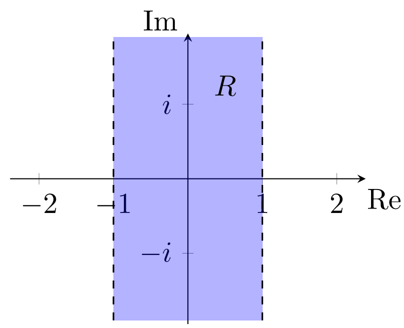
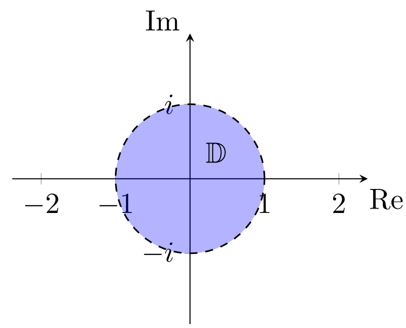
Section 2 Friday, September 16, 2022

Material covered: Outcomes 1.1–1.5.

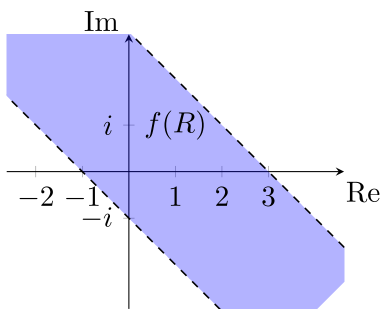
Solutions

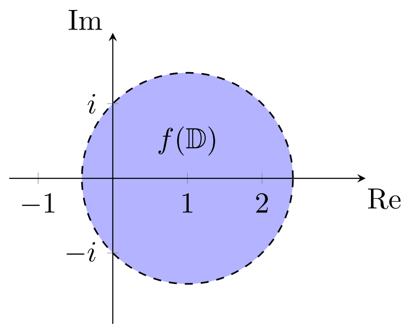
1. For each of the following complex functions $f\colon \mathbb{C} \to \mathbb{C}$, find the image under f of the strip $R \coloneqq \{z \in \mathbb{C} \mid -1 < \operatorname{Re} z < 1\}$ and the unit disk $\mathbb{D} \coloneqq \{z \in \mathbb{C} \mid |z| < 1\}$ (pictured below).
   a) $f(z) = (1+i)z + 1$ b) $f(z) = z^2$.





Solution.
a) Recall that multiplication by a complex number $re^{i\theta}$ is equivalent to a scaling by a factor r and rotation by an angle θ. Since $1+i = \sqrt{2}e^{i\frac{\pi}{4}}$, the map $z \mapsto (1+i)z$ is a scaling by a factor $\sqrt{2}$ and a rotation by an angle $\pi/4$. This map is composed with a translation $z \mapsto z+1$ of one unit to the right. The sets $f(R)$ and $f(\mathbb{D})$ are shown below.





A more algebraic way to find $f(R)$ is as follows: Let $z \in R$ so that $z = x + iy$ with $-1 < x < 1$ and $y \in \mathbb{R}$.

$$f(z) = (1+i)(x+iy) + 1 = (x-y+1) + i(x+y).$$

For convenience, let us write

$$u = \operatorname{Re} f(z) = x - y + 1, \quad v = \operatorname{Im} f(z) = x + y.$$

Let us solve for x, y in terms of u, v. We have

$$x = \frac{1}{2}(u + v - 1)$$
$$y = \frac{1}{2}(-u + v + 1)$$

Since $y \in \mathbb{R}$ is arbitrary and $-1 < x < 1$, we have found that

$$f(R) = \{u + iv \in \mathbb{C} \mid -2 < u + v - 1 < 2\}.$$

We can also find $f(\mathbb{D})$ using this approach. Let $z = x + iy \in \mathbb{D}$, with $x^2 + y^2 < 1$. And define u, v as before. Then

$$\begin{aligned} x^2 + y^2 &= \frac{1}{4}\left[(u+v-1)^2 + (-u+v+1)^2\right] \\ &= \frac{1}{4}\left[2u^2 - 4u + 2 + 2v^2\right] = \frac{1}{2}\left[(u-1)^2 + v^2\right]. \end{aligned}$$

We conclude that

$$f(\mathbb{D}) = \{u + iv \mid (u-1)^2 + v^2 < 2\}$$

which is a disk of radius $\sqrt{2}$ centered at $z = 1$.

b) In contrast to the map from part a), this map is not injective. Indeed, for any $z \in \mathbb{C}$,

$$f(z) = z^2 = (-z)^2 = f(-z).$$

Our approach in part a) of inverting the map (solving $u + iv = f(x + iy)$ for x, y) is not as simple for this map, so we use a slightly different approach.

Let us start with the strip R. Since $f(-z) = f(z)$, it suffices to consider $z = x + iy \in R$ with $x \geq 0$, $y \in \mathbb{R}$. Define $u = \operatorname{Re} f(z)$, $v = \operatorname{Im} f(z)$ for convenience. We have

$$f(z) = (x + iy)^2 = (x^2 - y^2) + 2ixy.$$

Assuming for the moment that $x > 0$, we have

$$u = x^2 - y^2 = x^2 - \frac{1}{4x^2}(2xy)^2 = x^2 - \frac{1}{4x^2}v^2.$$

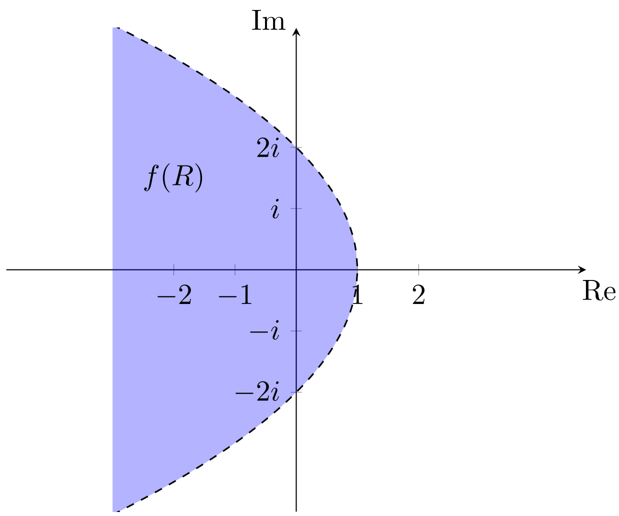
The right hand side is a continuous function of x for $0 < x < 1$, and

$$\lim_{x \searrow 0} \left(x^2 - \frac{1}{4x^2}v^2 \right) = -\infty,$$
$$\lim_{x \nearrow 1} \left(x^2 - \frac{1}{4x^2}v^2 \right) = 1 - \frac{1}{4}v^2.$$

(For the first limit, note that $v \neq 0$ since $x > 0$ by assumption.) Thus, for a given $v \neq 0$, u varies between $-\infty$ and $1 - v^2/4$.

It remains to consider the case $x = 0$, i.e. the image of the imaginary axis. This is straightforward: $f(iy) = -y^2$, so f maps the imaginary axis to the set $\{u \in \mathbb{R} \mid u \leq 0\}$. We conclude that

$$f(R) = \{u + iv \in \mathbb{C} \mid u < 1 - \frac{1}{4}v^2\}.$$



Next consider the unit disk $\mathbb{D}$. It seems plausible that $f(z) = z^2$ maps $\mathbb{D}$ to itself, $f(\mathbb{D}) = \mathbb{D}$. But how do we prove it? One approach is to show separately that $f(\mathbb{D}) \subset \mathbb{D}$ and that $\mathbb{D} \subset f(\mathbb{D})$.

For $z \in \mathbb{D}$, we have

$$|f(z)| = |z^2| = |z|^2 < 1,$$

so $f(z) \in \mathbb{D}$. We conclude that the image $f(\mathbb{D})$ is contained in $\mathbb{D}$, i.e. $f(\mathbb{D}) \subset \mathbb{D}$.

To show that $\mathbb{D} \subset f(\mathbb{D})$, note that any $z \in \mathbb{D}$ can be written as $z = re^{i\theta}$ with $0 \le r < 1$ and $\theta \in \mathbb{R}$. Then the point $\sqrt{r}e^{i\theta/2}$ is in $\mathbb{D}$ and

$$f(\sqrt{r}e^{i\theta/2}) = z.$$

Thus we have shown that $f(\mathbb{D}) = \mathbb{D}$.

2. Let $a, b \in \mathbb{C}$ be complex numbers and define $f \colon \mathbb{C} \to \mathbb{C}$ by

$$f(z) = az + b\overline{z}.$$

a) Determine sufficient and necessary conditions on a and b for f to be bijective.

   Can f be injective without being surjective, or vice versa? Justify your answer.

b) Assuming that f is bijective, find its inverse.

Solution.
a) By considering $z = x + iy$ as a vector (x, y) with two real components, we can identify f with a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ and use results from linear algebra. To see this, compute:

$$\begin{aligned} f(x+iy) &= a(x+iy) + b(x-iy) \\ &= \operatorname{Re}(a+b)x - \operatorname{Im}(a-b)y + i\Big(\operatorname{Im}(a+b)x + \operatorname{Re}(a-b)y\Big). \end{aligned}$$

We see that

$$f(x+iy) = A\begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$A = \begin{bmatrix} \operatorname{Re}(a+b) & -\operatorname{Im}(a-b) \\ \operatorname{Im}(a+b) & \operatorname{Re}(a-b) \end{bmatrix}.$$

Since this is a linear map $\mathbb{R}^2 \to \mathbb{R}^2$, we know that the following are equivalent:

(i) f is injective,

(ii) f is surjective,

(iii) $\operatorname{rank} A = 2$,

(iv) $\det A \neq 0$.

Now,

$$\begin{aligned} \det A &= \operatorname{Re}(a+b)\operatorname{Re}(a-b) + \operatorname{Im}(a+b)\operatorname{Im}(a-b) \\ &= (\operatorname{Re} a)^2 - (\operatorname{Re} b)^2 + (\operatorname{Im} a)^2 - (\operatorname{Im} b)^2 \\ &= |a|^2 - |b|^2 \end{aligned}$$

so f is bijective if and only if

$$\det A = |a|^2 - |b|^2 \neq 0,$$

or equivalently if and only if $|a| \neq |b|$.

b) Now assuming $|a| \neq |b|$, the inverse of A is

$$A^{-1} = \frac{1}{|a|^2 - |b|^2} \begin{bmatrix} \mathrm{Re}(a-b) & \mathrm{Im}(a-b) \\ -\mathrm{Im}(a+b) & \mathrm{Re}(a+b) \end{bmatrix}.$$

The inverse of f is the linear map whose matrix is A^{-1}. Thus

$$\begin{aligned} f^{-1}(x+iy) &= \\ &= \frac{1}{|a|^2 - |b|^2}\Big[\mathrm{Re}(a-b)x + \mathrm{Im}(a-b)y + i\Big(-\mathrm{Im}(a+b)x + \mathrm{Re}(a+b)y\Big)\Big] \\ &= \frac{1}{|a|^2 - |b|^2}\Big[\overline{a}(x+iy) - b(x-iy)\Big] \end{aligned}$$

That is,

$$f^{-1}(z) = \frac{1}{|a|^2 - |b|^2}\Big(\overline{a}z - b\overline{z}\Big).$$

3. Let $z = x + iy$ where $x, y \in \mathbb{R}$. Find the real and imaginary parts of the following expressions in terms of x and y:
   a) e^{z^2}
   b) $\cosh(z)$

Solution.

a) We have

$$e^{(x+iy)^2} = e^{(x^2-y^2)+2xyi} = e^{(x^2-y^2)}\Big[\cos(2xy) + i\sin(2xy)\Big].$$

Thus

$$\mathrm{Re}\big(e^{z^2}\big) = e^{(x^2-y^2)}\cos(2xy), \quad \mathrm{Im}\big(e^{z^2}\big) = e^{(x^2-y^2)}\sin(2xy)$$

b) Recall that

$$\cosh(z) = \frac{e^z + e^{-z}}{2}.$$

So,

$$\begin{aligned}
\cosh(x+iy) &= \frac{e^{x+iy} + e^{-x-iy}}{2} \\
&= \frac{1}{2}\Big[e^x\big(\cos(y) + i\sin(y)\big) + e^{-x}\big(\cos(y) - i\sin(y)\big)\Big] \\
&= \frac{e^x + e^{-x}}{2}\cos(y) + i\frac{e^x - e^{-x}}{2}\sin(y) \\
&= \cosh(x)\cos(y) + i\sinh(x)\sin(y).
\end{aligned}$$

4. Find all solutions $z \in \mathbb{C}$ of the equation:

$$e^{z^2} = i$$

Solution.
First we solve $e^w = i$ for w and then we solve $z^2 = w$ for z. The equation $e^w = i$ implies that $|e^w| = |i| = 1$ so w is purely imaginary. Let us write $w = i\theta$ for $\theta \in \mathbb{R}$. Then the equation implies

$$\cos(\theta) + i\sin(\theta) = i.$$

Equating real and imaginary parts, we see that the solutions are

$$\theta = (2n + \frac{1}{2})\pi, \quad n \in \mathbb{Z}.$$

Now for a given n, we need to solve

$$z^2 = i\Big(2n + \frac{1}{2}\Big)\pi = \Big(2n + \frac{1}{2}\Big)\pi e^{i\frac{\pi}{2}}. \tag{1}$$

Each nonzero complex number has two distinct square roots. Therefore, we obtain two solutions for each n:

$$z = \pm\sqrt{(2n + 1/2)\pi e^{i\frac{\pi}{4}}},$$

or, in Cartesian form,

$$z = \pm\sqrt{(n+1/4)\pi}(1+i).$$

Note: It may be tempting conclude from (1) that

$$z = \pm\sqrt{(2n+1/2)\pi i},$$

However, we have not defined "$\sqrt{z}$" for general complex numbers z, only for positive real numbers. Indeed, each $z \neq 0$ has two square roots, and unless specified, it is not clear which one is meant by "$\sqrt{z}$". Later we will define the *principal branch of the square root* and denote it by $\sqrt{z}$.

5. Define $f\colon \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} \frac{(z+i\bar{z})^2}{z^2}, & z \neq 0 \\ 2i, & z = 0 \end{cases}.$$

Evaluate the limit

$$\lim_{z\to 0} f(z)$$

or prove that the limit does not exist. Is f continuous at $z = 0$?

Solution.
We show that the function f is discontinuous at the origin by showing that the limit

$$\lim_{t\to 0^+} f(tu) \tag{2}$$

depends on the direction u from which we approach the origin (here $t \in \mathbb{R}$). For $u = 1$, we have

$$f(t) = \frac{(t+it)^2}{t^2} = \frac{(1+i)^2 t^2}{t^2} = (1+i)^2 = 2i,$$

so

$$\lim_{t\to 0} f(t \cdot 1) = 2i.$$

In words, the limit is $2i$ if we approach 0 along the real axis. On the other hand, along the imaginary axis, we have

$$f(it) = \frac{(it + i(-it))^2}{(it)^2} = \frac{(1+i)^2 t^2}{-t^2} = -2i,$$

and thus

$$\lim_{t\to 0} f(t \cdot i) = -2i.$$

Since these two limits are different, f is discontinuous at the origin.

Warning: The condition that the limits (2) exist and are equal for all $u \in \mathbb{C}$, $u \neq 0$ is necessary for f to be continuous at 0 but *not sufficient*, as we will see in the next problem.

6. Consider the complex function f defined by

$$f(z) = f(x+iy) = \begin{cases} \frac{x^2 y}{x^4+y^2}, & z \neq 0 \\ 0 & z = 0 \end{cases}.$$

where $x, y \in \mathbb{R}$.

a) Let $u = a + ib$ where a, b are not both zero. Compute the limit

$$\lim_{t\to 0} f(tu) = \lim_{t\to 0} f(ta + itb).$$

b) Show that the limit $\lim_{z\to 0} f(z)$ does not exist.

Hint: Show that the limit

$$\lim_{t\to 0} f(t + it^2)$$

is different from the limit you found in (a).

Solution.
a) For any $t \in \mathbb{R}$, $t \neq 0$, we have

$$f(ut) = f((a+ib)t) = \frac{(at)^2(bt)}{(at)^4 + (bt)^2} = a^2 b \frac{t}{a^4 t^2 + b^2}.$$

First note that if $b = 0$, then $f(ut) = 0$ for all t, so $\lim_{t\to 0} f(ut) = 0$ in this case. If $b \neq 0$, then the denominator $a^4t^2 + b^2$ converges to $b^2 \neq 0$ as $t \to 0$ while the numerator converges to 0, so

$$\lim_{t\to 0} f(ut) = 0.$$

Thus we have shown that the limit along any straight line exists and equals 0.

b) To show that the function is not continuous at 0, it suffices to find a curve $z(t)$ so that $z(t) \to 0$ as $z \to 0$ so that

$$\lim_{t\to 0} f(z(t))$$

doesn't exist or is different from the one we found in (a). As suggested by the hint, we try the curve $z(t) = t + it^2$. For any $t \in \mathbb{R}$, $t \neq 0$, we have

$$f(t + it^2) = \frac{t^2 t^2}{t^4 + t^4} = \frac{t^4}{2t^4} = \frac{1}{2}.$$

Therefore,

$$\lim_{t\to 0} f(z(t)) = \frac{1}{2}.$$

Since this limit is different from the limit obtained when approaching the origin along a straight line, the limit

$$\lim_{z\to 0} f(z)$$

does not exist.