2.4 The Covariant Derivative

Summary of local flatness:

Everything is locally flat.

Given any space, no matter how curved it may be, at any point there exists local Cartesian coordinates so that the metric is δ_{mn} in these local coordinates, the first derivatives of the metric w.r.t. these local coordinates vanish, and the second derivatives of the metric w.r.t. these local coordinates do not vanish.

Mathematically, in general, given an arbitrary space, it is always possible, at any point on the space, to choose local Cartesian coordinates x^m such that

1. $g_{mn} = \delta_{mn}$ at this point;

2. $\dfrac{\partial g_{mn}}{\partial x^r} = 0$ for all m, n, r, i.e. all first derivatives of every component of the metric with respect to every local coordinate variable can be set to zero;

3. $\dfrac{\partial^2 g_{mn}}{\partial x^r \partial x^s} \neq 0$

1 The problem with derivatives: coordinate derivatives are not tensors

We know that the laws of physics are written in tensor equations, as discussed in note 2.2. This is the reflection of the fact that laws of physics should be the same in all frames.

An important construction for equations is the derivative. The derivative encodes how quantities change w.r.t. ("with respect to") space or time, and is obviously necessary for writing physics equations.

The natural question follows: is the derivative a tensor? If the derivative is not a tensor, then it cannot be used to write equations for laws of physics.

There's a quick check for whether some quantity is a tensor. In 2.2 we found that if a tensor is zero in one frame (meaning all of its components are zero in one frame), then it's zero in all frames. Therefore, if a quantity is zero in one frame but non-zero in some other frame, then that quantity can't be a tensor.

Consider a constant vector field V^m on the 2D Cartesian plane. Vectors are just tensors of rank 1 and are easier to visualize. For this reason I will sometimes refer to vector fields as "tensor fields" as well. For now let's not associate any coordinates with the plane, but just the vector field in its purity [1].

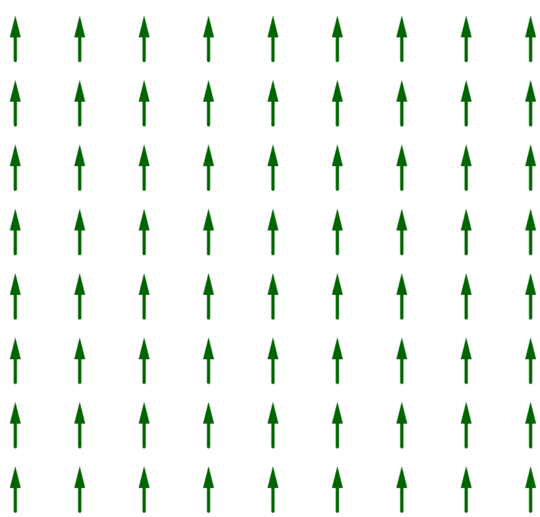


Figure 1: A constant vector field V^m on the 2D Cartesian plane, not associated to any coordinates.

[1] Try to suppress your urge to think of this vector field as $\begin{pmatrix}0\\1\end{pmatrix}$, since we are not using Cartesian coordinates. We are not using any coordinates here.

Because the space is two dimensional, and the vector has two components, a "derivative tensor" will have four components. If we denote the two possible coordinate variables as (x^1, x^2) [2], then naively, such a "derivative tensor" might be given as

$$\begin{pmatrix} \dfrac{\partial V^1}{\partial x^1} & \dfrac{\partial V^2}{\partial x^1} \\ \dfrac{\partial V^1}{\partial x^2} & \dfrac{\partial V^2}{\partial x^2} \end{pmatrix}$$

In other words, we construct this two-index object (which may or may not be a tensor; we don't know yet) with entries simply being the coordinate derivatives, i.e. the components of the tensor field being differentiated w.r.t. the coordinate variables. This is the simplest notion of "derivative of tensors" that we can think of.

As a notational note, it is customary to denote the derivative of a component of a tensor w.r.t. a coordinate variable as $\partial_r V^m \equiv \dfrac{\partial V^m}{\partial x^r}$. In this notation, the above two-index object is denoted as

$$\begin{pmatrix} \partial_1 V^1 & \partial_1 V^2 \\ \partial_2 V^1 & \partial_2 V^2 \end{pmatrix}$$

Notice that this notation preserves the upstairs-ness and downstairs-ness of indices.

Let's now associate some coordinates to this vector field. First, we equip the 2D plane with Cartesian coordinates (x, y).

[2] Not to be confused with (x, y); here (x^1, x^2) denotes any two possible coordinate variables on the plane, like maybe (r, θ)

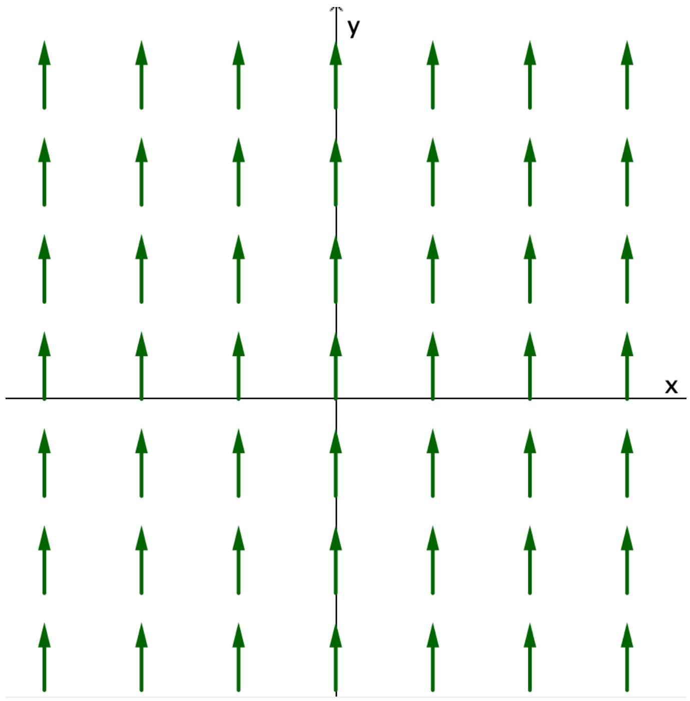


Figure 2: A constant vector field V^m on the 2D Cartesian plane, in Cartesian coordinates (x, y).

The vector field has no x component and a constant y component. For simplicity let's choose the unit distance to be the length of the vector field. Thus, the explicit components of the vector field in Cartesian coordinates are

$$V^m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and thus the components of the "derivative tensor" are

$$\begin{pmatrix} \partial_1 V^1 & \partial_1 V^2 \\ \partial_2 V^1 & \partial_2 V^2 \end{pmatrix} = \begin{pmatrix} \partial_x 0 & \partial_x 1 \\ \partial_y 0 & \partial_y 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus the "derivative tensor" is zero (meaning all of its components are zero) in one frame. If it truly were a tensor, then if we switch to polar coordinates, all components of this tensor would still all be zero.

But it's not that simple. Inspect the vector field under polar coordinates:

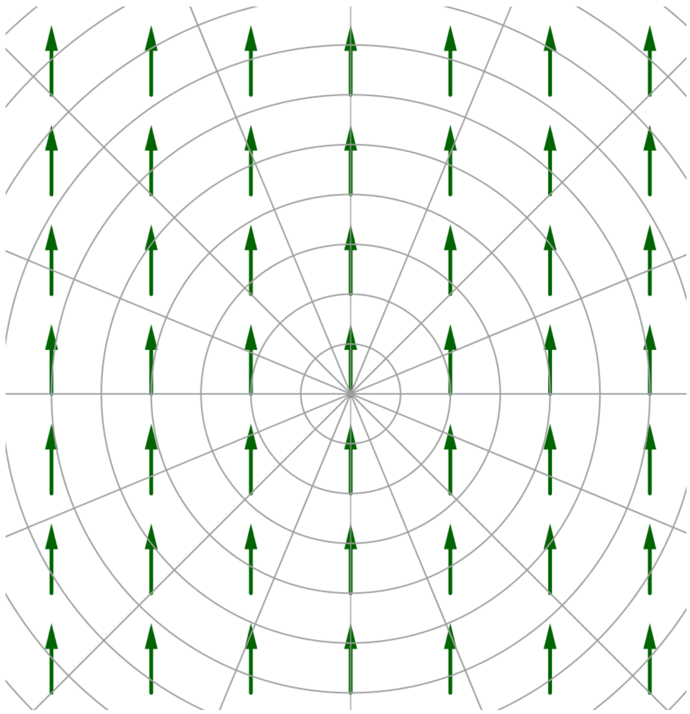
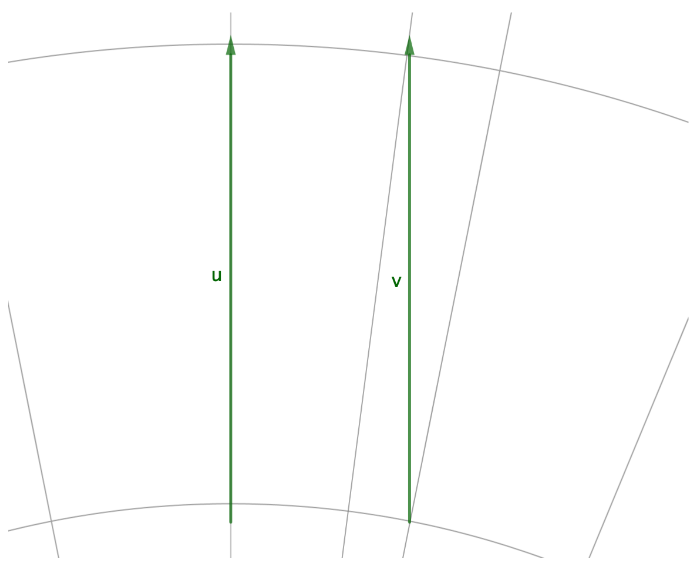


Figure 3: A constant vector field V^m on the 2D Cartesian plane, in polar coordinates (r, θ).

Let's zoom in into two vectors in this constant vector field: one on the y-axis, denoted by u, and another that's infinitesimally to the right of the y-axis, denoted by v:



For the vector u on the y-axis, it has no θ component, since both its start point and its end point are on the line of the same θ. For the vector v a little to the right of the y-axis, however, it does have some θ component, since its start point and its end point are on lines of different θ. Therefore, if we take a little step to the right, then the θ component of the vector field does change. In this case it changes from 0 to some small positive number.

In other words, under polar coordinates, there exists directions in which if we take an infinitesimal step, then the θ component of the vector field changes. This means that **there exists directions s in which $\partial_s V^\theta$, the derivative of the θ component of the vector field w.r.t. that direction, is non-zero**.

Therefore, it is impossible for all the coordinate derivatives of the vector's components to vanish! Suppose the "derivative tensor" is really a tensor. Since it's zero in Cartesian coordinates, then it has to be zero in polar coordinates as well, i.e.

$$\begin{pmatrix} \partial_1 V^1 & \partial_1 V^2 \\ \partial_2 V^1 & \partial_2 V^2 \end{pmatrix} = \begin{pmatrix} \partial_r V^r & \partial_r V^\theta \\ \partial_\theta V^r & \partial_\theta V^\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

But by chain rule, the derivative of the θ component of the vector field w.r.t. any direction s is given by [3]

$$\partial_s V^\theta = \left(\partial_r V^\theta\right)\left(\partial_s r\right) + \left(\partial_\theta V^\theta\right)\left(\partial_s \theta\right)$$

Plugging in $\partial_r V^\theta = 0$ and $\partial_\theta V^\theta = 0$ gives $\partial_s V^\theta = 0$, w.r.t. any direction s. But it directly contradicts our analysis earlier: we just saw that there exists directions s in which $\partial_s V^\theta$ is non-zero! Thus, at least one of $\partial_r V^\theta$ and $\partial_\theta V^\theta$ must be non-zero, meaning that it is impossible for the components of this "derivative tensor" to be all zero in polar coordinates, despite they were all zero in Cartesian coordinates!

Therefore, our postulated "derivative tensor" $\partial_r V^m$, with components simply being the coordinate derivatives, is **not** a tensor.

This creates a serious problem. To write tensor equations for laws of physics, we cannot freely use the good old coordinate derivatives. Terms like $\dfrac{\partial(\text{something})}{\partial x}$ or $\dfrac{\partial(\text{something})}{\partial \phi}$, which we are so comfortable with, are unfortunately banned!

We need to find how to take derivatives that are themselves tensors.

[3] I will help you out here: in the more familiar notation its $\dfrac{\partial V^\theta}{\partial x^s} = \dfrac{\partial V^\theta}{\partial r}\dfrac{\partial r}{\partial x^s} + \dfrac{\partial V^\theta}{\partial \theta}\dfrac{\partial \theta}{\partial x^s}$

2 Designing a tensor derivative

At first glance it seems like an incredibly complicated task. How would we even go about designing a derivative that is itself a tensor?

A good starting point would be to first try inventing a notation for such a tensor. Consider V_m, a tensor field of rank-1, and we want to take its tensor derivative. How should we denote this derivative? Well, it of course matters which component of the tensor field is being differentiated, so we would expect V_m to show up in the derivative notation. It also matters which coordinate we are differentiating w.r.t., i.e. whether it's a ∂_x or a ∂_y [4]. Thus a candidate notation might be

$$\partial_r V_m$$

However, $\partial_r V_m$ is reserved for the usual coordinate derivatives of the components of our tensor field. Let's indicate that our tensor derivative is, excuse my rudeness, more superior than the coordinate derivative by using a CAPITAL D instead of a small del [5]:

$$D_r V_m$$

If this $D_r V_m$ were to be a tensor, then just from the notation it's going to be a tensor with two lower indices.

Without knowing anything about this $D_r V_m$, the most general thing we can write down is that it is the usual coordinate derivative plus some unknown correction terms:

$$\boxed{\boldsymbol{D_r V_m = \partial_r V_m - \Gamma^t_{rm} V_t}}$$

The correction terms, $\Gamma^t_{rm} V_t$, is a sum over t, indicating that without really knowing anything, we let every single component of our tensor field V_m to carry some possible contribution. The correction coefficients Γ^t_{rm} will of course, apart from t dependence, also depend on r, the coordinate w.r.t. which we are differentiating, and m, the component of the tensor field being differentiated. Both r and m in $\Gamma^t_{rm} V_t$ are real indices, so that the index on both sides balances.

The minus instead of plus for the correction terms here is a convention.

[4] Yes, ∂_x stands for $\dfrac{\partial}{\partial x}$.

[5] Note: some books use $\boldsymbol{\nabla}_r V_m$, for example Sean Carroll's.

This is about as far as we can go with just symbol-pushing. It's time to get down to the details. There are two properties that we would like this tensor derivative to have:
(1) When differentiating a constant vector field (and tensor field), the tensor derivative should give zero. Moreover, it needs to give zero in every frame.
(2) This thing is here to do one job: it's here to be a tensor. So it should satisfy the tensor transformation rule.

We now see what we can learn from these two properties.

2.1 Tensor derivative on a constant vector field should give zero in every frame

Suppose V_m is an arbitrary constant vector field. We thus want

$$D_r V_m = \partial_r V_m - \Gamma^t_{rm} V_t = 0$$

in every frame.

Now, in 2.3 we learned that given any arbitrary space, at any point there exists a set of local Cartesian coordinates. Therefore, at any point on any space, there is some local Cartesian coordinates available to us.

There's nothing stopping us from evaluating the tensor derivative in the local Cartesian coordinates. Derivatives from a point are well-defined within the small vicinity of that point, and the local Cartesian coordinates do cover that small vicinity. Therefore,

$$(D_r V_m)\Big|_{\text{local Cartesian frame}} = (\partial_r V_m)\Big|_{\text{local Cartesian frame}} - (\Gamma^t_{rm} V_t)\Big|_{\text{local Cartesian frame}} = 0$$

Since V_m is a constant vector field, its usual coordinate derivatives in Cartesian coordinates all vanish, i.e. $(\partial_r V_m)\big|_{\text{local Cartesian frame}} = 0$. Therefore,

$$(\Gamma^t_{rm} V_t)\Big|_{\text{local Cartesian frame}} = 0$$

But V_m was an arbitrary constant vector field, and in general non-zero. Thus, we must have that in local Cartesian coordinates, the correction coefficients $\Gamma^t_{rm} = 0$.

We have thus obtained our first result:

$$\boxed{\textbf{In local Cartesian coordinates, } \boldsymbol{\Gamma^t_{rm} = 0}}$$

2.2 Tensor derivative is a tensor

Warning: some hardcore index-surgery straight ahead.

This tensor derivative is here to do one job: it's here to be a tensor. So it should satisfy the tensor transformation rule. Thus, we have

$$D_{r'}V_{m'} = \frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}D_r V_m$$

Substituting the definition for the tensor derivative $D_r V_m = \partial_r V_m - \Gamma^t_{rm}V_t$:

$$\begin{aligned} D_{r'}V_{m'} &= \frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}(\partial_r V_m - \Gamma^t_{rm}V_t) \\ &= \frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\partial_r V_m - \frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^t_{rm}V_t \qquad (1) \end{aligned}$$

But, another way to evaluate $D_{r'}V_{m'}$ is simply to prime all the indices in the definition:

$$D_{r'}V_{m'} = \underbrace{\partial_{r'}V_{m'}}_{\mathrm{I}} - \underbrace{\Gamma^{t'}_{r'm'}V_{t'}}_{\mathrm{II}} \qquad (2)$$

Now,

$$\begin{aligned} \mathrm{I} &= \partial_{r'}V_{m'} \\ &= \frac{\partial x^r}{\partial x^{r'}}\frac{\partial}{\partial x^r}V_{m'} \quad \text{(by chain rule, } \frac{\partial}{\partial x^{r'}} = \frac{\partial x^r}{\partial x^{r'}}\frac{\partial}{\partial x^r}\text{)} \\ &= \frac{\partial x^r}{\partial x^{r'}}\frac{\partial}{\partial x^r}\left(\frac{\partial x^m}{\partial x^{m'}}V_m\right) \quad \text{(transforming } V_{m'}\text{)} \\ &= \frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial V_m}{\partial x^r} + \frac{\partial x^r}{\partial x^{r'}}V_m\frac{\partial^2 x^m}{\partial x^r \partial x^{m'}} \quad \text{(product rule)} \end{aligned}$$

$$\begin{aligned} \mathrm{II} &= \Gamma^{t'}_{r'm'}V_{t'} \\ &= \Gamma^{t'}_{r'm'}\frac{\partial x^t}{\partial x^{t'}}V_t \quad \text{(transforming } V_{t'}\text{)} \end{aligned}$$

Therefore,

$$(2) = D_{r'}V_{m'} = \frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial V_m}{\partial x^r} + \frac{\partial x^r}{\partial x^{r'}}V_m\frac{\partial^2 x^m}{\partial x^r \partial x^{m'}} - \Gamma^{t'}_{r'm'}\frac{\partial x^t}{\partial x^{t'}}V_t$$

Equating (1) and (2):

$$\begin{aligned}&\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\partial_r V_m - \frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^t_{\ rm}V_t\\ =&\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial V_m}{\partial x^r} + \frac{\partial x^r}{\partial x^{r'}}V_m\frac{\partial^2 x^m}{\partial x^r \partial x^{m'}} - \Gamma^{t'}_{\ r'm'}\frac{\partial x^t}{\partial x^{t'}}V_t\end{aligned}$$

The first term on both sides $\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\partial_r V_m$ cancels, leaving

$$-\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^t_{\ rm}V_t = \frac{\partial x^r}{\partial x^{r'}}V_m\frac{\partial^2 x^m}{\partial x^r \partial x^{m'}} - \Gamma^{t'}_{\ r'm'}\frac{\partial x^t}{\partial x^{t'}}V_t \tag{3}$$

Inspect the first term on the right:

$$\begin{aligned}\frac{\partial x^r}{\partial x^{r'}}V_m\frac{\partial^2 x^m}{\partial x^r \partial x^{m'}} &= V_m\frac{\partial x^r}{\partial x^{r'}}\frac{\partial}{\partial x^r}\frac{\partial x^m}{\partial x^{m'}}\\ &= V_m\frac{\partial}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}} \qquad \text{(chain rule on } x^r\text{)}\\ &= V_m\frac{\partial^2 x^m}{\partial x^{r'}\partial x^{m'}}\\ &= V_t\frac{\partial^2 x^t}{\partial x^{r'}\partial x^{m'}} \qquad \text{(rename a dummy from } m \text{ to } t\text{)}\end{aligned}$$

Plugging back into (3):

$$-\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^t_{\ rm}V_t = V_t\frac{\partial^2 x^t}{\partial x^{r'}\partial x^{m'}} - \Gamma^{t'}_{\ r'm'}\frac{\partial x^t}{\partial x^{t'}}V_t$$

The V_t thus cancels:

$$-\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^t_{\ rm} = \frac{\partial^2 x^t}{\partial x^{r'}\partial x^{m'}} - \Gamma^{t'}_{\ r'm'}\frac{\partial x^t}{\partial x^{t'}}$$

Rearranging:

$$\Gamma^{t'}_{\ r'm'}\frac{\partial x^t}{\partial x^{t'}} = \frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^t_{\ rm} + \frac{\partial^2 x^t}{\partial x^{r'}\partial x^{m'}}$$

Multiplying by $\dfrac{\partial x^{t'}}{\partial x^t}$:

$$\Gamma^{t'}_{r'm'} = \frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^t_{rm} + \frac{\partial^2 x^t}{\partial x^{r'}\partial x^{m'}}\frac{\partial x^{t'}}{\partial x^t} \tag{4}$$

By declaring that $D_r V_m$ is a tensor, we have obtained (4), the transformation rule for these Γ^t_{rm} coefficients [6]. Turns out that Γ^t_{rm} is **not** a tensor, since its transformation rule is not the transformation rule for tensors! This is why when writing Γ^t_{rm}, we are free to be careless about the index placement. There's no need to worry whether t or r is the first index, since it's not a tensor.

The extra term $\dfrac{\partial^2 x^t}{\partial x^{r'}\partial x^{m'}}$ prompts us to conjecture that there might be some sort of symmetry between the r and m index, since mixed second partial derivatives commute. Let's apply (4) on Γ^t_{mr}:

$$\Gamma^{t'}_{m'r'} = \frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^r}{\partial x^{r'}}\Gamma^t_{mr} + \frac{\partial^2 x^t}{\partial x^{m'}\partial x^{r'}}\frac{\partial x^{t'}}{\partial x^t} \tag{5}$$

(4) − (5) gives

$$\Gamma^{t'}_{r'm'} - \Gamma^{t'}_{m'r'} = \frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^t_{rm} - \frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^r}{\partial x^{r'}}\Gamma^t_{mr}$$

where the second term cancels since $\dfrac{\partial^2 x^t}{\partial x^{r'}\partial x^{m'}} = \dfrac{\partial^2 x^t}{\partial x^{m'}\partial x^{r'}}$, due to the commutativity of second partial derivatives.

Factoring out:

$$\Gamma^{t'}_{r'm'} - \Gamma^{t'}_{m'r'} = \frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}(\Gamma^t_{rm} - \Gamma^t_{mr}) \tag{6}$$

[6] If you are reading Sean Carroll's text at the same time, you will notice that he derives a somewhat different transformation for Γ^t_{rm}. This is because he starts his narration with $D_r V^m$ instead of $D_r V_m$ like us, so he arrives at a manifestly different result. The two results can be shown, although somewhat complicatedly, to be equivalent. This issue shouldn't raise any anxiety: the explicit transform for Γ^t_{rm} is not that important on its own. I didn't even box this result.

Recall that any two-index object can be decomposed into a symmetric part and an anti-symmetric part (2.2 exercise 1(a)). Let's fix t, so that Γ^t_{mr} is only a two-index object. Let $\Gamma^t_{mr} = S_{mr} + A_{mr}$, where $S_{mr} = S_{rm}$ and $A_{mr} = -A_{rm}$.

Therefore,

$$
\begin{aligned}
\Gamma^t_{rm} - \Gamma^t_{mr} &= (S_{rm} + A_{rm}) - (S_{mr} + A_{mr}) \\
&= A_{rm} - (-A_{rm}) \\
&= 2A_{rm}
\end{aligned}
$$

Plug back into (6):

$$
\Gamma^{t'}_{r'm'} - \Gamma^{t'}_{m'r'} = \frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}2A_{rm} \tag{7}
$$

$$
= \frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^m}{\partial x^{r'}}\frac{\partial x^r}{\partial x^{m'}}2A_{mr} \tag{8}
$$

$$
= -\frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^m}{\partial x^{r'}}\frac{\partial x^r}{\partial x^{m'}}2A_{rm} \tag{9}
$$

From (7) to (8) we renamed the dummies: m is renamed to r, and r is renamed to m. From (8) to (9) we used $A_{mr} = -A_{rm}$. We now equate (7) and (9), canceling a factor of 2:

$$
\frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}}A_{rm} = -\frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^m}{\partial x^{r'}}\frac{\partial x^r}{\partial x^{m'}}A_{rm}
$$

$$
\therefore \left(\frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^m}{\partial x^{m'}} + \frac{\partial x^{t'}}{\partial x^t}\frac{\partial x^m}{\partial x^{r'}}\frac{\partial x^r}{\partial x^{m'}}\right)A_{rm} = 0
$$

In general, there's no reason to believe that the stuff in the bracket is zero. For example, consider a simple transform on a 1D space $x^{1'} = 2x^1$. Every derivative in the bracket is either $\dfrac{\partial x^{1'}}{\partial x^1} = 2$ or $\dfrac{\partial x^1}{\partial x^{1'}} = \dfrac{1}{2}$, so they can't add to zero.

Since the bracket is non-zero, we can divide it out and get

$$A_{rm} = 0$$

But A_{rm} was the antisymmetric part of Γ^t_{rm}. Since $A_{rm} = 0$, this says Γ^t_{rm} carries no antisymmetric part, and is thus symmetric in the lower two indices!

Thus, by demanding that $D_r V_m$ is a tensor, we have obtained the second important property of these Γ^t_{rm} coefficients:

$$\boxed{\boldsymbol{\Gamma^t_{rm}} \textbf{ is symmetric in the two lower indices, i.e. } \boldsymbol{\Gamma^t_{rm} = \Gamma^t_{mr}}}$$

It's time to introduce the proper names to these things.
· D_r is called the **covariant derivative**.
· Γ^t_{rm} are called the **connection coefficients**, or the **Christoffel symbols**.

The two properties we have derived are:
(1) The Christoffel symbols vanish in local Cartesian coordinates.
(2) The Christoffel symbols are symmetric in the lower two indices.

The fact that D_r is called the "covariant derivative" has nothing to do with covariant vectors or gradients. It's simply a way to say that the index on this derivative itself is downstairs. In addition, D_{mn} is not a thing: you can't take a derivative simultaneously w.r.t. two directions at once.

What about the covariant derivative on tensors of rank larger than one? Since it's a derivative, we would want to design it to obey the product rule. Consider a rank-2 tensor constructed by the tensor product [7] $T_{mn} = V_m W_n$. The covariant derivative, expanded using product rule, would be

$$\begin{aligned} D_s T_{mn} &= D_s(V_m W_n) \\ &= V_m(D_s W_n) + W_n(D_s V_m) \qquad \text{(product rule)} \\ &= V_m(\partial_s W_n - \Gamma^t_{sn} W_t) + W_n(\partial_s V_m - \Gamma^t_{sm} V_t) \\ &= \underbrace{V_m \partial_s W_n + W_n \partial_s V_m}_{= \partial_s(V_m W_n) \text{ by product rule}} - V_m \Gamma^t_{sn} W_t - W_n \Gamma^t_{sm} V_t \\ &= \partial_s T_{mn} - \Gamma^t_{sn} T_{mt} - \Gamma^t_{sm} T_{tn} \end{aligned}$$

Therefore,

$$\boxed{\boldsymbol{D_s T_{mn} = \partial_s T_{mn} - \Gamma^t_{sn} T_{mt} - \Gamma^t_{sm} T_{tn}}}$$

We can see the product-rule like nature of the covariant derivative on higher-rank tensors from the explicit form. Recall that for vectors, the covariant derivative was $D_r V_m = \partial_r V_m - \Gamma^t_{rm} V_t$. Apart from the coordinate derivative term (which is easy to remember), the correction term is constructed by changing the m on V_m into a dummy t. The Christoffel symbol's upstairs index has to be a t to complete the dummy pair in V_t, so the two downstairs indices have to be the remaining rm. It is quite convenient that the Christoffel symbols are symmetric in the lower two indices, as now we don't have to remember the order of the two lower indices.

Looking at $D_s T_{mn} = \partial_s T_{mn} - \Gamma^t_{sn} T_{mt} - \Gamma^t_{sm} T_{tn}$, it can be remembered similarly, with a product-rule like touch. The first term is the coordinate derivative term, as usual. For the connection terms, first keep one index on T_{mn} fixed and pretend the other is a dummy, and then pretend the other index is fixed and treat the first as a dummy. Here we first pretend m is fixed, so n turns into a dummy t. This dummy t needs an upstairs t, and the Γ provides that. The lower n is replaced by the dummy t, and the only place it can escape to is one of the downstairs spots on the Γ going with T_{mt}, hence the second term. Then we pretend n is fixed and turn m into a dummy, giving us the third term.

The product-rule like nature of the covariant derivative also holds for tensors with even higher ranks. You will be asked to play with this in the exercises.

[7] See 2.25

3 Metric compatibility

We've really just seen one tensor: the metric g_{mn}. Let's take its covariant derivative and see what we can find.

$$D_s g_{mn} = \partial_s g_{mn} - \Gamma^t_{sn} g_{mt} - \Gamma^t_{sm} g_{tn} \tag{10}$$

This is a tensor equation, so we can evaluate it in any frame. Let's evaluate (10) in the local Cartesian coordinates associated with some point.

In the local Cartesian coordinates, we know that the Christoffel symbols all vanish. Therefore the two terms $-\Gamma^t_{sn} g_{mt} - \Gamma^t_{sm} g_{tn}$ are zero.

In addition, we know that in local Cartesian coordinates, all first derivatives of every component of the metric with respect to every local coordinate variable also vanish. Thus, $\partial_s g_{mn} = 0$.

Plugging into (10), we get $D_s g_{mn} = 0$ in local Cartesian coordinates around some point.

But here's the catch: $D_s g_{mn}$ is a tensor. If it's zero in one frame, then it's zero in every frame! Therefore, locally around any point (since local Cartesian coordinates exist for any point), we have $D_s g_{mn} = 0$ in every possible coordinate frame! Since this property is true for all points, it is thus a global property!

This is called **metric compatibility**. The covariant derivative of the metric vanishes everywhere in every frame:

$$\boxed{\boldsymbol{D_s g_{mn} = 0}}$$

We are now in a position to finally derive the explicit form of the Christoffel symbols Γ^t_{rm}. We take different index permutations of metric compatibility to get

$$D_s g_{mn} = \partial_s g_{mn} - \Gamma^t_{sn} g_{mt} - \Gamma^t_{sm} g_{tn} = 0 \tag{11}$$

$$D_m g_{sn} = \partial_m g_{sn} - \Gamma^t_{mn} g_{st} - \Gamma^t_{ms} g_{tn} = 0 \tag{12}$$

$$D_n g_{sm} = \partial_n g_{sm} - \Gamma^t_{nm} g_{st} - \Gamma^t_{ns} g_{tm} = 0 \tag{13}$$

(12) + (13) − (11) gives

$$\partial_m g_{sn} - \Gamma^t_{mn} g_{st} - \Gamma^t_{ms} g_{tn} + \partial_n g_{sm} - \Gamma^t_{nm} g_{st} - \Gamma^t_{ns} g_{tm} - \partial_s g_{mn} + \Gamma^t_{sn} g_{mt} + \Gamma^t_{sm} g_{tn} = 0$$

Now we cancel terms due to symmetry of the metric and the Christoffel symbols. The two dot-underlined terms cancel because $\Gamma^t_{ms} = \Gamma^t_{sm}$. The two dash-underlined terms cancel because $\Gamma^t_{ns} = \Gamma^t_{sn}$, and $g_{tm} = g_{mt}$. This leaves us with

$$\partial_m g_{sn} - \Gamma^t_{mn} g_{st} + \partial_n g_{sm} - \Gamma^t_{nm} g_{st} - \partial_s g_{mn} = 0$$

And again, $\Gamma^t_{mn} = \Gamma^t_{nm}$, so

$$\partial_m g_{sn} + \partial_n g_{sm} - \partial_s g_{mn} = 2\Gamma^t_{mn} g_{st}$$

Multiplying both sides by the inverse metric g^{as} and dividing out the 2:

$$\Gamma^t_{mn} g^{as} g_{st} = \frac{1}{2} g^{as} (\partial_m g_{sn} + \partial_n g_{sm} - \partial_s g_{mn})$$

On the left we have $g^{as} g_{st} = \delta^a_t$ (see 2.2). Therefore the left is $\Gamma^t_{mn} \delta^a_t$. Summing out the delta gives Γ^a_{mn}. Therefore,

$$\Gamma^a_{mn} = \frac{1}{2} g^{as} (\partial_m g_{sn} + \partial_n g_{sm} - \partial_s g_{mn})$$

Renaming a to t, we have thus obtained the explicit formula for the Christoffel symbols:

$$\boxed{\boldsymbol{\Gamma^t_{mn} = \frac{1}{2} g^{ts} \left(\partial_m g_{sn} + \partial_n g_{sm} - \partial_s g_{mn}\right)}}$$

Note: since the inverse metric is also symmetric, sometimes I will use

$$\Gamma^t_{mn} = \frac{1}{2} g^{st}(\partial_m g_{sn} + \partial_n g_{sm} - \partial_s g_{mn})$$

There's really no difference between the two versions. In fact they aren't even two versions. They are really the same. The difference is merely aesthetic.

From the explicit form of the Christoffel symbols we can quickly see that the two properties derived earlier are true:

(1) $\Gamma^t_{mn} = \Gamma^t_{nm}$ indeed.
If we swap m and n, the g^{st} doesn't have m or n as indices so it's unaffected; the first two terms $\partial_m g_{sn} + \partial_n g_{sm}$ simply swap into each other into $\partial_n g_{sm} + \partial_m g_{sn}$; and the last term is unaffected since $g_{nm} = g_{mn}$.

(2) In Cartesian coordinates, $\Gamma^t_{mn} = 0$.
In Cartesian coordinates, all metric components are 1 or 0, so all the derivatives $\partial_a g_{bc}$ are zero. The bracket is thus zero, and the Christoffel symbols are thus zero.

Note: while earlier we were claiming that the Christoffel symbols are zero in local Cartesian coordinates, we looked at local frames simply because a local Cartesian frame would always be available. If the Cartesian coordinates were global, all metric components are still 1 or 0, so the Christoffel symbols still vanish.

4 Enough, what really *is* the covariant derivative?

That was a lot of index surgery. However, even with the explicit formulas for a tensor derivative (we called that "the covariant derivative"), which we can now use in our equations for the laws of physics, we still don't feel like we really understand what the covariant derivative *is*.

We know that it behaves like the usual familiar coordinate derivative in some ways: on a constant vector field, the covariant derivative is zero. It also obeys the product rule of differentiation. So it can't be that different than the usual coordinate derivative.

In fact, a good starting point would be to consider the covariant derivative in Cartesian coordinates. Looking at

$$D_r V_m = \partial_r V_m - \Gamma^t_{rm} V_t$$

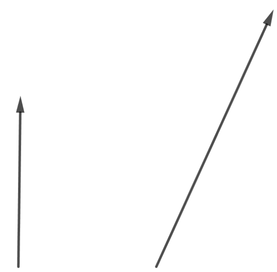
and realizing that $\Gamma^t_{rm} = 0$ in Cartesian coordinates, we find that **in Cartesian coordinates, the covariant derivative is simply the usual coordinate derivative.**

Thus, whatever the usual coordinate derivative does in Cartesian coordinates, that is what the covariant derivative does for all coordinates. Think about it: the covariant derivative is a tensor, so it performs the same functions in all coordinates [8]. Thus, whatever it does in Cartesian coordinates, it should do that in all coordinates. And quite fortunately, in Cartesian coordinates, covariant derivative is just the usual coordinate derivative.

Let's now inspect what the usual coordinate derivative does in Cartesian coordinates.

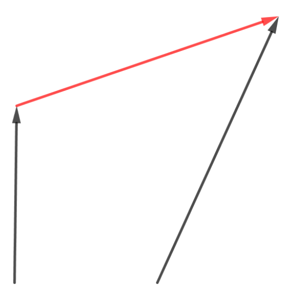
[8] By now you should already have grown accustomed to the intuition that, the correctness of tensor statements is independent of the choice of (or even the need for) coordinates. "Function" in this sense is not a mathematical function; instead it simply refers to the purpose of the covariant derivative, i.e. the use of the word "function" in an everyday sense.

Consider some vector field V^m on the usual 2D Cartesian plane with Cartesian coordinates, and we zoom into into a small local neighborhood. Let's focus on two vectors in this neighborhood, whose root positions differ by some small x.

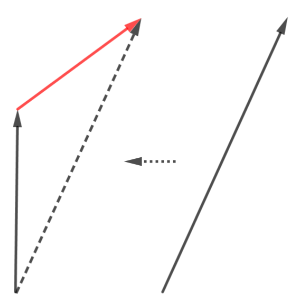


The vector field is the vector on the left at the point (x, y), and is the vector on the right at the point $(x + dx, y)$. Their vectorial difference is thus the derivative $\partial_x V^m$.

Notice that, when evaluating their vectorial difference, the difference is not this:



but this:



In other words, the second vector is **parallel transported** so that the two roots coincide, then take the vectorial difference.

This procedure is necessary for taking vector derivatives. Imagine a constant vector field, so the left and right vectors would be the same. If their difference is taken without this parallel transport, the difference vector would be $(dx, 0)$, which is not correct. It is only after we parallel transport the second vector to coincide the roots, then we can get the difference vector as $(0, 0)$.

So, the vector derivative is the infinitesimal difference vector between two neighboring points in a vector field, but with the vectors parallel transported to coincide the roots.

This infinitesimal difference vector is a *vector*, so it transforms. In Cartesian coordinate, the components of this infinitesimal difference vector are simply the coordinate derivatives. **In other coordinates, the components of this infinitesimal difference vector are the covariant derivatives**, since we went to great pains to make sure that the covariant derivative obeys the transformation rules!

Therefore, it is really easy to see what the covariant derivative actually does. Conceptually, it is just like the usual vector derivative: it tells us the infinitesimal difference vector between two neighboring points in a vector field, with the difference vector being taken under parallel transport. The punchline is that the covariant derivative takes care of the tensor-y transformations, so it tells us this infinitesimal difference vector *in an arbitrary frame*!

Exercises 4 and 5 will give concrete examples.

5 Summary

1. The usual coordinate derivatives of a tensor, i.e. the derivatives of the components of a tensor w.r.t. the coordinate variables $\partial_s T_{mn}$, are not themselves components of tensors. They cannot appear standalone in equations that are frame-invariant.

2. The covariant derivative:

$$D_r V_m = \partial_r V_m - \Gamma^t_{rm} V_t$$

$$D_s T_{mn} = \partial_s T_{mn} - \Gamma^t_{sn} T_{mt} - \Gamma^t_{sm} T_{tn}$$

3. The Christoffel symbols:

$$\Gamma^t_{mn} = \frac{1}{2} g^{st} \left(\partial_m g_{sn} + \partial_n g_{sm} - \partial_s g_{mn} \right)$$

$$\Gamma^t_{mn} = \Gamma^t_{nm}$$

$$\Gamma^t_{mn} = 0 \quad \text{in Cartesian coordinates}$$

4. Metric compatibility:

$$D_s g_{mn} = 0$$

6 Exercises

1. On page 14 we saw how to generalize the covariant derivative to tensors with two lower indices with two methods: (1) rigorously with the tensor product, and (2) intuitively by index-surgery.

By using both of these two methods, find the covariant derivative of tensors with an arbitrary number of lower indices, i.e. find $D_\sigma T_{\nu_1\nu_2\ldots\nu_l}$.

2. Show that

$$D_m V^n = \partial_m V^n + \Gamma^n_{mr} V^r$$

Hence, find the covariant derivative of tensors with an arbitrary rank, i.e. find

$$D_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l}$$

Hint: use index raising.

3. Calculate all the Christoffel symbols in polar coordinates on the 2D Cartesian plane. There should be 6 of them (up to symmetry, i.e. if we count $\Gamma^r_{r\theta} = \Gamma^r_{\theta r}$ as just one).

4. A constant vector field on the 2D Cartesian plane is given by $V^m = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in Cartesian coordinates. Since the vector field is constant, we should see that all of its covariant derivatives vanish in both Cartesian coordinates and polar coordinates. Check this by calculating the covariant derivatives explicitly. There should be 4 covariant derivatives in each coordinate. Make sense of your results with parallel transport.

5. A vector field on the 2D Cartesian plane is given by $V^m = \begin{pmatrix} x \\ y \end{pmatrix}$ in Cartesian coordinates. Calculate all 8 covariant derivatives of this vector field in both Cartesian and polar coordinates. Make sense of your results with parallel transport.

6. A covariant vector V_n and a contravariant vector W^m is being parallel transported along some path x^s by an infinitesimal step. Show that

$$D_s(V_n W^n) = 0$$

Hint: use index lowering/raising and metric compatibility. What does this result physically mean?

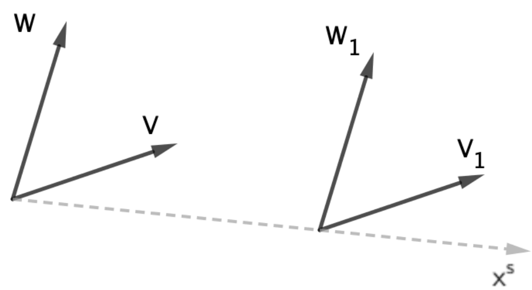


Figure 4: Two vectors V and W being parallel transported along some direction x^s. V is being transported to V_1, and W is being transported to W_1. The labels in this figure are **not** tensor notation, but rather just non-rigorous labels for illustrative purposes.