-of-fit provides the squared relative efficiency of the resulting portfolio. If, in other contexts, the goodness-of-fit of the raw means and betas is a more relevant metric, however, then one must simply recognize that such a metric need bear no relation to the relative mean-variance efficiency of portfolio p.

IV. Using Fitted Mean Returns: An Optimization Setting

In the absence of an exact linear relation between expected returns and betas, it seems useful to have an economic context in which one might, at a theoretical level, fit a linear relation and judge its goodness-of-fit. We consider here the simple context of mean-variance portfolio optimization, where the expected returns fitted from a linear cross-sectional relation are used as inputs to the problem of maximizing a portfolio's expected return for a given variance. The extent to which the portfolio constructed in the optimization differs from the efficient portfolio depends only on the differences between true and fitted expected returns.

Because the cross-section of mean returns can possibly be explained better by variables used in addition to, or even in place of, betas computed against an inefficient portfolio, we allow such variables to be included in the analysis. For $k < n$, let Z denote an $n \times k$ matrix of full column rank, where one column is ι. The matrix Z can simply be the $n \times 2$ matrix X defined previously, so that the results below include fitting the mean-beta relation as a special case. We consider linear cross-sectional relations that fit expected returns as

$$\hat{E} = Za, \tag{23}$$

for some $k \times 1$ vector a.

The quality of the approximation to expected returns in equation (23) is characterized by the results of a portfolio optimization that uses $\hat{E}$ instead of E as inputs. Let $w(\hat{E}; \sigma^2)$ denote the solution to the portfolio maximization problem,

$$\max_{w} w'\hat{E} \tag{24}$$

subject to the constraints

$$w'Vw = \sigma^2 \quad \text{and} \tag{25}$$

$$w'\iota = 1, \tag{26}$$

for a given $\sigma^2 > \sigma_g^2$.

Let δ denote the coefficient vector in a GLS regression of E on Z,

$$\delta = (Z'V^{-1}Z)^{-1}Z'V^{-1}E. \tag{27}$$

The fitted mean returns from the GLS regression are given by

$$E^{\dagger} = Z\delta. \tag{28}$$

Note that $E^{\dagger}$ is a special case of $\hat{E}$ in equation (23) with $a = \delta$. We see that this choice of a is best in the following sense.

PROPOSITION 6:

$$[w(E^{\dagger};\sigma^2)]'E \geq [w(\hat{E};\sigma^2)]'E \tag{29}$$

for all a.

In other words, the *true* expected return of the portfolio constructed using the GLS inputs is greater than or equal to the true expected return of a portfolio constructed using any other inputs of the form in equation (23).

As before, the goodness-of-fit for the GLS regression is given by

$$R_{GLS}^2 = 1 - \frac{(E - Z\delta)'V^{-1}(E - Z\delta)}{(E - \iota\bar{\mu})'V^{-1}(E - \iota\bar{\mu})}, \tag{30}$$

where $\bar{\mu}$ is defined as in equation (21). This goodness-of-fit is also the squared relative efficiency of the portfolio constructed using GLS inputs.

PROPOSITION 7: *For any $\sigma^2 > \sigma_g^2$, let q denote the portfolio with weights $w(E^{\dagger};\sigma^2)$. Then*

$$R_{GLS}^2 = \psi_q^2. \tag{31}$$

In other words, the portfolio constructed using the fitted expected returns from the GLS regression has a squared relative efficiency equal to that regression's goodness-of-fit. Note that R_{GLS}^2, and thus ψ_q^2, do not depend on the value for σ^2 specified in the portfolio optimization. In the special case where $Z = X$, it follows from Propositions 5 and 7 that the portfolio constructed with the GLS inputs has the same (squared) relative efficiency as portfolio p. In fact, it can also be shown in that case that the weights in portfolio p are equal to $w(E^{\dagger};\sigma_p^2)$.

V. Estimation and Inference

In many empirical investigations of asset pricing, cross-sectional regressions are typically estimated using sample estimates of expected returns, betas, and, in the case of GLS, the covariance matrix. Recent examples include Fama and French (1992) in the case of OLS and Amihud, Christensen, and Mendelson (1992) in the case of GLS. It is straightforward to show that the probability limits of these regression estimators equal their population counterparts, γ^* in the case of OLS (equation (2)) and ϕ in the case of GLS (equation (17)).

When portfolio p is exactly mean-variance efficient, then $\gamma^* = \phi$ for any repackaging of the assets, so the OLS and GLS estimators then have the same probability limits. In that case, Shanken (1992) shows that the GLS estimator is asymptotically efficient, even though the betas used as independent variables are first estimated in OLS regressions.[9] As Shanken cautions, however, one might be concerned about the GLS estimator in finite samples. Amsler and Schmidt (1985) conduct a Monte Carlo investigation and find that, with six years of monthly returns on fifteen or fewer assets, the GLS estimator of the zero-beta rate generally outperforms the OLS estimator in terms of variance, mean square error, and mean absolute error, but not in terms of bias.

When portfolio p is inefficient, there can be substantial divergence between the probability limits of the OLS and GLS estimators, given the possible differences between γ^* and ϕ demonstrated in this study. This potential divergence in probability limits complicates a comparison of these estimators in small samples. On one hand, OLS may very well perform better as an estimator of γ^* than does GLS as an estimator of ϕ. On the other hand, γ^* may be a less interesting quantity to estimate, since it need bear essentially no relation to the degree of inefficiency in portfolio p.

Propositions 4 and 5 also hold with the population moments replaced by their sample counterparts, which implies that the outcome of the GLS estimation is determined by the location of the index portfolio in sample mean-variance space. In other words, the GLS estimation summarizes the information in the sample covariance matrix by the index portfolio's sample mean-variance location. A portfolio's sample mean-variance location has been used in a variety of approaches to estimation and inference. Shanken (1985) shows that the outcome of a cross-sectional GLS regression of means on betas can be used to test mean-variance efficiency in the absence of a riskless asset, and Roll (1985) provides a geometric mean-variance interpretation of this test. Kandel and Stambaugh (1989) show that likelihood-ratio tests of mean-variance efficiency, with or without a riskless asset, can also be computed using the index portfolio's sample mean-variance location.[10] Inferences about the degree of inefficiency in a portfolio, formulated as a composite hypothesis instead of a point hypothesis of exact efficiency, have also been based on a portfolio's sample mean-variance location. Kandel and Stambaugh (1987) and

[9] Shanken's GLS estimator is defined using the covariance matrix of the residuals from the first-pass market-model regressions, but he shows in earlier work (Shanken (1985, footnote 16)) that the same estimator is obtained using the covariance matrix of returns.

[10] The mean-variance characterization for the likelihood-ratio test in the riskless-asset case is due to Gibbons, Ross, and Shanken (1989).

Shanken (1987) conduct such tests in a frequentist setting, while Kandel, McCulloch, and Stambaugh (1995) investigate a portfolio's degree of inefficiency in a Bayesian setting.

In contrast to all of the alternative approaches noted above, the outcome of an OLS regression of sample means on betas is essentially unrelated to the index portfolio's sample mean-variance location. This statement follows from the observation that Propositions 1 and 2 also hold with sample moments in place of population moments. Although it seems sensible that inferences about a portfolio's *population* mean-variance location should be based on its *sample* mean-variance location, perhaps finite-sample statistical considerations should temper such logic. Further investigation of these issues would no doubt be useful.

VI. Conclusions

As is well known, an exact linear relation between expected returns and betas with respect to a given portfolio p occurs if and only if portfolio p lies exactly on the minimum-variance boundary. If portfolio p is at all inefficient, however, a plot of expected returns versus betas bears essentially no relation to the position of portfolio p in mean-variance space. An OLS slope and R^2 arbitrarily close to zero can occur when portfolio p is arbitrarily close to the minimum-variance boundary. A near-perfect linear relation can occur, with any desired intercept and slope, if portfolio p is grossly inefficient.

Although OLS is inadequate to the task, the *exact* linear mean-beta relation implied by the efficiency of portfolio p can indeed be generalized to an *approximate* linear relation in the presence of inefficiency in portfolio p. If the linear relation is fitted as a GLS regression of expected returns on betas, using the variance-covariance matrix of returns, then that relation's coefficients and goodness-of-fit measure bear simple relations to the location of portfolio p in mean-variance space. If portfolio p is close to efficient, based on a relative efficiency measure that can be stated in terms of either means or variances, then the fitted relation will be close to the exact linear relation corresponding to an efficient portfolio whose mean and variance are close to those of portfolio p.

When portfolio p is inefficient, it may be useful to adopt an economic context in which to fit a linear relation between expected return and beta and characterize, at a theoretical level, that relation's goodness-of-fit. We consider a context in which the quality of the linear relation is judged by its ability to provide fitted expected returns that are useful substitutes for true expected returns as inputs to a standard one-period portfolio optimization. For a given set of cross-sectional independent variables, including but not limited to beta, using the expected returns fitted from a GLS regression produces a portfolio with a higher expected return than using any other linear combination of the independent variables. The (squared) relative efficiency of that portfolio is simply the goodness-of-fit for the GLS regression.

The absence of a relation between the index portfolio's relative efficiency and a plot of expected returns versus betas illustrates the difficulty in using and assessing any model that delivers multiple implications. For example, the Capital Asset Pricing Model of Sharpe (1964), Lintner (1965), and Black (1972) delivers two major implications: (i) the market portfolio is mean-variance efficient, and (ii) the relation between expected returns and betas is linear. Many finance academics prefer not to view these implications as separate, since either one implies the other, but such a strict view does not easily accommodate the fact that any financial model is at best a convenient and useful abstraction rather than an exact representation of reality.[11] That is, the strict view does not easily entertain the possibility that, for practical purposes, one implication can hold while the other fails. This study demonstrates that either implication can hold nearly perfectly while the other is grossly violated.

In some applications, the implication of interest may be that the market portfolio is mean-variance efficient or, in practical terms, very nearly so. This implication might lead, for example, to an "index fund" portfolio strategy or to the use of a market index as a performance benchmark against which to compare other portfolios of similar volatility. If the model's implication of interest is instead the cross-sectional mean-beta relation, then we see that the relative efficiency of the index portfolio offers little guidance as to the properties of such a relation. An additional problem with the mean-beta implication arises, however. Even if a linear mean-beta relation fits arbitrarily well (but not perfectly) for a given set of n assets that generate all portfolio opportunities, the same relation can still provide a poor approximation for the expected return on another asset (a repackaging of the n assets). Many applications of the model are likely to use a relation fitted with one set of assets to approximate the expected return on another asset, such as a project in a capital budgeting problem or a managed portfolio in a performance evaluation. Thus, unless one takes seriously the possibility that the linear mean-beta relation holds perfectly, this implication of the model seems to offer limited applicability.

Appendix

Proof of Proposition 1: Let F be a nonsingular $n \times n$ matrix whose first three columns are

$$f_1 = \iota, \tag{A1}$$

$$f_2 = \beta, \tag{A2}$$

and

$$f_3 = E - X\theta, \tag{A3}$$

[11] Such a view of modeling is advanced, for example, by Fama (1976).

respectively. Note that when portfolio p is inefficient, the above three vectors are linearly independent. Define the OLS coefficient vector

$$\gamma = (X'X)^{-1} X'E. \tag{A4}$$

Let Q be an $n \times n$ diagonal matrix whose diagonal elements are all ones except for the $(3,3)$ element, which satisfies

$$q_{(3,3)} < \frac{\varepsilon}{\left(f_3' X (X'X)^{-2} X' f_3\right)^{1/2}} \tag{A5}$$

when $\theta \neq \gamma$ and $q_{(3,3)} = 1$ when $\theta = \gamma$. In the latter case, note that $f_3' X = 0$. Define the nonsingular matrix $B \equiv FQF^{-1}$. It is easy to verify that the columns of F are eigenvectors of B, and that the diagonal elements of Q are the corresponding eigenvalues. Hence,

$$B\iota = Bf_1 = f_1 q_{(1,1)} = \iota, \tag{A6}$$

$$B\beta = Bf_2 = f_2 q_{(2,2)} = \beta, \tag{A7}$$

and

$$Bf_3 = f_3 q_{(3,3)}. \tag{A8}$$

Equations (A6) and (A7) can be rewritten as:

$$BX = X. \tag{A9}$$

Let G be a nonsingular $n \times n$ matrix whose columns are orthogonal to each other and whose first three columns are

$$g_1 = \iota, \tag{A10}$$

$$g_2 = \beta - \frac{(\iota'\beta)}{n}\iota, \tag{A11}$$

and

$$g_3 = BE - X(X'X)^{-1} X'BE. \tag{A12}$$

Note that g_3 is the vector of residuals from the regression of BE on BX $(= X)$. When portfolio p is inefficient, the vector E is not spanned by the columns of X, and, therefore, the above three columns of G are linearly independent and orthogonal to each other. Let $\Lambda \equiv (G'G)^{-1}$. Since the columns of G are orthogonal to each other, Λ is a diagonal matrix. Let H be an $n \times n$ diagonal matrix whose diagonal elements are all ones except for the $(3,3)$ element, which is given by

$$h_{(3,3)} = \left(\frac{(v'v)(1-\omega)}{(g_3' g_3)\,\omega} \right)^{1/2}, \tag{A13}$$

where

$$v \equiv X(X'X)^{-1} X'BE - \frac{(\iota'BE)}{n}\iota. \tag{A14}$$

Define the nonsingular matrix $C \equiv GH\Lambda G'$. It is easy to verify that the columns of G are eigenvectors of C as well as C', and the diagonal elements of H are the corresponding eigenvalues of both C and C'. Hence,

$$C'\iota = C\iota = Cg_1 = g_1 h_{(1,1)} = \iota, \tag{A15}$$

$$C'\beta = C\beta = C\left(g_2 + \frac{(\iota'\beta)}{n} g_1\right) = \beta, \tag{A16}$$

$$C'X = CX = X, \tag{A17}$$

and

$$Cg_3 = h_{(3,3)} g_3. \tag{A18}$$

Now let $A \equiv CB$, which is nonsingular. Equations (A9) and (A17) imply that

$$AX = CBX = CX = X, \tag{A19}$$

so $A\iota = \iota$. Substituting (A19) into the definition of γ^* in equation (2) and simplifying by using equations (A3), (A8), (A9), and (A17) gives:

$$\begin{aligned}\gamma^* &= (X'A'AX)^{-1}X'A'AE = (X'X)^{-1}X'CBE \\ &= (X'X)^{-1}X'BE = (X'X)^{-1}X'B(X\theta + f_3) \\ &= \theta + (X'X)^{-1}X'Bf_3 \\ &= \theta + q_{(3,3)}(X'X)^{-1}X'f_3.\end{aligned} \tag{A20}$$

If $\gamma = \theta$, then $\gamma^* = \theta$, since in that case $q_{(3,3)} = 1$ and $X'f_3 = 0$. When $\gamma \neq \theta$, inequality (5) is obtained by combining equation (A20) with equation (A5):

$$\|\gamma^* - \theta\| = \left(q_{(3,3)}^2 f_3' X(X'X)^{-2}X'f_3\right)^{1/2} < \varepsilon. \tag{A21}$$

Using equations (2), (3), (A17), and (A19) we get:

$$\begin{aligned}(E^* - X^*\gamma^*) &= AE - AX(X'A'AX)^{-1}X'A'AE \\ &= AE - X(X'X)^{-1}X'BE \\ &= C(BE - X(X'X)^{-1}X'BE) \\ &= Cg_3 = g_3 h_{(3,3)},\end{aligned} \tag{A22}$$

which implies that

$$AE = g_3 h_{(3,3)} + X(X'X)^{-1}X'BE. \tag{A23}$$

Using equations (A17), (A23), and the definition of v in (A14) we get:

$$\begin{aligned}E^* - \frac{(\iota'E^*)}{n}\iota &= AE - \frac{(\iota'AE)}{n}\iota \\ &= g_3 h_{(3,3)} + X(X'X)^{-1}X'BE - \frac{(\iota'BE)}{n}\iota \\ &= g_3 h_{(3,3)} + v.\end{aligned} \tag{A24}$$

Equation (6) is obtained by substituting equations (A22) and (A24) into equation (4), observing that $(v'g_3) = 0$, and using the definition of $h_{(3,3)}$ in equation (A13):

$$
\begin{aligned}
R^2_{OLS} &= 1 - \frac{(E^* - X^*\gamma^*)'(E^* - X^*\gamma^*)}{\left(E^* - \dfrac{(\iota' E^*)}{n}\iota\right)'\left(E^* - \dfrac{(\iota' E^*)}{n}\iota\right)} \\
&= 1 - \frac{(g_3' g_3)h^2_{(3,3)}}{(g_3' g_3)h^2_{(3,3)} + v'v} \\
&= \frac{v'v}{(g_3' g_3)h^2_{(3,3)} + v'v} \\
&= \omega. \qquad \text{(A25)}
\end{aligned}
$$

Proof of Proposition 2: In this proof we use matrix notation similar to that employed in the proof of Proposition 1, although the matrices are redefined here and may be different from those in that proof. Proposition 1 implies that there exists a nonsingular $n \times n$ matrix C, with $C\iota = \iota$, such that $R^2_{OLS} = \omega$ in the regression of CE on $C\beta$. Here we construct an additional repackaging, using a nonsingular matrix B, and then consider the regression of BCE on $BC\beta$. Define

$$\bar{\beta} = \frac{1}{n}\iota' C\beta, \qquad \text{(A26)}$$

$$\bar{E} = \frac{1}{n}\iota' CE, \qquad \text{(A27)}$$

$$\sigma_\beta^2 = \frac{1}{n}(C\beta - \bar{\beta}\iota)'(C\beta - \bar{\beta}\iota), \quad \text{and} \qquad \text{(A28)}$$

$$\sigma_E^2 = \frac{1}{n}(CE - \bar{E}\iota)'(CE - \bar{E}\iota). \qquad \text{(A29)}$$

The following construction of the matrix B applies to the case where

$$\sigma_\beta^2 \neq \sigma_{\beta^*}^2 \quad \text{and} \quad \sigma_E^2 \neq \sigma_{E^*}^2. \qquad \text{(A30)}$$

The special cases where one of these inequalities does not hold will be discussed later. Define

$$k_\beta = \frac{\bar{\beta}^*\sigma_\beta - \bar{\beta}\sigma_{\beta^*}}{\sigma_\beta - \sigma_{\beta^*}} \qquad \text{(A31)}$$

and

$$k_E = \frac{\bar{E}^*\sigma_E - \bar{E}\sigma_{E^*}}{\sigma_E - \sigma_{E^*}}. \tag{A32}$$

Let F be a nonsingular $n \times n$ matrix whose first three columns are

$$f_1 = \iota, \tag{A33}$$

$$f_2 = C\beta - k_\beta \iota, \tag{A34}$$

and

$$f_3 = CE - k_E \iota. \tag{A35}$$

Note that when portfolio p is inefficient, the above three vectors are linearly independent. Let Q be an $n \times n$ diagonal matrix whose diagonal elements are all ones except for the (2, 2) and (3, 3) elements, which are given by

$$q_{(2,2)} = \frac{\sigma_{\beta^*}}{\sigma_\beta} \quad \text{and} \tag{A36}$$

$$q_{(3,3)} = \frac{\sigma_{E^*}}{\sigma_E}. \tag{A37}$$

Define the nonsingular matrix B,

$$B \equiv FQF^{-1}. \tag{A38}$$

It is easy to verify that the columns of F are eigenvectors of B and that the diagonal elements of Q are the corresponding eigenvalues. Hence,

$$B\iota = \iota, \tag{A39}$$

$$Bf_2 = f_2 q_{(2,2)}, \tag{A40}$$

and

$$Bf_3 = f_3 q_{(3,3)}. \tag{A41}$$

Equations (A40) and (A41) can be rewritten as:

$$\begin{aligned} BC\beta &= q_{(2,2)}C\beta + k_\beta(1 - q_{(2,2)})\iota \\ &= \frac{1}{\sigma_\beta}\left[\sigma_{\beta^*}C\beta + \left(\beta^*\sigma_\beta - \bar{\beta}\sigma_{\beta^*}\right)\iota\right], \end{aligned} \tag{A42}$$

and

$$\begin{aligned} BCE &= q_{(3,3)}CE + k_E(1 - q_{(3,3)})\iota \\ &= \frac{1}{\sigma_E}\left[\sigma_{E^*}CE + \left(E^*\sigma_E - \bar{E}\sigma_{E^*}\right)\iota\right]. \end{aligned} \tag{A43}$$

Now let $A \equiv BC$, which is nonsingular. Using equations (A31), (A32), (A36), (A37), (A42), and (A43), we can verify equations (7) and (8):

$$\begin{aligned}\frac{1}{n}\iota' A\beta &= \frac{1}{n}\iota' BC\beta \\ &= \frac{1}{\sigma_\beta}\left[\sigma_{\beta^*} n\bar{\beta} + \left(\beta^*\sigma_\beta - \bar{\beta}\sigma_{\beta^*}\right)n\right] \\ &= \bar{\beta}^*, \end{aligned} \tag{A44}$$

$$\begin{aligned}\frac{1}{n}\iota' AE &= \frac{1}{n}\iota' BCE \\ &= \frac{1}{\sigma_E}\left[\sigma_{E^*} n\bar{E} + \left(E^*\sigma_E - \bar{E}\sigma_{E^*}\right)n\right] \\ &= \bar{E}^*. \end{aligned} \tag{A45}$$

From equations (A42) and (A43) observe that

$$BC\beta - \bar{\beta}\iota = \frac{\sigma_{\beta^*}}{\sigma_\beta}(C\beta - \bar{\beta}\iota) \quad \text{and} \tag{A46}$$

$$BCE - \bar{E}\iota = \frac{\sigma_{E^*}}{\sigma_E}(CE - \bar{E}\iota), \tag{A47}$$

which, combined with (A28) and (A29), can be used to verify equations (9) and (10):

$$\begin{aligned}\frac{1}{n}(A\beta - \bar{\beta}^*\iota)'(A\beta - \bar{\beta}^*\iota) &= \frac{1}{n}(BC\beta - \bar{\beta}^*\iota)'(BC\beta - \bar{\beta}^*\iota) \\ &= \frac{1}{n}\frac{\sigma_{\beta^*}^2}{\sigma_\beta^2}(C\beta - \bar{\beta}\iota)'(C\beta - \bar{\beta}\iota) \\ &= \sigma_{\beta^*}^2, \end{aligned} \tag{A48}$$

$$\begin{aligned}\frac{1}{n}(AE - \bar{E}^*\iota)'(AE - \bar{E}^*\iota) &= \frac{1}{n}(BCE - \bar{E}^*\iota)'(BCE - \bar{E}^*\iota) \\ &= \frac{1}{n}\frac{\sigma_{E^*}^2}{\sigma_E^2}(CE - \bar{E}\iota)'(CE - \bar{E}\iota) \\ &= \sigma_{E^*}^2. \end{aligned} \tag{A49}$$

To prove equation (11), note from equations (A42) and (A43) that the vector BCE is obtained from the vector CE by adding a constant to each element of the product of CE and a constant. Similarly, the vector $BC\beta$ is obtained from the vector $C\beta$ by adding a constant to each element of the product of $C\beta$ and a constant. The value of R^2_{OLS} is invariant with respect to such linear transformations of the dependent and independent regression variables, so we conclude that the value of R^2_{OLS} in the regression of BCE on $BC\beta$ is ω, the same as the value of R^2_{OLS} in the regression of CE on $C\beta$.

In the special case where one of the inequalities in equation (A30) does not hold, one can construct the matrix B as a product of two repackaging matrices, $B = B_2 B_1$. Consider, for example, the case where $\sigma_{\beta^*} = \sigma_\beta$. Construct B_1 using equations (A31) to (A38) but with σ_{β^*} changed to $\sigma_\beta \cdot c$ for some $c > 1$. The resulting vector of betas, $B_1 C\beta$, will have the desired mean $\bar{\beta}^*$. For the construction of the second matrix, B_2, use the vector $B_1 C\beta$ instead of $C\beta$ in equations (A26) and (A28), change σ_{β^*} to $\sigma_\beta \cdot (1/c)$, and then again follow equations (A31) to (A38). After this second repackaging, the cross-sectional variance of the betas is the desired value σ_{β^*}. The value of R^2_{OLS} does not change in either of these repackagings.

Proof of Proposition 3: We first define the 2×2 matrix,

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \iota' V^{-1} \iota & \iota' V^{-1} E \\ \iota' V^{-1} E & E' V^{-1} E \end{bmatrix} \tag{A50}$$

and its determinant,

$$D = LN - M^2. \tag{A51}$$

As is well known (Roll (1977)),

$$\mu_g = \frac{M}{L}, \tag{A52}$$

$$\sigma_g^2 = \frac{1}{L}, \tag{A53}$$

and if (μ, σ^2) is a point on the minimum variance boundary, then

$$\sigma^2 = \frac{(L\mu^2 - 2M\mu + N)}{D}. \tag{A54}$$

Equation (A54) can then be rewritten as:

$$\sigma^2 - \frac{1}{L} = \frac{L}{D}\left(\mu - \frac{M}{L}\right)^2. \tag{A55}$$

By construction, (μ_p, σ_y^2) and (μ_x, σ_p^2) are on the minimum variance boundary. Using equations (A52), (A53), and (A55) we get

$$\sigma_y^2 - \sigma_g^2 = \frac{L}{D}(\mu_p - \mu_g)^2 \quad \text{and} \tag{A56}$$

$$\sigma_p^2 - \sigma_g^2 = \frac{L}{D}(\mu_x - \mu_g)^2. \tag{A57}$$

Dividing equation (A56) by (A57) gives

$$\left(\frac{\mu_p - \mu_g}{\mu_x - \mu_g}\right)^2 = \left(\frac{\sigma_y^2 - \sigma_g^2}{\sigma_p^2 - \sigma_g^2}\right). \tag{A58}$$

Equation (13) follows from (A58) and the definition of ψ in equation (12).

Proof of Proposition 4: The geometric analysis of the GLS coefficients in Roll (1985) may be used as a starting point for this proof. For the sake of clarity, we provide a complete proof. Observe that

$$X = \begin{bmatrix} \iota & \frac{1}{w_p' V w_p} V w_p \end{bmatrix}, \tag{A59}$$

so

$$X'V^{-1}X = \begin{bmatrix} \iota' V^{-1} \iota & \frac{1}{w_p' V w_p} \\ \frac{1}{w_p' V w_p} & \frac{1}{w_p' V w_p} \end{bmatrix} = \frac{1}{\sigma_p^2} \begin{bmatrix} L\sigma_p^2 & 1 \\ 1 & 1 \end{bmatrix}, \tag{A60}$$

$$(X'V^{-1}X)^{-1} = \frac{\sigma_p^2}{L\sigma_p^2 - 1} \begin{bmatrix} 1 & -1 \\ -1 & L\sigma_p^2 \end{bmatrix}, \tag{A61}$$

and

$$X'V^{-1}E = \begin{bmatrix} \iota' V^{-1} E \\ \frac{w_p' E}{w_p' V w_p} \end{bmatrix} = \begin{bmatrix} M \\ \frac{\mu_p}{\sigma_p^2} \end{bmatrix}. \tag{A62}$$

Multiplying equations (A61) and (A62) gives

$$\phi = \frac{\sigma_p^2}{L\sigma_p^2 - 1} \begin{bmatrix} M - \frac{\mu_p}{\sigma_p^2} \\ L\mu_p - M \end{bmatrix}. \tag{A63}$$

Using equations (A52) and (A53), the second element of ϕ in equation (A63) can be written as

$$\phi_2 = \left(\frac{\sigma_p^2}{\sigma_p^2 - \sigma_g^2} \right) (\mu_p - \mu_g). \tag{A64}$$

This expression for ϕ_2 is presented also by Roll (1985). The expression in equation (18) is obtained by observing that, since portfolio x is on the minimum-variance boundary, we can write μ_g as

$$\begin{aligned} \mu_g &= \mu_{x0} + (\mu_x - \mu_{x0}) \frac{\mathrm{cov}\{r_g, r_x\}}{\sigma_x^2} \\ &= \mu_{x0} + (\mu_x - \mu_{x0}) \frac{\sigma_g^2}{\sigma_p^2}, \end{aligned} \tag{A65}$$

where the second line follows by substituting σ_p^2 for σ_x^2 (equal by construction) and from the property that every asset has covariance σ_g^2 with the global minimum-variance portfolio (Roll (1977)). Equation (A65) can be rewritten as

$$\sigma_p^2 = \sigma_g^2 \frac{(\mu_x - \mu_{x0})}{(\mu_g - \mu_{x0})} \tag{A66}$$

Substituting equation (A66) for σ_p^2 in (A64), simplifying, and using the definition of ψ_p in equation (12) gives equation (18). The expression for ϕ_1 in equation (19) follows directly by substituting equations (A52), (A53), and (A65) into the first element of ϕ in equation (A63) and simplifying.

Proof of Proposition 5: Equation (20) can be rewritten as:

$$1 - R_{GLS}^2 = \frac{(E - X\phi)'V^{-1}(E - X\phi)}{(E - \iota\bar{\mu})'V^{-1}(E - \iota\bar{\mu})}, \tag{A67}$$

Through straightforward algebra, using equations (A50) to (A63), one can express the numerator of $(1 - R_{GLS}^2)$ in equation (A67) as

$$\begin{aligned}(E - X\phi)'V^{-1}(E - X\phi) &= \frac{L\mu_p^2 - 2M\mu_p + N - D\sigma_p^2}{1 - L\sigma_p^2} \\ &= \frac{D}{L}\frac{\left(\sigma_p^2 - \sigma_y^2\right)}{\left(\sigma_p^2 - \sigma_g^2\right)}\end{aligned} \tag{A68}$$

where the second line makes use of the equation for the minimum-variance boundary in equation (A54),

$$\sigma_y^2 = \left(L\mu_p^2 - 2M\mu_p + N\right)/D. \tag{A69}$$

The denominator of $(1 - R_{GLS}^2)$ can be expressed, using equations (21), (A50), and (A51), as

$$(E - \iota\bar{\mu})'V^{-1}(E - \iota\bar{\mu}) = \frac{D}{L}. \tag{A70}$$

Taking one minus the ratio of (A68) to (A70) gives

$$R_{GLS}^2 = \frac{\left(\sigma_y^2 - \sigma_g^2\right)}{\left(\sigma_p^2 - \sigma_g^2\right)}, \tag{A71}$$

which is equal to ψ_p^2 using Proposition 3.

Proof of Proposition 6: Since $w(\hat{E};\sigma^2)$ is the solution to the portfolio maximization problem in equations (24) to (26), there exist scalars $\zeta_1 > 0$ and ζ_2 such that the following first-order condition is satisfied:

$$\begin{aligned}w(\hat{E};\sigma^2) &= \zeta_1 V^{-1}\hat{E} + \zeta_2 V^{-1}\iota \\ &= \zeta_1 V^{-1}Za + \zeta_2 V^{-1}\iota,\end{aligned} \tag{A72}$$

where the second line uses equation (23). The maximization problem's constraints imply that

$$\zeta_2 = \frac{1}{L}(1 - \zeta_1 \iota' V^{-1} Za), \quad \text{and} \tag{A73}$$

$$\zeta_1 = \left(\frac{L\sigma^2 - 1}{La'Z'V^{-1}Za - (\iota'V^{-1}Za)^2} \right)^{1/2}. \tag{A74}$$

Using equations (A50), (A73), and (A74), the expected return $[\omega(\hat{E}; \sigma^2)]'E$ can be written as

$$[w(\hat{E}; \sigma^2)]'E = \frac{M}{L} + \zeta_1 \left[(E'V^{-1}Za) - \frac{M}{L}(\iota'V^{-1}Za) \right]. \tag{A75}$$

Recall that $\delta = (Z'V^{-1}Z)^{-1}Z'V^{-1}E$. Let d_1 be an n-vector with 1 in the first element and 0 elsewhere. Noting that the first column of Z is ι, it is easily verified that

$$Zd_1 = \iota, \tag{A76}$$

$$d_1 = (Z'V^{-1}Z)^{-1} Z'V^{-1}\iota, \tag{A77}$$

$$\delta' Z'V^{-1}\iota = E'V^{-1}Zd_1 = E'V^{-1}\iota = M, \quad \text{and} \tag{A78}$$

$$\iota'V^{-1}Zd_1 = d_1'Z'V^{-1}Zd_1 = \iota'V^{-1}\iota = L. \tag{A79}$$

Define

$$K \equiv E'V^{-1}Z\delta = (E'V^{-1}Z)(Z'V^{-1}Z)^{-1}(Z'V^{-1}E), \tag{A80}$$

and note that $K \geq (M^2/L)$, using equations (A76) to (A79) and the Cauchy-Schwarz inequality. For any n-vector a there exist scalars c_1 and c_2 and an n-vector u such that

$$a = c_1\delta + c_2 d_1 + u, \tag{A81}$$

$$u'(Z'V^{-1}Z)d_1 = u'Z'V^{-1}\iota = 0, \quad \text{and} \tag{A82}$$

$$u'(Z'V^{-1}Z)\delta = u'Z'V^{-1}E = 0. \tag{A83}$$

Maximizing the expected return $[w(\hat{E}; \sigma^2)]'E$ with respect to a is, therefore, equivalent to maximizing the expected return with respect to c_1, c_2, and an n-vector u that satisfies equations (A82) and (A83). Using equations (A78), (A79), (A82), and (A83), we get

$$\begin{aligned} \iota'V^{-1}Za &= c_1(\iota'V^{-1}Z\delta) + c_2(\iota'V^{-1}Zd_1) + (\iota'V^{-1}Zu) \\ &= c_1 M + c_2 L, \end{aligned} \tag{A84}$$

$$\begin{aligned} E'V^{-1}Za &= c_1(E'V^{-1}Z\delta) + c_2(E'V^{-1}Zd_1) + (E'V^{-1}Zu) \\ &= c_1 K + c_2 M, \quad \text{and} \end{aligned} \tag{A85}$$

$$a'Z'V^{-1}Za = c_1^2 K + c_2^2 L + 2c_1c_2 M + u'Z'V^{-1}Zu. \tag{A86}$$

Substituting equations (A84) to (A86) into equations (A74) and (A75) gives

$$[w(\hat{E};\sigma^2)]'E = \frac{M}{L} + \frac{(L\sigma^2 - 1)^{1/2} c_1(KL - M^2)}{\left[c_1^2(KL - M^2) + (u'Z'V^{-1}Zu)\right]^{1/2} L}. \tag{A87}$$

Let $c_2 = 0$, since equation (A87) does not depend on c_2. The maximum must occur with $c_1 > 0$, since the denominator and the other factors in the numerator are positive. For any $c_1 > 0$, the maximum occurs at $u = 0$ and does not depend on c_1. Thus, let $c_1 = 1$, which implies the maximum occurs at $a = \delta$ or $\hat{E} = Z\delta = E^\dagger$.

Proof of Proposition 7: We first observe that $(E - Z\delta)$ can be written as

$$(E - Z\delta) = \left[I - Z(Z'V^{-1}Z)^{-1}Z'V^{-1}\right]E, \tag{A88}$$

which, when substituted into the numerator of $(1 - R^2_{GLS})$, using equation (30), provides

$$\begin{aligned}
&(E - Z\delta)'V^{-1}(E - Z\delta) && \text{(A89)}\\
&= E'\left[I - Z(Z'V^{-1}Z)^{-1}Z'V^{-1}\right]'V^{-1}\left[I - Z(Z'V^{-1}Z)^{-1}Z'V^{-1}\right]E \\
&= E'V^{-1}E - E'V^{-1}Z(Z'V^{-1}Z)^{-1}Z'V^{-1}E \\
&= N - K && \text{(A90)}
\end{aligned}$$

where the last line uses equations (A50) and (A80). The denominator of $1 - R^2_{GLS}$ can be expressed, using equations (21), (A50), and (A51), as

$$(E - \iota\bar{\mu})'V^{-1}(E - \iota\bar{\mu}) = \frac{D}{L}. \tag{A91}$$

Taking one minus the ratio of equations (A90) to (A91) and using equation (A51) gives

$$R^2_{GLS} = \frac{(KL - M^2)}{D}. \tag{A92}$$

Let μ_q and σ_q^2 denote the mean and variance of the return on the portfolio q, respectively. By Proposition 3,

$$\psi_q^2 = \frac{\sigma_y^2 - \sigma_g^2}{\sigma_q^2 - \sigma_g^2}, \tag{A93}$$

where σ_y^2 denotes here the variance of the minimum-variance portfolio with mean return μ_q. Equation (A56) implies that the numerator of equation

(A93) can be written as

$$\sigma_y^2 - \sigma_g^2 = \frac{L}{D}(\mu_q - \mu_g)^2. \tag{A94}$$

Rewriting the maximized value of equation (A87) in terms of μ_q, σ_q^2, and μ_g gives:

$$(\mu_q - \mu_g)^2 = \frac{\left(\sigma_q^2 - \sigma_g^2\right)(KL - M^2)}{L}, \tag{A95}$$

which implies that the denominator of equation (A93) can be written as:

$$\sigma_q^2 - \sigma_g^2 = \frac{L(\mu_q - \mu_g)^2}{KL - M^2}. \tag{A96}$$

Dividing equation (A94) by (A96) yields

$$\psi_q^2 = \frac{(KL - M^2)}{D} = R_{GLS}^2, \tag{A97}$$

where the second equality is based on equation (A92).

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