Applied Math 104 — School of Engineering and Applied Sciences
Fall 2022 — Harvard University

Homework 7 — Handed out: Wednesday, October 26, 2022
Due: Wednesday, November 2, 2022 by 11:59pm

Material covered: Outcomes 5.4–5.5.

Solutions

1. In this problem we compute the integral

$$I(a) = \int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\,\mathrm{d}x$$

where $a \in \mathbb{R}$.

a) Let

$$f(z) = \frac{e^{iaz}}{1+z^2}$$

so that

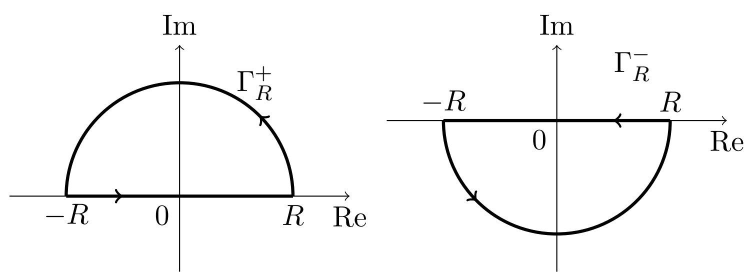
$$I(a) = \int_{-\infty}^{\infty} \mathrm{Re}(f(x))\,\mathrm{d}x.$$

Determine the poles of f and compute the residue at each pole.

b) Compute the contour integrals

$$\int_{\Gamma_R^+} f(z)\,\mathrm{d}z, \quad \int_{\Gamma_{R-}} f(z)\,\mathrm{d}z$$

where $R > 1$ and Γ_R^+, Γ_R^- are the following contours:



c) Does the integral of f along the two semicircles converge to 0 as $R \to \infty$? Does the answer depend on a?

d) Use the results of parts (a)–(c) to compute the integrals:

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\,\mathrm{d}x,$$
$$I_2 = \int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\,\mathrm{d}x$$

Solution.
a)

We factor the denominator:

$$f(z) = \frac{e^{iaz}}{(z-i)(z+i)}.$$

Since $z_1 = i$ and $z_2 = -i$ are first order zeros of the denominator and the numerator does not vanish at these points, they are simple poles of f. Let us compute the residues:

$$\mathrm{Res}(f, i) = \lim_{z \to i} \Big((z-i) f(z) \Big) = \lim_{z \to i} \frac{e^{iaz}}{z+i} = \frac{e^{-a}}{2i},$$

$$\mathrm{Res}(f, -i) = \lim_{z \to -i} \Big((z+i) f(z) \Big) = \lim_{z \to -i} \frac{e^{iaz}}{z-i} = -\frac{e^{a}}{2i}.$$

b) For $R > 1$, Γ_R^+ encloses the pole $z_1 = i$, and the residue theorem gives

$$\int_{\Gamma_R^+} f(z)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}(f, i) = \pi e^{-a}, \qquad R > 1.$$

Similarly, Γ_R^- encloses the pole $z_2 = -i$, and the residue theorem gives

$$\int_{\Gamma_R^-} f(z)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}(f, -i) = -\pi e^{a}, \qquad R > 1$$

c) Let C_R^+ be the semicircle of radius R in the upper half-plane and C_R^- be the semicircle of radius R in the lower half-plane (both centered at the origin). Let $g(z) = 1/(1+z^2)$ so that $f(z) = e^{iaz}g(z)$. We will use Jordan's lemma to show that, for $a > 0$,

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,\mathrm{d}z = 0, \qquad (a > 0)$$

and for $a < 0$,

$$\lim_{R\to\infty} \int_{C_R^-} f(z)\,\mathrm{d}z = 0 \qquad (a < 0).$$

Let us start with the integral over C_R^+. Assuming $a > 0$, we can bound this integral by

$$\begin{aligned} \left| \int_{C_R^+} f(z)\,\mathrm{d}z \right| &\le \int_{C_R^+} |f(z)||\mathrm{d}z| \le \max_{|z|=R} |g(z)| \int_{C_R^+} |e^{iaz}||\mathrm{d}z| \\ &< \frac{\pi}{a} \max_{|z|=R} |g(z)| \end{aligned} \tag{1}$$

where we used Jordan's lemma in the last step. Since the degree of the denominator of $g(z) = 1/(1+z^2)$ is greater than the degree of the numerator, it follows that

$$\lim_{R\to\infty} \max_{|z|=R} |g(z)| = 0. \tag{2}$$

We can also show this explicitly:

$$|g(z)| = \frac{1}{|z^2+1|} \le \frac{1}{|z|^2 - 1} = \frac{1}{R^2 - 1}.$$

Combining the results (1) and (2), we conclude that

$$\lim_{R\to\infty} \int_{C_R^+} f(z)\,\mathrm{d}z = 0, \qquad a > 0$$

For $a < 0$, the factor e^{iaz} becomes larger rather than smaller as $\operatorname{Im}(z)$ increases, and the above integral does not vanish in the limit $R \to \infty$.

The same argument can be used to show that for $a < 0$,

$$\lim_{R\to\infty} \int_{C_R^-} f(z)\,\mathrm{d}z = 0.$$

Finally, we consider the case $a = 0$. In that case, $f(z) = 1/(z^2+1)$ is a rational function and the degree of the denominator is 2 greater than the degree of the numerator. Therefore, the integrals over both C_R^+ and C_R^- converge. To see this concretely, (1) becomes

$$\begin{aligned}\left| \int_{C_R^+} f(z)\,\mathrm{d}z \right| &\le \int_{C_R^+} |f(z)||\mathrm{d}z| \le \max_{|z|=R} |g(z)| \int_{C_R^+} 1|\mathrm{d}z| \\ &\le \pi R \max_{|z|=R} |g(z)| \le \frac{\pi R}{R^2-1} \xrightarrow[R\to\infty]{} 0,\end{aligned} \tag{3}$$

and similarly for C_R^-.

d) We start with the first integral,

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\,\mathrm{d}x,$$

and let us assume $a > 0$. By part (c), we can "close the contour" in the upper half-plane, since the integral over C_R^+ vanishes in the limit $R \to \infty$. Concretely, we have (recall that $R > 1$)

$$\int_{\Gamma_R^+} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}(f, i) = \pi e^{-a}.$$

Rearranging, we obtain

$$\int_{-R}^{R} f(x)\,\mathrm{d}x = \int_{\Gamma_R^+} f(z)\,\mathrm{d}z - \int_{C_R^+} f(z)\,\mathrm{d}z = \pi e^{-a} - \int_{C_R^+} f(z)\,\mathrm{d}z.$$

By part (c), the second term on the right converges to zero as $R \to \infty$. Therefore, taking the limit $R \to \infty$ on both sides yields

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \pi e^{-a}, \qquad a > 0.$$

For $a < 0$, we must instead close the contour in the lower half-plane. Repeating the steps above, we find

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \pi e^{a}, \qquad a < 0.$$

For $a = 0$, we may close the contour in either the upper or lower half-plane; we get the same result either way:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \operatorname{Res}(f, i) = \pi.$$

We notice that the second integral,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} \operatorname{Re}(f(x))\,\mathrm{d}x = \operatorname{Re}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

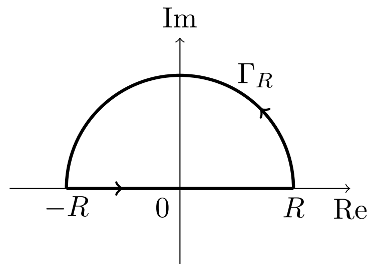
is precisely the real part of the first integral, which we just computed. But the first integral is already a real number, so we are done. To summarize our results,

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\,\mathrm{d}x = \pi e^{-|a|}, \quad a \in \mathbb{R}.$$

2. Compute the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x+x^2)^2}.$$

Solution.
We let
$$f(z) = \frac{1}{(1+z+z^2)^2}$$
for $z \in \mathbb{C}$. Then f is a rational function and the degree of the denominator is four higher than the degree of the numerator. Therefore, we can compute this integral by closing the contour in the complex plane. For $R > 0$ let Γ_R be the contour composed of the line segment $[-R, R] \subset \mathbb{R}$ and the semicircle with center 0 and radius R in the upper half-plane, which we denote C_R^+. The contour Γ_R is shown below; it is traversed clockwise.



Then
$$\int_{\Gamma_R} f(z)\,\mathrm{d}z = \int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z$$
or, after rearranging,
$$\int_{-R}^{R} f(x)\,\mathrm{d}x = \int_{\Gamma_R} f(z)\,\mathrm{d}z - \int_{C_R^+} f(z)\,\mathrm{d}z. \tag{4}$$
We evaluate the two terms on the right-hand side in the limit $R \to \infty$. The integral over the semicircle C_R^+ vanishes since the degree of the denominator of f is at least 2 greater than the degree of the numerator of f.

Thus, in the limit $R \to \infty$,
$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{z_i \in A} \mathrm{Res}(f, z_i) \tag{5}$$

where A is the set of all singular points of f in the upper half-plane. Let's identify the singular points and compute the residues.

Factoring the denominator, we have

$$(1+z+z^2)^2 = (z-z_1)^2(z-z_2)^2$$

where

$$z_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \qquad z_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

are the two roots of $1+z+z^2$. Only z_1 lies in the upper half-plane, and it is a pole of order 2. To compute the residue, let's define

$$g(z) = (z-z_1)^2 f(z) = \frac{1}{(z-z_2)^2},$$

so that $\operatorname{Res}(f, z_1) = g'(z_1)/1! = g'(z_1)$. We compute,

$$g'(z) = -\frac{2}{(z-z_2)^3}$$

and thus

$$g'(z_1) = -\frac{2}{(z_1-z_2)^3} = -\frac{2}{(i\sqrt{3})^3} = \frac{2}{3\sqrt{3}i}.$$

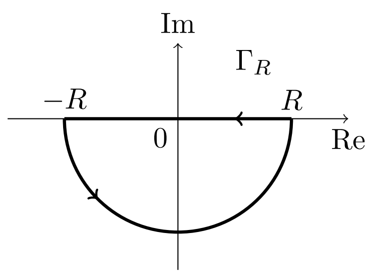
Therefore, the integral we were asked to compute is

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x+x^2)^2} = 2\pi i \operatorname{Res}(f, z_1) = \frac{4\pi}{3\sqrt{3}}.$$

3. Compute the integral
$$\int_{-\infty}^{\infty} \frac{xe^{-ikx}}{1+x^2}\,\mathrm{d}x.$$
where (a) $k > 0$, (b) $k < 0$.

Solution.
Let
$$f(z) = \frac{ze^{-ikz}}{1+z^2}$$
for $z \in \mathbb{C}$. We note that $f(z) = R(z)e^{-ikz}$ where $R(z) = z/(1+z^2)$ is a rational function, and the degree of the denominator is one higher than the degree of the numerator. Therefore, we can compute this integral by closing the contour in the complex plane. If $k > 0$, we must close it in the lower half-plane and if $k < 0$ we must close it in the upper half-plane.

a) Let us assume $k > 0$. For $R > 0$ let Γ_R be the contour composed of the line segment $[-R, R] \subset \mathbb{R}$ and the semicircle with center 0 and radius R in the *lower* half-plane, traversed clockwise. The contour Γ_R is shown below. We denote by C_R^- the semicircular part.



Notice that Γ_R traverses the line segment $[-R, R]$ in the negative direction. Therefore, we have
$$\int_{\Gamma_R} f(z)\,\mathrm{d}z = -\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{C_R^-} f(z)\,\mathrm{d}z$$
or, after rearranging,
$$\int_{-R}^{R} f(x)\,\mathrm{d}x = -\int_{\Gamma_R} f(z)\,\mathrm{d}z + \int_{C_R^+} f(z)\,\mathrm{d}z. \tag{6}$$
We evaluate the two terms on the right-hand side in the limit $R \to \infty$. The integral over the semicircle C_R^- vanishes by Jordan's lemma.

For the second term, we note that f has a simple pole at $z = -i$ and no other singular points in the lower half-plane. Therefore,

$$\int_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}(f, -i), \qquad R > 1 \tag{7}$$

and

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = -2\pi i\,\mathrm{Res}(f, -i) \tag{8}$$

Let's compute the residue:

$$\mathrm{Res}(f, -i) = \lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}\frac{ze^{-ikz}}{z-i} = \frac{e^{-k}}{2}.$$

Thus we find

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = -i\pi e^{-k}. \tag{9}$$

b) Now suppose that $k < 0$. Then we must close the contour in the upper half-plane and we see from the above derivation that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i\,\mathrm{Res}(f, i) = 2\pi i \cdot \frac{e^{k}}{2} = i\pi e^{k}. \tag{10}$$

Alternatively, we can see this from the following manipulations:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{xe^{-ikx}}{1+x^2}\,\mathrm{d}x &= -\int_{-\infty}^{\infty} \frac{(-x)e^{-i(-k)(-x)}}{1+(-x)^2}\,\mathrm{d}x \\ &= -\int_{-\infty}^{\infty} \frac{ue^{-i(-k)u}}{1+u^2}\,\mathrm{d}u = -\Big[-i\pi e^{-(-k)}\Big] = i\pi e^{k}.\end{aligned}$$

In the first step we did nothing but rewrite the integrand. In the second step, we changed variables to $u = -x$. In the third step, we used the result from (a), noting that $-k$ is positive.

We can summarize the results from (a) and (b) by writing

$$\int_{-\infty}^{\infty} \frac{xe^{-ikx}}{1+x^2}\,\mathrm{d}x = -i\pi e^{-|k|}\,\mathrm{sgn}(k)$$

where sgn is the sign function.

Note: Integrals of the kind

$$\int_{-\infty}^{\infty} g(x) e^{-ikx}\,\mathrm{d}x$$

appear when computing Fourier transforms, which play a central role in the remainder of this course. In fact, $\hat{g}(k) = -i\pi e^{-|k|}\,\mathrm{sgn}(k)$ is precisely the Fourier transform of the function $g(x) = x/(1+x^2)$.

4. In this problem, we compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,\mathrm{d}x.$$

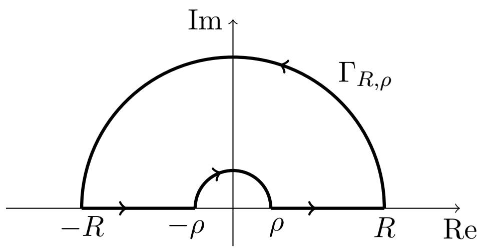
using the residue theorem. We cannot use the same method that we used in problem 1, because e^{iz}/z has a pole on the real line. We modify the method by integrating around the singularity, as shown in the figure below.

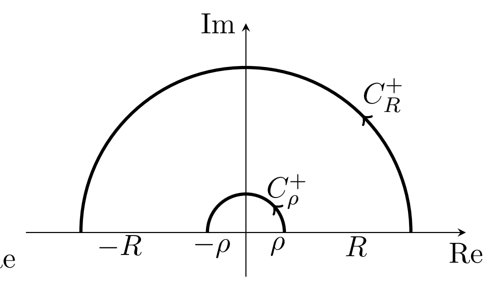
We start by proving the following result:

a) Let C_ρ^+ be the semicircle of radius $\rho > 0$ in the upper half-plane, traversed counter-clockwise. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an analytic function with a simple pole at $z = 0$. Show that

$$\lim_{\rho \to 0} \int_{C_\rho^+} f(z)\,\mathrm{d}z = i\pi\,\mathrm{Res}(f, 0).$$

Hint: Expand f in a Laurent series around 0 and integrate term by term.





Left: The contour $\Gamma_{R,\rho}$. Right: The semicircles C_R^+ and C_ρ^+.

b) Let $f(z) = e^{iz}/z$ and let $\Gamma_{R,\rho}$ be the contour shown above with $0 < \rho < R$, so that

$$\int_{\Gamma_{R,\rho}} f(z)\,\mathrm{d}z = \int_\rho^R f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z + \int_{-R}^{-\rho} f(x)\,\mathrm{d}x - \int_{C_\rho^+} f(z)\,\mathrm{d}z,$$

and thus, after rearranging,

$$\int_\rho^R \frac{e^{ix} - e^{-ix}}{x}\,\mathrm{d}x = \int_{\Gamma_{R,\rho}} f(z)\,\mathrm{d}z + \int_{C_\rho^+} f(z)\,\mathrm{d}z - \int_{C_R^+} f(z)\,\mathrm{d}z.$$

Evaluate each term on the right-hand side in the limit $R \to \infty$, $\rho \to 0$. Use this to compute

$$I_1 = \int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x$$

and

$$I_2 = \int_{-\infty}^\infty \frac{\sin(x)}{x}\,\mathrm{d}x.$$

Solution.

a) As suggested by the hint, we expand f in a Laurent series around 0,

$$f(z) = \sum_{n=-1}^{\infty} a_n z^n$$

and integrate term by term:

$$\int_{C_\rho^+} f(z)\,\mathrm{d}z = \sum_{n=-1}^{\infty} a_n \int_{C_\rho^+} z^n\,\mathrm{d}z \tag{11}$$

Let us evaluate the integrals in the limit $\rho \to 0$. We parameterize the semicircle by $\gamma\colon [0,\pi] \to \mathbb{C}$, $\gamma(\theta) = \rho e^{i\theta}$ to obtain

$$\int_{C_\rho^+} z^n\,\mathrm{d}z = \int_0^\pi \left(\rho e^{i\theta}\right)^n i\rho e^{i\theta}\,\mathrm{d}\theta = i\rho^{n+1} \int_0^\pi e^{i(n+1)\theta}\,\mathrm{d}\theta.$$

If $n \geq 0$, we obtain in the limit $\rho \to 0$,

$$\lim_{\rho\to 0} \int_{C_\rho^+} z^n \, \mathrm{d}z = i \int_0^\pi e^{i(n+1)\theta} \, \mathrm{d}\theta \cdot \lim_{\rho\to 0} \rho^{n+1} = i \int_0^\pi e^{i(n+1)\theta} \, \mathrm{d}\theta \cdot 0 = 0.$$

For $n = -1$, there is no ρ-dependence and we obtain

$$\lim_{\rho\to 0} \int_{C_\rho^+} z^n \, \mathrm{d}z = i \int_0^\pi \mathrm{d}\theta = i\pi.$$

Therefore, taking the limit $\rho \to 0$ in equation (11) yields

$$\int_{C_\rho^+} f(z) \, \mathrm{d}z = a_{-1} \int_{C_\rho^+} z^n \, \mathrm{d}z = a_{-1} \cdot i\pi = i\pi \operatorname{Res}(f, 0).$$

b) The function f has a simple pole at $z = 0$ and no other singular points. The contour $\Gamma_{R,\rho}$ does not enclose any singularities, so by the residue theorem

$$\int_{\Gamma_{R,\rho}} f(z) \, \mathrm{d}z = 0$$

for any $R > \rho > 0$. From part (a), we know that

$$\lim_{\rho\to 0} \int_{C_\rho^+} f(z) \, \mathrm{d}z = i\pi \operatorname{Res}(f, 0) = i\pi$$

Finally, the integral over C_R^+ converges to 0 as $R \to \infty$ by Jordan's lemma. Thus we conclude that

$$\int_0^\infty \frac{e^{ix} - e^{-ix}}{x} \, \mathrm{d}x = \lim_{\substack{\rho\to 0 \\ R\to\infty}} \int_\rho^R \frac{e^{ix} - e^{-ix}}{x} \, \mathrm{d}x = i\pi.$$

It follows immediately that

$$I_1 = \int_0^\infty \frac{\sin(x)}{x} \, \mathrm{d}x = \frac{1}{2i} \int_0^\infty \frac{e^{ix} - e^{-ix}}{x} \, \mathrm{d}x = \frac{\pi}{2}.$$

Since $\sin(x)/x$ is an even function, we conclude that

$$I_2 = \int_{-\infty}^\infty \frac{\sin(x)}{x} \, \mathrm{d}x = 2I_1 = \pi.$$