Homework 10 Handed out: Monday, November 21st, 2022
Due: Wednesday, November 30th, 2022 by 11:59pm

Material covered: Outcomes 9.1-9.3

1. **Filtering.** Consider a low-pass filter $h(t)$, defined by its Fourier transform in the frequency domain:

$$\hat{h}(k) = \begin{cases} 1 & \text{for } |k| \le c \\ 0 & \text{for } |k| > c \end{cases}$$

   This filter allows frequencies below the cutoff value c, and excludes frequencies above c.

   Now suppose that we have a time-domain signal:

$$x(t) = \frac{\sin(at)}{\pi t}$$

   a) Determine the convolution $y(t) = h(t) * x(t)$ in the case when $a < c$ and when $a > c$.
   b) Sketch $y(t)$ and $\hat{y}(k)$ for when $a < c$ and when $a > c$.

2. **Forced Damped Harmonic Oscillator.** Consider a block with mass m attached to a spring with spring constant k. A spring force $F_{spring} = -kx$, along with a drag force, $F_{drag} = -bx'(t)$, will oppose the motion of the block. So, without additional external forces, the position of the block will be governed by the second-order ODE,

$$mx''(t) = -kx(t) - bx'(t)$$

   or equivalently

$$mx''(t) + bx'(t) + kx(t) = 0$$

   Finally, suppose that there is also an external force $y(t)$ acting on the block. Altogether, we obtain the dynamics:

$$mx''(t) + bx'(t) + kx(t) = y(t)$$

The behavior of the oscillator will depend on the relative magnitudes of the damping force, spring force, and the mass of the block. In particular, if b is small, the system will oscillate around equilibrium prior to settling in response to a perturbation (underdamped). If b is large, the system will smoothly approach equilibrium without oscillating (overdamped). We will consider the latter case; specifically, assume that $b^2 > 4km$.

In this problem, you will use Fourier Transforms to solve for $x(t)$ for different forcing functions $y(t)$.

a) Determine the transfer function

$$H(w) = \frac{\hat{x}(w)}{\hat{y}(w)}$$

b) Suppose the external force is sinusoidal, $y(t) = \sin(at)$. What is the motion of the block $x(t)$?

c) Suppose the system experiences a unit impulse,

$$y(t) = \delta(t)$$

What is $x(t)$?

3. **Convolutions.** Let $f(x) = 1/(1+x^2)$. Determine $f * f$, the convolution of f with itself.

4. **2D Fourier Transforms (Theory).** Consider the function

$$f(x,y) = \cos(2\pi x)\sin(2\pi y a + \frac{b\pi}{4})$$

Find the 2D Fourier transform $\hat{f}(k_x, k_y)$. Then, for each of the following cases, sketch where it is nonzero in the (k_x, k_y) plane.

a) a = 0, b = 0

b) a = 0, b = 1

c) a = 1, b = 1

Hint: This function, and its Fourier Transform, is visualized for you in the Colab notebook for this assignment. You can change the "a" and "b" parameters and verify that your answer matches the computational result.

5. **2D Fourier Transforms (Computation).** Open the Python notebook here and follow the instructions.

6. **Discrete Fourier Transform.** Consider the function

$$f(x) = 4 + \sin(6\pi x) - 2\cos(2\pi x)$$

Compute the discrete Fourier transform by hand, for a 4Hz sampling rate on the interval [0, 1).