Section 8 Friday, November 4, 2022

Material covered: Outcomes 7.1–7.4.

Solutions

Notation. We will occasionally need to consider the Fourier series of several functions at the same time. It those cases, it is convenient to use the notation $a_n(f)$, $b_n(f)$, $c_n(f)$ to denote the Fourier coefficients of a function f.

1. Consider the 2π-periodic function f which on $[-\pi, \pi]$ is given by

$$f(x) = \pi - |x|, \qquad -\pi \le x \le \pi.$$

a) Sketch the graph of the function f and its derivative f' on the interval $[-3\pi, 3\pi]$.

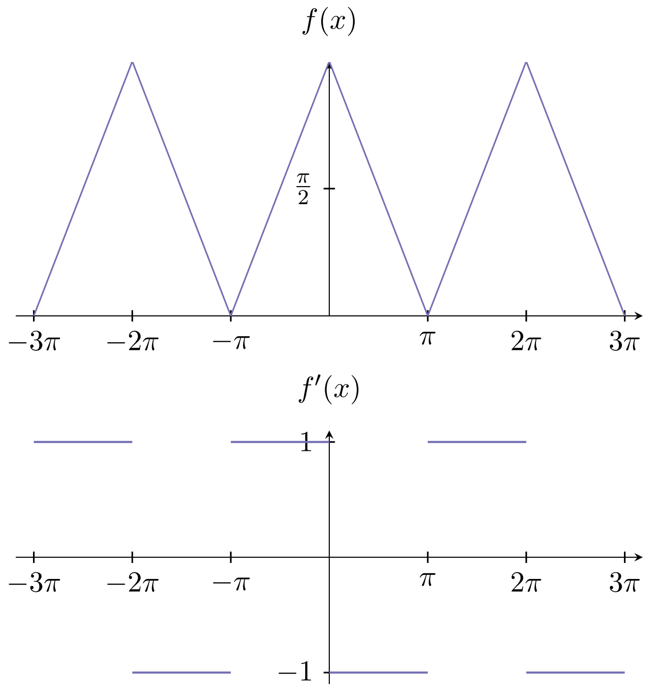
b) Find the Fourier series of f in trigonometric form; i.e. find the coefficients a_0, a_n and b_n in

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big)$$

c) Find the Fourier series of f in exponential form; i.e. find the coefficients c_n in

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \tag{1}$$

Solution.
a) The graphs of f and f' are shown below.



b) Since f is an even function, the sine coefficients b_n are all zero. To compute the remaining coefficients, we again exploit the symmetry of f and notice that the integral over $[-\pi, \pi]$ is twice the integral over $[0, \pi]$. We have:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \mathrm{d}x = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - |x|) \, \mathrm{d}x = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \, \mathrm{d}x = \pi,$$

and for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{d}x = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - |x|) \cos(nx) \, \mathrm{d}x \\ &= \int_{-\pi}^{\pi} \cos(nx) \, \mathrm{d}x - \frac{2}{\pi} \int_0^{\pi} x \cos(nx) \, \mathrm{d}x \end{aligned}$$

The first integral is zero and the second can be evaluated using integration by parts:

$$\int_0^\pi x\cos(nx)\,\mathrm{d}x = \left[\frac{1}{n}x\sin(nx)\right]_0^\pi - \frac{1}{n}\int_0^\pi \sin(nx)\,\mathrm{d}x$$
$$= 0 - \frac{1}{n}\left[-\frac{1}{n}\cos(nx)\right]_0^\pi = \frac{(-1)^n - 1}{n^2}$$

So the cosine coefficients are

$$a_n = \frac{2}{\pi}\frac{1-(-1)^n}{n^2}, \qquad n \geq 1.$$

Thus the Fourier series of f (in trigonometric form) is

$$\frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos(nx).$$

Noting that $a_n = 0$ whenever $n > 0$ is even, another way to write the Fourier series is

$$\frac{\pi}{2} + \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty}\frac{1}{n^2}\cos(nx).$$

c) Since we know the coefficients a_n and b_n, the easiest way to find the Fourier series in exponential form is to use the relations

$$\begin{aligned} a_0 &= 2c_0 \\ a_n &= c_n + c_{-n} \\ b_n &= i(c_n - c_{-n}) \end{aligned} \tag{2}$$

which can be inverted to find

$$\begin{aligned} c_0 &= \tfrac{1}{2}a_0 \\ c_n &= \tfrac{1}{2}(a_n - ib_n) \qquad (n \geq 1) \\ c_{-n} &= \tfrac{1}{2}(a_n + ib_n) \qquad (n \geq 1). \end{aligned} \tag{3}$$

In our case, $b_n = 0$ for all n so $c_n = c_{-n} = a_n/2$ for all n. Thus we see that $c_0 = \pi/2$,

$$c_n = \frac{2}{\pi}\frac{1-(-1)^n}{n^2}, \qquad n \neq 0$$

and the Fourier series in exponential form is

$$\frac{\pi}{2} + \frac{4}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1-(-1)^n}{n^2} e^{inx}.$$

Since f is piecewise continuously differentiable and continuous, the Fourier series converges pointwise to f. Therefore, we can write

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2}\cos(nx) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1-(-1)^n}{n^2} e^{inx}.$$

2. a) Assume that f and g are functions with the same period $2L$. Show that $f+g$ is $2L$-periodic and

$$c_n(f+g) = c_n(f) + c_n(g).$$

b) Assume that f is 2π-periodic, piecewise continuously differentiable and continuous. Show that

$$c_n(f') = inc_n(f), \quad a_n(f') = nb_n(f), \quad b_n(f') = -na_n(f).$$

How does the result change if f is $2L$-periodic?

Solution.
a) The fact that $f+g$ is $2L$-periodic follows immediately from the definition: For any $x \in \mathbb{R}$,

$$(f+g)(x+2L) = f(x+2L) + g(x+2L) = f(x) + g(x) = (f+g)(x).$$

The second statement follows from the linearity of integration For the Fourier coefficients:

$$\begin{aligned}
c_n(f+g) &= \frac{1}{2L}\int_{-L}^{L}(f+g)(x)e^{-in\pi x/L}\,\mathrm{d}x \\
&= \frac{1}{2L}\int_{-L}^{L}f(x)e^{-in\pi x/L}\,\mathrm{d}x + \frac{1}{2L}\int_{-L}^{L}g(x)e^{-in\pi x/L}\,\mathrm{d}x = c_n(f)+c_n(g).
\end{aligned}$$

b) Let's prove the result for the c_n coefficients of a $2L$-periodic function. Using integration by parts, we have

$$\begin{aligned}
\int_0^{2L} f'(x)e^{-in\pi x/L}\,\mathrm{d}x &= \Big[f(x)e^{-in\pi x/L}\Big]_0^{2L} + \frac{in\pi}{L}\int_0^{2L} f(x)e^{-in\pi x/L}\,\mathrm{d}x \\
&= \Big[f(2L)-f(0)\Big] + \frac{in\pi}{L}\int_0^{2L} f(x)e^{-in\pi x/L}\,\mathrm{d}x
\end{aligned}$$

But f is $2L$-periodic, so we are left with

$$c_n(f') = \frac{in\pi}{L}c_n(f).$$

The corresponding rules for a_n and b_n follow easily from the relations (2):

$$\begin{aligned}
a_n(f') = c_n(f') + c_{-n}(f') &= \frac{in\pi}{L}c_n(f) + \frac{i(-n)\pi}{L}c_{-n}(f) \\
&= \frac{n\pi}{L}i(c_n(f)-c_{-n}(f)) = \frac{n\pi}{L}b_n(f).
\end{aligned}$$

Similarly for $b_n(f')$.

3. Let f be the function from problem 1. Find the Fourier series of f' in trigonometric form and exponential form.

Solution.
Since f is piecewise continuously differentiable and continuous, we can obtain the Fourier series of f' by using the rules we proved in

problem 2(b), or equivalently by differentiating the Fourier series of f term by term. This is considerably easier than computing the integrals as we did in part (b). The Fourier series of f' is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\pi}{2}+\frac{4}{\pi}\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{n^2}\cos(nx)\right]=-\frac{4}{\pi}\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{n}\sin(nx).$$

Thus f' has the Fourier coefficients

$$\begin{aligned} a_n &= 0, \qquad n\geq 0,\\ b_n &= -\frac{4}{\pi n}, \qquad n\text{ odd}, b_n &= 0, \qquad n\text{ even}, \end{aligned}$$

Similarly for the Fourier series in exponential form:

$$\frac{4}{\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1-(-1)^n}{n^2}(in)e^{inx}.=\frac{4i}{\pi}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1-(-1)^n}{n}e^{inx}.$$

Since f' is piecewise continuously differentiable but *not* continuous, the Fourier series converges pointwise at x to

$$\frac{f'(x^+)+f'(x^-)}{2}=\lim_{h\searrow 0}\frac{f'(x+h)+f'(x-h)}{2}=\begin{cases}0, & x=n\pi, n\in\mathbb{Z}\\ f'(x), & \text{otherwise.}\end{cases}$$

4. Let f be the 2-periodic function defined by

$$f(x)=\frac{1}{6}x(x-1)(x+1), \qquad |x|\leq 1.$$

a) Compute f'' and sketch the graph of f'' on the interval $[-3,3]$.

b) Do we expect the Fourier series of f to exhibit the Wilbraham–Gibbs phenomenon? What about f' and f''?

c) Find the Fourier series of f by first computing the Fourier coefficients of f'' and integrating the Fourier series of f'' term by term twice.

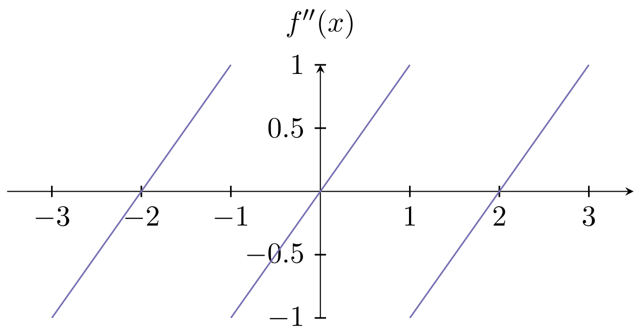
Solution.
a) Since f is 2-periodic, its derivatives are also 2-periodic, so it suffices to compute f'' on $[-1, 1]$. We have

$$f'(x) = \frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}x}\Big(x^3 - x\Big) = \frac{1}{6}(3x^2 - 1), \quad |x| \leq 1,$$

and

$$f''(x) = x, \quad |x| < 1,$$

Note that $f''(x)$ is undefined when x is an odd integer because f' is not differentiable there. However, the one-sided limits $f''(x^+)$ and $f''(x^-)$ exist at these points. The graph of f'' is shown below.

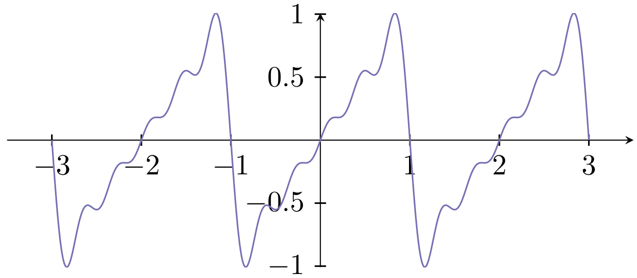


b) The Fourier series of a function g will exhibit the Wilbraham–Gibbs phenomenon if g has jump discontinuities. On the other hand, if g is piecewise continuously differentiable and continuous, then the Fourier series converges uniformly to g and thus does not exhibit the Wilbraham–Gibbs phenomenon. Since f and its derivatives are smooth on the interior of the interval $[-1, 1]$, it suffices to check whether they are continuous at $x = \pm 1$. Since they are 2-periodic, this is equivalent to asking whether they take the same value at 1 and -1.

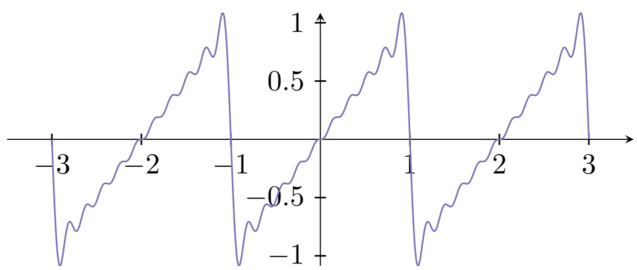
Now $f(1) = f(-1) = 0$, so f is continuous. Likewise, f' is continuous since $f'(1) = f'(-1) = 1/3$. On the other hand, $f''(1^-) = 1$ while $f''(1^+) = f''(-1^+) = -1$, so f'' has a jump discontinuity at

$x = \pm 1$. We conclude that the Fourier series of f'' will exhibit the Wilbraham–Gibbs phenomenon. The partial Fourier sum S_5, S_{10} and S_{20} of f'' are shown below. Notice that while the approximation is improving, the maximum pointwise error does not go to zero as we increase the number of points.

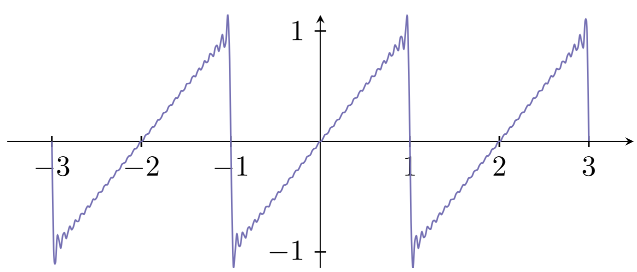
$S_5(x)$



$S_{10}(x)$



$S_{30}(x)$



c) Since f'' is odd, let us find the Fourier series in trigonometric form. The coefficients a_n all vanish, and the coefficients b_n are

$$b_n = \int_{-1}^{1} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = 2\ int_0^1 x \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2(-1)^{n+1}}{n\pi}$$

where we integrated by parts in the last step. Thus the Fourier series of f'' is

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

and it converges pointwise to $\frac{1}{2}(f''(x^+) + f''(x^-))$.

Let us use this to find the Fourier series of f. We note that f is also odd, so the coefficients $a_n(f)$ vanish. To find $b_n(f)$, we use the result of Problem 2 b) twice:

$$b_n(f'') = -n\pi a_n(f') = -(n\pi)^2 b_n(f)$$

Therefore,

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^3\pi^3} \sin(n\pi x),$$

where the equality is justified because the Fourier series converges pointwise to $f(x)$. In fact, the convergence is very fast because $|b_n| \propto n^{-3}$.

5. Find the error in the following incorrect "proof"

Let f be 2π-periodic and continuously differentiable. Then f is smooth, i.e. $f^{(n)}$ exists and is continuous for all n.

Proof: Since f is continuously differentiable, it is equal to its Fourier series, so we can write

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

for appropriately chosen c_n. Now we can extend f to the complex plane as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{inz}$$

and we see that f is analytic on the complex plane, since

$$\partial_{\bar{z}} f(z) = \sum_{n=-\infty}^{\infty} c_n \partial_{\bar{z}} e^{inz} = \sum_{n=-\infty}^{\infty} c_n \cdot 0 = 0.$$

Now, every analytic function is smooth, so we conclude that the restriction of f to the real line is smooth.

Solution.
Clearly the proof must be incorrect, because we have seen several examples of periodic functions which are continuously differentiable but not smooth, for example the function f in problem 4.

The mistake is that the series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{inz}$$

will generally not converge if $\operatorname{Im} z \neq 0$. To see this, note that if $z = x + iy$ then $|e^{inz}| = (e^{-y})^n$ is rapidly growing as $n \to \infty$ if $y < 0$, or as $n \to -\infty$ if $y > 0$. If f is not sufficiently smooth, then the terms $c_n e^{inz}$ in the above series will grow in magnitude rather than decay, and thus the series diverges. Therefore, the series does not define an analytic function on the complex plane.

In class, we discussed a connection between the smoothness of f and the rate of decay of the Fourier coefficients (equivalently, the rate of convergence of the Fourier series). If f has $k-1$ continuous derivatives and $f^{(k)}$ is discontinuous, then the Fourier coefficients of f decay polynomially in n,

$$\lim_{|n| \to \infty} n^k c_n(f) = 0,$$

while the quantity $|e^{inz}|$ is increasing geometrically in n.