Section 5 — Friday, October 7, 2022

Material covered: Outcomes 4.1–4.4.

Solutions

1. Let $f\colon \mathbb{C}\setminus\mathbb{R}_- \to \mathbb{C}$ be the principal value of the square root,

$$f(z) = z^{\frac{1}{2}} = e^{\frac{1}{2}\operatorname{Log} z}.$$

   a) Show that the function F defined by

$$F\colon \mathbb{C}\setminus\mathbb{R}_- \to \mathbb{C}, \quad F(z) = \frac{2}{3} z^{3/2}$$

   is an antiderivative for f on $\mathbb{C}\setminus\mathbb{R}_-$.

   Compute the integral

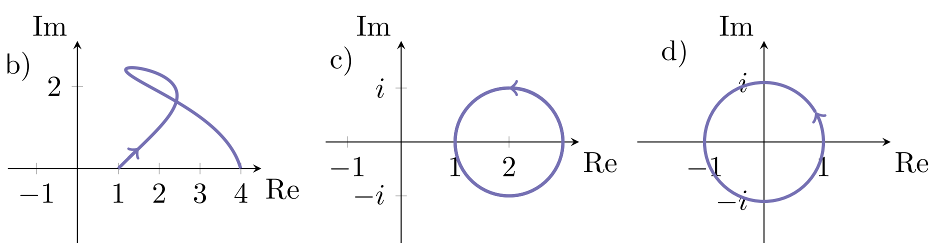
$$\int_C f(z)\,\mathrm{d}z$$

   for each of the following contours C (pictured below):

   b) C is the curve from 1 to 4 shown in figure (b),

   c) $C = C(2,1)$ is the circle with center 2 and radius 1, traversed counter-clockwise.

   d) $C = C(0,1)$ is the circle with center 0 and radius 1, traversed counter-clockwise.



Solution.
a) By definition,

$$F(z) = \frac{2}{3} e^{\frac{3}{2}\operatorname{Log} z}$$

so by the chain rule, F is analytic on $U \coloneqq \mathbb{C} \setminus \mathbb{R}_-$ (where Log is analytic) with derivative

$$\begin{aligned} F'(z) &= \frac{2}{3}\exp'\left(\frac{3}{2}\operatorname{Log} z\right)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{3}{2}\operatorname{Log} z\right) = \frac{2}{3}e^{\frac{3}{2}\operatorname{Log} z} \cdot \frac{3}{2z} \\ &= e^{\frac{3}{2}\operatorname{Log} z} e^{-\operatorname{Log} z} = e^{\frac{1}{2}\operatorname{Log} z} = f(z). \end{aligned}$$

This shows that F is an antiderivative of f on U.
b) Since the curve C is contained in U, the integral can be easily found using the antiderivative F:

$$\int_C f(z)\,\mathrm{d}z = F(4) - F(1) = \frac{3}{2}4^{2/3} - \frac{3}{2}1^{2/3} = \frac{14}{3}.$$

c) C is a simple closed contour and both C and its interior are contained in the open set U where f is analytic. It follows from the Cauchy integral theorem that

$$\int_C f(z)\,\mathrm{d}z = 0.$$

d) C is a simple closed contour but its interior is not contained in U, so we cannot use the Cauchy integral theorem. We'll show two ways to evaluate this integral. Firstly, we can compute the integral directly by parameterizing the contour:

$$\int_C f(z)\,\mathrm{d}z = \int_0^{2\pi} f(e^{it}) i e^{it}\,\mathrm{d}t = \int_0^{2\pi} \exp\left(\frac{1}{2}\operatorname{Log}(e^{it})\right) i e^{it}\,\mathrm{d}t$$

Now we must be careful to evaluate $\operatorname{Log}(e^{it})$ correctly. For $0 \le t < \pi$ we have

$$\operatorname{Log}(e^{it}) = it.$$

For $\pi \le t \le 2\pi$, we note that $t - 2\pi$ is between $-\pi$ and π so

$$\operatorname{Log}(e^{it}) = \operatorname{Log}(e^{i(t-2\pi)}) = i(t - 2\pi).$$

Thus the integrand is

$$\exp\Big(\frac{1}{2}\operatorname{Log}(e^{it})\Big)ie^{it} = \begin{cases} ie^{i\frac{3}{2}t}, & 0 \le t < \pi \\ -ie^{i\frac{3}{2}t}, & \pi < t < 2\pi \end{cases}.$$

Now we can compute:

$$\int_C f(z)\,\mathrm{d}z = \int_0^{\pi} ie^{i\frac{3}{2}t}\,\mathrm{d}t - \int_{\pi}^{2\pi} ie^{i\frac{3}{2}t}\,\mathrm{d}t = \left[\frac{2}{3}e^{i\frac{3}{2}t}\right]_0^{\pi} - \left[\frac{2}{3}e^{i\frac{3}{2}t}\right]_{\pi}^{2\pi} = -\frac{4}{3}i.$$

Note: We could have saved ourselves some work by parameterizing the contour differently, with $t \in [-\pi, \pi]$:

$$\int_C f(z)\,\mathrm{d}z = \int_{-\pi}^{\pi} f(e^{it})ie^{it}\,\mathrm{d}t = \int_{-\pi}^{\pi} e^{\frac{1}{2}it}ie^{it}\,\mathrm{d}t$$

An alternative way to compute the integral is using the antiderivative F. We have

$$\int_C f(z)\,\mathrm{d}z = \lim_{t \nearrow \pi} F(e^{it}) - \lim_{t \searrow -\pi} F(e^{it}).$$

In the first term, we are approaching the point -1 from the upper half-plane, where

$$\lim_{t\nearrow\pi} F(e^{it}) = \lim_{t\nearrow\pi} \frac{2}{3}e^{\frac{3}{2}\operatorname{Log}(e^{it})} = \frac{2}{3}e^{\frac{3}{2}i\pi} = -\frac{2}{3}i.$$

For the second term, we are approaching -1 from the lower half-plane, so

$$\lim_{t\searrow-\pi} F(e^{it}) = \lim_{t\searrow-\pi} \frac{2}{3}e^{\frac{3}{2}\operatorname{Log}(e^{it})} = \frac{2}{3}e^{\frac{3}{2}(-i\pi)} = \frac{2}{3}i.$$

Combining these results, we have

$$\int_C f(z)\,\mathrm{d}z = -\frac{4}{3}i.$$

2. Compute the integral

$$\int_{C(0,2)} \frac{z^2+3z+2}{z^2-1}\,\mathrm{d}z$$

where $C(0,2)$ denotes the circle with radius 2 centered at the origin.

Solution.
We start by factoring the numerator and denominator:

$$\frac{z^2+3z+2}{z^2-1} = \frac{(z+1)(z+2)}{(z+1)(z-1)}.$$

We see that $z=-1$ is a removable singularity. For all $z \neq 1$,

$$\frac{z^2+3z+2}{z^2-1} = \frac{(z+2)}{(z-1)}.$$

Since the contour encircles the point 1, Cauchy's integral formula with $f(z) = z+2$ gives

$$\int_C \frac{z^2+3z+2}{z^2-1}\,\mathrm{d}z \int_C = \frac{f(z)}{(z-1)}\,\mathrm{d}z = 2\pi i f(1) = 6\pi i.$$

3. For each integer $n \in \mathbb{Z}$, compute the integral

$$I_n = \int_C \frac{e^z}{z^{n+1}}\,\mathrm{d}z.$$

where $C = C(0,1)$ is the unit circle centered at the origin.

Solution.
For integers $n \leq -1$, the integrand is holomorphic on $\mathbb{C}$ so

$$\int_C \frac{e^z}{z^{n+1}}\,\mathrm{d}z = 0, \qquad n \leq -1$$

by the Cauchy integral theorem. For $n \geq 0$, we use Cauchy's integral formula with $f(z) = e^z$:

$$\int_C \frac{e^z}{z^{n+1}}\,\mathrm{d}z = \frac{2\pi i}{n!} f^{(n)}(0) = \frac{2\pi i}{n!}, \qquad n \geq 0.$$

4. Compute the integral

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{3 + \sin\theta}$$

by converting it to a contour integral over the unit circle. *Hint:* First show that if $z = \gamma(\theta)$ is a point on the unit circle parameterized by $\gamma(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$ then

$$\cos(\theta) = \frac{z + 1/z}{2}, \quad \sin(\theta) = \frac{z - 1/z}{2i}, \quad \mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta.$$

Solution.

If $z = \gamma(\theta) = e^{i\theta}$ is a point on the unit circle, then

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}(z - 1/z),$$

and $\mathrm{d}z = \gamma'(\theta)\,\mathrm{d}\theta = ie^{i\theta}\,\mathrm{d}\theta$. Therefore, we can write I as a contour integral over the unit circle $C = C(0,1)$,

$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}\theta}{3 + \sin\theta} &= \int_0^{2\pi} \frac{1}{3 + (e^{i\theta} - e^{-i\theta})/2i} \frac{\gamma'(\theta)}{ie^{i\theta}}\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{1}{i\gamma(\theta)} \frac{1}{3 + (\gamma(\theta) - 1/\gamma(\theta))/2i} \gamma'(\theta)\,\mathrm{d}\theta \\
&= \int_C \frac{1}{iz} \frac{1}{3 + (z - 1/z)/2i}\,\mathrm{d}z.
\end{aligned}$$

We can evaluate this integral using the Cauchy integral formula. First, let's simplify the integrand:

$$\frac{1}{iz} \frac{1}{3 + (z - 1/z)/2i} = \frac{2}{z^2 + 6iz - 1}.$$

We see that the integrand is a rational function, so it is analytic except where the denominator vanishes. We can find the roots of the denominator using the quadratic formula. We then have $z^2 + 6iz - 1 = (z - z_1)(z - z_2)$ where

$$z_1 = -3i - 2\sqrt{2}i, \qquad z_2 = -3i + 2\sqrt{2}i.$$

Notice that the root z_1 is outside C and z_2 is inside C. In order to apply Cauchy's integral formula, we rewrite the integral as

$$I = \int_C \frac{f(z)}{z - z_2}\,\mathrm{d}z$$

where

$$f(z) = \frac{2}{z - z_1}$$

is analytic in a domain which contains C and its interior. Therefore, Cauchy's integral formula gives

$$\begin{aligned} I &= 2\pi i f(z_2) = 2\pi i \frac{2}{z_2 - z_1} = \frac{4\pi i}{(-3i + 2\sqrt{2}i) - (-3i - 2\sqrt{2}i)} \\ &= \frac{4\pi i}{4\sqrt{2}i} = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

5. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function, let $z_0 \in \mathbb{C}$ and $r > 0$. Show that $f(z_0)$ is equal to the average value of f over the circle with center z_0 and radius r, i.e.

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t.$$

Hint: Convert the integral on the right into a contour integral over the unit circle and use Cauchy's integral formula.

Solution.
Let $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. We see that the integral on the right may be written

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\,\mathrm{d}t &= \frac{1}{2\pi} \int_0^{2\pi} f(\gamma(t))\,\mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} f(\gamma(t))\gamma'(t)\frac{1}{ire^{it}}\,\mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\gamma(t))}{i(\gamma(t) - z_0)}\gamma'(t)\,\mathrm{d}t = \frac{1}{2\pi i} \int_{C(0,1)} \frac{f(z)}{z - z_0}\,\mathrm{d}z. \end{aligned}$$

The result now follows from Cauchy's integral formula.

6. Let $U \subset \mathbb{C}$ be an open set, let $f = u + iv$ be a complex-valued map on U (not necessarily holomorphic), and assume that u, v have continuous partial derivatives. Let $\Omega \subset U$ be an open subset of U such that the boundary $\partial\Omega \subset U$ is a simple closed curve parametrized counter-clockwise by function γ.

   a) Use Green's theorem to show that
   $$\int_{\partial\Omega} f \,\mathrm{d}z = 2i \iint_{\Omega} \frac{\partial f}{\partial \overline{z}} \,\mathrm{d}x \,\mathrm{d}y$$
   where $z = x + iy$.

   Conclude that if f is holomorphic on U, then
   $$\int_{\partial\Omega} f \,\mathrm{d}z = 0.$$

   b) Show that the area of Ω is
   $$\mathrm{area}(\Omega) = \frac{1}{2i} \int_{\partial\Omega} \overline{z} \,\mathrm{d}z.$$

Solution.
a) We have

$$\int_{\partial\Omega} f(z) \,\mathrm{d}z = \int_{\partial\Omega} (u+iv)(\mathrm{d}x+i\,\mathrm{d}y) = \int_{\partial\Omega} (u\,\mathrm{d}x - v\,\mathrm{d}y) + i \int_{\partial\Omega} (v\,\mathrm{d}x + u\,\mathrm{d}y).$$

Now Green's theorem applied to the first integral gives

$$\int_{\partial\Omega} (u\,\mathrm{d}x - v\,\mathrm{d}y) = \iint_{\Omega} (-\partial_x v - \partial_y u) \,\mathrm{d}x \,\mathrm{d}y$$

and for the second integral

$$\int_{\partial\Omega} (v\,\mathrm{d}x + u\,\mathrm{d}y) = \iint_{\Omega} (\partial_x u - \partial_y v) \,\mathrm{d}x \,\mathrm{d}y.$$

Combining these results and rearranging,

$$\begin{aligned}\int_{\partial\Omega} f(z)\,\mathrm{d}z &= \iint_{\Omega} \Big[i(\partial_x u + i\partial_x v) - (\partial_y u + i\partial_y v)\Big]\,\mathrm{d}x\,\mathrm{d}y \\ &= 2i\iint_{\Omega}\Big[\frac{1}{2}\partial_x(u+iv) + i\partial_y(u+iv)\Big]\,\mathrm{d}x\,\mathrm{d}y \\ &= 2i\iint_{\Omega}\Big[\frac{1}{2}\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big]\,\mathrm{d}x\,\mathrm{d}y = 2i\iint_{\Omega}\frac{\partial f}{\partial \overline{z}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

In particular, if f is holomorphic then $\partial_{\overline{z}} f = 0$ and

$$\int_{\partial\Omega} f(z)\,\mathrm{d}z = 0$$

which is Cauchy's integral theorem.

b) This follows from choosing $f(z) = \overline{z}$ in the formula we proved in (a):

$$\int_{\partial\Omega} \overline{z}\,\mathrm{d}z = 2i\iint_{\Omega} 1\,\mathrm{d}x\,\mathrm{d}y = 2i\,\mathrm{area}(\Omega).$$