Homework 3 Handed out: Wednesday, September 21, 2022
Due: Wednesday, September 28, 2022 by 11:59pm

Material covered: Outcomes 2.4, 3.1–3.3.

Recall that Log denotes the principal branch of the logarithm, defined on $\mathbb{C} \setminus \mathbb{R}_-$ (the complex plane excluding the negative real axis) as follows: If $z = re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$, then

$$\operatorname{Log}(z) := \log(r) + i\theta.$$

1. For $a \in \mathbb{C}$, let z^a denote the principal branch of the a-th power, defined for $z \in \mathbb{C} \setminus \mathbb{R}_-$ by

   $$z^a = e^{a \operatorname{Log} z}.$$

   Find the real and imaginary parts of the following numbers:

   a) $\operatorname{Log}(1 - i)$ b) $(1 + i)^i$ c) $i^{1/\pi}$

2. Identify the domains of the following functions, show that they are holomorphic and compute their derivatives.

   a) $f(z) = \operatorname{Log}(z^2)$ b) $f(z) = \operatorname{Log}(e^z)$

3. Determine where on the complex plane the following functions are holomorphic. Compute the derivative where it exists.

   a) $f(z) = -\operatorname{Im} z + i \operatorname{Re} z$ b) $f(z) = \dfrac{1}{\sin(z)}$

4. Compute the Wirtinger derivatives $\partial_z f$ and $\partial_{\bar{z}} f$. Use this to determine where the functions are holomorphic, and find the derivative $f'(z)$ where it exists.

   a) $f(z) = z + 1/z$ b) $f(z) = z^2|z|^2$

5. Determine where the functions are holomorphic by checking whether the Cauchy–Riemann equations are satisfied.

   a) $f(x+iy) = \dfrac{x}{x^2+y^2} + \dfrac{iy}{x^2+y^2}$

   b) $f(x+iy) = e^{y^2-x^2}\cos(2xy) - ie^{y^2-x^2}\sin(2xy)$