Homework 1 Handed out: Wednesday, September 7, 2022
Due: Wednesday, September 14, 2022 by 11:59pm

Material covered: Outcomes 1.1–1.5.

Solutions

1. Simplify the following expressions, writing them in the form $x + iy$ where x, y are real numbers:

   a)
   $$(3+4i)^2$$

   b)
   $$\frac{1+i}{1-i}$$

   c)
   $$\mathrm{Im}(\overline{1+i}).$$

   Solution.
   a)
   $$(3+4i)^2 = 3^2 + 2(3)(4i) + (4i)^2 = 9 - 16 + 24i = -7 + 24i.$$

   b)
   $$\frac{1+i}{1-i} = \frac{(1+i)\overline{(1-i)}}{(1-i)\overline{(1-i)}} = \frac{(1+i)^2}{|1-i|^2} = \frac{2i}{2} = i.$$

   c)
   $$\mathrm{Im}(\overline{1+i}) = \mathrm{Im}(1-i) = -1.$$

   Note: Remember that both $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ are real numbers.

2. Write the following complex numbers in the form $x + iy$ where x, y are real numbers:

a)

$$\sqrt{2}e^{i\frac{\pi}{4}}.$$

b)

$$(1+i)^{2022}.$$

Write the following complex numbers in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$:

c)

$$(1+\sqrt{3}i)^2$$

d)

$$(1+\sqrt{3}i)\overline{(1+i)}$$

Solution.
a)

$$\sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1+i.$$

b) First rewrite $1+i$ in polar form: $1+i = \sqrt{2}e^{i\pi/4}$. We then find

$$(1+i)^{2022} = \left(\sqrt{2}e^{i\pi/4}\right)^{2022} = \left(\sqrt{2}\right)^{2022}\left(e^{i\pi/4}\right)^{2022} = 2^{1011}e^{i\frac{2022}{4}\pi}.$$

Now, $2022\frac{\pi}{4} = 252 \cdot 2\pi + \frac{3\pi}{2}$, so

$$(1+i)^{2022} = 2^{1011}e^{i\frac{3\pi}{2}} = -2^{1011}i.$$

c) We have $|1+\sqrt{3}i| = 2$ and $\mathrm{Arg}(1+\sqrt{3}i) = \arctan(\sqrt{3}) = \pi/3$. Thus

$$(1+\sqrt{3}i)^2 = \left(2e^{i\pi/3}\right)^2 = 4e^{i\frac{2\pi}{3}}$$

d) One approach is to multiply before converting to polar form. We have

$$(1+\sqrt{3}i)\overline{(1+i)} = (\sqrt{3}+1) + i(\sqrt{3}-1).$$

The modulus is

$$|(\sqrt{3}+1)+i(\sqrt{3}-1)| = \sqrt{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2} = \sqrt{8} = 2\sqrt{2},$$

and the argument is

$$\text{Arg}\Big((\sqrt{3}+1)+i(\sqrt{3}-1)\Big) = \arctan\Big(\frac{\sqrt{3}-1}{\sqrt{3}+1}\Big).$$

Thus we have found

$$(1+\sqrt{3}i)\overline{(1+i)} = 2\sqrt{2}e^{i\theta}, \qquad \text{where } \theta = \arctan\Big(\frac{\sqrt{3}-1}{\sqrt{3}+1}\Big).$$

With a bit more work, one can show that $\theta = \frac{\pi}{12}$.

Another approach, which turns out to be more prudent, is to convert each factor into polar form before multiplying. We have

$$\begin{aligned} 1+\sqrt{3}i &= 2e^{i\frac{\pi}{3}}, \\ \overline{(1-i)} &= \overline{\sqrt{2}e^{i\pi/4}} = \sqrt{2}e^{-i\frac{\pi}{4}}. \end{aligned}$$

Thus

$$(1+\sqrt{3}i)\overline{(1+i)} = 2\sqrt{2}e^{i(\frac{1}{3}-\frac{1}{4})\pi} = 2\sqrt{2}e^{i\frac{\pi}{12}}.$$

3. a) Find all third roots of -1.
   b) Find all fourth roots of i.
   c) Find all fifth roots of 32.

Solution.
a) We want to solve the equation $z^3 = -1$ for z. We have $|z|^3 = |z^3| = |-1| = 1$ from which we conclude that $|z| = 1$. Write z in polar form as $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$. By De Moivre's formula,

$$z^3 = \cos(3\theta) + i\sin(3\theta) = -1.$$

Therefore $3\theta = (2k+1)\pi$ for some $k \in \mathbb{Z}$. Keeping in mind that $e^{i\theta}$ and $e^{i(\theta+2\pi)}$ are the same complex number, we see that there are three solutions:

$$\begin{aligned}
z &= e^{i\frac{\pi}{3}} = \cos\Big(\frac{\pi}{3}\Big) + i\sin\Big(\frac{\pi}{3}\Big) = \frac{1}{2} + \frac{\sqrt{3}}{2}i,\\
z &= e^{i\pi} = -1,\\
z &= e^{i\frac{5\pi}{3}} = e^{-i\frac{\pi}{3}} = \cos\Big(\frac{\pi}{3}\Big) - i\sin\Big(\frac{\pi}{3}\Big) = \frac{1}{2} - \frac{\sqrt{3}}{2}i.
\end{aligned}$$

b) We solve $z^4 = i$ for z. We have $|z| = 1$ and

$$z^4 = \cos(4\theta) + i\sin(4\theta) = i.$$

Therefore $4\theta = (2k + 1/2)\pi$ for some $k \in \mathbb{Z}$. The four solutions have modulus 1 and arguments $\theta = \pi/8$, $\theta = 5\pi/8$, $\theta = 9\pi/8$ and $\theta = 13\pi/2$. We write the solutions in polar form for convenience:

$$\begin{aligned}
z &= e^{i\frac{\pi}{8}},\\
z &= e^{i\frac{5\pi}{8}},\\
z &= e^{i\frac{9\pi}{8}},\\
z &= e^{i\frac{13\pi}{8}}.
\end{aligned}$$

c) We solve $z^5 = 32$ for z. We have $|z|^5 = 32$ which implies $|z| = 2$. Writing $z = 2e^{i\theta}$, we have

$$z^5 = 32\Big[\cos(5\theta) + i\sin(5\theta)\Big] = 32.$$

Therefore $5\theta = 2k\pi$ for some $k \in \mathbb{Z}$. The five solutions are

$$z = 2e^{i\frac{2k\pi}{5}}, \qquad k \in \{0,1,2,3,4\}.$$

4. Find all $z \in \mathbb{C}$ that satisfy the equation and sketch the solution set:

a)

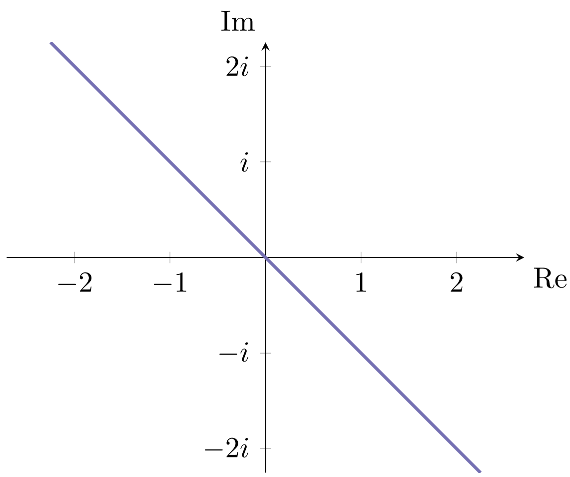
$$|z+1| = |z-i|.$$

b)

$$z + \overline{z} = 0.$$

Solution.
a) The equation states that z is equidistant from the points -1 and i. Therefore, z is a solution if and only if it lies on the diagonal line

$$\{x + iy \in \mathbb{C} \mid x + y = 0,\ x \in \mathbb{R}, y \in \mathbb{R}\}$$



If this is not geometrically obvious, one may use a more algebraic approach. Let $z = x + iy$ where x, y are real. Then z is a solution if and only if $|z+1|^2 = |z-i|^2$, i.e. if and only if

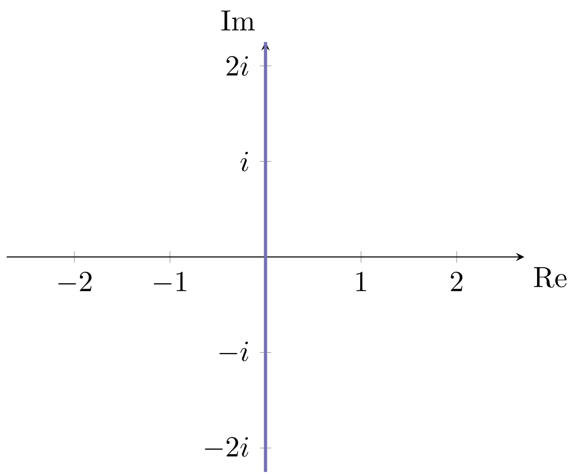
$$(x+1)^2 + y^2 = x^2 + (y-1)^2.$$

By expanding both sides and subtracting common terms, we see that this is equivalent to $x = -y$.

b) We observe that

$$z + \overline{z} = (\operatorname{Re} z + i \operatorname{Im} z) + (\operatorname{Re} z - i \operatorname{Im} z) = 2 \operatorname{Re} z.$$

Thus the equation is equivalent to $\operatorname{Re} z = 0$ and the solution set is the imaginary axis.



5. Find the roots of the polynomial $p(z)$. Then factor the polynomial.

   a)
   $$p(z) = z^2 - (1 + 2i)z - 1 + i.$$

   b)
   $$p(z) = z^5 - z.$$

Solution.
a) We can find the roots of this quadratic polynomial using the quadratic formula. Let

$$D = (-1 - 2i)^2 - 4(-1 + i) = (-3 + 4i) + 4 - 4i = 1.$$

Since $D \neq 0$, the equation has two distinct roots. They are

$$z = \frac{1+2i+\sqrt{D}}{2} = 1+i, z = \frac{1+2i-\sqrt{D}}{2} = i.$$

Therefore, the factorization is of the form

$$p(z) = a(z-i)(z-1-i)$$

for some complex number a. Expanding the right hand side, we see that we should take $a = 1$.

b) We can rewrite $p(z)$ as

$$p(z) = z(z^4 - 1).$$

We immediately see that $z = 0$ is a root. The remaining four roots as the fourth roots of 1, which are $z = 1$, $z = -1$, $z = i$ and $z = -i$. Thus we can write

$$z^4 - 1 = a(z-1)(z+1)(z-i)(z+i)$$

for some $a \in \mathbb{C}$. Expanding the right hand side, we see that $a = 1$. Thus the factorization of $p(z)$ is

$$p(z) = z(z-1)(z+1)(z-i)(z+i).$$