APM120, Homework #2

Applied Linear Algebra and Big Data

Last updated: Sunday 5th February, 2023, 20:48

Assigned Feb 7, due Feb 14, 1:00 pm, via gradescope, **in pdf $\leq$ 20Mb, $\leq$ 30 pages.**

Eigenvalues, eigenvectors I: networks, power method, PageRank, generalized eigenvalue problems

Show all steps in all calculations explicitly. Attach code used, well documented, and relevant plots and Matlab/python output, attaching code and figures *immediately following* the relevant question solution. A code printout is not a substitute for complete solutions, your solution should stand alone without the Matlab/python code or output. See needed python preliminaries at end of this HW[1]. It is fine to use Matlab/python unless a hand calculation (using only a simple calculator) is required *in orange*. For all questions, make sure you show all steps as if you are doing the problem by hand, and do not use library functions unless explicitly allowed in the question. Make sure you can do all calculations using no more than a hand-held calculator. See the end-note for guidelines and examples of hand-calculations.[2]

1. **Power methods:** In each of the following cases, use 50 iterations and show the full details of the first 5 and of the last one. When relevant, use an initial guess vector made of ones. Consider,

```
A=[ -5.868,  -7.551,   0.158;
    -7.551,    3.83,   0.426;
     0.158,   0.426,   6.038];
B=[  4.828,   2.053,  -0.002;
    -4.651,    3.21,   -7.63;
    -1.889,  -11.09,  -4.038];
```

   (a) Use the power method for calculating the largest eigenvalue and corresponding eigenvector of A. Perform the first iteration by hand.

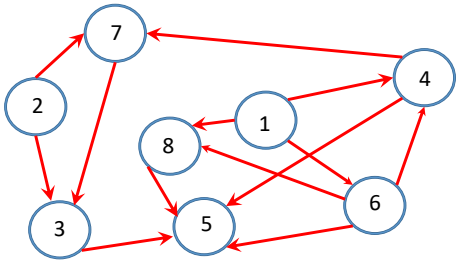
   (b) When can one use the *Hotelling deflation* to calculate the next-largest (in absolute value) eigenvalues and the corresponding eigenvectors? When can the *Wielandt deflation* be used for this purpose?

   (c) Can Hotelling deflation be used to calculate the second-largest eigenvalue of A? Of B? Why and how can you deduce that? Use Hotelling deflation to calculate the second and then the third-largest eigenvalue of A and the corresponding eigenvectors. Hand-calculate the matrix for Hotelling deflation and perform the first iteration by hand.

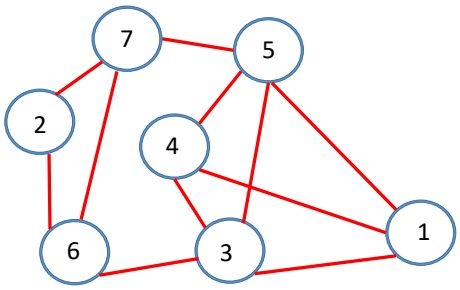
   (d) Can Wielandt deflation be used to calculate the eigenvalues/vectors of B? Why? Calculate the first eigenvalue/vector using Matlab/python. Then use Wielandt deflation to calculate the second-largest eigenvalue. Hand-calculate the matrix for Wielandt deflation and perform the first iteration by hand.

   (e) Compare your final solution to the correct eigenvalues obtained via Matlab, `[V1,D1]=eig(A)` or python, `D1,V1=np.linalg.eig(A)`. Discuss and explain the rate of convergence, or lack thereof, of the eigenvectors to the correct solution in each case.

2. **PageRank:** calculate the PageRank of the network below using the power method and assuming $\alpha = 0.85$.



3. **Spectral clustering (partitioning) of a network**. Consider the network shown in the figure



(A) Write the adjacency, degree and Laplacian matrices for this network. **(B)** Calculate eigenvalues and eigenvectors of the Laplacian matrix. **(C)** Show how the second eigenvector can be used to partition the network into two main sections. **(D)** Explain why this is expected in general.

4. ***Optional extra credit:** The **shifted inverse power method**: develop a method for calculating an eigenvalue of the following matrix A, `A=[-2.03,-4.73,-6.71;-4.73,0.767,-0.901;-6.71,-0.901,7.27]` that is closet to a specified number α: explain why the inverse power iterations of the matrix $\mathsf{A} - \alpha\mathsf{I}$ would lead to convergence to the eigenvector that is nearest to α. Demonstrate this for the above matrix, using 10 iterations with $\alpha = -7$. Compare your result for both the eigenvector and eigenvalue to those calculated by Matlab/python.

* [What's the point of ***optional extra credit challenge problems: apart from the fun of doing them, they may bring the total score of this HW assignment up to 110%, making up for problems you may have missed in this or other HW assignments...]

Python preliminaries & notes

1 python commands within the HW assume you have first used the followings: `import numpy as np; from numpy import linalg; import scipy as scipy; from scipy import linalg; import matplotlib.pyplot as plt; import matplotlib;`
Input a matrix A, column vector **b** and row vector **c** into python in the form
`A=np.array([[a11,a12,a13],[a21,a22,a23],[a31,a32,a33]]; b=np.array([[b1],[b2],[b3]]); c=np.array([[c1,c2,c3]]);` or convert Matlab arrays given in HW directly to python arrays using, e.g., `A=np.array(np.matrixlib.defmatrix.matrix('[1 2 3; 4 5 6]'));`

2 **Hand calculations**, in which you are asked to use only a simple hand-held calculator, are required only when we want to make sure you understand exactly what each step of an algorithm is doing. These also prepare you for the quizzes that involve similar hand calculations. **How much work to show?** Just don't use scratch paper, show all steps that you actually use for the hand calculations, but no more, carrying out calculations to **three significant digits**. Trust the graders to be reasonable, they were students in the course last year. **Examples: (i)** If you are asked, *not* in orange, to calculate the LU decomposition of a matrix, you may use Matlab/python as in `A(2,:)=A(2,:)-A(1,:)*(A(2,1)/A(1,1))` etc, but you may not use a library routine as in `[L,U,P]=lu(A)` except for checking your results. **(ii)** If required to calculate by hand the element $C_{2,3}$ of a matrix product $\mathsf{C} = \mathsf{AB}$, you need to explicitly multiply using a hand calculator the second row of A with the third column of B. You may not use Matlab/python to calculate `C=A*B` and then take the needed element from that product, nor to calculate `C23=A(2,:)*B(:,3)`.