2.1 What Is A Vector; Contravariant and Covariant Vectors

Since we want to write equations that are invariant in any coordinates, we will have to first figure out how various quantities transform between coordinates. Only then can we figure out how the equations built from these quantities will transform between coordinates, and then we can try to find an equation that does not transform.

Let's start with the most common and the most familiar quantity: vectors.

1 What is a vector?

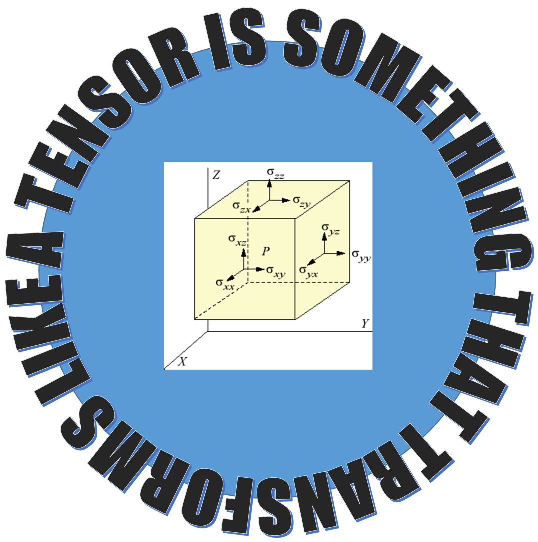

Figure 1: A very accurate, very precise, very useful, yet very useless definition: a tensor is something that transforms like a tensor. Vectors are just tensors with rank 1.

Try answering this question yourself: what is a vector?

The most common answer, I expect, is that "a vector is a list of numbers".

Well, that's not entirely wrong. A vector indeed consists of a list of numbers. However, that doesn't seem to tell the whole story. Consider this: you have a barrel of fruits, with N_1 apples, N_2 bananas, and N_3 coconuts [1]. For the sake of concreteness let's say you have 5 apples, 6 bananas and 1000 coconuts. If you like, you can write them as a "vector", the "number-of-fruits vector" $\vec{\mathbf{N}}$:

$$\vec{\mathbf{N}} = \begin{pmatrix} 5 \\ 6 \\ 1000 \end{pmatrix}$$

[1]Believe it or not, there aren't that many fruits that start with a C. I chose coconut because cranberry is hard to type, and who the hell knows what a calabash or a conkerberry or a citrofortunella is.

Alongside this number-of-fruits "vector", let's also throw in an actual honest-to-god vector, a position vector $\vec{\mathbf{r}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ on the 2-dimensional Cartesian plane. Maybe you can imagine an actual barrel containing 5 apples, 6 bananas and 1000 coconuts sitting at $\vec{\mathbf{r}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

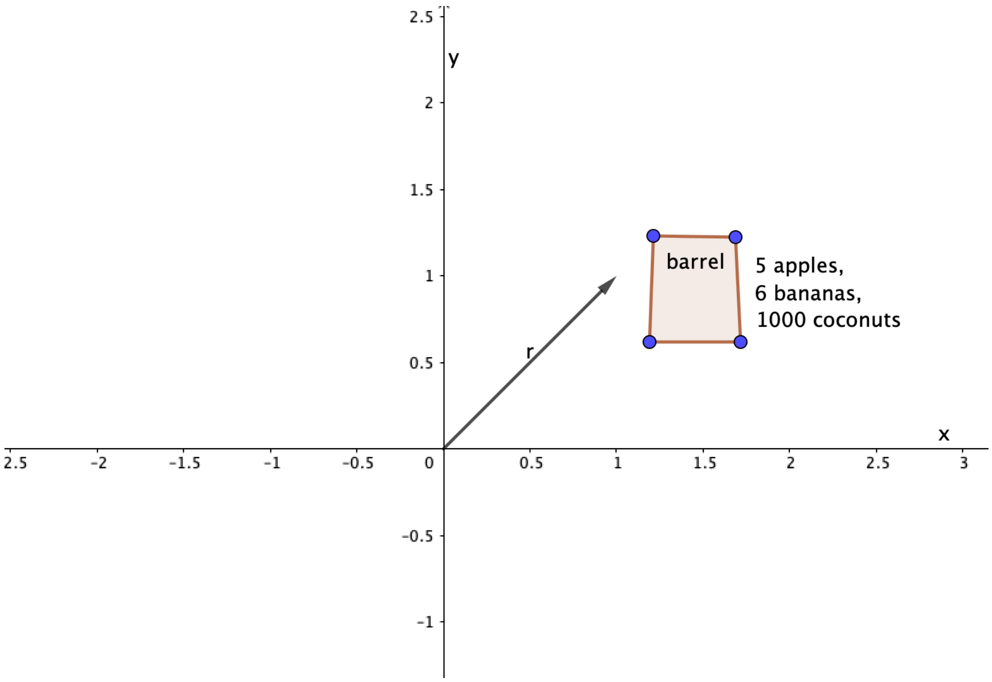

Figure 2: A barrel containing 5 apples, 6 bananas and 1000 coconuts sitting at $\vec{\mathbf{r}} = (1, 1)$.

Let's now apply a coordinate transform. For simplicity let's choose an easy transform: counterclockwise rotation by 90 degrees. This will rotate the frame into $x' = y, y' = -x$.

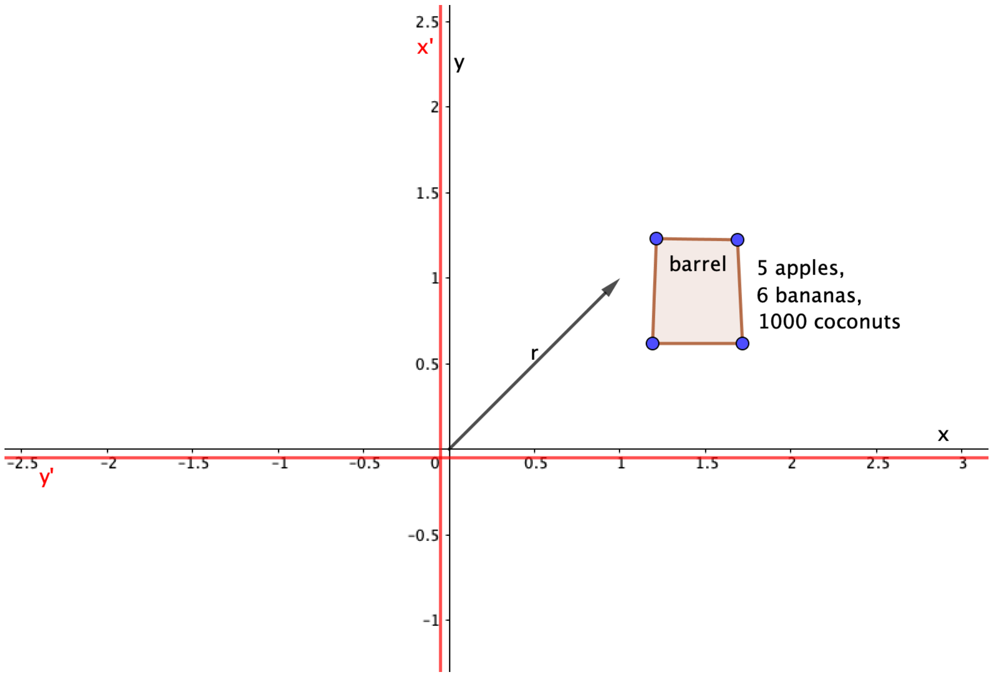

Figure 3: A barrel containing 5 apples, 6 bananas and 1000 coconuts. The new coordinate frame is shown in red. The origin of the two frames should coincide; when graphing them I nudged the new frame slightly so that it would be clear that there are two frames.

How do the two "vector"s transform? We are well-aware of how the position vector transforms. Either by the explicit transform $x' = y, y' = -x$ or just by inspection, we know the position vector of the barrel in the primed new frame is $\vec{\mathbf{r}}' = (1, -1)$. Under a coordinate transform, the numerical values of the components of this vector will change.

But how about the the "number-of-fruits vector" $\vec{\mathbf{N}}$? Well, after the rotation of the frame, there are still 5 apples, 6 bananas, and 1000 coconuts in the barrel. Nothing has changed about that; therefore, $\vec{\mathbf{N}}$ does not transform when we transform to different coordinates. $N_1 = 5$ still, $N_2 = 6$ still, and $N_3 = 1000$ still. So there are some differences between this "number-of-fruits vector" that is simply a list of numbers, and the honest-to-god position vector.

Quantities that do not transform when we make a coordinate transform are called **scalars**. For example, the number of each kind of fruit in a barrel, or the number of each kind of atom in a barrel (and hence the mass of the barrel, or the mass of whatever thing you have), or the temperature at a location, these quantities do not change their numerical values when you switch to another coordinate frame. No matter whether you point your x axis due east or due north, and no matter whether you place you origin in New York or in California, there will still be 1000 coconuts in the barrel, and the temperature in New York will still be 300K [2] on a relatively sunny day.

Quantities that do change their numerical values when you switch to another coordinate frame are called **tensors**. A vector, like our position vector, is a kind of tensor. It's called a tensor *of rank one*, though we won't go into what that means today. All that concerns us right now is that actual vectors are quantities whose (components') numerical values change when we make a coordinate transform. Position, velocity, acceleration, forces, all of these vectors in classical physics fall into this category. The numerical values of their components will change if we change coordinates (for example, if we rotate our reference frame).

So no, a vector is not just a list of numbers. It's a list of numbers, but a list *that transforms*. The "number-of-fruits vector" $\vec{\mathbf{N}}$ is thus not a vector; rather it's simply a list of scalars being written near each other.

[2] Kelvin; 300K is about 27 degrees Celsius or 80 degrees Fahrenheit.

2 How vectors transform: contravariant and covariant

Now that we know vectors are quantities that transform when we change coordinates, the natural question that follows is *how* vectors transform.

This question turns out to be more complicated than it looks, because as it turns out, there are two kinds of vectors that transform somewhat differently.

Consider a simple coordinate transform: changing the unit of length from meters to centimeters. If the unit distance is 1 centimeter in x'-y' frame instead of the 1 meter in x-y frame, then the point $(x, y) = (1, 1)$ would be described in the primed frame as $(x', y') = (100, 100)$. (1,1) in unprimed frame means 1 unprimed unit distance, which is 1 meter away from the origin (in both x and y). If the unit distance were 1 centimeter, then the point would be 100 unit distances, which is 100 centimeters, away from the origin. The coordinate transform is thus

$$\begin{cases} x' = 100x \\ y' = 100y \end{cases}$$

The position *vector* associated with the point thus transforms from $\vec{\mathbf{r}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (meters) to $\vec{\mathbf{r}}\,' = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$ (centimeters). The vector itself hasn't changed: before and after the change of units the vector points to the same location. It's just that the numerical values we associate to the components of this vector has changed, because we have changed our coordinates.

All quantities that have [meter to the positive one power] in their units transform like this. Displacement vectors and position vectors are kind of the same thing. A velocity of $\vec{\mathbf{v}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (meters per second) transforms to $\vec{\mathbf{v}}\,' = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$ (centimeters per second). An acceleration of $\vec{\mathbf{a}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (meters per second per second) transforms to $\vec{\mathbf{a}}\,' = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$ (centimeters per second per second).

These vectors, whose components' numerical values scale up when you scale down the unit from meters to centimeters, are called **contravariant vectors**. It's a very suitable name: "contra-" means "opposite", reminding ourselves that these vectors transform oppositely with respect to the base unit. When the base unit scales down, the components of these contravariant vectors scale up. Incidentally, this also means that **the base coordinates**

themselves are contravariant: when the unit scales down from meters to centimeters, then base coordinates transform like $x' = 100x$, i.e. scales up.

There are, however, another kind of vectors who carry [meter to the negative one power] in their units. For example, forces carry units [joules per meter] [3].

Consider a force $\vec{\mathbf{F}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (joules per meter). It means that $\vec{\mathbf{F}}$ is the force that does 1 joule of work when moving something by 1 meter in the x direction. What does it transform to when we scale the units down from meters to centimeters? Well, if $\vec{\mathbf{F}}$ does 1 joule of work when moving something by 1 meter, then it would do 0.01 joule of work when moving something by 0.01 meter, or 1 centimeter. Therefore $\vec{\mathbf{F}}\,' = \begin{pmatrix} 0.01 \\ 0 \end{pmatrix}$ (joules per centimeter). Again, the vector itself hasn't changed. It's the same force. It's just that the numerical values we associate to the components of this vector has changed, because we have changed our coordinates.

These vectors, whose components' numerical values scale down when you scale down the unit from meters to centimeters, are called **covariant vectors**. It's a very suitable name: "co-" means "together with", reminding ourselves that these vectors transform together with respect to the base unit. When the base unit scales down, the components of these covariant vectors also scale down.

So, contravariant vectors are vectors that transform like displacements (so they carry $[m^1]$ as units), and covariant vectors are vectors that transform like gradients (so they carry $[m^{-1}]$ as units).

Let's box up these realizations:

Contravariant vectors are vectors that transform like displacements.

and

Covariant vectors are vectors that transform like gradients.

Our task is now easy: we need to figure out how displacements and gradients transform.

[3]Reminder: work [joule] = force [newton] × distance [meter]. Another reminder: force is the negative gradient of potential energy. If you are unfamiliar with the latter, think about how the electric field is the negative gradient of the electric potential. The latter is just this statement's non-electric analog. Intuitively, it says things move towards lower potential energy. This should be intuitive: when a ball falls to the ground, it's moving to places with lower potential energy (mgh).

3 Notation

There's a small issue regarding notation. Traditionally, a vector is denoted by a bold font with an arrow, like $\vec{\mathbf{v}}$, and the components of a vector are denoted by subscripts, like v_i, with $i = 1, 2, 3, ...$

However, now that we know there are two kinds of vectors that transform differently, this notation becomes troublesome. When we write $\vec{\mathbf{v}}$, are we referring to a covariant vector, or a contravariant vector? There's no way to tell.

The resolution, used by Einstein, is simple: we discard the arrow notation $\vec{\mathbf{v}}$ altogether, and let the component notation v_i denote the vector directly. We let subscripts denote covariant vectors, and let superscripts denote contravariant vectors.

Therefore, $\boldsymbol{A_i}$ denotes a covariant vector, and $\boldsymbol{A^i}$ denotes a contravariant vector.

It might feel weird seeing something like A^i not meaning A to the i^{th} power, but over time you will get familiar with this notation.

Since the base coordinates themselves are contravariant, from now on we will denote them as $\{x^1, x^2, x^3, ...\}$ instead of $\{x, y, z, ...\}$.

An infinitesimal displacement vector is therefore denoted as dx^m. In the old literature it would be denoted as

$$\boldsymbol{d\vec{s}} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

but now since we are using upstairs indices to denote the base coordinates, this would become

$$\boldsymbol{d\vec{s}} = \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}$$

and since we are discarding the whole bold-and-arrow thing and using components directly, this would further become dx^m. It carries an upstairs index, just like how contravariant vectors should. You don't need to specify the range of m. It will almost always be obvious what the range of the indices is. In this case, m goes from 1 to 3.

What about gradients? Consider a scalar field $s(x, y, z)$, or in our new language, $s(x^1, x^2, x^3)$, or to further shorthand just $s(x^m)$.

The gradient of this scalar field is, written in old notation,

$$\boldsymbol{\nabla} s = \begin{pmatrix} \dfrac{\partial s}{\partial x} \\ \\ \dfrac{\partial s}{\partial y} \\ \\ \dfrac{\partial s}{\partial z} \end{pmatrix}$$

In our new notation it is

$$\boldsymbol{\nabla} s = \begin{pmatrix} \dfrac{\partial s}{\partial x^1} \\ \\ \dfrac{\partial s}{\partial x^2} \\ \\ \dfrac{\partial s}{\partial x^3} \end{pmatrix}$$

Again, since we are using component notation to represent the vector itself, this gradient would be denoted as $\dfrac{\partial s}{\partial x^m}$.

Notice that gradients are covariant, so they should carry downstairs indices. Therefore when we see an index that is a superscript but in the denominator, like $\dfrac{\partial s}{\partial x^m}$, we still consider it as a downstairs index.

When being transformed to another frame, we let the indices carry the primes. Therefore dx^m transforms to $dx^{m'}$, and $\dfrac{\partial s}{\partial x^m}$ transforms to $\dfrac{\partial s}{\partial x^{m'}}$.

4 Transform

We are now in a position to derive the transformation rules. We want to know how displacements and gradients transform.

Consider two sets of coordinates x^m and $x^{m'}$ on the same space [4]. The transformation between these two sets of coordinates is known, and is given by $x^m = x^m(x^{1'}, x^{2'}, ...)$ and $x^{m'} = x^{m'}(x^1, x^2, ...)$.

Displacement:

Let a differential displacement dx^m be known in the unprimed frame. Concretely, this means that given two points infinitesimally close, every component of the differential displacement vector between these two points in the unprimed frame is known, i.e. all the dx^m values are known. We seek the components of the same differential displacement vector between these two points in the primed frame, i.e. the $dx^{m'}$ values.

The displacement in the primed frame is, by chain rule,

$$dx^{m'} = \sum_m \frac{\partial x^{m'}}{\partial x^m} dx^m$$

Now let's introduce the **Einstein summation convention**: **when an index appears in a single term as an upstairs-downstairs pair, the index is assumed to be summed over.**

To see this convention in action, we can rewrite the above equation as

$$dx^{m'} = \frac{\partial x^{m'}}{\partial x^m} dx^m$$

[4] Or spacetime. We will develop our geometry theory with space so that we can draw familiar examples from Cartesian spaces or surface of a sphere, but all of our geometry theory will apply to spacetime as well. Keep in mind that a spacetime is just like a space: it's a space with one of its coordinates being a time coordinate. The technical word for a space and a spacetime is that they are both what's called a "manifold". I will not go deep it because it is not absolutely essential, but just know that the fact that spacetime and space share the same word in the technical literature shows that they are the same thing. For a in-depth discussion at manifolds you can read Sean Caroll's chapter 2.

In the single term $\dfrac{\partial x^{m'}}{\partial x^m} dx^m$, the index m appears once upstairs in dx^m and once downstairs as $\dfrac{\partial x^{m'}}{\partial x^m}$ (reminder that a contravariant index (i.e. superscript) in the denominator is still considered a downstairs index), so there is an implied sum over m.

There is *no* summation when they appear across different terms. For example, there are no sums implied by $a^m + b^m$ or $a^m = b^m$. Only when the same index appears once upstairs and once downstairs in a single term, for example things like $a^m b_m$, is there an implicit sum.

Also we know summation indices are dummy indices and can be renamed to whatever we like. Hence we can have things like $a^m b_m = a^p b_p = a^\mu b_\mu$. They all mean the same thing: $a^1 b_1 + a^2 b_2 + a^3 b_3 + ...$

So there we have it: transformation of displacements. Since all contravariant vectors transform like displacements, we now have the transformation rule for contravariant vectors:

$$V^{m'} = \frac{\partial x^{m'}}{\partial x^m} V^m$$

We got the above by identifying dx^m as an arbitrary contravariant vector V^m, and hence $dx^{m'}$ as $V^{m'}$.

Gradient:

Let the gradient $\dfrac{\partial s}{\partial x^m}$ of some scalar field s be known in the unprimed frame.

The gradient in the primed frame is, by chain rule,

$$\frac{\partial s}{\partial x^{m'}} = \frac{\partial s}{\partial x^m}\frac{\partial x^m}{\partial x^{m'}}$$

Since all covariant vectors transform like gradients, we now have the transformation rule for covariant vectors:

$$V_{m'} = V_m \frac{\partial x^m}{\partial x^{m'}}$$

We got the above by identifying $\dfrac{\partial s}{\partial x^m}$ as an arbitrary covariant vector V_m, and hence $\dfrac{\partial s}{\partial x^{m'}}$ as $V_{m'}$.

To keep in accordance with the contravariant transform, let's do a little reordering:

$$V_{m'} = \frac{\partial x^m}{\partial x^{m'}} V_m$$

These transformation rules are truly essential. Let's box them up.

SUMMARY OF VECTOR TRANSFORM RULES

Contravariant vectors: $V^{m'} = \dfrac{\partial x^{m'}}{\partial x^m} V^m$

Covariant vectors: $V_{m'} = \dfrac{\partial x^m}{\partial x^{m'}} V_m$

These transformation rules are in fact quite easy to remember. It couldn't really be anything else, given the placement of the indices.

Consider the covariant transform $V_{m'} = \dfrac{\partial x^m}{\partial x^{m'}} V_m$. Say you have forgotten the transform and is trying to figure it out what $V_{m'}$ should be given a known V_m:

$$V_{m'} = \frac{??}{??} V_m$$

The left hand side has a single lower index m', so the right hand side also needs to have a single lower index m' [5]. The m in V_m right now is a lower index, but there isn't a lower m on the left, so there has to be an upper m to make this m into a dummy index, so that it's not a real index:

$$V_{m'} = \frac{\partial x^m}{??} V_m$$

Since the right hand side also needs to have a single lower index m', it has to go to the denominator:

$$V_{m'} = \frac{\partial x^m}{\partial x^{m'}} V_m$$

Einstein notation is a really powerful notation. It guides you and tells you what to do. This is one of the many places where it guides you.

I will leave two important results in the exercises. I am leaving them in the exercises because I want them to help you get a feel of using the Einstein notation. Answers to exercises will be in a separate note. However, since these results are important, I'll list them out here in the main text.

1. If a vector (either covariant or contravariant) is zero in one frame, then it's zero in all frames. A vector is said to be zero if all of its components are zero.

2. The **inner product** $A^m B_m$ between a contravariant vector and a covariant vector is the same in all frames, i.e. inner products are frame-invariant.

[5] Of course both sides of an equation need to have their indices balanced.

5 Exercise

1. We have seen many times that when the length scale is scaled down from meters to centimeters, the coordinate transform is

$$\begin{cases} x^{1'} = 100x^1 \\ x^{2'} = 100x^2 \end{cases}$$

Using the formal transformation rules, explicitly show that under this transformation, the components of contravariant vectors scale up, and the components of covariant vectors scale down.

2. We have seen how to "derive" the covariant transform from matching indices if you forgot it. Say you have forgotten the contravariant transform. Derive it.

3. Show that the sum of two contravariant vectors is also contravariant. i.e. given A^m and B^m, both satisfying the contravariant transform, show that $V^m \equiv A^m + B^m$ also satisfies the contravariant transform [6].

4. Show that the sum of two covariant vectors is also covariant. i.e. given A_m and B_m, both satisfying the covariant transform, show that $V_m \equiv A_m + B_m$ also satisfies the covariant transform.

5. Show that if $V_m = 0 \,\forall\, m$ in some frame [7], then given an arbitrary transform $x^{m'} = x^{m'}(x^1, x^2, ...)$, we have $V_{m'} = 0 \,\forall\, m'$, i.e. if a covariant vector is zero in one frame, then it's zero in all frames.

6. Show that if $V^m = 0 \,\forall\, m$ in some frame , then given an arbitrary transform $x^{m'} = x^{m'}(x^1, x^2, ...)$, we have $V^{m'} = 0 \,\forall\, m'$, i.e. if a contravariant vector is zero in one frame, then it's zero in all frames.

7. Show that given a contravariant vector A^m and a covariant vector B_m, under an arbitrary transform $x^{m'} = x^{m'}(x^1, x^2, ...)$, we have $A^m B_m = A^{m'} B_{m'}$, i.e. inner product between a contravariant vector and a covariant vector is the same in all frames.

Hint: the partial derivatives of the base coordinates with each other is zero, and the partial derivative of a base coordinate with itself is one. i.e. in the old language, on a Cartesian plane, we have

$$\frac{\partial x}{\partial x} = \frac{\partial y}{\partial y} = 1, \frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = 0$$

[6] I will use $\equiv$ to mean "define".

[7] I will use $\forall$ to mean "for all"

8. We know the transformation between polar coordinates and Cartesian coordinates is

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

or, equivalently [8],

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \\ \theta = \tan^{-1}(\frac{y}{x}) \end{cases}$$

(a) Consider a contravariant vector in polar coordinates $A^m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This means its r-component is 0 and the θ-component is 1. Find its components $A^{m'}$ in Cartesian coordinates. Does your result make sense?

(b) Consider a covariant vector in polar coordinates $B_m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This means its r-component is 0 and the θ-component is 1. Find its components $B_{m'}$ in Cartesian coordinates. Does your result make sense?

(c) Explicitly verify that the inner product is frame-invariant, i.e. $A^m B_m = A^{m'} B_{m'}$. Physically, what does this result mean?

[8] Actually since arctangent goes from $-\pi/2$ to $\pi/2$, the full transform is $\theta = \tan^{-1}(y/x) + k\pi$, where $k = 0$ for the right half plane and $k = 1$ for the left half plane, but since adding constants don't do anything to derivatives, we can gladly take the shortened version.