2.2 What Is A Tensor; The Metric Tensor

A reminder of the vector transformation rules we learned in 2.1:

SUMMARY OF VECTOR TRANSFORM RULES

Contravariant vectors: $V^{m'} = \dfrac{\partial x^{m'}}{\partial x^m} V^m$

Covariant vectors: $V_{m'} = \dfrac{\partial x^m}{\partial x^{m'}} V_m$

1 What is a tensor?

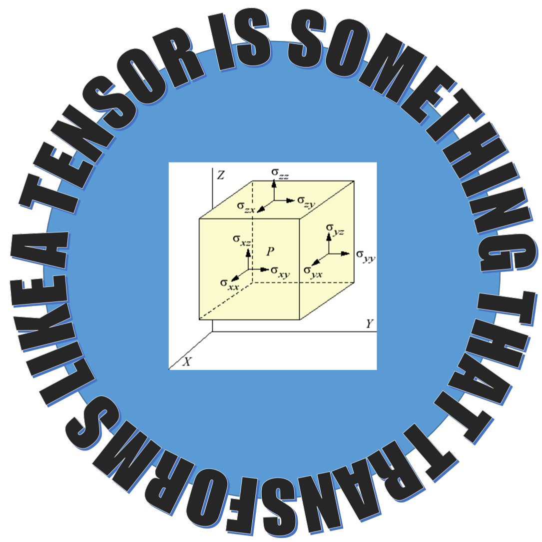


Figure 1: A tensor is something that transforms like a tensor.

Explaining what a tensor is to a general audience has always been a pain. The primary reason is that, there are no intuitive tensors in classical physics to serve as an anchoring example. For vectors, the clear example is the position vector. However, there are no such clear intuitive tensors in classical physics. The two most common tensors in classical physics are the stress tensor (the picture on the meme) and the moment of inertia tensor, both of which require their own hundred-page chapters if one wishes to explain them clearly. In addition, they both somehow involve complex rotations in three dimensional space, which can be really confusing for beginners [1].

However, tensors are not scary at all. Through the exercises on vectors in 2.1, which you hopefully have done and whose solutions you hopefully have read, we now have a pretty good grasp on how to describe vectors in different coordinate systems, and we now have a pretty good understanding of the phrase "a vector is something that transforms like a vector". Tensors are no different than vectors: a tensor is a quantity that obeys certain transformation rules.

[1] Describing rotations in 3D may not sound hard, but it is. Here's a taste of how it feels. Don't worry, this isn't required material for general relativity.

We've seen how to "guess" the transformation rules for contravariant and covariant vectors (2.1, main text page 13 and exercise 2). In essence, we have seen the procedure to guess the transformation rule for a quantity with one index (regardless of whether it's a covariant index or a contravariant index). With the same procedure, we can boldly, with no motivation whatsoever, guess a transformation rule for quantities with more than one index:

$$T^{m'n'} = \frac{\partial x^{m'}}{\partial x^m}\frac{\partial x^{n'}}{\partial x^n}T^{mn}$$

Both sides have two real upstairs indices m' and n' (the m and n on the right hand side are dummies that are summed over). Notice that when we have more than one index, the order of the indices of course matters: the first index of the quantity T is m' and m, so they go together as a pair in a single fraction. The second pair is n' and n.

As another example,

$$T_{m'n'} = \frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^n}{\partial x^{n'}}T_{mn}$$

We can even have one upstairs index and one downstairs index:

$$T^{m'}{}_{n'} = \frac{\partial x^{m'}}{\partial x^m}\frac{\partial x^n}{\partial x^{n'}}T^{m}{}_{n}$$

(Note: we write $T^{m}{}_{n}$, instead of T^{m}_{n} , to explicitly denote that the first index is upstairs and the second index is downstairs.)

These quantities with indices are what we call **tensors**. They are quantities that obey the above transformation rules. A tensor with n indices is called a tensor of **rank** n. It doesn't matter whether these indices are upstairs or downstairs. All three tensors above are tensors of rank 2.

In fact, we don't have to limit ourselves to quantities with two indices. We can postulate a general transformation rule for tensors of an arbitrary rank $k+l$, with k upstairs indices and l downstairs indices:

$$\boxed{T^{m'_1 \dots m'_k}{}_{n'_1 \dots n'_l} = \frac{\partial x^{m'_1}}{\partial x^{m_1}} \cdots \frac{\partial x^{m'_k}}{\partial x^{m_k}}\frac{\partial x^{n_1}}{\partial x^{n'_1}} \cdots \frac{\partial x^{n_l}}{\partial x^{n'_l}}T^{m_1 \dots m_k}{}_{n_1 \dots n_l}}$$

Quick calculation: check that when there's only one upstairs index or only one downstairs index, the above tensor transformation rule reduces to the transformation rules for contravariant and covariant vectors.

Having postulated a transformation rule for tensors, we immediately see that many properties we proved for vectors (in the exercises for 2.1) are nothing more than tensor properties, but only with one index. There are three essential properties that we looked at:

(1) How to extract the transformation rule given the positioning of the indices of a tensor;
(2) Adding two tensors of the same type gives you a third tensor of the same type;
(3) If a tensor is zero (meaning that all the components of this tensor is zero) in one frame, then it's zero in all frames.

We have used (1) to postulate the tensor transformation rule. It is obvious that (2) and (3) still holds for a tensor of arbitrary type. You are welcomed to write out the proof. The proof is exactly the same as the proof for vectors in the solutions to 2.1, with some extra derivatives being factored out.

One note on wording: you might have noticed that in (2) I'm using the word "type" instead of "rank". This is because two tensors of the same rank don't necessarily have the same index placements. For example, ${A^m}_n$ and ${B_m}^n$ are both rank-2, since they both have two indices. However, they obey different transformation rules and thus do not add together:

$$
\begin{aligned}
{A^{m'}}_{n'} &= \frac{\partial x^{m'}}{\partial x^m}\frac{\partial x^n}{\partial x^{n'}}\,{A^m}_n \\
{B_{m'}}^{n'} &= \frac{\partial x^{m}}{\partial x^{m'}}\frac{\partial x^{n'}}{\partial x^{n}}\,{B_m}^n
\end{aligned}
$$

If you try to add them, you can't factor out anything and just get a meaningless clump.

Two tensors are of the same type when they not only have the same rank, but also have the exact same index placements. So for example, ${A^m}_n$ and ${B^m}_n$ are of the same type.

$$
\begin{aligned}
{A^{m'}}_{n'} &= \frac{\partial x^{m'}}{\partial x^m}\frac{\partial x^n}{\partial x^{n'}}\,{A^m}_n \\
{B^{m'}}_{n'} &= \frac{\partial x^{m'}}{\partial x^m}\frac{\partial x^n}{\partial x^{n'}}\,{B^m}_n
\end{aligned}
$$

If you add them, you can factor out the derivatives as

$$
{A^{m'}}_{n'} + {B^{m'}}_{n'} = \frac{\partial x^{m'}}{\partial x^m}\frac{\partial x^n}{\partial x^{n'}}\left({A^m}_n + {B^m}_n\right)
$$

from which we can see that $({A^m}_n + {B^m}_n)$ is also a tensor that transforms as ${T^m}_n$, hence it's of the same type [2].

[2] I do want to point out that the "type" of a tensor is not standard wording; it's just how I call it.

Why do we care about these tensor quantities? The answer lies in property (3): **if a tensor is zero (meaning that all the components of this tensor is zero) in one frame, then it's zero in all frames**.

Recall that one of the main objectives of general relativity is that we want to write equations that hold true in all frames. The good old $F = ma$ is not such an equation [3].

Now, say that we have derived a **tensor equation** for some physical system

$$A^m{}_n = B^m{}_n$$

(I'm using this specific rank and type only for illustration purposes; all types of tensors will obey this property.)

We can move the right hand side to the left to get

$$A^m{}_n - B^m{}_n = 0$$

By property (2), $A^m{}_n - B^m{}_n$ is a tensor of the same type; let's call it $T^m{}_n \equiv A^m{}_n - B^m{}_n$ and thus we get [4]

$$T^m{}_n = 0$$

But since $T^m{}_n$ is a tensor, and it's zero in one frame, so by property (3) it will be zero in all frames! So we have

$$T^m{}_n = A^m{}_n - B^m{}_n = 0$$

in all frames! Therefore, our original tensor equation we derived for the physical system

$$A^m{}_n = B^m{}_n$$

thus holds true in all frames! This is exactly what we want our laws of physics to do: we want these laws to be written in equations that hold true in all frames.

[3] See 1.2

[4] The symbol $\equiv$ means "defined as".

We have, therefore, recovered some motivation for defining our tensors with such transformation rules. Specifically,

<u>THE TENSOR TRANSFORM RULE</u>

$$T^{m_1'...m_k'}{}_{n_1'...n_l'} = \frac{\partial x^{m_1'}}{\partial x^{m_1}} \cdots \frac{\partial x^{m_k'}}{\partial x^{m_k}} \frac{\partial x^{n_1}}{\partial x^{n_1'}} \cdots \frac{\partial x^{n_l}}{\partial x^{n_l'}} T^{m_1...m_k}{}_{n_1...n_l}$$

and

Tensor equations hold true in every frame.
Therefore, laws of physics are written as tensor equations.

Since laws of physics are written as tensor equations, we of course want to seek physical quantities that are tensorial [5]. Don't worry; there's no need to start learning what the stress tensor is or how to describe rotations in 3D. The most important physical tensor quantity shows up in the most unexpected place. Enter the metric tensor.

[5]The adjective for "tensor" is "tensorial". And the adjective for "vector" is "vectorial".

2 The metric tensor

In exercise 7 of 2.1 we discovered an important property: that the inner product between a contravariant vector and a covariant vector $A^m B_m$ is the same in every frame.

An important frame-invariant quantity is the magnitude (squared) of a vector. If the vector were a displacement between two points, its magnitude (squared) would be the distance (squared) between the two points; and that distance should be agreed by all observers. For example, the distance between the Earth and the moon is the same regardless of whether you are viewing the Earth-moon system from the Earth, from the moon, from Mars, or from some Asgardian spaceship passing by.

Let's consider a displacement vector. Recall that displacements are contravariant (so upstairs indices), so a displacement vector could be denoted by

$$\Delta x^m \equiv \begin{pmatrix} \Delta x^1 \\ \Delta x^2 \\ ... \\ \Delta x^n \end{pmatrix}$$

in a space with n spatial dimensions.

And again, Einstein notation is serving us well. Recall that the base coordinates themselves carry contravariant (so upstairs) indices. Thus, Δx^m simply means the difference in the value of each x^m, i.e. the difference in the value of each coordinate, so that's indeed a displacement.

As physicists we love to speak of infinitesimal quantities, so let's consider an infinitesimal displacement

$$dx^m \equiv \begin{pmatrix} dx^1 \\ dx^2 \\ ... \\ dx^n \end{pmatrix}$$

What is the magnitude of this displacement vector? In other words, what is the distance between the two infinitesimally close points?

We know that the magnitude squared of a vector is the inner product between that vector and itself, so we would expect the magnitude squared of this infinitesimal displacement to be $dx^m dx_m$. And it does serve our purposes: $dx^m dx_m$ is an inner product between an upper index and a lower index, so it is indeed frame-invariant. However, it is not very clear what dx_m means. Instead, let's make the following definition.

The most general form of a infinitesimal distance squared would be simply *assigning* a coefficient to all possible quadratic terms:

$$\boxed{ds^2 \equiv g_{mn} dx^m dx^n}$$

Here, g_{mn} denotes these coefficients, and ds^2 denotes the infinitesimal distance squared. Since we are keeping things as general as possible, we let these coefficients carry possible position dependencies. Although I will use the notation g_{mn}, it is important to realize that **these coefficients are not assumed to be constants**, i.e. they are really $g_{mn}(x)$.

Let's unpack this definition with an example. Consider the two-dimensional Cartesian plane with the usual coordinates (x^1, x^2) (in the usual language they are called (x, y)).

The definition for an infinitesimal distance squared becomes

$$ds^2 = g_{11} dx^1 dx^1 + g_{12} dx^1 dx^2 + g_{21} dx^2 dx^1 + g_{22} dx^2 dx^2$$

Since the left hand side has dimensions of length squared, the right hand side also needs to have dimensions of length squred. This is why we use quadratic terms. The most general way to form a distance squared is to just write out all possible quadratic combinations, and figure out what coefficient should go with each quadratic term.

Now, the two middle terms $g_{12} dx^1 dx^2$ and $g_{21} dx^2 dx^1$ are really the same term, so they clump together into $(g_{12} + g_{21}) dx^1 dx^2$. Therefore it is safe to impose the condition that $g_{12} = g_{21}$. This constraint isn't really constraining anything: there are really just 3 independent terms $dx^1 dx^1$, $dx^1 dx^2$ and $dx^2 dx^2$ in the definition, so really only 3 independent coefficients are needed, and hence an independent g_{21} is redundant if you know g_{12}. This idea extends naturally to spaces with higher dimensions as well. Therefore we have

$$\boxed{g_{mn} = g_{nm}}$$

In other words, these coefficients are symmetric. If simply counting degrees of freedom doesn't convince you, there's a more mathy proof in the exercises.

We know the infinitesimal distance in the Cartesian plane with Cartesian coordinates is

$$ds^2 = (dx^1)^2 + (dx^2)^2$$

Comparing with

$$ds^2 = g_{11}dx^1dx^1 + g_{12}dx^1dx^2 + g_{21}dx^2dx^1 + g_{22}dx^2dx^2$$

we see that $g_{11} = 1\,, g_{22} = 1\,, g_{12} = g_{21} = 0$

We can write these coefficients into a matrix as

$$g_{mn} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The matrix packaging these coefficients g_{mn} is called **the metric tensor**, or in short **the metric**. The metric is a rank-2 tensor: it has two indices and hence can be written as a matrix. Since $g_{mn} = g_{nm}$, the metric is also a symmetric matrix.

Note that ds^2 should NOT be interpreted as $(ds)^2$, i.e. it is not the square of ds. Rather, ds^2 is one whole piece of notation. This is because although this quantity will always be positive for space, in spacetime g_{mn} can be negative. For example, the metric for the spacetime without any gravitational source, under a coordinate (t, x, y, z), is [6]

$$g_{mn} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and thus

$$ds^2 = -(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

which means ds^2 can be negative. For example, a particle as rest would have $dx = dy = dz = 0$, but since time doesn't stop, we have $dt \neq 0$, so $ds^2 < 0$.

Therefore it is not useful to think about what ds is; instead, we look at ds^2 as a quantity in its entirety on its own. Though it's called "distance squared", the information it encodes is really just the infinitesimal distances. The "squared" really just means this quantity has dimensions of length squared and does not mean that it is the square of some other more fundamental quantity. We will from now on just refer to ds^2 as the infinitesimal distance. Just keep in mind that it has dimensions of length squared.

[6] We haven't really talked about how to deal with time yet; here it is enough to know that one of the metric components for a coordinate system on a spacetime can be negative.

Just calling g_{mn} the metric tensor won't make it a tensor; we have to show that these coefficients satisfy the tensor transformation rule. We use the fact that the differential distance [7] is the same in all frames. Thus if we switch coordinates to another primed frame, we will have

$$g_{mn}dx^m dx^n = g_{m'n'}dx^{m'}dx^{n'}$$

Now, dx^m and dx^n are differential displacements, so they are themselves contravariant. They transform as

$$dx^{m'} = \frac{\partial x^{m'}}{\partial x^m}dx^m \, , \, dx^{n'} = \frac{\partial x^{n'}}{\partial x^n}dx^n$$

Plugging in, we have

$$g_{mn}dx^m dx^n = g_{m'n'}\frac{\partial x^{m'}}{\partial x^m}dx^m \frac{\partial x^{n'}}{\partial x^n}dx^n$$

The $dx^m dx^n$ cancels and leaves us with

$$g_{mn} = g_{m'n'}\frac{\partial x^{m'}}{\partial x^m}\frac{\partial x^{n'}}{\partial x^n}$$

At this point the proof is really already complete, but if you like we can switch the order of the terms [8] and get the more familiar form

$$g_{mn} = \frac{\partial x^{m'}}{\partial x^m}\frac{\partial x^{n'}}{\partial x^n}g_{m'n'}$$

and we see that the metric g_{mn} satisfies the tensor transformation rule, and hence is indeed a tensor. (The tensor transform rule listed on page 6 is transforming from unprimed frame to primed frame; here this equation is transforming from primed frame to unprimed frame. These equations are equivalent, since, well, who decided that you are the primed frame?)

[7] "Differential distance" is a synonym for "infinitesimal distance".

[8] Unlike matrix-vector equations, in Einstein notation there's no need to worry about order of multiplication, since it's just summation. For example, $\sum_i \sum_j (a_i b_j c_i d_j) = \sum_i \sum_j (d_j b_j c_i a_i)$, i.e. the summation terms can arbitrarily commute.

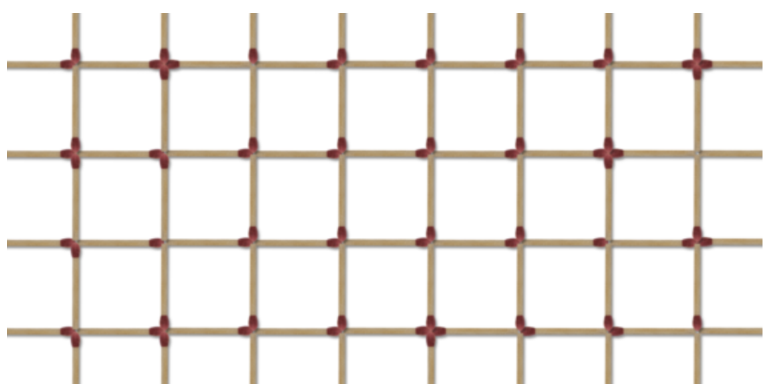
3 What do you really know when you know the metric?

Having gone through the song and dance on infinitesimal distances and the metric tensor, you might be wondering what the metric is used for.

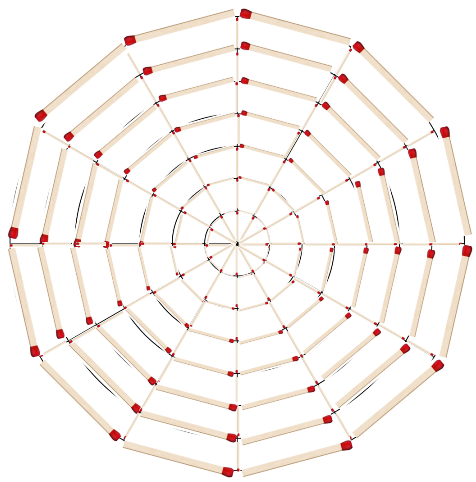
The metric gives us the formula for all the infinitesimal distances between all pairs of infinitesimally close points on a space under some coordinate. Concretely, given a space equipped with some coordinate system $(x^1, x^2, ...)$, we consider a point $(x^1, x^2, ...)$ on this space and an immediately neighboring point $(x^1 + dx^1, x^2 + dx^2, ...)$. Given that differential displacement dx^m, the metric tells you that the distance is $ds^2 = g_{mn}dx^m dx^n$ between these two neighboring points. Since the point $(x^1, x^2, ...)$ we chose was arbitrary, the metric tells you the differential distance between every possible pair of neighboring points.

In fact, knowing all the neighboring distances between neighboring points is enough to tell you **everything about a space and the coordinate being used**. As an analogy, consider you have a bunch of short little matchsticks, with each matchstick having some fixed, prescribed length, just like how the metric gives us a set of known infinitesimal lengths. You task is to lay out the matchsticks into a grid that covers some surface. You will find that with each set of matchsticks, there's one and only one kind of surface they can lay out, and there's only one way in which you can construct your grid.

If the matchsticks you are handed are all of equal length, then you can lay them out on a flat plane in a Cartesian fashion:

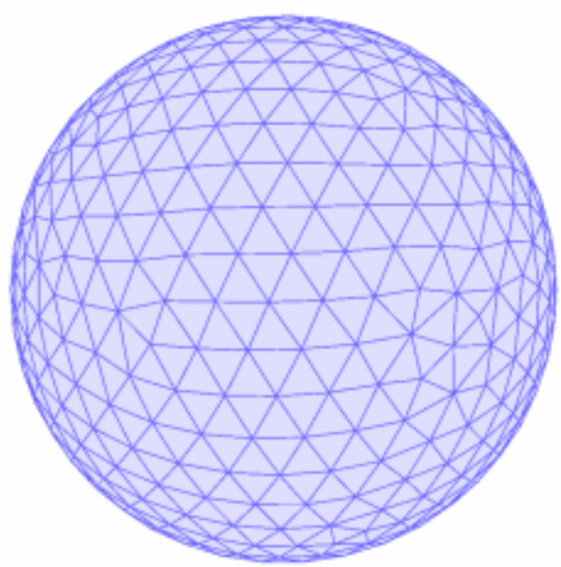


You might get handed a set of matchsticks that are not of equal length, but can nevertheless be laid out together on a flat plane in a polar fashion:



(Notice that the length of a matchstick (i.e. the length of a differential displacement, hence the metric) depends on the position.)

You might even get handed a set of matchsticks that can't be laid out on a flat plane, but can be laid out on the surface of a sphere [9]:



[9]If you want, you can try this yourself. Get some matchsticks and cut them into pieces so that you can lay the pieces out nicely on the surface of an orange. You might want to draw out the segments on the orange first and cut the matchsticks according to the traces you drew. Now try to lay the same pieces out on a flat table in some sort of regular grid. You won't be able to do it!

If you are handed a set of equal-length matchsticks, there's no way to construct a polar grid on a plane or a grid on the surface of a sphere [10]. Similarly, if you are handed a set of polar matchsticks, there's no way to construct a planar Cartesian grid. Thus, if these distances are handed to us by a metric, then we know both the space and the coordinates encoded by this metric. So, in short,

The metric tells you everything about a space and the coordinates in use.

We have seen (on page 9) that for a 2D Cartesian plane with Cartesian coordinates, the metric is simply the Kronecker delta

$$g_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{mn}$$

The metric is 1 when $m = n$ and 0 when $m \neq n$.

However, **this does not mean that the metric is the Kronecker delta on flat spaces!** The reason is very simple: the metric is a tensor, and so it transforms if we switch coordinates. For example, if we switch to polar coordinates, the metric will transform to something that's not the Kronecker delta. Exactly what it transforms to is left for you to figure out in the exercises. This is like how the polar matchsticks still lay out a flat plane, but they are not of equal length.

So, to our disappointment, **just because a metric is not the Kronecker delta doesn't mean the space is curved.** The polar metric is not δ_{mn}, but the polar metric is still a metric on a flat plane.

Therefore, we will need some other kind of criteria to judge whether a given metric is a flat space or a curved space [11]. For now just keep this in mind. We won't touch on that criteria until a couple of notes later.

Similarly, at most what we can say about a flat space is that ***there exists*** **coordinates on a flat space such that the metric is the Kronecker delta everywhere**. There's no guarantee that the metric will *actually be* the Kronecker delta in the working coordinates. The coordinates on that flat space in which $g_{mn} = \delta_{mn}$ is, of course, the Cartesian coordinates. In other words, if there doesn't exist coordinates on a space such that the metric is δ_{mn} everywhere, then the space is curved.

[10]In fact the word "sphere" already refers to "the surface of a sphere"; the entire "sphere", with the interior, is called a "ball". Therefore, the phrase "the surface of a sphere" is actually ungrammatical.

[11]Spoiler: it's the Riemann curvature tensor.

Remember how in motivation 1.2 we discussed the flatness and curvature of spacetime? When there are gravitational sources (like black holes), spacetime is itself intrinsically curved, and only curved coordinates can be used. When there are no gravitational sources, spacetime is itself flat, and we can use either flat coordinates or curved coordinates.

Now we can unpack this statement by a little bit: for the spacetime around gravitational sources (like black holes), no matter what coordinates we use, the metric will never be

$$\eta_{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

But in a spacetime without any gravitational source (i.e. flat spacetime, for example Alice and Bob simply moving past each other while carrying their lightclocks), there exists coordinates in which the metric is $\eta_{\mu\nu}$.

Note: $\eta_{\mu\nu}$ is the symbol for that $\mathrm{diag}(-1,1,1,1)$ matrix, just like how δ_{mn} is the symbol for that $\mathrm{diag}(1,1)$ matrix. In spacetime, the all-mighty position of δ_{mn} is taken by $\eta_{\mu\nu}$, as we will eventually see when we get to special relativity. As a convention, indices that only cover space coordinates are expressed by the English alphabet, while indices that cover both space and time coordinates are expressed by the Greek alphabet.

4 Index lowering and index raising

It's time to return to the issue of inner products. We have seen that the frame-invariant differential distance is given by the metric

$$ds^2 = g_{mn} dx^m dx^n$$

But we also know that the inner product between a covariant and a contravariant vector is also frame-invariant. For the infinitesimal distance, this would be the inner product between an infinitesimal displacement with itself. Thus we would like to have something like

$$ds^2 = dx_n dx^n$$

Comparing these two equations, it screams the following relationship:

$$dx_n = g_{mn} dx^m$$

First let's check indices: left hand side has a single downstairs n, and right hand side has a single downstairs n with a pair of dummy m, so we are good index-wise.

This is called **index lowering**. The metric "acting" on dx^m pulls down the upstairs index into a downstairs index. The metric can act on everything in this fashion, not just displacements.

Given any contravariant vector V^m, we *define* a corresponding covariant dual [12] by

$$\boxed{V_n \equiv g_{mn} V^m}$$

Since vectors are just special kind of tensors, the metric can thus act on any kind of tensors to lower indices. For example,

$$g_{an} T^{mnpq} = T^{m\ \ pq}_{\ \ a}$$
$$g_{an}\, g_{bq}\, T^{mnpq} = T^{m\ \ p}_{\ \ a\ \ b}$$

Notice that lowering indices does not change the position of an index relative to the other indices. Also, as usual, real indices are the same on both side of an equation, while dummy indices only appear on one side.

[12] It's best to not think about what the dual actually is. It's more of a mathematical wizardry we can use with the Einstein notation so that our math will be easier. Really, the dual vector is the vector expressed in a sort of "dual coordinate", but knowing that is not essential, so I'm omitting.

The benefit of defining such duals to tensors is mainly notation-wise. Say we want to take the inner product of a vector with itself, i.e. we want $A^m A_m$. With the index lowering operation, we can substitute $A_m = g_{mn}A^n$ and easily compute this as $g_{mn}A^m A^n$, once we know the metric and the contravariant vector itself. If the contravariant vector were the differential displacement dx^m, then we retrieve the definition for the differential distance $dx^m dx_m = g_{mn}dx^m dx^n$.

It is worth pointing out that, since the metric is symmetric, so $V_n = g_{mn}V^m$ can also be written as $V_n = g_{nm}V^m$, i.e. either index on the metric can serve as the dummy. Also, since the dummy index can be renamed arbitrarily, all of the following equations are equivalent:

$$V_n = g_{mn}V^m\, ,\, V_n = g_{nm}V^m\, ,\, V_n = g_{np}V^p\, ,\, V_n = g_{qn}V^q$$

A similar operation that works in the exact same way is **index raising:**

$$\boxed{V^n = g^{mn}V_m}$$

And like index lowering, index raising can act on tensors of arbitrary type. I'll proceed in this section with acting on vectors just for the simplicity of typesetting [13].

To figure out what g^{mn} is, we use the obvious fact that, if you first raise an index and then lower it, you get back where you started. In equations:

$$V^m = g^{mn}V_n = g^{mn}(V_n) = g^{mn}(g_{np}V^p) = g^{mn}g_{np}V^p$$

Looking at $V^m = g^{mn}g_{np}V^p$, we realize that

$$\boxed{g^{mn}g_{np} = \delta^m_p}$$

This is because $\delta^m_p V^p = V^m$. In the summation over p, every term with $p \neq m$ vanishes due to the Kronecker delta, and the remaining term is thus the $p = m$ term, whose Kronecker delta is one. In equations, $\delta^m_p V^p = \delta^m_m V^m = V^m$.

[13] "Typesetting" means the ease of typing out things in Latex.

A note on notation: the Kronecker delta is **not** a tensor. It doesn't transform if we switch coordinates. It still one if its two indices are the same and zero if they are different. In effect, the Kronecker delta is just a *symbol* to keep track of a set of fixed, non-transforming constants. Therefore we don't need to label its indices as a tensor, and we can be somewhat haphazard with the index placement. Based on how Einstein summation convention needs it in the exterior equation, it can be written as δ^m_p, δ^{mp}, or δ_{mp}. In $g^{mn}g_{np} = \delta^m_p$, left hand side has m as an upstairs real index and p as a downstairs real index, so we keep this arrangement on the right hand side.

What is $g^{mn}g_{np} = \delta^m_p$ telling us? The left hand side, $g^{mn}g_{np}$, has two real indices m and p, so it's a matrix. The Kronecker delta is telling us that the components of this matrix is 1 if the indices are the same (i.e. diagonal elements), and 0 if the indices are different (i.e. non-diagonal elements). This is just the identity matrix!

The left hand side $g^{mn}g_{np}$ says the (m, p)-element of this matrix is

$$g^{mn}g_{np} = g^{m1}g_{1p} + g^{m2}g_{2p} + ...$$

The g^{mn} are elements on the m-th row of the matrix g^{mn}, and the g_{np} are elements on the p-th column of the matrix g_{np}. Therefore, the (m, p)-element of $g^{mn}g_{np}$ is just the inner product between the m-th row of the matrix g^{mn} and the p-th column of the matrix g_{np}. In other words, this is just the good old matrix multiplication! The two matrices multiply to the identity matrix, so they are inverses of each other!

Let's box this:

$$\boxed{\textbf{The matrix inverse of } \boldsymbol{g_{mn}} \textbf{ is } \boldsymbol{g^{mn}}\textbf{, i.e. } \boldsymbol{g^{mn}g_{np} = \delta^m_p}}$$

A note on notation: when g^{mn} is written as a standalone term, we have every right to call the indices whatever we want. However, when we are multiplying the metric and the inverse metric together to get the identity matrix, because there's only a single summation in matrix multiplication, we write $g^{mn}g_{np}$. Had we written $g^{mn}g_{mn}$, we would have written a thing with two dummy summation indices, and that's not matrix multiplication anymore. Exactly what $g^{mn}g_{mn}$ is is left for you to figure out in the exercises.

5 Summary

This note carries quite a lot of information, so let's do a summary.

1. Tensor transformation rule:

$$T^{m'_1 \dots m'_k}{}_{n'_1 \dots n'_l} = \frac{\partial x^{m'_1}}{\partial x^{m_1}} \cdots \frac{\partial x^{m'_k}}{\partial x^{m_k}} \frac{\partial x^{n_1}}{\partial x^{n'_1}} \cdots \frac{\partial x^{n_l}}{\partial x^{n'_l}} \, T^{m_1 \dots m_k}{}_{n_1 \dots n_l}$$

2. If a tensor is zero in one frame (meaning all of its components are zero in one frame), then it's zero in all frames.

3. Tensor equations hold true in every frame. Therefore, laws of physics are written as tensor equations.

4. The metric tensor defines the position-dependent differential distances on a space under some coordinate:

$$ds^2 = g_{mn} dx^m dx^n$$

5. The metric tells you everything about the geometry of a space and the coordinates in use.

6. The metric is symmetric:

$$g_{mn} = g_{nm}$$

7. On flat spaces, there exists Cartesian coordinates such that $g_{mn} = \delta_{mn}$ everywhere. In general, an arbitrary coordinate on a flat space will not have $g_{mn} = \delta_{mn}$.

8. On curved spaces, there doesn't exist coordinates such that $g_{mn} = \delta_{mn}$ everywhere.

9. The metric has an inverse:

$$g^{mn} g_{np} = \delta^m_p$$

10. The metric and the inverse metric can be used to lower and raise indices:

$$V_n = g_{mn} V^m$$

$$V^n = g^{mn} V_m$$

6 Exercise

1. In this exercise we show that the metric g_{mn} is symmetric rigorously.
(a) Show that any 2-index object can be decomposed into a symmetric and an anti-symmetric part, i.e. given a 2-index object F_{mn}, it can be written as $F_{mn} = S_{mn} + A_{mn}$, where $S_{mn} = S_{nm}$ and $A_{mn} = -A_{nm}$.
(b) Show that $A_{mn}dx^m dx^n = 0$, where $A_{mn} = -A_{nm}$ is an arbitrary antisymmetric quantity. Why does this imply that g_{mn} has no antisymmetric part?

2. By using basic geometry alone, find the metric on:
(a) Cartesian 2D plane with Cartesian coordinates (x, y);
(b) Cartesian 2D plane with polar coordinates (r, θ);
(c) Cartesian 3D space with Cartesian coordinates (x, y, z);
(d) Cartesian 3D space with spherical coordinates (r, θ, ϕ).
Convince yourself that in general, the metric components expressed in some arbitrary coordinate is not constant and is position-dependent.

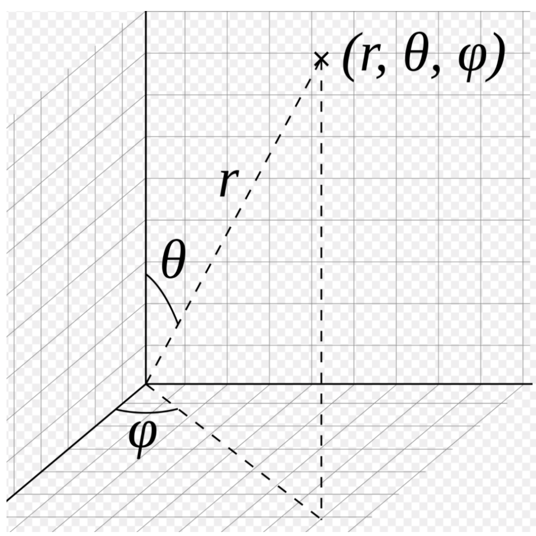


Figure 2: Convention for spherical coordinates. θ is the angle from the north pole, and ϕ is the angle along the equator.

3. Check that the metrics you found in question 2 obey the transformation rule for tensors.

4. Find the metric on the surface of a sphere of fixed radius R. Note that this is a two-dimensional space. Use the same angular convention as that in spherical coordinates for a 3D Cartesian space.

5. With raising and lowering indices, show that $A^m B_m = A_n B^n$.

6. Simplify the following:

$$g^{\mu\gamma}\, T^{\alpha\beta}{}_{\gamma\delta} = ?$$

$$g_{\mu\alpha}\, T^{\alpha\beta}{}_{\gamma\delta} = ?$$

$$g_{\mu\alpha}\, g_{\nu\beta}\, g^{\rho\gamma}\, g^{\sigma\delta}\, T^{\alpha\beta}{}_{\gamma\delta} = ?$$

$$g^{st}\, g_{st} = ?$$

For the last one, think carefully before proceeding. What is the type of this object? Is it a scalar or a tensor? Why? If it were a tensor, what is its rank?

7. A vector in polar coordinates on the 2D Cartesian plane is given by

$$A^m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let unprimed indices denote polar coordinates and primed indices denote Cartesian coordinates.
(a) Find $A_{m'}$ by (1) first finding A_m and then transforming it to the primed frame, and (2) first transforming to $A^{m'}$ and then lowering the index. Your answer should agree with each other.
(b) Explicitly verify that $A^m A_m = A^{m'} A_{m'}$, i.e. inner-product is frame-invariant. What does this inner product represent? Does it make physical sense? Checking the last question in the exercises for 2.1 might be helpful for this question.

8. Consider the usual Cartesian coordinates (x^1, x^2) on the usual Cartesian 2D plane. A second coordinate system is obtained by stretching the vertical axis by a factor of 2:

$$\begin{cases} x^{1'} = x^1 \\ x^{2'} = \dfrac{1}{2} x^2 \end{cases}$$

Verify that the differential distance is the same in both frames, i.e.

$$g_{mn} dx^m dx^n = g_{m'n'} dx^{m'} dx^{n'}$$

Reminder: the metric tensor transforms like, well, a tensor.

9. In this question we look at the inverse metric g^{mn} from another point of view.
(a) Show that the metric g_{mn} cannot have zero as an eigenvalue. Hint: consider $dx_m dx^m$.
(b) Show that if a matrix doesn't have zero as an eigenvalue, then it has an inverse.
Combining (a) and (b), we can see that g_{mn} has an inverse.

10. We already know that the metric is symmetric: $g_{mn} = g_{nm}$. Show that the inverse metric is also symmetric, i.e.

$$g^{mn} = g^{nm}$$

11. This question is a prelude to the next note.
Its solutions will not be in the "Solutions to 2.2" note, and will be discussed in the body of note 2.3 itself.

We know that for flat spaces, there exists a Cartesian coordinate such that $g_{mn} = \delta_{mn}$ everywhere on the space. This corresponds to the fact that $ds^2 = dx^2 + dy^2$ on the Cartesian 2D plane, regardless of your position. This means that if we can't find globally Cartesian coordinates, then the space is curved.

Unless you are a member of the flat-earth society, you would agree that the surface of the earth is a curved 2 dimensional space: a sphere with some fixed radius R. In question 4 you already derived the metric on such a sphere. You should have found out that the metric is not δ_{mn}.

The (θ, ϕ) coordinate system on a sphere is a global coordinate: it covers every position on the sphere. Under this global coordinate, the metric is not δ_{mn}. However, for us living on the surface of the earth, we rarely use these global coordinates (what's the last time you used latitude and longitude in your life?). Rather, we use a set of very "Cartesian-like" coordinates: we give directions using north, south, east and west. East is effectively the positive x axis, and north is effectively the positive y axis.

How can this be? Shouldn't there be no Cartesian coordinates on curved surfaces? The answer is that the north-south-west-east coordinate we use are **local** coordinates as opposed to global coordinates. From our point of view, the world immediately around us does indeed look pretty flat [14]. Therefore, in the local vicinity around us, we can use Cartesian coordinates. This is the idea that **everything is locally flat.** Although there are no globally Cartesian coordinates on the curved surface of the earth, locally at any position, there does exist local Cartesian coordinates.

(a) Alice lives on the surface of the earth. Without loss of generality, let's call her home the North pole (i.e. $\theta = 0$). Find the set of local Cartesian coordinates Alice uses in her everyday life in the vicinity around her. You might find the next couple of figures helpful.
Reminder: a set of new coordinate variables is nothing but a function of the old coordinate variables.

[14]...and hence why there are so many flat-earthers.

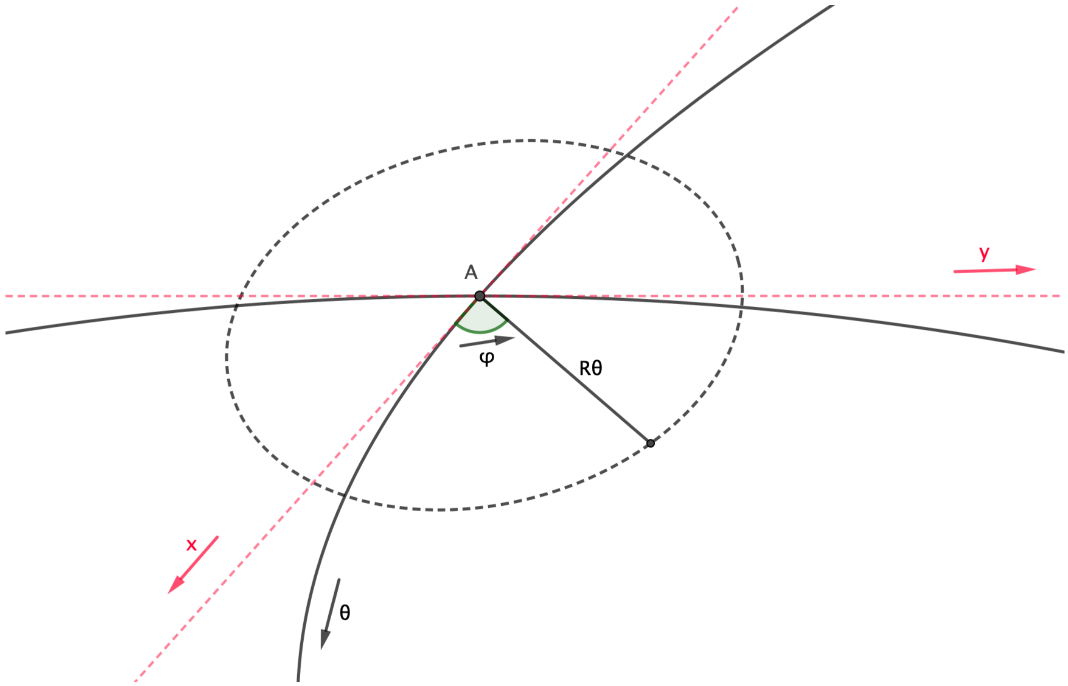


Figure 3: Alice (point A) living at the north pole. The global coordinates (θ, ϕ) are shown in black, and the local Cartesian coordinates (x, y) are shown in red and dashed.

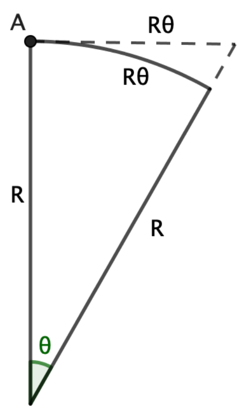


Figure 4: Alice (point A) living at the north pole. The radius of the earth is R. A small local arc around Alice has the same length as a small flat segment near her (since they are small), so both are $R\theta$. The small flat segment near her is the radius of a small circle around her in her flat local Cartesian coordinates.

(b) Find the metric in the local Cartesian coordinate around Alice. What do you notice?

(c) Find the first derivatives of all the metric components with respect to the local coordinates around Alice, i.e. find $\dfrac{\partial g_{mn}}{\partial x}$ and $\dfrac{\partial g_{mn}}{\partial y}$ for all $m, n = \theta, \phi$. Evaluate the derivatives at Alice's location (north pole). What do you notice? Does your result make sense?
Reminder: R, the radius of the sphere, is a fixed constant.

(d) Find $\dfrac{\partial^2 g_{\phi\phi}}{\partial x^2}$, again evaluated at Alice's location (north pole). What do you notice? Does your result make sense?

(e) Consider an arbitrary space with an arbitrary global coordinate system y^m, i.e. we are considering an arbitrary metric g_{mn}. Show that it is always possible, at any point on the space, to define local coordinates $x^m \equiv y^m + C^m_{nr} y^n y^r$ such that:
1. $g_{mn} = \delta_{mn}$ at this point;
2. $\dfrac{\partial g_{mn}}{\partial x^r} = 0$ for all m, n, r, i.e. all first derivatives of every component of the metric with respect to every local coordinate variable can be set to zero;
3. $\dfrac{\partial^2 g_{mn}}{\partial x^r \partial x^s} \neq 0$ in general, unless the space is flat.

Note that $x^m \equiv y^m + C^m_{nr} y^n y^r$ is simply expanding the local coordinates x^m to the second order of the known global coordinates y^m. Since we are only interested in sufficiently nearby places, higher orders in the expansion can be thrown away. The C^m_{nr} are simply the unknown expansion coefficients. The lack of zero-th order terms means that the origin of two sets of coordinates coincide, i.e. when $y^m = 0$ for all m, $x^m = 0$ for all m, too.

Hint: if you have N unknowns, then you need N equations to lock them down. If you have more equations than unknowns, however, then in general the unknowns can't be solved for. If the equations you have are first-order differential equations, you would additionally need some sort of initial condition for the value of the thing being differentiated.