Homework 10 Handed out: Monday, November 21st, 2022
Due: Wednesday, November 30th, 2022 by 11:59pm

Material covered: Outcomes 9.1-9.3

1. **Filtering.** Consider a low-pass filter $h(t)$, defined by its Fourier transform in the frequency domain:
$$\hat{h}(k) = \begin{cases} 1 & \text{for } |k| \leq c \\ 0 & \text{for } |k| > c \end{cases}$$
This filter allows frequencies below the cutoff value c, and excludes frequencies above c.

Now suppose that we have a time-domain signal:
$$x(t) = \frac{\sin(at)}{\pi t}$$

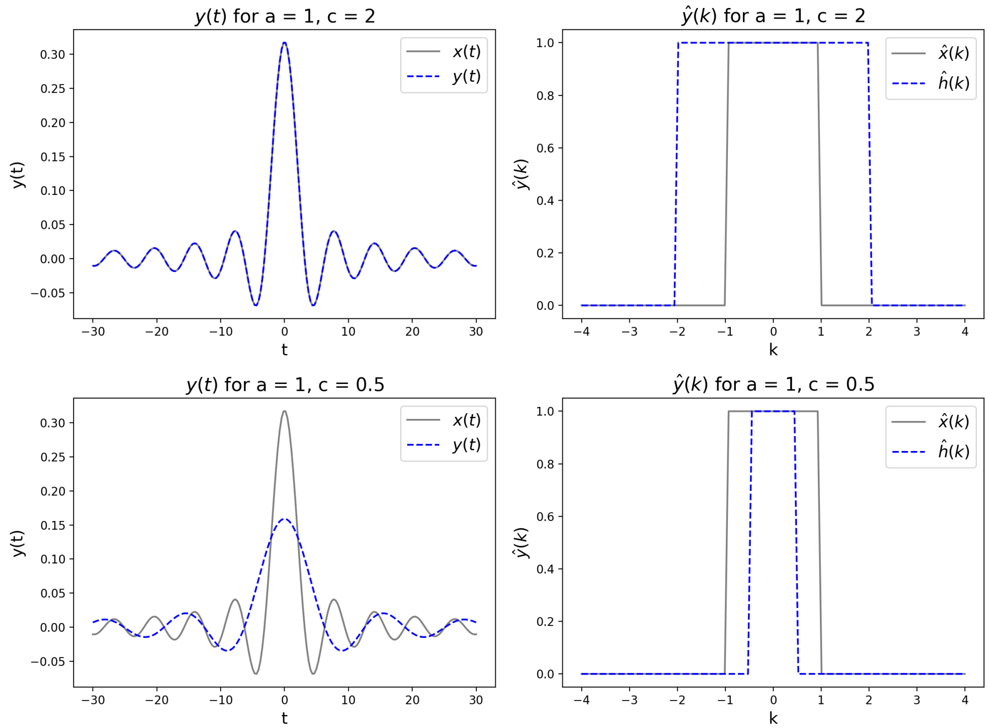
a) Determine the convolution $y(t) = h(t) * x(t)$ in the case when $a < c$ and when $a > c$.

b) Sketch $y(t)$ and $\hat{y}(k)$ for when $a < c$ and when $a > c$.

Solution

a) The convolution of two functions is natural to calculate in the frequency domain, since:
$$y(t) = h(t) * x(t) \quad \rightarrow \quad \hat{y}(w) = \hat{h}(w)\hat{x}(w)$$
and therefore
$$y(t) = \mathscr{F}^{-1}[\hat{h}(w)\hat{x}(w)]$$
We start by calculating $\hat{x}(w)$:
$$\hat{x}(w) = \begin{cases} 1 & \text{for } |w| \leq a \\ 0 & \text{for } |w| > a \end{cases}$$
So:
$$\hat{h}(w)\hat{x}(w) = \begin{cases} 1 & \text{for } |w| \leq \min(a, c) \\ 0 & \text{for } |w| > \min(a, c) \end{cases}$$
And therefore:
$$y(t) = \frac{\sin(\min(a, c)t)}{\pi t}$$

b) From part (a), we see that there are two different scenarios. If the cutoff frequency c is larger than a (the maximum frequency in our signal), then the signal is unchanged by filtering. On the other hand, if the cutoff frequency is smaller, then the signal is transformed into a sinc function with parameter c. These cases are illustrated below, with $a = 1$, $c = 2$ on the left and $a = 1$, $c = 0.5$ on the right.



2. **Forced Damped Harmonic Oscillator.** Consider a block with mass m attached to a spring with spring constant k. A spring force $F_{spring} = -kx$, along with a drag force, $F_{drag} = -bx'(t)$, will oppose the motion of the block. So, without additional external forces, the position of the block will be governed by the second-order ODE,

$$mx''(t) = -kx(t) - bx'(t)$$

or equivalently

$$mx''(t) + bx'(t) + kx(t) = 0$$

Finally, suppose that there is also an external force $y(t)$ acting on the block. Altogether, we obtain the dynamics:

$$mx''(t) + bx'(t) + kx(t) = y(t) \tag{1}$$

The behavior of the oscillator will depend on the relative magnitudes of the damping force, spring force, and the mass of the block. In particular, if b is small, the system will oscillate around equilibrium prior to settling in response to a perturbation (underdamped). If b is large, the system will smoothly approach equilibrium without oscillating (overdamped). We will consider the latter case; specifically, assume that $b^2 > 4km$.

In this problem, you will use Fourier Transforms to solve for $x(t)$ for different forcing functions $y(t)$.

a) Determine the transfer function

$$H(w) = \frac{\hat{x}(w)}{\hat{y}(w)}$$

b) Suppose the external force is sinusoidal, $y(t) = \sin(at)$. What is the motion of the block $x(t)$?

c) Suppose the system experiences a unit impulse,

$$y(t) = \delta(t)$$

What is $x(t)$?

Solution

a) By the derivative property,

$$\mathscr{F}[x'(t)] = iw\,\mathscr{F}[x(t)]$$

So, taking the Fourier Transform of both sides of Eq. 1:

$$-mw^2\hat{x}(w) + iwb\hat{x}(w) + k\hat{x}(w) = \hat{y}(w)$$

Equivalently:

$$\hat{x}(w)(-mw^2 + iwb + k) = \hat{y}(w)$$

So the transfer function is:

$$H(w) = \frac{1}{-mw^2 + iwb + k}$$

b) If

$$y(t) = \sin(at) = \frac{1}{2i}(e^{iat} - e^{-iat})$$

then

$$\hat{y}(w) = \frac{1}{2i}(\delta(w-a) - \delta(w+a))$$

Therefore,

$$\hat{x}(w) = H(w)\hat{y}(w) = \frac{\delta(w-a) - \delta(w+a)}{(2i)(-mw^2 + iwb + k)}$$

Recall that for a function $f(t)$, the delta function $\delta(t)$ satisfies:

$$\int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a)$$

This is useful for calculating the inverse Fourier transform of $\hat{x}(w)$:

$$\begin{aligned} x(t) = \mathscr{F}^{-1}[\hat{x}(w)] &= \frac{1}{2\pi}\int_{\infty}^{\infty} e^{iwt}\frac{\delta(w-a) - \delta(w+a)}{(2i)(-mw^2 + iwb + k)}dw \\ &= \frac{1}{4\pi i}\left(\frac{e^{iat}}{-ma^2 + iab + k} - \frac{e^{-iat}}{-ma^2 - iab + k}\right) \end{aligned}$$

We can further simplify this expression to show clearly that $x(t)$ is a real-valued function. Note that $e^{iat} = \cos(at) + i\sin(at)$, and let $u = -ma^2 + k$ and $v = ab$. Then:

$$\begin{aligned} \frac{e^{iat}}{-ma^2 + iab + k} - \frac{e^{-iat}}{-ma^2 - iab + k} &= \frac{\cos(at) + i\sin(at)}{u + iv} - \frac{\cos(at) - i\sin(at)}{u - iv} \\ &= \frac{(\cos(at) + i\sin(at))(u - iv)}{u^2 + v^2} - \frac{(\cos(at) - i\sin(at))(u + iv)}{u^2 + v^2} \\ &= \frac{-2iv\cos(at) + 2iu\sin(at)}{u^2 + v^2} \end{aligned}$$

So, altogether:

$$x(t) = \frac{1}{4\pi i}\frac{-2iv\cos(at) + 2iu\sin(at)}{u^2+v^2} = \frac{(k-ma^2)\sin(at) - ab\cos(at)}{2(k-ma^2)^2 + 2(ab)^2}$$

c) If $y(t) = \delta(t)$ then $\hat{y}(w) = 1$ so

$$\hat{x}(w) = \frac{1}{-mw^2 + iwb + k}$$

We are looking for

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}(w)e^{iwt}dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iwt}}{-mw^2 + iwb + k}dw \tag{2}$$

We can solve this integral by converting it to a complex integral over an appropriate contour. First, factor the denominator to find the poles of the integrand:

$$w_\pm = \frac{1}{-2m}(-ib \pm \sqrt{-b^2 + 4km}) = \frac{i}{2m}(b \pm \sqrt{b^2 - 4km})$$

Substituting into Equation 2 above, we have:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iwt}}{-m(w-w_+)(w-w_-)}dw$$

If $b^2 > 4km$, then $\sqrt{b^2-4km}$ is real. Further, $b - \sqrt{b^2-4km} > 0$ because $k, m > 0$. So, the integrand has two simple poles, both on the imaginary axis in the upper half-plane. For ease of notation, let $c = \sqrt{b^2 - 4km}$ so that $w_\pm = \frac{i}{2m}(b \pm c)$.

We can now solve this integral by choosing a contour in the complex plane and applying the residue theorem. In the case that $t < 0$, we can consider the closed contour consisting of the lower half-circle of radius R and the real line segment from $-R$ to R. Since there are no poles in the lower half-plane, the overall integral over the closed contour is 0. Additionally, since the denominator is of order two, then by Jordan's Lemma, the integral over the half-circle will approach 0 as $R \to \infty$, leaving only the integral over the real line. Therefore, $x(t) = 0$ for $t < 0$.

On the other hand, for $t > 0$, consider the closed contour consisting of the upper half-circle of radius R and the real line segment from $-R$ to R. Once again, since the denominator is of order two, then by Jordan's Lemma, the integral over the half-circle will approach 0 as $R \to \infty$, leaving only the integral over the real line.

Calculate the residues at the poles:

$$\text{Res}(w_+) = \frac{e^{iw_+t}}{-m(w_+ - w_-)} = \frac{e^{iw_+t}}{-m\frac{i}{2m}(2c)} = \frac{e^{-\frac{b+c}{2m}t}}{-ic} \tag{3}$$

$$\text{Res}(w_-) = \frac{e^{iw_-t}}{-m(w_- - w_+)} = \frac{e^{iw_-t}}{-m\frac{i}{2m}(-2c)} = \frac{e^{-\frac{b-c}{2m}t}}{ic} \tag{4}$$

Adding the residues and multiplying by $2\pi i$:

$$2\pi i(\frac{e^{-\frac{b+c}{2m}t}}{-ic} + \frac{e^{-\frac{b-c}{2m}t}}{ic}) = \frac{2\pi}{c}e^{-\frac{b}{2m}t}(e^{\frac{c}{2m}t} - e^{\frac{-c}{2m}t})$$

Finally, incorporating the factor of 2π from the inverse Fourier transform, we obtain:

$$x(t) = \begin{cases} \frac{1}{c} e^{-\frac{b}{2m}t}(e^{\frac{c}{2m}t} - e^{\frac{-c}{2m}t}) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

or equivalently

$$x(t) = \frac{1}{c}\Theta(t) e^{-\frac{b}{2m}t}(e^{\frac{c}{2m}t} - e^{\frac{-c}{2m}t})$$

where $c = \sqrt{b^2 - 4km}$ and $\Theta(t)$ is the Heaviside step function.

3. **Convolutions.** Let $f(x) = 1/(1 + x^2)$. Determine $f * f$, the convolution of f with itself.

Solution By the convolution property,

$$\mathscr{F}[f * f] = \mathscr{F}[f]^2$$

so

$$f * f = \mathscr{F}^{-1}[\mathscr{F}[f]^2]$$

From a previous homework, we know that

$$\mathscr{F}[\frac{1}{x^2+1}] = \pi e^{-|k|}$$

So,

$$\mathscr{F}[f]^2 = \pi^2 e^{-2|k|}$$

Taking the inverse transform, we obtain:

$$f * f = \frac{2\pi}{x^2 + 4}$$

4. **2D Fourier Transforms (Theory).** Consider the function

$$f(x, y) = \cos(2\pi x)\sin(2\pi ya + \frac{b\pi}{4})$$

Find the 2D Fourier transform $\hat{f}(k_x, k_y)$. Then, for each of the following cases, sketch where it is nonzero in the (k_x, k_y) plane.

a) a = 0, b = 0
b) a = 0, b = 1
c) a = 1, b = 1

Hint: This function, and its Fourier Transform, is visualized for you in the Colab notebook for this assignment. You can change the "a" and "b" parameters and verify that your answer matches the computational result.

Solution The 2D Fourier Transform of $f(x)$ is given by:

$$\begin{aligned}\hat{f}(k_x,k_y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(2\pi x)\sin(2\pi ya+\frac{b\pi}{4})e^{-i(k_x x+k_y y)}dxdy\\ &= \int_{-\infty}^{\infty}\cos(2\pi x)e^{-ik_x x}dx\int_{-\infty}^{\infty}\sin(2\pi ya+\frac{b\pi}{4})e^{-ik_y y}dy\end{aligned}$$

Recall that the Fourier Transform of the delta function is a complex exponential:

$$\mathscr{F}[\delta(t-c)] = \int_{-\infty}^{\infty}\delta(t-c)e^{-iwt}dt = e^{-iwc}$$

So by duality:

$$\mathscr{F}[e^{-itc}] = \mathscr{F}[\mathscr{F}[\delta(t-c)]] = 2\pi\delta(-t-c) = 2\pi\delta(t+c)$$

The first integral is the 1D Fourier Transform of cosine:

$$\int_{-\infty}^{\infty}\cos(2\pi x)e^{-ik_x x}dx = \frac{1}{2}\int_{-\infty}^{\infty}[e^{i(2\pi-k_x)x}+e^{-i(2\pi+k_x)x}]dx = \pi(\delta(k_x-2\pi)+\delta(k_x+2\pi))$$

The second integral is:

$$\begin{aligned}\int_{-\infty}^{\infty}\sin(2\pi ay+\frac{b\pi}{4})e^{-ik_y y}dy &= \frac{1}{2i}\int_{-\infty}^{\infty}\left(e^{i(2\pi ay+\frac{b\pi}{4}-k_y y)}-e^{-i(2\pi ay+\frac{b\pi}{4}-k_y y)}\right)dy\\ &= \frac{\pi}{i}\left(e^{i\frac{b\pi}{4}}\delta(k_y-2\pi a)-e^{-i\frac{b\pi}{4}}\delta(k_y+2\pi a)\right)\end{aligned}$$

Therefore, the 2D Fourier Transform is

$$\begin{gathered}\frac{\pi^2 e^{i\frac{b\pi}{4}}}{i}\left(\delta(k_y-2\pi a)\delta(k_x-2\pi)+\delta(k_y-2\pi a)\delta(k_x+2\pi)\right)-\\ \frac{\pi^2 e^{-i\frac{b\pi}{4}}}{i}\left(\delta(k_y+2\pi a)\delta(k_x-2\pi)+\delta(k_y+2\pi a)\delta(k_x+2\pi)\right)\end{gathered}$$

Notice that the product of the delta functions in k_x and k_y is a point impulse in (k_x,k_y) space.

a) When $a=b=0$, the function $f(x,y)$ is identically 0, and correspondingly the Fourier Transform is:

$$\hat{f}(k_x,k_y) = \frac{\pi^2}{i}\left(\delta(k_y)\delta(k_x-2\pi)+\delta(k_y)\delta(k_x+2\pi)\right)-\frac{\pi^2}{i}\left(\delta(k_y)\delta(k_x-2\pi)+\delta(k_y)\delta(k_x+2\pi)\right) = 0$$

b) When $a=0$ and $b=1$, we obtain the function $f(x,y)=\cos(2\pi x)/\sqrt{2}$, and correspondingly the Fourier Transform is:

$$\begin{aligned}\hat{f}(k_x,k_y) =& \frac{\pi^2 e^{i\frac{\pi}{4}}}{i}\left(\delta(k_y)\delta(k_x-2\pi)+\delta(k_y)\delta(k_x+2\pi)\right)-\frac{\pi^2 e^{-i\frac{\pi}{4}}}{i}\left(\delta(k_y)\delta(k_x-2\pi)+\delta(k_y)\delta(k_x+2\pi)\right)\\ =& \frac{\pi^2}{i}\left(e^{i\frac{\pi}{4}}-e^{-i\frac{\pi}{4}}\right)\left(\delta(k_y)\delta(k_x-2\pi)+\delta(k_y)\delta(k_x+2\pi)\right)\\ =& \sqrt{2}\pi^2\delta(k_y)\left(\delta(k_x-2\pi)+\delta(k_x+2\pi)\right)\end{aligned}$$

This expression is nonzero at the points $(k_x,k_y)=(2\pi,0)$ and $(-2\pi,0)$.

c) When $a = b = 1$, we obtain the function $f(x, y) = \cos(2\pi x)\sin(2\pi y + \frac{\pi}{4})$, and correspondingly the Fourier Transform is:

$$\begin{aligned}\hat{f}(k_x, k_y) =& \frac{\pi^2 e^{i\frac{\pi}{4}}}{i}\left(\delta(k_y - 2\pi)\delta(k_x - 2\pi) + \delta(k_y - 2\pi)\delta(k_x + 2\pi)\right) - \\ & \frac{\pi^2 e^{-i\frac{\pi}{4}}}{i}\left(\delta(k_y + 2\pi)\delta(k_x - 2\pi) + \delta(k_y + 2\pi)\delta(k_x + 2\pi)\right)\end{aligned}$$

This expression is nonzero at the points $(k_x, k_y) = (2\pi, 2\pi), (-2\pi, 2\pi), (2\pi, -2\pi)$, and $(-2\pi, -2\pi)$.

5. **2D Fourier Transforms (Computation).** Open the Python notebook here and follow the instructions.

Solution

a) The chessboard image is comprised of a grid that can be thought of as the product of separable periodic "box"-like functions in x and y dimensions. As we have seen, sharp edges, as in a box function, will correspond to a broadband spectrum. Consequently, the Fourier Transform is comprised of peaks along the k_x-axis and peaks along the k_y-axis.

b) The images of the bricks is similar to the chessboard, but while it is periodic in the y-direction, it is only weakly periodic in the x-dimension. So, in the Fourier domain, we still observe the peaks along the k_y-axis, but not along the k_x-axis.

c) Honeycombs are composed of a hexagonal tessellation, so there are now six axes in which the image has periodic box-like edges. Consequently, the Fourier Transform is comprised of collection of peaks corresponding to the broadband spectra for each of these six directions.

6. **Discrete Fourier Transform.** Consider the function

$$f(x) = 4 + \sin(6\pi x) - 2\cos(2\pi x)$$

Compute the discrete Fourier transform by hand, for a 4Hz sampling rate on the interval [0, 1).

Solution With a 4Hz sampling rate on the interval [0, 1), the samples will be calculated at points $x \in [0, 1/4, 1/2, 3/4]$.

At these points, the values of $f(x)$ are:

$$\begin{aligned}f(0) &= 4 + \sin(0) - 2\cos(0) = 2 \\ f(1/4) &= 4 + \sin(3\pi/2) - 2\cos(\pi/2) = 3 \\ f(1/2) &= 4 + \sin(3\pi) - 2\cos(\pi) = 4 + 2 = 6 \\ f(3/4) &= 4 + \sin(9\pi/2) - 2\cos(3\pi/2) = 5\end{aligned}$$

Since the DFT is defined as

$$\hat{f}[k] = \sum_{n=0}^{N-1} f[x] e^{\frac{-2\pi ikn}{N}}$$

We can express the calculation of the discrete Fourier transform as a multiplication of the sampled signal $f[x]$ by a matrix M containing the values:

$$M_{k,n} = e^{-2\pi i \frac{kn}{N}}$$

For $N = 4$, this matrix would be:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

Performing the multiplication, we find

$$\hat{f}[k] = \begin{bmatrix} 16 \\ 2i - 4 \\ 0 \\ -2i - 4 \end{bmatrix}$$