APM120, Homework #5

Applied Linear Algebra and Big Data

Last updated: Tuesday 28th February, 2023, 15:13

Assigned Feb 28, **due Thursday, March 9**, 1:00 pm, via gradescope, **in pdf $\leq$ 20Mb, $\leq$ 30 pages.**

Singular Value Decomposition: polar decomposition, under/over determined systems

Show all steps in all calculations explicitly. Attach code used, well documented, and relevant plots and Matlab/python output, attaching code and figures *immediately following* the relevant question solution. A code printout is not a substitute for complete solutions, your solution should stand alone without the Matlab/python code or output. See needed python preliminaries at end of this HW[1]. It is fine to use Matlab/python unless a hand calculation (using only a simple calculator) is required *in orange*. For all questions, make sure you show all steps as if you are doing the problem by hand, and do not use library functions unless explicitly allowed in the question. Make sure you can do all calculations using no more than a hand-held calculator. See the end-note for guidelines and examples of hand-calculations.[2]

1. **Least-squares solution of over-determined systems** (e.g., medical tomography with more rays than pixels): (**A**) Derive the expression for the least square solution to a general set of equations $\mathsf{A}\mathbf{x} = \mathbf{b}$ with more equations than unknowns **(B)** Use this expression it to solve the problem based on the following matrix, you may use built-in Matlab/python functions for the matrix inversion step,

   ```
   A=[ 4, 1, 3;  -4, -2, -5; 3, -1, -4; 0, 4, -1];
   b=[ 0; 5;  -1;  -2];
   ```

   (**C**) Calculate the residual $\mathbf{r} = \mathsf{A}\mathbf{x} - \mathbf{b}$, why is it nonzero? (**D**) Would you consider the residual large or small? Quantify your answer by comparing $\|\mathbf{r}\|$ and $\|\mathbf{b}\|$, why is this a good comparison? (**E**) Compare your answer to that using Matlab's backslash operator `x=A\b`, or python's `x,residuals,rank,sigma=np.linalg.lstsq(A,b,rcond=None);`.

2. **SVD solution of under-determined systems** (e.g., medical tomography with fewer rays than pixels): consider $\mathsf{A}\mathbf{x} = \mathbf{b}$ with

   ```
   A=[-0.4, 0.2, -0.7; 0.8, -0.7, -0.1];
   b=[-1; 0.5];
   ```

   (A) Hand-calculate $\mathsf{A}^{\dagger}$, the pseudo inverse of A; you may use Matlab/python to calculate the SVD of A for this purpose. **(B)** Solve for $\mathbf{x}$ using the pseudo inverse. **(C)** Compare both the solution and its norm to that obtained using Matlab/python's pseudo-inverse of A,

   ```
   % Matlab:
   x2=pinv(A)*b
   ```

   ```
   # python:
   x2=scipy.linalg.pinv(A)@b;
   ```

   and to the solution obtained using Matlab's backslash operator `x3=A\b;` or, python's: `x3,residuals,rank,sigma=np.linalg.lstsq(A,b,rcond=None);`

3. *****optional extra credit: Polar decomposition and computer animation:** read about this in the notes. (**A**) Calculate and write the polar decomposition of A,

   ```
   A=[-1.3, -0.3; 0.75, -0.52];
   ```

(**B**) Create a polygon from a set of vectors whose corners are at

```
Corners=[1, 7, 3, 4,-1, 2;
        -5, 2, 1, 4, 0, 0];
```

Apply the matrix transformation A to the polygon and plot the effects of the stretching and rotation separately and together, by modifying `SVD_applications_polar_decomposition_example.m/py` from the Sources directory. Use the results to geometrically interpret each of the two matrices involved in the polar decomposition. (**C**) Calculate the area of the two polygons before and after the transformation and compare their ratio to the determinant of A. *Hint:* The Shoelace formula. (**D**) **Computer animation:** create the matrix M corresponding to a rotation/stretching by A and translation by $(u, v) = (5, -0.5)$. Plot a sequence of 10 images representing the animation of the above polygon by the matrix M in two ways: using direct scaling of M ($t\mathsf{M} + (1-t)\mathsf{I}$ for $t = 0, \ldots, 1$) and using polar decomposition. You may want to start from `SVD_applications_polar_decomposition_animation.m/py`.

* [What's the point of *****optional extra credit** challenge problems: apart from the fun of doing them, they may bring the total score of this HW assignment up to 110%, making up for problems you may have missed in this or other HW assignments...]

Python preliminaries & notes

1 python commands within the HW assume you have first used the followings: `import numpy as np; from numpy import linalg; import scipy as scipy; from scipy import linalg; import matplotlib.pyplot as plt; import matplotlib;`
Input a matrix A, column vector **b** and row vector **c** into python in the form
`A=np.array([[a11,a12,a13],[a21,a22,a23],[a31,a32,a33]]; b=np.array([[b1],[b2],[b3]]); c=np.array([[c1,c2,c3]]);` or convert Matlab arrays given in HW directly to python arrays using, e.g., `A=np.array(np.matrixlib.defmatrix.matrix('[1 2 3; 4 5 6]'));`

2 **Hand calculations**, in which you are asked to use only a simple hand-held calculator, are required only when we want to make sure you understand exactly what each step of an algorithm is doing. These also prepare you for the quizzes that involve similar hand calculations. **How much work to show?** Just don't use scratch paper, show all steps that you actually use for the hand calculations, but no more, carrying out calculations to **three significant digits**. Trust the graders to be reasonable, they were students in the course last year. **Examples: (i)** If you are asked, *not* in orange, to calculate the LU decomposition of a matrix, you may use Matlab/python as in `A(2,:)=A(2,:)-A(1,:)*(A(2,1)/A(1,1))` etc, but you may not use a library routine as in `[L,U,P]=lu(A)` except for checking your results. **(ii)** If required to calculate by hand the element $C_{2,3}$ of a matrix product $\mathsf{C} = \mathsf{AB}$, you need to explicitly multiply using a hand calculator the second row of A with the third column of B. You may not use Matlab/python to calculate `C=A*B` and then take the needed element from that product, nor to calculate `C23=A(2,:)*B(:,3)`.