2.3 Everything is locally flat

Often in physics we speak about infinitesimal quantities. For example, the all-mighty chain rule

$$df = \frac{df}{dt}dt$$

is a statement about infinitesimal quantities. It states that the change in some dependent variable f is directly proportional to the change in some independent variable t, with the derivative df/dt being the ratio, if the changes were infinitesimally small.

Why are infinitesimal quantities so useful? This is because no matter how complicated the global structure is, **if we only look at the immediate local vicinity around some location, then the relationships between variables will reduce to a linear relationship**. In the above example, $f(t)$ could have been literally *any* function, and it can be very non-linear. However, the linear relationship of chain rule holds between the infinitesimal quantities regardless of how non-linear $f(t)$ globally is.

The same idea of "local = linear" lends itself naturally to the study of spaces. A space could be globally curved, but we would expect locally around any point on the space, the immediate vicinity around that point should look pretty flat.

This notion of "everything is locally flat" is easily confirmed by everyday experience. Unless you are a member of the flat-earth society, you would agree that the surface of the earth is a curved 2 dimensional space: a sphere with some fixed radius R. In the exercises for 2.2 we derived the metric on such a spherical surface:

$$g_{mn} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}$$

The two coordinate variables are (θ, ϕ), with $g_{11} = g_{\theta\theta} = R^2$ and $g_{22} = g_{\phi\phi} = R^2 \sin^2\theta$. θ is the angle from the north pole, and ϕ is the angle along the equator.

The (θ, ϕ) coordinate system on a sphere is a global coordinate: it covers every position on the sphere. Under this global coordinate, the metric is not δ_{mn}. However, for us living on the surface of the earth, we rarely use these global coordinates (what's the last time you used latitude and longitude in your life?). Rather, we use a set of very "Cartesian-like" coordinates: we give directions using north, south, east and west. East is effectively the positive x axis, and north is effectively the positive y axis. In the immediate vicinity around us, in our day-to-day lives, we have no need for these global curved coordinates, and there exists a set of local Cartesian coordinates we can use. The immediate vicinity around us looks pretty flat.

What does this mean mathematically? We know that for flat spaces, there exists a Cartesian coordinate such that $g_{mn} = \delta_{mn}$ everywhere on the space. This corresponds to the fact that $ds^2 = dx^2 + dy^2$ on the Cartesian 2D plane, regardless of your position. This means that if we can't find globally Cartesian coordinates, then the space is curved.

There are no coordinates that cover the entire radius-R sphere such that $g_{mn} = \delta_{mn}$ everywhere, because the sphere is a curved space. However, from our discussion above, it seems that the immediate neighborhood around any point on the sphere is flat. There should, therefore, exist coordinates on the immediate neighborhood around some point, such that $g_{mn} = \delta_{mn}$ everywhere in that neighborhood [1].

Before proving this, let's see this idea of local flatness with an example (the last question of 2.2).

(a) Alice lives on the surface of the earth. Without loss of generality, let's call her home the North pole (i.e. $\theta = 0$). Find the set of local Cartesian coordinates Alice uses in her everyday life in the vicinity around her. You might find the next couple of figures helpful. Reminder: a set of new coordinate variables is nothing but a function of the old coordinate variables.

Soln. From the next two figures, it is easily seen that in Alice's daily life, the local Cartesian coordinates she uses on the immediate neighborhood around her is

$$\begin{cases} x = R\theta\cos\phi \\ y = R\theta\sin\phi \end{cases}$$

[1]The technical name for these Cartesian-like coordinates in the local neighborhood is **Gaussian normal coordinates**. I will continue with "local Cartesian-like coordinates" (or sometimes just "local Cartesian coordinates") to keep the amount of new terminology to a minimum.

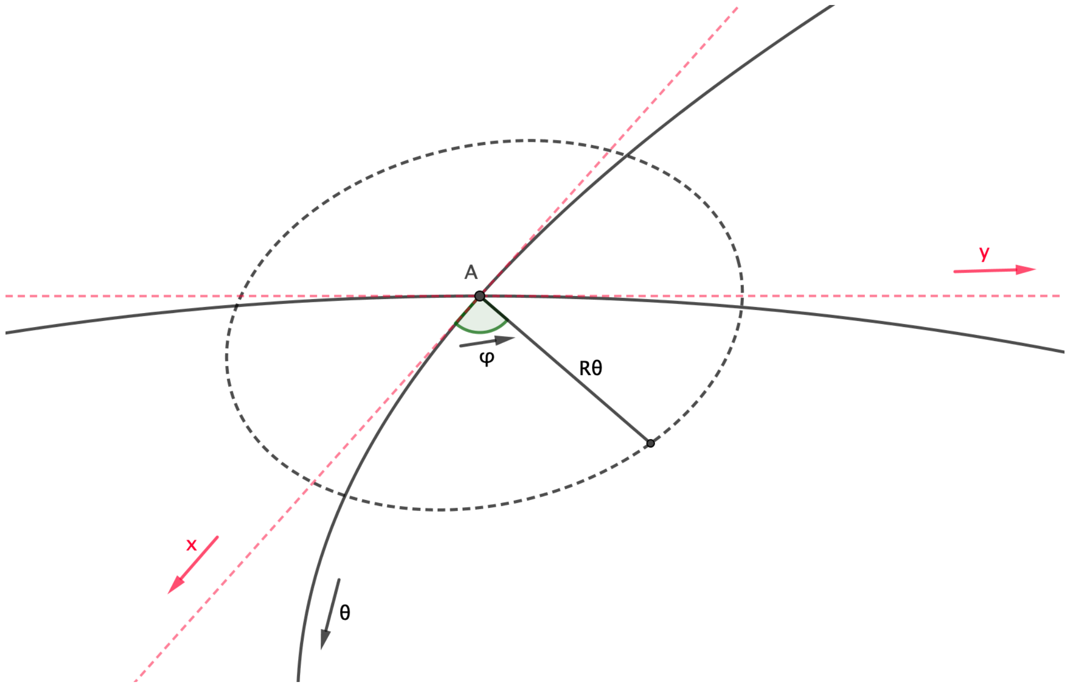

Figure 1: Alice (point A) living at the north pole. The global coordinates (θ, ϕ) are shown in black, and the local Cartesian coordinates (x, y) are shown in red and dashed.

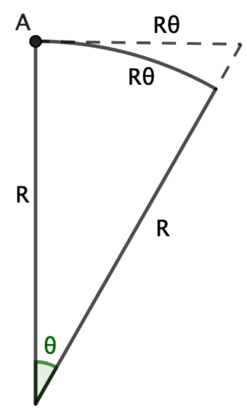

Figure 2: Alice (point A) living at the north pole. The radius of the earth is R. A small local arc around Alice has the same length as a small flat segment near her (since they are small), so both are $R\theta$. The small flat segment near her is the radius of a small circle around her in her flat local Cartesian coordinates.

(b) Find the metric in the local Cartesian coordinate around Alice. What do you notice?

Soln. Having been through the exercises in 2.2 (of course you've done them, right?), this should now be a piece of cake for us. The tensor transformation rule is

$$g_{m'n'} = \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^n}{\partial x^{n'}} g_{mn}$$

Let the unprimed frame be the global frame (θ, ϕ), and the primed frame be the local frame (x, y). The first step is to find the derivatives. The necessary derivatives are of the form "d (unprimed) / d (primed)", i.e. stuff like $\dfrac{\partial \theta}{\partial x}$. Therefore we need the inverse coordinate transform first. Since

$$\begin{cases} x = R\theta \cos\phi \\ y = R\theta \sin\phi \end{cases}$$

so obviously we have

$$\begin{cases} \theta = \dfrac{1}{R}\sqrt{x^2 + y^2} \\ \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

Remembering that the derivative of $\tan^{-1}(x)$ is $\dfrac{1}{1+x^2}$, we have the derivatives:

$$\frac{\partial \theta}{\partial x} = \frac{1}{R}\frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial \theta}{\partial y} = \frac{1}{R}\frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial \phi}{\partial x} = \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial \phi}{\partial y} = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2}$$

Thus, plugging in $g_{11} = g_{\theta\theta} = R^2$, $g_{22} = g_{\phi\phi} = R^2 \sin^2\theta$, and $g_{12} = g_{21} = 0$, we have

$$\begin{aligned}
g_{1'1'} &= g_{xx} \\
&= \frac{\partial x^m}{\partial x}\frac{\partial x^n}{\partial x} g_{mn} \\
&= \frac{\partial \theta}{\partial x}\frac{\partial \theta}{\partial x} g_{\theta\theta} + \frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial x} g_{\phi\phi} \\
&= \left(\frac{1}{R}\frac{x}{\sqrt{x^2+y^2}}\right)^2 R^2 + \left(-\frac{y}{x^2+y^2}\right)^2 R^2\sin^2\theta \\
&= \frac{x^2}{x^2+y^2} + \frac{y^2}{(x^2+y^2)^2} R^2 \sin^2\theta
\end{aligned}$$

Since we are in a small neighborhood around $\theta = 0$, we have $\sin\theta \approx \theta$, so

$$\begin{aligned}
g_{xx} &= \frac{x^2}{x^2+y^2} + \frac{y^2}{(x^2+y^2)^2} R^2\theta^2 \\
&= \frac{x^2}{x^2+y^2} + \frac{y^2}{(x^2+y^2)^2}(x^2+y^2) \\
&\quad (\text{since } x^2+y^2 = R^2\theta^2) \\
&= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} \\
&= 1
\end{aligned}$$

As you can imagine, the local approximation we used here will be repeatedly used, so let's single this result out: $R^2\sin^2\theta \approx R^2\theta^2 = x^2 + y^2$

Moving on:

$$\begin{aligned}
g_{1'2'} &= g_{xy} \\
&= \frac{\partial x^m}{\partial x}\frac{\partial x^n}{\partial y} g_{mn} \\
&= \frac{\partial \theta}{\partial x}\frac{\partial \theta}{\partial y} g_{\theta\theta} + \frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y} g_{\phi\phi} \\
&= \left(\frac{1}{R}\frac{x}{\sqrt{x^2+y^2}}\right)\left(\frac{1}{R}\frac{y}{\sqrt{x^2+y^2}}\right) R^2 + \left(-\frac{y}{x^2+y^2}\right)\left(\frac{x}{x^2+y^2}\right) R^2\sin^2\theta \\
&= \frac{xy}{x^2+y^2} - \frac{xy}{(x^2+y^2)^2}(x^2+y^2) \\
&= 0 \\
&= g_{yx} \qquad \text{(by symmetry of metric)}
\end{aligned}$$

$$
\begin{aligned}
g_{2'2'} &= g_{yy} \\
&= \frac{\partial x^m}{\partial y} \frac{\partial x^n}{\partial y} g_{mn} \\
&= \frac{\partial \theta}{\partial y} \frac{\partial \theta}{\partial y} g_{\theta\theta} + \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial y} g_{\phi\phi} \\
&= \left(\frac{1}{R} \frac{y}{\sqrt{x^2 + y^2}} \right)^2 R^2 + \left(\frac{x}{x^2 + y^2} \right)^2 R^2 \sin^2 \theta \\
&= \frac{y^2}{x^2 + y^2} + \frac{x^2}{(x^2 + y^2)^2} (x^2 + y^2) \\
&= 1
\end{aligned}
$$

Magically, in the local (x, y) frame around the immediate neighborhood of Alice, the metric is $g_{m'n'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{m'n'}$! Since locally around Alice, there exists a coordinate system in which the metric is the Kronecker delta, we conclude that **space is locally flat around Alice!**

There is another perspective when viewing local flatness. For example, imagine a parabola, like $y = x^2 + 1$. The vertex of this parabola $(0, 1)$ is locally flat. The value of the parabola (which is $y = 1$) is not what characterizes the local flatness, for that if we are talking about a different parabola (like $y = x^2 + 2$), the value at the vertex would be different (which is $y = 2$). Rather, the defining trait of a parabola's flatness at its vertex is that **the first derivative vanishes, and the second derivative doesn't vanish**.

The requirement that the first derivative vanishes is not hard to understand: it is the requirement that the value of the parabola stays constant (i.e. the parabola staying flat) in a small neighborhood around the vertex. The requirement that the second derivative doesn't vanish is also important, because as you know, second derivative encodes curvature. If the second derivative also vanishes, then there would be no curvature, and our "parabola" would stay constant not just in a small neighborhood around the vertex, but also globally as well. Therefore, if we seek *local* flatness instead of global flatness, both criteria have to hold.

Therefore, to check local flatness around Alice, apart from explicitly seeing that the metric really is δ_{mn}, we can also just check the first and second derivatives.

How should we proceed to check these derivatives? We have the metric in two forms: in the global frame (θ, ϕ) it is $\begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}$, and in the local frame (x, y) it is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Since the metric components are all constant in the local frame, checking their derivatives is pretty pointless [2]. Therefore, let's check the first and second derivatives of the metric in the global frame.

What should we differentiate the global metric with respect to? Well, differentiating the global metric with respect to the global coordinates are meaningless if we want to look at local properties. Therefore, the only remaining choice that makes sense is to take the derivative of the global metric with respect to the local coordinates. So let's do that now.

(c) Find the first derivatives of all the metric components with respect to the local coordinates around Alice, i.e. find $\frac{\partial g_{mn}}{\partial x}$ and $\frac{\partial g_{mn}}{\partial y}$ for all $m, n = \theta, \phi$. Evaluate the derivatives at Alice's location (north pole). What do you notice? Does your result make sense?
Reminder: R, the radius of the sphere, is a fixed constant.

Soln. Three components, namely $g_{\theta\theta} = R^2$ and $g_{\theta\phi} = g_{\phi\theta} = 0$ are constants, so their derivatives to everything vanishes.

Therefore, we only need to check $\frac{\partial g_{\phi\phi}}{\partial x}$ and $\frac{\partial g_{\phi\phi}}{\partial y}$:

$$\begin{aligned}
\frac{\partial g_{\phi\phi}}{\partial x} &= \frac{\partial}{\partial x}(R^2 \sin^2\theta) \\
&= \frac{\partial}{\partial \theta}(R^2 \sin^2\theta)\frac{\partial \theta}{\partial x} \qquad \text{(chain rule)} \\
&= R^2\, 2\sin\theta\cos\theta \frac{1}{R}\frac{x}{\sqrt{x^2+y^2}} \\
&= R\sin 2\theta \frac{x}{\sqrt{x^2+y^2}} \qquad (1)
\end{aligned}$$

Evaluating at Alice's position, which is $x = y = 0$, we see that $\frac{\partial g_{\phi\phi}}{\partial x} = 0$, as expected.

[2]Another way to look at it is that, since the metric in the (x, y) frame is only valid locally, its second derivative is not a meaningful quantity at all. Local quantities are, roughly speaking, valid within one small step of the point of interest (Alice), whereas second derivatives are information about things that are two small steps away from the point of interest.

Similarly,

$$\begin{aligned}
\frac{\partial g_{\phi\phi}}{\partial y} &= \frac{\partial}{\partial y}(R^2 \sin^2 \theta) \\
&= \frac{\partial}{\partial \theta}(R^2 \sin^2 \theta)\frac{\partial \theta}{\partial y} \qquad \text{(chain rule)} \\
&= R^2\, 2 \sin\theta \cos\theta \frac{1}{R}\frac{y}{\sqrt{x^2+y^2}} \\
&= R \sin 2\theta \frac{y}{\sqrt{x^2+y^2}}
\end{aligned}$$

Evaluating at Alice's position, which is $x = y = 0$, we see that $\dfrac{\partial g_{\phi\phi}}{\partial y} = 0$, as expected.

Therefore, the first derivatives of all metric components to any of the local Cartesian coordinate variables vanish, as expected. Explicitly,

$$\frac{\partial g_{mn}}{\partial x^s} = 0 \ \forall \ (m, n) = (\theta, \phi), x^s = (x, y)$$

Even more explicitly, this says

$$\frac{\partial g_{\theta\theta}}{\partial x} = \frac{\partial g_{\theta\theta}}{\partial y} = \frac{\partial g_{\theta\phi}}{\partial x} = \frac{\partial g_{\theta\phi}}{\partial y} = \frac{\partial g_{\phi\theta}}{\partial x} = \frac{\partial g_{\phi\theta}}{\partial y} = \frac{\partial g_{\phi\phi}}{\partial x} = \frac{\partial g_{\phi\phi}}{\partial y} = 0$$

Reminder: $\forall$ is "for all".

(d) Find $\dfrac{\partial^2 g_{\phi\phi}}{\partial x^2}$, again evaluated at Alice's location (north pole). What do you notice? Does your result make sense?

Soln. Continuing from (1):

$$\begin{aligned}
\frac{\partial^2 g_{\phi\phi}}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial g_{\phi\phi}}{\partial x} \\
&= \frac{\partial}{\partial x}\left(R\sin 2\theta \frac{x}{\sqrt{x^2+y^2}}\right) \\
&\approx \frac{\partial}{\partial x}\left(R\,2\theta \frac{x}{\sqrt{x^2+y^2}}\right) \qquad \text{(since small angle)} \\
&= \frac{\partial}{\partial x}\left(R\,2\frac{1}{R}\sqrt{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}}\right) \qquad \text{(plugging in } \theta\text{)} \\
&= \frac{\partial}{\partial x}(2x) \\
&= 2 \neq 0
\end{aligned}$$

Therefore, the second derivatives of the metric doesn't vanish in the vicinity around Alice [3]. Both conditions for local flatness are now checked.

Having gone through the example on the surface of a sphere, we now know what local flatness really is. Given any space, no matter how curved it may be, at any point [4] there exists local Cartesian coordinates so that the metric is δ_{mn} in these local coordinates, the first derivatives of the metric w.r.t. [5] these local coordinates vanish, and the second derivatives of the metric w.r.t. these local coordinates do not vanish. We are ready to prove this result in its generality.

[3]For first derivative, we wanted to check that all first derivatives vanishes, so we had to check every single one of them. Here we want to check that the second derivatives don't vanish, so only one non-zero entry is enough. You are welcomed to check whether other second derivatives like $\dfrac{\partial^2 g_{\phi\phi}}{\partial y^2}$ are non-zero. Personally I've never done these checks.

[4]We never restricted where Alice had to live; we simply used a degree of freedom in choosing the origin of the global coordinates and *called* Alice's home "the north pole".

[5]w.r.t. = with respect to

(e) Consider an arbitrary space with an arbitrary global coordinate system y^m, i.e. we are considering an arbitrary metric g_{mn}. Show that it is always possible, at any point on the space, to define local coordinates $x^m \equiv y^m + C^m_{nr} y^n y^r$ such that:

1. $g_{mn} = \delta_{mn}$ at this point;
2. $\dfrac{\partial g_{mn}}{\partial x^r} = 0$ for all m, n, r, i.e. all first derivatives of every component of the metric with respect to every local coordinate variable can be set to zero;
3. $\dfrac{\partial^2 g_{mn}}{\partial x^r \partial x^s} \neq 0$ in general, unless the space is flat.

Note that $x^m \equiv y^m + C^m_{nr} y^n y^r$ is simply expanding the local coordinates x^m to the second order of the known global coordinates y^m (notice the summation over n and r). Since we are only interested in sufficiently nearby places, higher orders in the expansion can be thrown away. The C^m_{nr} are simply the unknown expansion coefficients. The lack of zero-th order terms means that the origin of two sets of coordinates coincide, i.e. when $y^m = 0$ for all m, $x^m = 0$ for all m, too.

Hint: if you have N unknowns, then you need N equations to lock them down. If you have more equations than unknowns, however, then in general the unknowns can't be solved for. If the equations you have are first-order differential equations, you would additionally need some sort of initial condition for the value of the thing being differentiated.

Soln.
For simplicity, consider a 4-dimensional space [6]. The local coordinates will be known if we can solve for the unknown expansion coefficients C^m_{nr}. As a reminder, the local coordinates are nothing more than functions of the global coordinates. In the Alice example, the local coordinates (x, y) were just functions of the global coordinates (θ, ϕ). These functions can, like all functions, be expanded.

How many expansion expansion coefficients C^m_{nr} are there? In 4-dimension space, there are 4 choices for m, and for each m there's 10 pairs of nr. This 10 is $4+3+2+1$. Explicitly, the nr pairs are $(00, 01, 02, 03, 11, 12, 13, 22, 23, 33)$. We don't count, say, 21 once we've counted 12, since $C^m_{21} y^2 y^1$ and $C^m_{12} y^1 y^2$ are really the same term and hence just involves one unknown coefficient.

Therefore, the total number of unknown expansion coefficients is $10 * 4 = 40$. We have 40 unknowns, so we need 40 equations. Do we have 40 equations?

[6] We choose 4-dimensional because eventually the spacetime we study will be 4-dimensional: 3 dimensions of space, plus 1 dimension of time. When the space is really a spacetime, we start our indices at 0 instead of 1. The time dimension carries 0 as its index.

If we require that the first derivatives vanish, i.e. $\dfrac{\partial g_{mn}}{\partial x^r} = 0$ for all m, n, r, then we do.

Again, there are 4 choices of r, and 10 pairs of mn. The metric g_{mn} is symmetric, so once the 10 entries in the upper triangular region are known, the lower 6 entries are no longer independent. Hence in 4-dimensional space g_{mn} really has 10 independent entries.

$$\begin{pmatrix} \checkmark & \checkmark & \checkmark & \checkmark \\ \times & \checkmark & \checkmark & \checkmark \\ \times & \times & \checkmark & \checkmark \\ \times & \times & \times & \checkmark \end{pmatrix}$$

The $\times$ entries are not independent entries; they simply follow the $\checkmark$ entries.

Therefore we also have $10 * 4 = 40$ equations. 40 equations and 40 unknowns, so we're good! Thus, the local coordinates that satisfies

$$\frac{\partial g_{mn}}{\partial x^r} = 0 \ \forall \ m, n, r$$

can be solved for, and therefore exists.

If you are worried that the equations we have are first-order differential equations and thus need initial conditions, we are also in good shape. We have the condition $g_{mn} = \delta_{mn}$ at $x^r = y^r = 0$ (which is the point of interest) serving as an initial condition as well.

What if we instead require that the second derivatives are zero? i.e. What if we require that

$$\frac{\partial^2 g_{mn}}{\partial x^r \partial x^s} = 0$$

instead?

There are still 10 independent choices for mn, but now there are 4 independent choices of r *and* 4 independent choices of s, tallying up to a total of $10 * 4 * 4 = 160$ equations. There are more equations than unknowns, so the unknowns in general cannot be solved for.

In other words, the requirement that the second derivatives vanish

$$\frac{\partial^2 g_{mn}}{\partial x^r \partial x^s} = 0$$

in general can't be satisfied, so the local coordinates that vanishes the second derivatives of the metric in general do not exist.

Summary

Everything is locally flat.

Given any space, no matter how curved it may be, at any point there exists local Cartesian coordinates so that the metric is δ_{mn} in these local coordinates, the first derivatives of the metric w.r.t. these local coordinates vanish, and the second derivatives of the metric w.r.t. these local coordinates do not vanish.

Mathematically, in general, given an arbitrary space, it is always possible, at any point on the space, to choose local Cartesian coordinates x^m such that

1. $g_{mn} = \delta_{mn}$ at this point;

2. $\dfrac{\partial g_{mn}}{\partial x^r} = 0$ for all m, n, r, i.e. all first derivatives of every component of the metric with respect to every local coordinate variable can be set to zero;

3. $\dfrac{\partial^2 g_{mn}}{\partial x^r \partial x^s} \neq 0$

These results, specifically the second result (that the first derivative of the metric vanishes in local Cartesian coordinates) will be important for the derivation of the covariant derivative and the Christoffel symbols in 2.4.