Homework 6 Handed out: Wednesday, October 19, 2022
Due: Wednesday, October 26, 2022 by 11:59pm

Material covered: Outcomes 5.1–5.3.

1. Let $\alpha \in \mathbb{C}$ be a complex number and let

$$f\colon U \to \mathbb{C}, \quad f(z) = (1+z)^\alpha = e^{\alpha \operatorname{Log}(1+z)}$$

where $U = \mathbb{C} \setminus \{x + iy \in \mathbb{C} \mid y = 0, x \leq -1\}$. Determine the radius of convergence of the Taylor series of f around $z = 0$ and find the first four terms.

You may assume that α is not a positive integer or zero.

2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n.$$

Compute the following:

a) $f^{(4)}(0)$

b) $\displaystyle\int_C \frac{f(z)}{z^6}\, \mathrm{d}z$ where C is the unit circle.

3. Find the Laurent series expansion of the function

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

on each of the following annuli:

a) $D = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$

b) $D = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$

c) $D = \{z \in \mathbb{C} \mid |z| > 2\}$

4. For each of the following functions f, identify the singular points and classify them into removable singularities, poles and essential singularities. Calculate the residue at each singular point and determine the order of each pole.
   a) $f(z) = \dfrac{1}{z^4+1}$
   b) $f(z) = \dfrac{\cos(z)-1}{z^2}$
   c) $f(z) = z^3 e^{\frac{1}{z^2}}$
   d) $f(z) = \dfrac{\sin(z)}{(z-\pi)^4}$

5. Let
$$f(z) = \frac{e^{iz}}{(1+z^2)^2}$$
For $R > 0$, compute the contour integral
$$\int_{C_R} f(z)\,\mathrm{d}z$$
where C_R is the contour composed of a line segment from $-R$ to R and a semicircle of radius R in the upper half-plane, traversed counter-clockwise (shown below). How does the answer depend on R?