Note on Wigner Functions

Contents

1 Displacement Operators

Consider the usual one dimensional continuous quantum system with canonical variables $\{x, p\}$.

Suppose we have a quantum state $|\psi\rangle$. Suppose it is translated in position space by amount x. The new state is obviously given by the translation operator

$$e^{-i\hat{p}x/\hbar}|\psi\rangle$$

What is the expectation of the translation operator? This question is rarely asked in standard quantum curriculum, but just for fun let's compute it. We seek

$$\left\langle e^{-i\hat{p}x/\hbar}\right\rangle = \langle\psi|e^{-i\hat{p}x/\hbar}|\psi\rangle$$

which does not make much sense for the moment, but imagine that we have a momentum eigenstate $|p_0\rangle$, with $\hat{p} = p_0|p_0\rangle$. Then the expectation of the translation operator is

$$\langle p_0|e^{-i\hat{p}x/\hbar}|p_0\rangle = \langle p_0|e^{-ip_0x/\hbar}|p_0\rangle \propto e^{-ip_0x/\hbar}$$

where we wave our hands and ignore the fact that $\langle p_0|p_0\rangle$ is unnormalizable [1], and treat it as just a constant.

We see that, for a momentum eigenstate $|p_0\rangle$, the expectation of the translation-by-x operator is just $e^{-ip_0x/\hbar}$. Viewing it as a function of x (the amount we translated by), the expectation of the translation operator is an oscillation with frequency p_0.

What if we have a superposition of momentum eigenstates? Say $|\psi\rangle = c_1|p_1\rangle + c_2|p_2\rangle$. Then

$$\begin{aligned}\langle\psi|e^{-i\hat{p}x/\hbar}|\psi\rangle &= (c_1^*\langle p_1| + c_2^*\langle p_2|)\, e^{-i\hat{p}x/\hbar}\,(c_1|p_1\rangle + c_2|p_2\rangle)\\ &= (c_1^*\langle p_1| + c_2^*\langle p_2|)\,(c_1e^{-ip_1x/\hbar}|p_1\rangle + c_2e^{-ip_2x/\hbar}|p_2\rangle)\\ &\propto |c_1|^2e^{-ip_1x/\hbar} + |c_2|^2e^{-ip_2x/\hbar}\end{aligned}$$

We see that now the expectation of the translation operator is a superposition of oscillations, and the strength of each oscillation is precisely the probability to measure the corresponding momentum!

[1] Yeah yeah I know we can put space in a very big box/torus and blah blah but that's not the main point here, so why don't we just move on

In fact we can repeat this exercise for a general $|\psi\rangle = \int f(p)\,|p\rangle\,dp$:

$$\begin{aligned}
\langle\psi|e^{-i\hat{p}x/\hbar}|\psi\rangle &= \iint dp'dp''f^*(p')\,\langle p'|\,e^{-i\hat{p}x/\hbar}f(p'')\,|p''\rangle \\
&= \iint dp'dp''f^*(p')f(p'')\,\langle p'|\,e^{-ip''x/\hbar}\,|p''\rangle \\
&\propto \iint dp'dp''f^*(p')f(p'')e^{-ip''x/\hbar}\delta(p'-p'') \\
&= \int dp'|f(p')|^2e^{-ip'x/\hbar} \\
&= \int dp|f(p)|^2e^{-ipx/\hbar}
\end{aligned}$$

Therefore, we see that $\langle\psi|e^{-i\hat{p}x/\hbar}|\psi\rangle$, when viewed as a function of x (which was the amount of translation in position space), is a sum of exponentials, each with frequency p (technically $-p/\hbar$), and weight given by the probability of $|p\rangle$ in the original state $|\psi\rangle$. In other words, the frequency spectrum of $\langle\psi|e^{-i\hat{p}x/\hbar}|\psi\rangle$ is simply $|f(p)|^2$, the momentum PDF [2] of $|\psi\rangle$!

The expectation of translation operator, $\langle\psi|e^{-i\hat{p}x/\hbar}|\psi\rangle$, actually has a proper name. It's called the **characteristic function**. The frequency spectrum of the characteristic function is the momentum PDF.

Therefore, the Fourier transform of the characteristic function

$$\int dx\,\langle\psi|e^{-i\hat{p}x/\hbar}|\psi\rangle\,e^{-ipx/\hbar}$$

simply recovers $|f(p)|^2$, the momentum PDF of $|\psi\rangle$! (Recall that the characteristic function is being viewed as a function of x)

Similarly, if the state is translated by amount p in momentum space, i.e. the translation operator is $e^{i\hat{x}p/\hbar}$, then the Fourier transform of $\langle\psi|e^{i\hat{x}p/\hbar}|\psi\rangle$ recovers the position PDF of $|\psi\rangle$. This time viewed as a function of p, of course.

Note: while we were haphazard about the normalization in this one dimensional introductory motivation, we will be careful once we go to the full harmonic oscillator phase space. See the last page for our conventions. The reader who is unfamiliar with phase space quadrature operators is highly encouraged to look at it right now before proceeding.

[2]Probability Density Function

So what about forming a "joint PDF" of the position and momentum distribution of $|\psi\rangle$?

The state $|\psi\rangle$ carries the probabilistic distribution information about both its momentum and position. We would imagine a joint position-momentum PDF to be a function on the (x,p) phase space. However, this approach is problematic, as it implies that simultaneous specifications of position and momentum are possible.

The Wigner function is a solution to this problem. Whereas the dynamics of a classical harmonic oscillator can be clearly visualized and illustrated on the (x,p) phase space of position and momentum, the quantum mechanical version is not as straightforward. Representing the state of the oscillator with a point in phase space is inherently impossible, since a simultaneous specification of position and momentum is forbidden. Moreover, position and momentum eigenstates are themselves unphysical to begin with. The Wigner function offers a solution to visualize quantum states in phase space. It describes quantum states as a pseudo probability distribution in phase space, and can be thought of as a blurred probability cloud around classical trajectories. The Wigner function offers a direct and convenient visual way to intuitively interpret states of the harmonic oscillator, including coherent and squeezed states, which are extremely useful in photonic quantum computing schemes.

Define the **displacement operator**

$$\hat{D}(\vec{r}) = \exp\left(\frac{i}{2}(\hat{X}p - \hat{P}x)\right) \tag{1}$$

where $\vec{r} = \begin{pmatrix} x \\ p \end{pmatrix}$ is the (dimensionless) phase space coordinate. Physically, this is displacement in phase space by $\vec{r}$.

Note that the quadrature operators $\hat{X}$ and $\hat{P}$ are dimensionless. From now on when we speak of "position" and "momentum", we mean the position and momentum quadratures.

Suppose the quantum state is given by the density operator $\hat{\rho}$. The **characteristic function** is again the expectation of the displacement operator

$$\left\langle \hat{D}(\vec{r}) \right\rangle = \mathrm{tr}\left(\hat{\rho}\hat{D}(\vec{r})\right) \tag{2}$$

where again we remind ourselves that this is a function of $\vec{r}$, the amount of displacement.

By the insight we gained in the one dimensional example, the Fourier transform of the characteristic function should give us a "joint PDF" for position and momentum distributions of the state!

The **Wigner function** of state $\hat{\rho}$ is defined as the Fourier transform of the characteristic function:

$$\boxed{W_\rho(\vec{r}) = \int_{\mathbb{R}^2} \frac{d^2\vec{r}\,'}{(4\pi)^2} \exp\left(-\frac{i}{2}(xp' - px')\right) \left\langle \hat{D}(\vec{r}\,') \right\rangle} \tag{3}$$

If you find it hard to see that this is indeed the Fourier transform of the characteristic function, consider writing it in an alternate form. The displacement operator can also be written as

$$\hat{D}(\vec{r}) = \exp\left(\frac{i}{2}(\hat{X}p - \hat{P}x)\right) = \exp\left(\frac{i}{2}\hat{\vec{R}}^T\Omega\vec{r}\right)$$

where $\hat{\vec{R}} = \begin{pmatrix} \hat{X} \\ \hat{P} \end{pmatrix}$ is a vector of the quadrature operators, and $\Omega \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the so-called "symplectic form".

The "symplectic inner product" is very straight forward:

$$\vec{x}^T\Omega\vec{y} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} = x_1y_2 - x_2y_1$$

i.e. the 2-dimensional "cross product". Thus we see that $\hat{\vec{R}}^T\Omega\vec{r} = \hat{X}p - \hat{P}x$.

The Wigner function can be written as

$$W_\rho(\vec{r}) = \int_{\mathbb{R}^2} \frac{d^2\vec{r}\,'}{(4\pi)^2} \exp\left(-\frac{i}{2}\vec{r}^T\Omega\vec{r}\,'\right) \left\langle \hat{D}(\vec{r}\,') \right\rangle$$

And now it is obvious that this is a Fourier transform. The $\exp\left(-\frac{i}{2}\vec{r}^T\Omega\vec{r}\,'\right)$ picks out the oscillation with frequency $\vec{r}$ in the characteristic function $\left\langle \hat{D}(\vec{r}\,') \right\rangle$ (which is a function of $\vec{r}\,'$). Since the characteristic function is given by symplectic oscillations, the Fourier kernel needs also to be a symplectic oscillation, since the sign of the Fourier kernel's exponent needs to be opposite to the sign of the to-be-transformed function's exponent.

We will now prove a couple properties of the Wigner function to make our claim concrete. We will see that the Wigner function almost has all the desired properties of a joint PDF, except that it can take on negative values.

2 Properties of the Wigner Function

The above discussion of expectation of displacement operators is not Wigner's original formulation. In order to derive various properties of the Wigner function, we need the assistance of the Weyl Transform, which is Wigner's original formulation.

Define the **Weyl Transform** $\tilde{A}$ of an operator $\hat{A}$ as

$$\tilde{A}(\vec{r}) = \tilde{A}(x,p) = \int \frac{dy}{4\pi} \exp\left(-\frac{i}{2}py\right) \left\langle x + \frac{y}{2} \middle| \hat{A} \middle| x - \frac{y}{2} \right\rangle \tag{4}$$

where $\langle x'| \hat{A} |x\rangle$ are the matrix elements in position quadrature basis.

Let me give a road map of our results:
0. The Wigner function of a state $\hat{\rho}$ is the Weyl Transform of $\hat{\rho}$.
1. The expectation of an observable $\hat{A}$ is given by the average of $\tilde{A}(\vec{r})$ over phase space with "probability density" $W_\rho(\vec{r})$.
2. When integrated over one phase space variable, the Wigner function gives the marginal probability distribution in the other one.
3. The Wigner function is real.
4. The Wigner function is normalized.
5. The Wigner function is bounded (i.e. does not blow up).
6. The Wigner function can be negative.

We see that the Wigner function almost has all the desired properties of a joint PDF, except that it can take on negative values.

0. The Wigner function of a state $\hat{\rho}$ is the Weyl Transform of $\hat{\rho}$.

Proof.

$$
\begin{aligned}
W_\rho(\vec{r}) &= \int_{\mathbb{R}^2} \frac{d^2\vec{r}\,'}{(4\pi)^2} \exp\left(-\frac{i}{2}\vec{r}^{\,T}\Omega\vec{r}\,'\right) \left\langle \hat{D}(\vec{r}\,')\right\rangle \\
&= \int_{\mathbb{R}^2} \frac{d^2\vec{r}\,'}{(4\pi)^2} \exp\left(-\frac{i}{2}\vec{r}^{\,T}\Omega\vec{r}\,'\right) \operatorname{tr}\left(\hat{\rho}\hat{D}(\vec{r}\,')\right) \\
&= \int_{\mathbb{R}^2} \frac{dx'dp'}{(4\pi)^2} \exp\left(\frac{i}{2}(x'p - xp')\right) \operatorname{tr}\left(\hat{\rho}\exp\left(\frac{i}{2}(p'\hat{X} - x'\hat{P})\right)\right)
\end{aligned} \tag{5}
$$

We now use the Baker-Campbell-Hausdorff formula for the exponent of operators

$$
\exp\left(\hat{A} + \hat{B}\right) = \exp\left(\hat{B} + \hat{A}\right) = \exp\left(-\frac{1}{2}[\hat{A}, \hat{B}]\right)\exp\left(\hat{A}\right)\exp\left(\hat{B}\right) \quad \text{if } [\hat{A}, \hat{B}] \text{ is a number}
$$

With $\hat{A} = -\frac{i}{2}x'\hat{P}$, $\hat{B} = \frac{i}{2}p'\hat{X}$ and $[\hat{X}, \hat{P}] = 2i$, we have $[\hat{A}, \hat{B}] = \frac{x'p'}{4}(-2i) = -\frac{i}{2}x'p'$, so

$$
\exp\left(\frac{i}{2}(p'\hat{X} - x'\hat{P})\right) = \exp\left(\frac{i}{4}x'p'\right)\exp\left(-\frac{i}{2}x'\hat{P}\right)\exp\left(\frac{i}{2}p'\hat{X}\right)
$$

Another useful fact is

$$
\begin{aligned}
\exp\left(-\frac{i}{2}y\hat{P}\right)|x\rangle &= \int_{\mathbb{R}} \exp\left(-\frac{i}{2}y\hat{P}\right)|p\rangle\langle p|x\rangle\, dp \\
&= \int_{\mathbb{R}} \exp\left(-\frac{i}{2}yp\right)|p\rangle \frac{1}{\sqrt{4\pi}}\exp\left(-\frac{i}{2}xp\right)dp \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi}}\exp\left(-\frac{i}{2}(x+y)p\right)|p\rangle\, dp \\
&= |x+y\rangle
\end{aligned}
$$

This is just the action of position translation operator on a position eigenket, so the result is expected. We were simply careful about normalization.

The trace in (5) can be expanded in the position eigenbasis, and we get

$$
\begin{aligned}
W_\rho(\vec{r}) &= \int_{\mathbb{R}^2} \frac{dx'dp'}{(4\pi)^2} \exp\left(\frac{i}{2}(x'p\underline{-xp'})\right) \int_{\mathbb{R}} dy\, \langle y|\hat{\rho}\underline{\exp\left(\frac{i}{4}x'p'\right)} \exp\left(-\frac{i}{2}x'\hat{P}\right) \exp\left(\frac{i}{2}p'\hat{X}\right)|y\rangle \\
&= \int_{\mathbb{R}} \frac{dx'}{4\pi} \exp\left(\frac{i}{2}x'p\right) \int_{\mathbb{R}} dy \int_{\mathbb{R}} \frac{dp'}{4\pi} \underline{\exp\left(\frac{i}{2}p'(-x+\frac{x'}{2})\right)} \langle y|\hat{\rho} \exp\left(-\frac{i}{2}x'\hat{P}\right) \exp\left(\frac{i}{2}p'y\right)|y\rangle \\
&= \int_{\mathbb{R}} \frac{dx'}{4\pi} \exp\left(\frac{i}{2}x'p\right) \int_{\mathbb{R}} dy \underbrace{\int_{\mathbb{R}} \frac{dp'}{4\pi} \exp\left(\frac{i}{2}p'(y-x+\frac{x'}{2})\right)}_{=\delta(y-x+x'/2)} \langle y|\, \hat{\rho}\, \underbrace{\exp\left(-\frac{i}{2}x'\hat{P}\right)|y\rangle}_{=|y+x'\rangle} \\
&= \int_{\mathbb{R}} \frac{dx'}{4\pi} \exp\left(\frac{i}{2}x'p\right) \left\langle x-\frac{x'}{2}\middle|\, \hat{\rho}\, \middle| x+\frac{x'}{2}\right\rangle \\
&= \int_{\mathbb{R}} \frac{dy}{4\pi} \exp\left(-\frac{i}{2}yp\right) \left\langle x+\frac{y}{2}\middle|\, \hat{\rho}\, \middle| x-\frac{y}{2}\right\rangle = \tilde{\rho}(x,p)
\end{aligned}
$$

where we used the substitution $x' = -y$ in the last step. $\square$

We wanted to express the Wigner function as a Weyl transform, because of the following useful property of Weyl transforms.

Trace Property of Weyl Transform

Let $\hat{A}$, $\hat{B}$ be operators. Then,

$$
\operatorname{tr}\left(\hat{A}\hat{B}\right) = 4\pi \iint_{\mathbb{R}^2} dxdp\, \tilde{A}(x,p)\tilde{B}(x,p) \tag{6}
$$

Proof. We have

$$
\tilde{A}(x,p) = \int \frac{dy}{4\pi} \exp\left(-\frac{i}{2}py\right) \left\langle x+\frac{y}{2}\middle|\, \hat{A}\, \middle| x-\frac{y}{2}\right\rangle, \tilde{B}(x,p) = \int \frac{dy'}{4\pi} \exp\left(-\frac{i}{2}py'\right) \left\langle x+\frac{y'}{2}\middle|\, \hat{B}\, \middle| x-\frac{y'}{2}\right\rangle
$$

Therefore right hand side is

$$
\begin{aligned}
&4\pi \iint_{\mathbb{R}} dxdp\, \tilde{A}(x,p)\tilde{B}(x,p) \\
&= \iiint \underbrace{\int \exp\left(-\frac{i}{2}p(y+y')\right)\frac{dp}{4\pi}}_{=\delta(y+y')} \left\langle x+\frac{y}{2}\middle|\, \hat{A}\, \middle| x-\frac{y}{2}\right\rangle \left\langle x+\frac{y'}{2}\middle|\, \hat{B}\, \middle| x-\frac{y'}{2}\right\rangle dy'dxdy \\
&= \iint \left\langle x+\frac{y}{2}\middle|\, \hat{A}\, \middle| x-\frac{y}{2}\right\rangle \left\langle x-\frac{y}{2}\middle|\, \hat{B}\, \middle| x+\frac{y}{2}\right\rangle dxdy
\end{aligned}
$$

Let $u = x - y/2$, $v = x + y/2$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1/2 & 1/2 \end{vmatrix} = 1$$

so $dudv = dxdy$. Right hand side thus becomes

$$\iint \langle v| \hat{A} |u\rangle \langle u| \hat{B} |v\rangle \, dudv = \int \langle v|\hat{A}\hat{B}|v\rangle \, dv = \operatorname{tr}\left(\hat{A}\hat{B}\right) \qquad \square$$

This is an amazing result. We can now obtain the expectation of observables very easily using the trace property.

1. The expectation of an observable $\hat{A}$ is given by the average of $\tilde{A}(\vec{r})$ over phase space with "probability density" $W_\rho(\vec{r})$.

Proof. This is now almost automatic: we just integrate the product of the Weyl transforms. For state $\hat{\rho}$, the expectation of an observable $\hat{A}$ is

$$\operatorname{tr}\left(\hat{\rho}\hat{A}\right) = 4\pi \iint \tilde{\rho}(x,p)\tilde{A}(x,p)dxdp$$

But the Weyl transform of the state operator $\hat{\rho}$ is just the Wigner function, as we so arduously proved. Thus,

$$\operatorname{tr}\left(\hat{\rho}\hat{A}\right) = 4\pi \iint W_\rho(x,p)\tilde{A}(x,p)dxdp \tag{7}$$

$\square$

(7) instructs us to identify the Wigner function as the probability distribution in phase space coordinates! In fact, if one were allowed to be hand-wavy, this can be viewed as the correct intuitive interpretation of the Wigner function. This interpretation is further confirmed by the fact that the Weyl transform of $\hat{X}$ and $\hat{P}$ is just $\tilde{X}(x,p) = x/4\pi$ and $\tilde{P}(x,p) = p/4\pi$, respectively.

Exercise (and potential final exam question!!). Show that the Weyl transform of $\hat{X}$ and $\hat{P}$ is just $\tilde{X}(x,p) = x/4\pi$ and $\tilde{P}(x,p) = p/4\pi$, respectively. Hint: the position one is straightforward. For the momentum one, first write the Weyl transform in momentum basis. You should find that it takes on a "conjugate" form as the Weyl transform in the position basis in (4), as expected.

At this point, it will be helpful to express the Wigner function in terms of the actual wavefunction $\psi(x) = \langle x|\psi\rangle$. Let the system be in the state $\hat{\rho} = \sum_i P_i |\psi_i\rangle \langle\psi_i|$, i.e. a probability P_i to be in pure state $|\psi_i\rangle$. The Wigner function of the state is

$$\begin{aligned} W_\rho(x,p) = \tilde{\rho}(x,p) &= \int \frac{dy}{4\pi} \exp\left(-\frac{i}{2}py\right) \left\langle x+\frac{y}{2}\right| \hat{\rho} \left|x-\frac{y}{2}\right\rangle \\ &= \sum_i P_i \int \frac{dy}{4\pi} \exp\left(-\frac{i}{2}py\right) \left\langle x+\frac{y}{2}\middle|\psi_i\right\rangle \left\langle\psi_i\middle|x-\frac{y}{2}\right\rangle \\ &= \sum_i P_i \int \frac{dy}{4\pi} \exp\left(-\frac{i}{2}py\right) \psi_i(x+\frac{y}{2})\psi_i^*(x-\frac{y}{2}) \end{aligned} \tag{8}$$

(8) is usually the most useful formula in practice, since it links the Wigner function to the wavefunction, both being just simple functions that computers can understand, unlike bras and kets.

2. When integrated over one phase space variable, the Wigner function gives the marginal probability distribution in the other one.

Proof.

$$\begin{aligned} \int W_\rho(x,p)dp &= \sum_i P_i \int dy \underbrace{\int \frac{dp}{4\pi} \exp\left(-\frac{i}{2}py\right)}_{=\delta(y)} \psi_i(x+\frac{y}{2})\psi_i^*(x-\frac{y}{2}) \\ &= \sum_i P_i\, \psi_i(x)\psi_i^*(x) = \sum_i P_i\, |\psi_i(x)|^2 \end{aligned}$$

which is the correct position PDF of the state $\hat{\rho}$. The momentum PDF is similar, using the momentum basis Weyl transform. □

3. The Wigner function is real

Proof.
This is easily seen: (8) is invariant under complex conjugation. Just make the substitution $z = -y$ in the integral. □

4. The Wigner function is normalized, i.e.

$$\iint W_\rho(x,p)dxdp = 1 \tag{9}$$

Proof.
The Weyl transform of the identity operator is

$$\tilde{I}(x,p) = \int \frac{dy}{4\pi}\exp\left(-\frac{i}{2}py\right)\underbrace{\left\langle x+\frac{y}{2}\middle|\,\hat{I}\,\middle|x-\frac{y}{2}\right\rangle}_{=\delta((x+y/2)-(x-y/2))=\delta(y)} = \frac{1}{4\pi}$$

Therefore, by the trace property (6), we have

$$\begin{aligned}\iint W_\rho(x,p)dxdp &= 4\pi\iint W_\rho(x,p)\frac{1}{4\pi}dxdp\\ &= 4\pi\iint \tilde{\rho}(x,p)\tilde{I}(x,p)dxdp\\ &= \mathrm{tr}\left(\hat{\rho}\hat{I}\right) = \mathrm{tr}(\hat{\rho}) = 1 \qquad \square\end{aligned}$$

5. The Wigner function is bounded (i.e. does not blow up). In equations,

$$\iint [W_\rho(x,p)]^2 dxdp \le \frac{1}{4\pi} \tag{10}$$

Proof.

$$\iint [W_\rho(x,p)]^2 dxdp = \iint \tilde{\rho}\tilde{\rho}\, dxdp = \frac{1}{4\pi}\,\mathrm{tr}\left(\hat{\rho}^2\right) \le \frac{1}{4\pi}$$

since $\mathrm{tr}(\hat{\rho}^2) \le 1$ for density operators. □

All five properties above screams that the Wigner function is nothing but the probability distribution in phase space. However, as usual, quantum mechanics does not allow for such a naive and straightforward interpretation to flourish $--$ at least, not without a shadow of uncomfortableness.

6. <u>The Wigner function can be negative</u>

Proof.
Consider two pure states $\hat{\rho}_1 = |\psi_1\rangle \langle\psi_1|$ and $\hat{\rho}_2 = |\psi_2\rangle \langle\psi_2|$. Let the two pure states be orthogonal, i.e. $\langle\psi_1|\psi_2\rangle = 0$. We thus have

$$\mathrm{tr}(\hat{\rho}_1\hat{\rho}_2) = 4\pi \iint W_{\rho_1}(x,p)W_{\rho_2}(x,p)\,dxdp$$

On the other hand,

$$\mathrm{tr}(\hat{\rho}_1\hat{\rho}_2) = \mathrm{tr}(|\psi_1\rangle \langle\psi_1|\psi_2\rangle \langle\psi_2|) = \langle\psi_2|\psi_1\rangle \langle\psi_1|\psi_2\rangle = 0$$

where we used $\mathrm{tr}\Big(\hat{A}\,|\psi\rangle \langle\psi|\Big) = \langle\psi|\hat{A}|\psi\rangle$. Thus we see that

$$\iint W_{\rho_1}(x,p)W_{\rho_2}(x,p)\,dxdp = 0$$

If the Wigner functions were non-negative for all (x,p), then the only way this result holds is that one of $W_{\rho_1}(x,p)$ and $W_{\rho_2}(x,p)$ is zero for all (x,p), which is impossible since Wigner functions are normalized.

[The more pedantic reader might argue that W_{ρ_1} and W_{ρ_2} only need to be zero on complementary regions of phase space. While this is true mathematically, physically this is highly unlikely, as the above integral needs to hold for ANY two orthogonal pure states.]

Thus, some, and indeed most Wigner functions will attain negative values on some part of phase space. $\square$

3 Time Evolution of Wigner Functions

First, a bit of massaging is needed to transform the usual $\hat{q}_{usual}, \hat{p}_{usual}$ (dimensionful) Schrodinger equation into $\hat{X}, \hat{P}$ (dimensionless) quadrature coordinates. The Schrodinger equation in quadrature coordinates for the harmonic oscillator is

$$\begin{aligned}
i\hbar\frac{\partial\psi}{\partial t} =& \frac{\hat{p}_{usual}^2}{2m}\psi + \frac{1}{2}m\omega^2\hat{q}_{usual}^2\psi \\
=& \frac{p_{zpf}^2\hat{P}^2}{2m}\psi + \frac{1}{2}m\omega^2 q_{zpf}^2\hat{X}^2\psi \\
=& \frac{\hbar\omega m}{2}\frac{\hat{P}^2}{2m}\psi + \frac{1}{2}m\omega^2\frac{\hbar}{2\omega m}\hat{X}^2\psi \\
=& \frac{\hbar\omega}{4}\hat{P}^2\psi + \frac{\hbar\omega}{4}\hat{X}^2\psi
\end{aligned}$$

where $\psi = \psi(x,p)$, and (x,p) is the (dimensionless) quadrature coordinates we've been using thus far. Note that $x = q_{usual}/q_{zpf}$, $p = p_{usual}/p_{zpf}$.

The quadrature operators expressed in quadrature coordinates are

$$\hat{X} = \frac{\hat{q}_{usual}}{q_{zpf}} = \frac{q_{usual}}{q_{zpf}} = \frac{q_{zpf}x}{q_{zpf}} = x$$

$$\hat{P} = \frac{\hat{p}_{usual}}{p_{zpf}} = \sqrt{\frac{2}{\hbar\omega m}}\frac{\hbar}{i}\frac{\partial}{\partial q_{usual}} = \sqrt{\frac{2}{\hbar\omega m}}\frac{\hbar}{i}\frac{\partial}{\partial\sqrt{\frac{\hbar}{2\omega m}}x} = \sqrt{\frac{2}{\hbar\omega m}}\frac{\hbar}{i}\sqrt{\frac{2\omega m}{\hbar}}\frac{\partial}{\partial x} = -2i\frac{\partial}{\partial x}$$

So the Schrodinger equation in quadrature coordinate is

$$i\frac{\partial\psi}{\partial t} = \frac{\omega}{4}(\hat{P}^2 + \hat{X}^2)\psi = \frac{\omega}{4}\bigg(-4\frac{\partial^2\psi}{\partial x^2} + x^2\psi\bigg)$$

or

$$\frac{\partial\psi(x)}{\partial t} = i\omega\frac{\partial^2\psi(x)}{\partial x^2} + \frac{\omega}{4i}x^2\psi(x)$$

We start with (8)

$$W_\rho(x,p) = \sum_i P_i \int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\psi_i(x+\frac{y}{2})\psi_i^*(x-\frac{y}{2})$$

$$\begin{aligned}
\frac{\partial W_\rho(x,p)}{\partial t} &= \sum_i P_i \int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\Big[\frac{\partial\psi_i(x+\frac{y}{2})}{\partial t}\psi_i^*(x-\frac{y}{2}) + \psi_i(x+\frac{y}{2})\frac{\partial\psi_i^*(x-\frac{y}{2})}{\partial t}\Big] \\
&= \sum_i P_i \int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\Big\{ \\
&\Big[i\omega\frac{\partial^2\psi_i(x+\frac{y}{2})}{\partial x^2} + \frac{\omega}{4i}(x+\frac{y}{2})^2\psi_i(x+\frac{y}{2})\Big]\psi_i^*(x-\frac{y}{2}) \\
&+ \Big[-i\omega\frac{\partial^2\psi_i^*(x-\frac{y}{2})}{\partial x^2} - \frac{\omega}{4i}(x-\frac{y}{2})^2\psi_i^*(x-\frac{y}{2})\Big]\psi_i(x+\frac{y}{2})\Big\} \\
&= \sum_i P_i \int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\Big\{ \\
&i\omega\Big(\frac{\partial^2\psi_i(x+\frac{y}{2})}{\partial x^2}\psi_i^*(x-\frac{y}{2}) - \frac{\partial^2\psi_i^*(x-\frac{y}{2})}{\partial x^2}\psi_i(x+\frac{y}{2})\Big) \qquad (11) \\
&+ \frac{\omega}{4i}\psi_i^*(x-\frac{y}{2})\psi_i(x+\frac{y}{2})\Big((x+\frac{y}{2})^2 - (x-\frac{y}{2})^2\Big)\Big\} \qquad (12)
\end{aligned}$$

where we plugged in the Schrodinger equation for the time derivatives of the wavefunctions.

The two terms can now be massaged a bit. For the first term of (11), noticing that chain rule gives

$$\frac{\partial f(x+ay)}{\partial y} = \frac{\partial f(x+ay)}{\partial(x+ay)}\frac{\partial(x+ay)}{\partial y} = a\,\frac{\partial f(x)}{\partial x}$$

so we have

$$
\begin{aligned}
&\int dy \exp\Big(-\frac{i}{2}py\Big)\frac{\partial^2\psi_i(x+\frac{y}{2})}{\partial x^2}\psi_i^*(x-\frac{y}{2}) \\
&= \ 2\int dy \exp\Big(-\frac{i}{2}py\Big)\frac{\partial^2\psi_i(x+\frac{y}{2})}{\partial y\partial x}\psi_i^*(x-\frac{y}{2}) \\
&= \underbrace{2\exp\Big(-\frac{i}{2}py\Big)\psi_i^*(x-\frac{y}{2})\frac{\partial\psi_i(x+\frac{y}{2})}{\partial x}}_{=0 \text{ since } \psi \text{ dies at } y=\infty} -2\int dy\frac{\partial\psi_i(x+\frac{y}{2})}{\partial x}\frac{\partial}{\partial y}\Big[\exp\Big(-\frac{i}{2}py\Big)\psi_i^*(x-\frac{y}{2})\Big] \\
&= ip\int dy\frac{\partial\psi_i(x+\frac{y}{2})}{\partial x}\exp\Big(-\frac{i}{2}py\Big)\psi_i^*(x-\frac{y}{2}) - 2\int dy\frac{\partial\psi_i(x+\frac{y}{2})}{\partial x}\exp\Big(-\frac{i}{2}py\Big)\frac{\partial\psi_i^*(x-\frac{y}{2})}{\partial y} \\
&= ip\int dy \exp\Big(-\frac{i}{2}py\Big)\frac{\partial\psi_i(x+\frac{y}{2})}{\partial x}\psi_i^*(x-\frac{y}{2}) + \int dy \exp\Big(-\frac{i}{2}py\Big)\frac{\partial\psi_i(x+\frac{y}{2})}{\partial x}\frac{\partial\psi_i^*(x-\frac{y}{2})}{\partial x}
\end{aligned}
$$

Similarly, the second term of (11) is

$$
\begin{aligned}
&\int dy \exp\Big(-\frac{i}{2}py\Big)\frac{\partial^2\psi_i^*(x-\frac{y}{2})}{\partial x^2}\psi_i(x+\frac{y}{2}) \\
&= \ -2\int dy \exp\Big(-\frac{i}{2}py\Big)\frac{\partial^2\psi_i^*(x-\frac{y}{2})}{\partial y\partial x}\psi_i(x+\frac{y}{2}) \\
&= \underbrace{-2\exp\Big(-\frac{i}{2}py\Big)\psi_i(x+\frac{y}{2})\frac{\partial\psi_i^*(x-\frac{y}{2})}{\partial x}}_{=0 \text{ since } \psi \text{ dies at } y=\infty} +2\int dy\frac{\partial\psi_i^*(x-\frac{y}{2})}{\partial x}\frac{\partial}{\partial y}\Big[\exp\Big(-\frac{i}{2}py\Big)\psi_i(x+\frac{y}{2})\Big] \\
&= -ip\int dy\frac{\partial\psi_i^*(x-\frac{y}{2})}{\partial x}\exp\Big(-\frac{i}{2}py\Big)\psi_i(x+\frac{y}{2}) + 2\int dy\frac{\partial\psi_i^*(x-\frac{y}{2})}{\partial x}\exp\Big(-\frac{i}{2}py\Big)\frac{\partial\psi_i(x+\frac{y}{2})}{\partial y} \\
&= -ip\int dy \exp\Big(-\frac{i}{2}py\Big)\frac{\partial\psi_i^*(x-\frac{y}{2})}{\partial x}\psi_i(x+\frac{y}{2}) + \int dy \exp\Big(-\frac{i}{2}py\Big)\frac{\partial\psi_i^*(x-\frac{y}{2})}{\partial x}\frac{\partial\psi_i(x+\frac{y}{2})}{\partial x}
\end{aligned}
$$

Thus (11) is the difference of the above two terms. The second term of the above two terms cancel and we are left with

$$
\begin{aligned}
&\sum_i P_i i\omega \int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\Big[\frac{\partial^2 \psi_i(x+\frac{y}{2})}{\partial x^2}\psi_i^*(x-\frac{y}{2}) - \frac{\partial^2 \psi_i^*(x-\frac{y}{2})}{\partial x^2}\psi_i(x+\frac{y}{2})\Big] \\
=&\sum_i P_i i\omega \Big[ip\int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\frac{\partial \psi_i(x+\frac{y}{2})}{\partial x}\psi_i^*(x-\frac{y}{2}) + ip\int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\frac{\partial \psi_i^*(x-\frac{y}{2})}{\partial x}\psi_i(x+\frac{y}{2})\Big] \\
=&-\sum_i P_i \omega p \int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\frac{\partial}{\partial x}\Big(\psi_i(x+\frac{y}{2})\psi_i^*(x-\frac{y}{2})\Big) \\
=&-\omega p \frac{\partial}{\partial x}\Big(\sum_i P_i \int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\psi_i(x+\frac{y}{2})\psi_i^*(x-\frac{y}{2})\Big) \\
=&-\omega p \frac{\partial}{\partial x} W_\rho(x,p)
\end{aligned}
$$

Note that the dimensions are good: this is supposed to be a term for $\frac{\partial W}{\partial t}$, and x, p are dimensionless.

For (12),

$$
\begin{aligned}
&\sum_i P_i \int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\frac{\omega}{4i}\psi_i^*(x-\frac{y}{2})\psi_i(x+\frac{y}{2})\Big((x+\frac{y}{2})^2 - (x-\frac{y}{2})^2\Big) \\
=&\sum_i P_i \int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\frac{\omega}{4i}\psi_i^*(x-\frac{y}{2})\psi_i(x+\frac{y}{2})2xy \\
=&2x\frac{\omega}{4i}\sum_i P_i \int \frac{dy}{4\pi} y \exp\Big(-\frac{i}{2}py\Big)\psi_i^*(x-\frac{y}{2})\psi_i(x+\frac{y}{2}) \\
=&2x\frac{\omega}{4i}\frac{2}{-i}\frac{\partial}{\partial p}\Big(\sum_i P_i \int \frac{dy}{4\pi} \exp\Big(-\frac{i}{2}py\Big)\psi_i^*(x-\frac{y}{2})\psi_i(x+\frac{y}{2})\Big) \\
=&\omega x \frac{\partial}{\partial p} W_\rho(x,p)
\end{aligned}
$$

Again dimensions are good.

Assembling (11) and (12), we get the time evolution of the Wigner function for the harmonic oscillator in quadrature coordinates:

$$
\boxed{\frac{\partial W_\rho(x,p)}{\partial t} = \omega\Big(-p\frac{\partial W_\rho(x,p)}{\partial x} + x\frac{\partial W_\rho(x,p)}{\partial p}\Big)} \tag{13}
$$

What a beautiful equation (13) is.

(13) is simply the analog of the Liouville theorem for the harmonic oscillator in classical mechanics. It says that the time evolution of the Wigner function is that **it simply rotates clockwise with angular velocity ω in phase space, *with no distortion whatsoever***. The time evolution of Wigner functions for a harmonic oscillator is incredibly simple.

In equations, the solution to (13) is simply

$$W_\rho(x,p,t) = W_\rho(x\cos(\omega t) - p\sin(\omega t), p\cos(\omega t) + x\sin(\omega t), t=0) \tag{14}$$

First of all let's quickly check the chain rule gives the correct answer:

$$\begin{aligned}\frac{\partial W}{\partial t} &= \frac{\partial W}{\partial x}\frac{\partial}{\partial t}(x\cos(\omega t) - p\sin(\omega t)) + \frac{\partial W}{\partial p}\frac{\partial}{\partial t}(p\cos(\omega t) + x\sin(\omega t)) \\ &= \frac{\partial W}{\partial x}\,\omega(-x\sin(\omega t) - p\cos(\omega t)) + \frac{\partial W}{\partial p}\,\omega(-p\sin(\omega t) + x\cos(\omega t)) \\ &= \omega(-p\frac{\partial W}{\partial x} + x\frac{\partial W}{\partial p})\end{aligned}$$

which is the correct time evolution given by (13).

The solution (14) says that in time, the Wigner function for the harmonic oscillator rotates clockwise in phase space with angular velocity ω. (14) can be recast in the form

$$W_\rho(\vec{r},t) = W_\rho(L\vec{r}, t=0), \quad \text{where } L = \begin{pmatrix}\cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t)\end{pmatrix}$$

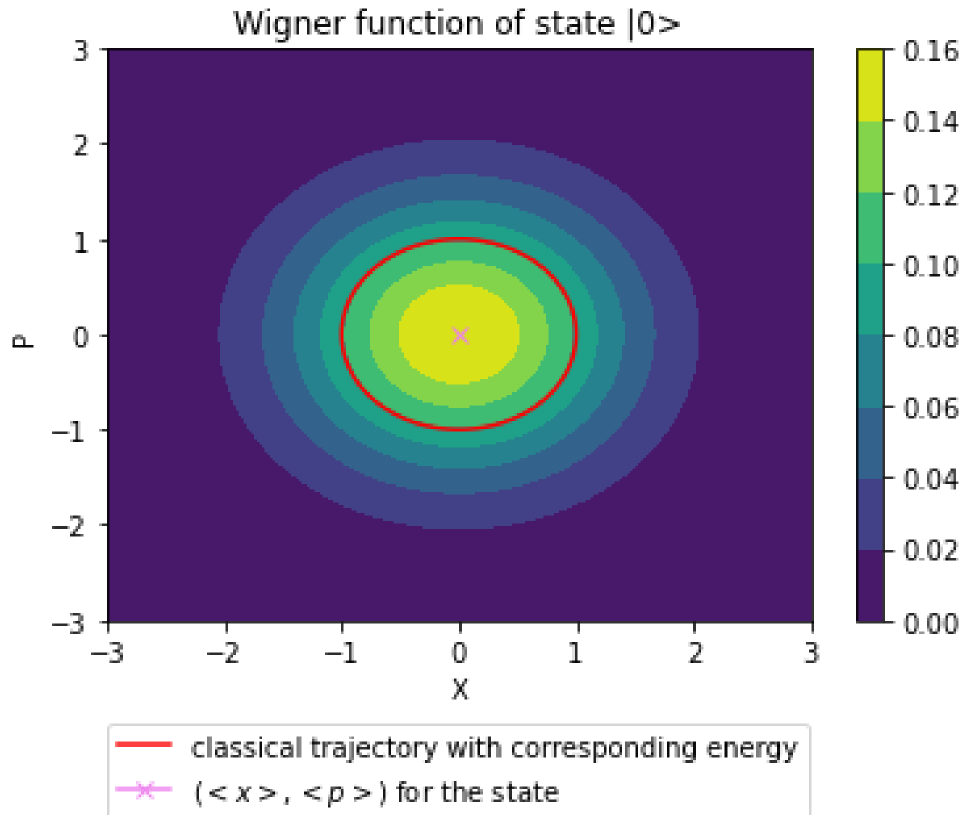
In other words, the value of W at position $\vec{r}$ at some later time t is given by the value of W at a counterclockwise-rotated position $L\vec{r}$ at some earlier time. Thus, in time, the Wigner function simply rotates clockwise in time.

The gallery showcases the time evolution of some Wigner functions for the harmonic oscillator.

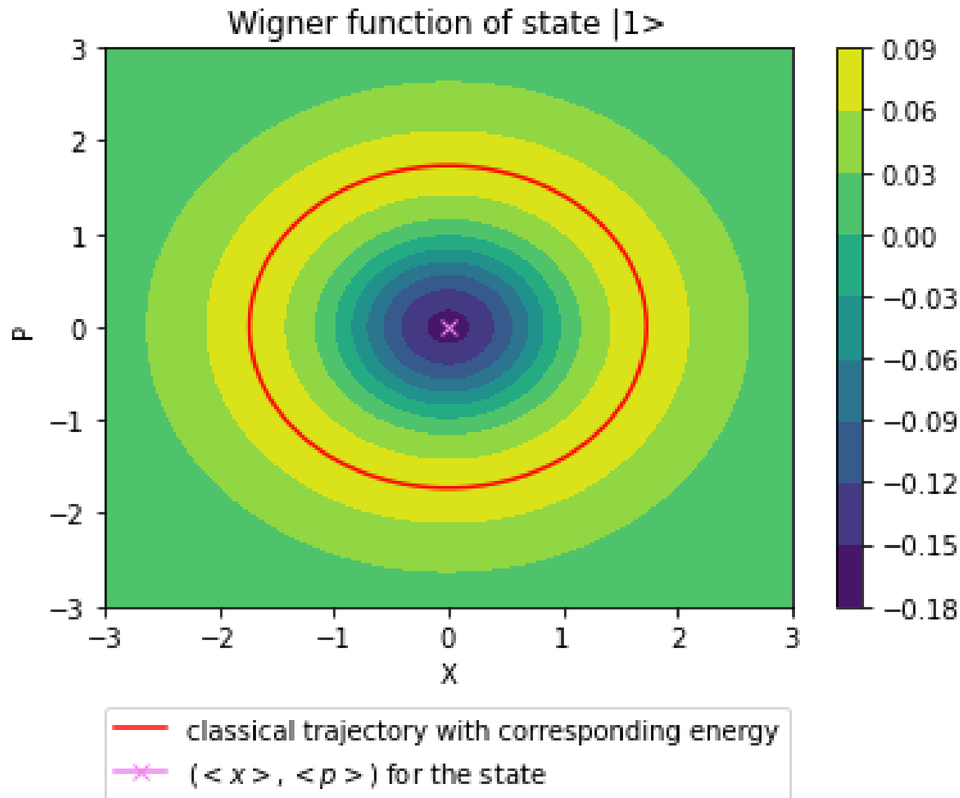
4 Coherent States

One particularly interesting class of states for the harmonic oscillator is the so-called coherent state. The displacement operator (1) can be recast into the following form with a complex-valued parameter

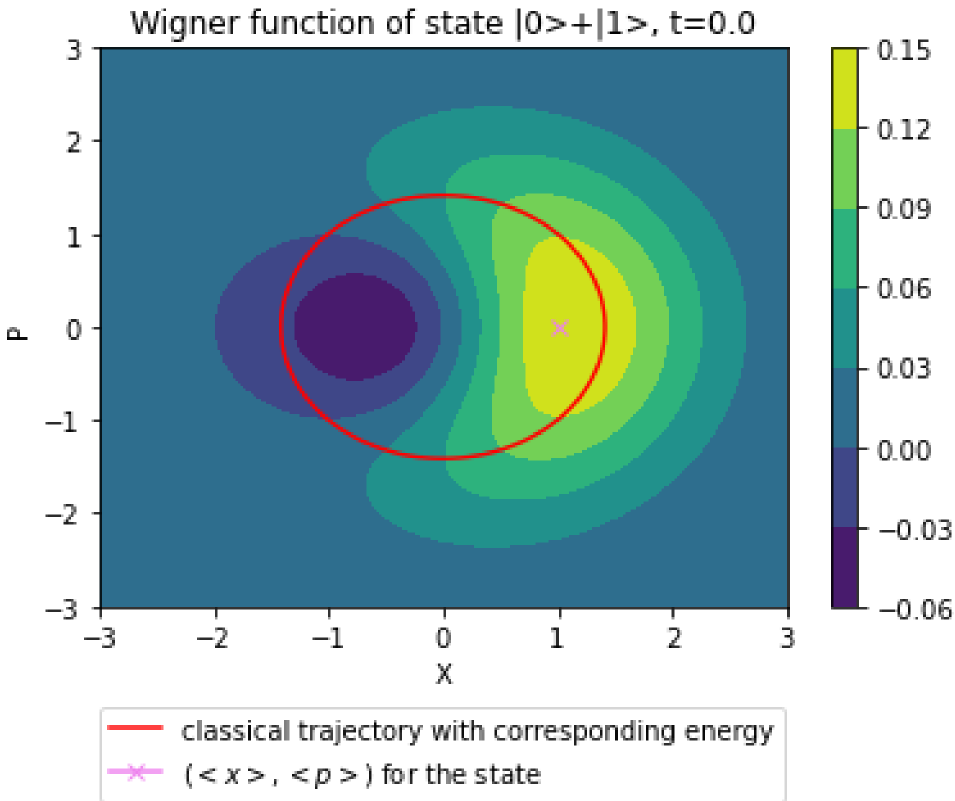
5 Gallery

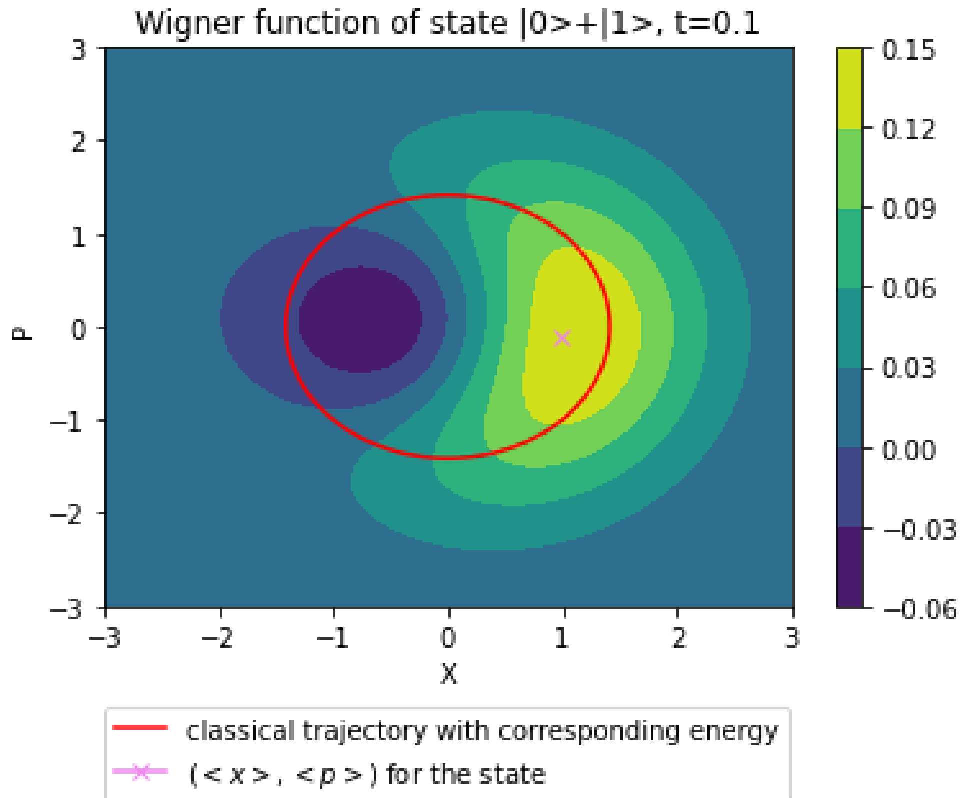


Wigner function of state |0>
3
2
1
0
−1
−2
−3
P
−3
−2
−1
0
1
2
3
X
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
classical trajectory with corresponding energy
(< x > , < p >) for the state

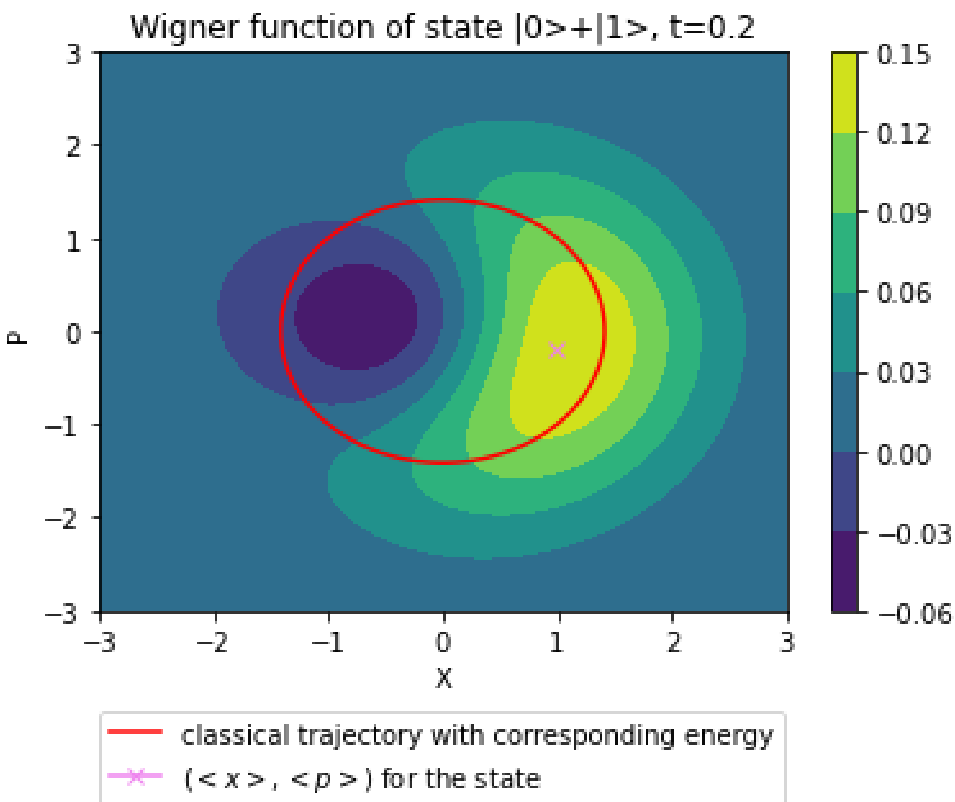


Wigner function of state |1>
P
X
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
0.09
0.06
0.03
0.00
−0.03
−0.06
−0.09
−0.12
−0.15
−0.18
classical trajectory with corresponding energy
(< x > , < p >) for the state

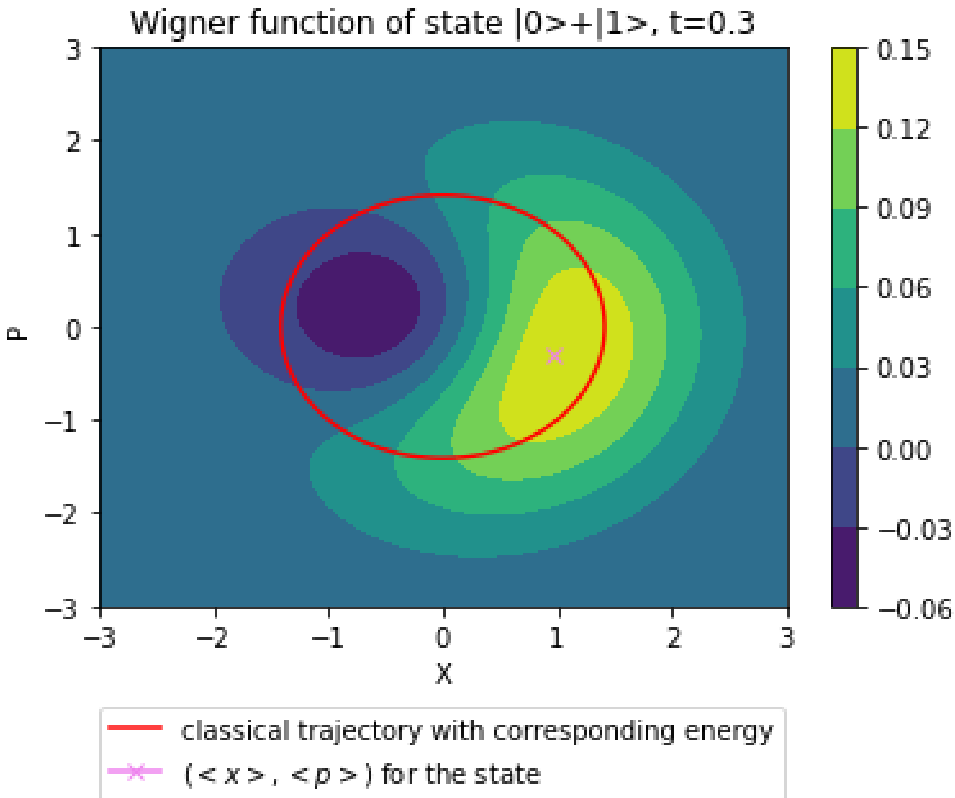

Wigner function of state |0>+|1>, t=0.0
3
2
1
0
−1
−2
−3
P
−3
−2
−1
0
1
2
3
X
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x > , < p >) for the state



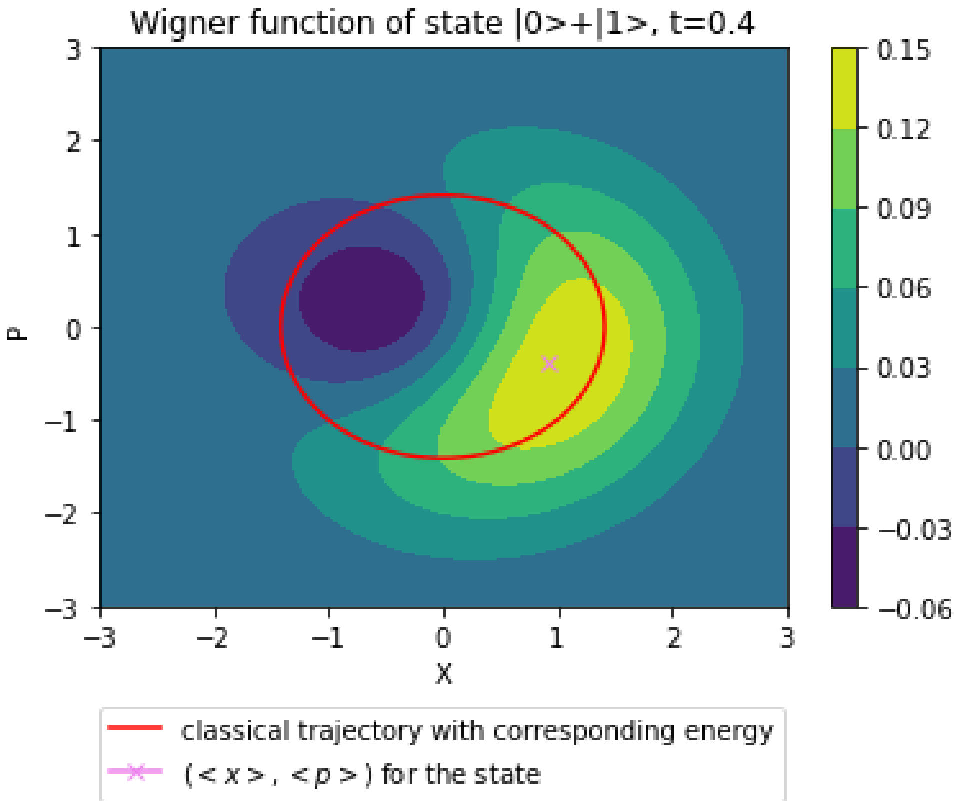
Wigner function of state |0>+|1>, t=0.1
P
X
3
2
1
0
−1
−2
−3
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x >, < p >) for the state



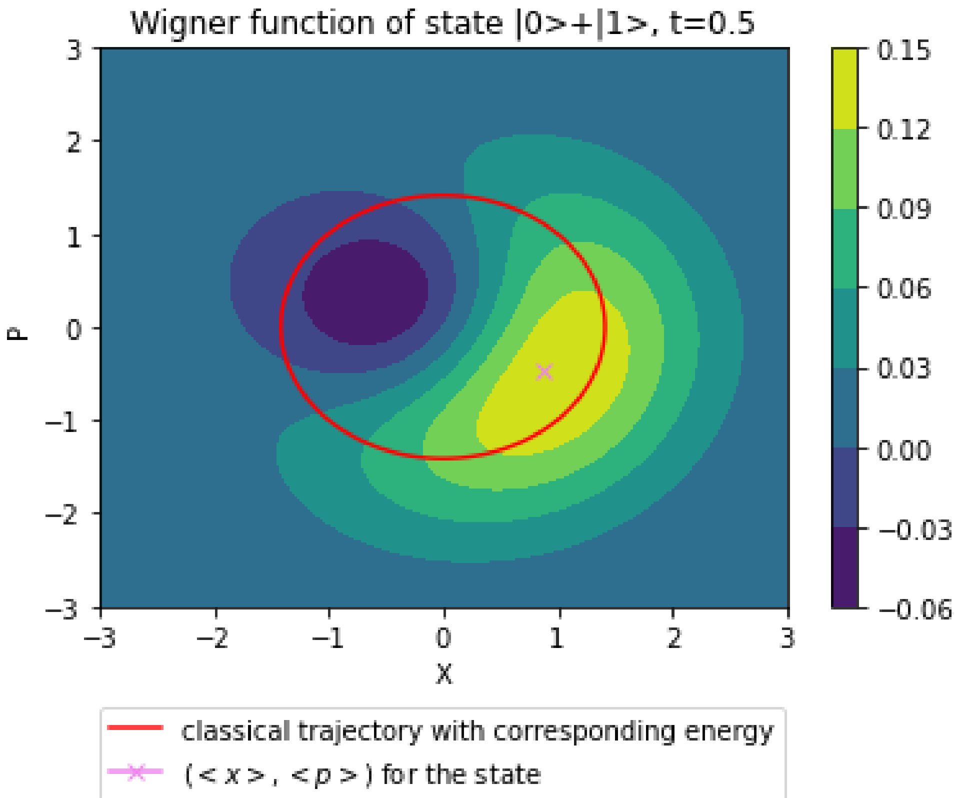
Wigner function of state |0>+|1>, t=0.2
P
X
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x > , < p >) for the state

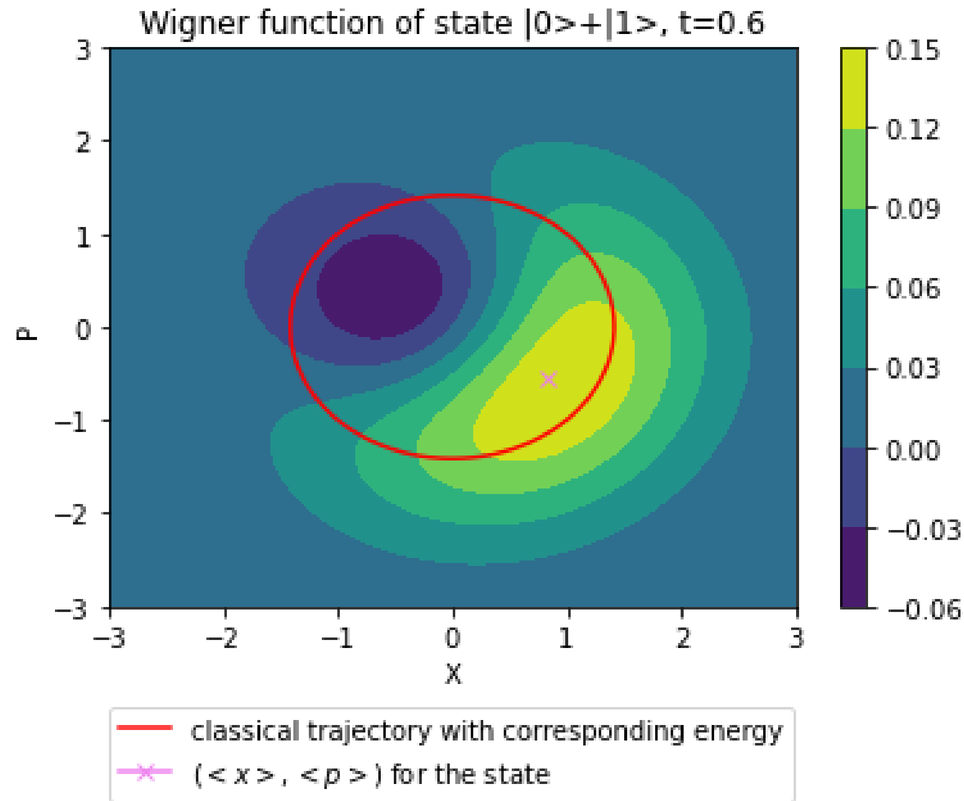


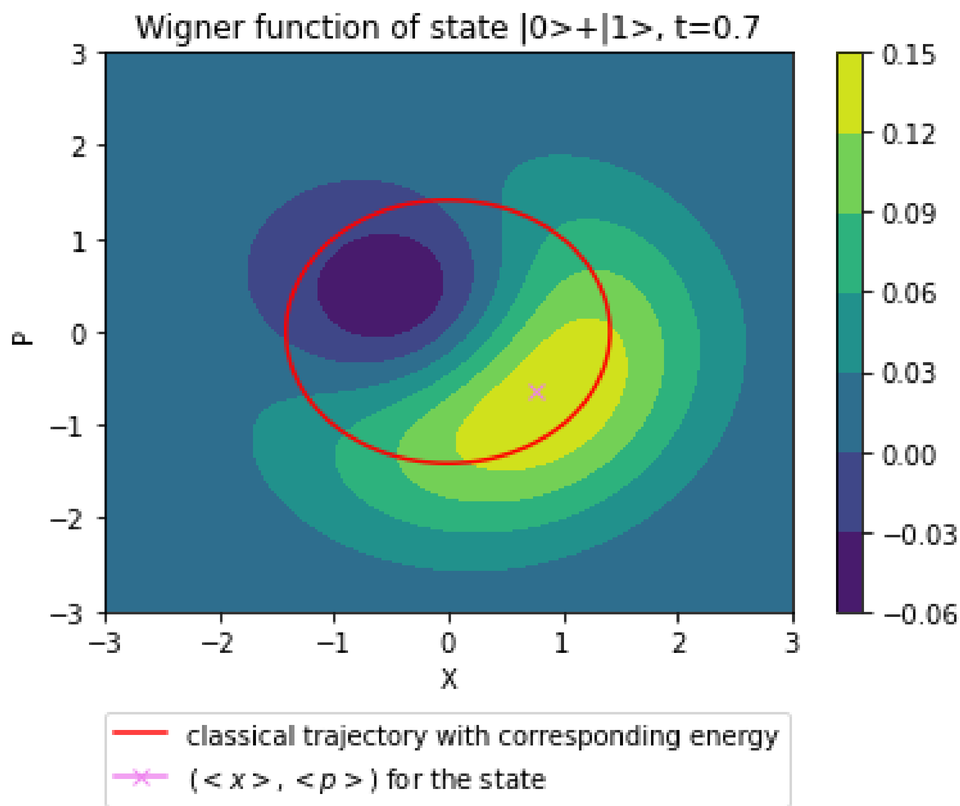
Wigner function of state |0>+|1>, t=0.3
P
X
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x > , < p >) for the state

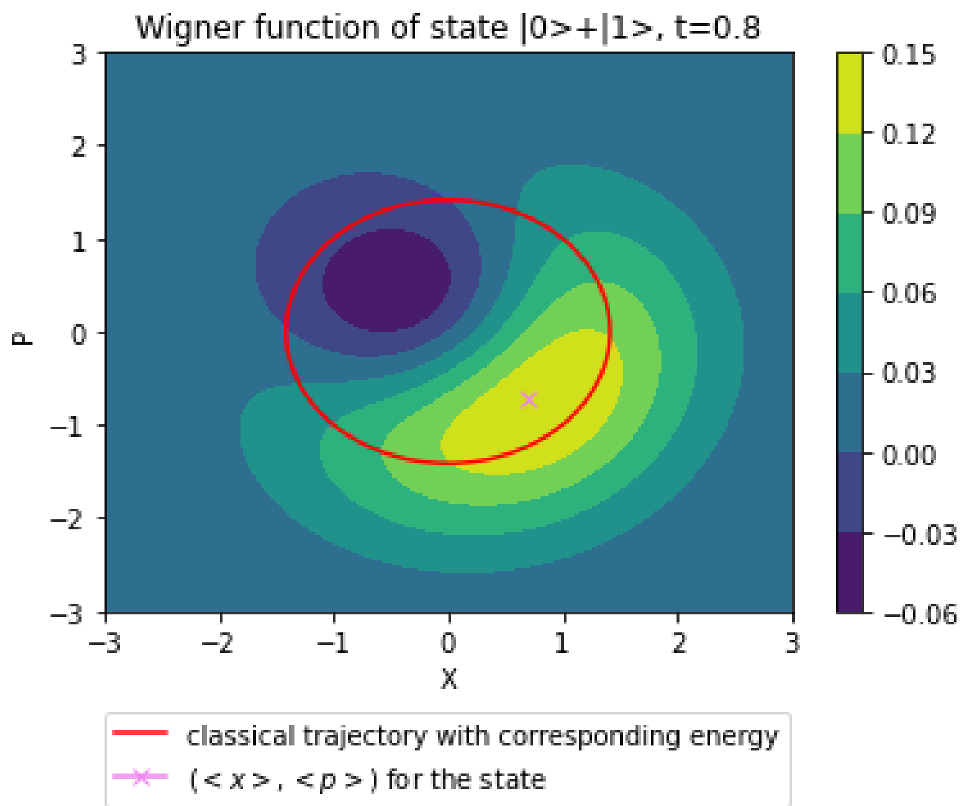


Wigner function of state |0>+|1>, t=0.4
P
X
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x >, < p >) for the state

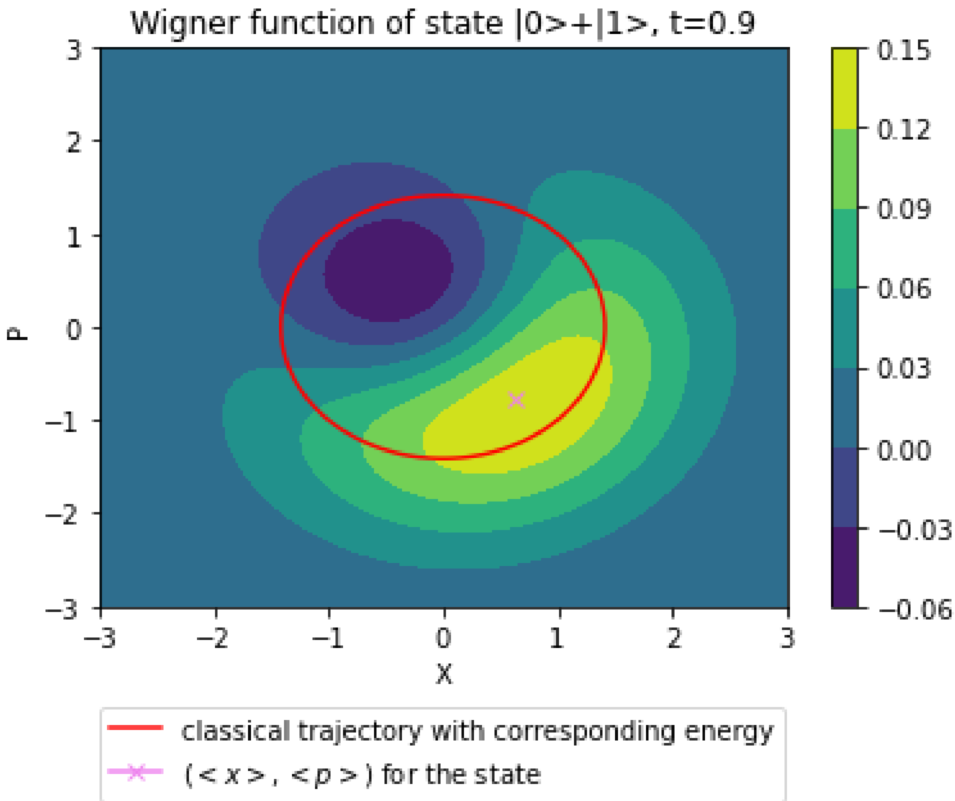

Wigner function of state |0>+|1>, t=0.5
P
X
classical trajectory with corresponding energy
(< x >, < p >) for the state


Wigner function of state |0>+|1>, t=0.6
P
X
3
2
1
0
−1
−2
−3
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x >, < p >) for the state

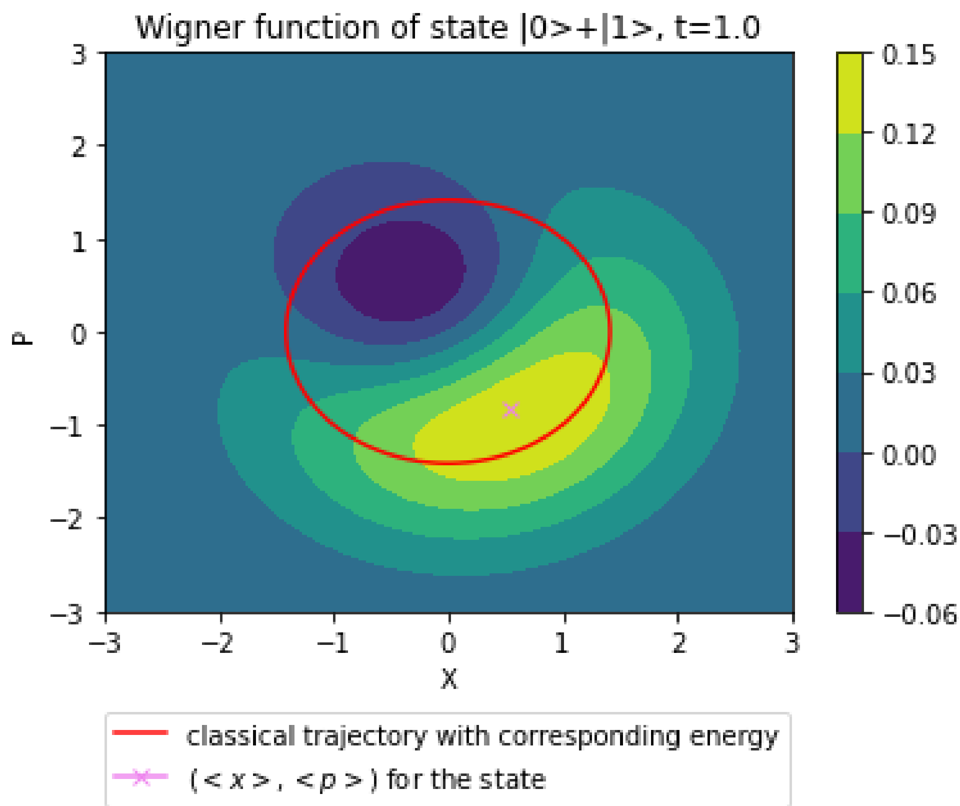

Wigner function of state |0>+|1>, t=0.7
P
X
3
2
1
0
−1
−2
−3
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x > , < p >) for the state



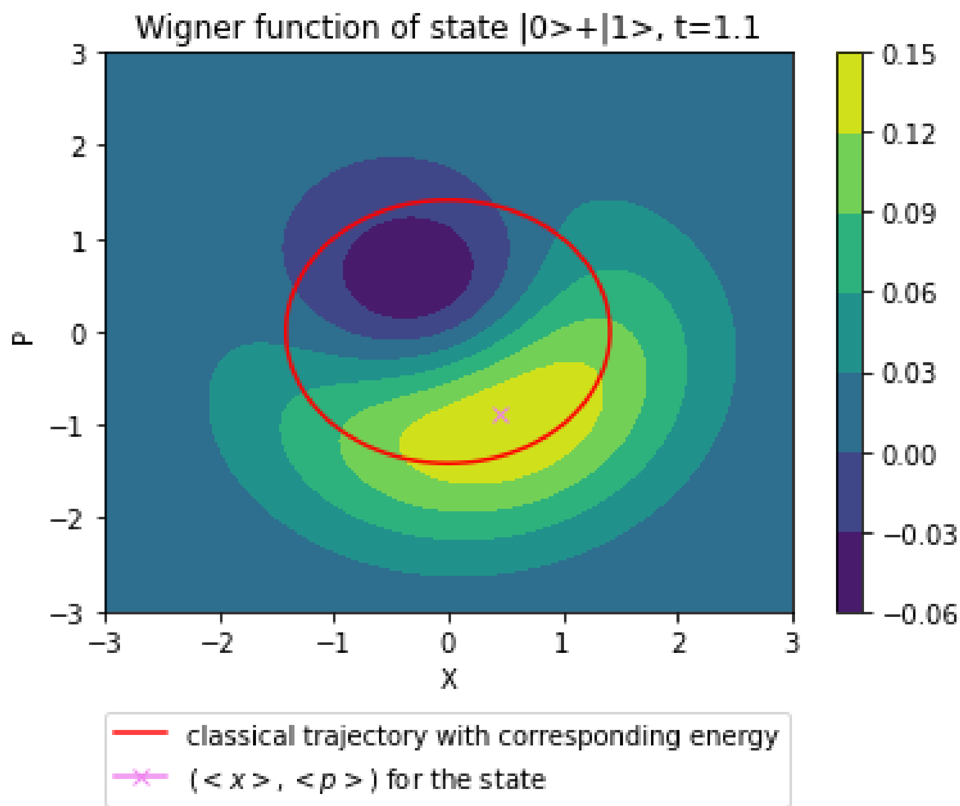
Wigner function of state |0>+|1>, t=0.8
3
2
1
0
−1
−2
−3
P
−3
−2
−1
0
1
2
3
X
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x > , < p >) for the state

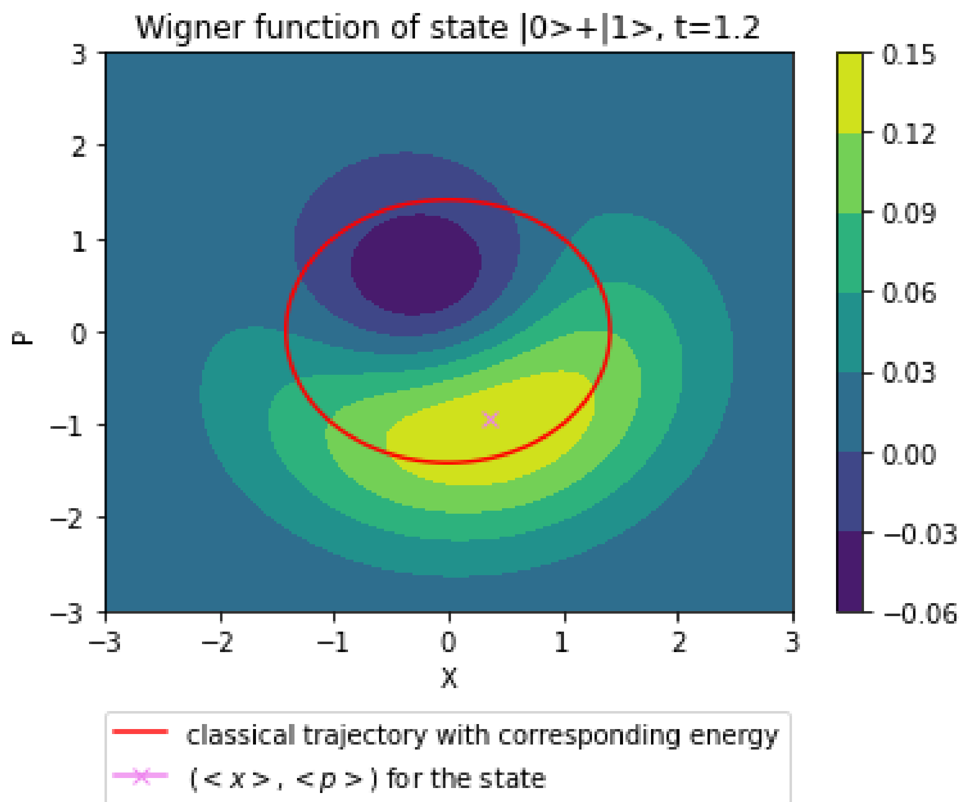


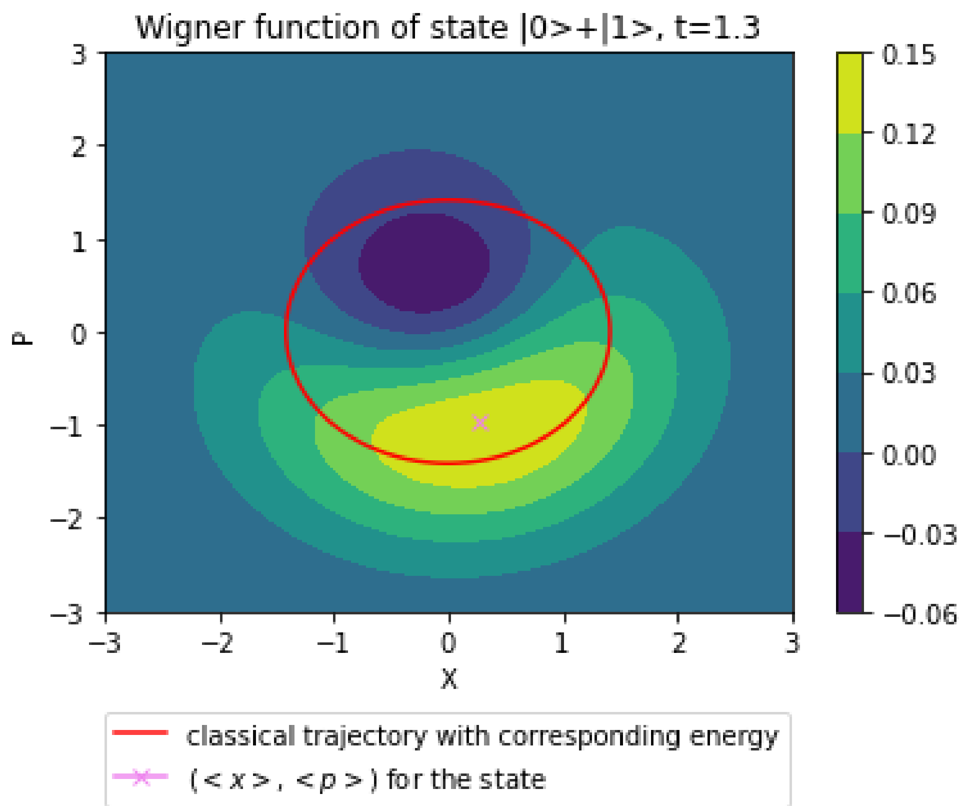
Wigner function of state |0>+|1>, t=0.9
P
X
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x > , < p >) for the state

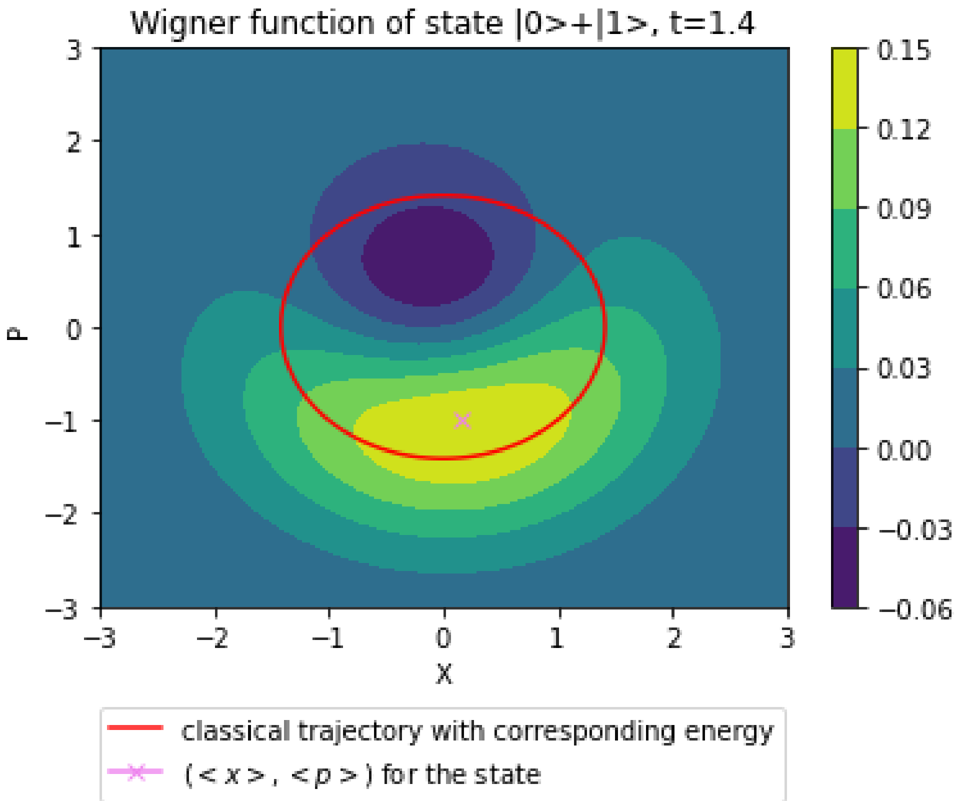


Wigner function of state |0>+|1>, t=1.0
3
2
1
0
−1
−2
−3
P
−3
−2
−1
0
1
2
3
X
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x >, < p >) for the state

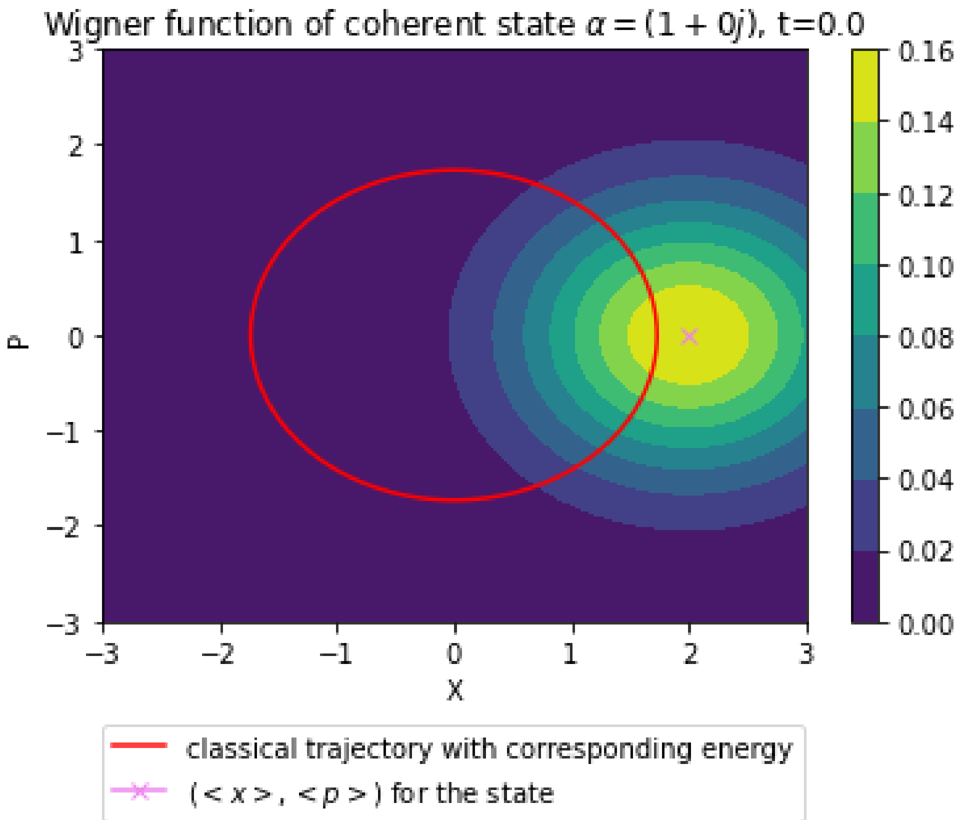

Wigner function of state |0>+|1>, t=1.1
P
X
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x > , < p >) for the state

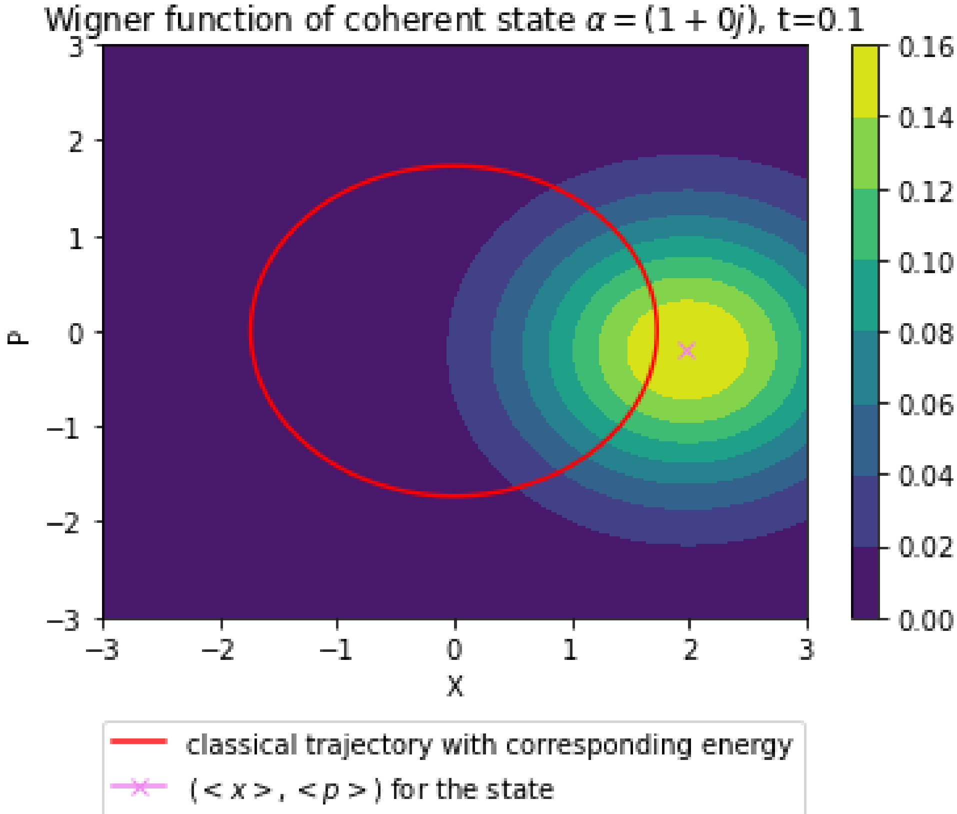

Wigner function of state |0>+|1>, t=1.2
P
X
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x >, < p >) for the state

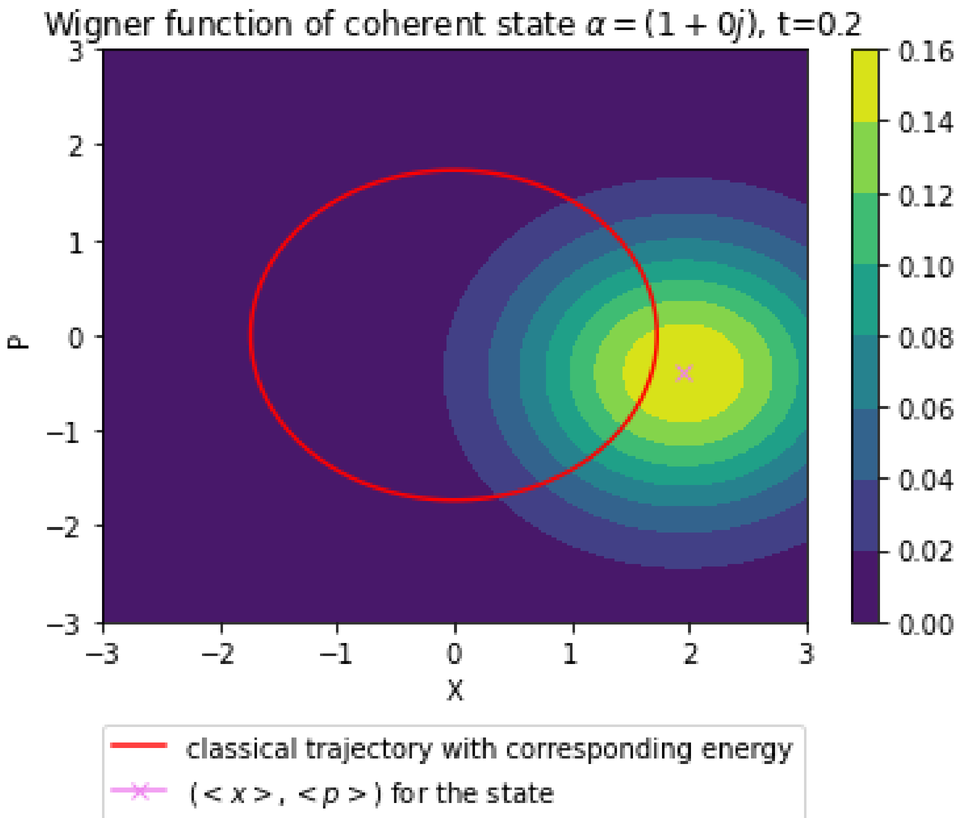

Wigner function of state |0>+|1>, t=1.3
P
X
3
2
1
0
−1
−2
−3
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x > , < p >) for the state

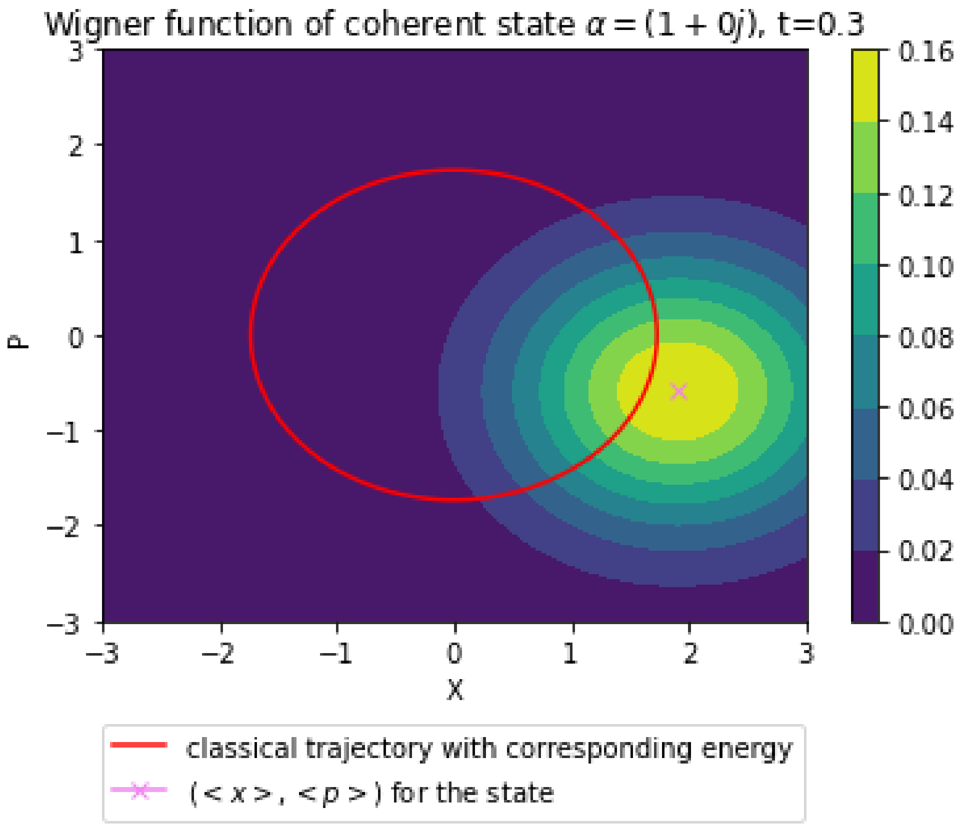


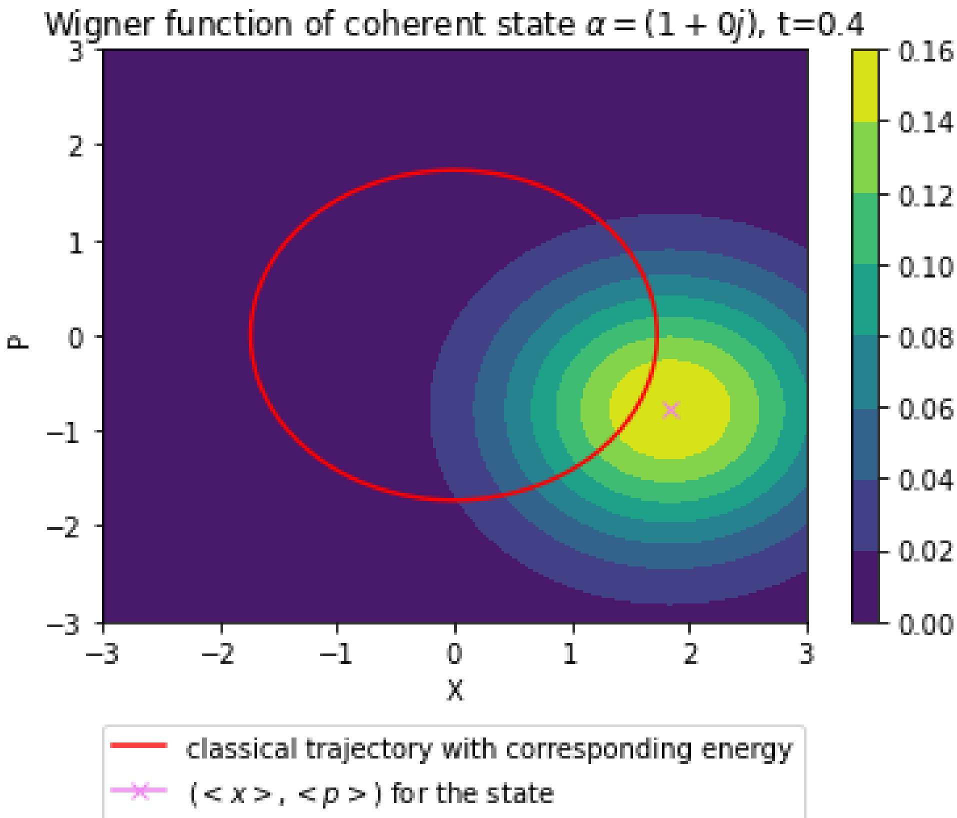
Wigner function of state |0>+|1>, t=1.4
P
X
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
0.15
0.12
0.09
0.06
0.03
0.00
−0.03
−0.06
classical trajectory with corresponding energy
(< x >, < p >) for the state


Wigner function of coherent state $\alpha = (1 + 0j)$, t=0.0
P
X
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state

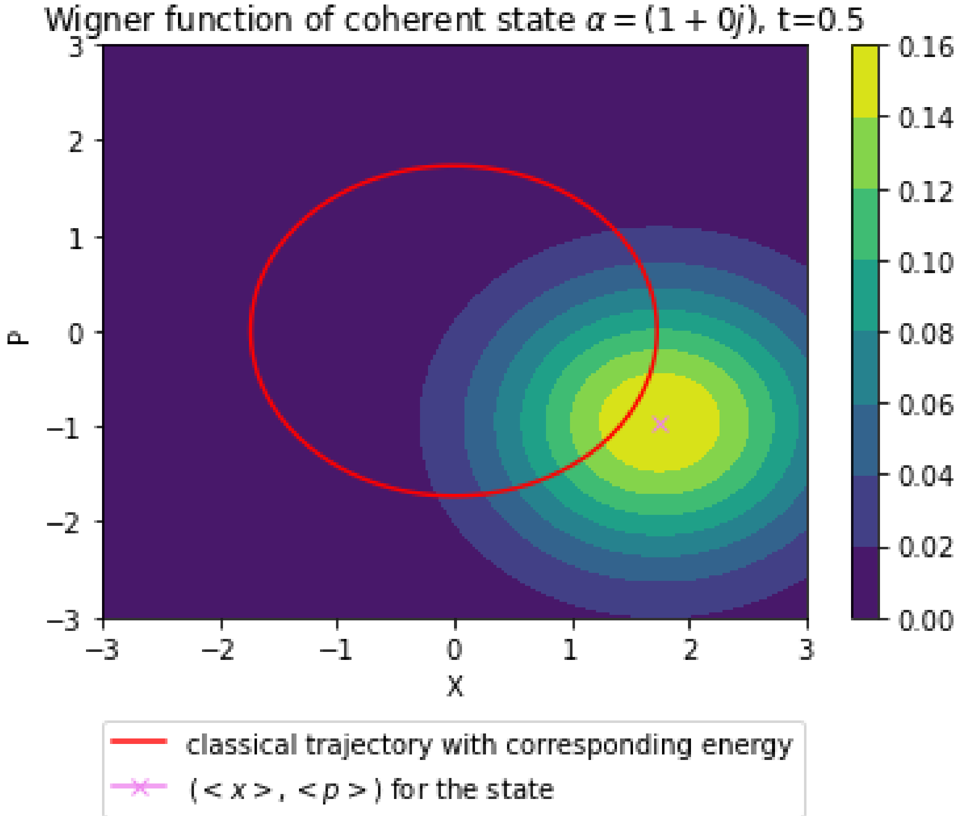

Wigner function of coherent state $\alpha = (1 + 0j)$, t=0.1
P
X
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state

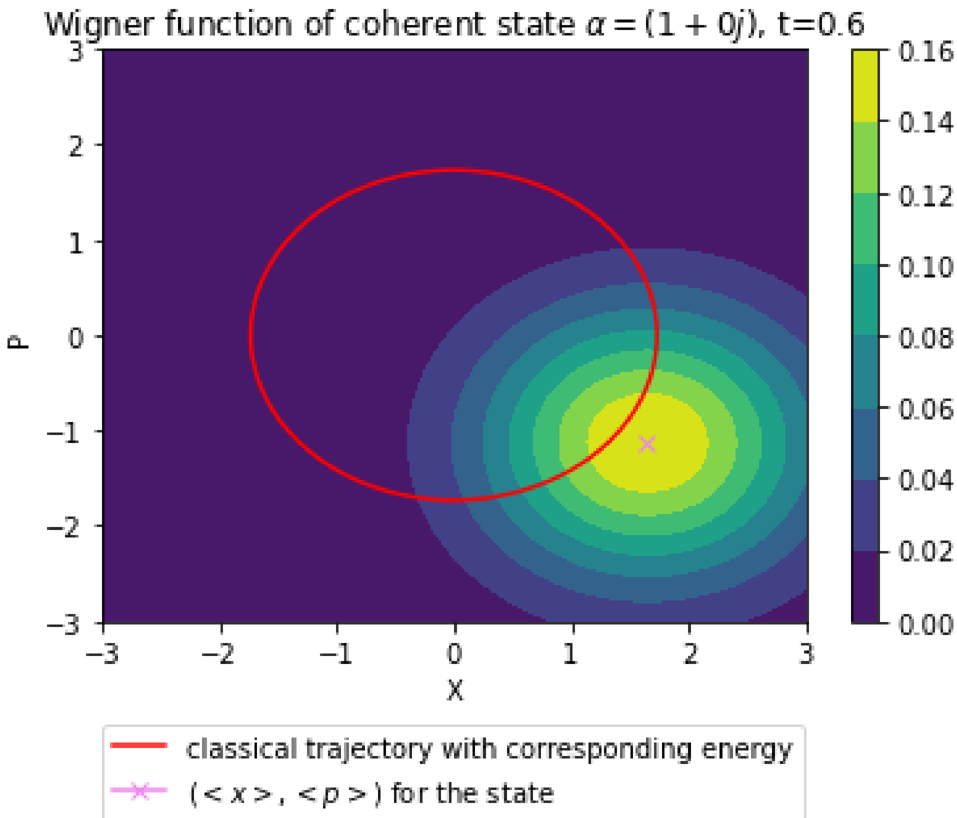

Wigner function of coherent state $\alpha = (1 + 0j)$, t=0.2
P
X
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state

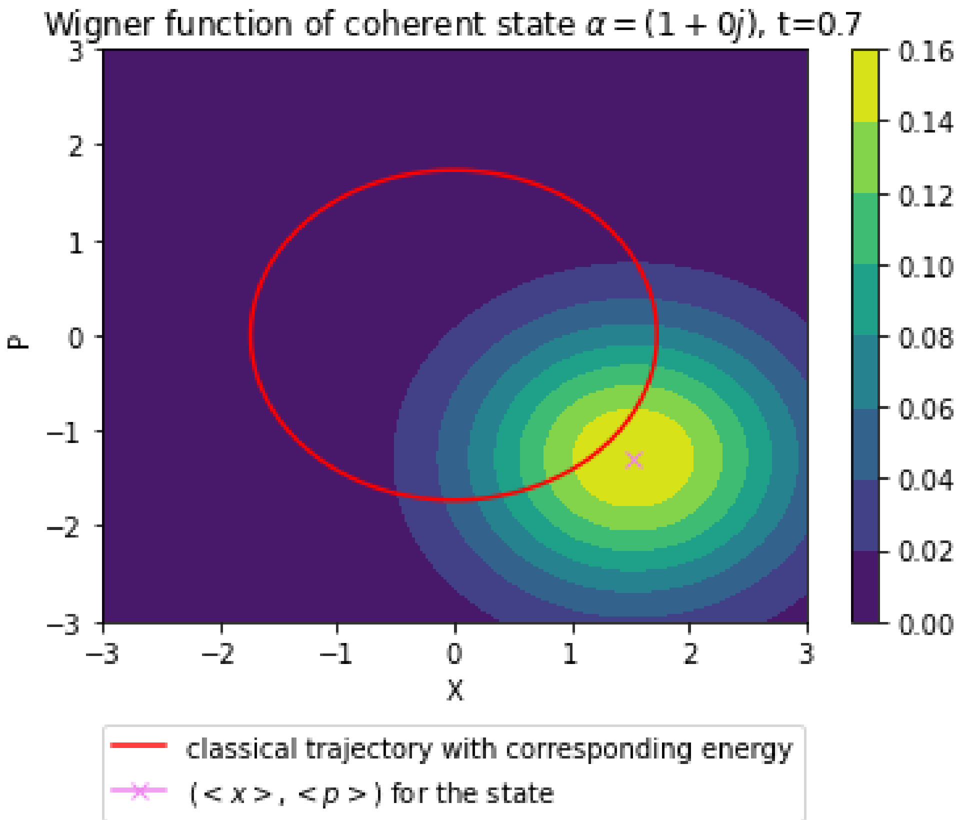

Wigner function of coherent state $\alpha = (1 + 0j)$, t=0.3
P
X
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state



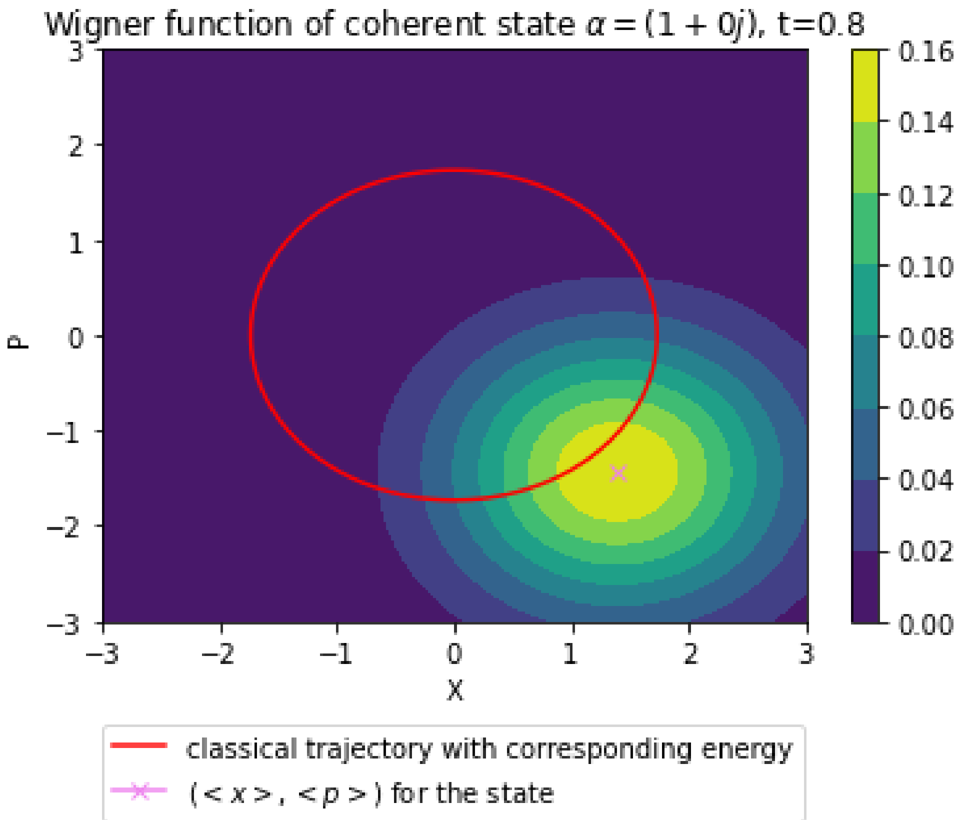
Wigner function of coherent state $\alpha = (1 + 0j)$, t=0.4
P
X
−3
−2
−1
0
1
2
3
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state


Wigner function of coherent state $\alpha = (1 + 0j)$, t=0.5
P
X
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state

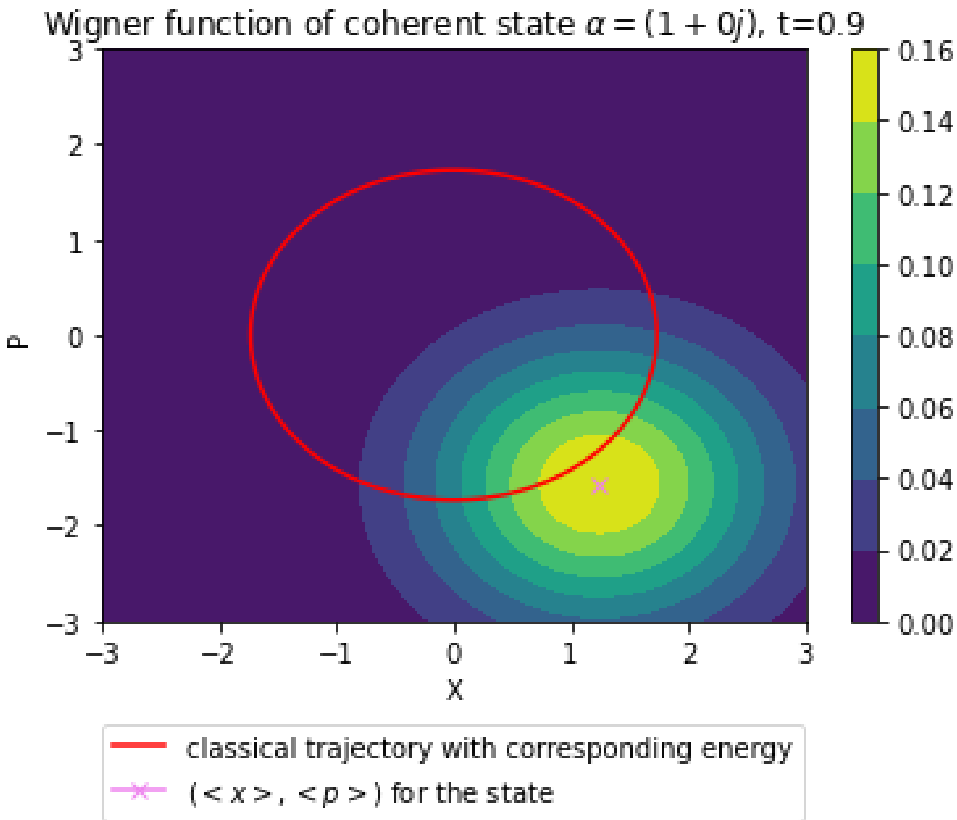

Wigner function of coherent state $\alpha = (1 + 0j)$, t=0.6
P
X
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state



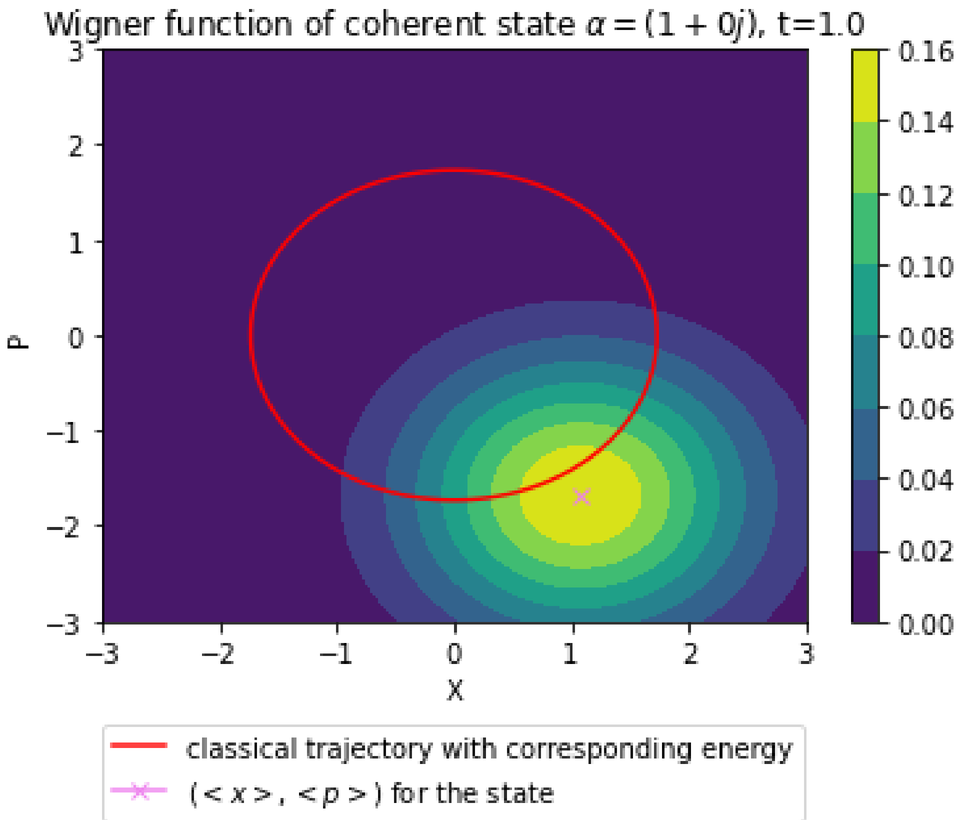
Wigner function of coherent state $\alpha = (1 + 0j)$, t=0.7
P
X
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
classical trajectory with corresponding energy
$(< x >, < p >)$ for the state

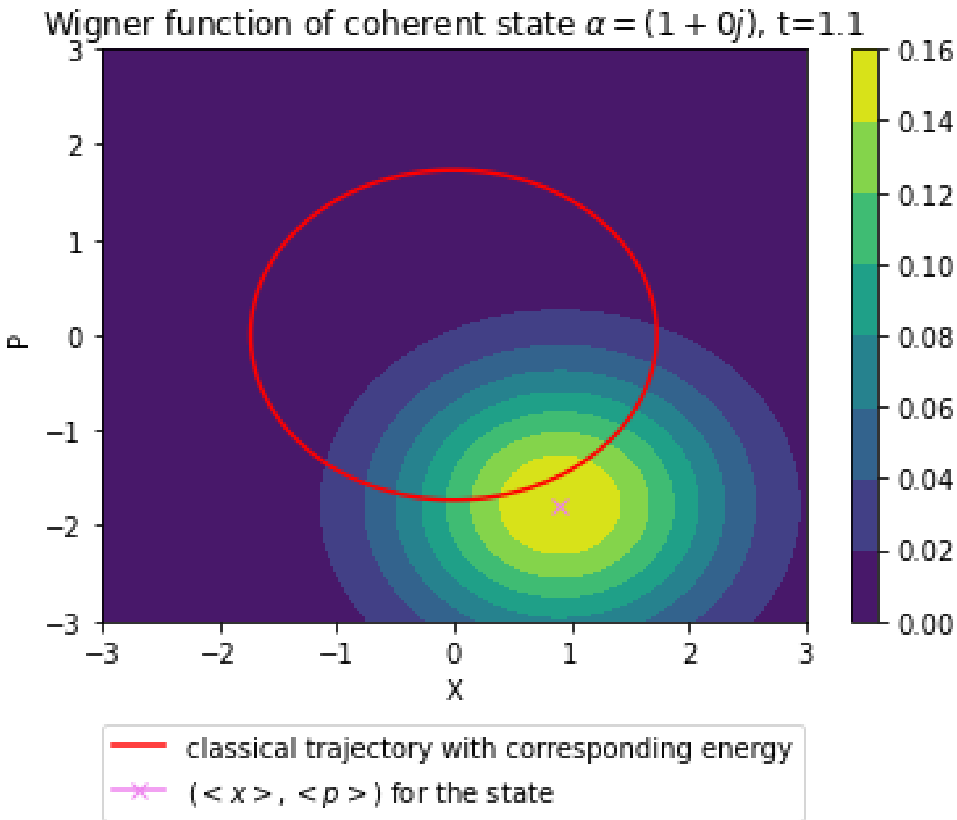


Wigner function of coherent state $\alpha = (1 + 0j)$, t=0.8
P
X
−3
−2
−1
0
1
2
3
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state

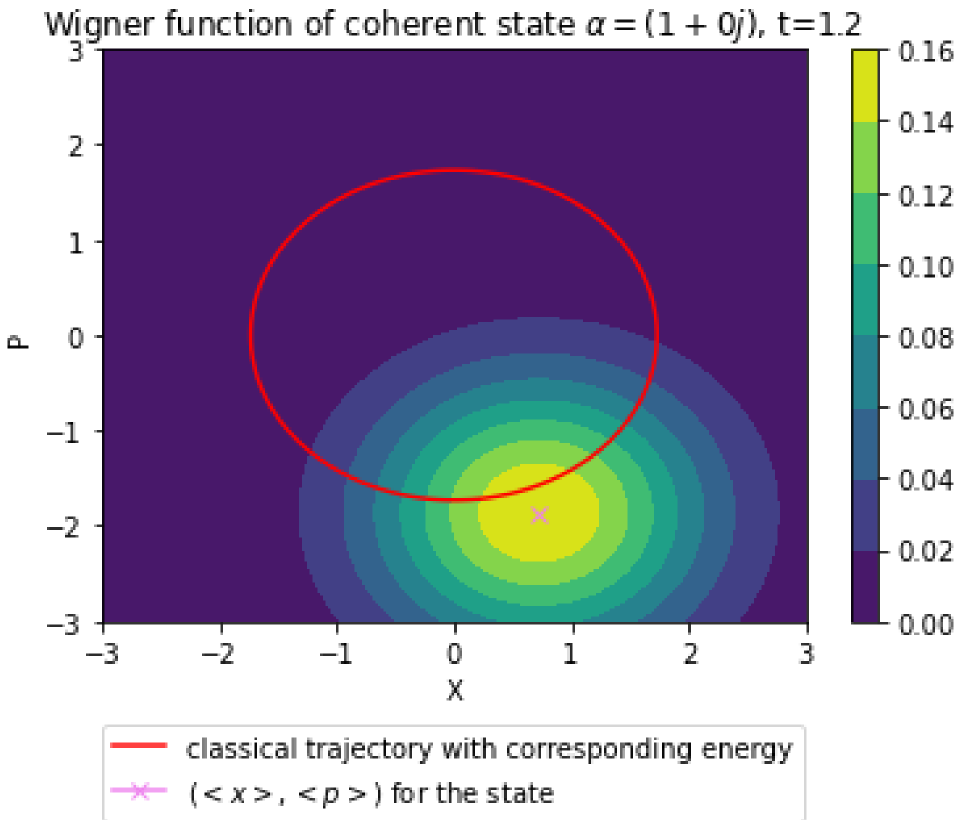


Wigner function of coherent state $\alpha = (1 + 0j)$, t=0.9
P
X
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state

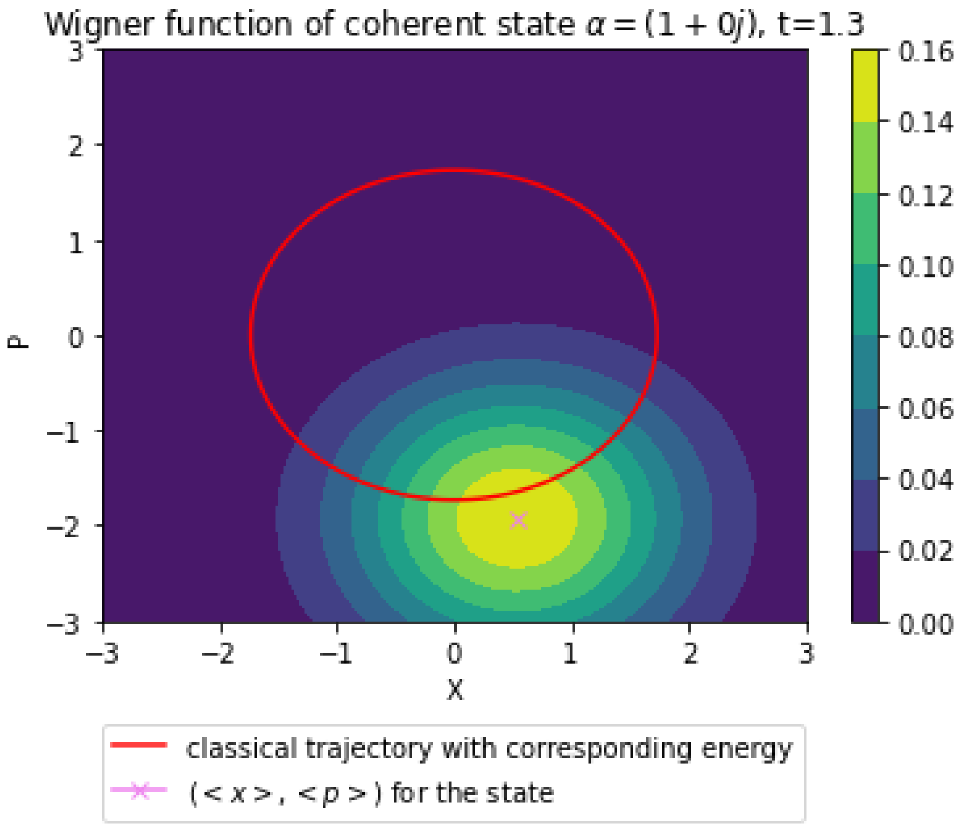

Wigner function of coherent state $\alpha = (1 + 0j)$, t=1.0
P
X
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state



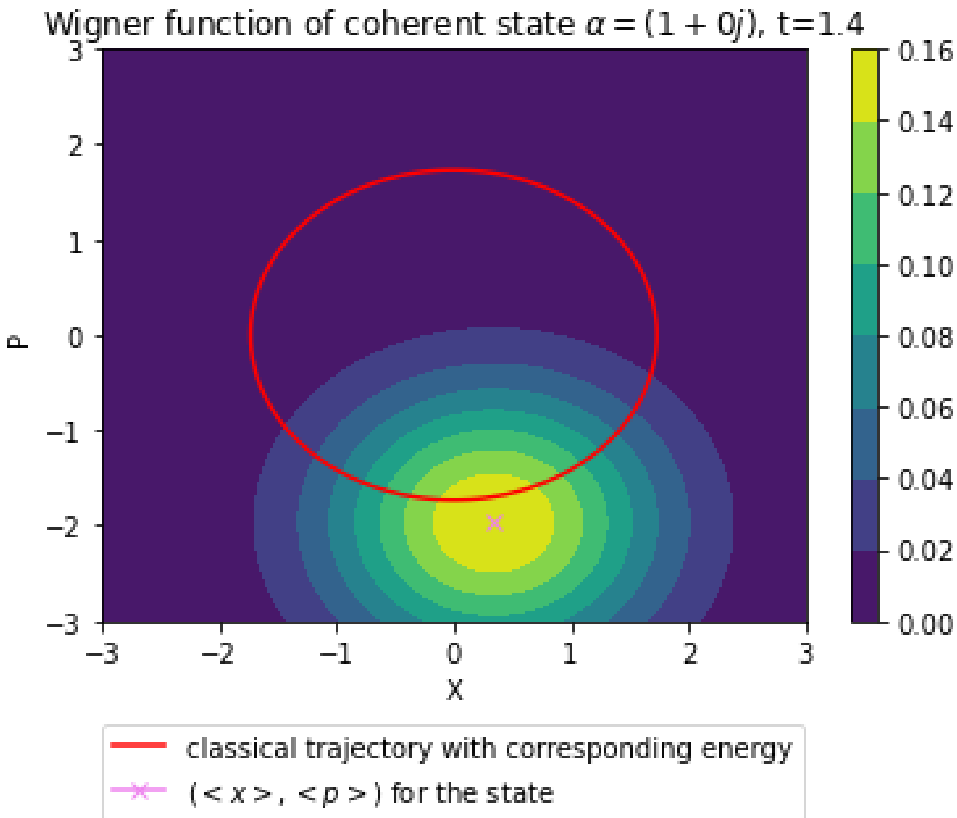
Wigner function of coherent state $\alpha = (1 + 0j)$, t=1.1
P
X
-3
-2
-1
0
1
2
3
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state



Wigner function of coherent state $\alpha = (1 + 0j)$, t=1.2
P
X
classical trajectory with corresponding energy
(< x > , < p >) for the state



Wigner function of coherent state $\alpha = (1 + 0j)$, t=1.3
P
X
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state



Wigner function of coherent state $\alpha = (1 + 0j)$, t=1.4
P
X
−3
−2
−1
0
1
2
3
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
classical trajectory with corresponding energy
$(<x>, <p>)$ for the state

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.lines as mlines
from scipy import special

cos = np.cos
sin = np.sin
pi = np.pi
sqrt = np.sqrt
exp = np.exp


# define the frequency of the oscillator
omega = 1
# we use hbar=m=omega=1


def genNumberState(n):
    # generates the normalized |n> wavefunction
    # in position space for the harmonic oscillator
    def n_(x,t=0):
        h = special.hermite(n)
        h_ = h(x/sqrt(2))
        leading = 1/sqrt(  2**(n+1/2) * sqrt(pi)  * np.math.
    factorial(n) )
        e = exp(-x**2/4)
        phase = exp(-1j * t * omega*(n+1/2))
        return phase*leading*h_*e
    return n_


def genWignerFcn(wf):
    # generates the Wigner function based on the
    # wavefunction of a pure state
    # input is a function
    def wigner(x,p):
        accum = 0
        yy = np.linspace(-10, 10, 1000)
        dy = yy[1]-yy[0]
        for i in range(len(yy)):
            y = yy[i]
```

```
            term = dy*exp( -1j*p*y/2 ) * wf(x+y/2) *  np.conjugate(wf(x-y/2))  /(4*pi)
            accum += term
        return accum
    return wigner

def energy(x,p):
    return p**2/2+x**2/2
    #return p**2/4+x**2/4

def expval_position(wf):
    # takes in a wavefunction, returns position expectation
    accum = 0
    xx = np.linspace(-5,5,1000)
    dx = xx[1]-xx[0]
    for i in range(len(xx)):
        #accum+= dx* np.conjugate(wf(xx[i])) * xx[i] * wf(xx[i])
        accum+= dx* np.conjugate(wf(xx[i])) * xx[i] * wf(xx[i])
    return accum.real

def expval_momentum(wf):
    # takes in a wavefunction, returns momentum expectation
    accum = 0
    xx = np.linspace(-5,5,1000)
    dx = xx[1]-xx[0]
    for i in range(len(xx)-1):
        accum+= dx* np.conjugate(wf(xx[i])) * (-1j) * (wf(xx[i+1]) - wf(xx[i]))/dx
    print(accum)
    return 2*accum.real


def genCoherent(alpha,tt):
    # alpha complex
    # generates the wf for coherent state |alpha>
    def f_(x,t=tt):
        accum = 0
        for i in range(20):
            wf = genNumberState(i)
```

```python
            accum += exp(-np.abs(alpha)/2) * wf(x,t) * (alpha**i) / sqrt(np.math.factorial(i))
        return accum
    return f_



def run(state, alpha=None):
    xx = np.linspace(-3,3,100)
    pp = np.linspace(-3,3,100)
    XX,PP = np.meshgrid(xx,pp)
    ENERGY = energy(XX,PP)

    if state == "01":
        for tt in range(0,15):
            wf0 = genNumberState(0)
            wf1 = genNumberState(1)
            def wf(x,t=tt*0.1):
                return (wf0(x,t) + wf1(x,t))/sqrt(2)
            wigner01 = genWignerFcn(wf)
            W01_complex = wigner01(XX,PP)
            W01 = W01_complex.real

            this_energy = 0.5*(omega*(0+1/2) + omega*(1+1/2))
            C = plt.contour(XX,PP, ENERGY, levels=[this_energy], colors="r")
            plt.clabel(C, levels=[], inline=False)
            labels = ["classical trajectory with corresponding energy"]
            for i in range(len(labels)):
                C.collections[i].set_label(labels[i])

            plt.contourf(XX, PP, W01, cmap='viridis')  # 'viridis' is just an example colormap; you can choose any other
            plt.colorbar()

            x,p = expval_position(wf), expval_momentum(wf)
            expval_xp, = plt.plot(x,p, color='violet', marker='x', label="$(<x>, <p>)$ for the state")

```

```
            # Add labels and a title
            plt.xlabel('X')
            plt.ylabel('P')
            plt.title(f"Wigner function of state |0>+|1>, t={round(tt*0.1,2)}")

            red_line = mlines.Line2D([],[],color="red", label = "classical trajectory with corresponding energy")
            lgd = plt.legend(handles=[red_line, expval_xp], loc=(0,-0.35))

            # Show the plot
            plt.savefig(f"01_t={round(tt*0.1,2)}.png", format="png", bbox_extra_artists=(lgd,), bbox_inches='tight')
            plt.show()

    elif state=="coherent":
        this_energy = 0
        for i in range(0,20):
            this_energy += omega*(i+1/2) *exp(-np.abs(alpha)**2) * np.abs(alpha**i)**2 / np.math.factorial(i)

        for tt in range(0,15):
            coherent = genCoherent(alpha, tt*0.1)
            wignerC = genWignerFcn(coherent)
            WC_complex = wignerC(XX,PP)
            WC = WC_complex.real


            C = plt.contour(XX,PP, ENERGY, levels=[this_energy], colors="r")
            plt.clabel(C, levels=[], inline=False)
            labels = ["classical trajectory with corresponding energy"]
            for i in range(len(labels)):
                C.collections[i].set_label(labels[i])

            plt.contourf(XX, PP, WC, cmap='viridis')  # 'viridis' is just an example colormap; you can choose any other
            plt.colorbar()
```

```python

            x,p = expval_position(coherent), expval_momentum(coherent)
            expval_xp, = plt.plot(x,p, color='violet', marker='x', label="$(<x>, <p>)$ for the state")

            # Add labels and a title
            plt.xlabel('X')
            plt.ylabel('P')
            plt.title(f"Wigner function of coherent state $\\alpha={alpha}$, t={round(tt*0.1,2)}")

            red_line = mlines.Line2D([],[],color="red", label = "classical trajectory with corresponding energy")
            lgd = plt.legend(handles=[red_line, expval_xp], loc=(0,-0.35))

            # Show the plot
            plt.savefig(f"coherent_t={round(tt*0.1,2)}.png", format="png", bbox_extra_artists=(lgd,), bbox_inches='tight')
            plt.show()

    else:
        ket0 = genNumberState(state)
        wigner0 = genWignerFcn(ket0)
        W0_complex = wigner0(XX,PP)
        W0 = W0_complex.real


        this_energy = omega*(state+1/2)
        C = plt.contour(XX,PP, ENERGY, levels=[this_energy], colors="r")
        plt.clabel(C, levels=[], inline=False)
        labels = ["classical trajectory with corresponding energy"]
        for i in range(len(labels)):
            C.collections[i].set_label(labels[i])


```

```
        plt.contourf(XX, PP, W0, cmap='viridis')  # 'viridis'
    is just an example colormap; you can choose any other
        plt.colorbar()

        x,p = expval_position(ket0), expval_momentum(ket0)
        expval_xp, = plt.plot(x,p, color='violet', marker='x',
    label="$(<x>, <p>)$ for the state")


        # Add labels and a title
        plt.xlabel('X')
        plt.ylabel('P')
        plt.title(f"Wigner function of state |{state}>")
        red_line = mlines.Line2D([],[],color="red", label = "
    classical trajectory with corresponding energy")
        lgd = plt.legend(handles=[red_line, expval_xp], loc
    =(0,-0.35))

        # Show the plot
        plt.savefig(f"{state}.png", format="png",
    bbox_extra_artists=(lgd,), bbox_inches='tight')
        plt.show()



# main
run(0)
run(1)
run("01")
run("coherent", alpha=1+0*1j)


"""
xx = np.linspace(-5,5,10000)

ket0 = genNumberState(0)
wf0 = ket0(xx)

ket1 = genNumberState(1)
wf1 = ket1(xx)

```

```
ket2 = genNumberState(2)
wf2 = ket2(xx)


plt.plot(xx,np.abs(wf0)**2, label="|0>")
plt.plot(xx,np.abs(wf1)**2, label="|1>")
plt.plot(xx,np.abs(wf2)**2, label="|2>")
plt.legend(loc=(1.04, 0))

'''
# test normalization
accum = 0
dx = xx[1]-xx[0]
for i in range(len(xx)):
    accum += dx*(np.abs(wf0[i])**2)
print(accum)
'''
"""


"""
# coherent state test
xx = np.linspace(-5,5,10000)

for tt in range(10):
    coherent = genCoherent(1+0*1j, tt*0.5)
    this_energy = 0
    for i in range(0,20):
        this_energy += omega*(i+1/2) *exp(-np.abs(1+0*1j)**2) * np.abs((1+0*1j)**i)**2 / np.math.factorial(i)
    #plt.plot(xx, np.abs(coherent(xx))**2)
    #print(tt, expval_position(coherent), expval_momentum(coherent))
'''
accum = 0
dx = xx[1]-xx[0]
for i in range(len(xx)):
    accum += dx*(np.abs(coherent(xx[i]))**2)
print(accum)
'''
"""
```

6 Appendix: The Big Beautiful Sheet of Quantum Optics Quadrature Conventions

The standard harmonic oscillator with dynamical variables $\{q, p\}$ as position and momentum is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2, \ \ [\hat{q}, \hat{p}] = i\hbar$$

The zero point fluctuations are

$$q_{zpf} \equiv \sqrt{\frac{\hbar}{2\omega m}}, \quad p_{zpf} \equiv \sqrt{\frac{\hbar\omega m}{2}}$$

Define **quadrature operators** as

$$\hat{X} \equiv \frac{\hat{q}}{q_{zpf}}, \quad \hat{P} \equiv \frac{\hat{p}}{p_{zpf}}$$

The standard results become

$$[\hat{X}, \hat{P}] = 2i, \quad \Delta X \Delta P \geq 1$$

Ladder operators are

$$\hat{X} = \hat{a}^\dagger + \hat{a}, \quad \hat{P} = i(\hat{a}^\dagger - \hat{a}), \quad [\hat{a}, \hat{a}^\dagger] = 1$$
$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle, \ \ \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

Hamiltonian becomes

$$\hat{H} = \frac{\hbar\omega}{4}(\hat{X}^2 + \hat{P}^2) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

The eigens of $\hat{X}|x\rangle = x|x\rangle$ and $\hat{P}|p\rangle = p|p\rangle$ are normalized as

$$\langle x|p\rangle = \frac{1}{\sqrt{4\pi}}e^{ixp/2}$$
$$|p\rangle = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{4\pi}}e^{ixp/2}|x\rangle, \ \ |x\rangle = \int_{-\infty}^{\infty}\frac{dp}{\sqrt{4\pi}}e^{-ixp/2}|p\rangle$$

Delta function integrals:

$$\int_{\mathbb{R}}\frac{dx}{2\pi}e^{ixp} = \delta(p), \ \ \int_{\mathbb{R}}\frac{dx}{4\pi}e^{ixp/2} = \delta(p)$$