Section 3 Friday, September 23, 2022

Material covered: Outcomes 1.1–1.5.

Solutions

1. For positive real numbers $t, s > 0$, the identity

$$\log(ts) = \log(t) + \log(s)$$

holds for the usual logarithm on the real numbers. But the analogous identity does *not* hold for the complex logarithm and in this problem we will find a counterexample.

Let Log denote the principal branch of the logarithm. Find two complex numbers z, $w \in \mathbb{C}$ such that $\mathrm{Log}(z)$, $\mathrm{Log}(w)$ and $\mathrm{Log}(zw)$ are defined, but $\mathrm{Log}(zw) \neq \mathrm{Log}(z) + \mathrm{Log}(w)$.

Solution. Recall that Log has a branch cut along the negative real axis. Therefore, we must pick z, w so that none of the numbers z, w, zw are purely real and negative. In that case, we have

$$e^{\mathrm{Log}(z)} = z, \quad e^{\mathrm{Log}(w)} = w, \quad e^{\mathrm{Log}(zw)} = zw.$$

Therefore,

$$e^{\mathrm{Log}(zw)} = zw = e^{\mathrm{Log}(z)} e^{\mathrm{Log}(w)} = e^{\mathrm{Log}(z)+\mathrm{Log}(w)}.$$

This does *not* imply that $\mathrm{Log}(zw) = \mathrm{Log}(z) + \mathrm{Log}(w)$, but we can conclude that

$$\mathrm{Log}(zw) = \mathrm{Log}(z) + \mathrm{Log}(w) + i2\pi n,$$

for some integer $n \in \mathbb{Z}$. By using $\mathrm{Log}(\zeta) = \log(|\zeta|) + i\,\mathrm{Arg}(\zeta)$, this is equivalent to

$$\mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w) + i2\pi n.$$

This suggests choosing z, w so that $0 < \operatorname{Arg}(z), \operatorname{Arg}(w) < \pi$ and $\operatorname{Arg}(z) + \operatorname{Arg}(w) > \pi$. For example, choose

$$z = w = e^{i\frac{3\pi}{4}}.$$

Then

$$z^2 = e^{i\frac{3\pi}{2}} = -i = e^{-i\frac{\pi}{2}}$$

so

$$\operatorname{Log}(z^2) = -i\frac{\pi}{2}.$$

but

$$\operatorname{Log}(z) + \operatorname{Log}(w) = i\frac{3\pi}{4} + i\frac{3\pi}{4} = i\frac{3\pi}{2} = \operatorname{Log}(z^2) + 2\pi.$$

This also serves as a counterexample for the false identity

$$\operatorname{Log}(z^2) = 2\operatorname{Log}(z)$$

which does not generally hold for complex z.

2. Identify and explain the error in the following "proof" that $e^i = 1$:

$$e^i = e^{i\frac{2\pi}{2\pi}} = \left(e^{i2\pi}\right)^{\frac{1}{2\pi}} = (1)^{\frac{1}{2\pi}} = 1.$$

Solution. The equality

$$e^{i\frac{2\pi}{2\pi}} = \left(e^{i2\pi}\right)^{\frac{1}{2\pi}}$$

is not correct. While the identity $(x^a)^b = x^{ab}$ $(a, b \in \mathbb{R})$ holds for real numbers x, it is not generally true for complex numbers, as the above counterexample shows. Let's try to understand why it does not hold.

Recall that the principal branch of the a-th power is defined

$$z^a = \exp\left(a\operatorname{Log} z\right) = e^{a\operatorname{Log} z}$$

Therefore,

$$(z^a)^b = \exp\left(b \operatorname{Log}\left(e^{a \operatorname{Log} z}\right)\right)$$

If $-\pi < \operatorname{Im}(a \operatorname{Log} z) < \pi$, then $\operatorname{Arg} e^{a \operatorname{Log} z}$ is between $-\pi$ and π so

$$\operatorname{Log}\left(e^{a \operatorname{Log} z}\right) = a \operatorname{Log} z \tag{1}$$

and thus

$$(z^a)^b = \exp\left(b \cdot a \operatorname{Log} z\right) = e^{ab \operatorname{Log} z} = z^{ab}.$$

However, if $\operatorname{Arg} e^{a \operatorname{Log} z}$ is not in this range, then equation (1) does not hold; the left and right sides will differ by some multiple of $2\pi i$. Try it for yourself!

3. Let Log denote the principal branch of the Logarithm. Identify the branch points and the branch cuts of the following functions:
   a) $f(z) = \operatorname{Log}(i(1-z))$ b) $f(z) = \operatorname{Log}(z^4)$

Solution.
a) Recall that Log has a branch point at $z = 0$ and a branch cut along the negative real axis, $\mathbb{R}_-$. Therefore, to find the branch point, we have to find which points get mapped to 0 under the function $g(z) = i(1-z)$. There is only one branch point, namely $z = 1$ (since $g(1) = 0$). To find the branch cut, we need to find which points get mapped to $\mathbb{R}_-$. Let's write $z = x + iy$ with $x, y \in \mathbb{R}$. Then

$$g(z) = i(1 - x - iy) = y + i(1 - x)$$

is real and negative if and only if $x = 1$ and $y < 0$. Thus the branch cut is

$$\{z = x + iy \in \mathbb{C} \mid x = 1, y < 0\}$$

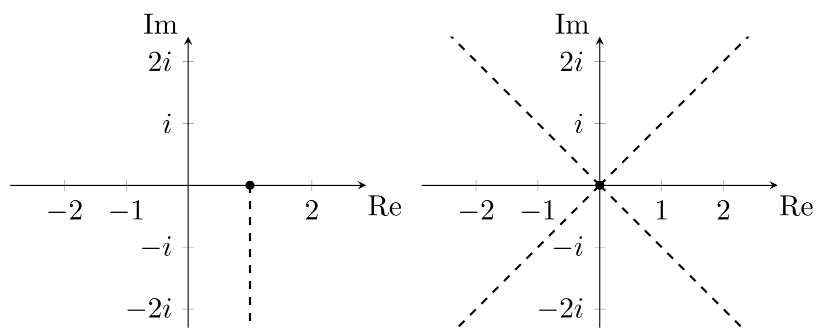
and the branch point is at $z = 1$ since $g(1) = 0$. The branch point and branch cut are shown below.

b) To find the branch point, notice that z^4 is zero if and only if $z = 0$. Therefore, the branch point is at $z = 0$. To find the branch cut, we need to find which points get mapped to $\mathbb{R}_-$. In other words, for which z is z^4 real and negative? Let's write $z = re^{i\theta}$ with $r > 0$. Then

$$z^4 = r^4 e^{4i\theta}$$

which is real and negative if and only if $4\theta = \pi + n2\pi,\ n \in \mathbb{Z}$, i.e. if $\theta = \frac{\pi}{4} + n\frac{\pi}{2}$, $n \in \mathbb{Z}$. $x = 1$ and $y < 0$. Thus the branch cuts are the four diagonal half-lines from the origin, as shown below. Precisely, they are

$$\{re^{i\theta} \in \mathbb{C} \mid r > 0, \theta = \frac{\pi}{4}(1 + 2n)\}, \qquad n = 0, 1, 2, 3.$$



4. Let $f(z) = z^{1/2}$ be the principal branch of the square root. Compute the derivative of f in two different ways:
   a) Using the identity $z^a = e^{a \operatorname{Log} z}$ for $a \in \mathbb{C}$, $z \in \mathbb{C} \setminus \mathbb{R}_-$ and the chain rule.
   b) Define $g(z) = z^2$ so that $g(f(z)) = z$ for all z where the left side is defined. Then use the inverse function theorem.

Solution.
a) We have

$$f(z) = e^{\frac{1}{2} \operatorname{Log} z} = g(h(z))$$

where

$$g(z) = e^z, \qquad h(z) = \frac{1}{2}\operatorname{Log} z.$$

The function h is differentiable on $\mathbb{C} \setminus \mathbb{R}_-$ with derivative $h'(z) = \frac{1}{2z}$. The function g is differentiable on $\mathbb{C}$ with derivative $g'(z) = e^z$. By the chain rule, $f = g \circ h$ is differentiable on $\mathbb{C} \setminus \mathbb{R}_-$ with derivative

$$f'(z) = g'(h(z))h'(z) = e^{\frac{1}{2}\operatorname{Log} z}\frac{1}{2z} = \frac{z^{\frac{1}{2}}}{2z}.$$

We can simplify this by noting that for $z \in \mathbb{C} \setminus \mathbb{R}_-$ (the domain of f), $\frac{1}{z} = z^{-1} = e^{-\operatorname{Log} z}$ and thus

$$\begin{aligned} f'(z) &= \frac{1}{2}e^{\frac{1}{2}\operatorname{Log} z}e^{-\operatorname{Log} z} = \frac{1}{2}e^{\frac{1}{2}\operatorname{Log} z - \operatorname{Log} z} \\ &= \frac{1}{2}e^{-\frac{1}{2}\operatorname{Log} z} = \frac{1}{2}z^{-\frac{1}{2}} = \frac{1}{2z^{\frac{1}{2}}}. \end{aligned}$$

b) We note that g is differentiable on $\mathbb{C}$. Let $z_0 \in \mathbb{C} \setminus \mathbb{R}_-$ be a point in the domain of f. Since f is continuous at z_0, g is differentiable at $f(z_0)$ and the composition $h(z) = g(f(z)) = z$ is differentiable at z_0, then f is differentiable at z_0 by the inverse function theorem. Since z_0 was arbitrary, f is differentiable everywhere on its domain. The derivative is given by

$$f'(z) = \frac{h'(z)}{g'(f(z))} = \frac{1}{2f(z)} = \frac{1}{2z^{\frac{1}{2}}}.$$

5. Determine where on the complex plane the following functions are holomorphic (analytic). Compute the derivative where it exists.

   a) $f(z) = \frac{z}{z^2+1}$

   b) $f(z) = \begin{cases} \frac{|z|^2}{\bar{z}}, & z \neq 0 \\ 0, & z = 0. \end{cases}$

   c) $f(z) = e^{\sin(z)}$

d) $f(z) = \lambda(z)$ where λ is the complex logarithm which has a branch cut along the negative imaginary axis and which satisfies $\lambda(x) = \log(x) + 2\pi i$ for positive real x.

Solution.
a) Since f is the ratio of two holomorphic functions, it is differentiable wherever the denominator is nonzero. The denominator vanishes at $z = i$ and $z = -i$, so f is holomorphic on

$$\mathbb{C} \setminus \{i, -i\},$$

and the derivative is given by the quotient rule:

$$f'(z) = \frac{1 \cdot (z^2+1) - z(2z)}{(z^2+1)^2} = \frac{1-z^2}{(z^2+1)^2}.$$

b) We note that for $z \neq 0$,

$$\frac{|z|^2}{\overline{z}} = \frac{z\overline{z}}{\overline{z}} = z.$$

Since $f(0) = 0$ at $z = 0$, we see that $f(z) = z$ for all z and thus f is holomorphic with $f'(z) = 1$.

c) Both $\sin(z)$ and e^z are holomorphic on $\mathbb{C}$, so their composition f is holomorphic on $\mathbb{C}$ and $f'(z)$ is given by the chain rule:

$$f'(z) = e^{\sin z} \cos z.$$

d) Recall that "λ is a complex logarithm" means that λ is continuous and $e^{\lambda(z)} = z$ for z in the domain of λ.

Let U denote the complex plane excluding the negative imaginary axis. Let $z \in U$ and write $z = re^{i\theta}$ where $-\pi/2 < \theta < 3\pi/2$. Then

$$\lambda(z) = \log(r) + i(\theta + 2\pi).$$

To see this, you can check that $\lambda(x) = \log(x) + 2\pi i$ for $x > 0$ and for all $z \in U$,

$$e^{\lambda(z)} = e^{\log r + i(\theta + 2\pi)} = e^{\log r} e^{i(\theta+2\pi)} = re^{i\theta} = z.$$

Since $e^{\lambda(z)} = z$ it follows from the inverse function theorem that λ is holomorphic on U and

$$\lambda'(z) = \frac{1}{e^{\lambda(z)}} = \frac{1}{z}.$$

This shows that $\lambda'(z) = z^{-1}$ for any logarithm, not just the principal branch.

6. Let f be a complex function and write the real and imaginary parts of f in terms of polar coordinates:

$$f(z) = u(r,\theta) + iv(r,\theta), \qquad \text{where } z = re^{i\theta}.$$

Show that in polar coordinates, the Cauchy–Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Use this to check where the function $f = u + iv$ with

$$u(r,\theta) = r^2\cos(\theta), \quad v(r,\theta) = 2r^2(1+\sin(\theta)).$$

is analytic.

Solution Let us write $u(x, y)$ for $u(x + iy)$ as usual. Recall the definitions of $\partial/\partial r$ and $\partial/\partial\theta$:

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r} u(r\cos\theta, r\sin\theta) = \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta$$

and similarly

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta} u(r\cos\theta, r\sin\theta) = \frac{\partial u}{\partial x}\cdot(-r\sin\theta) + \frac{\partial u}{\partial y}\cdot r\cos\theta.$$

We can likewise write $\partial u/\partial x$, $\partial u/\partial y$ in terms of $\partial u/\partial r$, $\partial u/\partial \theta$:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \cos\theta\frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial u}{\partial \theta} \\ \frac{\partial u}{\partial y} &= \sin\theta\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial u}{\partial \theta}. \end{aligned} \tag{2}$$

Now, the Cauchy–Riemann equations say that $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. We rewrite these equations in terms of partial derivatives with respect to r and θ using (2). We obtain:

$$\begin{aligned} \cos\theta\frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial u}{\partial \theta} &= \sin\theta\frac{\partial v}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial v}{\partial \theta} \\ \sin\theta\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial u}{\partial \theta} &= -\cos\theta\frac{\partial v}{\partial r} + \frac{1}{r}\sin\theta\frac{\partial v}{\partial \theta}. \end{aligned}$$

Solving for $\partial u/\partial r$ and $\partial u/\partial \theta$, we find,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r},$$

which is what we were asked to show.

Next, we check whether the function

$$f(re^{i\theta}) = r^2\cos(\theta) + 2r^2(1+\sin(\theta))i$$

satisfies the Cauchy–Riemann equations. We have

$$\begin{aligned} \frac{\partial u}{\partial r} &= 2r\cos(\theta) \\ \frac{1}{r}\frac{\partial v}{\partial \theta} &= 2r\cos(\theta), \end{aligned}$$

so the equation

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

is satisfied. On the other hand,

$$\begin{aligned}
\frac{\partial v}{\partial r} &= 4r(1+\sin(\theta)) \\
-\frac{1}{r}\frac{\partial u}{\partial \theta} &= r\sin(\theta),
\end{aligned}$$

and we can see that the second Cauchy–Riemann equation is not satisfied anywhere on the complex plane, except at the single point $z = 0$ where $r = 0$. Therefore, the function is nowhere analytic.