Solutions to the exercises in 2.2

Useful equations:
1. Tensor transformation rule:

$$T^{\,m'_1 \ldots m'_k}{}_{n'_1 \ldots n'_l} = \frac{\partial x^{m'_1}}{\partial x^{m_1}} \cdots \frac{\partial x^{m'_k}}{\partial x^{m_k}} \frac{\partial x^{n_1}}{\partial x^{n'_1}} \cdots \frac{\partial x^{n_l}}{\partial x^{n'_l}}\, T^{\,m_1 \ldots m_k}{}_{n_1 \ldots n_l}$$

2. The metric tensor defines the position-dependent differential distances on a space under some coordinate:

$$ds^2 = g_{mn} dx^m dx^n$$

3. The metric is symmetric:

$$g_{mn} = g_{nm}$$

4. The metric has an inverse:

$$g^{mn} g_{np} = \delta^m_p$$

5. The metric and the inverse metric can be used to lower and raise indices:

$$V_n = g_{mn} V^m$$

$$V^n = g^{mn} V_m$$

1. In this exercise we show that the metric g_{mn} is symmetric rigorously.
(a) Show that any 2-index object can be decomposed into a symmetric and an anti-symmetric part, i.e. given a 2-index object F_{mn}, it can be written as $F_{mn} = S_{mn} + A_{mn}$, where $S_{mn} = S_{nm}$ and $A_{mn} = -A_{nm}$.
(b) Show that $A_{mn} dx^m dx^n = 0$, where $A_{mn} = -A_{nm}$ is an arbitrary antisymmetric quantity. Why does this imply that g_{mn} has no antisymmetric part?

Soln.
(a) Any 2-index object F_{mn} can be split as

$$F_{mn} = \frac{1}{2}(F_{mn} + F_{nm}) + \frac{1}{2}(F_{mn} - F_{nm})$$

And it can be seen that $S_{mn} \equiv \frac{1}{2}(F_{mn} + F_{nm})$ is symmetric, and $A_{mn} \equiv \frac{1}{2}(F_{mn} - F_{nm})$ is antisymmetric.

For example, if $F_{mn} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $F_{nm} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. Therefore,

$$S_{mn} = \frac{1}{2}(F_{mn} + F_{nm}) = \frac{1}{2}\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\right] = \begin{pmatrix} 1 & 2.5 \\ 2.5 & 4 \end{pmatrix}$$

and

$$A_{mn} = \frac{1}{2}(F_{mn} - F_{nm}) = \frac{1}{2}\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\right] = \begin{pmatrix} 0 & -0.5 \\ 0.5 & 0 \end{pmatrix}$$

and it can be seen that indeed, S_{mn} is symmetric, A_{mn} is antisymmetric, and they add up to the original F_{mn}.

(b) We have

$$\begin{aligned} & A_{mn} dx^m dx^n && \text{(1a)} \\ &= -A_{nm} dx^m dx^n && \text{(1b)} \\ &= -A_{mn} dx^n dx^m && \text{(1c)} \\ &= -A_{mn} dx^m dx^n && \text{(1d)} \end{aligned}$$

From (1a) to (1b) we used $A_{mn} = -A_{nm}$, i.e. antisymmetricity of A_{mn}.

From (1b) to (1c) we renamed the dummy indices, where m is renamed to n and n is renamed to m. If this kind of "exchange-renaming" makes you feel uncomfortable, simply imagining first renaming these dummies in (1b) to something else, say $-A_{pq} dx^q dx^p$ (with (n, m) renamed to (p, q)), then renaming them to $-A_{mn} dx^n dx^m$ (with (p, q) renamed to (m, n)).

From (1c) to (1d) we simply swapped the last two terms, i.e. $dx^n dx^m = dx^m dx^n$. Everything in Einstein summation convention commutes. See footnote 8 on page 10 of note 2.2.

So we have (1a) and (1d) giving $A_{mn} dx^m dx^n = -A_{mn} dx^m dx^n$, which means

$$A_{mn} dx^m dx^n = 0$$

Why does this mean the metric has no antisymmetric part? From part (a) we know that any 2-index object can be decomposed into a symmetric and an anti-symmetric part. The metric g_{mn} is a 2-index object, so let's denote its decomposition into symmetric and antisymmetric parts by $g_{mn} = S_{mn} + A_{mn}$, where $S_{mn} = S_{nm}$ and $A_{mn} = -A_{nm}$. Therefore the differential distances are

$$ds^2 = g_{mn} dx^m dx^n = S_{mn} dx^m dx^n + A_{mn} dx^m dx^n = S_{mn} dx^m dx^n$$

where we drop the second term because $A_{mn} dx^m dx^n = 0$.

Thus, we have $ds^2 = S_{mn} dx^m dx^n$. Since S_{mn} is a symmetric quantity that takes the position of the metric in the definition for differential distances, it is thus the metric. Hence the metric is symmetric. Another way to look at it is that, even if the metric had some antisymmetric part, the antisymmetric part wouldn't contribute to the differential distances anyway, so we simply declare that it has none.

---

2. By using basic geometry alone, find the metric on:
(a) Cartesian 2D plane with Cartesian coordinates (x, y);
(b) Cartesian 2D plane with polar coordinates (r, θ);
(c) Cartesian 3D space with Cartesian coordinates (x, y, z);
(d) Cartesian 3D space with spherical coordinates (r, θ, ϕ) [1].
Convince yourself that in general, the metric components expressed in some arbitrary coordinate is not constant and is position-dependent.

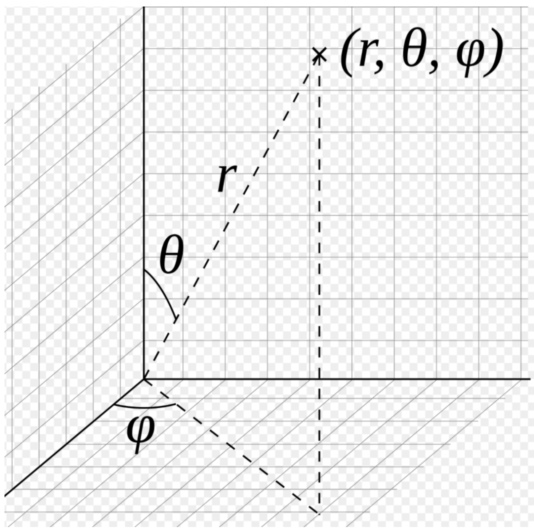

Figure 1: Convention for spherical coordinates. θ is the angle from the north pole, and ϕ is the angle along the equator.

[1]Note: while for 2D polar coordinates I don't have a preference for whether to call the polar angle by θ or by ϕ, the convention for 3D spherical coordinates is quite fixed, as shown in the figure. θ is the angle from the north pole, and ϕ is the angle along the equator.

Soln.

(a) Differential distances in Cartesian 2D plane with Cartesian coordinates (x, y) is simply

$$ds^2 = (dx)^2 + (dy)^2 = (dx^1)^2 + (dx^2)^2$$

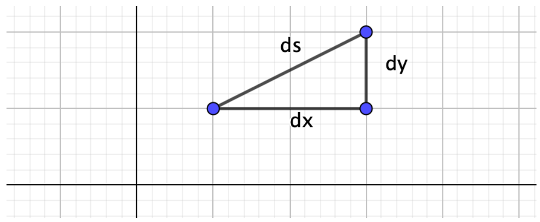

Figure 2: Differential distances in Cartesian 2D plane with Cartesian coordinates (x, y). x^1 stands for the x coordinate, and x^2 stands for the y coordinate.

which means $g_{11} = 1$, $g_{22} = 1$, and $g_{12} = g_{21} = 0$, so metric is

$$g_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) Differential distances in Cartesian 2D plane with polar coordinates (r, θ) is

$$ds^2 = (dr)^2 + r^2(d\theta)^2 = (1)drdr + (r^2)d\theta d\theta$$

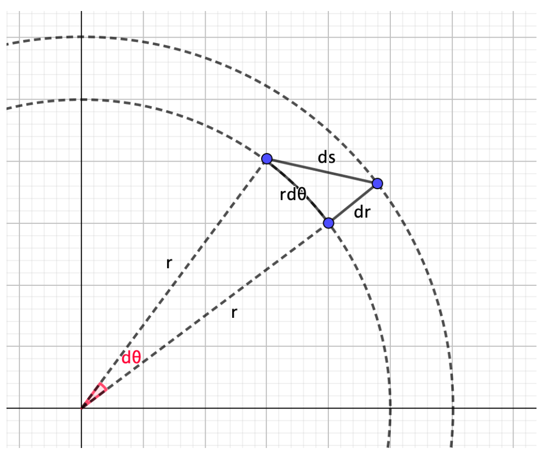

Figure 3: Differential distances in Cartesian 2D plane with polar coordinates (r, θ).

As per convention, the first metric index stands for r and the second metric index stands for θ, so we have $g_{11} = 1$, $g_{22} = r^2$, and $g_{12} = g_{21} = 0$, so metric is

$$g_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

(c) Differential distances in Cartesian 3D space with Cartesian coordinates (x, y, z) is simply

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

so the metric is

$$g_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) Differential distances in Cartesian 3D space with spherical coordinates (r, θ, ϕ) is

$$ds^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta(d\phi)^2$$

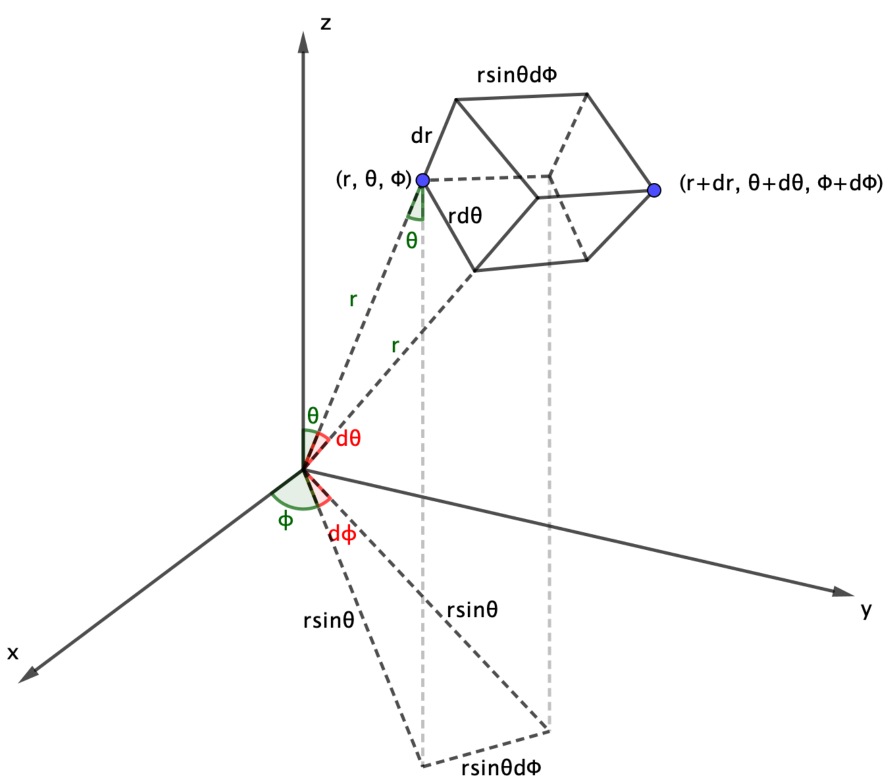

Figure 4: Differential distances in Cartesian 3D space with spherical coordinates (r, θ, ϕ).

As per convention, the first metric index stands for r, the second metric index stands for θ, and the third metric index stands for ϕ. Thus the metric is

$$g_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

3. Check that the metrics you found in question 2 obey the transformation rule for tensors.

Soln.

(a) *From 2D Cartesian coordinates to 2D polar coordinates, in Cartesian 2D plane*

Let the unprimed frame be the Cartesian coordinates, and primed frame be the polar coordinates, we have

$$g_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \, g_{m'n'} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

The necessarily derivatives between polar and Cartesian coordinates are (see last question of solutions to 2.1)

$$\frac{\partial x}{\partial r} = \cos\theta \, , \, \frac{\partial x}{\partial \theta} = -r\sin\theta \, , \, \frac{\partial y}{\partial r} = \sin\theta \, , \, \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \, , \, \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2} \, , \, \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}$$

We simply plug the values into the transformation

$$g_{m'n'} = \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^n}{\partial x^{n'}} g_{mn}$$

while noticing that the only nonzero g_{mn} terms are $g_{11} = g_{xx} = 1$ and $g_{22} = g_{yy} = 1$:

$$\begin{aligned} g_{1'1'} &= g_{rr} \\ &= \frac{\partial x^m}{\partial r} \frac{\partial x^n}{\partial r} g_{mn} \\ &= \frac{\partial x}{\partial r} \frac{\partial x}{\partial r} g_{xx} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial r} g_{yy} \\ &= (\cos\theta)(\cos\theta)(1) + (\sin\theta)(\sin\theta)(1) \\ &= 1 \end{aligned}$$

$$
\begin{aligned}
g_{2'2'} &= g_{\theta\theta} \\
&= \frac{\partial x^m}{\partial \theta}\frac{\partial x^n}{\partial \theta}\, g_{mn} \\
&= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta}\, g_{xx} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta}\, g_{yy} \\
&= (-r\sin\theta)(-r\sin\theta)(1) + (r\cos\theta)(r\cos\theta)(1) \\
&= r^2
\end{aligned}
$$

$$
\begin{aligned}
g_{1'2'} &= g_{r\theta} \\
&= \frac{\partial x^m}{\partial r}\frac{\partial x^n}{\partial \theta}\, g_{mn} \\
&= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta}\, g_{xx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta}\, g_{yy} \\
&= (\cos\theta)(-r\sin\theta)(1) + (\sin\theta)(r\cos\theta)(1) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{2'1'} &= g_{\theta r} \\
&= \frac{\partial x^m}{\partial \theta}\frac{\partial x^n}{\partial r}\, g_{mn} \\
&= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial r}\, g_{xx} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial r}\, g_{yy} \\
&= (-r\sin\theta)(\cos\theta)(1) + (r\cos\theta)(\sin\theta)(1) \\
&= 0
\end{aligned}
$$

And we have verified all components of $g_{m'n'}$. Notice that $g_{r\theta} = g_{\theta r}$, as expected. There was, in fact, no need to calculate $g_{\theta r}$ explicitly once we have calculated $g_{r\theta}$. If the metric g_{mn} were symmetric in one frame, it would be symmetric in all other frames. This is of course expected: $g_{mn} = g_{nm}$ is a tensor equation, and we know that tensor equations are true in every frame. Should you wish, you can easily prove this explicitly: if $g_{mn} = g_{nm}$, then $g_{m'n'} = g_{n'm'}$. This can be easily seen as follows:

$$
\begin{aligned}
g_{m'n'} &= \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^n}{\partial x^{n'}} g_{mn} \\
&= \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^n}{\partial x^{n'}} g_{nm} \\
&\quad \text{(since metric is symmetric in the unprimed frame, i.e. } g_{mn} = g_{nm}\text{)} \\
&= \frac{\partial x^n}{\partial x^{n'}} \frac{\partial x^m}{\partial x^{m'}} g_{nm} \\
&\quad \text{(changing order of terms is fine; everything commutes in Einstein notation)} \\
&= g_{n'm'}
\end{aligned}
$$

(b) *From 3D Cartesian coordinates to 3D spherical coordinates, in Cartesian 3D space*
We only calculate the upper triangular entries, since the lower ones can be obtained from symmetry.

Let the unprimed frame be the Cartesian coordinates, and primed frame be the spherical coordinates, we have

$$
g_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; g_{m'n'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}
$$

Spherical coordinates:

$$
\begin{cases}
x = r \sin\theta \cos\phi \\
y = r \sin\theta \sin\phi \\
z = r \cos\theta
\end{cases}
$$

I trust that you know how to obtain the derivatives.

Plugging in (again, only nonzero terms are $g_{11} = g_{xx} = 1$, $g_{22} = g_{yy} = 1$ and $g_{33} = g_{zz} = 1$):

$$
\begin{aligned}
g_{1'1'} &= g_{rr} \\
&= \frac{\partial x^m}{\partial r} \frac{\partial x^n}{\partial r} g_{mn} \\
&= \frac{\partial x}{\partial r} \frac{\partial x}{\partial r} g_{xx} + \frac{\partial y}{\partial r} \frac{\partial y}{\partial r} g_{yy} + \frac{\partial z}{\partial r} \frac{\partial z}{\partial r} g_{zz} \\
&= (\sin\theta\cos\phi)(\sin\theta\cos\phi)(1) + (\sin\theta\sin\phi)(\sin\theta\sin\phi)(1) + (\cos\theta)(\cos\theta)(1) \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
g_{1'2'} &= g_{r\theta} \\
&= \frac{\partial x^m}{\partial r}\,\frac{\partial x^n}{\partial \theta}\,g_{mn} \\
&= \frac{\partial x}{\partial r}\,\frac{\partial x}{\partial \theta}\,g_{xx} + \frac{\partial y}{\partial r}\,\frac{\partial y}{\partial \theta}\,g_{yy} + \frac{\partial z}{\partial r}\,\frac{\partial z}{\partial \theta}\,g_{zz} \\
&= (\sin\theta\cos\phi)(r\cos\theta\cos\phi)(1) + (\sin\theta\sin\phi)(r\cos\theta\sin\phi)(1) + (\cos\theta)(-r\sin\theta)(1) \\
&= r\sin\theta\cos\theta\cos^2\phi + r\sin\theta\cos\theta\sin^2\phi - r\sin\theta\cos\theta \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{1'3'} &= g_{r\phi} \\
&= \frac{\partial x^m}{\partial r}\,\frac{\partial x^n}{\partial \phi}\,g_{mn} \\
&= \frac{\partial x}{\partial r}\,\frac{\partial x}{\partial \phi}\,g_{xx} + \frac{\partial y}{\partial r}\,\frac{\partial y}{\partial \phi}\,g_{yy} + \frac{\partial z}{\partial r}\,\frac{\partial z}{\partial \phi}\,g_{zz} \\
&= (\sin\theta\cos\phi)(-r\sin\theta\sin\phi)(1) + (\sin\theta\sin\phi)(r\sin\theta\cos\phi)(1) + (\cos\theta)(0)(1) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{2'2'} &= g_{\theta\theta} \\
&= \frac{\partial x^m}{\partial \theta}\,\frac{\partial x^n}{\partial \theta}\,g_{mn} \\
&= \frac{\partial x}{\partial \theta}\,\frac{\partial x}{\partial \theta}\,g_{xx} + \frac{\partial y}{\partial \theta}\,\frac{\partial y}{\partial \theta}\,g_{yy} + \frac{\partial z}{\partial \theta}\,\frac{\partial z}{\partial \theta}\,g_{zz} \\
&= (r\cos\theta\cos\phi)(r\cos\theta\cos\phi)(1) + (r\cos\theta\sin\phi)(r\cos\theta\sin\phi)(1) + (-r\sin\theta)(-r\sin\theta)(1) \\
&= r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta \\
&= r^2
\end{aligned}
$$

$$
\begin{aligned}
g_{2'3'} &= g_{\theta\phi} \\
&= \frac{\partial x^m}{\partial \theta}\,\frac{\partial x^n}{\partial \phi}\,g_{mn} \\
&= \frac{\partial x}{\partial \theta}\,\frac{\partial x}{\partial \phi}\,g_{xx} + \frac{\partial y}{\partial \theta}\,\frac{\partial y}{\partial \phi}\,g_{yy} + \frac{\partial z}{\partial \theta}\,\frac{\partial z}{\partial \phi}\,g_{zz} \\
&= (r\cos\theta\cos\phi)(-r\sin\theta\sin\phi)(1) + (r\cos\theta\sin\phi)(r\sin\theta\cos\phi)(1) + (-r\sin\theta)(0)(1) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g_{3'3'} &= g_{\phi\phi} \\
&= \frac{\partial x^m}{\partial \phi}\frac{\partial x^n}{\partial \phi}\, g_{mn} \\
&= \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \phi}\, g_{xx} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \phi}\, g_{yy} + \frac{\partial z}{\partial \phi}\frac{\partial z}{\partial \phi}\, g_{zz} \\
&= (-r\sin\theta\sin\phi)(-r\sin\theta\sin\phi)(1) + (r\sin\theta\cos\phi)(r\sin\theta\cos\phi)(1) + (0)(0)(1) \\
&= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi \\
&= r^2\sin^2\theta
\end{aligned}
$$

And we have verified all components of $g_{m'n'}$.

Notice that indeed, the metric components are functions of position.

---

4. Find the metric on the surface of a sphere of fixed radius R. Note that this is a two-dimensional space. Use the same angular convention as that in spherical coordinates for a 3D Cartesian space.

Soln.
The surface of a sphere of fixed radius R is simply spherical coordinates on 3D Cartesian space [2] with $dr = 0$. The differential distance in spherical coordinates is

$$ds^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$$

Thus, on a 2D subspace with constant $r = R$, i.e. $dr = 0$, the differential distance reduces to

$$ds^2 = R^2(d\theta)^2 + R^2\sin^2\theta(d\phi)^2$$

So the metric is

$$g_{mn} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

Notice that since the space is 2-dimensional, the metric is a 2-by-2 matrix, as opposed to a 3-by-3 matrix for spherical coordinates on a 3-dimensional space. The two coordinate variables are (θ, ϕ), with $g_{11} = g_{\theta\theta} = R^2$ and $g_{22} = g_{\phi\phi} = R^2\sin^2\theta$.

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[2] Note that smaller case r is the coordinate variable for spherical coordinates on 3D Cartesian space, and upper case R is the fixed radius (like maybe 24) of a given sphere.

5. With raising and lowering indices, show that $A^m B_m = A_n B^n$.

Soln.

$$\begin{aligned} A^m B_m &= (g^{mn} A_n)(g_{pm} B^p) \\ &= g^{mn} g_{pm} A_n B^p \\ &= \delta^n_p A_n B^p \\ &= A_n B^n \end{aligned}$$

In the last equal sign we summed over the Kronecker delta: every term with $p \neq n$ vanishes since the delta is zero, and the term with $p = n$ has the delta being one. This is the last time I will explicitly explain how to sum over a Kronecker delta.

Also, make sure to use different dummy variables (n and p in this case) for different summations.

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6. Simplify the following:

$$g^{\mu\gamma}\, T^{\alpha\beta}{}_{\gamma\delta} = ?$$

$$g_{\mu\alpha}\, T^{\alpha\beta}{}_{\gamma\delta} = ?$$

$$g_{\mu\alpha}\, g_{\nu\beta}\, g^{\rho\gamma}\, g^{\sigma\delta}\, T^{\alpha\beta}{}_{\gamma\delta} = ?$$

$$g^{st}\, g_{st} = ?$$

For the last one, think carefully before proceeding. What is the type of this object? Is it a scalar or a tensor? Why? If it were a tensor, what is its rank?

Soln.

$$g^{\mu\gamma}\, T^{\alpha\beta}{}_{\gamma\delta} = T^{\alpha\beta\mu}{}_{\delta}$$

$$g_{\mu\alpha}\, T^{\alpha\beta}{}_{\gamma\delta} = T_{\mu}{}^{\beta}{}_{\gamma\delta}$$

$$g_{\mu\alpha}\, g_{\nu\beta}\, g^{\rho\gamma}\, g^{\sigma\delta}\, T^{\alpha\beta}{}_{\gamma\delta} = T_{\mu\nu}{}^{\rho\sigma}$$

For the last one, notice that in $g^{st}\, g_{st}$, there are no real indices. There are two pairs of dummy indices s and t. Therefore, this quantity doesn't have any real index and is thus not a set of numbers, but just one single number. Hence it's not a tensor but a scalar.

We can see that $g^{st}\, g_{st}$ is a scalar in two ways. The first way is to observe that its dummy indices look like an inner product. Therefore we can use the same technique as those we used when showing the inner product between vectors are frame-invariant (2.1, question 7).

The other way is more straightforward: we can just explicitly calculate what this number is.

$$\begin{aligned} g^{st}\, g_{st} &= g^{st}\, g_{ts} \\ &= \delta^s_s \\ &= \delta^1_1 + \delta^2_2 + \delta^3_3 + ... \\ &= 1 + 1 + 1 + ... \\ &= N \end{aligned}$$

where N is the dimensionality of the space in question.

In the first equal sign we used the symmetry of the metric: $g_{st} = g_{ts}$.

In the second equal sign we are using the definition of the inverse metric $g^{mn} g_{np} = \delta^m_p$, while noticing that in our case we plug in $m = s$ and $p = s$.

In the third equal sign we are expanding the summation over s. s is still a dummy summation index: it occurs as a upstairs-downstairs pair.

In the forth equal sign we are evaluating the Kronecker deltas.

In the fifth equal sign we add up all the ones. The range of s is the dimensionality of the metric (since s was the row (and the column) index on the metric), therefore in a two-dimensional space with a 2-by-2 metric, s would go from 1 to 2, giving two terms and hence two ones. The sum would be 2. In a three-dimensional space, the sum would be 3. In a four-dimensional space (like the spacetime in our universe), the sum would be 4.

Since $g^{st}\, g_{st}$ is just a pure number (like 4), it of course does not transform between coordinates, and is thus a scalar.

If you wish, you can check this with an explicit example. In 2-D space with Cartesian coordinates, the metric is the identity matrix, so the inverse metric is again the identity matrix. We would thus have

$$g^{st}\, g_{st} = g^{11}\, g_{11} + g^{22}\, g_{22} = (1)(1) + (1)(1) = 2$$

where all the non-diagonal terms vanish since this specific metric is diagonal.

Or, with polar coordinates:

$$g^{st}\, g_{st} = g^{11}\, g_{11} + g^{22}\, g_{22} = (1)(1) + (\frac{1}{r^2})(r^2) = 2$$

Reminder: metric in polar coordinates is

$$g_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

so inverse metric is

$$g^{mn} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

To invert a diagonal matrix, simply take the reciprocal of the diagonal elements.

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7. A vector in polar coordinates on the 2D Cartesian plane is given by

$$A^m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let unprimed indices denote polar coordinates and primed indices denote Cartesian coordinates.
(a) Find $A_{m'}$ by (1) first finding A_m and then transforming it to the primed frame, and (2) first transforming to $A^{m'}$ and then lowering the index. Your answer should agree with each other.
(b) Explicitly verify that $A^m A_m = A^{m'} A_{m'}$, i.e. inner-product is frame-invariant. What does this inner product represent? Does it make physical sense? Checking the last question in the exercises for 2.1 might be helpful for this question.

Soln.
(a) We are on 2D polar coordinates on Cartesian plane, so the metric and inverse metric are

$$g_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \; g^{mn} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

(1) We first lower the index by $A_n = g_{mn} A^m$:

$$\begin{aligned} A_1 &= g_{m1} A^m \\ &= g_{11} A^1 + g_{21} A^2 \\ &= (1)(0) + (0)(1) \\ &= 0 \end{aligned}$$

$$\begin{aligned} A_2 &= g_{m2} A^m \\ &= g_{12} A^1 + g_{22} A^2 \\ &= (0)(0) + (r^2)(1) \\ &= r^2 \end{aligned}$$

Therefore $A_m = \begin{pmatrix} 0 \\ r^2 \end{pmatrix}$.

We now transform A_m into Cartesian coordinates to obtain $A_{m'}$ (see page 7 for the derivatives):

$$\begin{aligned} A_{1'} &= \frac{\partial x^m}{\partial x^{1'}} A_m \\ &= \frac{\partial x^1}{\partial x^{1'}} A_1 + \frac{\partial x^2}{\partial x^{1'}} A_2 \\ &= \frac{\partial r}{\partial x} A_r + \frac{\partial \theta}{\partial x} A_\theta \\ &= (\frac{x}{\sqrt{x^2+y^2}})(0) + (-\frac{y}{x^2+y^2})(r^2) \\ &= -y \end{aligned}$$

$$\begin{aligned} A_{2'} &= \frac{\partial x^m}{\partial x^{2'}} A_m \\ &= \frac{\partial x^1}{\partial x^{2'}} A_1 + \frac{\partial x^2}{\partial x^{2'}} A_2 \\ &= \frac{\partial r}{\partial y} A_r + \frac{\partial \theta}{\partial y} A_\theta \\ &= (\frac{y}{\sqrt{x^2+y^2}})(0) + (\frac{x}{x^2+y^2})(r^2) \\ &= x \end{aligned}$$

where we canceled r^2 with (x^2+y^2). Therefore, we have $A_{m'} = \begin{pmatrix} -y \\ x \end{pmatrix}$.

(2) We first transform to Cartesian coordinates:

$$\begin{aligned}
A^{1'} &= \frac{\partial x^{1'}}{\partial x^m} A^m \\
&= \frac{\partial x^{1'}}{\partial x^1} A^1 + \frac{\partial x^{1'}}{\partial x^2} A^2 \\
&= \frac{\partial x}{\partial r} A^r + \frac{\partial x}{\partial \theta} A^\theta \\
&= (\cos\theta)(0) + (-r\sin\theta)(1) \\
&= -y
\end{aligned}$$

$$\begin{aligned}
A^{2'} &= \frac{\partial x^{2'}}{\partial x^m} A^m \\
&= \frac{\partial x^{2'}}{\partial x^1} A^1 + \frac{\partial x^{2'}}{\partial x^2} A^2 \\
&= \frac{\partial y}{\partial r} A^r + \frac{\partial y}{\partial \theta} A^\theta \\
&= (\sin\theta)(0) + (r\cos\theta)(1) \\
&= x
\end{aligned}$$

Therefore, we have $A^{m'} = \begin{pmatrix} -y \\ x \end{pmatrix}$.

We now lower the index, keeping in mind that in the primed frame (Cartesian), the metric is $g_{m'n'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Using $A_{n'} = g_{m'n'} A^{m'}$:

$$\begin{aligned}
A_{1'} &= g_{m'1'} A^{m'} \\
&= g_{1'1'} A^{1'} + g_{2'1'} A^{2'} \\
&= (1)(-y) + (0)(x) \\
&= -y
\end{aligned}$$

$$\begin{aligned}
A_{2'} &= g_{m'2'} A^{m'} \\
&= g_{1'2'} A^{1'} + g_{2'2'} A^{2'} \\
&= (0)(-y) + (1)(x) \\
&= x
\end{aligned}$$

Therefore, we have $A_{m'} = \begin{pmatrix} -y \\ x \end{pmatrix}$. The results we obtained from the two routes do agree with each other!

If you have been wondering why you haven't heard of the division between covariant and contravariant vectors throughout all these years of physics study, here you can see the answer. In Cartesian coordinates, the components of a contravariant vector and its covariant dual are the same! Therefore if you are doing classical physics in Cartesian coordinates, there's really no need to make the distinction between the two kind of vectors. The inner product for the length of a vector $A^m A_m$ simply reduces to the familiar $\sum_m (A_m)^2$ (i.e. the Pythagorean theorem [3]), since A_m and A^m are the same.

(b) The results we obtained are

$$A^m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; A_m = \begin{pmatrix} 0 \\ r^2 \end{pmatrix}, \; A^{m'} = \begin{pmatrix} -y \\ x \end{pmatrix}, \; A_{m'} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Therefore,

$$A^m A_m = A^1 A_1 + A^2 A_2 = (0)(0) + (1)(r^2) = r^2$$

$$A^{m'} A_{m'} = A^{1'} A_{1'} + A^{2'} A_{2'} = (-y)(-y) + (x)(x) = y^2 + x^2 = r^2$$

so they are indeed the same.

Physical meaning: the original vector $A^m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is contravariant, so it's a displacement expressed in polar coordinates. Its magnitude $A^m A_m$ is the distance covered by this displacement. Since the vector is the same (only expressed in different coordinates), the distance covered should also be the same regardless of whether we use polar or Cartesian coordinates. The distance covered is r^2, as expected (see 2.1, question 8(a)).

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[3] It often amazes me how much of modern science and technology hinges on the seemingly-random fact that the sum of the squares of the two sides of a right-angled triangle equals the square of the hypotenuse.

8. Consider the usual Cartesian coordinates (x^1, x^2) on the usual Cartesian 2D plane. A second coordinate system is obtained by stretching the vertical axis by a factor of 2:

$$\begin{cases} x^{1'} = x^1 \\ x^{2'} = \dfrac{1}{2}x^2 \end{cases}$$

Verify that the differential distance is the same in both frames, i.e.

$$g_{mn}dx^m dx^n = g_{m'n'}dx^{m'}dx^{n'}$$

Reminder: the metric tensor transforms like, well, a tensor.

Soln.
We simply need the metric in the primed frame. Using $g_{m'n'} = \dfrac{\partial x^m}{\partial x^{m'}}\dfrac{\partial x^n}{\partial x^{n'}}g_{mn}$, and again noting that the only non-zero terms in g_{mn} are $g_{11} = g_{22} = 1$:

$$\begin{aligned} g_{1'1'} &= \frac{\partial x^m}{\partial x^{1'}}\frac{\partial x^n}{\partial x^{1'}}g_{mn} \\ &= \frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{1'}}g_{11} + \frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{1'}}g_{22} \\ &= (1)(1)(1) + (0)(0)(1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} g_{2'2'} &= \frac{\partial x^m}{\partial x^{2'}}\frac{\partial x^n}{\partial x^{2'}}g_{mn} \\ &= \frac{\partial x^1}{\partial x^{2'}}\frac{\partial x^1}{\partial x^{2'}}g_{11} + \frac{\partial x^2}{\partial x^{2'}}\frac{\partial x^2}{\partial x^{2'}}g_{22} \\ &= (0)(0)(1) + (2)(2)(1) \\ &= 4 \end{aligned}$$

$$\begin{aligned} g_{1'2'} &= \frac{\partial x^m}{\partial x^{1'}}\frac{\partial x^n}{\partial x^{2'}}g_{mn} \\ &= \frac{\partial x^1}{\partial x^{1'}}\frac{\partial x^1}{\partial x^{2'}}g_{11} + \frac{\partial x^2}{\partial x^{1'}}\frac{\partial x^2}{\partial x^{2'}}g_{22} \\ &= (1)(0)(1) + (0)(2)(1) \\ &= 0 \\ &= g_{2'1'} \quad \text{(by symmetry)} \end{aligned}$$

Therefore,

$$\begin{aligned}
g_{m'n'}dx^{m'}dx^{n'} &= g_{1'1'}dx^{1'}dx^{1'} + g_{2'2'}dx^{2'}dx^{2'} \\
&\quad \text{(since the only non-zero components of } g_{m'n'} \text{ are } g_{1'1'} \text{ and } g_{2'2'}) \\
&= dx^{1'}dx^{1'} + 4dx^{2'}dx^{2'} \\
&= dx^1dx^1 + 4(\frac{1}{2}dx^2)(\frac{1}{2}dx^2) \\
&= (dx^1)^2 + (dx^2)^2 \\
&= (1)dx^1dx^1 + (0)dx^1dx^2 + (0)dx^2dx^1 + (1)dx^2dx^2 \\
&= g_{mn}dx^mdx^n
\end{aligned}$$

where we used the linearity of differential quantities: since $x^{2'} = \frac{1}{2}x^2$, so $dx^{2'} = \frac{1}{2}dx^2$.

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9. In this question we look at the inverse metric g^{mn} from another point of view.
(a) Show that the metric g_{mn} cannot have zero as an eigenvalue. Hint: consider $dx_m dx^m$.
(b) Show that if a matrix doesn't have zero as an eigenvalue, then it has an inverse.
Combining (a) and (b), we can see that g_{mn} has an inverse.

Soln.
(a) If the metric g_{mn} indeed has zero has an eigenvalue, let's denote the eigenvector with eigenvalue zero by dx^n. Thus, $g_{mn}dx^n = 0dx^n = 0$, by the definition of eigenvectors.

Now, since dx^n is an eigenvector, it is by definition non-zero. Thus it has some nonzero magnitude, i.e. $dx_n dx^n \neq 0$. Renaming the dummy index from n to m, we have $dx_m dx^m \neq 0$.

But on the other hand, $dx_m dx^m = g_{mn}dx^n dx^m = (g_{mn}dx^n)dx^m = 0dx^m = 0$, which leads to a contradiction.

Therefore, the metric g_{mn} cannot have zero as an eigenvalue.

(b) We know that the determinant of a matrix is the product of its eigenvalues [4]. Since g_{mn} doesn't have zero as an eigenvalue, it thus has a non-zero determinant, thus it has an inverse.

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[4] Or this proof is quite good too.

10. We already know that the metric is symmetric: $g_{mn} = g_{nm}$. Show that the inverse metric is also symmetric, i.e.

$$g^{mn} = g^{nm}$$

Soln.

$$g^{mn} \tag{2a}$$
$$=g^{mp}g^{nq}g_{pq} \tag{2b}$$
$$=g^{nq}g^{mp}g_{pq} \tag{2c}$$
$$=g^{nq}g^{mp}g_{qp} \tag{2d}$$
$$=g^{nm} \tag{2e}$$

From (2a) to (2b): we look at g_{pq} as the tensor whose indices are being raised. The lower p is raised into an upper m by g^{mp}, and the lower q is raised into an upper n by g^{nq}.

From (2b) to (2c): multiplication commutes. We are simply swapping g^{mp} and g^{nq}.

From (2c) to (2d): metric with lower indices is symmetric, i.e. $g_{pq} = g_{qp}$.

From (2d) to (2e): raising indices.

* By the same procedure, it is obvious that if any tensor with two lower indices is symmetric, then its dual with two upper indices is also symmetric, i.e. if $T_{mn} = T_{nm}$, then $T^{mn} = T^{nm}$.

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11. *Everything is locally flat.*

As said in 2.2, the solution to this question will be discussed in the body of note 2.3.