Section 6 Friday, October 21, 2022

Material covered: Outcomes 5.1–5.3.

Solutions

1. List the singular points of f. Determine the radius of convergence of the Taylor series of f centered at the point a without computing the series.
   a) $f(z) = \frac{e^z}{z(z-i)^3}, \quad a = 1 + i.$
   b) $f(z) = \frac{\sin(z)}{z^5}, \quad a = \frac{\pi}{2}.$
   c) $f(z) = \sin(\frac{1}{1-z}), \quad a = i.$

Solution. The radius of convergence R is simply the distance $|a - z_*|$ where z_* is the singular point closest to a. Thus it suffices to find the singular points z_1, ..., z_n and compute the distances $|a - z_i|$, $1 \leq i \leq n$.

Although it is not necessary to complete the problem, we also classify the singular points.

a) The function f is the ratio of analytic functions, so it is analytic except possibly at $z = 0$ and $z = i$ where the denominator vanishes. Since the numerator e^z does not vanish at either point, they are poles of f. The radius of convergence R of the Taylor series of f centered at $a = 1+i$ is the distance between a and the nearest singular point, which is

$$R = |a - i| = |1 + i - i| = 1.$$

Let's identify the orders of the poles. We can write

$$f(z) = \frac{g(z)}{(z-0)^1}, \quad g(z) = \frac{e^z}{(z-i)^3}$$

where $g(z)$ is analytic in $\mathbb{C} \setminus \{i\}$ and $g(0) \neq 0$. Therefore, $z = 0$ is a simple pole (pole of order 1) of f. Similarly, we can write

$$f(z) = \frac{h(z)}{(z-i)^3}, \quad h(z) = \frac{e^z}{z}$$

where $h(z)$ is analytic in $\mathbb{C} \setminus \{0\}$ and $h(i) \neq 0$. Therefore, $z = i$ is a pole of order 3.

b) The function f is the ratio of analytic functions, so it is analytic except possibly at $z = 0$ where the denominator vanishes. To determine the radius of convergence, we need to check whether this singular point is removable. To that end, let's find the Laurent series of f centered at 0.

The Taylor series of sin around 0 is

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots .$$

Then the Laurent series of f is easily computed as

$$f(z) = \frac{1}{z^5} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \frac{1}{z^4} - \frac{1}{3! z^2} + \frac{1}{5!} - \frac{z^2}{7!} + \cdots$$

from which we see that $z = 0$ is a pole of order 4 (the largest negative power that appears in the Laurent series).

The radius of convergence R of the Taylor series of f centered at $a = \frac{\pi}{2}$ is the distance to the singular point, which is

$$R = |a - 0| = \frac{\pi}{2}.$$

c) The function f is the composition of the entire function sin and the function $1/(1-z)$ which is analytic on $\mathbb{C} \setminus \{1\}$. Thus f is analytic except possibly at $z = 1$. To determine the type of singularity at

$z = 1$, we expand f in a Laurent series around $z = 1$. Using the Taylor series expansion for sin, we have

$$\sin\Big(\frac{1}{1-z}\Big) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{(1-z)^{2n+1}}$$
$$= \sum_{n=-\infty}^{0} \frac{(-1)^n}{(1-2n)!} (1-z)^{2n-1}.$$

Since the principal part of the Laurent series is an infinite sum (in other words, the Laurent series has infinitely many negative degree terms), $z = 1$ is an essential singularity of f.

The radius of convergence R of the Taylor series of f centered at $a = i$ is the distance to the singular point:

$$R = |a - 1| = |i - 1| = \sqrt{2}.$$

2. Consider the function

$$f(z) = \frac{1}{z^2(z-1)}$$

Find the Laurent series expansion of f centered at 0 in each of the following annuli:

a) $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$

b) $\{z \in \mathbb{C} \mid 1 < |z| < \infty\}$

Solution.

a) Recall that for $|z| < 1$,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots .$$

Then, for z in the annulus $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$, we may write

$$f(z) = -\frac{1}{z^2}\frac{1}{1-z} = -\frac{1}{z^2}\sum_{n=0}^{\infty} z^n = -\sum_{n=-2}^{\infty} z^n.$$

b) To find a power series which converges for $|z| > 1$, we rewrite $f(z)$ as

$$f(z) = \frac{1}{z^2}\frac{1}{z-1} = \frac{1}{z^3}\frac{1}{1-z^{-1}}.$$

The second factor can now be written as a geometric series

$$\frac{1}{1-z^{-1}} = \sum_{n=0}^{\infty} z^{-n}$$

which converges for $|z^{-1}| < 1$, i.e. for $|z| > 1$. Therefore,

$$f(z) = \frac{1}{z^3}\sum_{n=0}^{\infty} z^{-n} = \sum_{n=-\infty}^{-3} z^n$$

is a Laurent series for f which converges in the annulus $\{z \in \mathbb{C} \mid |z| > 1\}$.

3. Determine the Laurent series of the function

$$f(z) = \frac{\cos(z) - 1}{z^5}$$

centered at 0. Use the result to find the residue $\operatorname{Res}(f, 0)$ and compute the contour integral

$$\int_C f(z)\,\mathrm{d}z$$

where C is the unit circle traversed counter-clockwise.

Solution. The Taylor series of cos centered at 0 is

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots .$$

which converges for every $z \in \mathbb{C}$. Therefore, the Laurent series of f centered at the origin is

$$f(z) = \frac{1}{z^5} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = -\frac{1}{2!} z^{-3} + \frac{1}{4!} z^{-1} - \frac{1}{6!} z + \cdots ,$$

which converges for all $z \in \mathbb{C} \setminus \{0\}$.

The residue at $z = 0$ is the coefficient of z^{-1} in the Laurent series:

$$\operatorname{Res}(f, 0) = \frac{1}{4!} = \frac{1}{24}.$$

The integral is now easily computed. The contour C encloses the pole of f, so

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}(f, 0) = \frac{i\pi}{12}.$$

4. For each of the following functions f, identify the singular points and classify them into removable singularities, poles and essential singularities. Calculate the residue at each singular point and determine the order of each pole.

   a) $f(z) = \frac{z^4}{1-z^4}$

   b) $f(z) = \frac{\tan(z)}{z^3}$

   c) $f(z) = z^5 e^{-1/z}$

   Solution.

   a) The function f is rational, so it is analytic except at the 4th roots of unity $z_k = e^{ik2\pi/4}$, $0 \le k \le 3$, where the denominator $1 - z^4$ vanishes. We can also write these roots as $z_0 = 1$, $z_1 = i$, $z_2 = -1$, $z_3 = -i$. Since the numerator z^4 does not vanish at any of these points, each root of unity is a pole. Since they are simple roots of the denominator (each factor $z - z_k$ appears once in the factorization of $1 - z^4$), they are all simple poles.

Let us compute the residues. At $z = 1$, the residue is

$$\operatorname{Res}(f, 1) = \lim_{z \to 1} \left[-\frac{z^4}{(z-i)(z+1)(z+i)} \right] = -\frac{1^4}{(1-i)(1+1)(1-i)} = -\frac{1}{4}.$$

We can compute the remaining residues in the same way.

Alternatively, we can compute all four residues at the same time as follows: The residue at z_k is

$$\begin{aligned} \operatorname{Res}(f, z_k) &= \lim_{z \to z_k} (z - z_k) f(z) = \lim_{z \to z_k} z^4 \frac{z - z_k}{1 - z^4} \\ &= (z_k)^4 \lim_{z \to z_k} \frac{1}{-4z^3} = (z_k)^4 \frac{1}{-4(z_k)^3} = -\frac{1}{4} z_k. \end{aligned}$$

where we used L'Hôpital's rule in the third step. Thus the residues are

$$\operatorname{Res}(f, e^{ik\frac{2\pi}{4}}) = -\frac{1}{4} e^{ik\frac{\pi}{2}}.$$

b) Since f is the ratio of two analytic functions,

$$\frac{\tan(z)}{z^3} = \frac{\sin(z)}{z^3 \cos(z)},$$

f is analytic except at the singular points where the denominator vanishes. The singular points are $z = 0$ and the zeros of cos, which are

$$z_k = (2k+1)\frac{\pi}{2}, \qquad k \in \mathbb{Z}.$$

First consider the point $z = 0$. For the numerator, we have

$$\sin(0) = 0, \quad \sin'(0) = 1 \neq 0$$

so there exists an analytic function h such that $\sin(z) = zh(z)$ for all $z \in \mathbb{C}$ and $g(0) \neq 0$. Another way to see this is to write the Taylor series for sin:

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n} = zh(z).$$

Then,

$$f(z) = \frac{zh(z)}{z^3 \cos(z)} = \frac{h(z)/\cos(z)}{z^2} = \frac{g(z)}{z^2}$$

where the numerator

$$g(z) = \frac{h(z)}{\cos(z)}$$

is analytic in a neighborhood of $z = 0$ and nonzero at $z = 0$. Thus $z = 0$ is a pole of f of order 2. Let us compute the residue there:

$$\mathrm{Res}(f, 0) = \frac{1}{1!} g'(0) = g'(0).$$

The derivative of g can be found using the quotient rule:

$$g'(0) = \frac{h'(0)\cos(0) - h(0)\sin(0)}{(\cos(0))^2} = \frac{0 \cdot 1 - 1 \cdot 0}{1} = 0.$$

Thus the residue at the origin is

$$\mathrm{Res}(f, 0) = 0.$$

Next we consider the zeros of cosine. They are all simple poles of f, which can be seen by writing

$$f(z) = \frac{\sin(z)/z^3}{\cos(z)}.$$

The numerator is analytic on $\mathbb{C} \setminus \{0\}$ and nonzero at each zero of cos. Since the zeros of cos are all simple, meaning

$$\cos'(z_k) = -\sin(z_k) \neq 0$$

we see that the points (4) are simple poles of f. The residues are

$$\begin{aligned} \mathrm{Res}(f, z_k) &= \lim_{z \to z_k} (z - z_k) f(z) = \lim_{z \to z_k} \frac{z - z_k}{\cos(z)} \frac{\sin(z)}{z^3} \\ &= \frac{\sin(z_k)}{z_k^3} \lim_{z \to z_k} \frac{z - z_k}{\cos(z)}. \end{aligned}$$

The limit on the right can be evaluated using L'Hôpital's rule:

$$\lim_{z \to z_k} \frac{z - z_k}{\cos(z)} = \lim_{z \to z_k} \frac{1}{-\sin(z)} = -\frac{1}{\sin(z_k)}.$$

Thus the result is

$$\text{Res}(f, z_k) = -\frac{1}{z_k^3} = -\frac{1}{\left((2k+1)\frac{\pi}{2}\right)^3}.$$

c) The function f has a single isolated singularity at $z = 0$. To determine the type of singularity, we write down the Laurent series of f centered at $z = 0$. Recall that the Taylor series of e^z is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C},$$

so the Laurent series of $e^{-1/z}$ is

$$e^{-1/z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{-n}, \qquad z \neq 0.$$

Then the Laurent series of f is

$$z^5 e^{-1/z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{-n+5}$$

which has infinitely many negative degree terms, so $z = 0$ is an essential singularity of f.

The residue of f at 0 is the coefficient of z^{-1} in the Laurent series:

$$\text{Res}(f, 0) = \frac{(-1)^6}{6!} \frac{1}{6!}.$$

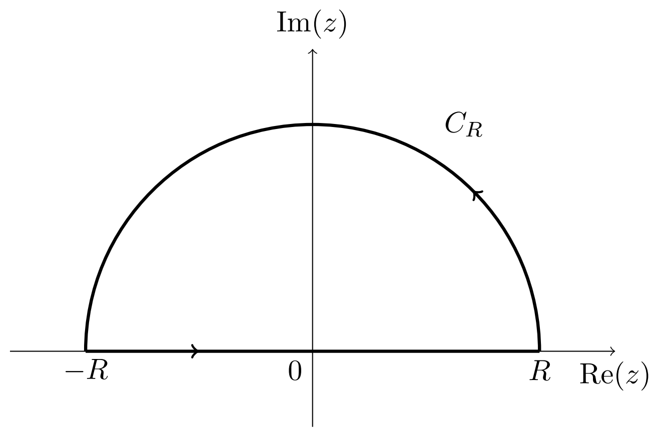
5. Let

$$f(z) = \frac{z^2 + 7}{(z^2+1)^2(z^2+4)}.$$

For $R > 0$, compute the contour integral

$$\int_{C_R} f(z)\,\mathrm{d}z$$

where C_R is the contour composed of a line segment from $-R$ to R and a semicircle of radius R in the upper half-plane, traversed counter-clockwise (shown below). How does the answer depend on R?



Solution. In order to use the residue theorem, we have to identify the isolated singular points of f and compute the residues at the ones enclosed by C_R. Since f is a rational function, the singular points are the zeros of the denominator. Let us factor the denominator:

$$f(z) = \frac{z^2 + 7}{(z-i)^2(z+i)^2(z-2i)(z+2i)}$$

The denominator has two simple roots, $2i$ and $-2i$. Since the numerator does not vanish there, these are simple poles of f. The denominator also has two roots of multiplicity 2, namely i and $-i$. The numerator does not vanish at these points, so they are second order poles of f. Out of these, only the roots $z_1 = 2i$ and $z_2 = i$ are in the upper half-plane. The other roots will not be enclosed by C_R no matter what the radius R is, so we do not need to compute the residues there.

Let us compute the residues of the poles in the upper half-plane, starting with the simple pole z_1:

$$\begin{aligned}\operatorname{Res}(f,2i) &= \lim_{z\to 2i}\Big((z-2i)f(z)\Big) = \lim_{z\to 2i}\frac{z^2+7}{(z-i)^2(z+i)^2(z+2i)}\\ &= \frac{(2i)^2+7}{i^2(3i)^2(4i)} = \frac{1}{12i}.\end{aligned}$$

For the second order pole z_2, we have

$$\begin{aligned}\operatorname{Res}(f,i) &= \frac{1}{1!}\lim_{z\to i}\frac{d}{dz}\Big((z-i)^2f(z)\Big)\\ &= \lim_{z\to i}\frac{d}{dz}\frac{z^2+7}{(z+i)^2(z-2i)(z+2i)}\end{aligned}$$

Let us compute the derivative using the quotient rule:[1]

$$\begin{aligned}\frac{d}{dz}\frac{z^2+7}{(z+i)^2(z-2i)(z+2i)} &= \frac{2z}{(z+i)^2(z-2i)(z+2i)}\\ &\quad - \frac{(z^2+7)\left(\frac{2}{z+i}+\frac{1}{z-2i}+\frac{1}{z+2i}\right)}{(z+i)^2(z-2i)(z+2i)}\end{aligned}$$

which evaluated at $z=i$ is

$$\begin{aligned}\frac{d}{dz}\frac{z^2+7}{(z+i)^2(z-2i)(z+2i)}\bigg|_{z=i} &= \frac{2i}{(2i)^2(-i)(3i)} - \frac{(i^2+7)\left(\frac{2}{2i}+\frac{1}{-i}+\frac{1}{3i}\right)}{(2i)^2(-i)(3i)}\\ &= \frac{1}{6i}+\frac{1}{6i} = \frac{1}{3i}.\end{aligned}$$

[1] A useful trick when evaluating the derivative of the product of several terms, e.g. $f(z)=u(z)v(z)$ is to use $\frac{f'}{f}=\frac{u'}{u}+\frac{v'}{v}$.

To summarize,

$$\operatorname{Res}(f, 2i) = \frac{1}{12i}, \qquad \operatorname{Res}(f, i) = \frac{1}{3i}.$$

The integral is now easily computed using the residue theorem. For $R < 1$, the contour encloses no singular points, so

$$\int_{C_R} f(z)\,\mathrm{d}z = 0, \qquad R < 1.$$

For $1 < R < 2$, the contour encloses only the pole $z_2 = i$, and the residue theorem gives

$$\int_{C_R} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}(f, i) = \frac{2\pi}{3}, \qquad 1 < R < 2.$$

Finally, for $R > 2$, C_R encloses both $z_1 = 2i$ and $z_2 = i$, and the residue theorem gives

$$\int_{C_R} f(z)\,\mathrm{d}z = 2\pi i\big(\operatorname{Res}(f, 2i) + \operatorname{Res}(f, i)\big) = \frac{5\pi}{6}, \qquad R > 2.$$