Homework 9 Handed out: Wednesday, November 9, 2022
Due: Wednesday, November 16, 2022 by 11:59pm

Material covered: Outcomes 8.1–8.5.

1. Recall the Fourier transform pair derived in class:

$$\mathscr{F}\left[e^{-a|x|}\right] = \frac{2a}{k^2 + a^2}$$

a) Using Fourier transform properties, calculate the Fourier transform:

$$\mathscr{F}\left[\frac{1}{x^2 - 2x + 10}\right]$$

b) Using your solution to (a) along with Fourier transform properties, calculate the Fourier transform:

$$\mathscr{F}\left[\frac{xe^{3ix} - e^{3ix}}{(x^2 - 2x + 10)^2}\right]$$

Solution

a) By the Fourier inversion property, we know that if $\hat{f}(k) = \mathscr{F}[f(x)]$, then $\mathscr{F}[\hat{f}(x)] = 2\pi f(-k)$.

Therefore,

$$\mathscr{F}\left[\frac{1}{x^2 + a^2}\right] = \frac{2\pi}{2a} e^{-a|-k|} = \frac{\pi}{a} e^{-a|k|}$$

Notice that

$$x^2 - 2x + 10 = x^2 - 2x + 1 + 9 = (x-1)^2 + 3^2$$

So,

$$\mathscr{F}[\frac{1}{x^2 + 9}] = \frac{\pi}{3} e^{-3|k|}$$

and incorporating the shift property:

$$\mathscr{F}\left[\frac{1}{x^2-2x+10}\right] = \frac{\pi}{3}e^{-3|k|}e^{-ik} = \frac{\pi}{3}e^{-(3|k|+ik)}$$

b) Notice that

$$\frac{d}{dx}\left[\frac{1}{x^2-2x+10}\right] = -\frac{(2x-2)}{(x^2-2x+10)^2}$$

With the derivative property, we know that

$$\mathscr{F}\left[-\frac{1}{(x^2-2x+10)^2}(2x-2)\right] = ik\frac{\pi}{3}e^{-(3|k|+ik)}$$

Divide by -2, and incorporate a shift:

$$\mathscr{F}\left[\frac{e^{3ix}(x-1)}{(x^2-2x+10)^2}\right] = -\frac{\pi i(k-3)}{6}e^{-(3|k-3|+i(k-3))} = \pi i(\frac{1}{2}-\frac{k}{6})e^{-3|k-3|-ik+3i}$$

2. In this problem, we will use an alternative approach to solve 1(a):

$$\mathscr{F}\left[\frac{1}{x^2-2x+10}\right]$$

Rewrite the Fourier transform expression as a complex contour integral for an appropriate half-circle, then solve the integral using the residue theorem.

Solution The Fourier Transform will be given by

$$\int_{-\infty}^{\infty}\frac{e^{-ikx}}{x^2-2x+10}dx$$

To evaluate this integral, we consider the complex-valued function

$$f(z) = \frac{e^{-ikz}}{z^2-2z+10}$$

First, factor the polynomial to find simple poles at $z = 1 + 3i$ and $z = 1 - 3i$. The residues at these poles are then:

$$\frac{e^{-i(1+3i)k}}{6i} = \frac{1}{6i}e^{(3-i)k}$$

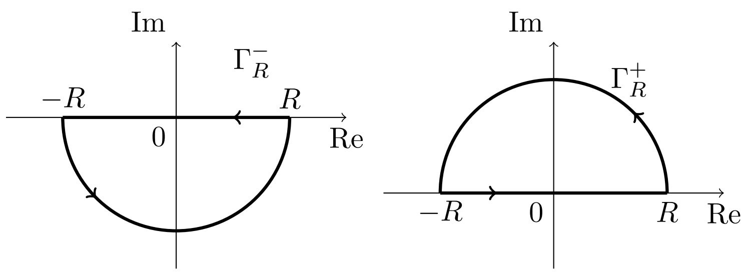
and

$$\frac{e^{-i(1-3i)k}}{-6i} = -\frac{1}{6i}e^{(-3-i)k}$$

respectively.

Since convergence of the term $e^{-ikz} = e^{-ik(x+iy)} = e^{-ikx+ky}$ at a large radius R depends on the sign of ky, we will consider a contour in the bottom half-plane for the case $k > 0$, and a contour in the top half-plane for the case $k < 0$.

Consider $R > 3$, and let Γ_R^+, Γ_R^- be the following contours:



For $k > 0$, consider the integral

$$\oint_{\Gamma_R^-} f(z)dz = \int_R^{-R} f(x)dx + \int_{C_R^-} f(x)dx$$

By Jordan's Lemma, the integral over the half-circular arc C_R would approach 0 as $R \to \infty$, leaving the integral over the real line. Additionally, by the residue theorem,

$$\oint_{\Gamma_R^-} f(z)dz = -\frac{\pi}{3}e^{(-3-i)k}$$

So, taking the limit as $R \to \infty$:

$$\mathscr{F}[f(x)] = \frac{\pi}{3} e^{-(3+i)k} \text{ for } k > 0$$

Similarly, for $k < 0$, consider the integral

$$\oint_{\Gamma_R^+} f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R^+} f(x)dx$$

Again, by Jordan's Lemma, the integral over the half-circular arc C_R^+ would approach 0 as $R \to \infty$, leaving the integral over the real line. By the residue theorem,

$$\oint_{\Gamma_R^+} f(z)dz = \frac{\pi}{3} e^{(3-i)k}$$

So, taking the limit as $R \to \infty$:

$$\mathscr{F}[f(x)] = \frac{\pi}{3} e^{-(3(-k)+ik)} \text{ for } k < 0$$

And writing both cases together:

$$\mathscr{F}[f(x)] = \frac{\pi}{3} e^{-(3|k|+ik)}$$

3. The Plancherel Theorem states that for functions $f(x), g(x)$ with Fourier transforms $\hat{f}(k)$, $\hat{g}(k)$,

$$\langle \hat{f}(k), \hat{g}(k) \rangle = 2\pi \langle f(x), g(x) \rangle$$

where $\langle \dots \rangle$ indicates the functional inner product,

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

Prove that

$$\int_{-\infty}^{\infty} \frac{\sin(at)\sin(bt)}{t^2} dt = \pi \min(a, b)$$

for real-valued $a, b > 0$.

Solution In lecture, we derived that the Fourier Transform of the boxcar function B_a is the sinc function:

$$\mathscr{F}^{-1}\left[\frac{\sin(ak)}{ak}\right] = B_a(x) = \begin{cases} \frac{1}{2a} & |x| < a \\ 0 & |x| \geq a \end{cases}$$

Therefore, by Plancherel's Theorem,

$$\int_{-\infty}^{\infty} \frac{\sin(at)}{t}\frac{\sin(bt)}{t} dt = 2\pi \int_{-\infty}^{\infty} ab\ B_a(x)B_b(x)dx$$

Since $B_a(x)B_b(x)$ is identically 0 on $[-\min(a,b), \min(a,b)]$, this integral becomes

$$2\pi \int_{-\min(a,b)}^{\min(a,b)} \frac{1}{4} dx = \frac{\pi}{2} \cdot 2 \cdot \min(a,b) = \pi \min(a,b)$$

4. Let a be a positive integer, and consider the function $f(x) = e^{-x}x^a$ for $x > 0$, $f(x) = 0$ for $x \leq 0$. Find $\mathscr{F}[f(x)]$.

Solution By the derivative property, we know that

$$\mathscr{F}[xf(x)] = i(\hat{f})'(k)$$

Consequently,

$$\mathscr{F}[x^2 f(x)] = i((i\hat{f}')'(k) = i^2 \hat{f}^{(2)}(k)$$

Which generalizes to

$$\mathscr{F}[x^a f(x)] = i^a \hat{f}^{(a)}(k)$$

In this case, we can consider $f(x) = x^a g(x)$ where $g(x) = e^{-x}$ for $x > 0$, $f(x) = 0$ for $x \leq 0$. Since the Fourier transform of $g(x)$ is simply

$$\hat{g}(k) = \int_0^\infty e^{-x}e^{-ikx}dx = \int_0^\infty e^{-x(1+ik)}dx = -\frac{e^{-x(1+ik)}}{1+ik}|_0^\infty = \frac{1}{1+ik}$$

The a'th derivative of $\hat{g}(k)$ is

$$\hat{g}^{(a)}(k) = a!(-1)^a i^a (1 + ik)^{-a-1}$$

So, in all, we find

$$\mathscr{F}[e^{-x} x^a] = i^a a!(-1)^a i^a (1 + ik)^{-a} = \frac{a!}{(1 + ik)^{a+1}}$$

5. *Spectrograms.* Open the Python notebook here and follow the instructions.

Solutions

a) Can you guess what/who this is an audio recording of? This is of course not graded, but it is a recording of a Musician Wren in the Amazon rainforest.

b) What frequency has the greatest amplitude in the signal? The spectrum has the highest peak at 1900 Hz.

c) Do the spectra above differ from each other? Why or why not? Yes, they are different because the frequencies present in the recording are varying over time.

d) What is the purpose of using a window function? Similarly to a Fourier series, the discrete Fourier transform considers a periodic extension of the given signal. Consequently, if the value of the first time sample is very different from the last time sample, the resulting discontinuity can distort the calculated spectrum. The window functions mitigate this issue by tapering the edges of the segment to 0, and are designed to minimize distortion in the frequency domain.

e) Screenshot and remaining questions depend on the students' individual recordings.