HOMEWORK 1: Math review

Reference: Lectures 1-7 **Due: 2018 September 12**

This problem set is intended as a review of some mathematics useful for solving inverse problems. The problems should be fairly simple but may require you to review some basic concepts in linear algebra. Some problems will also require you to become proficient with a program to perform numerical computations (probably Python or Matlab). If you are not familiar with such a program, now is the time to become so!

You are encouraged to consult with other students on this and any of the problem sets (not on exams, though). However, you should always hand in your own unique version of the solutions.

Problem 1. Consider a mathematical model of the form $G(m) = d$, where m is a column vector of length m and d is a column vector of length n. Suppose the model G is linear. Show that $G(m)$ can be written as Γm, where Γ is an n×m matrix. Relate the elements of the matrix Γ to the function G. Hint: start be expanding the vector m in terms of the standard basis (i.e. the set of vectors with one nonzero element and all the rest zeros). Use the superposition and scaling laws. Finally, recall the definition of matrix-vector multiplication, and identify the elements of Γ.

Problem 2. This problem concerns the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 3 \\ 4 & 6 & 7 & 11 \end{pmatrix}$$

Find the nullspace and range (column space) of the matrix and also of its transpose. What are the dimensions of these four subspaces? Hint: It's easiest to begin by reducing the matrix to reduced row echelon form (RREF). The nullspace is made up of vectors x that solve $Ax = 0$. The column space is made up of the linearly independent columns of A (the pivot columns). The sum of the dimensions of the column space and nullspace is equal to the number of columns. The nullspace of the transpose of A is called the left nullspace of A, and the range of the transpose of A is called the row space of A.

Also work this simple problem: If the nonsingular matrix A can be diagonalized as $P\Lambda P^{-1}$, what is the diagonalization of A^{-1}. The answer should be quite brief.

Problem 3. Overdetermined problems generally have no solution. Can you think of a situation where an overdetermined problem has one solution? How about many solutions? Likewise, underdetermined problems generally have multiple solutions. Can you think of a circumstance when an underdetermined problem has one solution? No solutions?

Problem 4. Write a program to begin to investigate the vertical seismic profiling problem. Take the depth of the bore hole to be 40 m. Take the slowness to increase linearly with depth, from 2 s/km at the top of the borehole to 5 s/km at the bottom. Between 3/8 and 5/8 of the way down, however, let the slowness fall to 2 s/km to make things interesting.

Let the model of the slowness be specified on 80 points, uniformly distributed from the top to the bottom of the borehole, i.e. $m = 80$. Consider two cases, where the data are uniformly distributed and even-determined ($n = 80$) and overdetermined ($n = 160$).

a) Generate a model vector representing the true slowness.

b) Generate the G matrix that relates these to the data vector.

c) Predict the data vector.

d) Now, add normally distributed noise, $\mathcal{N}(0, 0.1^2)$ (zero mean, standard deviation 0.1 ms) to the data. Since the data are numbers of the order of 100 ms, this corresponds to something like only 0.1% relative RMS error.

e) Recover the model slowness two ways, using direct differentiation of the data and using the minimum length inverse. Compare the results for the two cases and comment.

Be sure to plot the results with descriptive labels where appropriate and include your program listing, along with a description of what's going on, in your solution. Try experimenting with different model/data sizes and noise levels. Retain your code, as you'll build on it later.

