2.25 An Interlude on Tensor Products

So far, apart from the tensor transformation rule, we have looked at just one operation regarding tensors: if you add two tensors of the same type, you get another tensor of the same type.

A second operation regarding tensors is multiplying two tensors together. This is known as the **tensor product**. We will not have explicit use of tensor products in our journey of general relativity [1], except for just one very specific derivation, so I am introducing them here in an interlude.

Here's the question: we know what V_m is. It's a covariant vector. Let's take two covariant vectors V_m and W_n and "multiply" them together. The result is written as $V_m W_n$.

In the old days, multiplying vectors were kind of a serious issue. You need to specify whether it's the dot product or the cross product, and the formula for dot/cross products takes on more complicated forms in non-Cartesian coordinates. You need to carefully define the product between vectors.

Luckily for us, since Einstein notation deals with the components explicitly, there's no need for such intensive care. The meaning of $V_m W_n$ is not ambiguous at all: it's a two-index object whose (m, n) component is the product of the m-th component of the vector V and the n-th component of the vector W.

As a running example, consider two covariant vectors $V_m = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $W_n = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. In component form this really just means

$$V_1 = 1\,,\ V_2 = 2\,,\ W_1 = 3\,,\ W_2 = 4$$

Therefore, the object $V_m W_n$ has components

$$V_1 W_1 = (1)(3) = 3$$

$$V_1 W_2 = (1)(4) = 4$$

$$V_2 W_1 = (2)(3) = 6$$

$$V_2 W_2 = (2)(4) = 8$$

[1] Tensor products are more heavily used in quantum mechanics. The state space for any quantum system that's a combination of smaller subsystems is the tensor product of the state space of these smaller subsystems. For example, the state space of the orbital-spin coupled electron in the hydrogen atom is $|nlm\rangle \otimes |\pm\rangle$.

..and that's it! No fancy definitions for dot/cross products needed, once again we are saved by the godsend that is Einstein notation.

Since $V_m W_n$ has two indices, we can write its components as a matrix, with the first index m as a row index and the second index n as a column index:

$$V_m W_n = \begin{pmatrix} V_1W_1 & V_1W_2 \\ V_2W_1 & V_2W_2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$$

If you've drilled through the tensor exercises in 2.2, you might start to feel that this $V_m W_n$ object has a strong similarity to the metric tensor, which also has two indices and can be written as a matrix. Let's prove that $V_m W_n$ indeed is a tensor with two downstairs indices. To prove that something is a tensor, we simply need to show that it satisfies the transformation rule for tensors:

$$\begin{aligned} V_{m'} W_{n'} &= \left(\frac{\partial x^m}{\partial x^{m'}} V_m \right) \left(\frac{\partial x^n}{\partial x^{n'}} W_n \right) \\ &\qquad \text{(transformation rule for covariant vectors)} \\ &= \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^n}{\partial x^{n'}} V_m W_n \end{aligned}$$

Comparing with the tensor transformation rule $T_{m'n'} = \dfrac{\partial x^m}{\partial x^{m'}} \dfrac{\partial x^n}{\partial x^{n'}} T_{mn}$, we see that $V_m W_n$ indeed is a tensor with two downstairs indices! You are welcomed to check that our specific example $V_m W_n = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$ satisfies the tensor transformation rule, say between Cartesian and polar coordinates.

This result can be easily generalized to products not just between two covariant vectors, but really products between any two types of tensors. All that's different in the proof is simply more derivatives being factored out. For example, you are welcomed to write out the proof for the following:

$$\begin{aligned} V^m W_n &= T^m{}_n \\ V^m W^n &= T^{mn} \\ A_{mn} B_p &= T_{mnp} \end{aligned}$$

Thus, **multiplying two arbitrary tensors (not necessarily of the same type) gives you a new tensor.** The type of the new tensor is simply the "sum" of the types of the constituent tensors, as the above examples suggest.

Why do we care about such tensor products? Since the laws of physics are written in tensor equations, eventually we will want to develop some notion of a tensor derivative (since laws of physics will be about how things move and change). I'm not talking about the derivative of tensors; rather, **we need some sort of derivative that is itself a tensor**, so that we can use it as a term in the tensor equations we write down.

This task turns out to be quite difficult, and as you can imagine getting that tensor derivative will require us to do some careful thinking. Thus it would be helpful to first consider how to define a tensor derivative on vectors (since vectors are just tensors that are easier to visualize), and then generalize the result from these vector derivatives to a tensor derivative.

This particular generalization can be hugely simplified by using this tensor product. If we have developed the base case for a vector derivative, i.e.

$$\text{derivative of } V_m$$

then we can use the tensor product to write the derivative of a tensor as

$$\text{derivative of } T_{mn} = \text{derivative of } (V_m W_n)$$

and at this point we can use the product rule to get

$$\begin{aligned}\text{derivative of } T_{mn} &= \text{derivative of } (V_m W_n) \\ &= V_m(\text{derivative of } W_n) + (\text{derivative of } V_m)W_n\end{aligned}$$

and our generalization is easily completed.

As I said, this is the only usage for tensor products in our coverage of general relativity.