Section 9 Friday, November 11, 2022

Material covered: Outcomes 8.1–8.5.

Solutions

1. Use the derivative property of the Fourier transform and the Fourier inversion theorem

$$(\mathcal{F}\,\mathcal{F}\,f)(x) = 2\pi f(-x)$$

to prove that

$$\mathcal{F}\left(x^n f(x)\right)(k) = i^n \frac{\mathrm{d}^n}{\mathrm{d}k^n}\,\mathcal{F}(f)(k) = i^n \hat{f}^{(n)}(k). \tag{1}$$

whenever f and $x^n f(x)$ are integrable.

Solution.
Let us prove the claim for $n = 1$:

$$\mathcal{F}\left(xf(x)\right)(k) = i\frac{\mathrm{d}}{\mathrm{d}k}\,\mathcal{F}(f)(k) = i\hat{f}'(k).$$

The general case then follows by induction.
Let $g = \mathcal{F}^{-1} f$ (so that $f = \mathcal{F}\, g$) and notice that the Fourier inversion theorem implies that

$$g(k) = \left(\mathcal{F}^{-1}\, f\right)(k) = \frac{1}{2\pi}(\mathcal{F}\, f)(-k). \tag{2}$$

Start with the derivative property, applied to the function g:

$$\mathcal{F}(g')(x) = ix\,\mathcal{F}(g)(x) = ixf(x).$$

We now take the Fourier transform of both sides:

$$\mathcal{F}\left(\mathcal{F}(g')\right)(k) = \mathcal{F}\left(ixf(x)\right)(k).$$

Using the Fourier inversion theorem,

$$\mathcal{F}\Big(ixf(x)\Big)(k) = 2\pi g'(-k). \tag{3}$$

To find $g'(-k)$ in terms of $(\mathcal{F}f)'(k)$, differentiate equation (2) and use the chain rule:

$$g'(k) = -\frac{1}{2\pi}\big(\mathcal{F}f\big)'(-k).$$

Substituting into (3), we obtain

$$\mathcal{F}\Big(ixf(x)\Big)(k) = \frac{1}{2\pi}\cdot\Big(-\frac{1}{2\pi}\big(\mathcal{F}f\big)'(k)\Big) = -\big(\mathcal{F}f\big)'(k),$$

which by linearity is equivalent to

$$\mathcal{F}\Big(xf(x)\Big)(k) = i\big(\mathcal{F}f\big)'(k).$$

2. a) Show that if $f \in L^1(\mathbb{R})$ is an even function, then

$$\hat{f}(k) = 2\int_0^\infty \cos(kx)f(x)\,\mathrm{d}x.$$

Use this to conclude that $\hat{f}$ is even.

b) For $f \in L^1(\mathbb{R})$ odd, derive a similar expression for $\hat{f}(k)$ in terms of an integral over the positive real line. Use this to conclude that $\hat{f}$ is odd.

Solution.
a) Start from the definition of $\hat{f}(k)$ and use Euler's formula:

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^\infty f(x)e^{-ikx}\,\mathrm{d}x \\ &= \int_{-\infty}^\infty f(x)\cos(kx)\,\mathrm{d}x - i\int_{-\infty}^\infty f(x)\sin(kx)\,\mathrm{d}x\end{aligned} \tag{4}$$

If f is even, then $f(x)\sin(kx)$ is odd and therefore

$$\int_{-\infty}^{\infty} f(x)\sin(kx)\,\mathrm{d}x = 0.$$

The other integrand $f(x)\cos(kx)$ is even, and therefore,

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\cos(kx)\,\mathrm{d}x = 2\int_{0}^{\infty} f(x)\cos(kx)\,\mathrm{d}x.$$

Since cos is even, it follows easily that $\hat{f}$ is even:

$$\hat{f}(-k) = 2\int_{0}^{\infty} f(x)\cos(-kx)\,\mathrm{d}x = 2\int_{0}^{\infty} f(x)\cos(kx)\,\mathrm{d}x = \hat{f}(k).$$

b) Starting from equation (4) and assuming f is odd, we see that the first integral is zero while the second is

$$\hat{f}(k) = -i\int_{-\infty}^{\infty} f(x)\sin(kx)\,\mathrm{d}x = -2i\int_{0}^{\infty} f(x)\sin(kx)\,\mathrm{d}x.$$

It follows that $\hat{f}$ is odd.

3. a) Compute the Fourier transform of

$$f(x) = \frac{1}{(1+x^2)^2}$$

using residue calculus.

b) Compute the Fourier transform of the following functions:

$$f_1(x) = \frac{x}{(1+x^2)^2}$$

$$f_2(x) = \frac{5x^2-1}{(1+x^2)^4} = \frac{1}{4}f''(x).$$

Solution.

a) Notice that f is even, so $\hat{f}$ is also even (recall problem 2). Therefore, it suffices to compute $\hat{f}(k)$ for, say, $k < 0$. Then the values of $\hat{f}(k)$ for $k > 0$ are given by

$$\hat{f}(k) = -\hat{f}(-k).$$

Since f is analytic, we can evaluate the integral using residue calculus. For $k < 0$, we can close the contour with a semicircle in the upper half-plane. It follows from the usual argument[1] (using Jordan's lemma to bound the integral over the semicircle), that

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\,\mathrm{d}x = 2\pi i \sum_{z_i \in A} \operatorname{Res}\bigl(f(z)e^{-ikz}, z_i\bigr)$$

where A is the set of singular points of f in the upper half-plane. Now

$$f(z) = \frac{1}{(z+i)^2(z-i)^2}$$

so the only pole of f in the upper half-plane is a double pole at $z = i$. Let us compute the residue. If we let $g(z) = (z-i)^2 f(z) e^{-ikz}$, then

$$\begin{aligned}\operatorname{Res}\bigl(f(z)e^{-ikz}, i\bigr) = \frac{1}{1!}g'(i) &= \left[\frac{-ike^{-ikz}}{(z+i)^2} - \frac{2e^{-ikz}}{(z+i)^3}\right]_{z=i} \\ &= -k\frac{e^k}{4i} + \frac{2e^k}{8i} = (1-k)\frac{e^k}{4i}.\end{aligned}$$

So for $k < 0$,

$$\hat{f}(k) = \frac{\pi}{2}(1-k)e^k = \frac{\pi}{2}(1+|k|)e^{-|k|},$$

[1] This is a routine argument by now so we omit the details. If you cannot fill in the steps, the reasoning is explained in the solution to Problem 3 on Homework 7, for example.

where we have written $\hat{f}(k)$ in terms of $|k|$ in anticipation of the next step. Since $\hat{f}$ is even, we have for $k > 0$

$$\hat{f}(k) = \hat{f}(-k) = \frac{\pi}{2}(1 + |k|)e^{-|k|}.$$

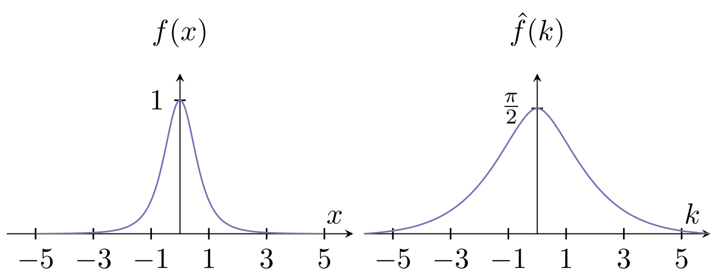
What is the value of $\hat{f}(0)$? Since f is integrable, $\hat{f}(k)$ must be continuous and so

$$\hat{f}(0) = \lim_{k \to 0} \frac{\pi}{2}(1 + |k|)e^{-|k|} = \frac{\pi}{2}.$$

To summarize, the formula

$$\hat{f}(k) = \frac{\pi}{2}(1 + |k|)e^{-|k|}$$

is valid for all k. The graphs of f and $\hat{f}$ are shown below.



b) Computing Fourier transforms using the definition as in part (a) can be labor intensive, so we use shortcuts whenever possible. From the property we proved in Exercise 1,

$$\hat{f}_1(k) = \mathcal{F}\left(xf(x)\right)(k) = i\hat{f}'(k).$$

How do we compute the derivative of a function which involves absolute values? Note that since $\hat{f}$ is even, $\hat{f}'$ is odd. Therefore, it is enough to compute the derivative for, say, $k > 0$:

$$\hat{f}'(k) = \frac{\mathrm{d}}{\mathrm{d}k}\left[\frac{\pi}{2}(1 + k)e^{-k}\right] = -\frac{\pi}{2}ke^{-k} \tag{5}$$

Here we replaced $|k|$ with k since $k > 0$ by assumption, and computed the derivative in the usual way. Since f is odd, it follows that for $k < 0$,

$$\hat{f}'(k) = -\hat{f}'(-k) = -\frac{\pi}{2} k e^{k}.$$

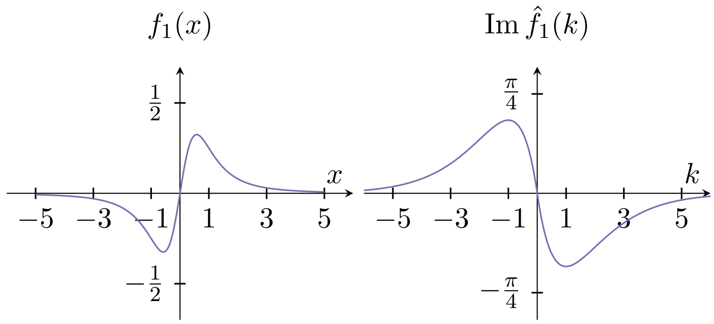
We see that we can write

$$\hat{f}'(k) = -\frac{\pi}{2} k e^{-|k|} \tag{6}$$

and this expression is valid for all k.[2] Thus

$$\hat{f}_1(k) = -i\frac{\pi}{2} k e^{-|k|}$$

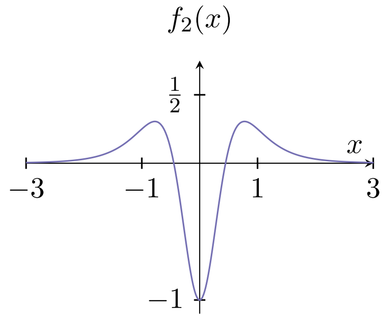
The graphs of f_1 and $\hat{f}_1$ are shown below.

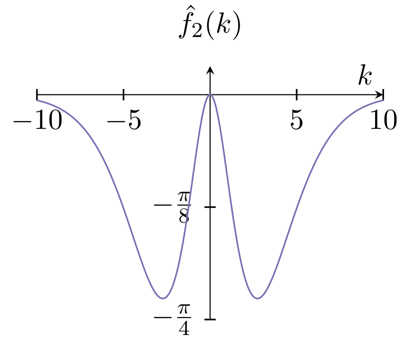


The Fourier transform of f_2 is easily computed using the derivative property:

$$\hat{f}_2(k) = \frac{1}{4}\mathcal{F}\left(f''\right)(k) = \frac{1}{4}(ik)^2 \hat{f}(k) = -\frac{\pi}{8} k^2 (1 + |k|) e^{-|k|}$$

[2] With practice, you will be able to deduce the final expression (6) directly from (5).





4. (Optional) Fourier analysis was originally developed to solve differential equations. In last week's homework, we saw how Fourier series can be used to obtain solutions of a linear differential equations with a periodic right-hand side ("driving force"). In this problem, we'll show how to find a solution to a linear differential equation with a general right-hand side using the Fourier transform.

   Consider the differential equation

   $$-u''(x) + \omega^2 u(x) = e^{-|x|}, \tag{7}$$

   where you may assume $\omega > 0$, $\omega \neq 1$.

   a) Take the Fourier transform of both sides of the equation to obtain an algebraic equation for $\hat{u}(k)$.

   b) Solve the equation to find $\hat{u}(k)$.

   c) Compute the inverse Fourier transform of $\hat{u}$ to find a solution $u(x)$ to the equation (7).

Solution.

a) We need to find the Fourier transform of the right hand side, $f(x) = e^{-|x|}$. We computed the Fourier transform of this function in class:

$$\hat{f}(k) = \frac{2}{1+k^2}.$$

Now we take Fourier transforms of each side of the differential equation. Using the rule

$$\mathcal{F}\big[f^{(n)}(x)\big](k) = (ik)^n \hat{f}(k)$$

we see that $\hat{u}$ satisfies

$$k^2\hat{u}(k) + \omega^2\hat{u}(k) = \frac{2}{1+k^2}.$$

b) This is a simple algebraic equation for $\hat{u}(k)$ and is easily solved:

$$\hat{u}(k) = \frac{1}{\omega^2+k^2}\frac{2}{1+k^2}$$

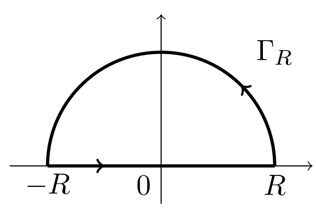
c) We compute the inverse Fourier transform

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{(\omega^2+k^2)(1+k^2)}e^{ixk}\,\mathrm{d}k \tag{8}$$

using contour integration and the residue theorem. We have

$$\hat{u}(k) = \frac{2}{(k-i\omega)(k+i\omega)(k-i)(k+i)}.$$

We see that $\hat{u}$ is analytic except at the points $k = \pm i\omega$ and $k = \pm i$ where it has simple poles. Before we start computing residues, let us note that $\hat{u}$ is an even real-valued function, so u is also even and real-valued. Therefore, it suffices to compute the integral (8) for $x > 0$. In that case, we can close the contour in the upper half-plane, i.e. we can use the following contour:



Jordan's lemma guarantees that that the integral over the semi-circular part goes to zero in the limit $R \to \infty$. Thus $u(x)$ is equal to the integral over the entire contour Γ_R in the limit $R \to \infty$, which is $2\pi i$ times the sum of residues in the upper half-plane:

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k) e^{ixk} \, \mathrm{d}k = i \operatorname{Res}(\hat{u}(k)e^{ixk}, i\omega) + i \operatorname{Res}(\hat{u}(k)e^{ixk}, i),$$

for $x > 0$. All that remains is to compute the residues:

$$\operatorname{Res}(\hat{u}(k)e^{ixk}, i\omega) = \left[\frac{2e^{ixk}}{(k+i\omega)(k^2+1)}\right]_{k=i\omega} = \frac{2e^{-\omega x}}{2i\omega(1-\omega^2)}$$

$$\operatorname{Res}(\hat{u}(k)e^{ixk}, i) = \left[\frac{2e^{ixk}}{(k^2+\omega^2)(k+i)}\right]_{k=i} = \frac{2e^{-x}}{2i(\omega^2-1)}$$

And we find (using the fact that u is even):

$$u(x) = \frac{e^{-\omega|x|}}{\omega(1-\omega^2)} + \frac{e^{-|x|}}{(\omega^2-1)} = \frac{1}{1-\omega^2}\left[\frac{1}{\omega}e^{-\omega|x|} - e^{-|x|}\right].$$

We have found a particular solution to the ODE. The general solution is

$$u(x) + c_1 \cosh(\omega x) + c_2 \sinh(\omega x)$$

where c_1, c_2 are constants.