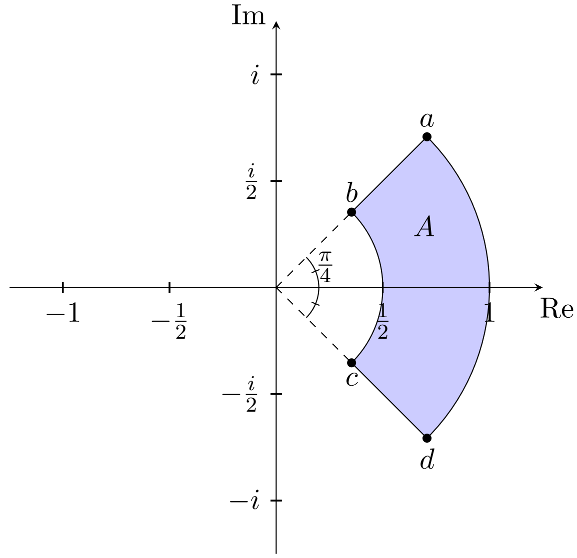
Homework 2 Handed out: Wednesday, September 14, 2022
Due: Wednesday, September 21, 2022 by 11:59pm

Material covered: Outcomes 2.1–2.3.

1. Let A be the annulus sector of points $z = re^{i\theta}$ such that $\frac{1}{2} \le r \le 1$ and $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ (pictured). For each of the following complex functions $f\colon \mathbb{C} \to \mathbb{C}$, sketch $f(A)$, i.e. the image of A under f. Label the images of the points a, b, c, d (i.e. label $f(a)$, $f(b)$, $f(c)$ $f(d)$).
   a) $f(z) = z^2$
   b) $f(z) = \frac{1}{z}$



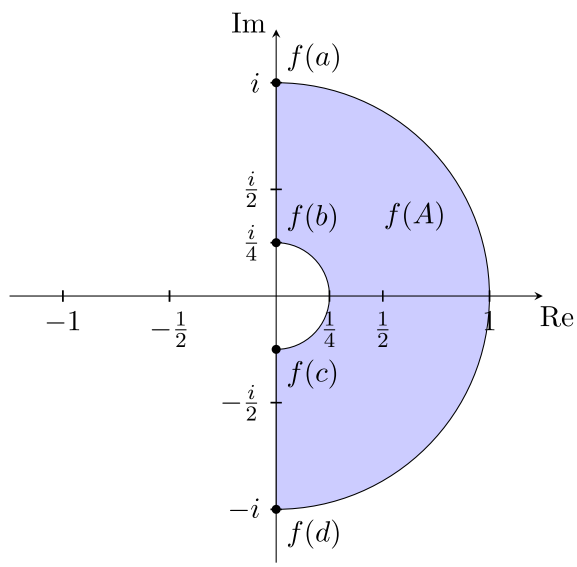
Solution.
a) Write $z = re^{i\theta}$ with $r > 0$ $-\pi < \theta < \pi$. Then

$$z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}.$$

A is the set of points with $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $\frac{1}{2} \leq r \leq 1$, and this set gets mapped to the annulus sector of points with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $\frac{1}{4} \leq r \leq 1$. This set is shown below. The points

$$
\begin{aligned}
f(a) &= (e^{i\frac{\pi}{4}})^2 = e^{i\frac{\pi}{2}} = i, \\
f(b) &= \left(\frac{1}{2}e^{i\frac{\pi}{4}}\right)^2 = \frac{1}{4}i, \\
f(c) &= \left(\frac{1}{2}e^{-i\frac{\pi}{4}}\right)^2 = -\frac{1}{4}i, \\
f(d) &= (e^{-i\frac{\pi}{4}})^2 = -i,
\end{aligned}
$$

are labeled.



b) Write $z = re^{i\theta}$ with $r > 0$ $-\pi < \theta < \pi$. Then

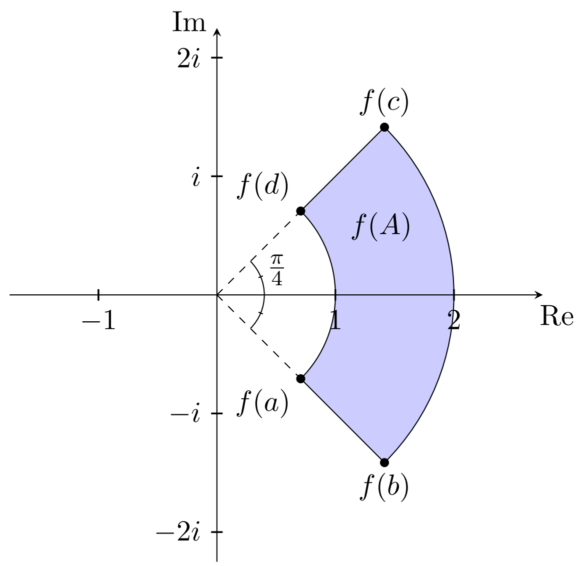
$$
\frac{1}{z} = (re^{i\theta})^{-1} = r^{-1}e^{-i\theta}.
$$

A is the set of points with $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $\frac{1}{2} \leq r \leq 1$, and this set gets mapped to the annulus sector of points with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and

$1 \le r \le 2$. This set is shown below. The points

$$f(a) = (e^{i\frac{\pi}{4}})^{-1} = e^{-i\frac{\pi}{4}},$$
$$f(b) = \left(\frac{1}{2}e^{i\frac{\pi}{4}}\right)^{-1} = 2e^{-i\frac{\pi}{4}},$$
$$f(c) = \left(\frac{1}{2}e^{-i\frac{\pi}{4}}\right)^{-1} = 2e^{i\frac{\pi}{4}},$$
$$f(d) = (e^{-i\frac{\pi}{4}})^{-1} = e^{i\frac{\pi}{4}},$$

are labeled.



2. For the following functions $f\colon \mathbb{C} \to \mathbb{C}$, evaluate the limit

$$\lim_{z \to z_0} f(z)$$

or prove that the limit does not exist. Is f continuous at z_0?

a)

$$f(z) = \begin{cases} \frac{z^5 - z}{z+i} & z \neq -i \\ 0 & z = -i \end{cases}, \qquad z_0 = -i.$$

b)

$$f(z) = \begin{cases} \frac{z^2+\bar{z}^2}{2i|z|} & z \neq 0 \\ 0 & z = 0 \end{cases}, \qquad z_0 = 0.$$

c)

$$f(z) = \begin{cases} \frac{x^2 y}{(x+iy)(x^2+y^2)} & z \neq 0 \\ 0 & z = 0 \end{cases}, \qquad z_0 = 0,$$

where $z = x + iy$, $x, y \in \mathbb{R}$.

Solution. **a)** We start by factoring the numerator:

$$z^5 - z = z(z^4 - 1) = z(z-1)(z+1)(z-i)(z+i).$$

Therefore, for $z \neq -i$, we have

$$f(z) = z(z-1)(z+1)(z-i).$$

Thus the limit is

$$\begin{aligned} \lim_{z \to -i} f(z) &= \lim_{z \to -i} \Big[z(z-1)(z+1)(z-i) \Big] \\ &= (-i)(-i-1)(-i+1)(-2i) = 4. \end{aligned}$$

The limit exists but does not equal $f(-i) = 0$, so f is discontinuous at $z = -i$.

We say that f has a *removable discontinuity* at $z = -i$ because there exists a function

$$\tilde{f}(z) = z(z-1)(z+1)(z-i)$$

which is continuous at $z = -i$ and coincides with f for $z \neq -i$. In other words, we can remove the discontinuity by changing the value of f at a single point.

b) We suspect that the limit is zero since the absolute value of the numerator scales like $O(|z|^2)$ while the absolute value of the numerator is $O(|z|)$ as $z \to 0$. Indeed, we have from the triangle inequality:

$$|z^2 + \overline{z}^2| \leq |z^2| + |\overline{z}^2| = |z|^2 + |\overline{z}|^2 = 2|z|^2.$$

Thus for $z \neq 0$, we have the bound

$$|f(z)| = \frac{|z^2 + \overline{z}^2|}{2|z|} \leq \frac{2|z|^2}{2|z|} = |z|.$$

The right hand side goes to zero as $z \to 0$, so we must have $|f(z)| \to 0$, $z \to 0$. Therefore,

$$\lim_{z \to 0} f(z) = 0 = f(0)$$

and f is continuous at $z = 0$.

c) The absolute value of the numerator and denominator both scale like $O(|z|^3)$ as $z \to 0$. Therefore, either the limit does not exist, or it exists and is a non-zero number.

To determine whether the limit exists, it is useful to write $z = re^{i\theta}$ with $r > 0$, $\theta \in \mathbb{R}$. Then $x = r\cos\theta$, $y = r\sin\theta$ and

$$\begin{aligned} x^2 y &= r^3 \cos(\theta)^2 \sin(\theta), \\ (x+iy)(x^2+y^2) &= re^{i\theta} \cdot r^2 = r^3 e^{i\theta}. \end{aligned}$$

so for $z \neq 0$ (equivalently $r > 0$),

$$\begin{aligned} f(z) &= \cos(\theta)^2 \sin(\theta) e^{-i\theta} \\ &= \cos(\theta)^2 \sin(\theta)(\cos\theta - i\sin\theta). \end{aligned}$$

We see that the limit as $z \to 0$ does not exist, since the limit

$$\lim_{r \searrow 0} f(re^{i\theta}) = \lim_{r \searrow 0} \Big[\cos(\theta)^2 \sin(\theta)(\cos\theta - i\sin\theta) \Big]$$

depends on the value of θ. For example, approaching $z = 0$ along the positive real axis (corresponding to $\theta = 0$) yields

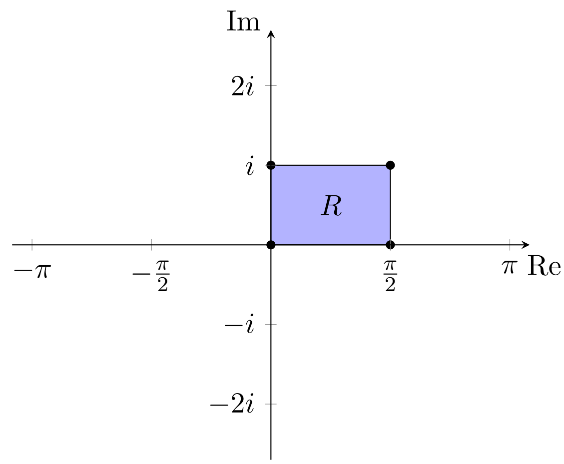
$$\lim_{r \searrow 0} f(r) = \lim_{r \searrow 0} \Big[\cos(0)^2 \sin(0)(\cos(0) - i\sin(0)) \Big] = \lim_{r \searrow 0} 0 = 0.$$

while approaching $z = 0$ along the diagonal defined by $\theta = \pi/4$, we find a different limit:

$$\begin{aligned} \lim_{r \searrow 0} f(re^{i\pi/4}) &= \lim_{r \searrow 0} \Big[\cos\Big(\frac{\pi}{4}\Big)^2 \sin\Big(\frac{\pi}{4}\Big)(\cos\Big(\frac{\pi}{4}\Big) - i\sin\Big(\frac{\pi}{4}\Big)) \Big] \\ &= \lim_{r \searrow 0} 2(1+i) = 2(1+i). \end{aligned}$$

Therefore, $\lim_{z \to 0} f(z)$ does not exist. This also implies that f is not continuous at 0.

3. Sketch the image of the rectangle R shown below under the map $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = \sin(z)$. Label the images of the corners, i.e. the points $f(0)$, $f(\pi/2)$, $f(\pi/2 + i)$, $f(i)$.



Solution.
Write $z = x + iy$ where $x, y \in \mathbb{R}$ and recall that

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

The image of R is

$$f(R) = \{\sin(x)\cosh(y) + i\cos(x)\sinh(y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le 1\}.$$

In order to visualize this set, let us find the boundary.

i. If we fix $y = 0$ and vary x, we see that the line segment on the real axis gets mapped to

$$\{\sin(t)\cosh(0) + i\cos(t)\sinh(0) \mid 0 \le t \le \frac{\pi}{2}\} =$$
$$\{\sin(t) + 0i \mid 0 \le t \le \frac{\pi}{2}\} = \{x + 0i \mid 0 \le x \le 1\}.$$

ii. Fixing $x = \pi/2$ and varying y, we find that the line segment between $\pi/2$ and $\pi/2 + i$ gets mapped to

$$\{\sin\left(\frac{\pi}{2}\right)\cosh(t) + i\cos\left(\frac{\pi}{2}\right)\sinh(t) \mid 0 \le x \le \frac{\pi}{2}\} =$$
$$\{\cosh(t) + 0i \mid 0 \le t \le 1\} = \{x + 0i \mid 1 \le x \le \cosh(1)\}.$$

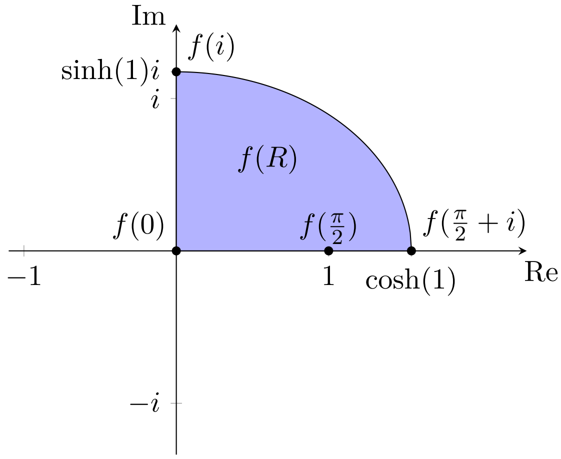
iii. Similarly, the line segment between 0 and i gets mapped to

$$\{\sin(0)\cosh(t) + i\cos(0)\sinh(t) \mid 0 \le t \le 1\} =$$
$$\{0 + i\sinh(t) \mid 0 \le t \le 1\} = \{0 + yi \mid 0 \le y \le \sinh 1\}.$$

iv. Finally, the line segment between i and $\pi/2 + i$ gets mapped to

$$\{\sin(t)\cosh(1) + i\cos(t)\sinh(1) \mid 0 \le t \le 1\}$$

which is a segment from an ellipse with semi-major axis $\cosh(1)$ and semi-minor axis $\sinh(1)$. The set $f(R)$ is shown below with the points $f(0)$, $f(\pi/2)$, $f(\pi/2 + i)$ and $f(i)$ labeled.



4. Let $z = x + iy$ where $x, y \in \mathbb{R}$. Find the real and imaginary parts of the following expressions in terms of x and y:
   a) $e^{1/z}$
   b) $\cos(z^2)$

Solution.
a) Let's start by writing the argument $1/z$ in terms of x, y:

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2},$$

Then

$$e^{1/z} = \exp\Big(\frac{x-iy}{x^2+y^2}\Big) = e^{\frac{x}{x^2+y^2}}\, e^{-i\frac{y}{x^2+y^2}}$$

which we can rewrite as

$$e^{1/z} = e^{\frac{x}{x^2+y^2}}\left[\cos\Big(\frac{y}{x^2+y^2}\Big) - i\sin\Big(\frac{y}{x^2+y^2}\Big)\right]$$

Thus

$$\mathrm{Re}\,(e^{1/z}) = e^{\frac{x}{x^2+y^2}}\cos\Big(\frac{y}{x^2+y^2}\Big),$$
$$\mathrm{Im}\,(e^{1/z}) = -e^{\frac{x}{x^2+y^2}}\sin\Big(\frac{y}{x^2+y^2}\Big)$$

b) We have $z^2 = (x^2 - y^2) + 2ixy$. Recall also that

$$\cos(w) = \cos(\operatorname{Re} w)\cosh(\operatorname{Im} w) - i\sin(\operatorname{Re} w)\sinh(\operatorname{Im} w).$$

Thus

$$\cos(z^2) = \cos(x^2 - y^2)\cosh(2xy) - i\sin(x^2 - y^2)\sinh(2xy)$$

from which we can read the real and imaginary parts:

$$\begin{aligned}\operatorname{Re}\cos(z^2) &= \cos(x^2 - y^2)\cosh(2xy),\\ \operatorname{Im}\cos(z^2) &= -\sin(x^2 - y^2)\sinh(2xy).\end{aligned}$$

5. Find all solutions $z \in \mathbb{C}$ of the following equations:
   a) $e^z = -1$
   b) $(\sin(z))^2 = 4$

Solution.
a) It is convenient to write $z = x + iy$ where $x, y \in \mathbb{R}$. We then see that $1 = |e^z| = |e^x e^{iy}| = e^x$ which implies that $x = 0$. The equation is thus equivalent to

$$\cos(y) + i\sin(y) = -1.$$

Equating real and imaginary parts, we see that y is an odd multiple of π. Thus the solutions are

$$z = (2n+1)\pi i, \quad n \in \mathbb{Z}.$$

b) Let $w = \sin(z)$ and start by solving $w^2 = 4$. The solutions are $w = \pm 2$. We now proceed to solve the equations $\sin(z) = 2$ and $\sin(z) = -2$ separately. Write $z = x + iy$ where $x, y \in \mathbb{R}$ and use

$$\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

To solve $\sin(z) = 2$, equate real and imaginary parts in

$$\sin(x)\cosh(y) + i\cos(x)\sinh(y) = 2$$

to conclude that $\cos(x)\sinh(y) = 0$ and $\sin(x)\cosh(y) = 2$. The first equation implies that

$$y = 0 \quad \text{or} \quad x = \frac{(n+1)\pi}{2},\ n \in \mathbb{Z}.$$

Suppose that $y = 0$. Then $\sin(z) = \sin(x)$ with x real, and $\sin(x) = 2$ does not have any solutions. Therefore, we must have

$$x = \frac{(n+1)\pi}{2},\ n \in \mathbb{Z},$$

and

$$\sin(z) = \sin\Big((n+1)\frac{\pi}{2}\Big)\cosh(y) = (-1)^n\cosh(y) = 2.$$

Since cosh is non-negative on the reals, we see that n must be even and $y = \pm\,\mathrm{arcosh}(2)$.

Similarly, we see that the solutions to $\sin(z) = -2$ are $z = (n+1)\pi/2 \pm i\,\mathrm{arcosh}(2)$ with n odd. Putting the two cases together, the solutions to $(\sin(z))^2 = 4$ are

$$z = \frac{(n+1)\pi}{2} \pm i\,\mathrm{arcosh}(2), \quad n \in \mathbb{Z}.$$