1.1 Motivation: Losing Track Of Time

What does physics study?

An apple falls from above Sir Newton's head, and we want to know how long before it hits him; the Earth moves around the sun, and we want to know how long before it completes a full circle; a rocket launches from the surface of our planet, and we want to know how fast we need to launch it so that it can escape our planet's gravity; an arrow shoots out towards a wildebeest for lunch, and we want to know what angle we need to shoot it out at so that it goes the farthest.

It is believed that physics originated in Ancient Greece. You look around the world, and you see it's full of stuff. What is that stuff, and how do these stuff behave? Aristotle believed that all things in the world had a place in nature where they belonged to: fire rises to the skies because fire belongs to higher places, earth (as in mud, not the planet) and rocks fall to the ground because they belong to lower places. Aristotle also made one of the more famous mistakes in the history of science: as it later turned out, force is not necessary to maintain motion.

Then Newton came along, unifying fire, rocks, apples and planets into a single framework, known today as "Newton's Laws of Motion". Now Newton's three powerful laws are taught in every highschool classroom.

Have you noticed it?

Physics is the study of how things *move*.

How does the falling apple, the revolving Earth, the rising rocket, and the flying arrow all move? For Aristotle, he was wondering why fire and mud moved the way they did, and why they kept moving. Newton's Laws are just outright called the laws of *motion*.

Many people forget that physics is the study of how things move when they are lost in a sea of derivatives and integrals, electricity and magnetism, pressure and buoyancy, protons and neutrons. But this theme is the heart of physics.

Therefore, it is important that we crisply understand exactly how to describe motion.

1 Describing Motion

Imagine a one-dimensional world. Usually such a world is represented by a single x-axis. Each position on the axis has some numerical label associated with it, and the position with $x = 0$ is called the origin. The variable x is known as a spatial *coordinate*.

What does it mean for a particle to move on this axis? The meaning is quite clear: at some instant in time, the particle has some position. At a later instant in time, the particle has some other position. In everyday language, we might say that the particle has “moved” to another position.

This relationship between position of the particle and time is represented by a function $x(t)$. If you want to know where the particle is at any given time instant t_0, just plug in your time and you will know that the particle is at $x(t_0)$.

If the world is two-dimensional, nothing’s really different: instead of using a single position coordinate x to label positions on the one-dimensional axis, we use two position coordinates, x and y, to label positions on the two-dimensional plane. The particle’s position is given by the vector function $(x(t), y(t))$.

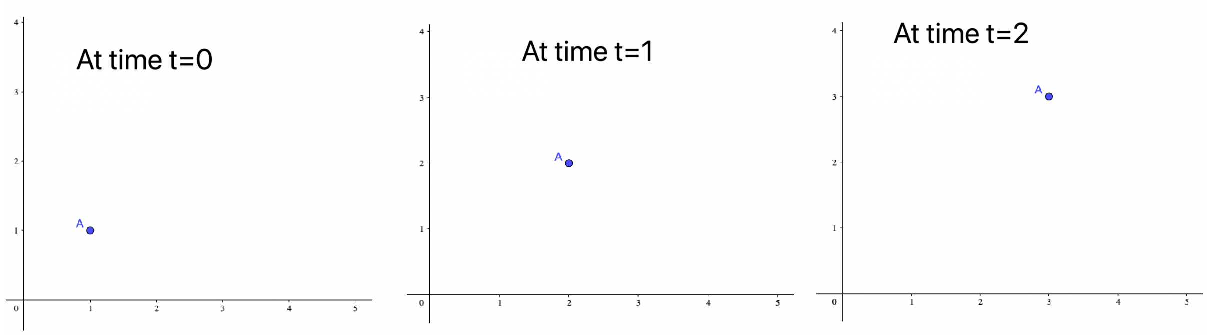


Figure 1: At some instant in time, the particle has some position. At a later instant in time, the particle has some other position. In everyday language, we might say that the particle has “moved” to another position. At time $t = 0, 1, 2$, the particle is at $(1, 1), (2, 2), (3, 3)$, respectively.

The path traced out by the particle $(x(t), y(t))$ is essentially a pair of parametric equations, with the time t as a parameter. Its path is really just a parametric curve.

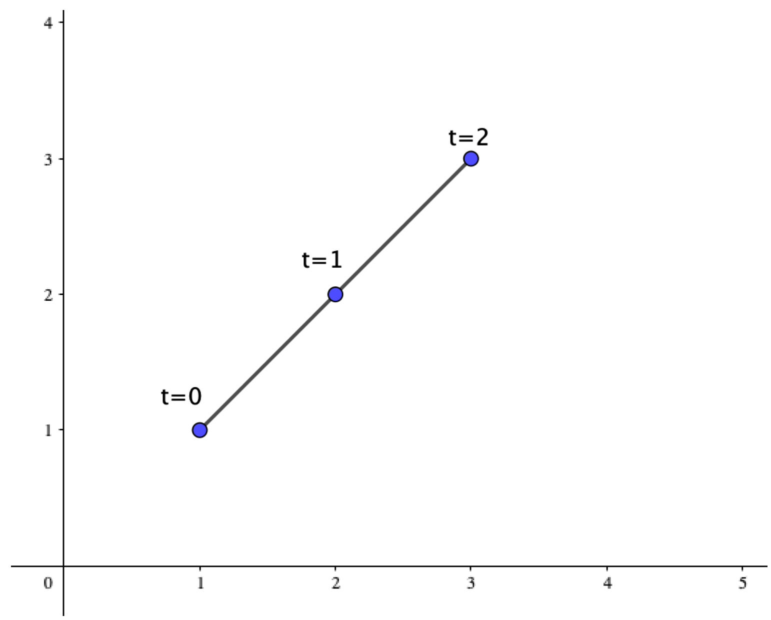


Figure 2: The path of the particle's motion is a parametric curve, with time t being the parameter. The path here is $(x(t), y(t)) = (t+1, t+1), 0 \leq t \leq 2$

It can be helpful to think about how this process is carried out in an experiment. Imagine you were to do an experiment to record the motion of this particle on the two dimensional plane. If it were a highschool physics experiment, you might be given a nice (albeit a little lame) form like this:

Recording The Motion Of A Ball On A Plane

High School of Earth

Step 1: After coming into the lab, grab a pair of rulers and a clock. You may use any time-keeping device, like a stopwatch, a pocket-watch, or a mechanical clock. We will refer to these time-keeping devices as "clocks".

Step 2: Place your pair of rulers at right angles to each other. The intersection of the rulers is the origin $(0, 0)$ of your spatial coordinate frame.

t	x	y
0		
0.5		
1		
1.5		
2		
2.5		

Step 3: At $t = 0$, i.e. before you start your clock, measure the position of the ball and record it into the chart on the $t = 0$ row.

Step 4: Start your clock. At $t = 0.5$, measure the position of the ball and record it into the chart on the $t = 0.5$ row.

Step 5: At $t = 1$, measure the position of the ball and record it into the chart on the $t = 1$ row.

Step 6: repeat for all time values.

At the end of the experiment, you will probably end up with a chart like this [1]:

t	x	y
0	1	1
0.5	1.5	1.5
1	2	2
1.5	2.5	2.5
2	3	3
2.5	3.5	3.5

[1] I know you don't care about experimental errors and units, so I'm making up perfectly precise data.

And that table gives us the relationship between the value of the parameter t and the value of the position vector $(x(t), y(t))$ of the ball. It agrees with our previous discussions of motion being described by a parametric curve, where time serves as the parameterizing parameter.

Although there were three numerical data for each row in the experimental data, they are not on the same footing. Two of them, namely the positions x and y, are *measurements*. You can't control their values, and you don't know their values prior to measuring them. The other one, the time t, is something you can independently control. It is **not** a measurement. To record the ball's position at $t = 1$, you simply wait for your clock to tick to $t = 1$, and measure the unknown positions. There is no need to measure time: time simply is ticking its own way. When the chart on the experimental instruction card was handed to you, the time column was already filled out: we were in possession of the time values even before going into the lab rooms in step one! No measurements were needed whatsoever.

Why is time not a measurement? This is because **everyone agrees on what time it is**. The instruction on that experimental guide "wait until $t = 0.5$ (seconds)" is an completely unambiguous instruction. To any student in any class, doing the experiment in any lab room, they all know what $t = 0.5$ means. Consider two students, Alice (subscript A) and Bob (subscript B), doing the experiment together. They start their clocks together, so when $t_A = 0$, $t_B = 0$ too. If they share the same spatial coordinate axis, then their measurements' data table would look completely identical. Alice would be 100-percent confident that if her clock reads 0.5 seconds, Bob's clock would also read 0.5 seconds. Because of this agreement, their's no *need* to measure time, since she already knows the time on Bob's clock, on the moving ball's clock if the ball carried one, and even on her other classmate Cindy's clock. They are all equal to 0.5 seconds, the same as her own clock. What measurements are there to be made? Isn't 0.5 seconds just 0.5 seconds?

It's like this scene from *Friends* Season 2 Episode 23 *The One with the Chicken Pox*:
Monica: You know what, tomorrow, I'm gonna do your clocks.
Richard: You're gonna do what to my clocks?
Monica: I'm going to set them to my time.
Richard: Well, I'm confused. ***I thought we shared time.***
Monica: No. No. See, in my bedroom I set my clock six minutes fast. You wanna know why?
Richard: Because it's in a slightly different time zone than the kitchen?

All jokes aside, in Newtonian physics we do share time. What could go wrong with that? That's just such an obvious thing that we don't even think about it. Well, all that's gonna change.

2 Laws Of Physics Are Universal

The principle of relativity is very simple [2]:

The laws of physics are the same in every reference frame.

Physicists love to talk about reference frames. And that makes sense: to quantitatively describe the position of something, you need to pick a location to call zero, and you need to pick a few directions to be the x, y and z axis.

An important principle of physics is that the laws of physics shouldn't be different just because I picked another direction to be the x axis, or just because I picked another location to call zero. Otherwise, someone in London would claim that the universe operates differently than someone in New York, which is clearly nonsense. In other words, we are free to set up our coordinate axes and origin however we want.

The reference frames can even be moving. If you've ever been on a plane and had a flight attendant pour you a drink, then you've already experienced it. When you are in the plane, flying at about 300 km/h, the origin of your reference frame (which is originated at you) moves with 300 km/h with respect to someone on the ground. However, the drink doesn't drift off its usual course and still lands in the cup, just like what would've happened had the drink been poured on the ground. The laws of physics (pouring liquid from a cup to another cup is an activity that nicely illustrates many laws of Newtonian physics) is the same regardless of whether you are static on the ground, or you are moving on the plane.

This principle makes perfect sense if you give it a little thought. From the ground's point of view, someone on the ground is static, and the person on the plane is moving with 300 km/h; but from the plane guy's point of view, himself (plane guy) is static, and it is the ground that is moving with 300 km/h (in the reverse direction). Since the plane guy sees the ground guy pouring drinks with no problem, the ground guy should also see the plane guy pour drinks with no problem.

[2] I will purposefully skip the discussion of inertial and non-inertial frames. Although special relativity and the constancy of the speed of light only holds for inertial frames, the tensor equations we will develop in general relativity will be valid for non-inertial frames as well. In fact, in general relativity it is in general impossible to define global coordinates that are inertial over all of spacetime, and there exists local inertial coordinates for every point. The laws of special relativity are simply the laws of general relativity, but evaluated with a local inertial reference frame (i.e. Minkowski spacetime with metric $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$). For this reason we will go directly to general relativity.

This principle wasn't anything new to Einstein; people had known about it since forever. The problem arises, however, when you combine it with Maxwell's equations [3].

Being a law of physics, Maxwell's equations are of course expected to hold true in every reference frame. The problem is that, Maxwell's equations produces a value for the speed at which electromagnetic waves travel (see Appendix A). That speed is denoted by c, with a numerical value of 299792458 m/s. Since light is just a kind of electromagnetic wave, Maxwell's equations says that light travels at c.

And since Maxwell's equations hold in every reference frame, hence **light travels at c in every reference frame**. This is more commonly known as **speed of light is constant**.

This innocent statement has some serious impact on time. To see why, let's build a new kind of clock: the lightclock.

[3] The reader is assumed to know classical electromagnetism. If not, watching Walter Lewin's lectures on E&M may be a good idea.

3 Losing Track of Time

As we've already seen, clocks are crucial to keeping track of motion, as time is the parameter that parametrizes the path of a particle through space. You also need clocks to carry out the experiment for recording the motion of some particle.

Clocks are not limited to mechanical clocks or the stopwatch on your iphone. In fact, anything that oscillates with a fixed period can be used as a clock. If an oscillator takes 1 second to do a period, then if you imagine a little screen displaying a number that increments by one each time the oscillator completes a period, effectively you've built yourself a nice little clock.

Actually all clocks work like this: the traditional mechanical clock uses a set of gears, with some clocks having a rod and a bob hanging below one of the gears, swinging back and forth, creating the classic mental picture of clocks. That rod and bob is the prime example of oscillators in classical physics and has been teaching students $\sin''(x) = -\sin(x)$ for decades. The digital clocks on your iphone uses the voltage oscillations in its circuits. If you've ever heard of atomic clocks, they use the oscillation of certain atoms between ground states and excited states. In fact the second is defined with oscillations: the second is defined as being equal to the time duration of 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the fundamental unperturbed ground-state of the caesium-133 atom [4].

[4] Of course, this is from Wikipedia: second.

Let's build our own oscillator with light.

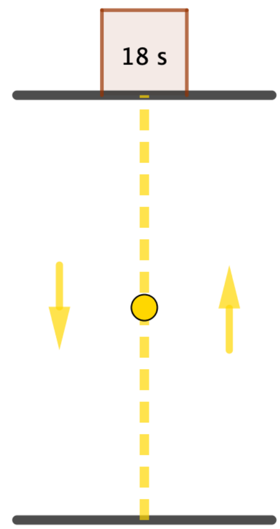


Figure 3: A light clock. A blip of light (yellow) bounces between the two mirrors (black). Each time the light blip hits the top mirror, the number on the monitor display (brown) increments by 1. The "18s" is merely an illustration showing that "this is a monitor that displays the time", and its numerical value of 18 carries no significance.

By all means, please go ahead and imagine that the two mirrors are separated by half a lightsecond [5]. Thus it takes light exactly 1 second to make a round trip and return to the top mirror.

Of course, the distance between the two mirrors aren't important; the important thing is that such a conceptual device can help us keep track of time, i.e. a clock. It is in fact the best clock there is. Since a meter is defined as the distance light would travel in 1/299792458th of a second (in vacuum), so if our light clock's mirrors are half a lightsecond away, the time it would take for light to complete this round trip would be, by definition, exactly one second.

[5]Similar to "lightyear", a lightsecond is a unit of length. It's the distance light travels in a second. Half a light second is 149896229 meters.

The interesting thing happens when you have two reference frames moving relative to each other, carrying two lightclocks.

Imagine Alice on a train platform with her lightclock. Her friend Bob, with his own lightclock, is on a train that's moving past the station with some velocity. When Alice's lightclock makes an entire round trip, Alice would think that one second has passed; similarly, when Bob's lightclock makes an entire round trip, Bob would think that one second has passed.

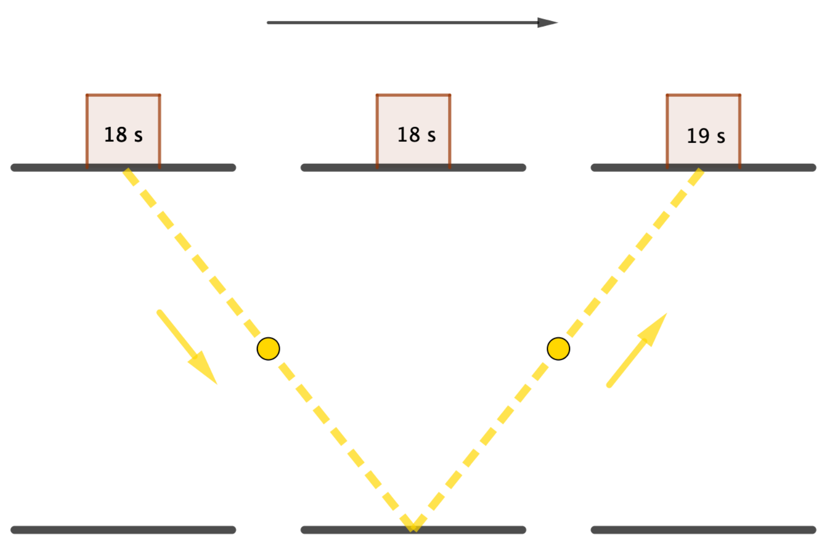


Figure 4: Bob's lightclock on the train, viewed from Alice on the stationary platform.

But now there's a problem. Viewed from Alice on the platform, Bob's blip of light is covering a longer distance for a round trip returning to the top mirror. Since speed of light is constant, a longer distance would need a longer time to complete! Thus, **viewed from Alice, when Alice's blip of light has already completed a straight-up-straight-down round trip, Bob's blip of light still hasn't reached the top mirror.**

In other words, **when Alice's lightclock registers one second, Bob's light clock would still need a while to register one second.** This is the famous time-dilation of special relativity, widely known as the slogan **moving clocks run slow**.

With some basic geometry, we can derive the famous factor $\dfrac{1}{\sqrt{1 - v^2/c^2}}$ here, but that's beside the point today. You are of course welcome to try. The answer is in Appendix B.

To lift your possible confusion, let's be absolutely clear on what "moving clocks run slow" means: when Alice thinks one second has passed, viewed from Alice on the stationary platform, Bob's clock would register a little less than one second. Alice's clock would register more time than Bob's clock. It is possible to choose a train velocity such that Alice's clock runs twice as fast as Bob's, i.e. when Alice's clock registers 10 seconds, Bob's registers 5 seconds; when Alice's 20, Bob's 10.

This is kind of a serious problem. The problem is not whether train clocks move faster or slower than platform clocks, or by what numerical factor. Sure, these questions might be slightly confusing to those just seeing this concept for the first time, but they are merely technical details and window-dressing.

The serious problem is this: **now different observers do not agree on time anymore.**

If different observers do not agree on time, then time would be a very bad parameter for describing motion. Remember our earlier particle who moved on $(x(t), y(t)) = (t+1, t+1), 0 \leq t \leq 2$ (figure 2)? Now, imagine a second observer whose clock do not agree with t. For example, let's imagine that the second observer's clock, denoted by t', runs twice as slow as t [6].

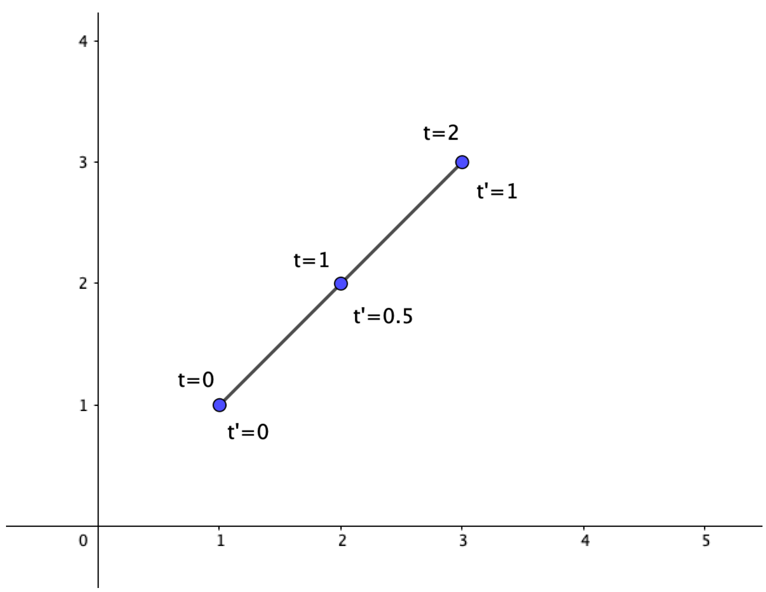


Figure 5: Two students recording the motion of the particle, but one clock t' runs twice as slow as the other t.

[6] I purposefully avoided using Alice and Bob here. I don't want you to think about the trains and whose clock is faster than whose anymore. All that matters is that there **can** be a different clock that runs at a different speed.

There are numerous problems with this. Although the two clocks were synchronized at the beginning (when $t = 0$, $t' = 0$ too), they don't stay synchronized (they don't register the same reading afterwards). This is like two teams starting a football game at the same time, but when one team thinks it's already 90 minutes and the game has finished, the other team thinks its still just half-time at 45 minutes. When Ronaldo is celebrating, Messi is still taking the half-time break. Just imagine the chaos in the press!

Apart from the football and media problems brought by clocks-running-at-different-rates on its own, more importantly, the ability for clocks to run at different speeds destroys the notion of "motion" in Newtonian physics. On the experimental instruction card (page 4), the form basically asks you where the particle is when time is 0, when time is 0.5, when time is 1, etc.. However, now that different observers don't agree on what "time is 1" means, the question "where is the particle when time is 1" loses its meaning. The observer using t as clock can say that the particle is at (2,2) when time is 1, whereas the observer using t' as clock can say that the particle is at (3,3) when time is 1. You can in fact engineer clocks running at all kinds of rates so that every position on the path is a possible answer to "where is the particle when time is 1"! Without knowing what clock the observer is using, describing the particle's motion is impossible. Since every position is a possible answer now, simply answering "it's at (2,2) when time is 1" conveys no useful information whatsoever.

Going back to the experimental room, since now clocks can do all kinds of crazy things, we have no information on time before carrying out the actual experiment. To find out what your own clock is going to do, you have to manually take the reading on the clock. In effect, **time is now a measurement**. Its status is downgraded from a frame-invariant parameter to just another coordinate to be recorded in the experiment.

Therefore, to record your particle's motion, you not only need the spatial coordinates x and y, you also need to record the time coordinate t. The space of possible states of the particle is now not just labeled by (x, y), but labeled by (t, x, y). In a 2-dimensional world, the state space of the particle is actually 3 dimensional.

Drawing 3 dimensional graphs are a pain, so let's dumb it down. Since all spatial dimensions are no different than each other, our theory can be extended to universes with an arbitrary number of spatial dimensions. Let's go to a world with just one spatial dimension x. It's state space is now 2-dimensional, i.e. (t, x).

Since the good-old Newtonian parameter t is now no good, we need to find a new parameter that is agreed by all observers. The new parameter is denoted by τ . We won't go into what it is at this point [7]. At this point it's simply a bookkeeping device for us to update our motion-recording graphs.

The previous (x, y) grid is called "space" because that's just what it is: a space that is labeled by two spatial coordinates. Now that the grid is (t, x) with one time coordinate, the grid is called, unsurprisingly, "spacetime".

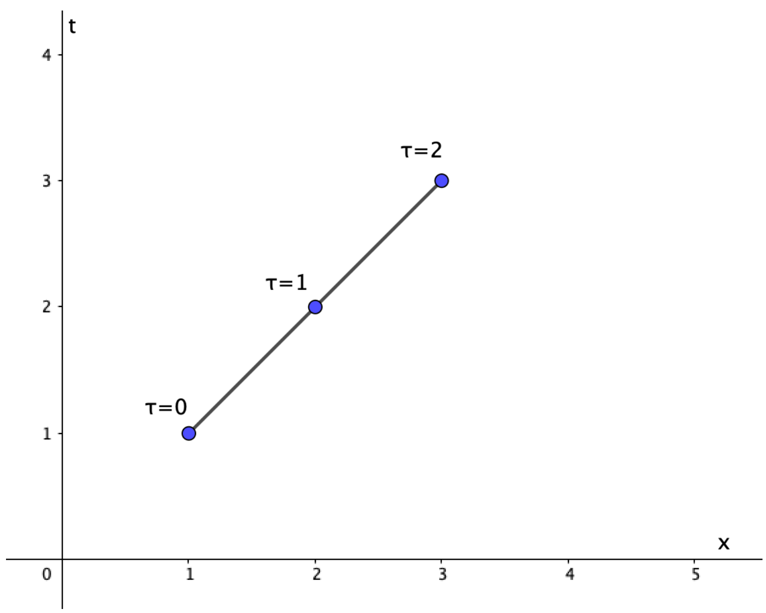


Figure 6: Motion of a particle in relativity. The particle is now moving in spacetime, instead of just space. The parameter of the curve is now not t, but this new guy τ. The path of the particle is $(t(\tau), x(\tau)) = (\tau + 1, \tau + 1), 0 \leq \tau \leq 2$

It is important to note that although the spacetime graph and the particle's path through spacetime is 2-dimensional, its actual path in everyday Newtonian spatial space is still one dimensional. The motion of this particle in spacetime corresponds to the following motion through space:

[7]Spoiler: it's the proper time

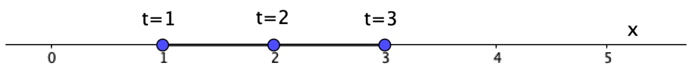
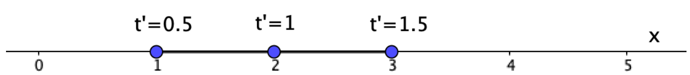
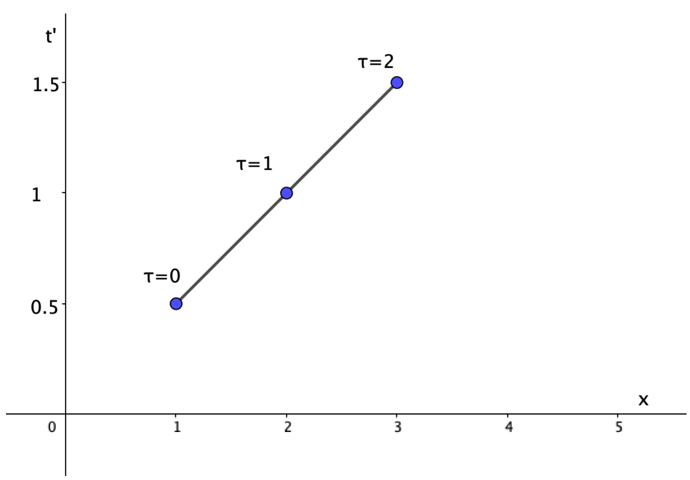


Figure 7: Actual motion of the particle through Newtonian spatial space. $t = 1$ means clock reads 1.

All usual Newtonian notions about coordinates in spatial space all apply to spacetime. For example, now a clock tunning half as fast won't mess with the parameter (since everyone agrees on τ by design), but it will simply stretch or compress the time axis by some factor. If our clock ran half as fast, in Newtonian space the motion is



But in spacetime it will be



Relativity is all about how things move in spacetime. It is in accordance with the thousand-year old topic of physics, of studying how things move; but now, since just plainly moving in space is a little undefined due to weird clocks, we study how things move in spacetime. If you feel more comfortable, you can imagine a sort of "imaginary particle" physically moving across some sort of "imaginary spacetime fluid", but really the "particle" in spacetime is just a mathematical point that encodes the space coordinate (reading on rulers) and the time coordinate (reading on clocks) together.

Other Newtonian notions also apply to this spacetime. For example velocity is defined in the same way, i.e. how much does the coordinate values change with respect to a small change in the parameter. In Newtonian physics this would be the familiar $\frac{dx}{dt}$; here it is $\frac{dt}{d\tau}$ and $\frac{dx}{d\tau}$. Once you have velocity defined, naturally follows the definition of momentum: just multiply velocity by mass. We will also see that definition for energy on this spacetime is the same as the definition for energy on the good old Newtonian space. When we want to find the particle's path in spacetime, we are really looking for an differential equation that relates (t, x) and τ, just like how in Newtonian space we were looking for an differential equation that relates (x, y) and t (one such equation is, of course, $F = m\frac{d^2x}{dt^2}$).

However, we would imagine that there are also aspects where this spacetime is different than the Newtonian space. Otherwise a (t, x) spacetime would be identical with the two dimensional Cartesian (x, y) space, which doesn't sound right. Although time and space are both coordinates that require measurements, they are, after all, different things. For example, in Newtonian space the distance between two points is $dx^2 + dy^2$, but in this lightclock spacetime the "distance" [8] is $-dt^2 + dx^2$. Most confusion in special relativity arise here: why, in some aspects spacetime resemble Newtonian space, but yet in other aspects it is so different?

Special relativity studies the particular spacetime illustrated by these moving lightclocks, i.e. clocks that are off-set by a factor of $\frac{1}{\sqrt{1 - v^2/c^2}}$. General relativity, on the other hand, develops a general theory on all spacetimes. Once we know the rationale behind general relativity, we can apply that to our lightclock spacetime, and the confusion in special relativity will go away. We will start with general relativity, as promised.

[8] "Distance" in quotes because as you can see from the explicit formula, this quantity can be negative.

Appendix A: Calculating speed of light with Maxwell's equations

We will use the differential form of Maxwell's equations. If you prefer the integral form, try Walter Lewin's derivation.

We will set both the vacuum permittivity ϵ_0 and the vacuum permeability μ_0 to one. When the calculation is done, we can restore them back into our results by matching units. Reminder that the product $\epsilon_0\mu_0$ has units $m^{-2} \cdot s^2$, therefore $1/\sqrt{\epsilon_0\mu_0}$ has units of $m \cdot s^{-1}$, or dimensions of speed.

Maxwell's equations:

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \vec{\mathbf{B}} &= 0 \\
\boldsymbol{\nabla} \cdot \vec{\mathbf{E}} &= \rho \\
\boldsymbol{\nabla} \times \vec{\mathbf{B}} &= \vec{\mathbf{j}} + \frac{\partial \vec{\mathbf{E}}}{\partial t} \\
\boldsymbol{\nabla} \times \vec{\mathbf{E}} &= -\frac{\partial \vec{\mathbf{B}}}{\partial t}
\end{aligned}$$

In vacuum, there are no sources, so both the charge density ρ and the current density $\vec{\mathbf{j}}$ are zero:

$$\boldsymbol{\nabla} \cdot \vec{\mathbf{B}} = 0 \tag{1}$$

$$\boldsymbol{\nabla} \cdot \vec{\mathbf{E}} = 0 \tag{2}$$

$$\boldsymbol{\nabla} \times \vec{\mathbf{B}} = \frac{\partial \vec{\mathbf{E}}}{\partial t} \tag{3}$$

$$\boldsymbol{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{4}$$

Set up our coordinate axes so that the direction of the plane wave's travel is in the positive z direction. Therefore we can assume that all components of both the electric and the magnetic fields are waves traveling in the positive z direction:

$$E_x = \epsilon_x \sin(kz - \omega t)$$

$$E_y = \epsilon_y \sin(kz - \omega t)$$

$$E_z = \epsilon_z \sin(kz - \omega t)$$

$$B_x = \beta_x \sin(kz - \omega t)$$

$$B_y = \beta_y \sin(kz - \omega t)$$

$$B_z = \beta_z \sin(kz - \omega t)$$

where E_x represents the x component of $\vec{\mathbf{E}}$, and ϵ_x is the unknown amplitude of E_x to be determined. The speed of the wave is thus ω/k.

Note that we also used the freedom in choosing the coordinate origin to choose the wave such that it is a sine without any phase, instead of a superposition of sine and cosine. If $\vec{\mathbf{E}}$ has sin, then $\vec{\mathbf{B}}$ must also have sin, since derivative of $\vec{\mathbf{E}}$ = derivative of $\vec{\mathbf{B}}$ ((3) and (4)). If $\vec{\mathbf{E}}$ has sin and $\vec{\mathbf{B}}$ has cos, then (3) and (4) becomes cos = sin, which is not good.

Now, (2):

$$\boldsymbol{\nabla} \cdot \vec{\mathbf{E}} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

Since there are no x and y dependence, the x and y derivatives drop out:

$$\frac{\partial E_z}{\partial z} = 0$$

Plugging in $E_z = \epsilon_z \sin(kz - \omega t)$:

$$\epsilon_z k \cos(kz - \omega t) = 0$$

Therefore $\epsilon_z = 0$. Similarly from (1), $\beta_z = 0$. Thus, $E_z = B_z = 0$.

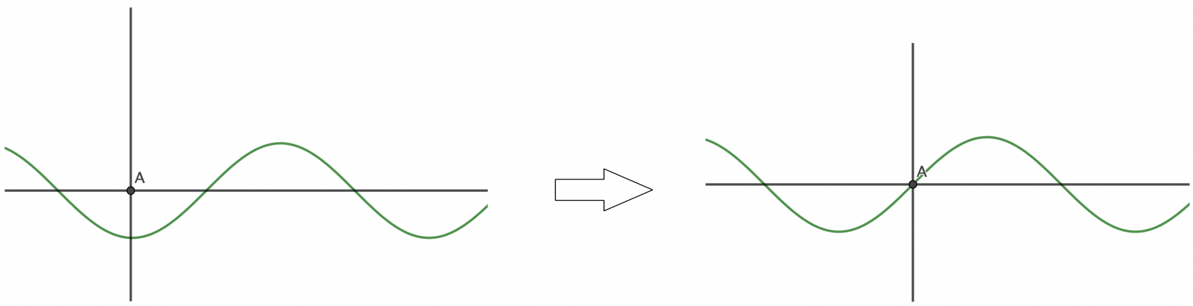


Figure 8: We can choose the location of the origin so that the wave is a pure sine with no phase. Think of the wave as fixed, and the coordinate axes as sliding around.

We can also choose the orientation of the x axis to be parallel to $\vec{\mathbf{E}}$. In this fashion $\vec{\mathbf{E}}$ only has a x component, i.e. $E_y = 0$.

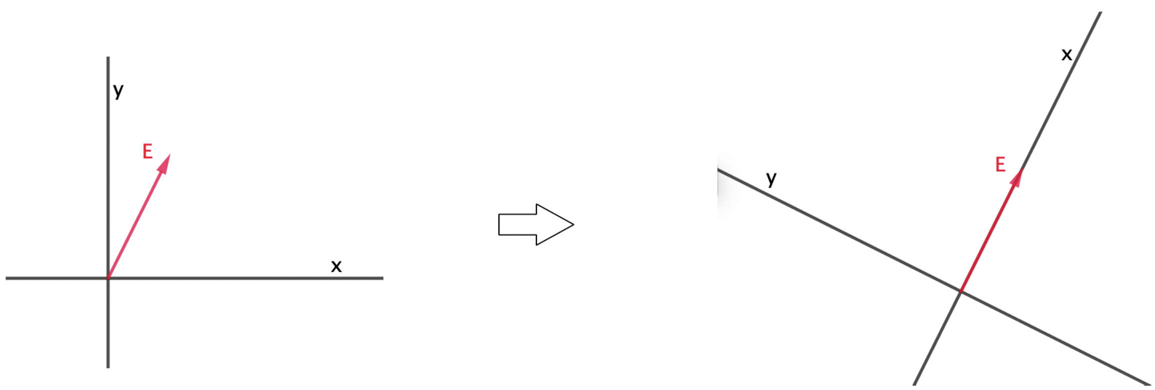


Figure 9: We can choose the orientation of the x axis to be parallel to $\vec{\mathbf{E}}$.

At this point we've used up our degrees of freedom: we have fixed an origin in order to set the phase to zero, we have chosen a z direction to lie along the direction of the wave's propagation [9], and we have chosen a x direction to lie along the direction of the electric field. Once the x and z directions are chosen, the y direction is automatically settled as $\hat{\mathbf{z}} \times \hat{\mathbf{x}}$. Therefore, to disappoint your impulsion, there are no furthermore degrees of freedom we can use to set B_y or B_x to zero manually.

[9] "Propagation" is the wave-word for "travel".

Now, (3):

$$\boldsymbol{\nabla} \times \vec{\mathbf{B}} = \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

The x component of this equation is

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \frac{\partial E_x}{\partial t}$$

The $\frac{\partial B_z}{\partial y} = 0$ since there is only z dependence (i.e. no y dependence).

Plugging in $B_y = \beta_y \sin(kz - \omega t)$ and $E_x = \epsilon_x \sin(kz - \omega t)$:

$$-\beta_y k \cos(kz - \omega t) = -\epsilon_x \omega \cos(kz - \omega t)$$

$$\frac{\epsilon_x}{\beta_y} = \frac{k}{\omega} \tag{5}$$

The y-component of (3) is

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \frac{\partial E_y}{\partial t}$$

The $\frac{\partial E_y}{\partial t} = 0$ since $E_y = 0$, and $\frac{\partial B_z}{\partial x} = 0$ since there's only z dependence. Thus,

$$\frac{\partial B_x}{\partial z} = 0$$

which, after plugging in $B_x = \beta_x \sin(kz - \omega t)$, gives $\beta_x = 0$ and hence $B_x = 0$.

Bonus: now that we have $E_z = B_z = 0$, $E_y = 0$, and $B_x = 0$, we can draw the wave in a nice picture. It propagates down the z direction, the electric field only has an x component, and the magnetic field only has a y component.

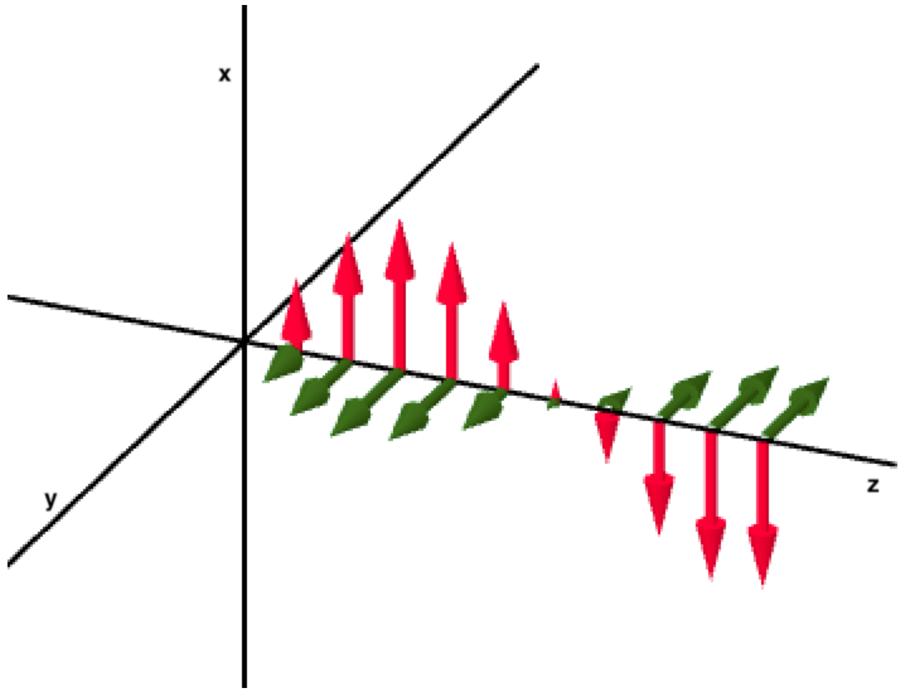


Figure 10: Electric (red) and magnetic (green) waves propagating along the positive z direction, snapshotted at $t = 0$. Only non-zero components are $E_x = \epsilon_x \sin(kz - \omega t)$ and $B_y = \beta_y \sin(kz - \omega t)$.

Now, (4):

$$\boldsymbol{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

Take its y component:

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}$$

The $\dfrac{\partial E_z}{\partial x} = 0$ since there is only z dependence.

Plugging in $E_x = \epsilon_x \sin(kz - \omega t)$ and $B_y = \beta_y \sin(kz - \omega t)$:

$$\epsilon_x k \cos(kz - \omega t) = -(-\beta_y \omega \cos(kz - \omega t))$$

$$\frac{\epsilon_x}{\beta_y} = \frac{\omega}{k} \tag{6}$$

Combining (5) and (6), we have

$$\frac{\epsilon_x}{\beta_y} = \frac{k}{\omega} = \frac{\omega}{k}$$

Therefore

$$\frac{\omega}{k} = 1$$

which is the speed c of our sine waves $\sin(kz - \omega t)$.

Putting back the permittivity ϵ_0 and the permeability μ_0 with the combination that gives the proper dimensions, we have the speed of light in SI units as

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

If you plug in the numbers, you get 299792458 m/s.

If you are confused about why the dimensionality trick works, think about it like this: our dimensionless result was $c = 1$. It was dimensionless because we didn't carry ϵ_0 and μ_0 (these are the guys with the lost dimensions, since we set them to one and "evicted" them from explicitly showing up in our equations) around during the calculations. To bring them back, there is only one way to combine them so that they give a quantity with a dimension of speed (meters over seconds, or length over time), namely $1/\sqrt{\epsilon_0 \mu_0}$ [10]. The actual dimension-ful result of c has to be our dimensionless result $c = 1$ multiplied by this quantity, so (1) the dimensions for c are correct, namely dimensionless result times the correct dimensions, and (2) in the final formula $c = 1 * 1/\sqrt{\epsilon_0 \mu_0}$, if we were to set ϵ_0 and μ_0 back to 1 again, we get back our dimensionless result $c = 1$.

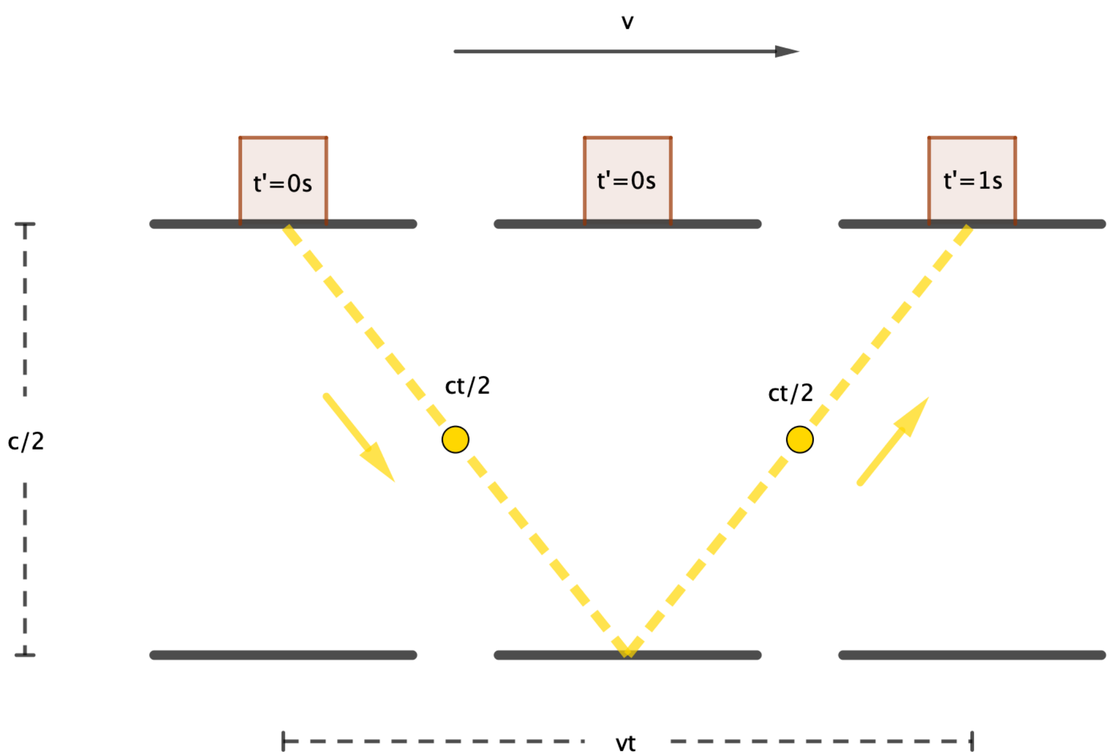
[10] Go ahead, I dare you to try to find another combination. No, multiplying by some constant (like 2) doesn't count.

Appendix B: Calculating time dilation from moving lightclocks

The question is, as viewed from Alice, when Bob's moving clock registers 1 second ($t' = 1$), how much time will have been registered on her stationary clock ($t =?$).

We know that for each period of the lightclock to take 1 second, the mirrors need to be separated half a lightsecond away, i.e. $c/2$.

Alice thinks that it took t seconds for Bob's clock to flip from $t' = 0$ to $t' = 1$. In this time the train has traveled vt in Alice's frame, and the blip of light has traveled ct in Alice's frame. The ct is 2-way, so the one-way distance covered by light in Alice's frame is $ct/2$.



Pythagorean:

$$(\frac{c}{2})^2 + (\frac{vt}{2})^2 = (\frac{ct}{2})^2$$

Simplifying gives $t = \dfrac{1}{\sqrt{1 - v^2/c^2}}$.

Note that since train speed is smaller than light speed, so $0 < \sqrt{1 - v^2/c^2} < 1$, which means $t > 1$, as expected: Alice's clock registers a longer time than Bob's 1 second, or in short "moving clocks run slow".