0 Prelude

General relativity is, in my mind, the most beautiful theory of physics.

Don't get me wrong, I do love quantum mechanics. However, I wouldn't call it "beautiful". While quantum mechanics certainly *works*, in the sense that it gives predictions that are well-confirmed by experiments, and in the sense that it is used to build excellent semiconductors and transistors and what not, its mathematical formulation is in fact quite ugly. There is no natural reason why nature should work with eigenstates of Hermitian operators, or why the fundamental object in nature should be those ghostly intangible "wave functions" that can somehow superposition and for some reason are complex-valued. These mathematical statements are stated as *axioms* in quantum mechanics, which means they are assumed to be true a priori to the development of the theory. There is no real good reason for stating these specific axioms, other than at the imperative of experiments. Typically, quantum mechanics textbooks will state 4 to 6 of these axioms, and you would need to take them by faith.

By contrast, the mathematical reasoning of general relativity is extremely simple, and extremely intuitive. When all the window dressing is throw away, the core idea of general relativity is this: **things move on straight lines**. Really, this is the only axiom. To quote Sean Carroll, "at heart it [general relativity] is a very simple subject, compared, for example, to anything involving quantum mechanics."

You often hear that general relativity is a paradigm shift; personally I disagree with this statement. Quantum machines is a paradigm shift, as it destroys the fundamental logic of classical (or *Newtonian*) physics and replaces it with a new system of quantum logic (operators instead of quantities, eigenvalues/states instead of just plain values/states, ...). General relativity, on the other hand, is more like a patch on Newtonian physics. It fixes some bugs, but the logic is still classical. For this reason, studying general relativity is actually easier than studying quantum mechanics.

But at the same time, general relativity is also often associated with the tag of being extremely difficult. To some extent, this is the fault of false and misdirecting advertising. Science fictions and movies have been flirting with general relativity for a long time, usually involving "curvature of spacetime" and "time travel through holes in spactime" and "black holes curving spacetime". While it is absolutely true that general relativity does involve the study of the curvature of spacetime and gravity, it is not where it's rooted. The theory is called "relativity" instead of "spacetime theory" or "gravitation theory" for a reason.

Because of the false advertising of general relativity as a study of curvature, many who wish to pursue it believe that they need to know differential geometry as a prerequisite. Some even go further and claim that you need to know Lagrangian mechanics to study general relativity, but this is just nonsense. Sure, knowing some differential geometry and/or Lagrangian mechanics would certainly *help*, but they are in no way *necessary*. The only real prerequisite is freshmen-level calculus (so things on the level of "derivative of $\sin(x)$ is $\cos(x)$"), and importantly you need to know the chain rule

$$df(x_0, x_1, ..., x_n) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i$$

but that's about the extent of it. We will develop our own version of "differential geometry" from a physicist's point of view, and we will also see how the usual Newtonian Lagrangian mechanics is nothing more than an approximation of the Lagrangian formulation of general relativity. It is in general relativity where the Lagrangian formulation starts to really shine and make sense. If you don't know Newtonian Lagrangian mechanics, that's even better: we will derive it from the simple and elegant principles of general relativity!

Another reason for why general relativity could potentially be difficult is the difficulty of its calculations. This is, I'm afraid, true. Fair warning: you will see A LOT OF Greek letter indices (plural of "index") flying around. Einstein did his best to simplify the notation, but it's still quite overwhelming.

But this difficulty is merely a facade: it's a difficulty in computations and derivations, difficulty in handling and massaging equations, not a difficulty in understanding the concepts. Over time this difficulty will fade away. The concepts themselves are, again, very straight forward.

I will of course center my narration around the development of the concepts. However, inevitably there will be times when the concepts need to pause for a while because there's a very long derivation standing in the way. When that occurs, I'll in general skip the derivation in the main text and relegate them to the end as an Appendix, and go back to the concepts in the main text. I will do the first couple of derivations in the main text for you just so you can get a feeling of how to deal with equations in general relativity, but as we go deeper I will do that less. You are, of course, encouraged to try out the derivations on your own to practice your GR (General Relativity) equation skills. If you can't find your way out of a derivation, the answer will be waiting for you in the appendix.

Many people struggle with general relativity because they don't have a good understanding of Newtonian physics. Sure, they might be able to use $F = ma$ to calculate things, like when will a stone hit the ground if you drop it from the roof of some building, but they don't really get what Newtonian physics studies. We will start with the Newtonian world, clearly lay out what it studies, and then build general relativity upon that. As I said, general relativity is merely a patch on Newtonian physics; therefore being fuzzy about the Newtonian world will be dangerous.

Along the way, we will pay a short visit to special relativity. Like the name suggests, special relativity is a special case of general relativity. Contrary to popular believes and most pedagogical methods, special relativity is, again, *not* a prerequisite to general relativity. Studying special relativity before general relativity creates the illusion that they are two independent subjects, but nothing could be further from the truth. While it is true that the low-hanging fruits of special relativity like time dilation and length contraction can be explained without general relativity, they are not the most important ideas, but merely that: low-hanging immediate results. They are best suited as a case study for general relativity, a special case.

We will first do a first-pass of general relativity, but the main purpose of that first-pass is to sort of learn the language and the vocabulary. Once that's done, we'll move on to formulate a Lagrangian approach with an action principle. Everything should start to click and make sense at this point.

Finally we will use the theory we developed and apply them to black holes.

So, let's dive in.