Homework 3 Handed out: Wednesday, September 21, 2022
Due: Wednesday, September 28, 2022 by 11:59pm

Material covered: Outcomes 2.4, 3.1–3.3.

Solutions

Recall that Log denotes the principal branch of the logarithm, defined on $\mathbb{C} \setminus \mathbb{R}_-$ (the complex plane excluding the negative real axis) as follows: If $z = re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$, then

$$\operatorname{Log}(z) \coloneqq \log(r) + i\theta.$$

1. For $a \in \mathbb{C}$, let z^a denote the principal branch of the a-th power, defined for $z \in \mathbb{C} \setminus \mathbb{R}_-$ by

$$z^a = e^{a \operatorname{Log} z}.$$

Find the real and imaginary parts of the following numbers:

a) $\operatorname{Log}(1-i)$ b) $(1+i)^i$ c) $i^{1/\pi}$

Solution.
a) We write $z = 1 - i$ in polar form with $-\pi < \theta < \pi$:

$$1 - i = \sqrt{2} e^{-i\frac{\pi}{4}}.$$

Then

$$\operatorname{Log}(1-i) = \log(\sqrt{2}) - i\frac{\pi}{4} = \frac{1}{2}\log(2) - i\frac{\pi}{4}.$$

b) We have $1 + i = \sqrt{2}e^{i\pi/4}$ (note that the argument is chosen between $-\pi$ and π), so

$$\operatorname{Log}(1+i) = \frac{1}{2}\log(2) + i\frac{\pi}{4}.$$

Then, by definition of z^i,

$$(1+i)^i = e^{i\,\mathrm{Log}(1+i)} = e^{-\frac{\pi}{4}+\frac{\log 2}{2}i} = e^{-\frac{\pi}{4}}\left(\cos\Big(\frac{\log 2}{2}\Big) + i\sin\Big(\frac{\log 2}{2}\Big)\right).$$

c) We have $i = e^{i\frac{\pi}{2}}$, so $\mathrm{Log}(i) = \log(1) + i\pi/2 = i\pi/2$ and

$$i^{\frac{1}{\pi}} = e^{\frac{1}{\pi}\mathrm{Log}(i)} = e^{\frac{1}{\pi}\cdot i\frac{\pi}{2}} = e^{i\frac{1}{2}} = \cos(\tfrac{1}{2}) + i\sin(\tfrac{1}{2}).$$

2. Identify the domains of the following functions, show that they are holomorphic and compute their derivatives.

   a) $f(z) = \mathrm{Log}(z^2)$ b) $f(z) = \mathrm{Log}(e^z)$

Solution.

a) Recall that Log is defined on $\mathbb{C}\setminus\mathbb{R}_-$ (the complex plane excluding the negative real axis). Therefore, $f(z)$ is defined for all z such that z^2 is *not* real and nonpositive. You can check that z^2 is real and nonpositive if and only if z is purely imaginary (including $z = 0$). Thus

$$U := \{z \in \mathbb{C} \mid \mathrm{Re}\, z \neq 0\}.$$

We know that Log is holomorphic with derivative $\mathrm{Log}'(z) = 1/z$, and $g(z) = z^2$ is holomorphic with derivative $g'(z) = 2z$. By the chain rule, $f(z) = \mathrm{Log}(g(z))$ is holomorphic with derivative

$$f'(z) = \mathrm{Log}'(g(z))g'(z) = \frac{1}{z^2}\cdot 2z = \frac{2}{z}.$$

Note: Even though the expression $2/z$ on the right hand side is defined and holomorphic on $\mathbb{C}\setminus\{0\}$, that does not mean that $f'(z)$ exists on $\mathbb{C}\setminus\{0\}$. As stated above, $f'(z)$ is only defined on the set $\{z \in \mathbb{C} \mid \mathrm{Re}\, z \neq 0\}$.

b) First we determine for which z the function e^z is real and negative. Writing $z = x + iy$, we have

$$e^z = e^x\Big(\cos(y) + i\sin(y)\Big)$$

Now, $e^x > 0$ for all x, so e^z is purely real and negative if and only if y is an odd multiple of π, $y = (2n+1)\pi$ for some $n \in \mathbb{Z}$. Hence the branch cut is

$$B \coloneqq \{x + iy \in \mathbb{C} \mid y = (2n+1)\pi,\ n \in \mathbb{Z}\}$$

and the domain of f is the complement of this set, $U \coloneqq \mathbb{C} \setminus B$.

We compute the derivative of f using the chain rule:

$$f'(z) = \operatorname{Log}'(\exp(z)) \exp'(z) = \frac{1}{e^z} e^z = 1.$$

Another way to see this is as follows: Consider $z = x + iy$ in a given connected component of the domain U, say $(2n+1)\pi < y < (2n+3)\pi$ for some $n \in \mathbb{Z}$. Then

$$e^z = e^x e^{iy} = e^x e^{iy - i(2n+2)\pi} = e^x e^{i\theta}$$

where $-\pi < \theta < \pi$. Thus

$$\begin{aligned} \operatorname{Log}(e^z) &= \log(|e^z|) + i \operatorname{Arg}(e^{iz}) = \log(e^x) + i\Big(y - (2n+2)\pi\Big) \\ &= x + iy - 2(n+1)\pi i = z - 2(n+1)\pi i. \end{aligned}$$

Differentiating yields $f'(z) = 1$. *Note:* It may be tempting to write "$\operatorname{Log}(e^z) = z$", but as we have just seen, this does not hold for all z. However, on each connected component of U, we have $\operatorname{Log}(e^z) = z + c$ for some constant c.

3. Determine where on the complex plane the following functions are holomorphic. Compute the derivative where it exists.

   a) $f(z) = -\operatorname{Im} z + i \operatorname{Re} z$ b) $f(z) = \dfrac{1}{\sin(z)}$

Solution.
a) We notice that

$$f(z) = -y + ix = i(x + iy) = iz$$

and thus f is holomorphic on the entire complex plane with derivative

$$f'(z) = i.$$

An alternative approach is to use the Wirtinger derivatives. We note that f has continuous partial derivatives and

$$\partial_{\overline{z}} f(z) = \Big(\partial_x + i\partial_y\Big)(-y) + \Big(\partial_x + i\partial_y\Big)(ix) = -i + i = 0.$$

Since $\partial_{\overline{z}} f$ vanishes at every point, f is holomorphic everywhere on $\mathbb{C}$.

A third, equally valid approach is to find the real and imaginary components of f and check that the Cauchy–Riemann equations are satisfied.

b) We see that $\sin(z)$ is holomorphic on $\mathbb{C}$, since

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

and the function e^z is holomorphic on $\mathbb{C}$. Thus f is differentiable wherever $\sin(z) \neq 0$. Now $\sin(z)$ vanishes at the points

$$z = n\pi, \qquad n \in \mathbb{Z}.$$

Thus f is holomorphic on

$$\mathbb{C} \setminus \{n\pi + 0i \mid n \in \mathbb{Z}\}$$

where $f'(z)$ is given by the quotient rule

$$f'(z) = \frac{0 \cdot \sin(z) - 1 \cdot \cos z}{(\sin z)^2} = -\frac{\cos z}{(\sin z)^2}$$

4. Compute the Wirtinger derivatives $\partial_z f$ and $\partial_{\overline{z}} f$. Use this to determine where the functions are holomorphic, and find the derivative $f'(z)$ where it exists.

   a) $f(z) = z + 1/z$ b) $f(z) = z^2|z|^2$

   Solution.
   a) The function f is defined on $\mathbb{C} \setminus \{0\}$ where it has continuous Wirtinger derivatives

$$\partial_z f(z) = \frac{\partial z}{\partial z} - \frac{1}{z^2}\frac{\partial z}{\partial z} = 1 - \frac{1}{z^2},$$
$$\partial_{\overline{z}} f(z) = \frac{\partial z}{\partial \overline{z}} - \frac{1}{z^2}\frac{\partial z}{\partial \overline{z}} = 0 - \frac{1}{z^2}0 = 0.$$

   Since $\partial_{\overline{z}} f = 0$ on $U = \mathbb{C} \setminus \{0\}$, f is holomorphic on U with derivative

$$f'(z) = \partial_z f(z) = 1 - \frac{1}{z^2}.$$

   b) Since $|z|^2 = z\overline{z}$, we can rewrite f as

$$f(z) = z^3\overline{z}.$$

   We compute the Wirtinger derivatives:

$$\partial_z f(z) = 3z^2\overline{z},$$
$$\partial_{\overline{z}} f(z) = z^3.$$

   We see that $\partial_{\overline{z}} f$ is nonzero except at $z = 0$. Therefore, f is differentiable at the single point $z = 0$ where $f'(0) = \partial_z f(0) = 3(0)^2\overline{0} = 0$. Since f is not differentiable in a neighborhood of any point, f is nowhere holomorphic.

5. Determine where the functions are holomorphic by checking whether the Cauchy–Riemann equations are satisfied.

   a) $f(x + iy) = \dfrac{x}{x^2 + y^2} + \dfrac{iy}{x^2 + y^2}$

   b) $f(x + iy) = e^{y^2 - x^2}\cos(2xy) - ie^{y^2 - x^2}\sin(2xy)$

Solution.
a) We have

$$u(x,y) \coloneqq \operatorname{Re} f(x+iy) = \frac{x}{x^2+y^2}, \quad v(x,y) \coloneqq \operatorname{Im} f(x+iy) = \frac{y}{x^2+y^2}.$$

We first note that u and v are singular at the origin, i.e. $|u(x,y)| \to \infty$ and $|v(x,y)| \to \infty$ as $(x,y) \to (0,0)$. Let us then consider points $(x,y) \neq (0,0)$. There, u and v are continuously differentiable, and we compute the partial derivatives:

$$\begin{aligned}
\frac{\partial u}{\partial x}(x,y) &= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} \\
\frac{\partial u}{\partial y}(x,y) &= -\frac{2xy}{(x^2+y^2)^2} \\
\frac{\partial v}{\partial x}(x,y) &= -\frac{2xy}{(x^2+y^2)^2} \\
\frac{\partial v}{\partial y}(x,y) &= \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}.
\end{aligned}$$

Substituting into the second Cauchy–Riemann equation,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

we obtain

$$-\frac{2xy}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

which holds if and only if $xy = 0$. This shows that f is nowhere differentiable except possibly on the real and imaginary axes (excluding $z = 0$). Therefore, f is nowhere holomorphic (any disk around a point on the axes contains points which are not on the axes). In this case, we did not have to use the other Cauchy–Riemann equation to determine that f is nowhere holomorphic.

Notice that f may be written as

$$f(z) = \frac{z}{z\overline{z}} = \frac{1}{\overline{z}},$$

and then it is simple to show that f is nowhere holomorphic (in fact, nowhere differentiable) using Wirtinger derivatives.

b) The real and imaginary parts of f are

$$u(x,y) = e^{y^2-x^2}\cos(2xy), \quad v(x,y) = -e^{y^2-x^2}\sin(2xy)$$

Both u and v are continuously differentiable on the entire plane. Let us compute the partial derivatives:

$$\begin{aligned}
\frac{\partial u}{\partial x}(x,y) &= -2e^{y^2-x^2}(x\cos(2xy) + y\sin(2xy)) \\
\frac{\partial u}{\partial y}(x,y) &= 2e^{y^2-x^2}(y\cos(2xy) - x\sin(2xy)) \\
\frac{\partial v}{\partial x}(x,y) &= -2e^{y^2-x^2}(x\cos(2xy) + y\sin(2xy)) \\
\frac{\partial v}{\partial y}(x,y) &= 2e^{y^2-x^2}(x\sin(2xy) - y\cos(2xy)).
\end{aligned}$$

We see that the Cauchy–Riemann equations hold at every point, so $f = u + iv$ is holomorphic on $\mathbb{C}$.

Indeed, you may notice that f can also be written as

$$f(z) = e^{-z^2},$$

a manifestly holomorphic function.