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Time and the Process of Security Price Adjustment

DAVID EASLEY and MAUREEN O'HARA*

ABSTRACT

This paper delineates the link between the existence of information, the timing of trades, and the stochastic process of prices. We show that time affects prices, with the time between trades affecting spreads. Because the absence of trades is correlated with volume, our model predicts a testable relation between spreads and normal and unexpected volume, and demonstrates how volume affects the speed of price adjustment. Our model also demonstrates how the transaction price series will be a biased representation of the true price process, with the variance being both overstated and heteroskedastic.

FEW TOPICS IN FINANCE are of broader interest than the time series properties of security prices. Fundamental to research on such diverse topics as security returns, market efficiency, investor trading strategies, option behavior, and security market design, the stochastic process of prices underlies much of the phenomena studied in financial economics. But how the stochastic process of prices behaves, or even what factors determine the movement between one security price and the next remains unclear. These theoretical questions have spurred extensive research on security price formation, much of it in the large, and growing area of security market microstructure.

The microstructure literature investigates how prices evolve by analyzing how traders learn from market data. This focus allows researchers to characterize the time series properties of prices as a function of the information trades reveal to the market. In the standard microstructure models, however, time *per se* plays no role. In the Kyle (1985) framework, for example, all trades are batched so that when individual orders arrive is not relevant (or even known) to the market maker. Similarly, in the Glosten and Milgrom (1985) sequential trade model, orders are assumed to arrive in some probabilistic fashion which is independent of any time parameters. In these models, the timing of trades is irrelevant for the behavior of prices because time itself has no information content.

*Easley is from the Department of Economics and O'Hara is from the Johnson Graduate School of Management, both at Cornell University. We would like to thank Doug Diamond, Joel Hasbrouck, Eric Hughson, Murugappa Krishnan, Andy Lo, and seminar participants at Boston College, Cornell, and the Wharton School for helpful comments. We would also like to acknowledge the extremely helpful comments on an anonymous referee. An earlier version of this paper was presented at the Western Finance Association Meetings, June 1990.

This specification makes sense if time is exogenous to the price process. But if time can be correlated with any factor related to the value of the asset, then the presence or absence of trade may provide information to market participants. In this paper we demonstrate that this correlation can arise from properties of the underlying information structure. In our model, traders learn from both trades and the lack of trades because each may be correlated with different aspects of information. In particular, while trade provides signals of the direction of any new information, the lack of trade provides a signal of the existence of any new information.

This latter effect we define as event uncertainty and it reflects the difficulty that uninformed traders face in even knowing whether new information exists. In the standard sequential trade framework used by Glosten and Milgrom (1985) and in Kyle (1985) this event uncertainty does not arise because an information event is assumed to have occurred. If information events are not certain, however, then whether trade occurs at all may provide a signal to the market. This suggests that the intervals between trades may have information content, and hence time *per se* is not exogenous to the price process.

The intuition that the absence of trade could provide information to market participants and that this induces a bias in transaction prices is not new to this paper. Diamond and Verrecchia (1987) used this insight to explain how short sale constraints could impart information to no-trading intervals, and hence affect the speed with which prices reflected adverse information. Moreover, the basic sequential trade approach we apply to develop our results is also not unique. Where the contribution of our paper lies is in delineating the link between the existence of information, the timing of trades, and the stochastic process of security prices.

Our results suggest that this link can explain a number of interesting phenomena in security price behavior. We show that time itself affects prices; while trades can cause price quotes to move, so too can periods of nontrading outcomes. Our model predicts that spreads will depend on the time between trades, with spreads decreasing as this time increases. Because the absence of trades will be correlated with volume, our model also predicts the relationship between spreads and both normal and unexpected volume. We demonstrate that these variables also affect the speed with which prices adjust to new information, yielding insights into how it is that markets become efficient.

Perhaps most important, our model provides a characterization of the underlying stochastic process of prices. Because event uncertainty is reflected in the intensity of trades, the sequence of trades provides information beyond that conveyed by individual transactions. For example, in our model, two sell transactions have very different information content if they occur contiguously in time than if they occur an hour apart. During the intervals between transactions, market makers (and market participants) may revise their beliefs about the value of the asset, a revision that will not be reflected in transactions prices until a trade occurs. But since the timing of trades is endogenous (depending in part on the existence of new information), the transactions price series will be a biased representation of the underlying "true" price process.

These biases have important implications for analyses not only of transactions data but for any data series in which nonsynchronous trading is present. Because time is not exogenous to the transactions price process, transactions prices cannot be Markov processes. Trades in our model are serially correlated because the probability of a trade, and thus a price observation, is positively correlated with volume. This has the important implication that the variance of the transaction price yields an overestimate of the variance of the true price process. Further, both the conditional value process and the transactions price process have nonconstant variances. These results may be of interest both to researchers using transactions data, and to the growing literature on security and option volatility.

In the next section we present a sequential trade model of security price formation that incorporates the effect of event uncertainty. In Section II we analyze the role of the time between trades, and characterize how time affects price quotes and spreads. We also examine the interaction of volume, quotes, and prices. Section III characterizes the stochastic process of prices, and in particular examines how the speed of price adjustment differs with respect to market parameters. Section IV then examines the implications of our results for empirical research. The final section discusses some extensions of our work.

I. The Model

We consider a sequential trade model similar to that of Glosten and Milgrom (1985) or Easley and O'Hara (1987). In this model, potential buyers and sellers trade an asset with a market maker who is responsible for quoting prices to buy and sell. Because we are interested in the effect of information on prices, we assume that the market maker is risk neutral and acts competitively. This assumption rules out any direct inventory effects on the market maker's prices, but does retain any information effects of inventory. For simplicity, we focus on the actions of a single market maker, but our assumption of competitive behavior implies the existence of at least potential competitors.[1]

We consider an asset whose eventual value is represented by a random variable V. We define an information event as the occurrence of a signal ψ about V. The signal can take on one of two values, L and H, with probabilities $\delta > 0$ and $1 - \delta > 0$. We let the expected value of the asset conditional on the signal be $E[V \mid \psi = L] = \underline{V}$ or $E[V \mid \psi = H] = \overline{V}$. If no information event has occurred, we denote this as $\psi = 0$ and the expected value of the asset simply remains at its unconditional level $V^* = \delta\underline{V} + (1 - \delta)\overline{V}$.

[1] More detailed discussion of these assumptions and their implications are given in Easley and O'Hara (1987).

In our model, information events need not occur, reflecting the realistic specification that since uninformed market participants do not receive any signals they may also not know whether any new information even exists.[2] If new information always exists (as is the case in the Glosten-Milgrom model), then the uninformed would know implicitly that others knew more and could act accordingly. Of course, in actual market settings if new information is known to exist it is a common practice to halt trading until the information is publically disseminated. Similarly, if new information never arose, then the issue of some traders having superior information is moot.

We capture this more natural "event uncertainty" that surrounds the existence of private information by assuming that the probability that an information event has occurred before the start of the current trading day is α, where $0 < \alpha < 1$. We then analyze the behavior of quotes and prices throughout the "day." Certainly, our specification of a day is arbitrary. In active markets, prices could adjust to new information in minutes and new information events could occur quite frequently. In inactive markets, there may not even be a single trade on some days. As our focus is on the effect of event uncertainty on price adjustment, what matters for our analysis is the learning problem confronting market participants. This is most easily characterized by adopting the fiction of a trading day and assuming that information events occur only between trading days. As we discuss in Section V, extensions to this simple framework are certainly possible and may lead to additional interesting insights.

Trade in this market can arise from uninformed and/or informed traders. We assume that informed traders are risk neutral and take prices as given. This assumption rules out strategic behavior by informed traders, but may be realistic given the trading mechanism and the potential existence of multiple informed traders. The uninformed's behavior is more problematic. We assume that some of the uninformed trade for liquidity reasons arising from the timing of consumption or portfolio considerations.[3] There may be other uninformed traders, however, whose demands reflect more complex motivations such as price sensitivity or individual-specific trading rules. These factors may influence the willingness of any uninformed trader to trade at any specific time or price. For the uninformed as a whole, we assume that a fraction γ are potential sellers and a fraction $1 - \gamma$ are potential buyers. If at time t an uninformed buyer checks the quote, the probability that he will

[2] In actual markets, this uncertainty over whether there is any new information is reflected in the existence of the Dow-Jones Rumor Wire. As its name suggests, the Rumor Wire prints rumors of new information. Since uninformed traders will lose to traders who have private information, the rumor wire essentially reflects the event uncertainty we model here.

[3] The presence of traders who are uninformed is necessary for the existence of trade by a rational market maker. If everyone who wants to trade with the market maker has superior information and is trading for speculative purposes, then the market maker loses on any trade he completes. Similarly, the uninformed, if they are rational, must not be trading solely for speculation.

trade is $\epsilon^B > 0$, with an uninformed seller's trading probability defined similarly as $\epsilon^S > 0$.[4]

Our assumptions on risk neutrality and competitive behavior for the market maker dictate that the market maker's price quotes yield zero expected profit conditional on a trade at the quote. Since informed traders will profit at the market maker's expense, the probability that a trade is actually information-based is clearly important for determining these prices. We assume that if an information event occurs the market maker expects the fraction of trades made by the informed to be μ.[5] This fraction of trades need not reflect the fraction of the trader population that observes the signal. Indeed, it may be that, following an information event virtually all trades come from informed traders, in which case μ will be close to one. However, specifying the mechanics of the order arrival process requires modelling both individual trader behavior and any frictions which might be present in the trading mechanism. The difficulty of this task led Glosten and Milgrom (1985) to adopt the simplifying convention of an exogenous arrival process whose parameters correspond to a simple probabilistic structure. We also adopt this structure. This framework is clearly an oversimplification, but since the probabilities can be viewed as the outcomes of the underlying problem, it does provide a reasonable way to characterize the arrival process. For our analysis here, we require that $0 < \mu < 1$. The case of $\mu = 1$ is easy to analyze (we address it in footnotes) and although the statement of some of our results change, none of the intuition is affected.

Trade occurs throughout the trading day. We divide the trading day into discrete intervals of time denoted $t = 1, 2, \cdots$. Each time interval is long enough to accommodate at most one trade.[6] This timing specification is

[4] We can also allow explicit price dependence in the uninformed's demands by having the probability of trade be a function of price or price and the expected value of the asset. With continuity, slope, and boundary conditions we can show via a fixed point theorem that a unique equilibrium exists. However, this generalization adds to the complexity of the analysis and it reduces our ability to characterize the equilibria. Further, in an asymmetric information economy, demand functions can be very badly behaved. Unless attention is restricted to special examples demand need not be downward sloping or even continuous. Our point of view is that over the relevant range of prices ($\underline{V}$, $\overline{V}$) price sensitivity of aggregate demand is probably not an important issue. In any case, since it is not clear what assumptions on price sensitive demand are reasonable, we have chosen to present the analysis with simple random demand.

[5] There is an alternative version of the details of the information and trading process which is consistent with our reduced form model. Suppose that if an information event occurs then at each time one trader selected at random becomes informed with probability μ. Upon becoming informed the trader makes a trade and leaves. If there has been no information event or if, as happens with probability $1 - \mu$, no one was selected to see the information, the trader who checks the quote is uninformed. This interpretation results in the same reduced form as in the text and so it generates the same analysis.

[6] The exact length of a trading interval is clearly arbitrary in our model. Inspection of the Institute for the Study of Securities Markets transactions data suggests that trades rarely occur more frequently than every five seconds and so this could be a reasonable specification. Since trading frequency differs dramatically between markets, however, for empirical work the appropriate length of an interval may differ across markets.

similar to that of Diamond and Verrecchia (1987) and allows us to capture the possibility that during some intervals no trades may occur.

In our model, trade takes place in a sequential fashion with traders randomly selected to trade according to the probabilities given above. In particular, at each time t the market maker announces the bid and ask prices at which he is willing to trade one unit of the asset. Similarly, at each time t a trader is selected to trade and has the option of buying one unit of stock at the market maker's ask price, selling one unit at the market maker's bid price, or not trading at all.

If the trader selected is an informed trader then she will buy if she has seen a high signal and the ask price is below $\overline{V}$; she will sell if she has seen a low signal and the bid price is above $\underline{V}$. Note that since the informed trader is risk neutral, she will always transact provided that prices are not at their full information value. If the trader selected is an uninformed trader, then whether he buys, sells, or doesn't trade at all depends on the trader's type and motivation for trading.

This trading structure can be understood most easily by reference to the tree diagram given in Figure 1. In our model, at the first node nature selects whether an information event occurs. If there is an information event, then the type of signal (either L or H) is determined at the second node. These two nodes are reached only at the beginning of the day. From this point, traders are selected at each time t to trade based on the probabilities described above. Thus, if an information event has occurred, an informed trader is selected with probability μ, and she then chooses either to buy or sell. Similarly, with probability $(1 - \mu)$ an uninformed trader is selected and he may choose to buy, sell, or not trade. If no information event has occurred, then all traders are uninformed and the trader selected may choose to buy, sell, or not trade with the indicated probabilities. For trade in the next time interval, only the trader selection process is repeated, so the game proceeds from the right of the dotted line on the tree diagram. This continues throughout the day.

There are two points to note about the differences between this structure and the approach found in other sequential trade models. First, the addition of the event uncertainty adds another "state" to the underlying game analyzed in Glosten and Milgrom (1985), the implications of which are discussed in more detail in Easley and O'Hara (1987). For our analysis here, what matters is how this affects the trade outcomes. As the diagram indicates, if there is no information event, then all trades are actually from the uninformed. Second, in our model the probability of a trader arriving in the next period is one. This differs (in interpretation, but not in implications) from the specification of Diamond and Verrecchia (1987) who assume that there is a $1 - \gamma$ $(\gamma > 0)$ probability that no trader will arrive. Our specification allows the trading intensity of the informed (the μ) to be quite large, while still retaining the ambiguity over the underlying information structure for the uninformed market participants.

Given this market structure, it may be that no trade actually occurs in some time interval. This can occur in our model only when an uninformed

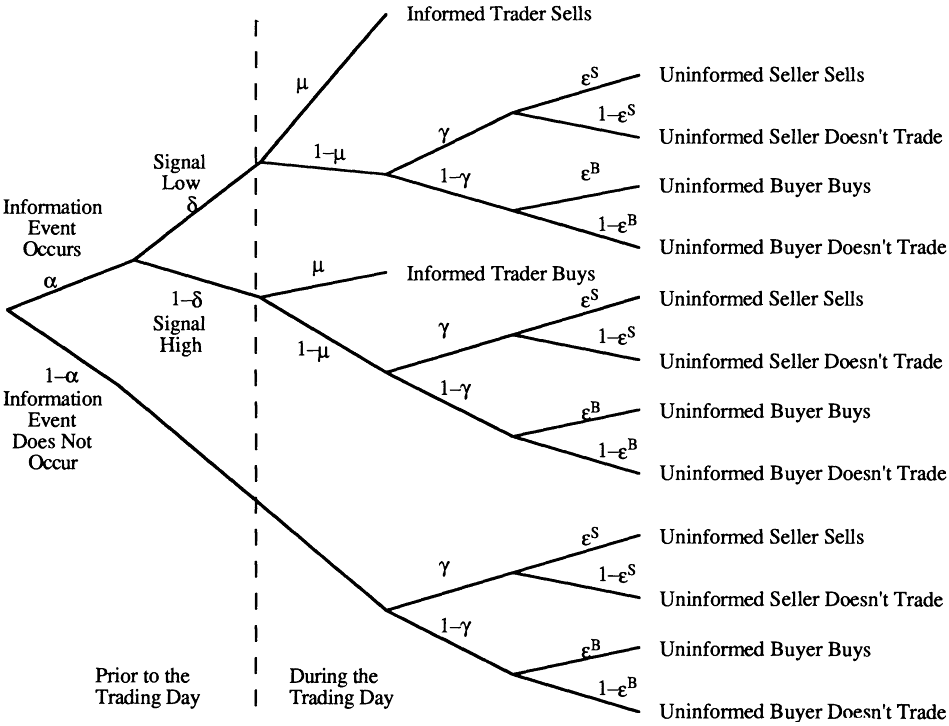

Figure 1. Tree Diagram of the Trading Process. α is the probability of an information event, δ is the probability of a low signal, μ is the probability that the trade comes from an informed trader, γ is the probability that an uninformed trader is a seller, and $\epsilon^S(\epsilon^B)$ is the probability that the uninformed trader will actually trade. Nodes to the left of the dotted line occur only at the beginning for the trading day; nodes to the right are possible at each trading interval.

trader checks the quotes and decides for portfolio reasons (as captured by the ϵ^S and ϵ^B probabilities) not to trade. Notice that this can occur both when there has been an information event and when there has not (since the no trade outcome can be found at the ends of each of these two branches). Hence, a no-trade observation does not in itself reveal whether there has been an information event.

What is important to stress, however, is that this no-trade outcome is more likely to occur when there is no new information. If there is no information event, then the probability of no trade is $\gamma(1 - \epsilon^S) + (1 - \gamma)(1 - \epsilon^B)$. Conversely, given that an information event has occurred, this probability falls to $(1 - \mu)[\gamma(1 - \epsilon^S) + (1 - \gamma)(1 - \epsilon^B)]$ because now there are both informed and uninformed traders in the market. A market maker observing a no-trade outcome, therefore, must consider the possibility that the lack of trade may signal that no new information exists.

A final issue to be addressed is the evolution of prices throughout the day. The market maker and the uninformed traders are Bayesians who know the structure of the market. What they do not know is whether an information event has occurred, whether it is good or bad news given that it has occurred, or whether any particular trader is informed. Each market participant can watch the market, however, and observe all trading activity. Over time, this allows the market maker (and the uninformed) to learn about the first two unknowns and revise their beliefs. It is this revision that causes quotes, and thus prices, to adjust.

We outline this quote-setting process for the first trade of the day. To determine his price quotes, the market maker must calculate the expected value of the asset conditioned on the type of trade that can occur. This requires determining the conditional probability of the low value $\underline{V}$. If no signal has occurred, this probability remains unchanged at δ. If an information event has occurred then $\Pr\{V = \underline{V}\}$ is one if the signal is low and zero if the signal is high. The market maker's updating formula is thus:

$$\delta(Q) = \Pr\{V = \underline{V} \mid Q\} = 1 \cdot \Pr\{\psi = L \mid Q\} + 0 \cdot \Pr\{\Psi = H \mid Q\} + \delta \Pr\{\Psi = 0 \mid Q\}, \tag{1}$$

where Q denotes the trade outcome. As the market maker is Bayesian, these conditional probabilities are given by Bayes rule:

$$\Pr\{\Psi = X \mid Q\} = \frac{\Pr\{\Psi = X\}\Pr\{Q \mid \Psi = X\}}{\Pr\{\Psi = L\}\Pr\{Q \mid \Psi = L\} + \Pr\{\Psi = H\}\Pr\{Q \mid \Psi = H\} + \Pr\{\Psi = 0\}\Pr\{Q \mid \Psi = 0\}}. \tag{2}$$

The explicit probabilities can be derived from the tree diagram given in Figure 1. Hence, to calculate the probability that a low signal occurred given a sale, $\Pr\{\psi = L \mid S\}$, note that $\Pr\{\psi = L\} = \alpha\delta$ and that $\Pr\{S \mid \psi = L\} = (\mu + (1 - \mu)\gamma\epsilon^S)$. The probabilities $\Pr\{S \mid \psi = H\}$ and $\Pr(S \mid \psi = 0\}$ can be calculated similarly so that:

$$\Pr\{\Psi = L \mid S\} = \frac{\delta(\alpha\mu + \alpha(1 - \mu)\gamma\epsilon^S)}{(\delta\alpha\mu + (1 - \alpha\mu)\gamma\epsilon^S)}. \tag{3}$$

To calculate $\Pr\{\psi = 0 \mid S\}$, note that $\Pr\{\psi = 0\} = 1 - \alpha$ and $\Pr\{S \mid \psi = 0\} = \gamma\epsilon^S$, so

$$\Pr\{\Psi = 0 \mid S\} = \frac{(1 - \alpha)\gamma\epsilon^S}{(\delta\alpha\mu + (1 - \alpha\mu)\gamma\epsilon^S)}. \tag{4}$$

The market maker's conditional δ given a sale, therefore, is then:

$$\delta_1(S_1) = \delta\left[\frac{\alpha\mu + \epsilon^S\gamma(1 - \alpha\mu)}{\delta\alpha\mu + \epsilon^S\gamma(1 - \alpha\mu)}\right] > \delta. \tag{5}$$

As is apparent, the market maker increases the probability he attaches to $\underline{V}$ given that someone wants to sell to him. The amount of this adjustment depends on the probability attached to an information event having occurred (α) and on the fraction of trades from the informed (μ). If there is no possibility of an information event or there are no trades by informed traders, then there is no adjustment to δ, i.e., $\delta = \delta(S_1)$. It is easy to demonstrate that in the case of a buy the market maker decreases the probability he attaches to $\underline{V}$, given the conditions on α and μ noted above.

Given these conditional expectations, the market maker can set initial bid and ask quotes. The initial bid is the expected value of the asset given that a sale takes place. Hence

$$E[V \mid S_1] = b_1 = \frac{\delta \underline{V}(\alpha\mu + \epsilon^S\gamma(1-\alpha\mu)) + (1-\delta)\overline{V}\epsilon^S\gamma(1-\alpha\mu)}{\delta\alpha\mu + \epsilon^S\gamma(1-\alpha\mu)}. \quad (6)$$

The initial ask is the expected value of the asset given that a buy occurs, so

$$E[V \mid B_1] = a_1$$
$$= \frac{\delta\underline{V}(\epsilon^B(1-\gamma)(1-\alpha\mu)) + (1-\delta)\overline{V}(\alpha\mu + \epsilon^B(1-\gamma)(1-\alpha\mu))}{(1-\delta)\alpha\mu + \epsilon^B(1-\gamma)(1-\alpha\mu)}. \quad (7)$$

As has been noted by several authors (see Glosten and Milgrom (1985), Copeland and Galai (1983)), these prices diverge from V^* to reflect the risk of information-based trading.

Having described the structure of the model and determined initial equilibrium quotes, we can now turn our attention to the question of how prices evolve in this market. Before proceeding to the analysis of our model, however, there are several aspects of our modeling approach that deserve comment. The sequential time-based framework we analyze is very different from the approach taken in many recent studies of asset prices. Following the work of Kyle (1985), numerous authors (see, for example, Admati and Pfleiderer (1988) and Kyle (1989) on strategic behavior; and Brown and Jennings (1989) and Grundy and McNichols (1989) on technical analysis) have modeled the trading process in a rational expectations framework in which orders are batched by the market maker and are cleared at a single price. In these papers either traders submit market orders which will be executed at whatever price it takes to clear the market, or traders are allowed to submit the price contingent orders that arise from negative exponential utility and normal distributions. In either of these frameworks it is possible to calculate linear rational expectations equilibria.[7]

[7] Outside of the examples used in the multiperiod rational expectations models, when price contingent demands are allowed it is typically not possible to calculate equilibria. Although equilibria have been shown to exist for the generic economy (see Jordan (1982)), they are so complex that it is hard to imagine how they could arise in any real market. Further, even the simple rational expectations examples have multiple equilibria. In addition to the usual nonrevealing equilibrium, these economies have a fully revealing equilibrium with traders conditioning on price and their own net trade. It is not obvious that this equilibrium is less plausible than the nonrevealing equilibrium, in fact it is more robust in that it always exists.

While this approach has provided a number of important insights into market phenomena, it is not amenable to the issue we study in this paper. One reason is that in a batch-clearing system, the aggregation of orders obliterates the information revealed by nontrading intervals. As we show, the lack of trade can affect the adjustment path of prices in potentially important ways. A second and related difficulty is that trades clear at a single price. In our model, as in actual markets, the bid-ask spread plays an important role and we demonstrate how the distinction between quotes and transaction prices may be crucial for empirical researchers. Finally, our focus on the adjustment path of prices requires a model of how individual prices are set and how, they, in turn, adjust over time. A sequential trading model captures this evolutionary process by focussing on how the market maker learns, how this affects the speed of price adjustment, and what this implies for the efficiency of markets. An added benefit of this focus on individual prices is that our model provides a framework for empirical investigations using transaction data. In the next section, we begin our investigation of the price adjustment process by determining how both trade and the absence of trade affect the revision of beliefs and hence the adjustment of prices.

II. Volume and Quotes

At each time t there are three possible outcomes for the trading process: the trader who checks the quotes may buy (B), sell (S) or choose not to trade (N). We let $Q_t \in [B, S, N]$ represent the outcome of the trading process at t. As the day proceeds the market maker observes and learns from the sequence of past trading outcomes. By the beginning of period t he has seen a history $Q^{t-1} = (Q_1, Q_2, \cdots, Q_{t-1})$ which may cause his beliefs to change. His beliefs at the beginning of period t are given by Bayes rule and are represented by $\rho_{Lt} = \Pr\{\psi = L \mid Q^{t-1}\}$, $\rho_{Ht} = \Pr\{\psi = H \mid Q^{t-1}\}$ and $\rho_{0t} = \Pr\{\psi = 0 \mid Q^{t-1}\}$. These are the conditional probabilities of the three possible events: a signal has occurred and it is low, a signal has occurred and it is high, and no signal has occurred.

The market maker's bid in period t is the expected value of the asset conditional on the history, Q^{t-1}, and a sale at t, $Q_t = S$. The bid at t is then

$$b_t = \Pr\{\psi = L \mid Q^{t-1}, S\}\underline{V} + \Pr\{\psi = H \mid Q^{t-1}, S\}\overline{V} + \Pr\{\psi = 0 \mid Q^{t-1}, S\}V^*.$$

The evolution of prices is determined by the evolution of beliefs. So to understand the stochastic process of prices, we need to analyze the stochastic process of beliefs.

One way to examine these beliefs is to consider how the existence of an information event affects the probability of a trade occurring in any time period. We know that if no information event has occurred, the probability of no trade in any time interval is $\gamma(1 - \epsilon^S) + (1 - \gamma)(1 - \epsilon^B)$. Conversely, if an information event has occurred the probability of no trade is $(1 - \mu)[\gamma(1 - \epsilon^S) + (1 - \gamma)(1 - \epsilon^B)]$, which is smaller because informed traders are sure to trade. Thus trade is positively correlated with the occurrence of an information event. The market maker uses this property to update his beliefs. As the following proposition shows, if no trade occurs in some time interval, the market maker raises his probability that no information event has occurred.

Proposition 1. *If there is no trade at time t then the probability of no information event rises and the probabilities of a high or a low signal fall, i.e.,* $\rho_{0,t+1} > \rho_{0,t}$, $\rho_{L,t+1} < \rho_{L,t}$ and $\rho_{H,t+1} < \rho_{H,t}$. *Further, the relative probability of a low to a high signal is unchanged, i.e.,*

$$\frac{\rho_{L,t}}{\rho_{H,t}} = \frac{\rho_{L,t+1}}{\rho_{H,t+1}}.$$

Proof: All proofs are given in the appendix.

One implication of Proposition 1 is that the market maker learns from the lack of trade, as well as from actual transactions. The information content of a no-trade observation differs from that of a transaction, however, because it conveys only information on the existence of a signal and not also on its direction. Consequently, the market maker does not change the relative probabilities of high and low signals, but does change their absolute probabilities.

This change in beliefs means that the market maker will change his quotes even though no transaction has occurred. Because the market maker believes it less likely that an information event has occurred, he moves his bid and ask closer to V^*. Proposition 2 demonstrates that this can cause quotes, and thus prices, to rise or fall depending on their position at time t.

Proposition 2: *Suppose there is no trade in period t. Then at time* $t + 1$:

A. *The bid rises if* $b_t < V^*$ *and falls if* $b_t > V^*$, *i.e.,* $b_{t+1} > b_t$ *if* $b_t < V^*$ *and* $b_{t+1} < b_t$ *if* $b_t > V^*$.
B. *The ask falls if* $a_t > V^*$ *and rises if* $a_t < V^*$, *i.e.,* $a_{t+1} < a_t$ *if* $a_t > V^*$ *and* $a_{t+1} > a_t$ *if* $a_t < V^*$.

As Proposition 2 demonstrates, both the bid and ask move in response to the absence of trade. What may seem paradoxical is that quotes do not always move in the same direction. The reason for this is that quotes are moving toward V^*, the unconditional expectation of V, and not toward the signal-based values of $\overline{V}$ or $\underline{V}$. Consequently, if the bid, for example, was above V^* it will fall; if it was below V^* it will rise.[8]

This movement, in turn, has implications for the bid-ask spread. At date t the spread is $\theta_t = a_t - b_t$. By Proposition 2 if $a_t > V^* > b_t$ the ask price falls,

[8] In each period, bid and ask prices bracket the current expected value of the asset. This differs, of course, from V^* (the prior expected value) as the market maker's expectation changes in response to trades.

the bid price rises, and thus the spread narrows (i.e., $\theta_{t+1} < \theta_t$) if there is no trade at date t. This narrowing of the spread reflects the information conveyed by a no-trade outcome. Specifically, the spread exists because of the possibility of trading with an informed trader. When no trade occurs, the market maker lowers the probability he attaches to an information event having occurred and so reduces the probability he attaches to a trade being from an informed trader. The effect this has on the spread is summarized in the following corollary.

Corollary 1: *Suppose $a_t > V^* > b_t$. If there is no trade in period t, then the spread in period $t + 1$ will be smaller than the spread in period t.*

Our result that the absence of trade leads to smaller spread is the opposite of the effect predicted by Diamond and Verrecchia (1987). In their model, the absence of trade in a period is bad news because it is more likely to occur when informed traders are precluded from selling by short sale prohibitions. However, if these prohibitions are relaxed, then the absence of trade in their model has no information content at all. By contrast, in our model, the absence of trade is more likely to occur when no information event (either good or bad) has occurred.[9] With trade now "safer" the market maker reduces his spread.

This result that the absence of trade can affect future prices and spreads suggests an interesting parallel to a result we derived in earlier work on the effects of trade size on subsequent prices. In Easley and O'Hara (1987) we demonstrated that transaction prices recover following block trades because the occurrence of a small transaction lowered the market maker's belief that new information existed. Consequently, order size was an important variable in the price process. In this paper, the absence of trade plays a similar role, causing the market maker to narrow his spread in response to increasing time between trades. What is intriguing about the effect demonstrated here is that it suggests that time between trades may play an important role in the behavior of prices. We return to this issue in Section IV where we discuss the empirical implications of our model.

Since the level of quotes is affected by both transactions and no-trade outcomes, an important issue to consider is how these variables affect the adjustment of quotes over time. To address this issue we need to characterize what variables the market maker uses (or keeps track of) in setting his quotes for time $t + 1$. In our model, the market maker's quotes are conditional expected values, so to determine his prices we must determine his beliefs. Suppose that in the past t trading intervals the market maker has observed n_t no trades, β_t buys, and s_t sells. The market maker's beliefs given

[9] There is another literature examining the link of public information and trading behavior (see, for example, Jain (1988) and Kim and Verrecchia (1991)). If traders know that an announcement is imminent, it is usually the case empirically that volume decreases as traders delay their trades until after the news arrival. As our focus is on private information whose very existence is uncertain, this literature is not directly relevant to our analysis.

this trading history are then given by (where the t subscripts have been dropped):

$$\Pr\{\psi = 0 \mid Q^t\} = (1-\alpha)(\gamma\epsilon^S)^s((1-\gamma)\epsilon^B)^\beta\Big[(1-\alpha)(\gamma\epsilon^S)^s((1-\gamma)\epsilon^B)^\beta + (1-\mu)^n\Big[\alpha\delta(\mu + (1-\mu)\gamma\epsilon^S)^s((1-\mu)(1-\gamma)\epsilon^B)^\beta + \alpha(1-\delta)((1-\mu)\gamma\epsilon^S)^s(\mu + (1-\mu)(1-\gamma)\epsilon^B)^\beta\Big]\Big]^{-1}. \quad (8)$$

with the probabilities of low and high signals calculated similarly.

What is important to notice is that since beliefs depend on (n_t, β_t, s_t), quotes will also depend on these variables. In particular, the bid at time $t+1$ can be written as:

$$b_{t+1} = \Pr\{\psi = L \mid n_t, s_t + 1, \beta_t\}\underline{V} + \Pr\{\psi = H \mid n_t, s_t + 1, \beta_t\}\overline{V} + \Pr\{\psi = 0 \mid n_t, s_t + 1, \beta_t\}V^*. \quad (9)$$

Expressing the bid, or ask, in this form reveals that quotes at time t depend not only on the most recent trade, but also on the total numbers of previous buys, sales, and no-trade outcomes. Indeed, to describe the stochastic processes of quotes, equation (9) demonstrates that we need only know the total numbers of no trades, buys, and sales; the process does not depend on any other variables.

From the perspective of a market observer, this result has two important implications. First, quotes at each point in time will depend in a specific way on the outcomes of previous trading periods. Consequently, since the sequence of trading outcomes matters for determining future quotes, watching past market outcomes is informative. Second, since the respective numbers of buys, sales, and no trades matters (indeed, determines prices), the total amount of trade or volume affects price behavior. In particular, because volume is related to the number of no-trade outcomes and the number of no-trade outcomes is related to the probability of an information event, prices at time $t+1$ depend on the volume of trade as of time t.

To explore these implications more fully, it is useful to delineate how specific variables affect the stochastic process of prices. We know from above that quotes at time $t+1$ depend solely on the information conveyed by the trading outcome (n_t, β_t, s_t). By definition, volume to time t is given by $v_t = s_t + \beta_t$ and, similarly, the market maker's inventory position at time t is given by $i_t = s_t - \beta_t$. It is easy to show that knowledge of (v_t, i_t, t) is equivalent to (n_t, β_t, s_t). Hence, if you know the total volume, the market maker's inventory position, and the time you can determine the market maker's quotes for trades at time $t+1$.

Suppose, however, that you knew only inventory and time. It is easy to show that (i_t, t) is a sufficient statistic for (β_t, s_t) but that it is not sufficient for (n_t, β_t, s_t). Hence, if it were never possible for a no-trade outcome to occur, knowledge of (i_t, t) would be sufficient to determine future quotes. In the

Glosten and Milgrom (1985) transaction-based model, for example, an observer of the market could simply track inventory (or more crudely, the imbalance of buys and sells) to know how the market maker would set quotes. Once we allow for the possibility of nontrading outcomes, however, this is no longer true: we need to follow volume as well as inventory to predict the level of quotes at time $t + 1$.

Alternatively, suppose that instead of tracking inventory and volume, you simply tracked the sequence of past prices. This is the approach taken in recent papers on technical analysis by Grundy and McNichols (1989) and Brown and Jennings (1989). They show that past prices carry information about future prices, so price-based technical analysis is valuable. Past prices also carry information in our framework, but they are not sufficient statistics for all past market information. It is easy to demonstrate that given any price sequence, the distribution of future quotes and prices will differ depending on volume. So although price-based technical analysis is valuable, price and volume-based technical analysis is even more valuable.[10]

One way to characterize our results on the role of volume is to recognize that the stochastic process (n_t, β_t, s_t) is Markov. Consequently, any variables with similar information content must also be Markov processes. In Easley and O'Hara (1987), we showed that prices alone and prices and inventory together would not in general be Markov processes. Hence, it is not surprising that these variables are not sufficient statistics for the price process derived in this paper. As the following proposition demonstrates, however, it is the case that (v_t, i_t, t) is a Markov process. Consequently, the volume of trade, inventory, and time all matter in adjustment of prices to information.

Proposition 3: *The stochastic processes* (n_t, s_t, β_t) *and* (i_t, v_t, t) *are Markov. That is, the distribution of* $(n_{t+1}, s_{t+1}, \beta_{t+1})$, *or* $(i_{t+1}, v_{t+1}, t + 1)$, *depends on* (n_t, s_t, β_t), *or* (i_t, v_t, t), *but is otherwise independent of the history of the process.*

It may seem paradoxical that allowing the market maker to learn from the lack of trades as well as from transactions means that volume now matters in the stochastic process of prices. Yet, it is precisely because no-trades can occur that the aggregate total of transactions provides information. It is easy to see that the probability the market maker assigns to no new information is increasing in the number of no trade outcomes n_t and decreasing in the volume v_t. Hence, the market maker interprets v_t as a signal of the *existence* of information. Since the market maker's inventory results from any imbal-

[10] One could ask whether this would also occur in the two period rational expectations examples with endownment uncertainty employed by Grundy and McNichols (1989) and Brown and Jennings (1989). In the Grundy and McNichols model even per capita volume has an infinite expectation, so conditioning on it is troublesome. In the Brown and Jennings model price and volume together reveal all information. So if traders can condition on contemporaneous market statistics (as they are doing with price conditioning) the equilibrium is revealing and technical analysis has no value.

ance in trades, watching i_t then provides a signal of the *direction* of any new information.

It is important to stress, however, that the stochastic process of prices also depends on time. Although volume per se is informative, it is volume as of a certain time t that affects the distribution of prices at time $t + 1$. This dependence means that where the market maker sets his quotes at time $t + 1$ depends upon the volume he has observed as of time t. Consequently, the spread a market maker sets at any point in the day will also depend on the volume in the market up to that point.

Characterizing exactly how this dependence affects the size of the spread is complicated, however, by the fact that the sequence of trades also affects the placement of the spread. In particular, a large number of buy orders may indicate both that a signal has occurred and that the signal was high. These two effects will move bid and ask prices toward $\overline{V}$, causing the spread to narrow as prices approach the upper bound. In this case, the narrowing of the spread reflects the market maker's increasing confidence in the direction of the signal. This convergence effect of volume is addressed in more detail in the next section.

For our purposes here, however, it is also important to delineate the pure effect on spreads that arises from volume alone. To see this, we must abstract from the directional effects of trades by considering how volume affects prices in the absence of any inventory or trade imbalance effects. This volume effect can be isolated by considering a trade sequence that does not affect the market maker's relative beliefs on the signal's direction. Hence, suppose that at time t the market maker has learned no new directional information. Thus, the relative probabilities of a high and low signal must be unchanged from their values at time 0. (Note that the absolute size of these probabilities need not be the same, however, as the market maker may have revised his beliefs about the existence of any new information).

As the following proposition demonstrates, this "existence" revision occurs as a result of volume. All else equal, the greater the volume the more likely it is that the market maker believes an information event has occurred. Hence the size of the spread at time $t + 1$ will be correlated with the volume up to time t.

Proposition 4: *Suppose low and high signals are equally likely and uninformed traders are equally likely to buy or sell, i.e.,* $(1 - \gamma)\epsilon^B = \gamma\epsilon^S$. *Suppose further that in period t the market maker's beliefs about the relative probability of a low or high signal are unchanged from period 1, i.e.,* $\rho_{Lt} = \rho_{Ht}$. *Then the spread at period t will be larger the greater is the volume up to period t.*

There are two reasons why this result is of interest. First, in the absence of event uncertainty it is not the case that volume affects spreads. For example, in the Glosten-Milgrom model the total volume to time t is irrelevant. If at time t the market maker's relative probabilities of a high and low signal were unchanged from time 0, then his spread would be identical to his initial spread no matter what the volume in the market. That is not the case here.

Because the market maker can revise his beliefs on the existence of information, the spread at time t can be very different from that at time 0.

A second aspect of this result is it illustrates the important role volume plays in the market maker's learning process and hence in the adjustment of prices to information. Because volume is providing information to the market maker on event uncertainty, the behavior of prices depends on the level of volume. The proposition illustrates these effects on the size of the spread, but it is also the case that volume affects the movement of prices. These price movements, in turn, reflect the efficiency of the market, and hence volume may be an important factor in the process of price adjustment. In the next section we investigate these issues by examining how volume influences the speed with which prices adjust to information.

III. Prices and Efficiency

To consider the dynamic behavior of the market, we first need to define the stochastic process of market prices. Because we distinguish between clock time and trade time, this price process must be specified with some care. In particular, we must incorporate the property that beliefs on the asset's value may change during nontrading intervals.

We define the stochastic process of conditional expected values $\{p_t^*\}$ by

$$p_t^* = \begin{bmatrix} a_t \text{ if } Q_t = B \\ b_t \text{ if } Q_t = S \\ E[V \mid Q^{t-1}, N] \text{ if } Q_t = N. \end{bmatrix}$$

This process is a martingale because it is a sequence of conditional expectations where $E[P_{t+1}^* \mid Q^t] = E[E[V \mid Q^{t+1}] \mid Q^t] = E[V \mid Q^t] = P_t^*$. Unfortunately, this process is not observable because we do not know the market maker's expectation in the absence of a trade.[11]

A transaction price arises only when a trader chooses to buy or sell. From a statistical perspective, we can view this transactions price process as being formed by an optional sampling of the sequence of conditional expected values $\{p_t^*\}$. In particular, we can define the sequence of transaction prices by:

$$p_j = p_{t_j}^* \text{ where } t_j = \min\{t: t > t_{j-1} \text{ and } Q_{t_j} \neq N\}.$$
$$\text{with } t_0 = 0.$$

This transaction price process is observable but the fact that it is formed by optional sampling leads to interesting and complex statistical properties. These complexities arise because the sampling times are not independently and identically distributed but are instead partially *chosen* by traders who

[11] Note that it is not legitimate to approximate $E[V \mid Q^{t-1}, N]$ by the average of the bid and ask at t. It is easy to show that the spread need not be centered on the expected value of the asset.

may be informed about the evolution of the price process. The time between trades, therefore, need not be independent of the evolution of the price process. Moreover, the variance of the process will also reflect this sampling bias as will the time series properties of security returns.

Other researchers have noted optional sampling problems in transactions data with respect to nonsynchronous issues (see for example Scholes and Williams (1977) and Lo and MacKinley (1990)). In these analyses, however, the probability that a security does not trade in any time period is assumed uncorrelated with the behavior of the underlying true process.[12] Nonsynchronous trading in these models introduces timing problems (and hence spurious time series correlations) but does not reflect any underlying bias in returns. The optional sampling problem in our analysis is more severe. Here the sampling problem is correlated with movements in the actual underlying price process. We discuss the implications of this dependence for empirical work in the next section. What needs to be determined for our characterization of the price process, however, is how this dependence affects the dynamic (or time series) behavior of prices. To address this issue we need to examine how the stochastic process of transactions prices differs from that of the "true" price process.

In the standard sequential trade model without event uncertainty, Glosten and Milgrom (1985) demonstrated that the sequence of transactions prices is a martingale with respect to the sequence of trades. In our model with event uncertainty, the same martingale result applies. Moreover, because any martingale is also a martingale with respect to past realizations of the random variable, prices in our model are a martingale with respect to all past prices. In standard finance terminology, our prices would be said to be weak-form efficient.

Transaction prices do not form a martingale, however, with respect to full information and so they are not strong form efficient. While prices can thus differ from full information values, over time prices converge to full information or strong-form efficient values. It is easy to demonstrate that transactions prices converge to the appropriate value: V^* if no signal has occurred. $\underline{V}$ if a low signal has occurred or $\overline{V}$ if a high signal has occurred.[13]

An interesting aspect of this convergence behavior is its relation to trading volume. As was demonstrated in the previous section, the stochastic process of prices depends on volume, and so not surprisingly does the stochastic process of transactions prices. Characterizing how this relationship affects the convergence of prices, however, requires specifying the components of

[12] Diamond and Verrecchia's (1987) analysis also involves a sampling problem induced by short sale constraints, but this sampling problem is a result of the trading mechanism and not of the underlying price process. Hence, their censored sampling problem differs from that here in that the sampling problem in our model arises endogenously from the information structure.

[13] The distribution of trades is different if no event has occurred, if a low signal has occurred or if a high signal has occurred. The market maker is a Bayesian who observes the distribution of trades. So convergence of beliefs, and thus prices, to the correct value is a standard Bayesian learning result.

volume in more detail. In particular, trading volume can be thought of as containing a component due to liquidity (or uninformed) trades and a component due to informed trades. When there has been no information event, then by definition there can be no informed trading but the amount of liquidity trades should be unaffected. We can thus define "normal" volume as this expected level of liquidity trading. Let $v_j = s_j + \beta_j$ be the volume to time j and let $\epsilon = [(1-\gamma)\epsilon^S + \gamma\epsilon^B]$ be the expected liquidity based volume per period. Then Proposition 5 shows that the level of volume and the price movement are related by the underlying event uncertainty.

Proposition 5: *In the absence of an information event, volume converges to its normal level and prices converge to V^*. More precisely, for any $\eta > 0$,*

$$\lim_{\tau\to\infty} \Pr\left\{\left|\frac{v_j}{j} - \epsilon\right| > \eta \text{ and } |p_j - V^*| < \eta, \text{for some } j \geq \tau\right\} = 0.$$

One way to interpret Proposition 5 is that "It takes volume to move prices." Indeed, the proposition actually provides the stronger result that in the absence of abnormal volume prices converge to a level contained in the interval of initial quotes, i.e., $V^* \in (b_1, a_1)$. In this sense, absent abnormal volume prices do not move. While prices can (and will) deviate from V^* and volume can (and will) deviate from its normal level, such movements can only be temporary in the absence of an information event. This suggests that event uncertainty and its effect on the price process may provide one explanation for the oft-observed empirical relationship between prices and volume (see Karpoff (1987) for a survey of the empirical work in this area).

While the proposition demonstrates that volume affects convergence and we know that prices do indeed converge to their "correct" level (either $\underline{V}$, $\overline{V}$, or V^*), it would also be useful to know how quickly this occurs. Since prices only converge in the limit, the obvious answer is that it takes an infinite amount of time. However, a more useful answer would be to determine how quickly prices approximately reach their strong-form efficient levels. One way to characterize this is to calculate first passage times, or essentially how long it takes on average for prices to first reach some prespecified bound around the strong-form level. This is the approach taken by Diamond and Verrecchia (1987) in their analysis of short sale constraints. An alternative approach is to calculate the actual rate of convergence of the stochastic process, and then use that as a benchmark in comparing how various parameters affect the rate of convergence. This is the approach taken by Easley and O'Hara (1991) in their analysis of the effects of order form on price behavior.

Both approaches provide a means of characterizing the speed of price adjustment; both are also related in the sense that the answers they give will be comparable. For our analysis here, we find it more intuitive to characterize the underlying stochastic process by its rate of price convergence. Since our concern is with the behavior of the process, this approach allows us to explicitly measure its behavior with respect to the underlying information structure.

To state our results, we need a few definitions. Since in our model transaction prices are a sample of quotes and by equations (6) and (7) quotes are linear combinations of beliefs, it is sufficient to provide rates of convergence for beliefs. For each signal ψ define the probability on trades $p^{\psi} = (P^{\psi}(N), P^{\psi}(B), P^{\psi}(S))$, with representative element $p^{\psi}(Q)$ the probability of trade Q given the signal ψ. Now for two probabilities on trades, p^{ψ} and $p^{\psi'}$, the entropy of $p^{\psi'}$ relative to p^{ψ} is defined by

$$I_{p^{\psi}}(p^{\psi'}) = \sum_{Q \in \{N,B,S\}} p^{\psi}(Q) \log\left(\frac{p^{\psi}(Q)}{p^{\psi'}(Q)}\right). \tag{10}$$

This standard measure in statistics has the property that it is always non-negative and is zero if and only if $p^{\psi} = p^{\psi'}$. It essentially measures the distance between p^{ψ} and $p^{\psi'}$.

Using these definitions we can state our rate of convergence results, the intuition of which can be conveyed by a simple example. Suppose that a low signal has occurred. We know that the posteriors converge almost surely, i.e., $\rho_{Lt} \to 1$, $\rho_{Ht} \to 0$, $\rho_{Ot} \to 0$ a.s. It also follows directly from the Strong Law of Large Numbers that the posteriors of a Bayesian observing an i.i.d. process converge exponentially to their almost sure limits. Consequently, we show that almost surely ρ_{Ht}/ρ_{Lt} converges to zero exponentially at rate $I_{p^L}(p^H)$ and almost surely ρ_{Ot}/ρ_{Lt} converges to zero exponentially at rate $I_{p^L}(p^O)$. The linearity of the equilibrium quotes in beliefs then implies that quotes converge to $\underline{V}$ exponentially at a rate equal to the minimum of $I_{p^L}(p^H)$ and $I_{p^L}(p^O)$. So transaction prices converge exponentially in clock time at this rate. Our second result is on the comparative behavior of price processes. We show that increasing the fraction of trades from the informed or reducing normal volume, increases the relative entropy between p^L and either p^H or p^O. These increased relative entropies increase the rate of convergence because they increase the information content of observing trades. These results are summarized in the following proposition.[14]

Proposition 6: *Quotes, and thus transaction prices, converge to their strong form efficient values at exponential rates (in clock time). If signal ψ occurs, the exponential rate of convergence is $r(\psi) = \mathrm{Min}\{I_{p^{\psi}}(p^{\psi'})\colon \psi' \neq \psi\}$, where $I_{p^{\psi}}(p^{\psi'})$ is the entropy of $p^{\psi'}$ relative to p^{ψ}. That is, if signal ψ occurs then $|a_t - E[V \mid \psi]| < \exp - r(\psi)$ and $|b_t - E[V \mid \psi]| < \exp - r(\psi)$ for all large t, almost surely. These rates of convergence are increasing in the fraction of trades*

[14] Proposition 7 applies to the case of $0 < \mu < 1$. If $\mu = 1$, then when no signal occurs the market maker learns in finite time almost surely. He knows that there is no information event as soon as he observes no trade or any sequence of trades with both buys and sells. Alternatively, if a low signal occurs his posterior probability on a low signal at time t is $\rho_{Ot} = \alpha\delta/[\alpha\delta + (1 - \alpha)(\gamma\epsilon^S)^t]$. This sequence of beliefs converges to one and the rate of convergence is decreasing in normal volume. The analysis for a high signal is similar.

from the informed, and, when the uninformed are equally likely to buy or sell, decreasing in normal volume.

The convergence results in Proposition 6 provide insights into how prices adjust in securities markets. Increasing the fraction of trades from the informed hastens this adjustment process because their trading activity reveals the underlying information. Perhaps not surprisingly, the greater the fraction of trades from the informed (when an information event occurs) the faster this adjustment occurs. The role of volume is more intriguing. As the proposition demonstrates, greater normal trading volume actually slows the adjustment of prices to information. This occurs because in markets with more uninformed or liquidity trading the trades from informed traders can be hidden more effectively. Consequently, it takes the market maker longer to learn the information and this slows the rate at which prices reflect full information.[15]

That volume can affect the placement and adjustment of prices across time may seem an intuitively obvious concept. One interesting aspect of this result, however, is that it differs from the predictions of earlier work by Kyle (1985). In Kyle's model, orders are batched and all trades clear at a single price. Kyle shows that altering normal volume has no effect on prices because the single informed trader strategically increases (or decreases) his trade size to maintain his expected relative share. Consequently, in his model increasing normal volume affects the profits of the informed, but not the adjustment of prices.

One obvious question that arises is whether our volume results continue to hold if the informed adjust their trading behavior. A complete analysis of this issue requires allowing either endogenous entry of informed traders or defining some explicit strategic link between μ and volume. As our model is a partial equilibrium competitive analysis, it is not amenable to this exercise. We can, however, characterize the polar case in which the informed adjust completely. That is, if following an information event, all trades are actually from the informed. This corresponds to assuming that $\mu = 1$ and has the practical implication that since the informed are already making all the trades there is no way that any strategic choice could result in a larger role for the informed. Proposition 7 demonstrates that even with this complete role for the informed our results on volume and prices remain.

Proposition 7: *Suppose $\mu = 1$. Then the initial spread is decreasing in normal volume, and the rate of convergence of quotes, and thus transactions prices, to their strong form efficient values is also decreasing in normal volume.*

[15] It is useful to contrast these dynamic results with our earlier analysis of the spread. Markets with a high probability of trade by the uninformed, and thus a large normal volume, have a small initial spread but a slower speed of convergence to full-information prices. The small spread occurs because there are many uninformed trades relative to informed trades. But this makes it more difficult to detect the presence of informed traders and to discover their information. Conversely, markets with low normal volumes have large initial spreads but faster spreads of convergence.

At first glance, the results of Proposition 7 may seem surprising. Even assuming that following an information event all trades are from the informed does not remove the effects of normal volume on the price path. The reason is that in our model there still remains the uncertainty over whether an information event has actually occurred.[16] As we have demonstrated throughout this paper, it is this underlying event uncertainty that provides a role for volume. And it is this underlying event uncertainty that will remain even given the trading intensity choices of the informed.

IV. Empirical Implications

Our analysis thus far has examined the effects of uncertainty over the existence of new information on market behavior. A strength of our approach is that we analyze the behavior of both prices and quotes. Since this structure corresponds well to the new transaction-based data increasingly being studied by researchers, this suggests that our work may be particularly relevant for transactions-based studies. In this section we explore this applicability by considering the empirical implications of our model. In particular, we demonstrate how our results on the role of time, volume, and the stochastic process of prices may be useful in predicting security price behavior, in designing statistical tests of security prices, and in improving our understanding of market volatility.

Perhaps the most fundamental prediction of our model is that time affects the behavior of prices. As we have shown, because the lack of trade may signal that no new information exists, the time between trades (or clock time) can itself affect prices. From an empirical perspective, this has a number of testable implications. For example, a simple prediction of our model is that quotes will change in the absence of trades. Cursory examination of the transactions data series confirms that this does, indeed, occur. While the transaction price series, by definition, changes only with trades, price quotes do change without intervening transactions.

A related, and perhaps more important, prediction of our model is that the spread will decrease the longer the time between transactions. Recent research by Hasbrouck (1991) provides empirical support for this effect. As Figure 3 in that paper demonstrates, following a large trade the spread narrows over time with the absence of trades. Hasbrouck also presents evidence that trades that arrive when the spread is wide have a greater impact on price than those which arrive when the spread is narrow. This is consistent with our results that the behavior of prices will differ depending on factors such as the volume and timing of trades.

Of course, a direct test of our time-based model is to investigate the explicit relationship between time and the price process. This is the approach taken

[16] Obviously, if an information event is known to have occurred, then even one trade will cause prices to instantly adjust to $\overline{V}$ or $\underline{V}$ because with $\mu = 1$ trade is revealing. In this case, the whole issue of asymmetric information is academic.

in a recent paper by Hausman and Lo (1990). Using an ordered probit statistical model, Hausman and Lo test whether the time interval between trades can be viewed as exogenous to the price process. They reject the exogeneity of time at conventional statistical levels, a finding directly in support of the predictions of our model. They note, however, that because of the small size of the parameter estimates, the economic significance of time may be small. This degree of importance is clearly an important question for future research.

One implication of these results is that the behavior of security prices may be seriously misspecified by applying standard statistical testing approaches to transaction data. Although most studies restrict attention to transaction price data, this stochastic process of prices can be viewed as an optional sampling of the process of conditional expected values for the asset. As we discussed in the previous section, this conditional expected value process is sampled at times which are neither fixed nor independently distributed. Trades are positively serially correlated, so the probability of a trade and thus a price observation is positively correlated with volume. This follows because if an information event occurs, trades are more likely to occur and volume will be high because the informed always trade. A trade at time t, or high volume up to time t, increases the likelihood that an information event has occurred. And this, in turn, increases the likelihood of a trade at time $t+1$.

One reason why this is important is that it affects the variance structure of the price process. Let $E[(P_t^* - P_{t-1}^*)^2 \mid Q^{t-1}]$ be the conditional variance of the expected asset value at time t. By Corollary 1 the variance at time $t+1$ is less than the variance at t if there is no trade at t. Hence, roughly, variances are positively correlated with volume. Now as trades are positively correlated, periods of low variances tend to be grouped and occur in periods where there is little trade. The transaction price is a sampling of the expected value process exactly when this process has high variance—at trade times. So, if we want the variance of the conditional value of the asset or equivalently of the price at which trades could have occurred, the variance of the transaction price yields an overestimate. Further, both the conditional value process and the transaction price process have nonconstant variances. The underlying problem is that although both price processes are martingales, they are not Markov, but rather are history dependent.

While the difficulties this poses for transactions based studies are apparent, this underlying problem also affects some standard techniques frequently applied to daily or weekly data. In particular, a common problem in studies calculating security returns is nonsynchronous trading (see Scholes-Williams (1977); Lo and MacKinley (1990)). Although adjustments to deal with this problem have been proposed, these adjustments all assume that the nontrading interval is *independent* of the true return process for the stock. If, as our analysis demonstrates, this is not correct then the nonsynchronous trading problem can induce statistical problems even using the adjustments proposed in the literature.

Given these difficulties, our analysis suggests that if we can follow only price data then the process of quotes may be the better data set. Quotes, unlike transaction prices, occur continually and so do not have the optional sampling problem. A researcher watching transaction prices (and ignoring clock time and quantities) is allowing the market participants to select random observations of the underlying price process with sampling times correlated with the evolution of the process. This yields severe and unnecessary statistical problems. Quotes are observable, and, at least in our model, carry strictly more information than do transaction prices.

Of course, even the process of quotes suffers from severe history dependence. Fortunately, there are simple sufficient statistics for the history of the process. It follows from Proposition 3 that either quotes or prices combined with inventory, volume, and clock time are Markov processes. Rather than using prices alone, therefore, it would be preferable to estimate the $\{n_t, s_t, \beta_t\}$ process and the functional relationships determining quotes (i.e., equation (6) and its analog for asks). Analyzing this process would provide more information than the price sequence, and would avoid the arbitrary restrictions on the price process that must of necessity be imposed if the researcher is restricted to examining only prices.[17] We are currently investigating this estimation in other research.

Our characterization of this underlying stochastic process suggests that another area where our results may be of interest is in the specification of option volatilities. Although the basic Black-Scholes option model requires using a volatility estimate, more complex stochastic option volatility models (for example, Wiggins (1987) and Hull and White (1987)) require estimates of the parameters of the variance distribution as well. This has led to extensive research as to whether the time series properties of the variance are better described by an AR(1) process (see Poterba and Summers (1986)), and IMA(3) process (see French, Schwert, and Stambaugh (1987)), or some more general GARCH model (see Duan (1990)).

Our research provides some insights into the properties of this underlying volatility distribution. In particular, our result that the price process is not Markovian has important implications for appropriate representations for these processes. For example, while the commonly assumed diffusion processes are Markovian, GARCH processes, in general, are not.[18] Hence, our research suggests that a GARCH framework may be a more appropriate specification than some of the more standard representations.

One reason why this is important is that a GARCH process can be motivated as resulting from time dependence in the rate of information arrival. Our model here provides an explanation of how such time depen-

[17] For example, Barclay and Litzenberger (1988) assume that the rate of return process is determined by a Brownian motion. This structure approximates the rate of return over nontrading periods. This approximation is inconsistent, however, with our results on the stochastic process of prices and, in particular, its variance structure.

[18] As Duan (1990) points out, the only Markovian GARCH process is GARCH (0, 1) or ARCH(1).

dence may arise, and suggests several properties that the resulting process may have. For example, our results on the dependence of the price process on volume suggest that volatility will be similarly volume-affected. This is supported by recent empirical research by Lamoreaux and Lastrapes (1990). Moreover, our convergence results in Proposition 5 dictate that since price movements (in the sense of prices converging to new values) require abnormal (or unexpected) volume, the underlying volatility will be affected by abnormal volume. Diz and Finucane (1991) tested this prediction of our model using transactions data and an expectational model for expected volume. Their results that approximately two-thirds of the volatility interventions can be explained by abnormal volume provide strong direct support for the implications of our model.

V. Conclusion

In this paper we have analyzed the effects of information event uncertainty for market behavior. If information events are not certain to have occurred, then as we have demonstrated the lack of trade may provide a signal to market participants. This imparts information content to the time between trades, causing time per se to no longer be exogenous to the price process. Our research has delineated how this affects the behavior of quotes, spreads, and transactions prices, and has characterized the underlying stochastic process of prices. We have also demonstrated important divergences that arise between transactions prices and this underlying stochastic process. As we have discussed, these divergences have important implications for empirical work both in analyses of transactions data and in more general investigations of security price and option behavior. With the growing interest in market microstructure research in general, and in transactions-based studies in particular, our results should be useful in a wide variety of research applications.

We believe that an important area for future research is to explore more fully the implications of what we have defined as event uncertainty. As our results here suggest, a number of market phenomena may be directly linked to this underlying problem. For example, one simple extension is to consider a multi-day version of the model in which information events are allowed to happen between days. In that case, the probability of an information event by the start of day N is

$$\phi_{N-1} + (1 - \phi_{N-1})\alpha_N,$$

where α_N is the probability of an informational event between days $N-1$ and N and ϕ_{N-1} is the probability of an informational event having occurred by the end of day $N-1$.

One implication of this is that the spread will go up overnight, since the probability of an informational event unambiguously rises. We conjecture that other interesting price effects may be found by examining this issue in more detail. Since it is also a frequent market practice to halt trading when it is known that new information exists, but is not yet public, incorporating event uncertainty may also allow theoretical models to capture more accurately the behavior of market institutions and prices.

Appendix

Proof of Proposition 1: Calculation shows that

$$\rho_{0,t+1} = \frac{\rho_{0,t}}{(1 - \rho_{0,t})(1 - \mu) + \rho_{0,t}} > \rho_{0,t},$$

$$\rho_{L,t+1} = \frac{\rho_{L,t}(1 - \mu)}{(1 - \rho_{0,t})(1 - \mu) + \rho_{0,t}} < \rho_{L,t},$$

and

$$\rho_{H,t+1} = \frac{\rho_{H,t}(1 - \mu)}{(1 - \rho_{0,t})(1 - \mu) + \rho_{0,t}} < \rho_{H,t} //.$$

Proof of Proposition 2: The bid at time $t + 1$ is

$$b_{t+1} = E[V \mid Q^{t+1} = (Q^t, S)] = \delta_{t+1}(Q^{t+1})\underline{V} + (1 - \delta_{t+1}(Q^{t+1}))\overline{V},$$

where

$$\delta_{t+1}(Q^{t+1}) = \frac{(1 - \mu)\rho_{L,t}(\mu + (1 - \mu)\epsilon^S\gamma) + \delta\rho_{0,t}\epsilon^S\gamma}{(1 - \mu)[\rho_{L,t}\mu + \epsilon^S\gamma(1 - \mu(1 - \rho_{0,t}))] + \mu\rho_{0,t}\epsilon^S\gamma}.$$

So we need to show that if $\delta_t(Q^t) > \delta$, i.e., $b_t < V^*$, then $\delta_{t+1}(Q^{t+1}) < \delta_t(Q^t)$, i.e., $b_{t+1} > b_t$. Calculation shows that $\delta_{t+1}(Q^{t+1}) < \delta_t(Q^t)$ if and only if

$$\delta_t(Q^T) = \frac{\rho_{L,t}(\mu + (1 - \mu)\epsilon^S\gamma) + \delta\rho_{0,t}\epsilon^S\gamma}{\rho_{L,t}\mu + \epsilon^S\gamma[1 - \mu(1 - \rho_{0,t})]} > \delta.$$

The remaining claims follow from similar calculations.

Proof of Proposition 3: Trades are, by assumption, independently and identically distributed. The processes in Proposition 3 are counting processes for trades and time and are thus Markov.

Proof of Proposition 4: Let $\rho = (1 - \gamma)\epsilon^B = \gamma\epsilon^S$. Then $\dfrac{\Pr\{\psi = L / Q^t\}}{\Pr\{\psi = H \mid Q^t\}}$ $= \left(\dfrac{\delta}{1 - \delta}\right)\left(\dfrac{\mu + (1 - \mu)\rho}{(1 - \mu)\rho}\right)^{s-\beta}$. Note that this ratio is independent of n and equals $\dfrac{\delta}{1 - \delta}$ only if $s = \beta$. So the probability of no signal can be written

$$\Pr\{\psi = 0 \mid Q^t\} = \frac{1 - \alpha}{1 - \alpha + (1 - \mu)^t\alpha\left(\dfrac{\mu + (1 - \mu)\rho}{(1 - \mu)\rho}\right)^s}.$$

The bid and ask at t can be written

$$b_t = (1 - \Pr\{\psi = 0 \mid (Q^t, S)\})E[V \mid \psi \in \{L, H\}, (Q^t, S)] + \Pr\{\psi = 0 \mid (Q^t, S)\} V^*,$$

and

$$a_t = (1 - \Pr\{\psi = 0 \mid (Q^t, B)\})E[V \mid \psi \in \{L, H\}, (Q^t, B)] + \Pr\{\psi = 0 \mid (Q^t, B)\} V^*.$$

So the spread in period t is

$$\begin{aligned} T_t = {} & (1 - \Pr\{\psi = 0 \mid (Q^t, B)\})E[V \mid \psi \in \{L, H\}, (Q^t, B)] \\ & - (1 - \Pr\{\psi = 0 \mid (Q^t, S)\})E[V \mid \psi \in \{L, H\}, (Q^t, S)] \\ & + (\Pr\{\psi = 0 \mid (Q^t, B)\} - \Pr\{\psi = 0 \mid (Q^t, S)\})V^*. \end{aligned}$$

An additional buy or sale has the same affect on the probability of an information event having occurred so

$$\begin{aligned} T_t = {} & (1 - \Pr\{\psi = 0 \mid (Q^t, B)) \\ & \cdot [E[V \mid \psi \in \{L, H\}, (Q^t, B)] \\ & - E[V \mid \psi \in \{L, H\}, (Q^t, S)]]. \end{aligned}$$

The difference in expected values is positive and unaffected by volume. Calculation shows that $\Pr\{\psi = 0 \mid (Q^t, B)\}$ is increasing in n. So the spread is decreasing in the days volume.

Proof of Proposition 5: Almost sure convergence of prices to the correct value is a standard Bayesian learning result. The sequence of prices $\{p_j\}$ converges almost surely to the random variable $\hat{p}$ where

$$\hat{p} = \begin{bmatrix} \underline{V} \text{ if } \psi = L \\ \overline{V} \text{ if } \psi = H \\ V^* \text{ if } \psi = 0. \end{bmatrix}$$

Almost sure convergence of average volume to its mean follows from the Strong Law of Large Numbers. So V_j/j converges almost surely to the random variable $\hat{V}$ where

$$\hat{V} = \begin{bmatrix} \epsilon \text{ if } \psi = 0 \\ \mu + (1 - \mu)\epsilon \text{ if } \psi \in \{L, H\}. \end{bmatrix}$$

So $\{p_j, V_j/j\}$ converges almost surely to $\{\hat{p}, \hat{V}\}$. The claim in the Proposition is an implication of this convergence.

Proof of Proposition 6: We prove the proposition for the case of $\psi = L$, the other cases are similar. Let $p_B = (1-\gamma)\epsilon^B$, $p_S = \gamma\epsilon^S$ and $p_N = 1 - (p_B + p_S)$. Applying Bayes Law to find posterior probabilities as in equation (8) we have, for each t,

$$\log\left(\frac{\rho_{Ot+1}}{\rho_{Lt+1}}\right) = \log\left(\frac{1-\alpha}{\alpha\delta}\right) + \left(n_t \log p_N + \beta_t \log p_B + s_t \log p_S\right)$$
$$-\left(n_t \log((1-\mu)p_N) + \beta_t \log((1-\mu)p_B)\right.$$
$$\left.+ s_t \log(\mu + (1-\mu)p_S)\right),$$

and

$$\log\left(\frac{\rho_{Ht+1}}{\rho_{Lt+1}}\right) = \log\left(\frac{1-\delta}{\delta}\right) + \left(n_t \log((1-\mu)p_N) + \beta_t \log(\mu + (1-\mu)p_B)\right.$$
$$\left.+ s_t \log((1-\mu)p_S)\right) - \left(n_t \log((1-\mu)p_N) + \beta_t \log((1-\mu)p_B)\right.$$
$$\left.+ s_t \log(\mu + (1-\mu)p_S)\right).$$

So by the Strong Law of Large Numbers

$$\frac{1}{t}\log\left(\frac{\rho_{Ot+1}}{\rho_{Lt+1}}\right) \underset{\text{a.s.}}{\rightarrow} \sum_Q p^L(Q)\log(p^N(Q)) - \sum_Q p^L(Q)\log(p^L(Q))$$
$$= -I_{p^L}(p^O) < 0,$$

and

$$\frac{1}{t}\log\left(\frac{\rho_{Ht+1}}{\rho_{Lt+1}}\right) \underset{\text{a.s.}}{\rightarrow} \sum_Q p^L(Q)\log(p^H(Q)) - \sum_Q p^L(Q)\log(p^L(Q))$$
$$= -I_{p^L}(p^H) < 0,$$

Thus, ρ_{Ot}/ρ_{Lt} converges almost surely to zero at exponential rate $-I_{p^L}(p^O)$. and ρ_{Ht}/ρ_{Lt} converges almost surely to zero at exponential rate $-I_{p^L}(p^H)$.

By the equilibrium quote equation (6)

$$b_t - \underline{V} = \rho_{Lt}\underline{V} + \rho_{Ht}\overline{V} + \rho_{Ot}V^* - \underline{V}$$
$$= \rho_{Ht}(\overline{V} - \underline{V}) + \rho_{Ot}(V^* - \underline{V})$$
$$= (\rho_{Ht} + \rho_{Ot})(1-\delta)(\overline{V} - \underline{V}).$$

The convergence results above and simple calculation shows that ρ_{Ht} converges almost surely to zero at exponential rate $-I_{P^L}(p^H)$ and ρ_{Ot} converges almost surely to zero at exponential rate $-I_{p^L}(p^O)$. So $b_t - \underline{V}$ converges almost surely to zero at exponential rate—$\text{Min}\{I_{p^L}(p^H), I_{p^L}(p^O)\}$.

For the comparative dynamics results on μ it is sufficient to show that $I_{p^L}(p^H)$ and $I_{p^L}(p^O)$ are increasing in μ. Calculation shows that

$$\frac{\partial I_{p^L}(p^H)}{\partial \mu} = (1 - p_S)\log\left(1 + \frac{\mu}{(1-\mu)p_S}\right) + p_B\log\left(+ \frac{\mu}{(1-\mu)p_B}\right) > 0,$$

and

$$\frac{\partial I_{p^L}(p^O)}{\partial \mu} = (1 - p_S)\log\left(1 + \frac{\mu}{(1-\mu)p_S}\right) > 0.$$

For the comparative dynamics results on normal volume, $p = p_S = p_B$, it is sufficient to show that $I_{p^L}(p^H)$ and $I_{p^L}(p^O)$ are decreasing in p. Calculation shows that

$$\frac{\partial I_{p^L}(p^H)}{\partial p} = -\frac{\mu^2}{(1-\mu)p}\left(1 + \frac{\mu}{(1-\mu)p}\right)^{-1} < 0,$$

and

$$\frac{\partial I_{p^L}(p^O)}{\partial p} = (1 - \mu)\log\left(1 + \frac{\mu}{(1-\mu)p}\right) - \frac{\mu}{p} < 0$$

$$\text{as } \log\left(1 + \frac{\mu}{(1-\mu)p}\right) < \frac{\mu}{(1-\mu)p}.$$

Proof of Proposition 7: The claim about the initial prices follows directly from equations (2) and (3). We prove the convergence claim for the case of low signal, the other cases are similar. If a low signal occurs the market maker's posterior probability on a low signal at time t is $\rho_{Ot} = \alpha\delta[\alpha\delta + (1 - \alpha)(\gamma \epsilon^S)^t]$. This sequence of beliefs converges to one and the rate of convergence is decreasing in normal volume.

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