Homework 6 Handed out: Wednesday, October 19, 2022
Due: Wednesday, October 26, 2022 by 11:59pm

Material covered: Outcomes 5.1–5.3.

Solutions

1. Let $\alpha \in \mathbb{C}$ be a complex number and let

$$f\colon U \to \mathbb{C}, \quad f(z) = (1+z)^\alpha = e^{\alpha \operatorname{Log}(1+z)}$$

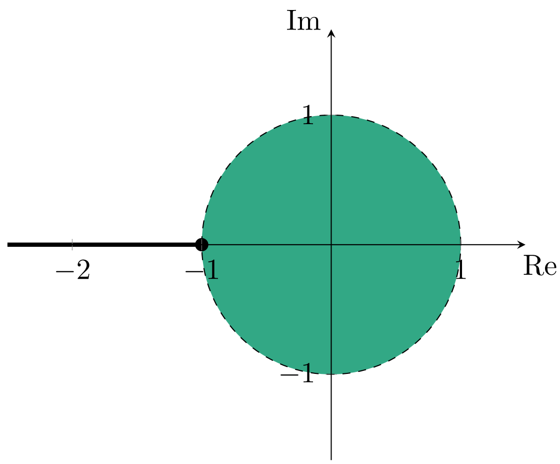
where $U = \mathbb{C} \setminus \{x + iy \in \mathbb{C} \mid y = 0, x \leq -1\}$. Determine the radius of convergence of the Taylor series of f around $z = 0$ and find the first four terms.

You may assume that α is not a positive integer or zero.

Solution. If α is a nonnegative integer, then f is a polynomial and thus an entire function. Therefore, the radius of convergence of the Taylor series is $R = +\infty$. Moreover, since f is a polynomial, it is equal to its Taylor series.

Let us therefore assume α is *not* a nonnegative integer. Then the radius of convergence R of the Taylor series is the radius of the largest disk around $z = 0$ on which f is analytic. Since f is analytic outside of the set $\{x + iy \in \mathbb{C} \mid y = 0, x \leq -1\}$, R is equal to the distance to the branch point at $z = -1$:

$$R = |(-1) - 0| = 1.$$



The branch cut and branch point of f are shown in black. The largest disk around the origin on which f is analytic is shown in green.

Next we compute the first four terms in the Taylor series. To do that, we will have to differentiate f four times. We recall from calculus that if α is a real number, then for $x \in \mathbb{R}$, $x > 1$,

$$f'(x) = \alpha(1+x)^{\alpha-1}$$

Let us check how this result generalizes to $\alpha \in \mathbb{C}$ and $z \in U$. Using the chain rule, we have

$$f'(z) = \alpha(1+z)^{-1}e^{\alpha \operatorname{Log}(1+z)}.$$

Now, $(1+z)^{-1} = e^{-\operatorname{Log}(1+z)}$, so we can rewrite this as

$$f'(z) = \alpha e^{(\alpha-1)\operatorname{Log}(1+z)} = \alpha(1+z)^{\alpha-1}. \tag{1}$$

In order to compute the second derivative, let's write $\beta = \alpha - 1$ so

$$f''(z) = \frac{\mathrm{d}}{\mathrm{d}z}f'(z) = \alpha\frac{\mathrm{d}}{\mathrm{d}z}(1+z)^{\beta}.$$

We can now use the result in equation (1) (with α replaced by β) to find

$$f''(z) = \alpha\beta(1+z)^{\beta-1} = \alpha(\alpha-1)(1+z)^{\alpha-2}.$$

By induction, we see that the nth derivative is

$$f^{(n)}(z) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+z)^{\alpha-n}.$$

and at $z = 0$, this is equal to

$$f^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-n+1).$$

The Taylor series of f around 0 is then

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!}\alpha(\alpha-1)\cdots(\alpha-n+1)z^n \\ = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}z^3 + \cdots\end{aligned}$$

and it converges to $f(z)$ for $|z| < 1$.[1]

2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n.$$

Compute the following:

a) $f^{(4)}(0)$

b) $\displaystyle\int_C \frac{f(z)}{z^6}\,\mathrm{d}z$ where C is the unit circle.

[1] If α is a nonnegative integer, then the series converges on $\mathbb{C}$ since all but finitely many terms are zero.

Solution.
a) Recall that if f is given by a convergent power series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

then the coefficients a_n are uniquely determined, and given by

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Therefore,

$$f^{(4)}(0) = 4!\frac{4^3}{3^4} = \frac{512}{27}.$$

b) The function $g(z) = f(z)/z^6$ has a pole of order 6 at the origin and no other singular points enclosed by C. The residue at 0 can be read from the Laurent series for g:

$$g(z) = \frac{1}{z^6}\sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n = \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^{n-6}.$$

The residue is the coefficient of z^{-1} in the series, which is

$$\operatorname{Res}(g, 0) = \frac{5^3}{3^5} = \frac{125}{243}.$$

The integral is now easily computed using the residue theorem:

$$\int_C \frac{f(z)}{z^6}\,\mathrm{d}z = 2\pi i \operatorname{Res}(g, 0) = \frac{250}{243}\pi i.$$

For completeness, we show that f is analytic on a disk of radius 3 around 0, so that the expression in (a) and the integral in (b) are well-defined.[2] We note that the series converges for $|z| < 3$, as can be seen from the ratio test:

$$\left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = \left|\frac{(n+1)^3/3^{n+1}}{n^3/3^n}\right| |z| = \frac{(n+1)^3}{n^3}\frac{|z|}{3} \xrightarrow[n\to\infty]{} \frac{|z|}{3}.$$

[2] This is not required to get full credit for the problem.

Thus the radius of convergence is $R = 3$. Since f is given by a convergence power series on $\{z \in \mathbb{C} \mid |z| < 3\}$, f is analytic there.

3. Find the Laurent series expansion of the function

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

on each of the following annuli:

a) $D = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$

b) $D = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$

c) $D = \{z \in \mathbb{C} \mid |z| > 2\}$

Solution.
a) We rewrite f as

$$f(z) = \frac{1}{z}\left[\frac{-1}{z-1} + \frac{1}{z-2}\right] = \frac{1}{z}\left[\frac{1}{1-z} - \frac{1/2}{1-z/2}\right]$$

For $|z| < 1$, we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

and for $|z| < 2$, we have

$$\frac{1/2}{1-z/2} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^n} z^n.$$

Combining these results, we find that for $|z| < 1$,

$$f(z) = \frac{1}{z}\left[\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^n} z^n\right] = \frac{1}{z}\sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right) z^n.$$

which can also be written

$$f(z) = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right) z^{n-1} = \sum_{n=-1}^{\infty}\left(1 - \frac{1}{2^{n+2}}\right) z^n.$$

b) In this domain, the above series for $1/(1-z)$ does not converge. Instead, we can rewrite it was

$$\frac{1}{1-z} = -\frac{1}{z}\frac{1}{1-1/z} = -\frac{1}{z}\sum_{n=0}^{\infty} z^{-n} = -\frac{1}{z}\sum_{n=-\infty}^{0} z^{n} = -\sum_{n=-\infty}^{-1} z^{n}$$

and this series converges for $|z| > 1$. Thus we obtain

$$f(z) = -\frac{1}{z}\Bigg[\sum_{n=-\infty}^{-1} z^n + \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n\Bigg] = -\sum_{n=-\infty}^{-2} z^n - \sum_{n=-1}^{\infty}\frac{1}{2^{n+2}}z^n.$$

c) Finally, for $|z| > 2$ the above series for $1/(1-z/2)$ does not converge. We rewrite the expression as before:

$$\begin{aligned}\frac{1}{z-2} &= \frac{1}{z}\frac{1}{1-2/z} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^{n+1} \\ &= \frac{1}{2}\sum_{n=-\infty}^{-1}\left(\frac{z}{2}\right)^n = \frac{1}{2}\sum_{n=-\infty}^{-1} 2^{-n}z^n\end{aligned}$$

which converges for $|z| > 2$. Hence, in this domain,

$$\begin{aligned} f(z) &= \frac{1}{z}\Bigg[-\sum_{n=-\infty}^{-1} z^n + \frac{1}{2}\sum_{n=-\infty}^{-1} 2^{-n}z^n\Bigg] \\ &= -\sum_{n=-\infty}^{-2} z^n + \frac{1}{4}\sum_{n=-\infty}^{-1} 2^{-(n-1)}z^{n-1} = \sum_{n=-\infty}^{-2}\Big(2^{-(n+2)}-1\Big)z^n.\end{aligned}$$

4. For each of the following functions f, identify the singular points and classify them into removable singularities, poles and essential singularities. Calculate the residue at each singular point and determine the order of each pole.

   a) $f(z) = \dfrac{1}{z^4+1}$

b) $f(z) = \dfrac{\cos(z) - 1}{z^2}$

c) $f(z) = z^3 e^{\frac{1}{z^2}}$

d) $f(z) = \dfrac{\sin(z)}{(z-\pi)^4}$

Solution.
a) Since f is a rational function, the singular points are the four roots of the denominator, which are the four 4th roots of -1:

$$z_k = e^{i\frac{2k+1}{4}\pi} = e^{ik\frac{\pi}{2}} e^{i\frac{\pi}{4}} = i^k e^{i\frac{\pi}{4}}, \qquad k \in \{0,1,2,3\}. \tag{2}$$

Since they are all simple roots (roots of multiplicity one) and since the numerator does not vanish at any of the roots, they are simple poles of f.

It remains to compute the residues. One approach is to compute them one at a time. For example, for the residue at z_0,

$$\begin{aligned}\operatorname{Res}(f, z_0) &= \lim_{z\to z_0}(z - z_0)f(z) = \lim_{z\to z_0} \frac{1}{(z-z_1)(z-z_2)(z-z_3)} \\ &= \frac{1}{(z_0-z_1)(z_0-z_2)(z_0-z_3)} = \frac{1}{(1-i)(1-i^2)(1-i^3)z_0^3} = \frac{1}{4z_0^3}\end{aligned}$$

In the second to last step, we used that $z_k = i^k z_0$ (see (2)). Now we plug in $z_0 = e^{i\pi/4}$ and simplify:

$$\operatorname{Res}(f, z_0) = \frac{1}{e^{i3\pi/4}\cdot 4} = -\frac{1}{4}e^{i\frac{\pi}{4}}.$$

The residues at z_1, z_2, z_3 can be computed in the same way.

Alternatively, all four residues can be computed at the same time as follows (using L'Hôpital's rule in the third step):

$$\begin{aligned}\operatorname{Res}(f, z_k) &= \lim_{z\to z_k}(z - z_k)f(z) = \lim_{z\to z_k}\frac{z - z_k}{z^4+1} \\ &= \lim_{z\to z_k}\frac{1}{4z^3} = \frac{1}{4z_0^3}.\end{aligned}$$

Thus:

$$\begin{aligned}
\mathrm{Res}(f, e^{i\frac{\pi}{4}}) &= -\frac{1}{4}e^{i\frac{\pi}{4}}, \\
\mathrm{Res}(f, e^{i\frac{3\pi}{4}}) &= -\frac{i}{4}e^{i\frac{\pi}{4}}, \\
\mathrm{Res}(f, e^{i\frac{5\pi}{4}}) &= \frac{1}{4}e^{i\frac{\pi}{4}}, \\
\mathrm{Res}(f, e^{i\frac{7\pi}{4}}) &= \frac{i}{4}e^{i\frac{\pi}{4}},
\end{aligned}$$

b) There is a single isolated singularity at $z = 0$. To classify it, we consider the leading behavior of $\cos(z)$ near $z = 0$:

$$\cos(z) = 1 - \frac{1}{2}z^2 + z^4 h(z)$$

for some analytic function h. Thus for $z \neq 0$,

$$f(z) = \frac{-z^2/2 + z^4 h(z)}{z^2} = -\frac{1}{2} + z^2 h(z)$$

and

$$\lim_{z \to 0} f(z) = -\frac{1}{2}.$$

We see that the singularity at $z = 0$ is removable. It follows that the residue is zero.

c) We use the Taylor series for e^z to expand f into a Laurent series centered at $z = 0$:

$$\begin{aligned}
f(z) &= z^3 \sum_{n=0}^{\infty} \frac{1}{n!} z^{-2n} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{3-2n} \\
&= z^3 + z + \frac{1}{2!}z^{-1} + \frac{1}{3!}z^{-3} + \cdots .
\end{aligned}$$

Since the Laurent series has infinitely many negative power terms, $z = 0$ is an essential singularity. The residue is the coefficient of the z^{-1} term:

$$\mathrm{Res}(f, 0) = \frac{1}{2!} = \frac{1}{2}.$$

d) As the quotient of two analytic functions, f is analytic except at $z = \pi$ where the denominator vanishes. Since $\sin(z)$ has a simple zero at $z = \pi$ (meaning $\sin(\pi) = 0$ and $\sin'(\pi) \neq 0$), and $(z-\pi)^4$ has a zero of multiplicity 4, we see that $z = \pi$ is a pole of order $4 - 1 = 3$. Another way to see this is to expand $\sin(z)$ in a Taylor series around $z = \pi$:

$$f(z) = \frac{1}{(z-\pi)^4}\left[-(z-\pi)+\frac{1}{6}(z-\pi)^3+\cdots\right] = \frac{1}{(z-\pi)^3}\left[-1+\frac{1}{6}(z-\pi)^2+\cdots\right]$$

We see that the largest negative power is $(z-\pi)^{-3}$, so $z = \pi$ is a pole of order 3. Moreover, we see that the coefficient of the $(z-\pi)^{-1}$ term is $\frac{1}{6}$, so

$$\mathrm{Res}(f, \pi) = \frac{1}{6}.$$

An alternative way to compute the residue is to let

$$g(z) = (z-\pi)^3 f(z) = \frac{\sin(z)}{z-\pi}.$$

Then

$$\mathrm{Res}(f, \pi) = \frac{1}{2!} g''(\pi).$$

However, this method can be quite tedius for higher order poles.

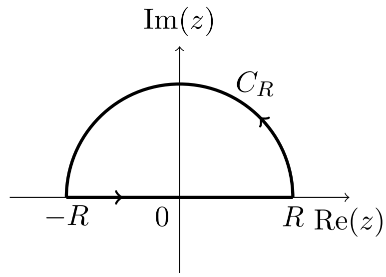
5. Let

$$f(z) = \frac{e^{iz}}{(1+z^2)^2}$$

For $R > 0$, compute the contour integral

$$\int_{C_R} f(z)\,\mathrm{d}z$$

where C_R is the contour composed of a line segment from $-R$ to R and a semicircle of radius R in the upper half-plane, traversed counter-clockwise (shown below). How does the answer depend on R?



Solution.
We compute the integral using the residue theorem. The integrand f has two poles, $z = \pm i$, both of order two. This can be seen by factoring the denominator:

$$f(z) = \frac{e^{iz}}{(z+i)^2(z-i)^2}.$$

The pole at $z = i$ is enclosed by C_R as long as $R > 1$. The pole at $z = -i$ is never enclosed by C_R, so it is unnecessary to compute the residue there.

We conclude that for $0 < R < 1$,

$$\int_{C_R} f(z)\,\mathrm{d}z = 0,$$

and for $R > 1$,

$$\int_{C_R} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}(f, i).$$

All that remains is to compute the residue at i. Let

$$g(z) = (z-i)^2 f(z) = \frac{e^{iz}}{(z+i)^2}.$$

Then

$$\operatorname{Res}(f, i) = \frac{1}{1!} g'(i) = \left[\frac{ie^{iz}}{(z+i)^2} - \frac{2e^{iz}}{(z+i)^3}\right]_{z=i} = \frac{1}{2ei}.$$

Thus for $R > 1$,

$$\int_{C_R} f(z)\,\mathrm{d}z = \frac{\pi}{e}.$$