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An Equilibrium Analysis of Hedging with Liquidity Constraints, Speculation, and Government Price Subsidy in a Commodity Market

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ABSTRACT

We develop a simple commodity model to analyze (i) the effects of hedging with liquidity constraints, due to producers' inability to bear unlimited trading losses, (ii) the role of speculation in the process of risk allocation between consumers and producers, and (iii) the equilibrium implications of government price subsidies to the producers. We find that (1) liquidity constraints can cause futures prices to exhibit mean reversion, which then makes speculation profitable; (2) speculation tends to make futures price volatility an increasing function of futures price; and (3) government price subsidy, if actively hedged by the producers, serves to lower the futures risk premium and reduce futures volatility.

COMMODITY SPOT AND FUTURES PRICES are determined by the following types of market participants: producers, consumers, speculators, and governments. Producers have generally taken short futures positions in order to hedge against price declines. Consumers have generally taken long futures positions in order to hedge against price increases. Speculators, who neither produce nor intend to consume the commodity, are present in the spot and futures markets for the purpose of making a profit. Governments play significant roles in the markets by providing producers with various support programs, and in some cases, by directly intervening in the markets to affect commodity prices or adjust inventory levels.

The traditional approach to commodity futures pricing has been based on the theory of storage.[1] As Fama and French (1987, 1988) show, this approach to commodity pricing is particularly useful for explaining the basis movement as a function of inventory levels, based on the hypothesis that the marginal convenience yield on inventory declines at a decreasing rate as

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[1] See Working (1948), Brennan (1958), Telser (1958), and Williams and Wright (1991) for references.

inventory increases. The alternative, but not competing, approach focuses on futures risk premium. See, for example, Keynes (1930), Hicks (1946), and Cootner (1960) for relating the futures risk premium to the positions taken by speculators. In essence, hedgers should expect to pay speculators in the form of futures risk premium for the service of providing insurance. In addition, Dusak (1973), Bodie and Rosansky (1980), Breeden (1980), Richard and Sundaresan (1981), and Hazuka (1984) use either static or intertemporal capital asset pricing models to examine the futures risk premium. This paper, which is a continuous-time general equilibrium model of commodity futures, stands closer to the second approach.

This paper does, however, differ from the existing literature by explicitly making the following two assumptions. First, we only consider those commodities that are indispensable goods in the following sense: when the commodity supply is low, a consumer is willing to pay a lot to have an adequate level of consumption. That is, when the supply is low, the commodity is necessary in the sense that the price elasticity of demand for the commodity is less than one.[2] Accordingly, we assume that a consumer's relative risk aversion is a decreasing function of consumption level.

Second, we assume that producers are faced with the liquidity constraint that their trading losses in the futures market cannot exceed a prespecified level. If the maximum level of tolerable trading losses is set to be zero, then the producers do not participate in the futures market. In general, the level of participation by the producers depends on the maximum level of tolerable trading losses. For brevity, we may refer to the case of no imposition of liquidity constraints as the case of complete hedging. Incomplete hedging is the case when the producers' trading losses cannot exceed a certain level in all states of nature. Note that the case of complete hedging does not imply that the producers always hedge the entire amount of expected output. As shown by Breeden (1984), when there is a risk premium, the risk-averse producers choose to take risky net positions in the commodities.

By making the first assumption, we recognize that the properties of a futures price process depend on the desirability of the commodity that underlies the futures contract. Thus, we should not expect that all commodity futures behave in the same way, which calls for a discriminatory analysis of commodity futures data. The first assumption is similar, in spirit, to the usual assumption (in the theory of storage) that the marginal convenience yield on inventory declines at a decreasing rate as inventory increases.

By making the second assumption, we recognize the fact that producers do not usually hedge as much as traditional theories would suggest. The debate over hedging with futures is probably as old as the futures exchanges themselves. The problem with complete hedging is not unique to farmers, who may not be able to tolerate unlimited trading losses in the futures markets. The problem also extends to multinational corporations in search of an answer to the question of foreign exchange hedging, or to domestic firms de-

[2] See Henderson and Quandt (1980) for a definition of necessary versus luxury commodity.

ciding whether to hedge the costs of inputs to production processes. Thus, certain portions of the paper may also interest scholars in the corporate finance area. Recently, there has been an interesting debate about hedging in the context of energy products (see, for example, Culp and Miller (1994) and Edwards and Canter (1995)). Given an exogenous futures price process, Deep (1996) considers, via a numerical method, dynamic hedging with futures under the liquidity constraint that the futures account cannot lose more than a prespecified amount of money. This paper, among other things, solves analytically for the optimal hedging strategy under the liquidity constraint by using a martingale approach. In the context of commodity markets, hedging strategies have been discussed in the absence of liquidity constraints (see Rolfo (1980), Anderson and Danthine (1983), and Ho (1984)).

The purposes of the paper are to: (1) show that liquidity constraints can cause the futures price to exhibit mean reversion, (2) examine the impact of speculation on the risk premium and volatility, (3) study the optimal speculative and hedging strategies, (4) analyze the equilibrium effects of government price support programs, and (5) explain the observed volatility skewness (for agricultural commodities, as well as for most other commodities, realized price volatility has been an increasing function of the price level). In the following, we summarize the results in the same order.

(1) *On mean reversion of futures prices.* Though there is some empirical discussion of mean reversion in spot commodity prices (see Bessembinder et al. (1995) for references), we have not seen rigorous empirical studies on mean reversion of commodity futures prices. Note that mean reversion of futures prices is very different from mean reversion of spot prices, because the former suggests profitability in rational speculation but the latter does not. In the context of incomplete hedging, our results suggest that when the futures price is high (or when the supply is low), the expected return on futures is negative; and when the futures price is low (or when the supply is high), the expected return is positive. Here is an intuition: When the supply is low and the futures price is high, the consumers' demand for hedging is high due to high risk aversion, but due to the lack of full participation by the producers, the speculators are induced to sell futures to the consumers. The negative return on futures is a form of reward for the insurance service provided by the speculators.

(2) *On the effect of speculation on the market price of risk and volatility.* In light of the growing importance of speculative capital in most liquid markets including commodity futures, we find it interesting to consider the effect of speculation on the futures price process. It is intuitive that the presence of speculative capital can only serve to drive down the market price of futures risk. The effect of speculation on futures volatility is not as obvious. In the context of this paper, we show analytically that speculation serves to increase futures volatility when the futures price is high, and acts to decrease volatility when the futures price is low. The examination of the futures risk premium through speculators' futures holdings can be traced back to Keynes (1930).

(3) *On speculative and hedging strategies.* We show that a risk-averse speculator's wealth is a decreasing function of the state price density (or pricing kernel in our parlance),[3] which implies that speculation is profitable if nothing dramatic happens to the commodity supply, but it is not profitable if the supply is too high or too low. Thus, the act of speculating is equivalent to writing an insurance policy on the supply of the commodity to guarantee that the supply will be normal. Although our result on speculative strategies generalizes Keynes (1930), who suggests that speculators are generally long in futures, the idea that speculators are essentially providers of insurance is the same as his. Our result on hedging strategies is consistent with such past research as that of Anderson and Danthine (1983) in the case of complete hedging, but it expands on previous research in that it handles the case of a liquidity constraint.

(4) *On government price subsidies.* We look at government price subsidies as a random number of put options on futures prices, as in Marcus and Modest (1986). We find that the effect of a government price subsidy on the equilibrium commodity prices depends on the extent to which producers can participate in futures trading. In the case of complete hedging, a price subsidy causes the futures risk premium to be smaller in size and reduces price volatility, particularly when the futures price is just at or below the government price support level. In Marcus and Modest (1986), the U.S. agricultural price support system is evaluated without considering the equilibrium implications of the price support system itself. If futures price volatility is indeed reduced in the presence of a price support system, then by option pricing theory, a partial equilibrium approach such as the one in Marcus and Modest could overestimate the true cost of a price support system. Crain and Lee (1996) provide evidence in the wheat markets in support of our findings that the government price subsidy serves to reduce futures price volatility.

(5) *On volatility skewness.* As far as futures volatility is concerned, there has been a lot of discussion about Samuelson's (1965) proposition that the standard deviation of a futures price is a decreasing function of futures maturity, and some discussion about the fact that futures prices are more variable during periods when significant amounts of supply and demand uncertainty are resolved. See, for example, Anderson and Danthine (1983), Anderson (1985), and Bobin (1990). Williams and Wright (1991) argue, via numerical simulations, that the theory of storage implies that the standard deviation of next-period price (P_{t+1}) is an increasing function of this-period price (P_t).[4] The volatility result in this paper bears resemblance to that of

[3] Under the assumption that the consumer's relative risk aversion is a decreasing function of consumption level, we can show (see Section IV.A) that the state price density is a bowl-shaped function of commodity supply. Thus, the speculator's wealth is a bell-shaped function of the commodity supply.

[4] Refer to Williams and Wright (1991, p. 173). Their intuition for the result that "with storage, the probability distribution of a commodity's price is skewed with long tail being toward high prices" is that "collectively, the market can always store whereas it cannot borrow from the future, storage is much more effective at supporting what would otherwise be very low prices than at reducing what would otherwise be very high prices," (page 2).

Williams and Wright, except that we can show analytically that price volatility, that is, the standard deviation of $(P_{t+1} - P_t)/P_t$, rather than the standard deviation of P_{t+1}, is an increasing function of P_t. This distinction is important for explaining the option-implied volatility skewness to be discussed below.

In fact, the original motivation of the paper was to explain the persistent implied volatility skewness in agricultural options: For a given option maturity, implied volatility is an increasing function of option strike price. This empirical fact is well known among agricultural option traders, but to the best of my knowledge it has received no attention from the academics, which is in sharp contrast to the kind of scrutiny placed on the implied volatility skewness of S&P 500 options (see Rubinstein (1994)). Table I lists some implied volatilities as of November 22, 1995.

Observe that for S&P 500 options, implied volatility is a decreasing function of option strike price, but for agricultural options, implied volatility is an increasing function of option strike price. Grossman and Zhou (1996) explain the implied volatility skewness in equity index options by modeling a proportion of market participants as users of portfolio insurance. According to that paper, volatility skewness follows essentially from the fact that the relative risk aversion in the economy increases as the stock market declines. To explain the kind of volatility skewness in agricultural options, it is then intuitive that we have to make use of the assumption that consumers are more risk averse when there is less commodity for consumption, that is, when the commodity price increases.

The organization of the paper is as follows. In Section I we state the objectives of all market participants except governments. In Section II we use an equivalent martingale method to solve the optimization problems faced by various agents, and we derive equilibrium conditions. In Section III we characterize the benchmark case in which there is no limit imposed on a producer's trading losses. Section IV presents the main results of the paper, including how speculation and incomplete hedging affect the market price of risk and price volatility, as well as optimal trading strategies. Section V discusses the equilibrium consequences of government price subsidies. Finally, in Section VI we conclude and present some stylized facts in support of incomplete hedging.

I. The Model

We consider a commodity futures market where trading takes place continuously between time 0 and time T. The traded securities are: (i) cash, which is risk-free, and (ii) a futures contract, which is a claim to one unit of commodity at time T. At time 0 the producer plants the seed for the commodity; at time T the crop is harvested. The information flow concerning

Table I

Implied Volatility Skewness in Call Options on Futures

This table shows the Black–Scholes implied volatilities of some call options on futures, based on closing prices on November 22 1995. The futures exchanges specify the option strike prices available for trading. The strike price of the At-the money call option is the strike price that is closest to the futures price. The strike price of the One strike in-the-money call option is the available strike price that is immediately below the strike price of the At-the-money call option. Similarly, the strike price of the One strike out-of-the-money call option is the available strike price that is immediately above the strike price of the At-the-money call option. Thus, the strike prices are in ascending order when we read the table from top to bottom.

Annual Volatility (%)	May Corn	May Wheat	May Beans	May Crude	May Silver	March S & P 500
Two strikes in-the-money	14.84	18.71	13.16	17.07	21.45	11.76
One strike in-the-money	16.29	18.88	13.49	17.61	21.55	11.45
At-the-money	16.85	19.05	14.49	18.03	22.90	11.11
One strike out-of-the-money	17.42	19.92	15.61	19.12	24.35	10.84
Two strikes out-of-the-money	17.55	20.86	16.69	19.89	25.55	10.72

Q_T, the output of the commodity at time T, is described by a filtration generated by a standard Brownian motion Z_t.[5]

The price of commodity futures is expressed in units of the price of cash. That is, the price of cash serves as a numéraire for the price of the commodity. We may think of cash as a basket of other goods that all market participants can consume. Since the price of cash serves as a numéraire for the price of the commodity, we will essentially study the price of the commodity *relative* to the price of the basket of other goods. By definition, the price of cash is always equal to one. In other words, the price numéraire is chosen in such a way that the cash earns a zero interest rate. We will use P_t to denote the commodity futures price at time t.[6]

[5] Note that Q_T is assumed to be exogenous. Thus, if the producer sees a sharp rise in the market price of the commodity, he cannot increase his output level by time T to benefit from the price increase. In the case of agricultural commodities, we may interpret T as the duration of one harvest cycle. Generally speaking, T can be thought of as the time needed to alter planned production level. The reader should also note that the uncertainty about Q_T is the only source of uncertainty in our model. That is, we only consider supply shocks. However, demand shocks could be accommodated analogously; after all, an increase in demand is equivalent to a decrease in supply.

[6] Note that in our model there is no spot but only a futures market, which makes it impossible to discuss futures basis. The theory of storage, which uses inventory to cushion demand and supply shocks, is more appropriate for basis analysis. We choose not to discuss the complex inventory problem mainly for analytical convenience. Having said that, our simple model does allow for an in-depth discussion of such topics as incomplete hedging, speculation, and government support programs. Moreover, as the referee points out, the demand function (used in this paper) of the consumer with decreasing relative risk aversion is similar to the demand function generated in a storage-based model, which is comforting in that our model is not incompatible with a storage-based model. For commodities subject to seasonal production, the basis problem becomes interesting only for futures contracts whose expiration dates occur after a new harvest takes place. For futures contracts based on the same crop as the spot commodity, the basis can be derived by the usual cash-and-carry formula. The choice of zero interest rate in the paper is innocuous (merely as the result of a choice of numéraire); see Grossman and Zhou (1996) for a similar treatment.

There are three representative agents: a producer, a consumer, and a speculator.

The **producer** is endowed with no cash but an uncertain amount of commodity at time T. He chooses a nonanticipative trading strategy in the risky and risk-free securities to maximize his expected utility of final wealth:[7]

$$\max E_0[U_f(W_T)], \tag{1}$$

subject to

$$W_T = \int_0^T C_t\, dP_t + Q_T \cdot P_T; \tag{2}$$

$$\int_0^T C_t\, dP_t \geq -K, \tag{3}$$

where E_0 is an expectation operator conditional on time-zero information, W_T denotes the final wealth which consists of trading revenue $\int_0^T C_t \cdot dP_t$ and the market value of his terminal endowment, $Q_T \cdot P_T$.[8] Here we use C_t to denote the number of futures contracts held at time t. The inequality (3) implies that the maximum trading loss at any time should never exceed a prespecified level K. We may simply think of K as the maximum amount of loan that the producer can obtain against his future revenue for the purpose of meeting margin requirements in his futures account. Although we only consider risk-averse producers in the paper, the main conclusions should hold for an equilibrium consideration of the hedging problems in a corporate finance environment.[9]

[7] By assuming that the producer's utility is a function of wealth only, we hold the view that the producer consumes only a negligible amount of the very commodity he produces.

[8] In the case of agricultural commodities, producers may benefit from various governmental subsidies. Values of those subsidies are not reflected in equation (1), but are discussed later in the paper.

[9] If instead of a risk-averse producer, we look at a firm's optimal hedging decision subject to liquidity constraints, then it may make sense to replace the producer's optimization problem with the following optimization problem for the firm: maximize $E_0[W_T \cdot \eta_T]$ subject to conditions (2) and (3), where η_T is the pricing kernel (to be discussed later in the paper). Since $E_0[W_T \cdot \eta_T]$ is the current value of the firm, maximizing $E_0[W_T \cdot \eta_T]$ means maximizing the market value of the firm. The firm's optimization problem, as well as the equilibrium problem, can be easily solved by the same martingale technology as used in the paper. Or better yet, we may treat the firm's optimization problem as the limiting case of maximizing $E_0[(W_T)^A \cdot \eta_T]$, where A approaches one from below. From this, we can show that the key equation for the producer's optimization problem, equation (18), can be replaced by $W_{fT}^* = \max[Q_T P_T - K, x_f]$ in the case of corporate hedging, where x_f is a positive constant. Curious readers can then continue to show that the essential results in the paper remain valid in the case of corporate hedging. In a very different setup, Mello and Parsons (1997) consider how the firm values are affected by liquidity constraints and different hedging strategies. I am grateful to René Stulz (the editor) for bringing up this point.

The **consumer** is only endowed with W_c in cash at time zero. He derives utility from both the consumption of the commodity and the final cash holdings. He maximizes the expected total utility by trading in the futures market:[10]

$$\max E_0[U_{1c}(C_T) + U_{2c}(B_T)], \tag{4}$$

subject to

$$W_T \equiv W_c + \int_0^T C_t\, dP_t = C_T \cdot P_T + B_T, \tag{5}$$

where final wealth W_T consists of the non–interest-bearing initial endowment and trading revenue. Note that $C_T \cdot P_T$ is the consumption expenditure, and the remaining wealth, $B_T = W_T - C_T \cdot P_T$, is in cash.

The **speculator** is only endowed with W_s in cash at time zero. His objective is to maximize the expected utility of his final wealth by trading in the futures market:

$$\max E_0[U_s(W_T)]; \tag{6}$$

subject to

$$W_T = W_s + \int_0^T C_t\, dP_t, \tag{7}$$

where W_T consists of the non–interest-bearing initial endowment and trading revenue.

We conclude this section by making the following assumptions. We assume:

$$Q_T = \exp\left(-\frac{1}{2}\sigma^2 \cdot T - \sigma Z_T\right), \tag{8}$$

where σ is a positive constant. A constant σ implies that the flow of information regarding Q_T arrives in the futures market homogeneously in time, which ignores the seasonality of the information process that is present for certain commodities such as soybeans. However, our main results would not be altered even if we modeled seasonality directly by making σ a function of time. Note that a high level of Z_T means a low level of output, which, as we will see, implies a high level of price.

[10] For simplicity, we assume that the consumer derives utility only from *terminal* consumption of the commodity. To study interim consumption between time zero and time T, one has to build a more complicated model to incorporate the cost of carrying a commodity forward in time. The assumption that the consumer also derives utility from final cash holdings allows for some substitution effect between the commodity and cash. The additive utility function, $U_{1c}(C_T) + U_{2c}(B_T)$, makes it easy to characterize the desirability of the commodity and compute the price elasticity of the consumer's demand function.

When we need to obtain exact solutions, we assume that all utility functions of cash holdings are logarithmic:

$$U_f(B) = U_s(B) = U_{2c}(B) = \ln(B). \tag{9}$$

Let $R(C)$ denote the consumer's relative risk aversion: $R(C) \equiv -[C \cdot U''_{1c}(C)]/U'_{1c}(C)$. We assume that $R(C)$ is a decreasing function of C. Moreover, we assume that there exists $\hat{C}$ such that when $C < \hat{C}$, $R(C) > 1$ and when $C > \hat{C}$, $R(C) < 1$. As we subsequently show, when C is below $\hat{C}$, the commodity is necessary, and when C is above $\hat{C}$, the commodity is a luxury. Under these assumptions, the consumer is motivated to keep the consumption level at or above $\hat{C}$. The monotonicity assumption about $R(C)$ is crucial for generating the kind of price volatility skewness mentioned in the introduction.[11]

II. Equilibrium Conditions

In this section, we first solve all agents' optimization problems by using an equivalent martingale approach. We then state the equilibrium conditions in terms of the equivalent martingale measure. Basically, the martingale approach enables us to transform dynamic programming problems into static optimization problems.

The martingale approach relies on the concept of a pricing kernel. For a price process $\{P_t\}$, a positive-valued adapted process $\{\eta_t\}$ is called a pricing kernel of $\{P_t\}$ if (i) $\eta_0 = 1$, (ii) η_t is a martingale, and (iii) $P_t \cdot \eta_t$ is a martingale. Sometimes we may simply call η_T a pricing kernel for $\{P_t\}$.

If the price process $\{P_t\}$ is a diffusion, $dP_t/P_t = \mu_t dt + \sigma_t \cdot dZ_t$, then by standard arguments the pricing kernel of $\{P_t\}$ is unique and is given by $\eta_t = E_t[\eta_T]$, where

$$\eta_T = \exp\left(-\frac{1}{2}\int_0^T \theta_t^2\, dt - \int_0^T \theta_t\, dZ_t\right), \tag{10}$$

where $\theta_t \equiv \mu_t/\sigma_t$ is the market price of risk. Note that in this case, we have $d\eta_t = -\,\theta_t \cdot \eta_t dZ_t$.

If η_t is the pricing kernel, then we can use the fact that η_t and $P_t \cdot \eta_t$ are martingales to show that the gains from any trade multiplied by the pricing kernel is also a martingale; that is, $\eta_t \cdot \int_0^t C_t dP_t$ is a martingale for any nonanticipative process C_t that satisfies certain regularity conditions. In particular, we have

$$E_0\left(\eta_T \cdot \int_0^T C_t dP_t\right) = 0. \tag{11}$$

[11] An extreme example of decreasing relative risk aversion is the following: $U_{1c}(C) = \sqrt{C}$ if $C \geq \hat{C}$, and $U_{1c}(C) = -\infty$ if $C < \hat{C}$. For this utility function, $\hat{C}$ is the subsistence level of consumption: The consumer is willing to pay any price to keep consumption above $\hat{C}$.

A (dynamically) complete market is such that the reverse is also true. In a complete market, for any terminal wealth W_T such that $E_0(\eta_T \cdot W_T) = 0$, we can represent W_T as a stochastic integral with respect to the price process $W_T = \int_0^T C_t dP_t$. Implicit in the assumption of a complete market is the assumption that there is no limit imposed on trading losses, since the time series of $\int_0^t C_s dP_s$ may not be bounded from below. In general, if $W_T = W_0 + \int_0^T C_t dP_t$ and C_t is a feasible strategy for a certain individual, then we say that W_T is attainable with W_0 for the individual.

Let us first consider the consumer's optimization problem. By equations (11) and (5), we infer that C_T and B_T must satisfy:

$$E_0[C_T \cdot (P_T \cdot \eta_T) + B_T \cdot \eta_T] = W_c. \tag{12}$$

We may now think of the consumer's optimization problem as one of choosing an optimal pair (C_T, B_T) to maximize the expected total utility subject to equation (12). This consideration leads to the following lemma, whose proof is given in the Appendix.

LEMMA 1: *If both U_{1c} and U_{2c} are strictly concave and twice differentiable, then the solution to the consumer's optimization problem is given by the following expressions for C^*_{cT}, the optimal consumption of the commodity at time T, and B^*_{cT}, the optimal cash holdings at time T:*

$$C^*_{cT} = I_{1c}(x_c \cdot P_T \cdot \eta_T); \tag{13}$$

$$B^*_{cT} = I_{2c}(x_c \cdot \eta_T), \tag{14}$$

provided that $W^*_{cT} \equiv C^*_{cT} \cdot P_T + B^*_{cT}$ is attainable, where I_{1c} is the inverse function of U'_{1c}, and I_{2c} is the inverse function of U'_{2c}. In addition, x_c is a positive constant chosen to satisfy the budget constraint $E_0[W^*_{cT} \cdot \eta_T] = W_c$.

The speculator's optimization problem is a well-known portfolio choice problem solved in financial economics. The speculator's problem can be seen as a special case of the consumer's optimization problem by setting $U_{1c} = 0$. Thus the solution to the consumer's problem can be applied to obtain the solution to the speculator's problem. We record the solution in the form of a corollary.

COROLLARY 1: *If U_s is strictly concave and twice differentiable, then the speculator's optimal final wealth W^*_{sT} is given by*

$$W^*_{sT} = I_s(x_s \cdot \eta_T), \tag{15}$$

provided that *W^*_{sT} is attainable, where I_s is the inverse function of U'_s. In addition, x_s is a positive constant chosen to satisfy the budget constraint $E_0[W^*_{sT} \cdot \eta_T] = W_s$.*

The producer's optimization problem may be thought of as one of choosing an optimal final wealth W_T to maximize his expected utility subject to

$$W_T \geq Q_T \cdot P_T - K \tag{16}$$

$$E_0(W_T \cdot \eta_T) = \overline{W}_f, \tag{17}$$

where $\overline{W}_f \equiv E_0[Q_T \cdot P_T \cdot \eta_T]$ is the producer's effective initial wealth. This consideration leads to the following lemma, whose proof is given in the Appendix.

LEMMA 2: *If U_f is strictly concave and twice differentiable, then the producer's optimal final wealth W^*_{fT} is given by*

$$W^*_{fT} = \max[Q_T \cdot P_T - K, I_f(x_f \cdot \eta_T)], \tag{18}$$

provided that W^*_{fT} is attainable, where I_f is the inverse function of U'_f. In addition, x_f is a positive constant chosen to satisfy the budget constraint $E_0[W^*_{fT} \cdot \eta_T] = \overline{W}_f$.

Let C^*_{ft}, C^*_{ct}, and C^*_{st} denote the optimal holdings of futures contracts of the producer, consumer, and speculator, respectively, at time t, when faced with a price process $\{P_t\}$. We define $\{P_t\}$ to be an equilibrium price process if (i) the goods market clears at time T, $C^*_{cT} = Q_T$, and (ii) the futures market clears at all times:

$$C^*_{ft} + C^*_{ct} + C^*_{st} = 0, \qquad \forall\, t \in [0,T). \tag{19}$$

Note that the market in the risk-free asset clears if the goods market and risky asset market clear.

Note that the optimal final wealth of a consumer can be written as $W^*_{cT} = C^*_{cT} \cdot P_T + B^*_{cT}$, where at equilibrium, $C^*_{cT} = Q_T$ and $B^*_{cT} = I_{2c}(x_c \cdot \eta_T)$ by equation (14); thus

$$Q_T \cdot P_T + I_{2c}(x_c \cdot \eta_T) = W_c + \int_0^T C^*_{ct}\, dP_t. \tag{20}$$

Similarly, we can write the optimal final wealth for the speculator and producer as follows:

$$I_s(x_s \cdot \eta_T) = W_s + \int_0^T C^*_{st}\, dP_t. \tag{21}$$

$$\max[Q_T \cdot P_T - K, I_f(x_f \cdot \eta_T)] = Q_T P_T + \int_0^T C^*_{ft}\, dP_t. \tag{22}$$

By summing equations (20), (21), and (22) and using equation (19), we obtain:

$$I_{2c}(x_c \cdot \eta_T) + I_s(x_s \cdot \eta_T) + \max[Q_T \cdot P_T - K, I_f(x_f \cdot \eta_T)] = W_c + W_s. \tag{23}$$

In addition, the goods market clearing and equation (13) imply

$$I_{1c}(x_c \cdot P_T \cdot \eta_T) = Q_T. \tag{24}$$

Equations (23) and (24) are the keys to determining the equilibrium price P_t. For given constants x_c, x_f, and x_s, equations (23) and (24) will determine P_T and η_T. On the other hand, given P_T and η_T, the three budget constraints for the consumer, producer, and speculator will determine x_c, x_f, and x_s:

$$E_0[Q_T \cdot P_T \cdot \eta_T + I_{2c}(x_c \cdot \eta_T) \cdot \eta_T] = W_c. \tag{25}$$

$$E_0[I_s(x_s \cdot \eta_T) \cdot \eta_T] = W_s. \tag{26}$$

$$E_0[\max(Q_T \cdot P_T - K, I_f(x_f \cdot \eta_T)) \cdot \eta_T] = E_0[Q_T \cdot P_T \cdot \eta_T]. \tag{27}$$

Therefore, equations (23) through (27) will jointly determine P_T, η_T, as well as the three constants x_c, x_f, and x_s. Having obtained P_T and η_T, we can then determine P_t. Note that by the definition of a pricing kernel, $P_t \cdot \eta_t$ is a martingale, which implies that $P_t \eta_t = E_t[P_T \cdot \eta_T]$. Using the martingale property of η_t, we have $\eta_t = E_t[\eta_T]$. Thus, $P_t \cdot E_t[\eta_T] = E_t[P_T \eta_T]$, which is tantamount to

$$P_t = \frac{E_t(P_T \cdot \eta_T)}{E_t(\eta_T)}. \tag{28}$$

The previous arguments can be summarized in the following theorem, whose proof is given in the Appendix.

THEOREM 1: *Let P_t be defined by equation (28). If P_T, η_T, x_c, x_f, and x_s satisfy equations (23) through (27), and if W^*_{cT}, W^*_{fT}, and W^*_{sT} as defined by equations (20), (22), and (21), respectively, are attainable, then P_t is an equilibrium price process. Moreover, P_t is a diffusion process with respect to the Brownian filtration. That is, there exist adapted processes μ_t and σ_t such that*

$$\frac{dP_t}{P_t} = \mu_t dt + \sigma_t dZ_t, \tag{29}$$

where μ_t and σ_t are dependent on the history $(Z_s)_{s=0}^t$.

III. Equilibrium with No Liquidity Constraints

In this section, we consider a benchmark case when $K = \infty$. In this case, we assume that the producer is able to obtain an infinite line of credit against his future revenue so that he can always meet margin requirements for his futures account. We first characterize properties of the equilibrium price and then turn to the analysis of optimal trading strategies.

Note that equilibrium condition (23) becomes

$$I_{2c}(x_c \cdot \eta_T) + I_s(x_s \cdot \eta_T) + I_f(x_f \cdot \eta_T) = W_c + W_s, \tag{30}$$

which implies that η_T is constant. Since $\eta_0 = E_0(\eta_T) = 1$, we infer $\eta_T = 1$. From equation (24) and $\eta_T = 1$, we have $P_T = x_c^{-1} \cdot U'_{1c}(Q_T)$. By equation (28) and $\eta_T = 1$, we obtain the equilibrium price in the benchmark case:

$$P_t^* = x_c^{-1} \cdot E_t[U'_{1c}(Q_T)], \tag{31}$$

which implies that futures price is a positive martingale. The constant x_c can be uniquely determined by equation (25), the consumer's budget constraint:

$$x_c = [1 + E_0(Q_T \cdot U'_{1c}(Q_T))]/W_c. \tag{32}$$

To obtain price volatility in this benchmark case, we take advantage of the following lemma, whose proof is given in the Appendix.

LEMMA 3: *Let F be a continuous and piecewise differentiable function. Then*

$$E_t[F(Q_T)] = E_0[F(Q_T)] - \sigma \int_0^t E_s[F'(Q_T) \cdot Q_T]\, dZ_s. \tag{33}$$

In particular, we have

$$F(Q_T) = E_0[F(Q_T)] - \sigma \int_0^T E_t[F'(Q_T) \cdot Q_T]\, dZ_t. \tag{34}$$

According to equation (33), we have:

$$E_t[U'_{1c}(Q_T)] = E_0[U'_{1c}(Q_T)] - \sigma \int_0^t E_s[Q_T \cdot U''_{1c}(Q_T)]\, dZ_s,$$

which, by virtue of equation (31), implies

$$dP_t^* = -x_c^{-1} \sigma \cdot E_t[Q_T U''_{1c}(Q_T)] \cdot dZ_t = x_c^{-1} \sigma \cdot E_t[R(Q_T) U'_{1c}(Q_T)]\, dZ_t. \tag{35}$$

Equation (35) shows that bad news about Q_T results in an increase in futures price.

If we write $dP_t^*/P_t^* = \sigma_t^* dZ_t$, then equations (35) and (31) together imply:

$$\sigma_t^* = \sigma \cdot \frac{E_t[R(Q_T) \cdot U'_{1c}(Q_T)]}{E_t[U'_{1c}(Q_T)]}, \tag{36}$$

which, by noting that

$$E_t[R(Q_T) \cdot U'_{1c}(Q_T)] = E_t[R(Q_T)] \cdot E_t[U'_{1c}(Q_T)] + \mathrm{cov}_t[R(Q_T), U'_{1c}(Q_T)],$$

yields

$$\sigma_t^* = \sigma \cdot E_t[R(Q_T)] + \sigma \cdot \frac{\mathrm{cov}_t[R(Q_T), U'_{1c}(Q_T)]}{E_t[U'_{1c}(Q_T)]}, \tag{37}$$

which implies that $\sigma_t^* > \sigma \cdot E_t[R(Q_T)]$ under the assumption that $R(Q_T)$ is a decreasing function of Q_T. Note that if $R(Q_T)$ were constant and equal to $\hat{R}$, then σ_t^* would be constant and equal to $\sigma\hat{R}$. Under the assumption that $R(Q_T)$ is high for low Q_T, equation (37) suggests that if $E_t(Q_T)$ is low, then σ_t^* is high. The following proposition, whose proof is given in the Appendix, shows that price volatility indeed goes up in reaction to bad news about the production level.[12]

PROPOSITION 1: *Under the assumption that $R(Q_T)$ is a decreasing function of Q_T, σ_t^* is an increasing function of Z_t. Or, equivalently, σ_t^* is a decreasing function of $E_t(Q_T)$.*

We now turn to the analysis of optimal trading strategies. Note that when $\eta_T = 1$, we have $W_{sT}^* = W_s$, and by equation (21) we conclude that $C_{st}^* = 0$ for all t. Intuitively, when P_t^* follows a martingale, there is no incentive for the speculator to participate in the market. Indeed, if the producer can fully participate in the futures market, we see no reason for the speculator to play any role in the process of risk allocation between the producer and the consumer.

[12] If we apply this paper's intuition to a model with both spot and futures markets, we would expect both the spot and futures prices to display similar positive correlations between prices and volatility. The exact spot price determination is very complex and idiosyncratic, and the spot price is not even unique. If we use the front-month futures price volatility as a proxy for the spot price volatility, then the empirical fact known among all agricultural option traders, that the persistent implied volatility skewness exists for both options on the front-month futures contracts and options on futures contracts based on a different harvest cycle, suggests that the positive correlation between price and volatility is shared by both spot and futures markets.

Note also that when $\eta_T = 1$, we have $W^*_{fT} = \overline{W}_f$. Thus, by equation (22) the only purpose of the producer's optimal trading strategy is to completely hedge away the risk in his future revenue. That is, C^*_{ft} is chosen to satisfy

$$\overline{W}_f = Q_T P_T + \int_0^T C^*_{ft}\, dP^*_t, \tag{38}$$

which shows that the producer's trading revenue is equal to $\overline{W}_f - Q_T P_T$. Since $P_T = x_c^{-1} \cdot U'_{1c}(Q_T)$, we may write

$$\int_0^T C^*_{ft}\, dP^*_t = \overline{W}_f - x_c^{-1} \cdot Q_T \cdot U'_{1c}(Q_T), \tag{39}$$

which implies that there does not exist a constant K such that $\int_0^T C^*_{ft}\, dP^*_t \geq -K$ almost surely. That is, the producer's trading losses can be unbounded.[13]

To extract the producer's optimal hedging strategy, we note that equation (34) gives

$$Q_T \cdot U'_{1c}(Q_T) = x_c \overline{W}_f - \sigma \cdot \int_0^T E_t[Q_T \cdot U'_{1c}(Q_T) + Q_T^2 \cdot U''_{1c}(Q_T)]\, dZ_t, \tag{40}$$

which, together with result (39), gives

$$\int_0^T C^*_{ft}\, dP^*_t = x_c^{-1} \sigma \cdot \int_0^T E_t[Q_T \cdot U'_{1c}(Q_T) + Q_T^2 \cdot U''_{1c}(Q_T)]\, dZ_t, \tag{41}$$

which implies $C^*_{ft} \cdot dP^*_t = x_c^{-1} \sigma \cdot E_t[Q_T \cdot U'_{1c}(Q_T) + Q_T^2 \cdot U''_{1c}(Q_T)] \cdot dZ_t$. By equation (35), we obtain

$$C^*_{ft} = -\frac{E_t[Q_T \cdot U'_{1c}(Q_T) + Q_T^2 \cdot U''_{1c}(Q_T)]}{E_t[Q_T \cdot U''_{1c}(Q_T)]}. \tag{42}$$

Under the assumption that $R(Q_T)$ is a decreasing function of Q_T, we can use equation (42) to show that $C^*_{ft} > -E_t(Q_T)$. That is, if a producer expects to harvest one bushel at time T, his optimal hedge in the futures market is less than one bushel. This result is consistent with past research such as Ho

[13] We want to show that for a necessary commodity there is no upper bound for $Q_T \cdot U'_{1c}(Q_T)$, and thus by equation (39) there does not exist a constant K such that $\int_0^T C^*_{ft} \cdot dP_t \geq -K$ almost surely. To see this, we note that there exists an ϵ such that $R(Q) > 1 + \epsilon$ for small Q (smaller than $\hat{C}$). We can then show that for small Q, $U'_{1c}(Q) > k \cdot Q^{-1-\epsilon}$ for a positive constant k, which implies that $Q \cdot U'_{1c}(Q) > k \cdot Q^{-\epsilon}$. The proof is complete by noting that $Q^{-\epsilon}$ is unbounded from above.

(1984). Because it is difficult to obtain an explicit expression for C^*_{ft} at this level of generality, it is helpful to compute C^*_{ft} in the hypothetical case when $R(Q_T)$ is constant and equal to $\hat{R}$. In this case, equation (42) becomes

$$C^*_{ft}(\hat{R}) = (\hat{R}^{-1} - 1)\cdot \exp[-\hat{R}\sigma^2(T-t)]\cdot E_t(Q_T), \tag{43}$$

which gives that $C^*_{ft}(\hat{R}) > -E_t(Q_T)$ for all levels of $\hat{R}$.

The hypothetical case of constant relative risk aversion can shed light on our case of interest. In states where Q_T is very high so that $R(Q_T)$ is lower than one, or, equivalently, when futures price and volatility are low, equation (43) implies that the producer can actually stay long in futures.[14] On the other hand, in states where Q_T is low and $R(Q_T)$ is higher than one, or, equivalently, when the futures price and volatility are high, the producer will short the commodity in the futures market.

The hypothetical case can also be used to argue that a higher σ implies a lower hedge ratio, where the hedge ratio is defined to be $[-C^*_{ft}/E_t(Q_T)]$. That is, if there is more uncertainty about the supply of the commodity, the producer tends to lower the hedge ratio. Note that as harvest time approaches (i.e., when $T - t$ decreases), the producer increases the hedge ratio (see also Anderson and Danthine (1983)).

We conclude this section with a note on the price elasticity of the consumer's demand for the commodity. Formally, price elasticity is defined to be $e(P) = -[dC(P)/dP]\cdot[P/C(P)]$, where $C(P)$ denotes the quantity demand as a function of price P. Let $T(P) \equiv C(P)\cdot P$ be the consumer's consumption expenditure, then $dT(P)/dP = C(P)\cdot[1 - e(P)]$, which implies that if $e(P) < 1$, the consumer spends more money on his consumption as the price goes up, and if $e(P) > 1$, the consumer spends less money on the consumption good as the price goes up.

Note that the consumer's consumption expenditure is equal to $T(P_T) = Q_T\cdot P_T = x_c^{-1}\cdot Q_T U'_{1c}(Q_T)$, where we note that Q_T is a decreasing function of P_T by the equation $P_T = x_c^{-1}\cdot U'_{1c}(Q_T)$. In states when Q_T is low so that $R(Q_T)$ is higher than one, we can show that $T(P_T)$ is an increasing function of P_T, or, equivalently, the consumer's elasticity of demand is less than one. That is, the commodity is a necessary commodity when its supply is tight. Similarly, in states when Q_T is high so that $R(Q_T)$ is less than one, the consumer's elasticity of demand is higher than one. That is, the commodity becomes a luxury commodity when its supply is abundant.

IV. Effect of Liquidity Constraints and Role of Speculation

In this section, we consider the case when $K < \infty$. In this case, the producer is not allowed to lose more than K in his futures account at any time. The main task in this section is to examine the equilibrium consequences of

[14] Our result that the producer may not want to short futures in the face of an abundant crop is similar to the result (in the theory of storage) that the producer's demand for storage goes up when he sees a big harvest ahead.

liquidity constraints and to understand the role of speculation. We find that with liquidity constraints, the futures price does not follow a martingale. In particular, we discover that the futures price exhibits mean reversion. This mean-reverting property of the futures price induces the speculator to buy at low prices and sell at high prices. However, as more speculative capital is at work, the futures price exhibits less mean-reversion as the market price of risk dwindles. In this sense, speculation improves upon market allocational inefficiency left by the producer's inability to bear unlimited trading losses.

We also find that speculation tends to skew price volatility in a certain way. Speculation causes volatility to rise in states when the futures price is high and the expected output is low, and it dampens volatility in states when the futures price is low and the expected output is high. That is, speculation makes price volatility an increasing function of the futures price.

Market Price of Risk

The pricing kernel, which represents the state price density, is crucial to understanding the market price of risk. Note that $d\eta_t = -\theta_t \eta_t \cdot dZ_t$ implies that θ_t, the market price of risk, is entirely determined by the properties of the pricing kernel. To derive η_T, we note the following equation in η_T obtained by eliminating P_T from equations (23) and (24):

$$I_{2c}(x_c \cdot \eta_T) + I_s(x_s \cdot \eta_T) + \max\left[\frac{Q_T U_{1c}'(Q_T)}{x_c \eta_T} - K, I_f(x_f \cdot \eta_T)\right] = W_c + W_s, \quad (44)$$

which shows that η_T is an increasing function of $Q_T \cdot U_{1c}'(Q_T)$. Moreover in states when the producer's total trading losses are less than K (i.e., when $I_f(x_f \cdot \eta_T) > [Q_T U_{1c}'(Q_T)] \cdot (x_c \eta_T)^{-1} - K$), equation (44) entails that η_T is constant.

Under the assumption that all utility functions of cash holdings are logarithmic, equation (44) and budget constraints imply

$$\eta_T = 1 + \max\left[\frac{x_f^{-1} - \overline{W}_f}{W_s + W_c}, \frac{-K - \overline{W}_f + (W_c - \overline{W}_f) Q_T U_{1c}'(Q_T)}{W_s + W_c + K}\right], \quad (45)$$

where the producer's effective wealth is given by

$$\overline{W}_f = \frac{E_0[Q_T U_{1c}'(Q_T)]}{1 + E_0[Q_T U_{1c}'(Q_T)]} \cdot W_c, \quad (46)$$

and x_f^{-1}, which is less than $\overline{W}_f$, may be thought of as the constant that makes $E_0(\eta_T - 1) = 0$ for the η_T given by equation (45).

Note that η_T is an increasing function of $[Q_T \cdot U'_{1c}(Q_T)]$. Thus, in order to understand the properties of η_T, we need to study the properties of $[Q_T \cdot U'_{1c}(Q_T)]$. The following lemma, whose proof is straightforward by direct differentiation, shows that the consumer's relative risk aversion determines the properties of $[Q_T \cdot U'_{1c}(Q_T)]$.

LEMMA 4: *The first-order derivative of $x \cdot U'_{1c}(x)$ is given by:*

$$\frac{d}{dx}[x \cdot U'_{1c}(x)] = c \cdot [1 - R(x)] \cdot \exp\left[-\int_{\epsilon}^{x} \frac{R(u)}{u}\, du\right], \tag{47}$$

where c and ϵ are positive constants.

The above lemma implies that when $Q_T < \hat{C}$ so that $R(Q_T) > 1$, $[Q_T \cdot U'_{1c}(Q_T)]$ is a decreasing function of Q_T. On the other hand, when $Q_T > \hat{C}$ so that $R(Q_T) < 1$, $[Q_T \cdot U'_{1c}(Q_T)]$ is an increasing function of Q_T. In other words, $[Q_T \cdot U'_{1c}(Q_T)]$ is a U-shaped function of Q_T.

The fact that $[Q_T \cdot U'_{1c}(Q_T)]$ is a U-shaped function of Q_T implies that η_T, which is given in equation (45), is made up of three components. In states when Q_T is very low, η_T is a decreasing function of Q_T; in states when Q_T is very high, η_T is an increasing function of Q_T; in the intermediate states, η_T is constant. Let us define a function $F(\cdot)$ by $\eta_T = F(Q_T)$. When $K = \infty$, we return to the benchmark case in which $\eta_T = 1$; when $K = 0$ so that the producer does not enter the futures market, the pricing kernel is a complete U-shaped function of Q_T; when $K \in (0, \infty)$, which is a case of incomplete hedging, we observe that η_T is a bowl-shaped function of Q_T.

Let us first consider the effect of incomplete hedging on the properties of the pricing kernel. Note that as K increases—that is, as the producer becomes more tolerant of his trading losses—equation (45) shows that the region of Q_T in which η_T is constant is widened. Also, as K increases, the slopes of other components decrease. In other words, the whole pricing kernel curve becomes flatter. To see this mathematically, we note that $(\eta_T - 1)$ is the maximum of a constant term and a nonconstant term. As K goes up, the nonconstant term goes down in value and becomes a flatter function of Q_T. To ensure that $E_0(\eta_T - 1) = 0$, the constant term has to rise. A higher constant term, together with a lower and flatter nonconstant term, implies that the region of Q_T in which η_T is constant has to be widened.

We now consider the effect of speculation on the properties of the pricing kernel. Note that as W_s increases(i.e., as more speculative capital is available), equation (45) shows that the region of Q_T in which η_T is constant remains the same. To see this, note that by using equation (45), $E_0(\eta_T - 1) = 0$ implies that

$$\left(\frac{x_f^{-1} - \overline{W}_f}{W_s + W_c}\right) \cdot (W_s + W_c + K) \equiv -L \tag{48}$$

is independent of W_s. Thus, equation (45) becomes

$$\eta_T = 1 + (W_s + W_c + K)^{-1} \cdot \max[-L, -K - \overline{W}_f + (W_c - \overline{W}_f) Q_T U'_{1c}(Q_T)], \quad (49)$$

which shows that the region in which η_T is flat; that is, the set

$$\{Q_T: -K - \overline{W}_f + (W_c - \overline{W}_f) Q_T U'_{1c}(Q_T) \leq -L\}, \quad (50)$$

is independent of W_s. Equation (49) also shows that the slopes of the other components of η_T decrease as W_s increases.

We are in a position to use the properties of the pricing kernel to infer the market price of risk. Note that in light of equation (33), $\eta_t = E_t[F(Q_T)]$ satisfies

$$\eta_t = 1 - \sigma \int_0^t E_s[F'(Q_T) \cdot Q_T] \, dZ_s, \quad (51)$$

which implies that $d\eta_t = -\sigma E_t[F'(Q_T) \cdot Q_T] \cdot dZ_t$. Since $d\eta_t = -\theta_t \eta_t \cdot dZ_t$, we obtain the following expression for the market price of risk:

$$\theta_t = \sigma \cdot \frac{E_t[F'(Q_T) \cdot Q_T]}{E_t[F(Q_T)]}. \quad (52)$$

Equation (52) shows that the slope of function F is important in determining the algebraic sign of θ_t. Roughly speaking, in states when $F'(Q_T)$ is expected to be positive, θ_t is positive; in states when $F'(Q_T)$ is expected to be negative, θ_t is negative.

Having just analyzed the effect of K and W_s on $F(Q_T)$, we are able to infer the effect of K and W_s on the market price of risk. In the presence of liquidity constraints, the futures price does not follow a martingale. In states when the supply of the commodity is expected to be very low or equivalently when P_t is very high, equation (52) implies that P_t is expected to fall; in states when the supply is expected to be very high or, equivalently, when P_t is very low, equation (52) implies that P_t is expected to rise. In the intermediate states, the futures price roughly follows a martingale. Therefore, liquidity constraints can cause the futures price to exhibit mean reversion.

However, as either K or W_s increases, the pricing kernel curve becomes flatter, and by equation (52), the mean-reverting property of P_t becomes less dramatic. In other words, $|\theta_t|$ decreases as K or W_s increases. In the extreme case when either $K = \infty$ or $W_s = \infty$, the mean-reverting property disappears.

Price Volatility

In this subsection, we analyze how price volatility is affected by the liquidity constraints and the presence of speculation. We find that speculation tends to make σ_t an increasing function of P_t. That is, in states when the expected output is low and futures price is high, speculation makes futures price more volatile. In states when the expected output is high and the futures price is low, speculation serves to dampen price volatility.

The following proposition, whose proof is given in the Appendix, relates σ_t to θ_t in a simple way. It allows us to use the properties of θ_t to analyze the properties of σ_t.

PROPOSITION 2: *Let σ_t^* denote the price volatility in the equilibrium without liquidity constraints. Then*

$$\sigma_t = \sigma_t^* + \theta_t. \tag{53}$$

In states when $E_t(Q_T)$ is very low so that P_t and σ_t^* are very high and θ_t is negative, equation (53) shows that in the presence of liquidity constraints, price volatility is lower than in the benchmark case. Similarly, in states when $E_t(Q_T)$ is very high so that P_t and σ_t^* are very low and θ_t is positive, equation (53) shows that in the presence of liquidity constraints, price volatility is higher than in the benchmark case. In the intermediate states, σ_t and σ_t^* are close.

As K or W_s increases, $|\theta_t|$ decreases and σ_t approaches σ_t^*. In other words, an increase in K or W_s tends to make price volatility an increasing function of futures price, and correspondingly, option-implied volatility tends to be an increasing function of option strike price (see Rubinstein (1994) for relating instantaneous volatility skewness to option-implied volatility skewness). Thus, the extent to which implied volatility is skewed is also an indirect measure of speculative activity in the futures market.

Optimal Trading Strategies

In this subsection, we discuss the nature of optimal trading strategies adopted by the speculator and the producer. The consumer's trading strategy, which is not explicitly discussed here, can be obtained by the market clearing condition. We find that the speculator buys at low prices and sells at high prices, and that the profitability of speculation decreases as either more speculative capital enters the market or the producer becomes more tolerant of trading losses. The producer's trading strategy is essentially to replicate the payoff of a put option on his future revenue.

Let us first examine the essence of speculation. By equation (15), the speculator's final wealth is a decreasing function of η_T, where η_T is a bowl-shaped function of Q_T. Thus, in states when Q_T is either very high or very low, the speculator loses money, and in intermediate states, speculation is profitable. Essentially, the act of speculating is equivalent to writing an insurance policy to guarantee that Q_T will be normal. The idea that speculators are providers of insurance goes back to Keynes (1930).

Under the assumption of a logarithmic utility function, equation (15) gives the speculator's optimal final wealth: $W_{sT}^* = (x_s \eta_T)^{-1}$, which implies that $W_{st}^* \cdot \eta_t = E_t(W_{sT}^* \cdot \eta_T) = x_s^{-1}$, where W_{st}^* denotes the optimal wealth at time t. Thus $x_s^{-1} = W_s$ and $W_{st}^* = W_s \cdot \eta_t^{-1}$. Ito's lemma entails that

$$dW_{st}^* = W_s \theta_t \eta_t^{-1} \cdot dZ_t + W_s \theta_t^2 \eta_t^{-1} \cdot dt, \tag{54}$$

where we have made use of $d\eta_t = -\theta_t \eta_t \cdot dZ_t$. Equation (54) can be rewritten as:

$$\frac{dW_{st}^*}{W_{st}^*} = \theta_t^2 \cdot dt + \theta_t \cdot dZ_t, \tag{55}$$

which shows that the expected rate of growth in the speculator's wealth is determined by $|\theta_t|$. As either W_s or K increases, $|\theta_t|$ decreases, which makes speculation less profitable.

By the definition of C_{st}^*, we have $dW_{st}^* = C_{st}^* dP_t = C_{st}^* P_t\ (\mu_t dt + \sigma_t dZ_t)$. According to equation (54), we obtain $C_{st}^* P_t \sigma_t = W_s \theta_t \eta_t^{-1}$, which gives $C_{st}^* = W_s \cdot (\theta_t / \sigma_t) \cdot (P_t \eta_t)^{-1}$. Note that equation (53) gives $\sigma_t = \sigma_t^* + \theta_t$. Note also that by (24), $P_t \eta_t = E_t[P_T \eta_T] = E_t[U_{1c}'(Q_T)]/x_c$, where x_c satisfies equation (25) and is given by $x_c = \{1 + E_0[Q_T U_{1c}'(Q_T)]\}/W_c$. Thus,

$$C_{st}^* = \frac{W_s}{W_c} \cdot \frac{\theta_t}{\sigma_t^* + \theta_t} \cdot \frac{1 + E_0[Q_T U_{1c}'(Q_T)]}{E_t[U_{1c}'(Q_T)]}. \tag{56}$$

Though there may be many factors that affect C_{st}^*, equation (56) shows that the algebraic sign of C_{st}^* is entirely determined by the algebraic sign of θ_t. Thus, in states when futures price is high but is expected to fall, the speculator stays short; in states when futures price is low but is expected to rise, the speculator stays long. In essence, the speculator is rewarded for taking advantage of mean reversion in the futures price. Note that as K increases—that is, as the producer becomes more tolerant of trading losses—$|\theta_t|$ decreases, which by equation (56) implies that the speculator takes a smaller position.

We now turn to the analysis of the producer's optimal trading strategy. Recall equation (22):

$$\max[Q_T \cdot P_T - K, I_f(x_f \cdot \eta_T)] = Q_T P_T + \int_0^T C_{ft}^* \, dP_t, \tag{57}$$

where the left-hand side is the producer's optimal final wealth. Note that in states when $I_f(x_f \cdot \eta_T) > Q_T \cdot P_T - K$, equation (23) implies that η_T is constant, and by equation (45), $\eta_T = 1 + (x_f^{-1} - \overline{W}_f)/(W_s + W_c)$. Thus, we can replace $I_f(x_f \cdot \eta_T)$ by a constant in the above equation:

$$\max[Q_T \cdot P_T - K, h] = Q_T P_T + \int_0^T C_{ft}^* \, dP_t, \tag{58}$$

where h is a positive constant defined by $h^{-1} = x_f + (1 - \overline{W}_f x_f)/(W_s + W_c)$.

Equation (58) can be rearranged as

$$\max[0,(h + K) - Q_T P_T] = K + \int_0^T C_{ft}^* \, dP_t, \tag{59}$$

which shows that the producer's trading strategy is to replicate the payoff of a put option on the revenue $Q_T P_T$ with strike equal to $(h + K)$. As a consequence, C_{ft}^* is equal to the 'delta' of the revenue put option. Equation (59) also shows that the cost of such a revenue put option is equal to K.

Equation (59) shows that in states when $Q_T P_T$ is expected to be very high so that the revenue put option is deep out-of-the-money, C_{ft}^* should be close to zero. In states when $Q_T P_T$ is expected to be very low so that the revenue put option is deep in-the-money, C_{ft}^* should be close to the 'delta' that would be used in order to completely hedge away $Q_T P_T$. In states when $Q_T P_T$ is expected to be close to $(h + K)$, the revenue put option is at-the-money and, as a result, C_{ft}^* should be between zero and the 'delta' that would be used in order to completely hedge away $Q_T P_T$.

By equation (23), we infer that $Q_T P_T$ can be seen as an increasing function of η_T. Recall that in Section A, η_T is shown to be high in extreme states when Q_T is either very high or very low. Thus, $Q_T P_T$ is high when Q_T is either very high or very low. Therefore, in states when the commodity supply is expected to be either very tight or abundant, C_{ft}^* should be close to zero. The producer's inaction in the market in these states helps us understand the existence of a commodity risk premium in such states.

Similarly, we can show that in the intermediate states when Q_T is expected to be moderate, $Q_T P_T$ is low enough so that the revenue put option is not deep out-of-the-money, and, as a result, C_{ft}^* is not close to zero, and it is between zero and the 'delta' that would be used in order to completely hedge away $Q_T P_T$. The producer's active participation in the market in these states serves to reduce the commodity risk premium in such states.

V. The Effect of Government Price Subsidy

In this section, we consider the equilibrium consequences of government price subsidy. A price subsidy is essentially a put option that pays to the producer an amount equal to $\max(0, \hat{P} - P_T)$ for each unit of commodity production, where $\hat{P}$ is the level of price support and is known at time zero. With price subsidies, the producer's revenue will be $Q_T \cdot \max(\hat{P}, P_T)$. In the United States, price support levels for agricultural products have been set by a formula based on the average prices of previous years.

Marcus and Modest (1986) look at the government price subsidy as a random number of put options on futures prices. In their paper, the U.S. agricultural price support system is evaluated without considering the equilibrium implications of the price support system itself. As we will see, the presence of price support can reduce price volatility, particularly in states when P_T is at or below $\hat{P}$. Thus, a partial equilibrium approach such as the one in Marcus and Modest (1986) could overestimate the true cost of agricultural support programs. See Crain and Lee (1996) for some empirical evidence that commodity futures volatility depends on government programs.

To analyze the equilibrium effect of a price subsidy, we have to know whether the producer can hedge in the futures market. In the extreme case when $K = 0$ so that the producer does not trade in the futures market, the government price subsidy has no effect at all on the equilibrium commodity price. If $K > 0$, then the government price subsidy can influence the commodity price because it affects the producer's trading decisions. Since $K = \infty$ corresponds to the case in which the producer has the most freedom to trade, we choose $K = \infty$ so that the producer can make the maximum use of the price subsidy, and in so doing, the equilibrium effect of price subsidy will be most transparent.

Note that equations (20) and (21) are still valid equations for the optimal final wealth of the consumer and the speculator, respectively. The optimal final wealth of the producer, originally given in equation (22), is now

$$I_f(x_f \cdot \eta_T) = Q_T \cdot \max(\hat{P}, P_T) + \int_0^T C_{ft}^* \, dP_t, \tag{60}$$

where C_{ft}^* denotes the producer's optimal futures holdings.

If we ignore how government subsidies are paid for,[15] then we can sum equations (20), (21), and (60) and use equation (19) to obtain:

$$I_{2c}(x_c \cdot \eta_T) + I_s(x_s \cdot \eta_T) + I_f(x_f \cdot \eta_T) = W_c + W_s + Q_T \cdot \max(\hat{P} - P_T, 0), \tag{61}$$

[15] In a more general model, one might assume that consumers end up paying taxes to the government, and the government uses taxes to pay for the subsidies. The resulting effect on equilibrium futures prices hinges on whether consumers use the futures markets to hedge potential tax liabilities. If the consumers choose to completely hedge away future tax liabilities in the futures markets, then the equilibrium will be the same as in the case of no government subsidy, *except that* the consumer's initial wealth is reduced by an amount equal to the present value of government subsidies. If the consumers decide not to hedge against tax liabilities, then the resulting equilibrium is the one studied in the paper. I am grateful to René Stulz (the editor) for bringing up this point.

which, after replacing P_T with $U'_{1c}(Q_T)/(x_c \cdot \eta_T)$ in light of equation (24), becomes

$$I_{2c}(x_c \eta_T) + I_s(x_s \eta_T) + I_f(x_f \eta_T) = W_c + W_s + Q_T \cdot \max\left[\hat{P} - \frac{U'_{1c}(Q_T)}{x_c \cdot \eta_T}, 0\right]. \quad (62)$$

The following lemma, whose proof is given in the Appendix, uses result (62) to represent η_T as a function of Q_T.

LEMMA 5: *The equilibrium pricing kernel η_T is a decreasing function of Q_T. In particular,*

$$\eta_T = \min\{k, h(Q_T)\}, \quad (63)$$

where $k \geq 1$ is a positive constant and $h(Q_T)$ is a positive and decreasing function of Q_T when $\eta_T < k$.

Equation (52), which states $\theta_t = \sigma \cdot E_t[F'(Q_T) \cdot Q_T]/E_t[F(Q_T)]$ where $\eta_T \equiv F(Q_T)$, can be applied here to derive properties of the market price of risk. According to equation (63), η_T is constant when Q_T is low enough to make $P_T > \hat{P}$, and in other states, η_T is a decreasing function of Q_T. Therefore, when $E_t(Q_T)$ is low or, equivalently, when the futures price is high, θ_t is slightly negative, and in states when $E_t(Q_T)$ is high or, equivalently, when the futures price is low, θ_t is negative.

To understand why the futures price is expected to fall when the expected output is high, we need to look at the producer's trading decisions. The producer's endowment is the sum of $Q_T P_T$ and the value of government price subsidy: $Q_T \max(0, \hat{P} - P_T)$. Compared to the previous benchmark case, the producer is now faced with the extra task of hedging an uncertain quantity Q_T of put options on price. When $E_t(Q_T)$ increases, we see dual effects: first, the number of possible put options increases, and, second, the delta of each put option becomes more negative. As a result, the producer is motivated to buy futures in order to reduce the total delta risk. Since the benchmark case gives zero risk premium, it is only reasonable to expect a negative risk premium in this case so that the futures market can clear—that is, speculators have to be induced to sell futures to the producers. We expect to see a more negative risk premium in states when the futures price is just at or below $\hat{P}$ because, by option theory, the delta of a put option is more sensitive in these states.

Equation (53), which states $\sigma_t = \sigma_t^* + \theta_t$, can be used to obtain price volatility in the presence of a government price subsidy. Since θ_t is slightly negative when $E_t(Q_T)$ is low, and more negative when $E_t(Q_T)$ is high, we conclude that σ_t is slightly lower than σ_t^* when $E_t(Q_T)$ is low, and σ_t is lower than σ_t^* when $E_t(Q_T)$ is high. In any circumstance, a price subsidy serves to reduce price volatility. The fact that, compared to the benchmark case, the producer is buying futures when the futures price gets lower[16] helps us understand the result that a price subsidy reduces volatility, especially in states when futures prices are low.

VI. Summary and Empirical Discussion

In this paper, we develop a simple framework to analyze the equilibrium consequences of liquidity constraints, speculation, and government price subsidy. When producers cannot fully participate in the futures market due to their inability to bear unlimited trading losses, the futures price tends to exhibit mean reversion, especially when the commodity output is either very high or very low. The mean-reverting property of the futures price induces speculators to buy at low prices and sell at high prices. Speculation tends to make volatility an increasing function of the price level.

The effect of government price subsidies on commodity prices depends on the extent to which producers can participate in futures trading. If producers are not faced with any trading loss constraints, then a price subsidy causes the expected return on the futures price to be negative and reduces price volatility, particularly in states when the futures price is just at or below the governmental price support level.

We conclude this paper by presenting some stylized facts, with the understanding that a more detailed test of the implications of this paper is needed. We compute daily returns based on the daily settlement price of nearby active soybean futures from the beginning of 1985 to the end of 1995. Having computed returns, we then adjust the futures price level by the GDP deflator to obtain the price of soybeans relative to the price of the basket of all other goods. To be more specific, let P_t denote the original price and P_t^a denote the adjusted price, then the price level adjustment is such that $P_t^a = P_t$ for January 2, 1987, and $P_t^a > P_t$ before January 2, 1987, and $P_t^a < P_t$ after January 2, 1987. Because the period under consideration is a period with very low GDP deflators, the price level adjustment has insignificant effect on our results.

We obtain a set of numbers denoted by $\{(P_t^a, R_{t,t+1})\}$, where $R_{t,t+1}$ is the daily futures return from t to $(t+1)$. We then sort the dataset by P_t^a in ascending order, and divide the whole sample into four subsamples with equal numbers of observations. For each subsample, we compute annualized mean and annualized standard deviation. Here are the results: the first subsample, which has the lowest P_t^a, has a mean of 12.51 percent and a standard deviation of 14.75 percent; the second subsample has a mean of 4.13 percent and standard deviation of 16.76 percent; the third subsample

[16] Or if we use the language in the theory of storage, we can say that the producer is motivated to store more commodities when the futures price gets lower, which gives rise to lower volatilities in these states.

has a mean of − 4.07 percent and standard deviation of 17.97 percent; and the fourth subsample, which has the highest P_t^a, has a mean of − 17.50 percent and standard deviation of 23.99 percent.

Based on the subsample mean results, we observe that the futures returns are positive when the futures prices are low, and negative when the futures prices are high. This observation is consistent with equilibrium predictions on the effect of liquidity constraints. The subsample standard deviation results suggest that futures price volatility is an increasing function of the price level, which justifies our assumption that the consumer's utility function exhibits decreasing relative risk aversion. Indeed, the dramatic pickup in volatility in the fourth subsample suggests that the relative risk aversion can be very high if consumption level is low, pointing to the conclusion that the consumer is willing to pay a lot to maintain an adequate level of consumption of indispensable commodities.

Appendix

Proof of Lemma 1: For any pair (C_T, B_T) that satisfies equation (12), we need to show

$$E_0[U_{1c}(C_T) + U_{2c}(B_T)] \leq E_0[U_{1c}(C_{cT}^*) + U_{2c}(B_{cT}^*)]. \tag{A1}$$

We use the standard support argument for concave functions. That is, for concave $U(\cdot)$, we have that for any a and b, $U(a) - U(b) \leq U'(b)\cdot(a - b)$. Observe that

$$\begin{aligned} &E_0[U_{1c}(C_T) + U_{2c}(B_T)] - E_0[U_{1c}(C_{cT}^*) + U_{2c}(B_{cT}^*)] \\ &\quad \leq E_0[U_{1c}'(C_{cT}^*)\cdot(C_T - C_{cT}^*)] + E_0[U_{2c}'(B_{cT}^*)\cdot(B_T - B_{cT}^*)] \qquad \text{(A2)} \\ &\quad = E_0[x_c\cdot P_T\cdot \eta_T\cdot(C_T - C_{cT}^*)] + E_0[x_c\cdot \eta_T\cdot(B_T - B_{cT}^*)] = 0. \end{aligned}$$

Q.E.D.

Proof of Lemma 2: For any terminal wealth W_T that satisfies $W_T \geq Q_T\cdot P_T - K$ and $E_0(W_T\cdot\eta_T) = \overline{W}_f$, we need to show $E_0[U_f(W_T)] \leq E_0[U_f(W_{fT}^*)]$, where W_{fT}^* is defined in equation (18).

We use the standard support argument for concave functions. That is, for concave $U(\cdot)$, we have that for any a and b, $U(a) - U(b) \leq U'(b)\cdot(a - b)$. Observe that

$$E_0[U_f(W_T) - U_f(W_{fT}^*)] \leq E_0[U_f'(W_{fT}^*)\cdot(W_T - W_{fT}^*)]. \tag{A3}$$

In states when $W_{fT}^* = I_f(x_f\cdot\eta_T)$, we have

$$U_f'(W_{fT}^*)\cdot(W_T - W_{fT}^*) = x_f\,\eta_T\cdot(W_T - W_{fT}^*). \tag{A4}$$

In states when $W_{fT}^{*} = Q_T P_T - K$, we have $W_{fT}^{*} \geq I_f(x_f \cdot \eta_T)$ and thus $U'_f(W_{fT}^{*}) \leq x_f \eta_T$, and $W_T \geq W_{fT}^{*}$, which together imply

$$U_f'(W_{fT}^{*}) \cdot (W_T - W_{fT}^{*}) \leq x_f \eta_T \cdot (W_T - W_{fT}^{*}). \tag{A5}$$

Therefore, we have

$$E_0[U_f(W_T) - U_f(W_{fT}^{*})] \leq E_0[x_f \eta_T \cdot (W_T - W_{fT}^{*})] = 0. \tag{A6}$$

Q.E.D.

Proof of Theorem 1: First let us show that P_t as defined by equation (28) is a diffusion process. According to the corollary in Protter (1990, p. 157), there exists a nonanticipating θ_t such that $\eta_t \equiv E_t(\eta_T)$ can be written as an exponential:

$$\eta_t = \exp\left(-\frac{1}{2}\int_0^t \theta_s^2 \, ds - \int_0^t \theta_s \, dZ_s\right), \tag{A7}$$

which implies that $d\eta_t/\eta_t = -\theta_t \cdot dZ_t$. Similarly, there exists a nonanticipating a_t such that

$$\frac{dE_t(P_T \cdot \eta_T)}{E_t(P_T \cdot \eta_T)} = a_t dZ_t. \tag{A8}$$

Applying Ito's lemma to $P_t = E_t(P_T \cdot \eta_T)/\eta_t$, we obtain

$$\frac{dP_t}{P_t} = (a_t + \theta_t) \cdot (\theta_t dt + dZ_t), \tag{A9}$$

which shows that P_t is a diffusion process.

To show that P_t is an equilibrium price process, we need to show that when all market participants are faced with P_t, the futures market clears at all times. Define $W_{ct} = E_t(W_{cT}^{*} \cdot \eta_T)/\eta_t$, $W_{st} = E_t(W_{sT}^{*} \cdot \eta_T)/\eta_t$, and $W_{ft} = E_t[(W_{fT}^{*} - Q_T P_T) \cdot \eta_T]/\eta_t$. Note that due to budget constraints $W_{c0} = W_c$, $W_{s0} = W_s$, and $W_{f0} = 0$. By the same argument used to derive $dP_t/P_t = (a_t + \theta_t) \cdot (\theta_t dt + dZ_t)$, we can show that there exist nonanticipating i_t, j_t, and k_t such that $dW_{ct}/W_{ct} = (i_t + \theta_t)\cdot(\theta_t dt + dZ_t)$, $dW_{st}/W_{st} = (j_t + \theta_t)\cdot(\theta_t dt + dZ_t)$, and $dW_{ft}/W_{ft} = (k_t + \theta_t)\cdot(\theta_t dt + dZ_t)$. As a result, we may write $dW_{ct} = C_{ct}dP_t$, $dW_{st} = C_{st}dP_t$, and $dW_{ft} = C_{ft}dP_t$. Or, equivalently,

$$W_{cT}^* = W_c + \int_0^T C_{ct}\,dP_t, \tag{A10}$$

$$W_{sT}^* = W_s + \int_0^T C_{st}\,dP_t, \tag{A11}$$

$$W_{fT}^* = Q_T P_T + \int_0^T C_{ft}\,dP_t. \tag{A12}$$

It remains to show that $C_{ct} + C_{st} + C_{ft} = 0$. Note that equation (23) can be rewritten as $W_{cT}^* + W_{sT}^* + (W_{fT}^* - Q_T P_T) = W_c + W_s$. Thus, $W_{ct} + W_{st} + W_{ft} = W_c + W_s$, which, through differentiation, implies that $C_{ct} + C_{st} + C_{ft} = 0$. Q.E.D.

Proof of Lemma 3: Define a function f by $f(t,Z_t) \equiv E_t[F(Q_T)]$. By definition, $f(t, Z_t)$ is a martingale. Ito's lemma gives:

$$f(t,Z_t) = f(0,Z_0) + \int_0^t \frac{\partial f}{\partial Z}\,dZ_s. \tag{A13}$$

Let us use $q(x)$ to denote the density function of $(Z_T - Z_t)$. Because $(Z_T - Z_t)$ and Z_t are stochastically independent, $q(x)$ does not depend on the value of Z_t. By noting that $Q_T = \exp[-\frac{1}{2}\sigma^2 T - \sigma(Z_T - Z_t) - \sigma Z_t]$, we can write:

$$f(t,Z_t) = \int_{-\infty}^{\infty} F\left[\exp\left(-\left(\frac{1}{2}\right)\sigma^2 T - \sigma Z_t - \sigma x\right)\right]\cdot q(x)\,dx, \tag{A14}$$

which, under the assumption that $F(\cdot)$ is continuous and piecewise differentiable, gives rise to:

$$\frac{\partial f}{\partial Z} = \int_{-\infty}^{\infty} F\left[\exp\left(-\left(\frac{1}{2}\right)\sigma^2 T - \sigma Z_t - \sigma x\right)\right]$$

$$\cdot(-\sigma)\cdot\exp\left(-\left(\frac{1}{2}\right)\sigma^2 T - \sigma Z_t - \sigma x\right)\cdot q(x)\,dx, \tag{A15}$$

which can be rewritten as

$$\frac{\partial f}{\partial Z} = -\sigma \cdot E_t[F'(Q_T)\cdot Q_T]. \tag{A16}$$

Therefore, $f(t,Z_t) = f(0,Z_0) - \sigma\int_0^t E_s[F'(Q_T)\cdot Q_T]\,dZ_s$, which is equation (33). Equation (34) follows by setting $t = T$ in equation (33). Q.E.D.

Proof of Proposition 1: Let us define $f(t,Z_t) = E_t[R(Q_T)U'_{1c}(Q_T)]$ and $g(t,Z_t) = E_t[U'_{1c}(Q_T)]$, then by equation (36), $\sigma_t^* = \sigma\cdot f(t,Z_t)\,/\,g(t,Z_t)$, which, by direct differentiation, implies that $\partial\sigma_t^*/\partial Z_t > 0$ provided that

$$\frac{\partial f(t,Z_t)}{\partial Z}\cdot g(t,Z_t) - f(t,Z_t)\cdot\frac{\partial g(t,Z_t)}{\partial Z} > 0. \tag{A17}$$

By an argument similar to the one used to establish Lemma 3, we obtain:

$$\begin{aligned}\frac{\partial f(t,Z_t)}{\partial Z} &= -\sigma\cdot E_t[R'(Q_T)U'_{1c}(Q_T)Q_T + R(Q_T)U''_{1c}(Q_T)Q_T] \\ &= -\sigma\cdot E_t[R'(Q_T)U'_{1c}(Q_T)Q_T - R^2(Q_T)\cdot U'_{1c}(Q_T)],\end{aligned} \tag{A18}$$

$$\frac{\partial g(t,Z_t)}{\partial Z} = -\sigma\cdot E_t[U''_{1c}(Q_T)Q_T] = \sigma\cdot E_t[R(Q_T)U'_{1c}(Q_T)]. \tag{A19}$$

Therefore, in order to establish $\partial\sigma_t^*/\partial Z_t > 0$, we need to show

$$\begin{aligned}0 < &-\sigma\cdot E_t[R'(Q_T)U'_{1c}(Q_T)Q_T - R^2(Q_T)U'_{1c}(Q_T)]\cdot E_t[U'_{1c}(Q_T)] \\ &- E_t[R(Q_T)U'_{1c}(Q_T)]\cdot\sigma\cdot E_t[R(Q_T)U'_{1c}(Q_T)].\end{aligned} \tag{A20}$$

Since $R'(Q_T) < 0$, we need only show that

$$0 \le E_t[R^2(Q_T)U'_{1c}(Q_T)]\cdot E_t[U'_{1c}(Q_T)] - \{E_t[R(Q_T)U'_{1c}(Q_T)]\}^2, \tag{A21}$$

which follows from the Cauchy–Schwarz inequality in the following way:

$$\begin{aligned}&\{E_t[R(Q_T)U'_{1c}(Q_T)]\}^2 \\ &\quad = \{E_t[R(Q_T)\sqrt{U'_{1c}(Q_T)}\cdot\sqrt{U'_{1c}(Q_T)}]\}^2 \\ &\quad \le E_t[R(Q_T)\sqrt{U'_{1c}(Q_T)}]^2\cdot E_t[\sqrt{U'_{1c}(Q_T)}]^2 \\ &\quad = E_t[R^2(Q_T)U'_{1c}(Q_T)]\cdot E_t[U'_{1c}(Q_T)].\end{aligned} \tag{A22}$$

Q.E.D.

Proof of Proposition 2: Note that $dP_t/P_t = \mu_t \cdot dt + \sigma_t \cdot dZ_t$ implies

$$P_T = P_0 \cdot \exp\left[\int_0^T \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt + \int_0^T \sigma_t \, dZ_t\right], \tag{A23}$$

which, together with relationship (10), suggests

$$P_T \cdot \eta_T = P_0 \cdot \exp\left[\int_0^T \left(\mu_t - \frac{1}{2}\sigma_t^2 - \frac{1}{2}\theta_t^2\right) dt + \int_0^T (\sigma_t - \theta_t)\, dZ_t\right], \tag{A24}$$

which, by noting $\mu_t = \sigma_t \cdot \theta_t$, can be restated as:

$$P_T \cdot \eta_T = P_0 \cdot \exp\left[-\frac{1}{2}\int_0^T (\sigma_t - \theta_t)^2 dt + \int_0^T (\sigma_t - \theta_t)\, dZ_t\right]. \tag{A25}$$

On the other hand, equation (24) entails that $P_T \cdot \eta_T = U'_{1c}(Q_T)/x_c$. Recall that in the benchmark case, P_t^* follows a martingale and by (31) we know that P_t^* is proportional to

$$E_t[P_T \cdot \eta_T] = P_0 \cdot \exp\left[-\frac{1}{2}\int_0^t (\sigma_s - \theta_s)^2 ds + \int_0^t (\sigma_s - \theta_s)\, dZ_s\right], \tag{A26}$$

which implies that $\sigma_t^* = \sigma_t - \theta_t$. Q.E.D.

Proof of Lemma 5: Note that equation (62) implies that when $P_T > \hat{P}$, η_T is a constant, and when $P_T < \hat{P}$, η_T is a function of Q_T. Under the assumption that utility functions of cash holdings are logarithmic, we can write

$$\eta_T = \min\left\{\frac{x_c^{-1} + x_s^{-1} + x_f^{-1}}{W_c + W_s}, \frac{x_c^{-1} + x_s^{-1} + x_f^{-1} + x_c^{-1} Q_T U'_{1c}(Q_T)}{W_c + W_s + \hat{P}Q_T}\right\}, \tag{A27}$$

where η_T is not constant only if $P_T < \hat{P}$, that is, only if $\hat{P} > x_c^{-1}\, U'_{1c}(Q_T)$. Let k be the constant such that $\eta_T = k$ when $P_T > \hat{P}$. Then we can write $\eta_T = \min\{k, h(Q_T)\}$, where

$$h(q) = \frac{k + b \cdot x_c^{-1} q \cdot U'_{1c}(q)}{1 + b\hat{P}q}, \tag{A28}$$

where $b \equiv (W_c + W_s)^{-1}$ is a positive constant. The fact that $k \geq 1$ follows from $E_0(\eta_T) = 1$ and $\eta_T \leq k$. To see that $h(q)$ is a decreasing function of q when $\hat{P} > x_c^{-1} U'_{1c}(q)$, we note that by direct differentiation, $h'(q) \leq 0$ if and only if

$$bx_c^{-1}[U'_{1c}(q) + q \cdot U''_{1c}(q)](1 + b\,\hat{P}q) \leq [k + b \cdot x_c^{-1} q \cdot U'_{1c}(q)] \cdot b\hat{P}, \qquad \text{(A30)}$$

which holds if $x_c^{-1} U'_{1c}(q) \leq \hat{P}$. Q.E.D.

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