Exact Bayesian solution of COR model

The weights for the different copies of y are decoupled. More precisely, the prior, likelihood, and hence posterior all factor into separate functions, one for each set of weights. Therefore we can solve the posterior separately for each set. Without loss of generality, we consider copy number 1, and we suppress this index.

The prior is given by

$$p(w) \propto \exp\left(-\tfrac{1}{2} w' \Sigma w\right)$$

where $\Sigma_{11} = 0$, $\Sigma_{ii} = \frac{1}{\eta^2}$ for $i > 1$, and $\Sigma_{ij} = 0$ for $i \neq j$. (Note that this is an improper prior, but it causes no problems with inference.)

The likelihood is given by

$$p(y|x, w) \propto \exp\left(-\frac{(xw - y)'(xw - y)}{2\sigma^2}\right).$$

Therefore the posterior is given by

$$\begin{aligned} p(w|x,y) &\propto \exp\left(-\tfrac{1}{2} w' \left(\Sigma + \tfrac{1}{\sigma^2} x'x\right) w + \tfrac{1}{\sigma^2} w'x'y\right) \\ &\propto \exp\left(-\tfrac{1}{2}\left(w - \left(\sigma^2\Sigma + x'x\right)^{-1} x'y\right)' \left(\Sigma + \tfrac{1}{\sigma^2} x'x\right)\left(w - \left(\sigma^2\Sigma + x'x\right)^{-1} x'y\right)\right) \\ &= \exp\left(-\tfrac{1}{2}\left(w - (\Lambda + x'x)^{-1} x'y\right)' \tfrac{1}{\sigma^2}(\Lambda + x'x)\left(w - (\Lambda + x'x)^{-1} x'y\right)\right) \end{aligned}$$

where $\Lambda_{11} = 0$, $\Lambda_{ij} = 0$ for $i \neq j$, and $\Lambda_{ii} = \frac{\sigma^2}{\eta^2} = \lambda$ (the penalty parameter) for $i > 1$.

Thus the posterior is a multivariate Gaussian with mean/median/mode at

$$w^* = (\Lambda + x'x)^{-1} x'y$$

and covariance matrix equal to

$$\sigma^2 (\Gamma + x'x)^{-1}.$$