Section 1 Friday, September 9, 2022

Material covered: Outcomes 1.1–1.5.

Solutions

1. Simplify the following expressions, writing them in the form $x + iy$ where x, y are real numbers:

   a)
   $$\frac{1}{(2+i)^2}$$

   b)
   $$\overline{2i - ie^{i\frac{\pi}{3}}}$$

   c)
   $$\omega + \omega^2 + \omega^3 + \omega^4 \quad \text{where } \omega = e^{i\frac{2\pi}{5}}.$$

   Solution.

   a) We multiply both numerator and denominator by $\overline{(2+i)^2} = 3 - 4i$ and simplify:

   $$\frac{1}{(2+i)^2} = \frac{3-4i}{(2+i)^2\overline{(2+i)^2}} = \frac{3-4i}{|2+i|^4} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i.$$

   b) We will make use of the following: For all complex numbers $z, w \in \mathbb{C}$, $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\overline{w}$. Thus

   $$\overline{2i - ie^{i\frac{\pi}{3}}} - 2i + i\overline{e^{i\frac{\pi}{3}}} = -2i + ie^{-i\frac{\pi}{3}} = -2i + e^{i\frac{\pi}{6}}.$$

   Here we have used $ie^{-i\frac{\pi}{3}} = e^{i\frac{\pi}{2}}e^{-i\frac{\pi}{3}} = e^{i\pi(\frac{1}{2}-\frac{1}{3})} = e^{i\frac{\pi}{6}}$. Now,

   $$-2i + e^{i\frac{\pi}{6}} = -2i + \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{3}{2}i.$$

c) One approach is to write the numbers $\omega, \omega^2, \omega^3, \omega^4$ in Cartesian form and then add the real and imaginary parts as usual. However, this is quite tedious. Instead, let us use the summation formula for a geometric progression. Let $S \coloneqq \omega + \omega^2 + \omega^3 + \omega^4$ and note that

$$\omega S = \omega^2 + \omega^3 + \omega^4 + \omega^5 = S - \omega + 1,$$

or $S(\omega - 1) = -(\omega - 1)$. Solving for S, we find that $S = -1$. We see that the five roots of unity sum to zero:

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0.$$

Can you generalize this to the nth roots of unity?

2. Write the following complex numbers in the form $x + iy$ where x, y are real numbers:

a)
$$\frac{3e^{i\frac{5\pi}{3}}}{\sqrt{3}e^{i\frac{5\pi}{6}}}$$

b)
$$\left(e^{i\frac{\pi}{8}}\right)^{1000}$$

Write the following complex numbers in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$:

c)
$$ie^{i\frac{2\pi}{9}}$$

d)
$$e^{i\frac{\pi}{4}} + e^{-i\frac{\pi}{4}}.$$

Solution.

a) It is wise to divide first before converting to Cartesian form:

$$\frac{3e^{i\frac{5\pi}{3}}}{\sqrt{3}e^{i\frac{5\pi}{6}}} = \sqrt{3}e^{i\pi\left(\frac{5}{3}-\frac{5}{6}\right)} = \sqrt{3}e^{i\frac{5\pi}{6}} = \sqrt{3}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\right) = -\frac{3}{2}+\frac{\sqrt{3}}{2}.$$

b)
$$\left(e^{i\frac{\pi}{8}}\right)^{1000} = e^{i\frac{\pi}{8}\cdot 1000} = e^{i125\pi} = -1.$$

c)
$$ie^{i\frac{2\pi}{9}} = e^{i\frac{\pi}{2}}e^{i\frac{2\pi}{9}} = e^{i\frac{13}{18}\pi}.$$

d) We convert the numbers to Cartesian form before adding:

$$e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}} = \frac{\sqrt{2}}{2}(1+i) - \frac{\sqrt{2}}{2}(1-i) = \sqrt{2}i = \sqrt{2}e^{i\frac{\pi}{2}}.$$

3. Find all $z \in \mathbb{C}$ that satisfy the equation and sketch the solution set:

a)
$$(1-i)z = (1+i)\overline{z}.$$

b)
$$z + i\overline{z} = 1+i.$$

Solution.

a) Let $z \in \mathbb{C}$ be a solution and write $z = x + iy$ where $x, y \in \mathbb{R}$. Then x, y satisfy

$$(1-i)(x+iy) = (1+i)(x-iy).$$

This equation is equivalent to a set of two coupled linear equations in the real variables x, y. To see this, rewrite the equation as

$$(x+y) + i(y-x) = (x+y) + i(x+y).$$

Equating real and imaginary parts, we obtain the equivalent linear system:

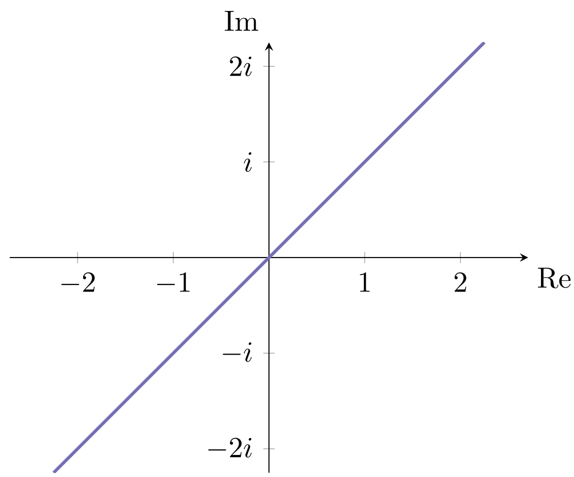
$$x + y = x + y$$
$$y - x = x - y.$$

The first equation is always true and the second is equivalent to $x = y$. Therefore, the solution set is

$$\{x + iy \in \mathbb{C} \mid x = y,\ x \in \mathbb{R}, y \in \mathbb{R}\}$$

which is a line in the complex plane, and may also be written

$$\{(1 + i)s \in \mathbb{C} \mid s \in \mathbb{R}\}.$$



b) Let $z \in \mathbb{C}$ be a solution and write $z = x + iy$ where $x, y \in \mathbb{R}$. Then x, y satisfy

$$(x + iy) + i(x - iy) = 1 + i,$$

or equivalently

$$(x + y) + i(x + y) = 1 + i.$$

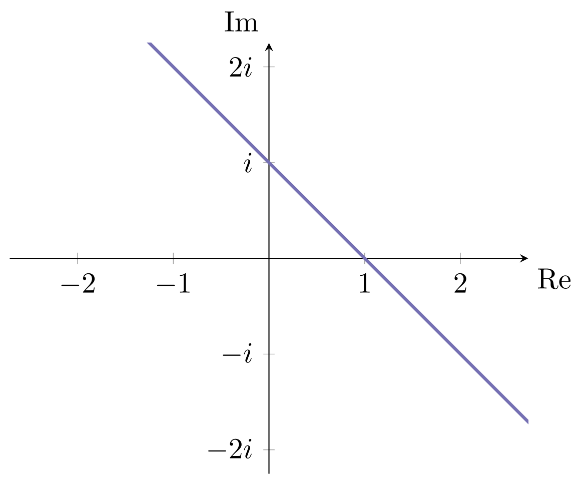
By equating real and imaginary parts (or simply by dividing by $1+i$ on both sides), we find that this is equivalent to

$$x + y = 1.$$

Thus the solution set is

$$\{x+iy \in \mathbb{C} \mid x+y = 1,\ x \in \mathbb{R}, y \in \mathbb{R}\} = \{s+(1-s)i \in \mathbb{C} \mid s \in \mathbb{R}\}.$$

which is a line in the complex plane.



4. a) Find all fourth roots of -1.
   b) Factor the polynomial $z^4 + 1$.
   c) Find the partial fraction decomposition of the rational fraction

$$R(z) = \frac{z^2}{z^4 + 1}.$$

Solution.

a) We solve the equation $z^4 = -1$ for z. We have $|z|^4 = |z^4| = |-1| = 1$ from which we conclude that $|z| = 1$. Write z in polar form as $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$. By De Moivre's formula,

$$z^4 = \cos(4\theta) + i\sin(4\theta) = -1.$$

Therefore $4\theta = (2k+1)\pi$ for some $k \in \mathbb{Z}$. Keeping in mind that $e^{i\theta}$ and $e^{i(\theta+2\pi)}$ are the same complex number, we see that there are four solutions:

$$\begin{aligned}
z_1 &= e^{i\frac{\pi}{4}} = \cos\Big(\frac{\pi}{4}\Big) + i\sin\Big(\frac{\pi}{4}\Big) = \frac{\sqrt{2}}{2}(1+i),\\
z_2 &= e^{i\frac{3\pi}{4}} = \frac{\sqrt{2}}{2}(-1+i),\\
z_3 &= e^{i\frac{5\pi}{4}} = -\frac{\sqrt{2}}{2}(1+i),\\
z_4 &= e^{i\frac{7\pi}{4}} = \frac{\sqrt{2}}{2}(1-i).
\end{aligned}$$

b) The factorization is of the form

$$z^4 - 1 = a(z-z_1)(z-z_2)(z-z_3)(z-z_4)$$

for some $a \in \mathbb{C}$. Equating the coefficients of z^4 on each side, we conclude that $a = 1$.

c) The numerator and denominator have no common factors, and the roots of the denominator $z^4 - 1$ all have multiplicity 1. Therefore, $R(z)$ has a partial fraction decomposition of the form

$$\frac{z^2}{z^4+1} = \frac{A_1}{z-z_1} + \frac{A_2}{z-z_2} + \frac{A_3}{z-z_3} + \frac{A_4}{z-z_4},$$

where the A_i are complex numbers that we need to determine. We start with A_1. Multiply both sides by $z - z_1$:

$$\frac{z^2}{(z-z_2)(z-z_3)(z-z_4)} = A_1 + A_2\frac{z-z_1}{z-z_2} + A_3\frac{z-z_1}{z-z_3} + A_4\frac{z-z_1}{z-z_4}.$$

We evaluate both sides at $z = z_1$ to obtain

$$\begin{aligned} A_1 &= \frac{z_1^2}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} = \frac{i}{\sqrt{2} \cdot \sqrt{2}(1+i) \cdot i\sqrt{2}} \\ &= \frac{1}{4} e^{-i\frac{\pi}{4}} = \frac{\sqrt{2}}{8}(1 - i). \end{aligned}$$

The remaining coefficients are computed similarly:

$$\begin{aligned} A_2 &= -\frac{1}{4} e^{i\frac{\pi}{4}} = -\frac{\sqrt{2}}{8}(1+i) \\ A_3 &= \frac{1}{4} e^{i\frac{3\pi}{4}} = \frac{\sqrt{2}}{8}(-1+i) \\ A_4 &= \frac{1}{4} e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{8}(1+i). \end{aligned}$$

Partial fraction decomposition will be very useful later in the course when we integrate rational functions.