Homework 8 Handed out: Wednesday, November 2, 2022
Due: Wednesday, November 9, 2022 by 11:59pm

Material covered: Outcomes 7.1–7.4.

Solutions

1. Let f be the $2L$-periodic function which on $[-L, L]$ is given by

$$f(x) = \begin{cases} x, & -L < x < L \\ 0, & x = \pm L. \end{cases}$$

   a) Compute the Fourier series of f in trigonometric form, i.e. find the coefficients a_0, a_n and b_n in

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right).$$

   b) Compute the Fourier series of f in complex exponential form, i.e. find the coefficients c_n in

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}.$$

Solution.
a) First note that f is odd on $[-L, L]$ and therefore the coefficients a_n are all zero. Moreover, we can simplify the computation of the coefficients b_n by using

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L}\int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

Therefore,

$$b_n = \frac{2}{L}\int_{0}^{L} x \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L}\frac{(-1)^{n+1}L^2}{n\pi} = \frac{2L}{\pi}\frac{(-1)^{n+1}}{n}.$$

Here we integrated by parts in the second step. Thus the Fourier series of f in trigonometric form is

$$\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right).$$

b) One way to find the coefficients c_n is to compute the integrals

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,\mathrm{d}x.$$

However, since we already know the coefficients of the Fourier series in trigonometric form, it is easier to use the relations

$$\begin{aligned} c_0 &= \tfrac{1}{2}a_0 \\ c_n &= \tfrac{1}{2}(a_n - ib_n) \qquad (n \geq 1) \\ c_{-n} &= \tfrac{1}{2}(a_n + ib_n) \qquad (n \geq 1). \end{aligned} \tag{1}$$

From this we see that $c_0 = 0$ and and for $n \geq 1$,

$$c_n = -c_{-n} = -\tfrac{1}{2}ib_n = i\frac{L}{\pi}\frac{(-1)^n}{n}.$$

Therefore, the Fourier series in exponential form is

$$-i\frac{L}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n}e^{in\pi x/L}.$$

Two partial sums of the Fourier series are shown below for $L = 1$. The Fourier series converges pointwise to $\frac{1}{2}(f(x^+) + f(x^-))$ but not uniformly.

2. Consider the 2π-periodic function f which on $[-\pi, \pi]$ is given by

$$f(x) = \begin{cases} x, & |x| \leq \frac{\pi}{2}, \\ \pi - x, & \frac{\pi}{2} \leq x \leq \pi, \\ -\pi - x, & -\pi \leq x \leq -\frac{\pi}{2}. \end{cases}$$

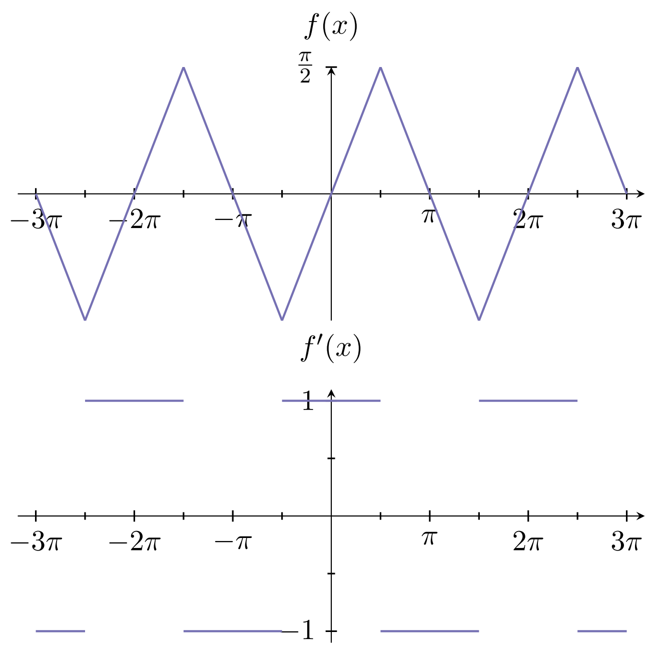
a) Sketch the graph of the function f and its derivative f' on the interval $[-\pi, 3\pi]$.

b) Find the Fourier series of f in trigonometric form; i.e. find the coefficients a_0, a_n and b_n in

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big(a_n \cos(nx) + b_n \sin(nx)\Big)$$

c) Find the Fourier series of f' in trigonometric form. What does the Fourier series of f' converge to at $x = \frac{\pi}{2}$?

Solution.
a) The graphs of f and f' are shown below. We note that f is continuous and piecewise smooth, and f' has a jump discontinuity at $x = n\pi/2$ where n is an odd integer.



b) Since f is odd on the interval $[-\pi, \pi]$, the coefficients a_n are all zero and the coefficients b_n are

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)\,\mathrm{d}x + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin(nx)\,\mathrm{d}x. \end{aligned}$$

The two integrals are easily evaluated using integration by parts. Alternatively, we might notice that

$$\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x - \tfrac{\pi}{2})\sin(n(x - \tfrac{\pi}{2}))\,\mathrm{d}x$$

where $f(x - \frac{\pi}{2})$ is even on $[-\pi/2, \pi/2]$ while $\sin(n(x - \frac{\pi}{2}))$ is an odd function for n even and an even function for n odd. Therefore, b_n is zero if n is even and for odd n,

$$\begin{aligned} b_n &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x)\sin(nx)\,\mathrm{d}x = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)\,\mathrm{d}x \\ &= \frac{4}{\pi}\frac{-n\pi\cos(n\frac{\pi}{2}) + 2\sin(n\frac{\pi}{2})}{2n^2} = \frac{4}{\pi}\frac{\sin(n\frac{\pi}{2})}{n^2}. \end{aligned}$$

Here we used that $\cos(n\frac{\pi}{2}) = 0$ for odd n. Thus, the Fourier series of f is

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\frac{\pi}{2})}{n^2}\sin(nx)$$

c) We can avoid computing any integrals by using the relations

$$a_n(f') = nb_n(f), \quad b_n(f') = -na_n(f)$$

which were proved in Problem 2 of Section 8. We see that the coefficients b_n are zero and for $n \geq 1$,

$$a_n(f') = nb_n(f) = \frac{4}{\pi}\frac{\sin(n\frac{\pi}{2})}{n}.$$

Finally, $a_0 = 0$ since the average value of f' on $[-\pi, \pi]$ is zero. The Fourier series of f' is then

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\frac{\pi}{2})}{n} \cos(nx)$$

Since f' is piecewise continuously differentiable, the Fourier series above converges pointwise to

$$\frac{f'(x^+) + f'(x^-)}{2}$$

In particular, at $x = \pi/2$, it converges to $((-1) + 1)/2 = 0$. This can also be seen directly from the series, since each term vanishes at $x = \pi/2$.

3. Consider the 2π periodic function

$$f(x) = \frac{\sin(x)}{5 + 4\cos(x)}.$$

Determine the Fourier series of f in trigonometric form.

Hint: One way to compute the integral

$$\int_0^{2\pi} f(\theta) \sin(n\theta)\, \mathrm{d}\theta$$

is to convert it to a contour integral over the unit circle and use the residue theorem.

Solution.
Since f is odd, the coefficients a_n are all identically zero. Let us compute the coefficients b_n:

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta)\, \mathrm{d}\theta = \frac{1}{\pi} \int_0^{2\pi} \frac{\sin(\theta)}{5 + 4\cos(\theta)} \sin(n\theta)\, \mathrm{d}\theta$$

Let us convert this into a contour integral over the unit circle $C = C(0,1)$ as usual, using the substitution

$$z = e^{i\theta}, \qquad \mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta$$

which implies

$$\begin{aligned}
\cos(\theta) &= \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1}) \\
\sin(\theta) &= \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - z^{-1}) \\
\sin(n\theta) &= \tfrac{1}{2i}(e^{in\theta} - e^{-in\theta}) = \frac{1}{2i}(z^n - z^{-n}).
\end{aligned}$$

We find

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_C \frac{(z - z^{-1})/2i}{5 + 4(z + z^{-1})/2}\,\frac{z^n - z^{-n}}{2i}\,\frac{1}{iz}\,\mathrm{d}z \\
&= -\frac{1}{4\pi i}\int_C \frac{(z - z^{-1})(z^n - z^{-n})}{2z^2 + 5z + 2}\,\mathrm{d}z \\
&= -\frac{1}{4\pi i}\int_C \frac{(z^2 - 1)(z^{2n} - 1)}{z^{n+1}(2z^2 + 5z + 2)}\,\mathrm{d}z
\end{aligned}$$

Clearly there is a pole of order $n+1$ at $z = z_1 = 0$. Moreover, $2z^2 + 5z + 2 = 2(z - z_2)(z - z_3)$ where $z_2 = -\frac{1}{2}$ and $z_3 = -2$. The contour C encloses the poles at z_1 and z_2, so

$$\begin{aligned}
b_n &= -\frac{1}{8\pi i}\int_C \frac{(z^2 - 1)(z^{2n} - 1)}{z^{n+1}(z - z_2)(z - z_3)}\,\mathrm{d}z \\
&= -\frac{1}{4}\Big[\operatorname{Res}(g, z_1) + \frac{1}{4}\operatorname{Res}(g, z_2)\Big]
\end{aligned}$$

where we define

$$g(z) = \frac{(z^2 - 1)(z^{2n} - 1)}{z^{n+1}(z - z_2)(z - z_3)}$$

for convenience. It remains to compute the two residues. Let us start with the easier one, at $z_2 = -\frac{1}{2}$:

$$\begin{aligned}\operatorname{Res}(g, z_2) &= \lim_{z\to z_2} g(z) = (-2)^{n+1}\frac{(\frac{1}{4}-1)(\frac{1}{4^n}-1)}{(-\frac{1}{2}+2)}\\ &= (-2)^n(4^{-n}-1) = (-1)^{n+1}(2^n - 2^{-n})\end{aligned}$$

For the residue at $z = 0$, we notice that the z^{2n} term will not contribute to the residue, so we have

$$\operatorname{Res}(g, 0) = \operatorname{Res}(\tilde{g}, 0)$$

where

$$\tilde{g}(z) = \frac{1 - z^2}{z^{n+1}(z - z_2)(z - z_3)}.$$

By dividing and then finding the partial fraction decomposition of the remainder, we find

$$\begin{aligned}\tilde{g}(z) &= \frac{1}{z^{n+1}}\left(\frac{-(z^2+\frac{5}{2}z+1)+\frac{5}{2}z+2}{z^2+\frac{5}{2}z+1}\right)\\ &= \frac{1}{z^{n+1}}\left(-1+\frac{5z+2}{2(z+1/2)(z+2)}\right) = \frac{1}{z^{n+1}}\left(\frac{2}{2+z}+\frac{1}{1+2z}-1\right)\end{aligned}$$

Now the simplest way to compute the residue is to expand in a Laurent series around $z = 0$ and read off the coefficient of z^{-1}. We have

$$\tilde{g}(z) = \frac{1}{z^{n+1}}\left[\sum_{k=0}^{\infty}\left(-\tfrac{z}{2}\right)^k + \sum_{k=0}^{\infty}(-2z)^k - 1\right]$$

from which we can read the coefficient of the z^{-1} term:

$$\operatorname{Res}(g, 0) = (-2)^{-n} + (-2)^n = (-1)^n(2^n + 2^{-n}).$$

Adding our results, we have

$$\begin{aligned}\operatorname{Res}(g, z_1) + \operatorname{Res}(g, z_2) &= (-1)^n(2^n + 2^{-n}) - (-1)^n(2^n - 2^{-n})\\ &= 2(-1)^n 2^{-n}.\end{aligned}$$

We finally obtain

$$b_n = -\frac{1}{2}(-1)^n 2^{-n}$$

so the Fourier series of f is

$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\sin(nx).$$

Notice that the coefficients of this series decay geometrically (exponentially) with n rather than polynomially. This is a consequence of f being analytic.

4. The differential equation

$$mx''(t) + kx(t) = F(t) \tag{2}$$

describes a mass-spring system acted on by a time-varying force $F(t)$. Recall that if $F(t) = 0$, then the general solution is

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

where $\omega_0 = \sqrt{k/m}$.

Suppose that the system is acted on by a periodic force of the form

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t),$$

and assume that ω_0 is *not* an integer multiple of the driving frequency ω. Show that there exists a particular solution x_p of equation (2) of the form

Show that there exists a particular solution x_p of equation (2) of the form

$$x_p(t) = \frac{\tilde{a}_1}{2} + \sum_{n=1}^{\infty} \tilde{a}_n \cos(n\omega t) \tag{3}$$

and determine the coefficients $\tilde{a}_0$, $\tilde{a}_n$.

Hint: Substitute the series (3) in the differential equation (2) and differentiate term by term.

Solution.
We look for a solution x_p given by a convergent Fourier series of the form (3). It follows from Problem 2 of Section 8 that the Fourier series of x_p'' is

$$-\sum_{n=1}^{\infty}(n\omega)^2\tilde{a}_n\cos(n\omega t)$$

and thus the Fourier series of $mx_p''+kx_p$ is

$$\begin{aligned}-m\sum_{n=1}^{\infty}(n\omega)^2\tilde{a}_n\cos(n\omega t)+k\frac{\tilde{a}_0}{2}+k\sum_{n=1}^{\infty}\tilde{a}_n\cos(n\omega t)\\ =\frac{k\tilde{a}_0}{2}+\sum_{n=1}^{\infty}\Big(k-m(n\omega)^2\Big)\tilde{a}_n\cos(n\omega t).\end{aligned}$$

Now we see that x_p solves the differential equation (2) if and only if this equals the Fourier series of F. This in turn holds if and only if

$$\tilde{a}_0=a_0/k,\quad \tilde{a}_n=\frac{a_n}{k-m(n\omega)^2}=\frac{a_n/m}{\omega_0^2-(n\omega)^2}.$$

The interpretation in this: The spring-mass system will oscillate with the same period $2\pi/\omega$ as the driving force $F(t)$. The shape of the waveform $x_p(t)$ will be different from $F(t)$: because of the factor $(\omega_0^2-n^2\omega^2)^{-1}$, the system has a stronger response to frequencies $n\omega$ which are closer to ω_0.

For completeness, let us consider what happens if $n_0\omega=\omega_0$ for some $n_0\in\mathbb{N}$ (this was not required to get full credit for the problem). In that case, the corresponding term in the Fourier series of x_p,

$$\frac{a_{n_0}/m}{\omega_0^2-(n_0\omega)^2}$$

is not defined since the denominator is zero. This term must be replaced by a particular solution of

$$mx''+kx=a_{n_0}\cos(\omega_0 t).$$

You can check that

$$\frac{a_{n_0}}{2m\omega_0} t \sin(\omega_0 t)$$

is a solution. Therefore,

$$x_p(t) = \frac{a_{n_0}}{2m\omega_0} t \sin(\omega_0 t) + \frac{a_0}{2k} + \frac{1}{m} \sum_{\substack{n=1 \\ n \neq n_0}}^{\infty} \frac{a_n/m}{\omega_0^2 - (n\omega)^2} \cos(n\omega t).$$

is a particular solution to (2).

We see that if the driving force has a component with frequency ω_0, then the motion is no longer periodic, because the amplitude of the corresponding term in the solution grows with time. This phenomenon is called *resonance* and ω_0 is called the resonance frequency of the system.