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On the Inverse of the Covariance Matrix in Portfolio Analysis

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ABSTRACT

The goal of this paper is the derivation and application of a direct characterization of the *inverse* of the covariance matrix central to portfolio analysis. Such a characterization, in terms of a few primitive constructs, provides the basis for new and illuminating expressions for key concepts as the optimal holding of a given risky asset and the slope of the risk-return efficiency frontier faced by the individual investor. The building blocks of the inverse turn out to be the regression coefficients and residual variance obtained by regressing the asset's excess return on the set of excess returns for all other risky assets.

THE GOAL OF THIS PAPER is the derivation and application of a direct characterization of the *inverse* of the covariance matrix of asset returns, $\boldsymbol{C} = [\sigma_{ij}]$, central to portfolio analysis. As argued below, the specifications of $\boldsymbol{C}^{-1}$ in terms of a few primitive constructs provides new and illuminating expressions for such key concepts as the optimal holding of a given risky asset and the slope of the risk-return efficiency locus. The building blocks of the inverse turn out to be the regression coefficients and residual variance obtained by regressing the asset's excess return on the set of excess returns for all other risky assets.

After setting up the problem in Section I, the desired characterization for $\boldsymbol{C}^{-1}$ is derived in Section II. Given this result, it is possible, as shown in Section III, to rewrite the familiar equation for the optimal holding of a given risky asset as proportional to the ratio of the intercept and residual variance of the aforementioned regression. The section provides intuitively appealing interpretations for these factors: the latter as that part of the asset's variance that is unhedgeable or nondiversifiable, the former as the asset's excess expected return over that of the linear combination of other assets that minimizes its unhedgeable variance. Section IV concludes the paper with a summary of its findings.

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I. Preliminaries

It is well known that every mean-variance optimizing investor will choose a portfolio falling on his or her risk-return efficiency frontier—the locus of portfolios of minimum variance conditional on a given expected return.[1] As illustrated below, $\boldsymbol{C}^{-1}$ is an important determinant of any portfolio on the frontier and of the risk-return trade-off along the frontier.

Assume markets for N risky assets, each with stochastic return $\tilde{r}_i$ and expected return $\bar{r}_i$, along with the opportunity for unlimited lending and borrowing of a riskless asset with return r_f. Let $\bar{\boldsymbol{m}}$ be the $N \times 1$ vector of the excess expected returns of each asset over the risk-free rate, with elements $\bar{r}_i - r_f$; and let $\boldsymbol{z}$ be a vector of nominal security holdings, with elements z_i. Minimizing the portfolio variance, $\boldsymbol{z}'\boldsymbol{C}\boldsymbol{z}$, subject to a predetermined portfolio expected return, yields the following expression for the vector of optimal holdings, $\boldsymbol{z}^*$, along the efficiency frontier:

$$\boldsymbol{z}^* = \lambda \boldsymbol{C}^{-1}\bar{\boldsymbol{m}}/2. \tag{1}$$

Although $\boldsymbol{z}^*$ depends on the unknown Lagrange multiplier λ—and, therefore, generally on the investor's utility function and the predetermined expected return—the *ratios* of the z_i^*s are preference-free and depend only on the investor's estimates of expected excess returns and, once again, the elements c_{ij}^{-1} of $\boldsymbol{C}^{-1}$ (Tobin's (1958) famous portfolio separation theorem). Finally, the trade-off between risk and expected return along the frontier also involves the inverse: $(\bar{\boldsymbol{m}}'\boldsymbol{C}^{-1}\bar{\boldsymbol{m}})^{1/2}$.

II. Derivation

The derivation of $\boldsymbol{C}^{-1}$ below adapts a useful partitioning technique used by Anderson and Danthine (1981) in their study of hedging in futures markets. Partition the set of the N first-order conditions leading to equation (1), above, between the first equation and an N-1 equation block; in matrix notation, the partitioned system appears as follows:

$$\begin{bmatrix} \sigma_{11} & \boldsymbol{\sigma}_{1j} \\ \boldsymbol{\sigma}_{j1} & \boldsymbol{C}_{N-1} \end{bmatrix} \begin{bmatrix} z_1 \\ \boldsymbol{z}_{N-1} \end{bmatrix} = \frac{\lambda}{2} \begin{bmatrix} \bar{m}_1 \\ \bar{\boldsymbol{m}}_{N-1} \end{bmatrix}, \tag{2}$$

where the scalars σ_{11}, z_1, *and* $\bar{m}_1$ are, respectively, the variance, asset level, and expected excess return for asset 1; $\boldsymbol{\sigma}_{1j}$ is the $1 \times N-1$ row vector of covariances between the first asset and the $N-1$ other assets, and $\boldsymbol{\sigma}_{j1}$ is its transpose. The matrix $\boldsymbol{C}_{N-1}$ in the bottom block is the $N-1$ square sub-

[1] See Elton and Gruber (1987), Copeland and Weston (1983), or Mossin (1973) for clear expositions of the points made in this section.

matrix of the covariance matrix $\boldsymbol{C}$ formed by eliminating its first row and column; finally, $\boldsymbol{z}_{N-1}$ and $\bar{\boldsymbol{m}}_{N-1}$ are the $N-1$ column vectors made up of all but the first elements of the original $\boldsymbol{z}$ and $\bar{\boldsymbol{m}}$ vectors, respectively.

Standard results on partitioned matrix inversion indicate that the inverse, with elements $\boldsymbol{A}_{ij}$, can be partitioned similarly to $\boldsymbol{C}$ in equation (2).[2] Multiplying the first row of the inverse by the second column of $\boldsymbol{C}$, we can solve for $\boldsymbol{A}_{12}$:

$$\boldsymbol{A}_{12} = -\boldsymbol{A}_{11}\boldsymbol{\sigma}_{1j}\boldsymbol{C}_{N-1}^{-1}. \tag{3}$$

Note that the $1 \times N-1$ matrix $\boldsymbol{A}_{12}$ equals the (scalar) inverse element $-\boldsymbol{A}_{11}$ times a term that is the row vector of *regression* coefficients, $\boldsymbol{\beta}_1'$, the result of regressing the returns from asset 1 on those of all the other $N-1$ risky assets. Using standard results on partitioned inverses, we can solve for $\boldsymbol{A}_{11}$:

$$\boldsymbol{A}_{11} = (\sigma_{11} - \boldsymbol{\sigma}_{1j}\boldsymbol{C}_{N-1}^{-1}\boldsymbol{\sigma}_{j1})^{-1}. \tag{4}$$

The last term inside the inverse can be shown to be equal to $\sigma_{11}R_1^2$, where the latter term is the *multiple regression coefficient* for the regression, noted above, of the returns from the first asset on those for all other assets. $\boldsymbol{A}_{11}$ thus becomes the reciprocal of $\sigma_{11}(1 - R_1^2)$—that part of the variance of asset 1 that is *unexplained* by the regression—that is, the variance of the residual or the error term of that regression, $\sigma_{\epsilon_1}^2$. After solving for $\boldsymbol{A}_{11}$ and substituting the result in equation (3), we have the final expression for the first row and, by symmetry, the first column of $\boldsymbol{C}^{-1}$:

$$\boldsymbol{A}_{11} = \frac{1}{\sigma_{11}(1 - R_i^2)} = \frac{1}{\sigma_{\epsilon_1}^2}, \tag{5}$$

$$\boldsymbol{A}_{12} = -\boldsymbol{A}_{11}\boldsymbol{\beta}_1' = \frac{-\boldsymbol{\beta}_1'}{\sigma_{11}(1 - R_1^2)} = \frac{-\boldsymbol{\beta}_1'}{\sigma_{\epsilon_1}^2}. \tag{6}$$

Although equations (5) and (6) are only part of $\boldsymbol{C}^{-1}$, they are sufficient, in conjunction with equation (1), to obtain the final expression for the optimal level of the first asset:

$$z_i^* = \left[\frac{1}{\sigma_{11}(1 - R_1^2)}\right]\frac{\lambda\bar{m}_1}{2} + \sum_{j=2}^{N}\left(\left[-\frac{\beta_{1j}}{\sigma_{11}(1 - R_1^2)}\right]\frac{\lambda\bar{m}_j}{2}\right). \tag{7}$$

[2] See, for example, Theil (1971, pp. 16–19), and, especially, Judge et al. (1985, p. 947) for the results on partitioned matrix inversion needed to derive equations (3) and (4).

In determining the remaining elements of $\boldsymbol{C}^{-1}$, rather than focusing on solving for $\boldsymbol{A}_{22}$, we can exploit equation (7) and the fact that the choice of a particular asset as the first or "y" variable is clearly an arbitrary one. Let us permute the rows and columns of $\boldsymbol{C}$ to move the moments of some other asset i to the first row and column—noting that the overall solution to this system of equations is unchanged. By repeating the partitioned inversion detailed above, we can derive an expression for the first element of the new system, which must be equal to the ith element, z_i^*, of the original system; this expression will be identical in form to equation (7) with subscripts i substituted for 1.

We can show that these new coefficients are indeed the sought-after elements of the ith row and column of $\boldsymbol{C}^{-1}$ by comparing the above expression for z_i^* to the alternative implied by equation (1), above. Since the excess expected returns, $\bar{m}_i$ and $\bar{m}_j$, may assume any value, we have the immediate implication that for any row i in $\boldsymbol{C}^{-1}$, c_{ii}^{-1} must equal the reciprocal of $\sigma_{ii}(1-R_i^2)$ and c_{ij}^{-1} must equal $-\beta_{ij}/\sigma_{ii}(1-R_i^2)$.

Collecting results, the direct characterization of $\boldsymbol{C}^{-1}$ for the general case becomes:[3]

$$\begin{bmatrix} \dfrac{1}{\sigma_{11}(1-R_1^2)} & -\dfrac{\beta_{12}}{\sigma_{11}(1-R_1^2)} & \cdots & -\dfrac{\beta_{1N}}{\sigma_{11}(1-R_1^2)} \\ -\dfrac{\beta_{21}}{\sigma_{22}(1-R_2^2)} & \dfrac{1}{\sigma_{22}(1-R_2^2)} & \cdots & -\dfrac{\beta_{2N}}{\sigma_{22}(1-R_2^2)} \\ \vdots & \vdots & \ddots & \vdots \\ -\dfrac{\beta_{N1}}{\sigma_{NN}(1-R_N^2)} & -\dfrac{\beta_{N2}}{\sigma_{NN}(1-R_N^2)} & \cdots & \dfrac{1}{\sigma_{NN}(1-R_N^2)} \end{bmatrix}. \tag{8}$$

[3] There are at least two questions on which one might want further verification: (1) proof that the inverse matrix (8) is indeed symmetric; and (2), further evidence that $\boldsymbol{CC}^{-1} = \boldsymbol{I}$. Neither is immediately obvious by multiplying the various rows and columns of the two matrices because we are forced to multiply estimated coefficients that relate to different regressions. However, one can relate the elements in matrix (8) to the standard definitions of the elements of $\boldsymbol{C}^{-1}$, the determinant $|\boldsymbol{C}|$ and the various cofactors of the elements in $\boldsymbol{C}$, $\boldsymbol{COF}_{ij}$. For example, by adapting the results of Johnston (1972, pp. 132 ff.), one can show that a diagonal element in matrix (8) equals $\boldsymbol{COF}_{ii}/|\boldsymbol{C}|$, where the first term is the cofactor of the diagonal element $\boldsymbol{c}_{ii}$. Moreover, the regression coefficient, β_{ij}, can be shown to equal $-\boldsymbol{COF}_{ij}/\boldsymbol{COF}_{ii}$. Putting these two results together, the element $\boldsymbol{c}_{ij}^{-1}$ equals $\boldsymbol{COF}_{ij}/|\boldsymbol{C}|$; similarly, we have $\boldsymbol{c}_{ji}^{-1} = \boldsymbol{COF}_{ji}/|\boldsymbol{C}|$. However, since $\boldsymbol{C}$ is symmetric, we also have $\boldsymbol{COF}_{ij} = \boldsymbol{COF}_{ji}$, and the two elements are equal—verifying the symmetry of the inverse.

The above expressions for the elements of $\boldsymbol{C}^{-1}$ facilitate showing that $\boldsymbol{CC}^{-1} = \boldsymbol{I}$. Recalling that by symmetry the columns are identical to the corresponding rows of the inverse, when multiplying the ith row of $\boldsymbol{C}$ with the ith column of the inverse, one gets the sum of products of the elements in the ith row of $\boldsymbol{C}$ each multiplied by its corresponding cofactor, all divided by $|\boldsymbol{C}|$—the net result being 1. All off-diagonal elements of $\boldsymbol{CC}^{-1}$ must be equal to zero, since they involve an expansion by alien cofactors.

III. Implications for Asset Holdings

On first glance, the inverse matrix (8), that is so central in equation (1) for the optimal holdings of risky assets, seems to contain a welter of intriguing but not particularly illuminating terms. It turns out, however, that these terms combine to yield understandable and intuitively attractive expressions for the optimal holding of a given asset, both in special cases and in the general case with arbitrary, nonzero covariances.

Consider first the special case of independent returns. With all off-diagonal elements of $\boldsymbol{C}$ equal to zero for this case, the inverse matrix (8) is also diagonal, with each element equal to the reciprocal of a given asset's variance, $1/\sigma_{ii}$. Using the implication of equation (1) that asset holdings are proportional to $\boldsymbol{C}^{-1}\bar{\boldsymbol{m}}$, the holding for any asset i for the independence case must be proportional to the ratio of its excess expected return to its variance, $\bar{m}_i/\sigma_{ii}$.

Although ostensibly much more complicated, the general case, in fact, is a natural generalization of the independence case. Let us consider in turn the denominator and numerator of the expression implied by equation (1) and the general form of the inverse matrix (8). Since equation (1) shows that the holding of risky asset i is proportional to the vector product of the ith row of $\boldsymbol{C}^{-1}$ with the column vector of excess expected returns, the denominator of the expression becomes $\sigma_{ii}(1 - R_i^2)$ instead of σ_{ii} in the independence case. The squared multiple regression coefficient appearing in the denominator, R_i^2, equals that maximum percentage of the variance of the return of asset i that can be explained by a linear combination of the returns of all other available risky assets; because the optimal linear combination minimizes the residual variance, $\sigma_{\epsilon_i}^2$, it is easily shown that the denominator is the minimum *nondiversifiable* or *unhedgeable* part of asset i's variance.[4] The coefficients of this optimal linear combination are calculated via a least-squares regression. In other contexts, this optimal combination has also been called the *pure hedge* or a *regression hedge.*[5]

[4] Consider the "portfolio" formed by a dollar in asset i and the amount $-\beta_{ik}$ in each of the other assets k, where β_{ik} is the appropriate coefficient, appearing in the inverse matrix (8), from the multiple regression of the excess return for the ith asset on the excess returns of all the other assets. By definition, for any sample period the value or observed return of this "portfolio" will be the residual from the least-squares multiple regression defined above, ϵ_i. Since a property of the regression is the minimization of the variance of this residual over the sample period, or the maximization of the explanation of the variance of the return of asset i, no other linear combination of these asset returns can reduce this residual variance further. The variance of this "portfolio" equals the variance of the residual from the multiple regression, $\sigma_{\epsilon_i}^2$, which also equals $\sigma_{ii}(1 - R_i^2)$.

[5] In Anderson and Danthine's 1981 study of hedging in futures markets, the optimal linear combination balancing their "cash" position was denoted as the pure hedge (p. 1187). In an international setting, Adler and Dumas (1980) identify an asset's currency risk exposure as a coefficient in a particular linear regression. Simultaneously and independently, Sercu (1980) developed related notions of regression hedges.

The numerator in the general expression, instead of $\bar{m}_i$ in the independence case, becomes $\bar{m}_i - \sum_{k \neq i} \beta_{ik} \bar{m}_k$. This, too, has intuitive appeal as soon as one recalls that the regression line to which the β_{ik}'s apply passes through the point of the means. Letting α_i equal the intercept of the regression for the ith asset, we have that $\alpha_i = \bar{m}_i - \sum_{k \neq i} \beta_{ik} \bar{m}_k$; hence the numerator of the general expression is equal to the *intercept* of this regression equation. As such, the numerator equals that part of the expected excess return of asset i that *cannot* be accounted for by the excess expected return of the *same* linear combination of assets that minimizes the residual variance of asset i's return—that is, the numerator equals the difference between $\bar{r}_i - r_f$ and the expected costs of the optimal hedge.

Thus, as contrasted with the raw or unadjusted expected returns and variances that determine optimal asset holdings in the independence case, the expression for holdings in the general case uses the same concepts, but in *adjusted* form—adjusted for that part of the asset's expected excess return and variance that can be explained by the optimal linear combination of other risky assets:[6]

$$z_i = (\lambda/2) \frac{\bar{m}_i - \sum_{k \neq i} \beta_{ik} \bar{m}_k}{\sigma_i^2 (1 - R_i^2)} = (\lambda/2) \frac{\alpha_i}{\sigma_{\epsilon_i}^2}. \tag{9}$$

IV. Summary

This paper derives and applies the inverse of the covariance matrix central to portfolio analysis. As shown in matrix (8), $\boldsymbol{C}^{-1}$ is composed of two key elements: (1) the set of coefficients obtained by regressing the excess return for a given asset on the excess returns of all other risky assets; and (2) the residual variance for that regression, which is equal to the nondiversifiable or unhedgeable part of each asset's variance of return $[\sigma_i^2(1 - R_i^2)]$. It is of some interest to note that *everything* in $\boldsymbol{C}^{-1}$ relates to the characteristics of the N regressions that minimize each asset's residual variance, which, for good reason, may be termed the optimal hedge regressions.

Knowledge of the inverse matrix leads to equation (9), an illuminating expression for the optimal holding of any given asset i. The numerator is proportional to the difference between asset i's expected excess return and the expected excess return of its optimal hedging combination (the intercept of its optimal hedge regression). The denominator is that part of asset i's variance that cannot be diversified away (the residual variance of the optimal hedge regression).

[6] The referee has pointed out that for practical purposes, such as determining the optimal holdings of an asset i that is perceived to be mispriced, the right-hand side of equation (9) allows a particularly compact expression in terms of just two characteristics of the associated linear regression.

Knowledge of $\boldsymbol{C}^{-1}$ also can contribute to the analysis of questions beyond those addressed in this paper.[7] For example, shifts in the investor's risk-return frontier, either because of changes in the underlying covariances or because of the introduction of new assets, can now be analyzed explicitly. And the notion of an asset's nondiversifiable risk, so important to $\boldsymbol{C}^{-1}$, can be related to the concept of its systematic risk, developed in the context of the Capital Asset Pricing Model.

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[7] See Stevens (1997) for work on the questions noted below.