Section 7 Friday, October 28, 2022

Material covered: Outcomes 5.4–5.5.

Solutions

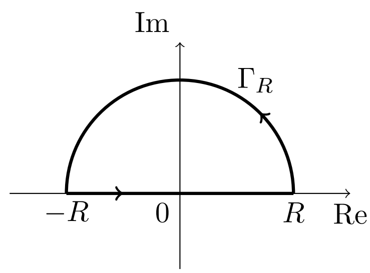
1. Compute the following integral:

$$I = \int_{-\infty}^{\infty} \frac{x \sin(x)}{x^2 - 2x + 10} \, \mathrm{d}x$$

Hint: Evaluate the integral of the function

$$f(z) = \frac{ze^{iz}}{z^2 - 2z + 10}$$

over the contour Γ_R shown below. Then take the imaginary part of the result.



Solution.
One way to evaluate the integral is to write $\sin(x)$ as a sum of complex exponentials,

$$\frac{x \sin(x)}{x^2 - 2x + 10} = \frac{1}{2i} \frac{xe^{ix}}{x^2 - 2x + 10} - \frac{1}{2i} \frac{xe^{-ix}}{x^2 - 2x + 10}$$

and integrating the two terms separately. Here we show a slightly quicker approach which only requires computing one contour integral.

Note that for $x \in \mathbb{R}$, we have

$$\frac{x\sin(x)}{x^2-2x+10} = \operatorname{Im}\frac{xe^{ix}}{x^2-2x+10} = \operatorname{Im} f(x)$$

where f is the function defined in the hint. Therefore,

$$I = \int_{-\infty}^{\infty} \operatorname{Im} f(x)\,\mathrm{d}x = \operatorname{Im}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

and the integral on the right can be evaluated by closing the contour in the upper half-plane.

For $R > 0$ let Γ_R be the contour composed of the line segment $[-R, R] \subset \mathbb{R}$ and the semicircle with center 0 and radius R in the upper half-plane, which we denote C_R^+. The contour Γ_R is shown above. Then

$$\int_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z$$

or, after rearranging,

$$\int_{-R}^{R} f(x)\,\mathrm{d}x = \int_{\Gamma_R} f(z)\,\mathrm{d}z - \int_{C_R^+} f(z)\,\mathrm{d}z. \tag{1}$$

Let's evaluate the first term on the right using the residue theorem.

The function f has two isolated singular points,

$$z_1 = 1+3i, \quad z_2 = 1-3i$$

as can be seen by factoring the denominator. Since the numerator vanishes at neither point, both are poles of order 1. Let's compute the residue at z_1, which lies in the upper half-plane:

$$\operatorname{Res}(f, z_1) = \lim_{z\to z_1}(z-z_1)f(z) = \lim_{z\to z_1}\frac{ze^{iz}}{z-z_2} = \frac{(1+3i)e^{-3+i}}{6i}.$$

Thus for R sufficiently large,

$$\int_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}(f, z_1) = \frac{\pi}{3e^3}(1+3i)e^i$$

Finally, we evaluate the second term on the right side of (1) in the limit $R \to \infty$. We note that $f(z) = \frac{P(z)}{Q(z)}e^{iz}$ where

$$P(z) = z, \qquad Q(z) = z^2 - 2z + 10,$$

Since $\deg(P) < \deg(Q)$, integral converges to 0 as $R \to \infty$ by Jordan's lemma. Therefore, taking the limit $R \to \infty$ in equation (1), we get

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x &= 2\pi i\,\mathrm{Res}(f, z_1) = \frac{\pi}{3e^3}(1+3i)e^i \\ &= \frac{\pi}{3e^3}\Big[(\cos(1) - 3\sin(1)) + i(3\cos(1) + \sin(1))\Big].\end{aligned}$$

Taking imaginary parts of both sides, we obtain the desired integral:

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^2 - 2x + 10}\,\mathrm{d}x = \frac{\pi}{3e^3}\left(3\cos(1) + \sin(1)\right).$$

By taking real parts instead, we also get the following integral for free:

$$\int_{-\infty}^{\infty} \frac{x\cos(x)}{x^2 - 2x + 10}\,\mathrm{d}x = \frac{\pi}{3e^3}\left(\cos(1) - 3\sin(1)\right).$$

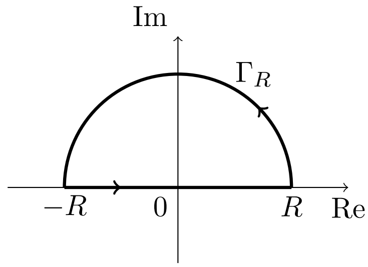
2. Compute the following integral:

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)(x^2+2)^2}.$$

Solution.
Let

$$f(z) = \frac{1}{(z^2+1)(z^2+2)^2}.$$

Then the integrand is a rational function, the numerator has degree 0 and the denominator has degree $6 > 0+2$. Therefore, we can close the contour in the complex plane as shown:



We have

$$\int_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z$$

where Γ_R, C_R^+ are defined as in the solution to problem 1. The integral over C_R^+ vanishes in the limit $R \to \infty$. Therefore, we obtain

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i \sum_{z_i \in A} \mathrm{Res}(f, z_i)$$

where A is the set of isolated singularities of f in the upper half-plane.

Factoring the denominator of f,

$$f(z) = \frac{1}{(z-i)(z+i)(z-\sqrt{2}i)^2(z+\sqrt{2}i)^2}$$

we see that f has simple poles at $z = \pm i$ and poles of order 2 at $z = \pm\sqrt{2}i$. Let's evaluate the residues at $z_1 = i$ and $z_2 = \sqrt{2}i$, starting with z_1:

$$\begin{aligned}\mathrm{Res}(f, i) &= \lim_{z\to i}(z-i)f(z) = \lim_{z\to i} \frac{1}{(z+i)(z^2+2)^2} \\ &= \frac{1}{(2i)(i^2+2)^2} = \frac{1}{2i}.\end{aligned}$$

For the residue at z_2, let's define

$$g(z) = (z - z_2)^2 f(z) = \frac{1}{(z^2+1)(z+\sqrt{2}i)^2},$$

so that $\mathrm{Res}(f, z_1) = g'(z_1)/1! = g'(z_1)$. This derivative is a bit messy to compute by hand. We can make things a little easier by writing $g(z) = a(z)^{-1} b(z)^{-2}$ where $a(z) = z^2 + 1$, $b(z) = z + \sqrt{2}i$. Then[1]

$$\frac{g'(z)}{g(z)} = -\frac{a'(z)}{a(z)} - 2\frac{b'(z)}{b(z)} = -\frac{2z}{z^2+1} - \frac{2}{(z+\sqrt{2}i)}$$

$$g'(\sqrt{2}i) = g(\sqrt{2}i)\Big[-\frac{2\sqrt{2}i}{(\sqrt{2}i)^2+1} - \frac{2}{(2\sqrt{2}i)}\Big] = \frac{1}{8}\cdot\frac{5i}{\sqrt{2}}.$$

Thus the residue is

$$\mathrm{Res}(f, \sqrt{2}i) = g'(\sqrt{2}i) = \frac{5i}{8\sqrt{2}}.$$

Combining these results, we find that

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)(x^2+2)^2} = 2\pi i\left(\frac{1}{2i} + \frac{5i}{8\sqrt{2}}\right) = \left(1 - \frac{5}{4\sqrt{2}}\right)\pi.$$

3. In this problem, we compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\,\mathrm{d}x$$

using the residue theorem. First note that the singular point at 0 is removable, so the integral is well-defined. However, we will run into trouble when we try to close the integral in the complex plane. To see why, let's expand $\sin^2(z)$ in complex exponentials:

$$\frac{\sin^2(z)}{z^2} = \frac{1}{z^2}\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 = \frac{1-e^{2iz}}{4z^2} + \frac{1-e^{-2iz}}{4z^2} = f(z) + f(-z),$$

[1] This useful technique is called logarithmic differentiation, see: https://en.wikipedia.org/wiki/Logarithmic_differentiation#Products.

where

$$f(z) = \frac{1 - e^{2iz}}{4z^2}.$$

We cannot use the same contour for both terms. The term containing e^{2iz} decays in magnitude in the upper half-plane ($\operatorname{Im} z \to +\infty$) and grows in the lower half-plane ($\operatorname{Im} z \to -\infty$); vice versa for the term containing e^{-2iz}.

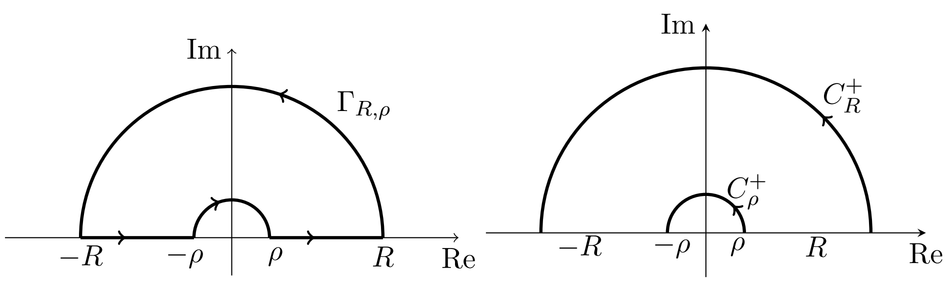
If we try to integrate the two terms separately, closing the contour in the upper half-plane for the first term and in the lower half-plane for the second term, then we have a problem: The integrands have simple poles on the real axis. The integrals

$$\int_{-\infty}^{\infty} \frac{1 - e^{2ix}}{x^2}\,\mathrm{d}x, \qquad \int_{-\infty}^{\infty} \frac{1 - e^{-2ix}}{x^2}\,\mathrm{d}x$$

do not exist because the integrand blows up as $x \to 0$. However, we observe that

$$\begin{aligned} I &= \lim_{\rho \to 0} \left[\int_{-\infty}^{-\rho} \Big(f(x) + f(-x) \Big)\,\mathrm{d}x + \int_{\rho}^{\infty} \Big(f(x) + f(-x) \Big)\,\mathrm{d}x \right] \\ &= 2 \lim_{\rho \to 0} \left[\int_{-\infty}^{-\rho} f(x)\,\mathrm{d}x + \int_{\rho}^{\infty} f(x)\,\mathrm{d}x \right] \end{aligned}$$

and these two integrals can be evaluated using the contour $\Gamma_{R,\rho}$ shown below.



Left: The contour $\Gamma_{R,\rho}$. Right: The semicircles C_R^+ and C_ρ^+.

a) Let $\Gamma_{R,\rho}$ be the contour shown above with $0 < \rho < R$, so that

$$\int_{\Gamma_{R,\rho}} f(z)\,\mathrm{d}z = \int_{\rho}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z + \int_{-R}^{-\rho} f(x)\,\mathrm{d}x - \int_{C_\rho^+} f(z)\,\mathrm{d}z,$$

and thus, after rearranging,

$$\int_{\rho}^{R} f(x)\,\mathrm{d}x + \int_{-R}^{-\rho} f(x) = \int_{\Gamma_{R,\rho}} f(z)\,\mathrm{d}z + \int_{C_\rho^+} f(z)\,\mathrm{d}z - \int_{C_R^+} f(z)\,\mathrm{d}z.$$

Evaluate each term on the right-hand side in the limit $R \to \infty$, $\rho \to 0$. Use this to compute

$$I = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\,\mathrm{d}x$$

Solution.
Let's start with the closed contour $\Gamma_{R,\rho}$. We note that f has a simple pole at $z = 0$ and no other singular points. Since $\Gamma_{R,\rho}$ does not enclose any singular points, the residue theorem gives

$$\int_{\Gamma_{R,\rho}} f(z)\,\mathrm{d}z = 0.$$

In problem 4 a) of Homework 7, you are asked to show that if f is an analytic function with a simple pole at $z = 0$, then

$$\lim_{\rho \to 0} \int_{C_\rho^+} f(z)\,\mathrm{d}z = i\pi\,\mathrm{Res}(f, 0).$$

Let's compute the residue. We have

$$\frac{1}{4z^2}(1 - e^{2iz}) = -\frac{1}{4z^2}\sum_{n=1}^{\infty}\frac{1}{n!}(2iz)^n = -\frac{i}{2z} + \sum_{n=2}^{\infty}\frac{1}{4n!}(2i)^n z^{n-2}$$

so

$$\mathrm{Res}(f, 0) = \frac{1}{2i}$$

Therefore, the contribution from the semiricle C_ρ^+ in the limit $\rho \to 0$ is

$$\lim_{\rho \to 0} \int_{C_\rho^+} f(z)\,\mathrm{d}z = i\pi \frac{1}{2i} = \frac{\pi}{2}.$$

Finally, we show that the integral over C_R^+ vanishes in the limit $R \to \infty$. We have

$$\int_{C_R^+} f(z)\,\mathrm{d}z = \int_{C_R^+} \frac{1}{4z^2}\,\mathrm{d}z - \int_{C_R^+} \frac{e^{2iz}}{4z^2}\,\mathrm{d}z.$$

The first term goes to zero in the limit $R \to \infty$ since the integrand is a rational function whose denominator has degree 2 higher than the numerator. The second term goes to zero by Jordan's lemma.

Combining these results, we find

$$\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} = 2 \lim_{R\to\infty, \rho\to 0} \left[\int_\rho^R f(x)\,\mathrm{d}x + \int_{-R}^{-\rho} f(x) \right] = 2 \cdot \frac{\pi}{2} = \pi.$$