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Imperfect Competition among Informed Traders

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ABSTRACT

We analyze competition among informed traders in the continuous-time Kyle (1985) model, as Foster and Viswanathan (1996) do in discrete time. We explicitly describe the unique linear equilibrium when signals are imperfectly correlated and confirm the conjecture of Holden and Subrahmanyam (1992) that there is no linear equilibrium when signals are perfectly correlated. One result is that at some date, and at all dates thereafter, the market would have been more informationally efficient had there been a monopolist informed trader instead of competing traders. The relatively large amount of private information remaining near the end of trading causes the market to approach complete illiquidity.

How will an investor with information superior to the market trade? One intuition is that he will trade very aggressively, limited only by risk aversion and margin requirements. Another is that he will moderate his trades so as not to tip his hand too early. The Kyle (1985) model of informed trading supports the latter intuition. The informed trader in Kyle's model, when trading in continuous time, does so in such a way that his information is revealed at a constant rate. The first intuition is supported by Holden and Subrahmanyam (1992). The difference between the two papers is that Holden and Subrahmanyam assume at least two traders have the same information. Each trader tries to beat the others to the market, with the result that their information is revealed almost immediately.

The purpose of this paper is to study the nature of competition between informed traders, and the rate at which information is revealed to the market, when traders may have diverse signals and can trade continuously. The assumption of continuous trading simplifies the solution of the model. We build upon the work of Foster and Viswanathan (1996) and Cao (1995), who study this model in discrete time.

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There is a unique linear equilibrium when traders have diverse signals, as the convergence analyses of Foster and Viswanathan and Cao suggest.[1] We obtain a closed-form solution for the equilibrium (if one accepts the inverse incomplete gamma function as a closed form). In the case of exactly two informed traders, the equilibrium is found in terms of elementary functions. The formula for the equilibrium shows how the nature of competition changes over time and how it depends on the number of informed traders and the correlation of their signals.

We show there is no linear equilibrium when traders' signals are perfectly correlated. This confirms a conjecture of Holden and Subrahmanyam. The aggressive trading in the many-period version of the model leads to a degeneracy in continuous time. In general, competition is more intense in the early stages of the model, and information is revealed more quickly, when the degree of correlation between signals is higher.

A stark contrast with the Holden–Subrahmanyam result is provided by the following. Suppose two traders have uncorrelated signals. Then each will trade less intensely than would a single trader with the same aggregate information. Furthermore, aggregate trading is less intense and the information is revealed to the market less quickly when there are two traders than when there is only one. Thus, whether competition leads to more intense trading and quicker revelation of information depends very much on how the information is distributed among the competing agents.

A somewhat surprising result is that, beyond some date, the market would have learned more from a monopolist informed trader than from competing traders, regardless of the correlation of the competitors' signals (provided the correlation is not perfect). Thus, no matter how closely we approximate the Holden–Subrahmanyam model by assuming signals are nearly perfectly correlated, the Holden–Subrahmanyam result that competition leads to immediate revelation is quite far from valid, in the sense that by some date competitors will not have revealed as much of the aggregate information as a monopolist would have. Admittedly, if the correlation is very high, then this date will be very close to the end of trading, and the market's learning will be very nearly complete, so the difference in informational efficiency will be small. However, we might have conjectured that information revelation by a single trader would never surpass the revelation that occurs when competing traders have nearly identical information.

The relatively large amount of private information remaining near the end of trading leads to an extreme adverse selection problem. This causes the market to become nearly completely illiquid. In contrast, there is generally a period of rising depth near the beginning of trading. These patterns in market depth reflect the changing nature of competition over time, as the

[1]This paper confirms and generalizes the formula Cao (1995) obtained for the continuous-time equilibrium. Cao derived the equilibrium by various means, including a convergence analysis.

correlation of agents' private information conditional on the market's information changes over time. This is a major point of Foster and Viswanathan (1996). We are able to demonstrate these patterns analytically, confirming the numerical calculations of Foster and Viswanathan.

The patterns in market depth are very different from what prevails when there is only one informed trader. Kyle (1985, p. 1329) explains why depth cannot increase deterministically when there is only one informed trader: the trader could acquire a large position when depth is lower and then liquidate it at more favorable average prices when depth is higher, leading to unbounded profits. This reasoning does not apply when there are multiple informed traders. The strategy of running up the price and then liquidating would be confounded by other traders selling at the inflated prices. More generally, there is a temporary component to the price effects of trades, and this has a significant influence on the optimal trading of each agent. This is in contrast to the model with a single informed trader, in which price effects of informed trades are permanent.

Related work includes Attiyeh (1994), Holden and Subrahmanyam (1994), Back and Pedersen (1998), and Baruch (1997). Back and Pedersen study the single-agent continuous-time Kyle model when the volatility of liquidity trading is time varying and the informed agent receives a continuous flow of information. They show that Kyle's conclusion of constant depth is robust to these generalizations.[2] Attiyeh assumes there is an exogenous probability of the information being announced early, in the single-agent continuous-time Kyle model. This creates an incentive to trade quickly, which is countered by the depth of the market rising over time. Holden and Subrahmanyam assume the single informed trader is risk averse. Risk aversion also creates an incentive to trade early, because the risk is that noise traders will eliminate profitable trading opportunities by moving the price in the direction of the asset value. This is also countered by the depth of the market rising over time. Baruch presents the continuous-time solution when the informed trader is risk averse.

The plan of the paper is as follows. The model is described in Section I. Section II describes how market makers and traders update their estimates of the asset value over time and gives some necessary conditions for equilibrium. Section III discusses the optimization problems of the informed traders and gives conditions on the dynamics of market depth that are necessary for equilibrium. Section IV presents the main results of the paper: the description of the unique linear equilibrium when traders have diverse information and the result that there is no linear equilibrium when traders have identical information. Section V describes the effects of the number of traders and the correlation of their signals on the intensity of trading, the rate

[2] More precisely, they show that the reciprocal of depth is in general a martingale and is constant when the asset value is normally distributed, as Kyle assumes. This is the same result obtained by Back (1992).

of information transmission, the depth of the market, and the expected profits of informed traders. Section VI concludes the paper. All proofs are in the Appendix.

I. The Model

We assume $N \geq 1$ risk-neutral informed agents continuously trade a single risky asset over the time period $[0,1)$. An announcement is made at time 1 that reveals the liquidation value of the asset. At each time t prior to time 1, the asset price $P(t)$ is set through competition by risk-neutral market makers. Market makers observe the sum of the orders of the informed traders and liquidity traders. The cumulative order process of the liquidity traders is assumed to be a Wiener process Z.[3] The risk-free rate is taken to be zero.

Each agent i receives a mean-zero signal $\tilde{s}^i$ at time 0. We assume the signals and the liquidation value of the asset have a nondegenerate joint normal distribution that is symmetric in the signals.[4] Let $\tilde{v}$ denote the expectation of the liquidation value conditional on the combined information of the informed traders. By normality, $\tilde{v}$ is an affine function of the $\tilde{s}^i$. By rescaling the $\tilde{s}^i$ if necessary, we can assume without loss of generality that

$$\tilde{v} = \bar{v} + \sum_{i=1}^{N} \tilde{s}^i, \tag{1}$$

for a constant $\bar{v}$. This is a normalization adopted by Foster and Viswanathan (1996, Sec. VI). For simplicity, we assume $\bar{v} = 0$.

Let

$$\phi = \frac{\operatorname{var}(\tilde{v})}{\operatorname{var}(N\tilde{s}^i)}. \tag{2}$$

This is a measure of the quality of each trader's information. Specifically, ϕ is the "R-squared" in the linear regression of $\tilde{v}$ on $\tilde{s}^i$; that is, it is the percentage of the variance in $\tilde{v}$ that is explained by the trader's information. If

[3] This implies that the variance of liquidity trades per unit of time is one. We could easily make this an arbitrary constant σ_Z. However, this parameter will affect market depth, the profits of informed traders, and so on, exactly the same as it does in the single-informed-trader model studied by Kyle (1985). Because we have nothing new to say about the parameter, we have set it equal to one.

[4] Symmetry means that the joint distribution of the asset value and the signals $\tilde{s}_1,\ldots,\tilde{s}_N$ is invariant to a permutation of the indices $1,\ldots,N$.

$\phi = 1$, then either $N = 1$ or the $\tilde{s}^i$ are perfectly correlated. In either case, each informed trader has perfect information about $\tilde{v}$. Letting ρ denote the correlation coefficient of $\tilde{s}^i$ with $\tilde{s}^j$ for $i \neq j$, one can compute ϕ for $N > 1$ as

$$\phi = \frac{1}{N} + \frac{N-1}{N}\rho. \tag{3}$$

We investigate linear equilibria, meaning that there are functions α, β, and λ such that the rate of trade of each agent i at each time t is

$$\alpha(t)P(t) + \beta(t)\tilde{s}^i \tag{4}$$

and the price changes according to

$$dP(t) = \lambda(t)\left\{dZ(t) + \sum_{i=1}^{N}[\alpha(t)P(t) + \beta(t)\tilde{s}^i]dt\right\}. \tag{5}$$

It might seem more natural to consider a trading strategy that is linear in a trader's estimate of the asset value rather than in his initial signal. However, this would make it difficult to compute the estimates of the asset value, because each trader's estimate would depend on other agents' trades, which depend on their estimates of the asset value, which depend on other agents' trades, and so forth. This is what is called the "forecasting the forecasts of others" problem. This problem is avoided by specifying strategies that depend on initial signals rather than on value estimates. This key idea is due to Foster and Viswanathan (1996). We will show that, in equilibrium, trading strategies can also be written as functions of value estimates rather than as functions of signals. In fact, the equilibrium rate of trade is proportional to the difference between the value estimate and the asset price (see Lemma 6 in Section III), just as it is in the single-agent model of Kyle (1985).

In general, we define a trading strategy for agent i to be a function $\theta : \Re \times [0,1) \times \mathcal{C} \to \Re$, where $\mathcal{C}$ denotes the set of continuous functions $x : [0,1) \to \Re$ such that $\lim_{t\to 1} x(t)$ exists and is finite. We interpret $\theta(s,t,x)$ as the rate of purchase of the risky asset at time t when $\tilde{s}^i = s$ and the price path is x.[5] Heuristically, $\theta(\tilde{s}^i, t, P)\,dt$ is the market order submitted by trader i at time t. We do not allow the market order to include a stochastic component (of the form $\sigma\,dB$ for some Brownian motion B). In equilibrium, it would be suboptimal to include such a component if it were uncorrelated with the infor-

[5] Note that we do not assume the price is continuous at $t = 1$, though we will prove this is true in equilibrium. We do assume the price path has a finite limit as $t \to 1$, because we interpret the limit as the price just before the announcement. If the limit did not exist, the price change at the announcement date, and hence the capital gain or loss, would not be well defined.

mation arriving via others' orders. It may be optimal to correlate with others' orders, but such correlation is not feasible in practice with market orders, so we prohibit it.[6]

We require the rate of trade $\theta(s,t,x)$ to depend on x only through the history $\{x(u)|u \le t\}$. The linear strategy (equation (4)) is written as

$$\theta^i(s,t,x) = \alpha(t)x(t) + \beta(t)s. \tag{6}$$

It has the special property that the rate of trade depends only on the contemporaneous price and not on the previous history. We do not impose this as a constraint but rather deduce that it is optimal behavior. We also require trading strategies to be such that the stochastic differential equation

$$dP(t) = \lambda(t)\left\{dZ(t) + \sum_{i=1}^{N} \theta^i(\tilde{s}^i,t,P)\,dt\right\}, \qquad P(0) = 0, \tag{7}$$

has a unique solution P. Furthermore, we require the solution P to have a finite second moment and to have paths belonging to $\mathcal{C}$. This is a restriction on the strategy sets of the traders: given that agents $i \neq j$ follow linear strategies of the form in equation (6), we require agent j to follow a strategy such that equation (7) has a solution with the desired properties.[7] The constraints on trading strategies are specified more precisely in Section III.

An equilibrium is defined by two conditions: $P(t)$ equals the conditional expectation of $\tilde{v}$, given the information generated by the aggregate order process up to time t, and each strategy θ^i is optimal, given the other strategies θ^j and given λ. These conditions are stated more precisely in Section IV.

A remark on the notation in the text: we use uppercase Roman letters to denote stochastic processes (with the exception of N, which is an integer) and lowercase Roman letters with tildes to denote random variables. We follow the usual convention of not writing the state of the world as an argument of a stochastic process or a random variable. All Greek letters, both upper- and lowercase, denote constants or functions of time, with the exception of θ, which is a more complicated type of function.

[6] Sandy Grossman has pointed out to us that it should be feasible to create correlation via limit orders. This is an interesting subject for future research.

[7] See, for example, Protter (1990, Sec. V.3) for conditions that guarantee the existence of unique solutions to stochastic differential equations. Our approach has the disadvantage of linking the feasible set for each trader to the strategies assumed to be chosen by the other traders and the market makers. In this respect, we are modeling a generalized game rather than a game. It would be better to define a feasible set for each trader and a set of λ functions for the market makers such that, given any vector of choices from these sets, the stochastic differential equation defining the price has a unique solution with the desired limits existing. This can surely be done, but it is a thicket of technicalities that we have chosen to avoid.

II. Value Estimates and Variances

In this section, we consider the filtering problems of traders and market makers and give some necessary conditions for equilibrium. Throughout the section, we take α, β, and λ to be given continuous functions on $[0,1)$. We make the harmless assumption that $\beta, \lambda > 0$.[8] Unless stated otherwise, we assume in this section that each trader i uses the linear strategy of equation (6). Throughout the remainder of the paper, we let P denote the unique solution of equation (5) with $P(0) = 0$.

Let $\boldsymbol{F} \equiv \{\mathcal{F}(t) | 0 \leq t < 1\}$ denote the filtration generated by the aggregate order process

$$Y(t) \equiv Z(t) + \sum_{i=1}^{N} \int_0^t [\alpha(u)P(u) + \beta(u)\tilde{s}^i]du. \tag{8}$$

We interpret $\boldsymbol{F}$ as the market makers' information structure. Define $\Sigma(t)$ by

$$\frac{1}{\Sigma(t)} = \int_0^t \beta(u)^2\, du + \frac{1}{\mathrm{var}(\tilde{v})}. \tag{9}$$

LEMMA 1: *Assume each trader i follows a linear strategy as in equation* (6). *Then* $\Sigma(t)$ *is the conditional variance of* $\tilde{v}$ *given* $\mathcal{F}(t)$. *Set* $V(t) = E[\tilde{v} | \mathcal{F}(t)]$ *and define*

$$W(t) = Z(t) + \int_0^t \beta(u)\{\tilde{v} - V(u)\}\, du. \tag{10}$$

The process W is a Wiener process on the market makers' information structure $\boldsymbol{F}$. *Furthermore,*

$$V(t) = \int_0^t \beta(u)\Sigma(u)\, dW(u). \tag{11}$$

The process W is called the "innovation" process for the market makers' estimation problem. The differential

$$dW \equiv dZ + \beta\{\tilde{v} - V\}\, dt \tag{12}$$

is the unpredictable part of the order flow: the difference between the actual order

$$dY \equiv dZ + [N\alpha P + \beta\tilde{v}]dt \tag{13}$$

[8] We have verified that β and λ must be positive in any equilibrium, but for the sake of simplicity we have omitted the verification from the paper.

and the expected order

$$[N\alpha P + \beta V]dt. \tag{14}$$

The lemma shows that the market's estimate of $\tilde{v}$ is revised according to $dV = \beta\Sigma\, dW$.

Now we need to make a similar analysis of the information of an informed trader. Consider an arbitrary trader j. Assume that each trader $i \neq j$ follows a linear strategy as in equation (6) and assume that j follows an arbitrary strategy θ. Let P^θ denote the solution (assumed to exist) of equation (7). The trader observes P^θ and his signal $\tilde{s}^j$. Let $\boldsymbol{F}^j \equiv \{\mathcal{F}^j(t) | 0 \le t < 1\}$ denote the filtration generated by P^θ and $\tilde{s}^j$. This is the trader's information structure. We want to describe the conditional expectation and variance of $\tilde{v}$ given this information. If $\phi = 1$ then the trader is perfectly informed of $\tilde{v}$ by definition. If $\phi \neq 1$, define $\Omega(t)$ by

$$\frac{1}{\Omega(t)} = \int_0^t \beta(u)^2\, du + \frac{1}{(1-\phi)\Sigma(0)}, \tag{15}$$

and, if $\phi = 1$, set $\Omega(t) = 0$.

LEMMA 2: *Consider an arbitrary trader j and assume each trader $i \neq j$ follows a linear strategy as in equation* (6). *Assume the price change is proportional to the order size as in equation* (5). *Then $\Omega(t)$ is the conditional variance of $\tilde{v}$ given $\mathcal{F}^j(t)$. Set $V^j(t) = E[\tilde{v}|\mathcal{F}^j(t)]$ and define*

$$W^j(t) = Z(t) + \int_0^t \beta(u)\{\tilde{v} - V^j(u)\}\, du, \tag{16}$$

The process W^j is a Wiener process on the information structure $\boldsymbol{F}^j$, and

$$V^j(t) = N\phi\tilde{s}^j + \int_0^t \beta(u)\Omega(u)\, dW^j(u). \tag{17}$$

The differential of the innovation process W^j is again the difference between the actual order and the expected order, but now we are computing the expected order using trader j's information. The order is not directly observed by the trader, but he observes the price, and we are maintaining the assumption that the price change is proportional to the order size, so the trader can infer the order. The lemma shows that his estimate of the asset value $\tilde{v}$ is revised as $dV^j = \beta\Omega\, dW^j$.

The variance Ω is smaller than the variance Σ. Therefore, the revision in trader j's value estimate tends to be smaller than the revision in the market's. In equilibrium, the price must equal the market's estimate of $\tilde{v}$, which implies the price must be more responsive to unanticipated orders than are traders' estimates of the asset value. This has important implications for the nature of competition between the traders. Two traders who agree the asset is underpriced but disagree on the extent of underpricing will initially compete to buy the asset. However, they will eventually be on opposite sides of the market as the price rises above the value estimate of the less optimistic trader. Foster and Viswanathan (1996) describe this phenomenon as the "rat race" of competition transforming into a "waiting game." We will return to this in the next section.

The following measure of the relative tightness of beliefs will be very convenient for describing the equilibrium. Define

$$\delta(t) = \frac{\Sigma(t) - \Omega(t)}{\Sigma(t)}. \tag{18}$$

This is the percentage of the market's uncertainty that is resolved by each informed trader's information. According to equation (15), $\delta(0) = \phi$. The following shows how δ evolves over time, in terms of either Σ or Ω.

LEMMA 3: *For all t,*

$$\delta(t) = \frac{\phi\Sigma(t)}{(1-\phi)\Sigma(0) + \phi\Sigma(t)} \tag{19}$$

and

$$\delta(t)\Omega(0) = \phi\Omega(t). \tag{20}$$

Lemma 3 implies $\delta(t)$ is monotonically decreasing. Thus, over time, each individual agent loses (relinquishes) his ability to predict $\tilde{v}$ significantly better than the market can. This implies that the revisions of the market's and the traders' estimates of $\tilde{v}$ become more nearly congruent over time.

A requirement for equilibrium is that $P(t) = V(t)$ for all t. By matching the dt and dZ coefficients in the dynamics of P and V, it is easy to show the following.

LEMMA 4: *Assume each trader plays a linear strategy as in equation* (6). *To have $P(t) = V(t)$ for all t, it is necessary and sufficient that $\lambda(t) = \beta(t)\Sigma(t)$ and $\alpha(t) = -\beta(t)/N$ for all t.*

It follows that the conditions $\lambda = \beta\Sigma$ and $\alpha = -\beta/N$ are necessary for equilibrium. These conditions are also obtained by Foster and Viswanathan (1996). Note that when $\alpha = -\beta/N$ and $P = V$, the market makers' conditional expectation of the aggregate order is zero—see equation (14). Having prices proportional to orders as in equation (5) therefore implies that price changes are unpredictable, as they must be when market makers are risk neutral and competitive.

Given the conditions $\lambda = \beta\Sigma$ and $\alpha = -\beta/N$, the entire equilibrium is determined by λ. To see this, note that equation (9) implies

$$\lambda(t)^2 = \beta(t)^2\Sigma(t)^2 = -\Sigma'(t). \tag{21}$$

Therefore, the function Σ is determined by λ. The conditions $\lambda = \beta\Sigma$ and $\alpha = -\beta/N$ then determine β and α.

To determine λ, equivalently $1/\lambda$, which Kyle (1985) calls "the depth of the market," we turn to the equilibrium condition that has not yet been exploited, namely, the requirement that each informed agent's trading strategy be optimal.

III. Optimal Trading and Market Depth

In this section, we focus on an arbitrarily chosen trader, say, trader j. Assume equation (6) holds for each $i \neq j$. We define a trading strategy θ to be feasible for trader j if there exists a unique solution P^θ to the stochastic differential equation (7) for the given λ and for the given α and β that characterize the other traders' strategies and if

$$\lim_{t\to 1} P(t) \text{ exists a.s.,} \tag{22}$$

$$\int_0^1 \theta(\tilde{s}^j, u, P^\theta)\, du \text{ exists a.s.,} \tag{23}$$

and

$$E\int_0^1 P^\theta(t)^2\, dt < \infty. \tag{24}$$

By "existence" of the limits in conditions (22) and (23)—note that condition (23) is the limit of the integral over $[0, t]$—we mean they exist and are finite. These limits define, respectively, the price and number of shares held by trader j just before the announcement. Condition (24) is the "no doubling strategies" condition introduced in Back (1992). Given the existence of the limits, the integral

$$\int_0^1 (\tilde{v} - P^\theta(t))\theta(\tilde{s}^j, t, P^\theta)\, dt \tag{25}$$

exists and equals the profit of trader j. One can verify the existence of the integral by integrating by parts as in Back (1992), where the formula for profits is derived from the Merton-type wealth equation. By the law of iterated expectations, the expected profit is

$$E\int_0^1 (V^j(t) - P^\theta(t))\theta(\tilde{s}^j, t, P^\theta)\, dt. \tag{26}$$

Before turning to the characterization of optimal strategies, it is worthwhile to describe the price dynamics more explicitly. Assuming all traders $i \neq j$ follow linear strategies with $\alpha = -\beta/N$ and assuming $\lambda = \beta\Sigma$ (recall that these are necessary conditions for equilibrium), we can write the price process P^θ as

$$\begin{aligned} &\int_0^t \lambda(u)\exp\left\{-\int_u^t \frac{N-1}{N}\lambda(a)\beta(a)\, da\right\} \\ &\quad \times \left(\beta(u)\sum_{i\neq j}\tilde{s}^i\, du + \theta(\tilde{s}^j, u, P^\theta)\, du + dZ(u)\right). \end{aligned} \tag{27}$$

This shows that the price process is driven by the components

$$\beta(u)\sum_{i\neq j}\tilde{s}^i\, du + \theta(\tilde{s}^j, u, P^\theta)\, du + dZ(u), \tag{28}$$

of the aggregate trades and that the price impacts of these trades depreciate over time at rate

$$\frac{N-1}{N}\lambda(a)\beta(a). \tag{29}$$

The depreciation is a result of the terms

$$\alpha(a)P(a) = -\frac{1}{N}\beta(a)P(a) \tag{30}$$

in the trades of the other $N - 1$ agents and the factor λ by which trades affect the price.

The price impacts of trades depreciate over time because (i) if other informed agents are on the same side of the market, the price impacts would have occurred anyway through the other agents' trading; thus, the price is only temporarily displaced by agent j's trades from where it would have been in the absence of those trades; and (ii) if other informed traders are on the opposite side of the market, the price impacts of agent j's trades will actually be partially reversed by others' trades. In either case (i) or case (ii), the temporary nature of the price impacts is a motive for aggressive trading. In case (i), agent j is trying to beat others to the market. In case (ii), agent j wants to trade aggressively because the more he trades the more the other agents will trade on the opposite side, creating more profitable opportunities for him. This desire to trade aggressively is offset in the first part of the trading interval by the depth rising over time. Rising depth creates an incentive to wait before trading. However, as we will see, in the later part of the trading interval the depth of the market falls.

The following describes how the market must behave just before the announcement and what the pattern of market depth must be for a linear strategy to be optimal for trader j. Technically, this is the key result of the paper. The characterization of equilibrium in Section IV follows directly from the necessary conditions in Lemma 4 (which form the hypothesis of this lemma), from equation (32), and from solving the differential equation (25).

LEMMA 5: *Assume each trader $i \neq j$ uses a linear trading strategy as in equation* (6). *Assume $\lambda = \beta\Sigma$ and $\alpha = -\beta/N$ and assume α, β, and λ are continuously differentiable on* (0,1). *The conditions*

$$\frac{d}{dt}\left(\frac{1}{\lambda(t)}\right) = \left(2N - 1 - \frac{1}{\delta(t)}\right)\frac{\beta(t)}{N} \tag{31}$$

and

$$\lim_{t\to 1}\Sigma(t) = 0 \quad \textit{or} \quad \lim_{t\to 1}\lambda(t) = \infty \tag{32}$$

are necessary and sufficient for

$$\theta^j(s,t,x) \equiv \alpha(t)x(t) + \beta(t)s \tag{33}$$

to create expected profit (expression (26)) that is as large as that obtained from any feasible strategy.

If $N = 1$, then the right-hand side of equation (31) is zero (because $\delta(t) = 1$); hence, depth must be constant, as shown by Kyle (1985). Because $\beta > 0$ and δ is decreasing, depth cannot be constant for larger N. In fact, equation (31) shows that depth must be rising when $\delta(t) > 1/(2N - 1)$, reach its maximum when $\delta(t) = 1/(2N - 1)$, and then fall, with the market approaching complete illiquidity if $\delta(t) \to 0$ (as will be shown to be true in equilibrium if there is more than one informed trader).

Condition (31) is a local condition for optimality at each $t < 1$, which we will discuss below. Condition (32) means there is no money "left on the table" an instant before the announcement. If the first part of condition (32) holds, then the market's information about $\tilde{v}$ is precise by the announcement date, and the asset will be correctly priced. If the second part holds, then the market is nearly illiquid an instant before the announcement, so there would be no profitable trades available even if the asset were mispriced. These conditions are not mutually exclusive. In fact, both conditions hold in equilibrium when there is more than one informed trader.

We will show that conditions (31) and (32) are incompatible if there is more than one informed trader and their signals are perfectly correlated. This implies there is no equilibrium in this circumstance.

To understand condition (31), we need to consider how the nature of competition between the traders changes over time.

LEMMA 6: *Assume each trader uses a linear trading strategy as in equation* (6) *and assume $\lambda = \beta\Sigma$ and $\alpha = -\beta/N$. Then, for all t and i,*

$$dP(t) = \lambda(t)\beta(t)[V^i(t) - P(t)]dt + \lambda(t)\,dW^i(t), \tag{34}$$

$$V^i(t) = [1 - \delta(t)]P(t) + \delta(t)N\tilde{s}^i, \tag{35}$$

and

$$\alpha(t)P(t) + \beta(t)\tilde{s}^i = \frac{\beta(t)}{N\delta(t)}[V^i(t) - P(t)]. \tag{36}$$

For each j, the expected value of

$$\sum_{i \neq j} \alpha(t)P(t) + \beta(t)\tilde{s}^i \tag{37}$$

conditional on trader j's information at time t is

$$\frac{\beta}{N\delta(t)}\big(N\delta(t) - 1\big)\big(V^j(t) - P(t)\big). \tag{38}$$

These conditions must hold in any equilibrium. Equation (34) shows that the price drifts toward each trader's estimate of the asset value. Equation (35) shows that the price and value estimate converge as $t \to 1$ if $\delta(t) \to 0$. Equation (36) shows that along the equilibrium path each agent's strategy is linear in his value estimate, in addition to being linear in his initial signal, as we discussed in Section I. A trader buys the asset if and only if it is undervalued relative to his value estimate.

Equation (36) and the last statement of the lemma show that the sum of an agent's trade and his expectation of others' trades always has the same sign as his own trade. This means that he always expects part of his order to be filled by market makers. Hence, he can expect to make money even when he believes himself to be on the opposite side of the market from other informed traders. Of course, the market makers recover their losses from the liquidity traders.

The last statement of the lemma is the most important for understanding how the nature of competition changes over time. Suppose trader j believes the asset is undervalued; that is, $V^j(t) > P(t)$. Then trader j believes other traders also believe the asset is undervalued and will be buying if and only if $\delta(t) > 1/N$. When $\delta(t) > 1/N$, the market is in what Foster and Viswanathan call the "rat race" phase.

As was stated in Section I, if ρ denotes the correlation of $\tilde{s}^i$ with $\tilde{s}^j$, then the initial value of δ is

$$\delta(0) = \phi = \frac{1 + (N-1)\rho}{N} > \frac{1}{N} \quad \Leftrightarrow \quad \rho > 0. \tag{39}$$

Positive correlation of signals would be the usual state of affairs. In this case, the market is initially in a rat race phase. It is of some interest to note that the above relation between δ and ρ generalizes to arbitrary t. For $i \neq j$, let $\rho(t)$ denote the correlation of $\tilde{s}^i$ with $\tilde{s}^j$ conditional on the market's information at time t. It can be shown via the Kalman filter that

$$\delta(t) = \frac{1 + (N-1)\rho(t)}{N}. \tag{40}$$

Therefore, the rat race phase is the phase in which the conditional correlation of the signals is positive. This is the way that Foster and Viswanathan define it.

Expecting other agents to be on the same side of the market provides an inducement to trade quickly. In fact, any continuous trading strategy could be improved by advancing trades forward in time unless some offsetting factor induces patience. This offsetting factor must be that the market becomes deeper over time. Indeed, equation (31) shows that the depth of the market rises throughout the rat race phase.

Over time, the fact that the price is more sensitive to unanticipated orders than are traders' value estimates results in the price moving to a level between the value estimates of the traders. Traders realize that after the price has incorporated most of their common information, they will eventually be on opposite sides of the market. They cannot know exactly when this occurs, but each expects the others to be on the opposite side of the market at the time when $\delta(t)$ falls below $1/N$. Foster and Viswanathan call this the "waiting game" phase of the market. In the case of negatively correlated signals,

$\delta(0) < 1/N$, and traders expect to be on the opposite side of the market at the outset. However, in the more usual circumstance of positively correlated signals, it is some date between zero and one at which traders expect to be on opposite sides of the market.

When traders are on opposite sides of the market, there are two forces at work. The first is that price impacts of trades have a temporary component, as described earlier. The second is that traders on the opposite side of the market push the price in a favorable direction, so trades can be made at better prices if one waits. This creates an incentive to wait to trade. When $\delta(t) = 1/N$, the second force is nonexistent, but the first force is operative. This means that the depth of the market must be rising when $\delta(t) = 1/N$ to offset the incentive for agents to trade infinitely aggressively. However as time passes and $\delta(t)$ falls, the second force becomes more important. When $\delta(t)$ falls below $1/(2N-1)$, the second force is more important than the first, and depth must be falling to induce agents to trade at all.[9] Thus, the market is deepest when $\delta(t) = 1/(2N-1)$.

IV. Equilibrium

We define an equilibrium to be α, β, and λ that are continuous on $[0,1)$ and continuously differentiable on $(0,1)$, with β and λ positive, and which satisfy (i) $P(t) = V(t)$ for all t (where P and V are as defined in Section II) and (ii) the trading strategy of equation (33) is feasible and maximizes expression (26) over the set of feasible strategies.[10] Our main result is the following and is obtained by combining Lemmas 4 and 5 and solving the differential equation (31).

THEOREM 1: *If there are multiple informed traders* ($N > 1$) *and their signals are perfectly correlated* ($\phi = 1$)*, then there is no equilibrium. Otherwise, there is a unique equilibrium. Set* $\Sigma(0) = \mathrm{var}(\tilde{v})$*, and consider the constant*

$$\kappa = \int_1^\infty x^{2(N-2)/N} e^{-2x(1-\phi)/N\phi}\, dx. \tag{41}$$

For each $t < 1$*, define* $\Sigma(t)$ *by*

$$\int_1^{\Sigma(0)/\Sigma(t)} x^{2(N-2)/N} e^{-2x(1-\phi)/N\phi}\, dx = \kappa t. \tag{42}$$

[9] Because the first factor is more important than the second when $\delta(t)$ is between $1/N$ and $1/(2N-1)$, it may be inappropriate to call this phase of the market part of the "waiting game."

[10] Hence, by definition, an equilibrium is symmetric and linear. We do not investigate the existence of asymmetric or nonlinear equilibria.

The equilibrium is

$$\beta(t) = \left(\frac{\kappa}{\Sigma(0)}\right)^{1/2}\left(\frac{\Sigma(t)}{\Sigma(0)}\right)^{(N-2)/N}\exp\left\{\frac{1}{N}\left(\frac{1-\phi}{\phi}\right)\frac{\Sigma(0)}{\Sigma(t)}\right\}, \tag{43}$$

$$\alpha(t) = -\beta(t)/N, \tag{44}$$

$$\lambda(t) = \beta(t)\Sigma(t). \tag{45}$$

Furthermore, $\Sigma(t)$ is the conditional variance of $\tilde{v}$ given the market makers' information at time t.

In the case $N = 1$, we recover the result of Kyle (1985). From equation (41), we have $\kappa = 1$, and equation (42) reduces to $\Sigma(t) = (1 - t)\Sigma(0)$. The equations (43)–(45) become

$$\beta(t) = \frac{1}{(1-t)\sqrt{\Sigma(0)}}, \qquad \alpha(t) = -\frac{1}{(1-t)\sqrt{\Sigma(0)}}, \qquad \lambda(t) = \sqrt{\Sigma(0)}. \tag{46}$$

If $N > 1$ (and $\phi \neq 1$), we can write the equilibrium as follows. Recall the definition of the incomplete gamma function:

$$\Gamma(a, b) \equiv \int_b^\infty u^{a-1}e^{-u}\,du. \tag{47}$$

Set $a_N = 3 - 4/N$ and $b_{N\phi} = 2(1 - \phi)/(N\phi)$. The constant κ defined in equation (41) and appearing in equation (43) equals

$$(b_{N\phi})^{-a_N}\Gamma(a_N, b_{N\phi}). \tag{48}$$

Furthermore, we can write equation (42) as

$$\Gamma\left(a_N, \frac{b_{N\phi}\Sigma(0)}{\Sigma(t)}\right) = (1-t)\Gamma(a_N, b_{N\phi}). \tag{42$'$}$$

This shows that $\Sigma(t)$ is obtained by inverting the incomplete gamma function in its second argument.

The formula for the equilibrium simplifies considerably when $N = 2$. The reason is that $a_N = 1$ when $N = 2$, and for any b, $\Gamma(1, b) = e^{-b}$, so the inverse of the incomplete gamma function in its second argument is simply the negative of the natural logarithm function.

COROLLARY 1: *Assume* $N = 2$. *Set* $\Sigma(0) = \mathrm{var}(\tilde{v})$ *and let* ρ *denote the correlation coefficient of* $\tilde{s}^1$ *with* $\tilde{s}^2$. *The unique equilibrium is given by*

$$\alpha(t) = -\frac{1}{2}\sqrt{\frac{1+\rho}{(1-\rho)\Sigma(0)(1-t)}}, \tag{49}$$

$$\beta(t) = \sqrt{\frac{1+\rho}{(1-\rho)\Sigma(0)(1-t)}}, \tag{50}$$

$$\lambda(t) = \frac{\sqrt{(1+\rho)(1-\rho)\Sigma(0)}}{[1-\rho-(1+\rho)\log(1-t)]\sqrt{1-t}}. \tag{51}$$

For each t, *the conditional variance of* $\tilde{v}$ *given the market makers' information at time* t *is*

$$\Sigma(t) = \frac{(1-\rho)\Sigma(0)}{1-\rho-(1+\rho)\log(1-t)}. \tag{52}$$

V. Comparative Dynamics

In this section, we analyze the intensity of informed trading, the residual uncertainty of market makers regarding the asset value, and the depth of the market. A benchmark case is that of a monopolist informed trader studied by Kyle (1985). We will illustrate the effects of competition between informed traders by comparing each variable to what it would be if there were a monopolist informed trader.

The parameters of the model are the variance of $\tilde{v}$, the number N of informed traders, and the correlation ρ of their signals (rather than ρ, one could use ϕ as the parameter of the signal distribution—see equation (3)). We will fix $\mathrm{var}\,\tilde{v} = 1$ and examine the effects of varying N and ρ.

The intensity of informed trading is measured by $\beta(t)/[N\delta(t)]$, because $\beta(t)/[N\delta(t)]$ is the coefficient of the information variable $V^i(t) - P(t)$ in each informed agent's trade—see Lemma 6. In the monopolist case with $\mathrm{var}\,\tilde{v} = 1$, the intensity of informed trading is $1/(1-t)$; the conditional variance of $\tilde{v}$ given the market makers' information, $\Sigma(t)$, is $1-t$; and market depth, $1/\lambda(t)$, equals one. The following shows how competition affects these variables.

COROLLARY 2: *Assume* $\mathrm{var}\,\tilde{v} = 1$. *If there are two informed traders and their signals are uncorrelated, then, at all times* $t > 0$, *the intensity of informed trading is less than that of a monopolist, that is,*

$$\frac{\beta(t)/[N\delta(t)]}{1/(1-t)} < 1, \tag{53}$$

and the market's residual uncertainty is greater than it would be if there were a monopolist informed trader, that is,

$$\frac{\Sigma(t)}{1-t} > 1. \tag{54}$$

For any number of traders $N > 1$ and any correlation of signals (provided the correlation is not perfect),

$$\frac{\beta(t)/[N\delta(t)]}{1/(1-t)} \to 0, \qquad \frac{\Sigma(t)}{1-t} \to \infty, \qquad \textit{and} \qquad 1/\lambda(t) \to 0 \tag{55}$$

as $t \to 1$.

In brief, competition leads to relatively low informed trading intensity and relatively informationally inefficient markets. This is true at all times when there are two informed traders with uncorrelated signals, and it is true near the date of the public announcement in general. Furthermore, in all cases, the market approaches complete illiquidity as the date of the public announcement approaches.

The following figures illustrate these and other facets of the equilibrium. Figure 1 shows the intensity of informed trading in relation to that of a monopolist. The intensity is greater when there are more traders, and it is greater when the signals are highly correlated. When there are two informed traders with uncorrelated signals, the intensity of informed trading is less than the intensity with which a monopolist would trade—the solid line in Panel B of Figure 1 lies below one—as Corollary 2 asserts. In general the intensity converges to zero in relation to that of a monopolist as the public announcement approaches.

Figure 2 shows the residual uncertainty $\Sigma(t)$ of the market makers in relation to what the residual uncertainty would be if there were a single informed trader. The residual uncertainty is lower when there are more informed traders and when the signals are highly correlated, as one would expect given the patterns for the intensity of informed trading. The ratio converges to infinity as the public announcement date approaches. This is a consequence of the relatively low intensity with which the competing informed traders trade as the announcement date approaches. The solid line in Panel B of Figure 2 lies above one, meaning that, when there are two competing traders with uncorrelated signals, the residual uncertainty is larger than it would be with a monopolist informed trader. This is in accordance with Corollary 2.

Figure 3 shows the depth of the market. The depth converges to zero as $t \to 1$. This contrasts with the constant depth that occurs when there is a single informed trader. The depth of the market depends on the amount of information in the order flow, which depends on the amount of information

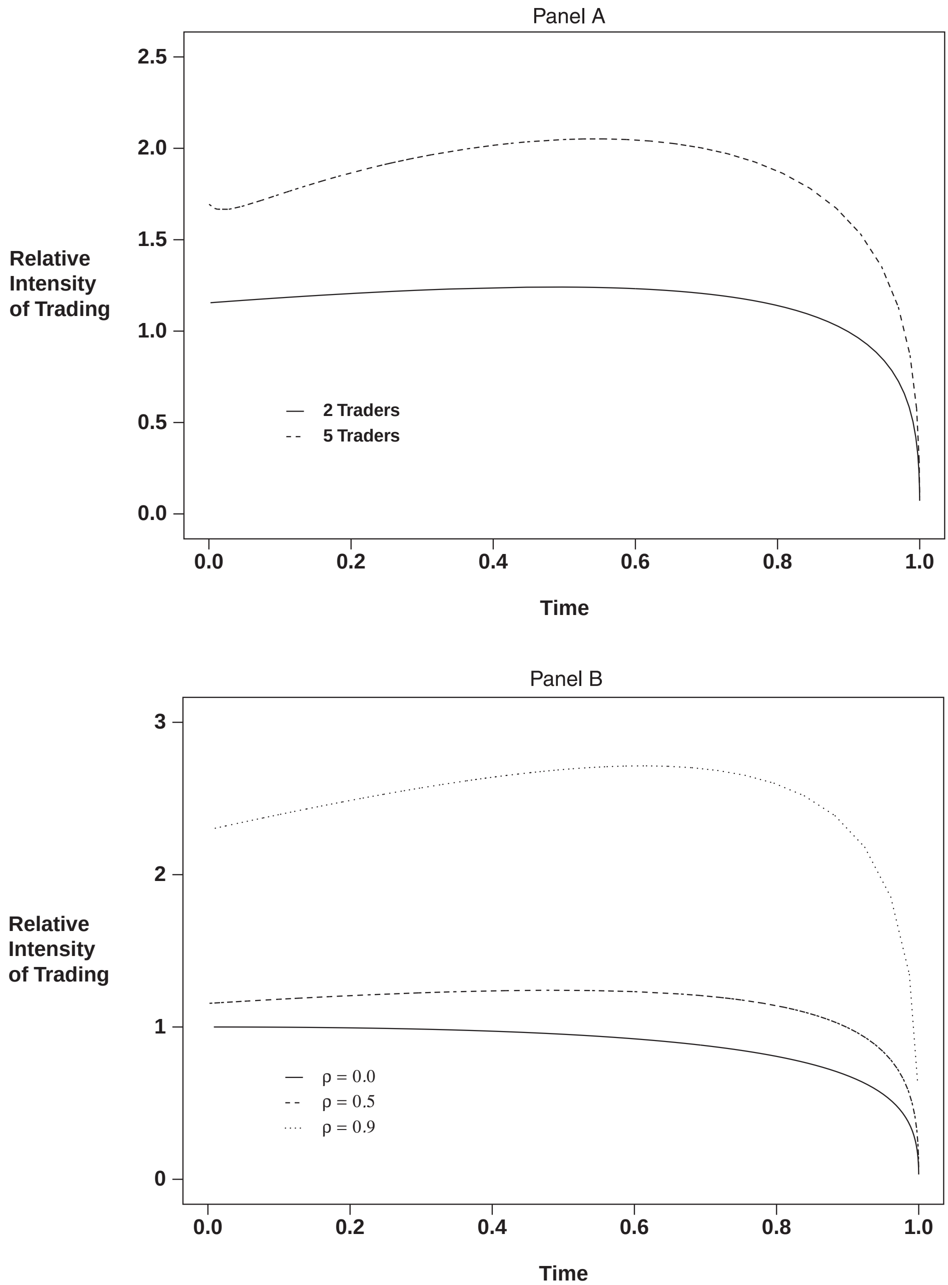

Figure 1. Intensity of informed trading. The ratio of an agent's trade to the mispricing he or she perceives is termed the "intensity" of trading. The figure shows the intensity of an agent's trading divided by the intensity with which a monopolist would trade. In Panel A, the correlation coefficient of the signals ρ is set equal to 0.5, and the number of traders is varied. In Panel B, there are two informed traders and the correlation coefficient is varied. The unconditional variance of the asset value is set equal to one.

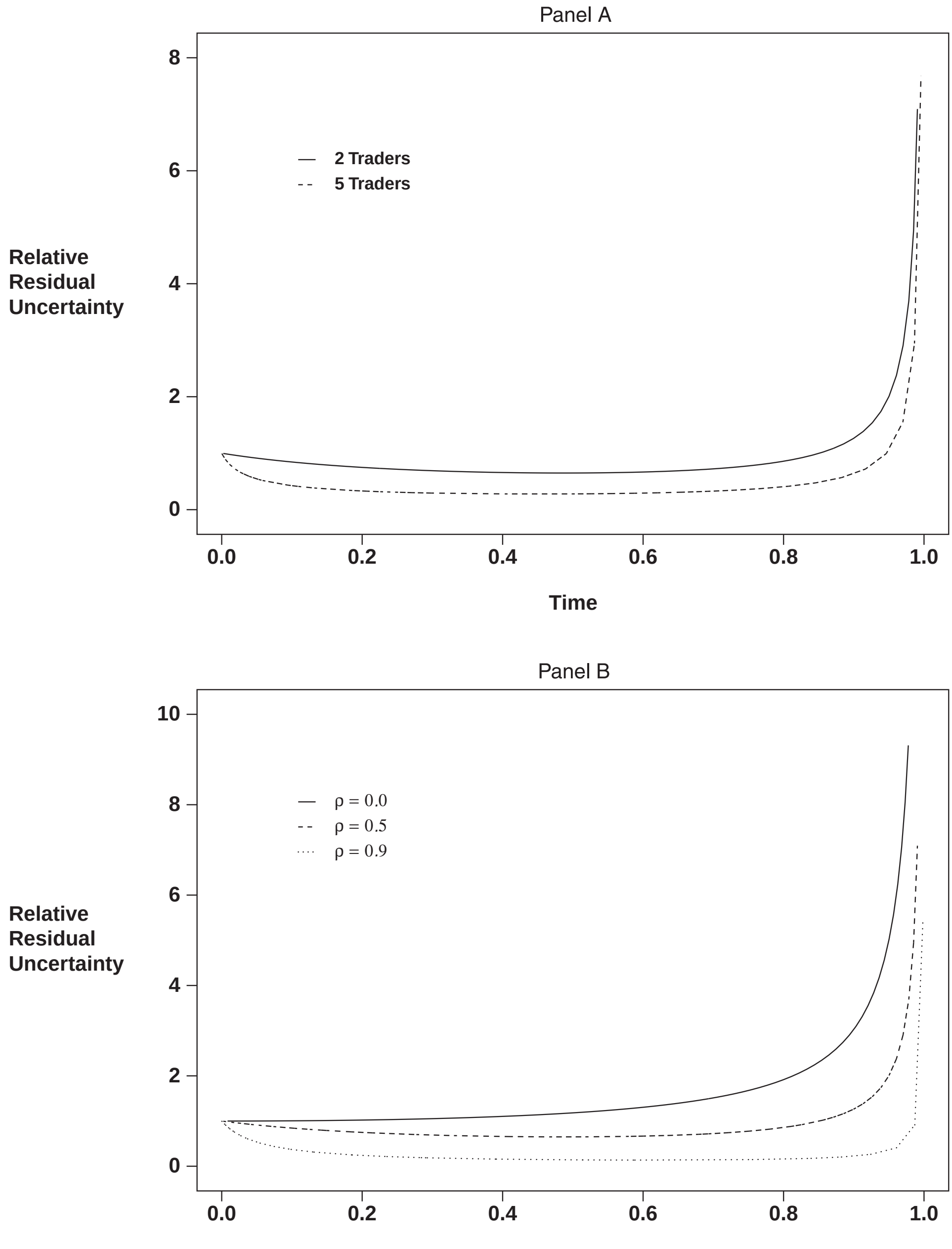

Figure 2. The market's residual uncertainty. This shows the variance of the asset value conditional on the market makers' information, divided by the conditional variance if instead there were a monopolist informed trader. In Panel A, the correlation coefficient of the signals ρ is set equal to 0.5, and the number of traders is varied. In Panel B, there are two informed traders and the correlation coefficient is varied. The unconditional variance of the asset value is set equal to one.

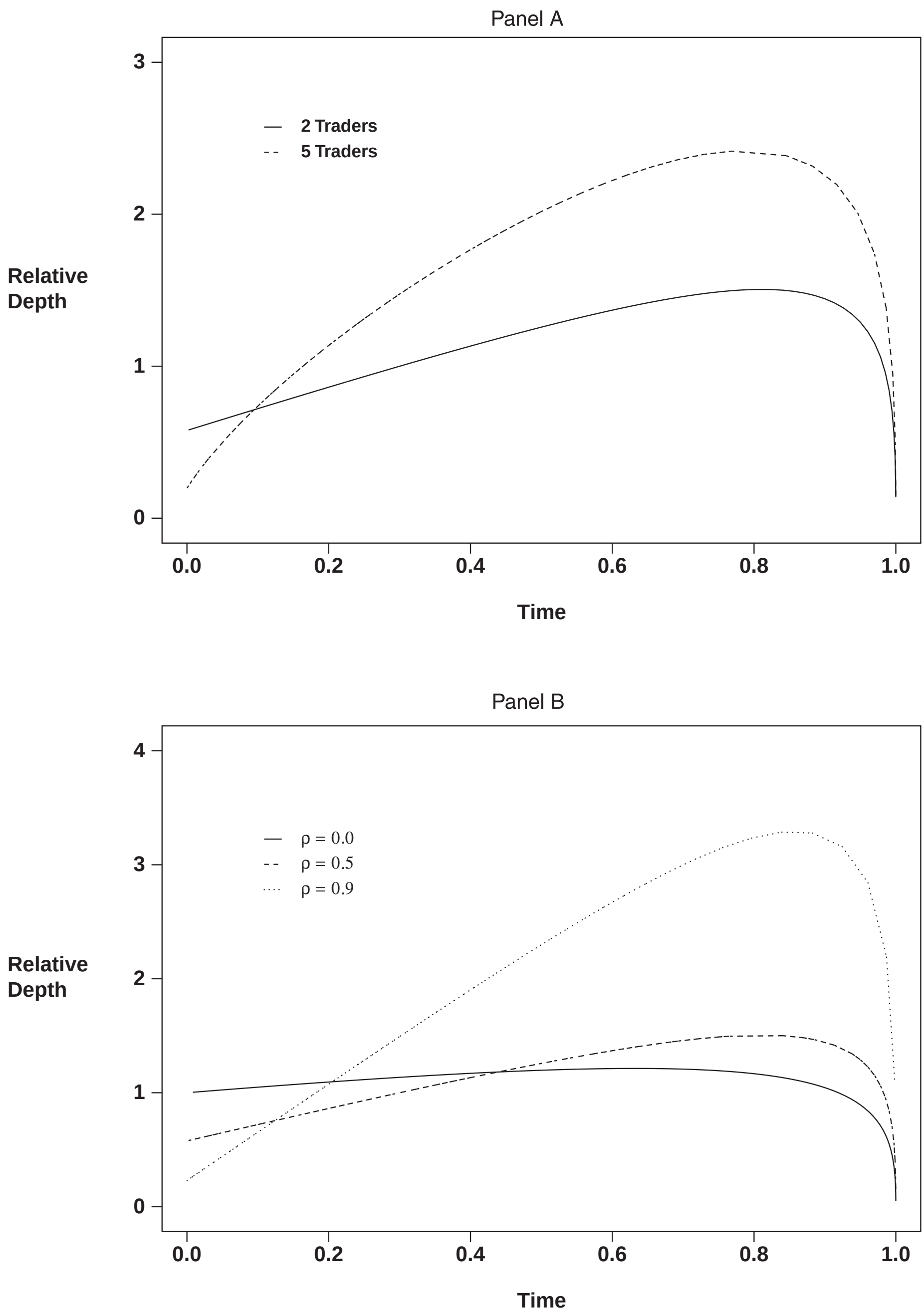

Figure 3. Market depth. This shows the depth of the market. The unconditional variance of the asset value is set equal to one. If there were a monopolist informed trader, the depth of the market would be equal to one. In Panel A, the correlation coefficient of the signals ρ is set equal to 0.5, and the number of traders is varied. In Panel B, there are two informed traders and the correlation coefficient is varied.

outstanding and the trading intensity of the informed traders. The former is measured by Σ and the latter by β (aggregating the informed trades, the rate of informed trading is $\beta(\tilde{v} - P)$). In fact, the depth is $1/(\beta\Sigma)$. The intensity β is relatively small near the announcement date when there are competing informed traders, but the residual uncertainty Σ is large. The latter variable dominates; thus there is a large amount of information in the order flow and consequently a very shallow market near the announcement date. As we have seen, the intensity of trading is greater when there are more informed traders. The greater intensity of trading implies more information in the order flow initially and hence a shallower market initially. This is shown in Panel A of Figure 3. However, the greater intensity of trading leads to greater learning by the market, and eventually the amount of information in the order flow is smaller. Consequently, the market is eventually deeper when there are more informed traders, as Panel A of Figure 3 shows. The effect of correlation, which is shown in Panel B of Figure 3, is exactly the same—greater correlation implies more intense trading and a shallower market initially but less information in the order flow and a deeper market eventually.

The last variable we consider is the expected profit of informed traders.

COROLLARY 3: *In equilibrium, the expected profit of each informed trader is*

$$\frac{1}{N}\int_0^1 \lambda(t)\,dt. \tag{56}$$

This equals

$$\frac{1}{N}\left(\frac{\mathrm{var}(\tilde{v})}{\kappa}\right)^{1/2}\int_1^\infty x^{-2/N}e^{-x(1-\phi)/N\phi}\,dx. \tag{57}$$

Note that expression (56) is consistent with the expected losses of liquidity traders being

$$\int_0^1 dP(t)\,dZ(t) = \int_0^1 \lambda(t)\,dZ(t)\,dZ(t) = \int_0^1 \lambda(t)\,dt \tag{58}$$

and with these losses being the gains of the informed traders, shared equally.

Table I shows the expected aggregate profits ($\int_0^1 \lambda(t)\,dt$) of the informed traders as a function of N and ρ, with the variance of $\tilde{v}$ set equal to one. The expected profit is less than one, which is what a single informed trader would earn, and for fixed ρ the expected profit is decreasing in the number of traders. Expected profits are quite small when there is a large number of traders with highly correlated signals. As Foster and Viswanathan (1996) observe, the expected profit is higher when there is some positive correlation

Table I

Expected Profits of Informed Traders

Aggregate expected profits of informed traders are shown as a function of the number of informed traders (N) and the correlation of the traders' signals. The unconditional variance of the asset value is set equal to one. The expected profit of a monopolist informed trader would be one. In all cases, competing informed traders earn lower profits than would a monopolist.

	Correlation					
N	0%	5%	10%	25%	50%	90%
2	0.923	0.930	0.936	0.947	0.938	0.727
3	0.912	0.922	0.930	0.932	0.876	0.528
5	0.904	0.920	0.924	0.894	0.764	0.362
10	0.899	0.917	0.902	0.796	0.590	0.232
15	0.898	0.909	0.871	0.716	0.495	0.183
20	0.897	0.898	0.838	0.653	0.434	0.156

between the signals than when the signals are uncorrelated. However, the correlation at which expected profits is maximized diminishes as the number of traders increases.

VI. Conclusion

Competition has a significant impact on the optimal trading of an informed agent. In the equilibrium of the continuous-time Kyle model with a single informed trader, the trader is actually indifferent among a large class of strategies. Any strategy in which trade is continuous and no money is "left on the table" at the end is optimal (see Back (1992)). This can be interpreted as the trader being extremely patient—it is costless to wait until later to trade, provided he does not wait until the information is publicly revealed. This patience implies that it is not an equilibrium for the depth of the market to vary systematically over time, because such variations could be manipulated. In contrast, an informed trader in the multiagent model has a unique optimal strategy in equilibrium. Furthermore, the changing nature of competition between the agents implies that depth must vary systematically over time. In particular, the market must approach complete illiquidity just before the public announcement of the information.

Competition may or may not lead to greater "efficiency" of prices. Whether the market price reflects private information more quickly when there are competing informed traders depends on how the information is distributed among the agents. Indeed, in this model, it is never the case that the market is *always* more efficient when information is distributed among competing traders than when the information is possessed by a single trader.

Appendix

Proof of Lemma 1: This is an application of the Kalman–Bucy filter. See Kallianpur (1980, Sec. 10.3). Q.E.D.

Proof of Lemma 2: This follows from the Kalman–Bucy filter, just as does Lemma 1, except that here we also need to verify the formulas for $\Omega(0)$ and $V^i(0)$ implied by equations (15) and (17), respectively. Because $\tilde{s}^j$ and $\tilde{v}$ are joint normal,

$$V^j(0) \equiv E[\tilde{v}|\tilde{s}^j] = \frac{\operatorname{cov}(\tilde{v}, \tilde{s}^j)}{\operatorname{var}(\tilde{s}^j)}\, \tilde{s}^j. \tag{A1}$$

This is the same as equation (17) at $t = 0$. Equation (15) evaluated at $t = 0$ is equivalent to

$$\phi = \frac{\Sigma(0) - \Omega(0)}{\Sigma(0)}. \tag{A2}$$

Because ϕ is the "R-squared" in the linear regression of $\tilde{v}$ on $\tilde{s}^j$, this implies $\Omega(0) = \operatorname{var}(\tilde{v}|\tilde{s}^j)$, as claimed. Q.E.D.

Proof of Lemma 3: If $\phi = 1$, then $\Omega \equiv 0$, and $\delta \equiv 1$, and both claims are trivially true. Assume $\phi \neq 1$. From definitions (9) and (15),

$$\frac{1}{\Omega(t)} - \frac{1}{\Sigma(t)} = \frac{1}{(1-\phi)\Sigma(0)} - \frac{1}{\Sigma(0)} = \left(\frac{\phi}{1-\phi}\right)\frac{1}{\Sigma(0)}. \tag{A3}$$

Multiplying by $\Sigma(t)$ and adding one to both sides gives

$$\frac{\Sigma(t)}{\Omega(t)} = \frac{(1-\phi)\Sigma(0) + \phi\Sigma(t)}{(1-\phi)\Sigma(0)}. \tag{A4}$$

Rearranging this yields equation (19). Furthermore, adding $1/\Sigma(t)$ to both sides of equation (A3), taking reciprocals, multiplying by ϕ, and then using equation (19) implies

$$\phi\Omega(t) = (1-\phi)\Sigma(0)\delta(t) = \Omega(0)\delta(t). \tag{A5}$$

Q.E.D.

The proof of Lemma 4 is straightforward and hence omitted. We will reverse the order of the proofs of Lemmas 5 and 6, because Lemma 6 is used in the proof of Lemma 5.

Proof of Lemma 6: Condition (34) follows from using definition (16) to substitute for $dZ(t)$ in equation (7). Condition (35) is trivially true if $\phi = 1$. Assume $\phi \neq 1$. Set $R(t) = P(t) - V^i(t)$ and

$$S(t) = \frac{\delta(t)}{1-\delta(t)}\left(V^i(t) - \frac{V^i(0)}{\phi}\right) = \frac{\Sigma(t)-\Omega(t)}{\Omega(t)}\left(V^i(t) - \frac{V^i(0)}{\phi}\right). \tag{A6}$$

The claim is that $R = S$. We have $R(0) = S(0)$ (recall $P(0) = 0$),

$$dR(t) = -\lambda(t)\beta(t)R(t)\,dt + [\lambda(t) - \beta(t)\Omega(t)]\,dW^i(t), \tag{A7}$$

and

$$dS(t) = \frac{d}{dt}\log\left(\frac{\Sigma(t)-\Omega(t)}{\Omega(t)}\right)S(t)\,dt + \beta(t)[\Sigma(t)-\Omega(t)]\,dW^i(t). \tag{A8}$$

These stochastic differential equations are obtained by using equations (17) and (34). These are linear stochastic differential equations; hence, each has a unique solution. We will show that the equations are the same. The uniqueness of the solution therefore implies $R = S$.

The coefficients of dW^i are the same, given the assumption $\lambda = \beta\Sigma$. Using the fact that

$$\frac{d}{dt}\left(\frac{1}{\Sigma(t)}\right) = \frac{d}{dt}\left(\frac{1}{\Omega(t)}\right) = \beta(t)^2 \tag{A9}$$

(see definitions (9) and (15)) one can compute

$$\frac{d}{dt}\log\left(\frac{\Sigma(t)-\Omega(t)}{\Omega(t)}\right) = -\beta(t)^2\Sigma(t) = -\lambda(t)\beta(t), \tag{A10}$$

which shows that the coefficient of $R(t)\,dt$ equals that of $S(t)\,dt$.

Condition (36) follows from equation (35) and the fact that $V^i(0) = N\phi\tilde{s}^i$. To obtain the last statement, note that trader j's expectation of $\sum_{i\neq j}\tilde{s}^i$ is $V^j(t) - \tilde{s}^j$. Substitute this and $\alpha = -\beta/N$ into expression (37) and then use equation (35) to eliminate $\tilde{s}^j$. Q.E.D.

Proof of Lemma 5: Let

$$\eta(t) = \frac{d}{dt}\left(\frac{1}{\lambda(t)}\right). \tag{A11}$$

Let $\pi(s,t,v)$ denote a function that is continuously differentiable in t and twice continuously differentiable in v. For the moment, we leave π otherwise undefined. Set

$$H(t,v,p,r) = \frac{1}{\lambda(t)}\left\{v[r-p] + \frac{p^2}{2} - \frac{r^2}{2}\right\} \tag{A12}$$

and

$$J(s,t,v,p) = H(t,v,p,\pi(s,t,v)). \tag{A13}$$

For the strategy $\theta \equiv 0$, we have

$$\begin{aligned} dP^0(t) &= \lambda(t)\,dZ(t) + \lambda(t)\beta(t)\left(\sum_{i\neq j}\tilde{s}^i - \frac{N-1}{N}P^0(t)\right)dt \\ &= \lambda(t)\,dW^j(t) + \lambda(t)\beta(t)\left(V^j(t) - \tilde{s}^j - \frac{N-1}{N}P^0(t)\right)dt. \end{aligned} \tag{A14}$$

The drift of $J\big(\tilde{s}^j,t,V^j(t),P^0(t)\big)$ is $\mu\big(\tilde{s}^j,t,V^j(t),P^0(t)\big)\,dt$, where $\mu(s,t,v,p)$ is defined by

$$\mu = J_t + \lambda\beta\left(v - s - \frac{N-1}{N}p\right)J_p + \frac{1}{2}\beta^2\Omega^2 J_{vv} + \lambda\beta\Omega J_{vp} + \frac{1}{2}\lambda^2 J_{pp}. \tag{A15}$$

Here, the subscripts denote partial derivatives, and the arguments of μ, J, λ, β, and Ω have been omitted for convenience.

For future reference, we remark here that μ is a quadratic function of p:

$$\mu(s,t,v,p) = a(t)p^2 + b(s,t,v)p + c(s,t,v), \tag{A16}$$

where

$$a = \frac{\eta}{2} - \frac{N-1}{N}\beta, \tag{A17}$$

and

$$b = -\eta v + \left(2 - \frac{1}{N}\right)\beta v - \beta s. \tag{A18}$$

Only the term $c(s,t,v)$ depends on the definition of π.

For a general strategy θ, we can write the drift of $J(\tilde{s}^j, t, V^j(t), P^\theta(t))$ as

$$\begin{aligned} &\mu(\tilde{s}^j, t, V^j(t), P^\theta(t))\, dt + \lambda(t)\theta(\tilde{s}^j, t, P^\theta) J_p(\tilde{s}^j, t, V^j(t), P^\theta(t))\, dt \\ &\quad = \mu(\tilde{s}^j, t, V^j(t), P^\theta(t))\, dt + [P^\theta(t) - V^j(t)]\theta(\tilde{s}^j, t, P^\theta)\, dt. \end{aligned} \tag{A19}$$

The "stochastic part" of $dJ(\tilde{s}^j, t, V^j(t), P^\theta(t))$ is

$$\sigma(\tilde{s}^j, t, V^j(t), P^\theta(t))\, dW^j(t), \tag{A20}$$

where $\sigma(s, t, v, p)$ is defined by $\sigma = \lambda J_p + \beta \Omega J_v$. Combining and integrating these yields

$$\begin{aligned} &\int_0^t [V^j(u) - P^\theta(u)]\theta(\tilde{s}^j, u, P^\theta)\, du \\ &\quad = \int_0^t \mu(\tilde{s}^j, u, V^j(u), P^\theta(u))\, du \\ &\qquad + \int_0^t \sigma(\tilde{s}^j, u, V^j(u), P^\theta(u))\, dW^j(u) \\ &\qquad - J(\tilde{s}^j, t, V^j(t), P^\theta(t)) + J(\tilde{s}^j, 0, V^j(0), P^\theta(0)). \end{aligned} \tag{A21}$$

The expectation of the limit as $t \to 1$ of the left-hand side of equation (A21) is the expected profit. By analyzing the right-hand side, we will show that this is maximized by the linear strategy θ^j in definition (33) if and only if conditions (31) and (32) are satisfied. We continue to denote the price process P^{θ^j} by P.

As an aside, we will relate our method of proof to the usual Bellman equation approach. The left-hand side of equation (A19) is an expression for the drift of J that is valid for an arbitrary function J, not just the special function we are considering. The Bellman equation states that the drift of the value function plus the instantaneous profit is maximized at zero by the optimal strategy. The instantaneous profit in this problem is $(V^j - P^\theta)\theta$. Therefore the left-hand side of equation (A19) plus $(V^j - P^\theta)\theta$ must be maximized at zero by the optimal θ. However, each of these terms is linear in θ. The sum of the θ coefficients must be zero for the maximum to be finite. This implies $\lambda J_p = P^\theta - V^j$. This should be true for all possible values of P^θ and V^j, so $J_p(s, t, v, p) = (p - v)/\lambda(t)$. All functions satisfying this equation are of the form given in definition (A13), up to the addition of a term depending only on (s, t, v). Hence, we have imposed part of the Bellman equation. This is the reason we get the formula given in equation (A21) for expected profit in terms of J. However, we have not used all of the Bellman equation, because we have not imposed that the maximum is zero. The drift of the value function plus the instantaneous profit must have the form given in definition (A15), if the θ terms cancel. The Bellman equation states that this is zero for all (s, t, v, p), which is a partial differential equation for the value function J. It turns out that there is no smooth function J satisfying both this partial differential equation and the requirement $J_p(s,t,v,p) = (p-v)/\lambda(t)$. Therefore, there is no smooth solution to the Bellman equation. To see this, note that if equation (A15) is identically zero, then its partial derivative with respect to p must be identically zero. We can differentiate equation (A15) with respect to p and use $J_p = (p-v)/\lambda$ to compute all of the derivatives. The result is

$$(p-v)\eta + \beta\left(v - s - \frac{N-1}{N}p\right) - \beta\left(\frac{N-1}{N}\right)(p-v) = 0, \tag{A22}$$

where, as before, η denotes the time derivative of $1/\lambda$. The left-hand side of equation (A22) is a linear function of (s,v,p) with nonzero coefficients (because $\beta \neq 0$). Therefore, it cannot be zero for all (s,v,p). However, Lemma 6 shows that, when the linear strategy θ^j is followed, the state variables $(\tilde{s}^j, t, V^j(t), P(t))$ always satisfy the linear equation

$$v = \big(1-\delta(t)\big)p + \delta(t)Ns. \tag{A23}$$

Given this, equation (31) implies the state variables $(\tilde{s}^j, t, V^j(t), P(t))$ also satisfy equation (A22). Thus, the Bellman equation does not give us the optimal control in the usual way—that is, as the optimizer of the drift plus instantaneous profit. In fact, any θ trivially optimizes this expression because it is independent of θ. However, the Bellman equation gives us the optimal control indirectly because the state variables must be controlled to remain on a time-dependent plane in (s,v,p)–space. This is possible because s is fixed over time and V^j and P^θ are locally perfectly correlated. We will not directly use any of the arguments of this paragraph in the remaining proof. However, we should note that the function J defined in equation (A13) is not the actual value function. To simplify the notation, we have subtracted a function of (s,t,v). In fact, we have subtracted the actual value of the problem, so we will have $J(\tilde{s}^j, t, V^j(t), P(t)) = 0$ for all t. As a result, even though the drift of the actual value function plus the instantaneous profit is zero at all times when following the strategy θ^j under condition (31), the function μ defined by equations (A13) and (A15) will be nonzero.

Sufficiency. Suppose conditions (31) and (32) hold. By condition (31),

$$a = -\frac{\beta(1-\delta)}{N\delta} \tag{A24}$$

and

$$b = \frac{\beta(v - N\delta s)}{N\delta}. \tag{A25}$$

Case 1: Consider the case $\phi \neq 1$. Set

$$\pi(s,t,v) = v + \frac{\delta(t)}{1-\delta(t)}(v - Ns). \tag{A26}$$

Then

$$J(s,t,v,p) = \frac{1}{2\lambda(t)}\left[(v-p)^2 - \frac{\delta(t)^2}{\big(1-\delta(t)\big)^2}(v-Ns)^2\right]. \tag{A27}$$

By Lemma 6, $P(t) = \pi\big(\tilde{s}^j, t, V^j(t)\big)$. Because $H(t,v,p,p) = 0$, this implies $J(\tilde{s}^j, t, V^j(t), P(t)) = 0$. Hence, $dJ(\tilde{s}^j, t, V^j(t), P(t)) = 0$. Specializing equation (A21) to this case yields

$$\int_0^t [V^j(u) - P(u)]\theta^j(\tilde{s}^j, u, P)\,du = \int_0^t \mu\big(\tilde{s}^j, u, V^j(u), P(u)\big)\,du. \tag{A28}$$

Consider an arbitrary θ. Because $J(\tilde{s}^j, t, V^j(t), P(t)) = 0$ for all t and $P^\theta(0) = P(0)$, we have $J(\tilde{s}^j, 0, V^j(0), P^\theta(0)) = 0$. We can calculate the diffusion term in equation (A21) as follows:

$$\begin{aligned}
\sigma = \lambda J_p + \beta\Omega J_v &= p - v + \frac{\beta\Omega}{\lambda}\left(\pi - p + \frac{v-\pi}{1-\delta}\right) \\
&= p - v + (1-\delta)\left(\pi - p + \frac{v-\pi}{1-\delta}\right) \\
&= -\delta(\pi - p).
\end{aligned} \tag{A29}$$

Therefore, equation (A21) reduces to

$$\begin{aligned}
&\int_0^t [V^j(u) - P^\theta(u)]\theta(\tilde{s}^j, u, P^\theta)\,du \\
&\quad = \int_0^t \mu(\tilde{s}^j, u, V^j(u), P^\theta(u))\,du \\
&\qquad - \int_0^t \delta(u)[\pi\big(\tilde{s}^j, u, V^j(u)\big) - P^\theta(u)]\,dW^j(u) \\
&\qquad - J(\tilde{s}^j, t, V^j(t), P^\theta(t)).
\end{aligned} \tag{A30}$$

Finally, equations (A24) and (A25) imply that the unique maximum of μ in p occurs at $p = \pi$. Because $P(t) = \pi\big(\tilde{s}^j, t, V^j(t)\big)$, this implies

$$\mu(\tilde{s}^j, t, V^j(t), P^\theta(t)) \le \mu\big(\tilde{s}^j, t, V^j(t), P(t)\big). \tag{A31}$$

Combining this with equations (A28) and (A30) yields

$$\begin{aligned}
&\int_0^t [V^j(u) - P^\theta(u)]\theta(\tilde{s}^j, u, P^\theta)\, du \\
&\quad \le \int_0^t [V^j(u) - P(u)]\theta^j(\tilde{s}^j, u, P)\, du \\
&\quad - \int_0^t \delta(u)[\pi\big(\tilde{s}^j, u, V^j(u)\big) - P^\theta(u)]\, dW^j(u) \\
&\quad - J(\tilde{s}^j, t, V^j(t), P^\theta(t)).
\end{aligned} \tag{A32}$$

The regularity condition (equation (24)) guarantees the existence of the stochastic integral in equation (A32) and guarantees it is an $\mathbf{F}^j$–martingale. Furthermore, condition (32) and equation (A27) imply

$$\liminf_{t\to 1} J(\tilde{s}^j, t, V^j(t), P^\theta(t)) = \liminf_{t\to 1} \frac{1}{2\lambda(t)}\,(V^j(t) - P^\theta(t))^2 \ge 0. \tag{A33}$$

Therefore,

$$E\int_0^1 [V^j(u) - P^\theta(u)]\theta(\tilde{s}^j, u, P^\theta)\, du \le E\int_0^1 [V^j(u) - P(u)]\theta^j(\tilde{s}^j, u, P)\, du, \tag{A34}$$

as desired.

Case 2: Suppose $\phi = 1$. We will show in this case that the expected value of equation (A21) is independent of θ. The last term on the right-hand side of equation (A21) is independent of θ because $P^\theta(0) = 0$ for all θ. Furthermore, it follows from equations (A24) and (A25) and the facts that $\delta(t) = 1$ and $V^j(t) = N\tilde{s}^j$ that $\mu(\tilde{s}^j, t, V^j(t), P^\theta(t))$ does not depend on $P^\theta(t)$ and hence does not depend on θ.

Define $\pi(s, t, v) = v$. Then

$$J(s, t, v, p) = \frac{1}{2\lambda(t)}\,(v - p)^2. \tag{A35}$$

We again have $\sigma = -\delta(\pi - p)$ and, the expectation of the stochastic integral in equation (A21) is zero. Obviously,

$$\liminf_{t\to 1} J(\tilde{s}^j, t, V^j(t), P^\theta(t)) \geq 0. \tag{A36}$$

Therefore the expected profit is independent of θ and hence is trivially maximized by θ^j.

Necessity. First we argue that

$$(\forall t)\, P(t) = \operatorname*{argmax}_p \mu(\tilde{s}^j, t, V^j(t), p) \tag{A37}$$

is a necessary condition for optimality. Suppose (A37) does not hold. Then for some $\gamma \neq 0$ and some stopping time τ,

$$\mu(\tilde{s}^j, \tau, V^j(\tau), P(\tau) + \gamma) > \mu(\tilde{s}^j, \tau, V^j(\tau), P(\tau)) \tag{A38}$$

on the non-null event $\tau < 1$. Let $\varepsilon > 0$ be sufficiently small that the event $\tau \leq 1 - \varepsilon$ is non-null and let τ' denote the smaller of $1 - \varepsilon$ and

$$\inf\{t \geq \tau \mid \mu(\tilde{s}^j, t, V^j(t), P(t)) \geq \mu(\tilde{s}^j, t, V^j(t), P(t) + \gamma)\}. \tag{A39}$$

There is a strategy θ that deviates from θ^j at time τ when $\tau \leq 1 - \varepsilon$, causing the price to move toward $P + \gamma$, to stay between $P(t)$ and $P(t) + \gamma$ between τ and τ', and to return to $P(t)$ before τ', and that follows θ^j after time τ'. Because μ is a quadratic function, having the price between P and $P + \gamma$ during (τ, τ') implies that μ is larger during this interval under the deviating strategy. Because the deviation from $P(t)$ is bounded and the deviating strategy returns to θ^j before time 1, the deviating strategy is feasible, and, by increasing μ, it increases the expected profit—see equation (A21). Therefore, equation (37) is a necessary condition for strategy (33) to be optimal.

We now argue that

$$\lim_{t\to 1} P(t) - V^j(t) = 0 \quad \text{or} \quad \lim_{t\to 1} \lambda(t) = \infty \tag{A40}$$

is a necessary condition for optimality. Suppose condition (A40) is violated on a non-null event. Recall that the candidate optimum θ^j satisfies equation (36). Let $\varepsilon > 0$ and $\gamma > 1$ and consider a strategy that deviates at time $1 - \varepsilon$ from θ^j by replacing $\beta/N\delta$ in equation (36) by $\gamma\beta/N\delta$. This causes the price to be pushed more aggressively toward V^j. If J is defined either by equation (A27) or equation (A35), then this will increase $-J$ and hence, by equation (A21), increase the expected profit in the event that condition (A40) is violated. A potentially offsetting effect is that μ will be changed. However, by taking ε sufficiently small, we can make the change in the integral of μ as small as desired. Therefore, the expected profit can be increased unless condition (A40) is satisfied.

We will now derive conditions (31) and (32) from conditions (A37) and (A40).

Case 1: Suppose $\phi \neq 1$. Then $P(t) = \pi(\tilde{s}^j, t, V^j(t))$, where π is defined by equation (A26). The necessary condition for π to be the argmax of μ in p is that $2a\pi + b = 0$. Substituting from equations (A26), (A17), and (A18) and simplifying gives

$$(1-\delta)(2a\pi + b) = \left[\delta\eta + \left(\frac{1+\delta}{N} - 2\delta\right)\beta\right]v + [-N\delta\eta + (2N\delta - \delta - 1)\beta]s. \tag{A41}$$

For this to equal zero for each t and all possible values of $v = V^j(t)$ and $s = \tilde{s}^j$, it is necessary and sufficient that the coefficients of v and s equal zero. For either coefficient to be zero is equivalent to equation (31). Therefore, condition (A37) implies equation (31).

It follows immediately from equation (35) that the first part of condition (A40) implies the first part of condition (32).

Case 2: Suppose $\phi = 1$. Then $V^j(t) = N\tilde{s}^j$. Hence,

$$2a(t)p + b(\tilde{s}^j, t, V^j(t)) = \left(\eta - \frac{2(N-1)}{N}\beta\right)(p - N\tilde{s}^j). \tag{A42}$$

Therefore, condition (A37) implies either equation (31) or $P(t) = N\tilde{s}^i$. The latter is inconsistent with the fact that $P = V$ and V is a nondegenerate Gaussian process (Lemma 1). This establishes that condition (A37) implies equation (31).

In the case $\phi = 1$, $\widetilde{V}^j(t) = \tilde{v}$. Therefore, the first part of equation (A40) implies that the **F**-martingale P converges to the random variable $\tilde{v}$ (a.s.). This implies equation (32) (Revuz and Yor (1991, Theorem II.3.1)). Q.E.D.

Proof of Theorem 1: From Lemmas 4 and 5, an equilibrium is defined by α, β, and λ satisfying $\alpha = -\beta/N$, $\lambda = \beta\Sigma$, the differential equation (31), and the limit condition (32), where Σ and β are related by equation (9) and which is such that the strategy given in equation (33)is feasible for each trader j. Ignoring the feasibility condition for a moment, note that α, β, and λ are uniquely defined by Σ, which is arbitrary except for the conditions $\Sigma(0) = \mathrm{var}(\tilde{v})$, $\Sigma'(t) < 0$, and $\Sigma(t) > 0$ for all $t < 1$ (which follow from equation (9)) and the differential equation (31) and limit condition (32).

Let $\Gamma(t) = 1/\Sigma(t)$ (this should not be confused with the incomplete gamma function appearing in Section IV). The necessary and sufficient conditions for equilibrium can be written in terms of Γ as follows:

$$\beta(t) = \sqrt{\Gamma'(t)}, \tag{A43}$$

$$\alpha(t) = -\frac{\sqrt{\Gamma'(t)}}{N}, \tag{A44}$$

$$\lambda(t) = \frac{\sqrt{\Gamma'(t)}}{\Gamma(t)}, \tag{A45}$$

$$\Gamma''(t) + \left(2 - \frac{4}{N}\right)\frac{\Gamma'(t)^2}{\Gamma(t)} - \frac{2(1-\phi)}{N\phi}\frac{\Gamma'(t)^2}{\Gamma(0)} = 0, \tag{A46}$$

$$\Gamma(0) = \frac{1}{\operatorname{var}(\tilde{v})}, \tag{A47}$$

$$(\forall t)\ \Gamma'(t) > 0, \tag{A48}$$

and

$$\lim_{t\to 1} \Gamma'(t) = \infty. \tag{A49}$$

Only two of these conditions, conditions (A46) and (A49), require explanation. Condition (A49) is equivalent to condition (32), given conditions (A45) and (A48). Equation (A46) is equivalent to equation (31), given equations (A43)–(A45). To derive equation (A46) from equation (31), use $\lambda = \beta\Sigma$ (from equations (A43) and (A45)) and then equation (A43) to compute

$$\frac{d}{dt}\left(\frac{1}{\lambda(t)}\right) = -\frac{1}{\beta(t)\Sigma(t)}\left(\frac{\beta'(t)}{\beta(t)} + \frac{\Sigma'(t)}{\Sigma(t)}\right) = -\frac{\beta'(t)}{\beta(t)^2\Sigma(t)} + \beta(t), \tag{A50}$$

and rearrange equation (31) to obtain

$$-\frac{2\beta'(t)}{\beta(t)^3} = \left(2 - \frac{4}{N}\right)\Sigma(t) - \frac{2(1-\phi)}{N\phi}\Sigma(0) \tag{A51}$$

and then use the fact that the left-hand side of this is the derivative of $1/\Gamma'(t)$.

On the set of functions Γ satisfying conditions (A47) and (A48), the differential operator in equation (A46) is Lipschitz, and hence the differential equation has at most one solution subject to the conditions (A47), (A48), and (A49). This shows there is at most one equilibrium.

Assume $N > 1$ and $\phi = 1$. Then from equation (A46) we have

$$\Gamma''(t) = -\left(2 - \frac{4}{N}\right)\frac{\Gamma'(t)^2}{\Gamma(t)} \leq 0. \tag{A52}$$

Hence, Γ is a concave function and therefore cannot satisfy condition (A49). This shows there is no symmetric linear equilibrium in this case.

Suppose $N = 1$ or $\phi \neq 1$. We need to show that that equations (41) and (42) define a solution of (A43)–(A49) and that the strategy (33) is feasible for each trader j. Note that the constant κ is finite and equation (42) implies conditions (A47) and (A48). It also implies $\lim_{t\to 1}\Gamma(t) = \infty$, which implies condition (A49). Condition (A43) follows from differentiating equation (42) with respect to t. Given this, equations (A44) and (A45) obviously follow from equations (44) and (45).

Now consider equation (A46). As already noted, given conditions (A43)–(A45), the differential equation (A46) is implied by equation (A51). The left-hand side of equation (A51) is the derivative of $1/\beta(t)^2$, so, substituting for β from equation (43), what we need to show is

$$\frac{d}{dt}\left\{\frac{\Sigma(0)}{\kappa}\left(\frac{\Sigma(0)}{\Sigma(t)}\right)^{(2N-4)/N} e^{-2(1-\phi)\Sigma(0)/N\phi\Sigma(t)}\right\} = \left(2 - \frac{4}{N}\right)\Sigma(t) - \frac{2(1-\phi)}{N\phi}\Sigma(0). \tag{A53}$$

Let $x(t)$ denote the expression in braces and let $y(t)$ denote the right-hand side, so the equation can be written as $x' = y$. Directly differentiating x yields

$$x' = -xy\frac{\Sigma'}{\Sigma^2}, \tag{A54}$$

so it remains to show that

$$-\,x\frac{\Sigma'}{\Sigma^2} = 1. \tag{A55}$$

This follows immediately from differentiating equation (42).

The final issue is the feasibility of equation (33). As an $\mathbf{F}$–martingale, V is closed on the right by the random variable $\tilde{v}$. Therefore it converges almost surely to a finite limit as $t \to 1$ (Revuz and Yor (1991, Theorem II.3.1)). Hence, the equality of P and V implies equation (22) holds. The equality of P and V also allows us to compute

$$E\int_0^1 P(t)^2\,dt = E\int_0^1 V(t)^2\,dt \leq \operatorname{var}(\tilde{v}), \tag{A56}$$

the inequality being due to the fact that $V(t)$ is a conditional expectation of $\tilde{v}$. This shows that condition (24) holds. To establish condition (23), note that $V(t) - P(t)/N$ is bounded a.s., so it suffices to show that

$$\int_0^1 \beta(t)\,dt < \infty. \tag{A57}$$

Make the change of variables $x = \Sigma(0)/\Sigma(t)$. Noting that equation (9) implies

$$dx = \Sigma(0)\beta(t)^2\,dt, \tag{A58}$$

we have

$$\int_0^1 \beta(t)\,dt \propto \int_1^\infty x^{(N-2)/N} e^{-x(1-\phi)/N\phi}\,dx < \infty. \quad \text{Q.E.D.} \tag{A59}$$

Proof of Corollary 1: This is a straightforward calculation. Q.E.D.

Proof of Corollary 2: Consider first the special case $N = 2$ and $\phi = 0$. We have

$$\beta(t) = \frac{1}{\sqrt{1-t}}, \qquad \Sigma(t) = \frac{1}{1-\log(1-t)},$$

and

$$\frac{1}{\delta(t)} = 1 + \frac{1}{\Sigma(t)} = 2 - \log(1-t). \tag{A60}$$

We have used Lemma 3 to obtain $1/\delta$. The first claim is that

$$(1-t)\,\frac{\beta(t)}{N\delta(t)} < 1. \tag{A61}$$

Making the change of variable $z = \sqrt{1-t}$, the left-hand side can be written as $z - z\log z$. This is equal to one when $z = 1$ ($t = 0$) and is an increasing function of z (decreasing function of t), which establishes the claim. The second claim is that $\Sigma(t)/(1-t) > 1$. Making the change of variable $z = 1 - t$, the left-hand side is the reciprocal of $z - z\log z$, and hence is larger than one.

We now turn to the general case. Make the change of variable $z(t) = \Sigma(0)/\Sigma(t)$. Equation (42) defines z as a strictly increasing function of t. Given the formula of equation (41) for κ, it is clear that $z \to \infty$ as $t \to 1$.

The first claim is that

$$(1-t)\,\frac{\beta(t)}{\delta(t)} \to 0 \tag{A62}$$

as $t \to 1$. Using the formula for δ in Lemma 3, we can write this as

$$(1-t)\beta(t) + (1-t)\,\frac{1-\phi}{\phi}\,\beta(t)z(t) \to 0. \tag{A63}$$

The second claim is that $\Sigma(t)/(1-t) \to \infty$, which is equivalent to $(1-t)z(t) \to 0$. The final claim is that $\lambda(t) \to \infty$, which is equivalent to $\beta(t)/z(t) \to \infty$.

The last claim is easiest, so we deal with it first. From equation (43), β is proportional to a power of z multiplied by the exponential of a positive constant times z. The ratio β/z has the same property. The exponential will dominate as $z \to \infty$ $(t \to 1)$, so the ratio will converge to infinity.

The first two claims involve three terms, each of which is proportional to $(1-t)$ multiplied by $\beta(t)$ or by $z(t)$ or by both. We need to show that all of the terms converge to zero. Because both $z(t)$ and $\beta(t)$ converge to infinity as $t \to 1$, the most difficult term is the one involving both β and z. Once we have demonstrated its convergence, convergence of the other terms follows. So, what we need to show is that $(1-t)\beta(t)z(t) \to 0$.

Dividing by κ in equation (42), we have

$$\begin{aligned}
1-t &= 1 - \frac{1}{\kappa}\int_1^z x^{2(N-2)/N} e^{-2x(1-\phi)/N\phi}\,dx \\
&= \frac{1}{\kappa}\left(\kappa - \int_1^z x^{2(N-2)/N} e^{-2x(1-\phi)/N\phi}\,dx\right) \\
&= \frac{1}{\kappa}\int_z^\infty x^{2(N-2)/N} e^{-2x(1-\phi)/N\phi}\,dx.
\end{aligned} \tag{A64}$$

We have used equation (41) to eliminate κ in the last step.

The previous argument concerning β/z also applies to βz; that is, the exponential will dominate as $z \to \infty$. All we need to show is that $1-t$ converges to zero exponentially as a function of z at a more rapid rate than β converges to infinity, as $z \to \infty$. In other words, it suffices to show that $1-t$ is dominated by a multiple of e^{-az} for some constant

$$a > \frac{1}{N}\left(\frac{1-\phi}{\phi}\right). \tag{A65}$$

Let

$$\frac{2}{N}\left(\frac{1-\phi}{\phi}\right) > a > \frac{1}{N}\left(\frac{1-\phi}{\phi}\right). \tag{A66}$$

For sufficiently large x,

$$x^{2(N-2)/N} e^{-2x(1-\phi)/N\phi} < e^{-ax}. \tag{A67}$$

It therefore follows from equation (A64) that

$$\kappa(1-t) < \int_z^\infty e^{-ax}\,dx = e^{-az} \tag{A68}$$

for sufficiently large z. Q.E.D.

Proof of Corollary 3: Substituting for the equilibrium trade $\alpha(t)P(t) + \beta(t)\tilde{s}^i$ of trader i from equation (36), the expected profit is

$$E\int_0^1 (V^i(t) - P(t))[\alpha(t)P(t) + \beta(t)\tilde{s}^i]dt = \frac{1}{N} E\int_0^1 \frac{\beta(t)}{\delta(t)} (V^i(t) - P(t))^2\, dt. \tag{A69}$$

Substituting for P(t) from equation (35), this equals

$$\frac{1}{N}\int_0^1 \frac{\beta(t)}{\delta(t)} E(V^i(t) - P(t))^2\, dt = \frac{1}{N}\int_0^1 \frac{\beta(t)}{\delta(t)}\left[\frac{\delta(t)}{1-\delta(t)}\right]^2 E(V^i(t) - N\tilde{s}^i)^2\, dt. \tag{A70}$$

Because

$$V^i(t) = N\phi\tilde{s}^i + \int_0^t \beta(u)\Omega(u)\, dW^i(u), \tag{A71}$$

and the two terms in this expression are independent with zero means,

$$\begin{aligned} E(V^i(t) - N\tilde{s}^i)^2 &= E[-N(1-\phi)\tilde{s}^i]^2 + E\left[\int_0^t \beta(u)\Omega(u)\, dW^i(u)\right]^2 \\ &= N^2(1-\phi)^2 \operatorname{var}(\tilde{s}^i) + \int_0^t \beta(u)^2\Omega(u)^2\, du \qquad \text{(A72)} \\ &= \frac{(1-\phi)^2}{\phi}\Sigma(0) + \Omega(0) - \Omega(t), \end{aligned}$$

where $\Sigma(0) \equiv \mathrm{var}(\tilde{v})$. To obtain the last equality above, we use the definition (equation (2)) of ϕ and the fact that

$$\beta(u)^2 = -\frac{\Omega'(u)}{\Omega(u)^2}, \tag{A73}$$

which follows from the definition (equation (15)) of Ω. Also from equation (15), $\Omega(0) = (1-\phi)\Sigma(0)$, and from equation (20),

$$\Omega(t) = \frac{1}{\phi}\,\delta(t)\Omega(0) = \frac{1-\phi}{\phi}\,\delta(t)\Sigma(0). \tag{A74}$$

Hence,

$$\begin{aligned} E(V^i(t) - N\tilde{s}^i)^2 &= \left[\frac{(1-\phi)^2}{\phi} + 1 - \phi - \frac{1-\phi}{\phi}\,\delta(t)\right]\Sigma(0) \\ &= \frac{1-\phi}{\phi}\big(1-\delta(t)\big)\Sigma(0). \end{aligned} \tag{A75}$$

This implies the expected profit is

$$\frac{1}{N}\left(\frac{1-\phi}{\phi}\right)\Sigma(0)\int_0^1 \beta(t)\,\frac{\delta(t)}{1-\delta(t)}\,dt. \tag{A76}$$

By equation (19),

$$\frac{\delta(t)}{1-\delta(t)} = \frac{\phi}{1-\phi}\left(\frac{\Sigma(t)}{\Sigma(0)}\right). \tag{A77}$$

Therefore, the expected profit is

$$\frac{1}{N}\int_0^1 \beta(t)\Sigma(t)\,dt = \frac{1}{N}\int_0^1 \lambda(t)\,dt, \tag{A78}$$

as claimed.

To establish the last part of the corollary, make the change of variable $x = \Sigma(0)/\Sigma(t)$ in the integral $\int_0^1 \beta(t)\Sigma(t)\,dt$. From equation (9), $dx = \Sigma(0)\beta(t)^2\,dt$. Therefore, the integral equals

$$\int_1^\infty \beta\Sigma\left(\frac{1}{\Sigma(0)\beta^2}\right)dx = \int_1^\infty \frac{1}{x}\,\beta^{-1}\,dx. \tag{A79}$$

Because

$$\beta = \left(\frac{\kappa}{\Sigma(0)}\right)^{1/2} x^{-(N-2)/N} e^{x(1-\phi)/N\phi}, \tag{A80}$$

this completes the proof. Q.E.D.

REFERENCES

Attiyeh, Gregory M., 1994, A theoretical examination of volatility persistence in strategic settings, Working paper, University of Arizona.

Back, Kerry, 1992, Insider trading in continuous time, *Review of Financial Studies* 5, 387–409.

Back, Kerry, and Hal Pedersen, 1998, Long-lived information and intraday patterns, *Journal of Financial Markets* 1, 385–402.

Baruch, Shmuel, 1997, Insider trading and risk aversion, Working paper, Washington University in St. Louis.

Cao, H. Henry, 1995, Imperfect competition in securities markets with diversely informed traders, Working paper, University of California at Berkeley.

Foster, F. Douglas, and S. Viswanathan, 1996, Strategic trading when agents forecast the forecasts of others, *Journal of Finance* 51, 1437–1478.

Holden, Craig W., and Avanidhar Subrahmanyam, 1992, Long-lived private information and imperfect competition, *Journal of Finance* 47, 247–270.

Holden, Craig W., and Avanidhar Subrahmanyam, 1994, Risk aversion, imperfect competition and long lived information, *Economics Letters* 44, 181–190.

Kallianpur, Gopinath, 1980, *Stochastic Filtering Theory* (Springer-Verlag, New York).

Kyle, Albert S., 1985, Continuous auctions and insider trading, *Econometrica* 53, 1315–1335.

Protter, Philip, 1990, *Stochastic Integration and Differential Equations* (Springer-Verlag, Berlin).

Revuz, Daniel, and Marc Yor, 1991, *Continuous Martingales and Brownian Motion* (Springer-Verlag, Berlin).