Homework 8 Handed out: Wednesday, November 2, 2022
Due: Wednesday, November 9, 2022 by 11:59pm

Material covered: Outcomes 7.1–7.4.

1. Let f be the $2L$-periodic function which on $[-L, L]$ is given by

$$f(x) = \begin{cases} x, & -L < x < L \\ 0, & x = \pm L. \end{cases}$$

   a) Compute the Fourier series of f in trigonometric form, i.e. find the coefficients a_0, a_n and b_n in

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

   b) Compute the Fourier series of f in complex exponential form, i.e. find the coefficients c_n in

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}.$$

2. Consider the 2π-periodic function f which on $[-\pi, \pi]$ is given by

$$f(x) = \begin{cases} x, & |x| \le \frac{\pi}{2}, \\ \pi - x, & \frac{\pi}{2} \le x \le \pi, \\ -\pi - x, & -\pi \le x \le -\frac{\pi}{2}. \end{cases}$$

   a) Sketch the graph of the function f and its derivative f' on the interval $[-\pi, 3\pi]$.

   b) Find the Fourier series of f in trigonometric form; i.e. find the coefficients a_0, a_n and b_n in

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

c) Find the Fourier series of f' in trigonometric form. What does the Fourier series of f' converge to at $x = \frac{\pi}{2}$?

3. Consider the 2π periodic function

$$f(x) = \frac{\sin(x)}{5 + 4\cos(x)}.$$

Determine the Fourier series of f in trigonometric form.

Hint: One way to compute the integral

$$\int_0^{2\pi} f(\theta) \sin(n\theta) \, \mathrm{d}\theta$$

is to convert it to a contour integral over the unit circle and use the residue theorem.

4. The differential equation

$$mx''(t) + kx(t) = F(t) \tag{1}$$

describes a mass-spring system acted on by a time-varying force $F(t)$. Recall that if $F(t) = 0$, then the general solution is

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

where $\omega_0 = \sqrt{k/m}$.

Suppose that the system is acted on by a periodic force of the form

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t),$$

and assume that ω_0 is *not* an integer multiple of the driving frequency ω. Show that there exists a particular solution x_p of equation (1) of the form

$$x_p(t) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty} \tilde{a}_n \cos(n\omega t) \tag{2}$$

and determine the coefficients $\tilde{a}_0$, $\tilde{a}_n$.

Hint: Substitute the series (2) in the differential equation (1) and differentiate term by term.

5. Open the Python notebook here and follow the instructions.
Note: This problem is not worth any points.