Homework 5 Handed out: Wednesday, October 5, 2022
Due: Wednesday, October 12, 2022 by 11:59pm

Material covered: Outcomes 4.1–4.4.

Solutions

1. Let $\mathrm{Log}\colon \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ be the principal value of the logarithm.

   a) Show that the function F defined by

   $$F\colon \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}, \quad F(z) = z\,\mathrm{Log}(z) - z$$

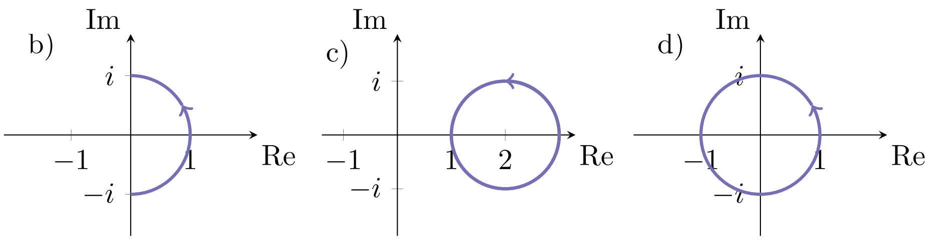
   is an antiderivative for Log on $\mathbb{C} \setminus \mathbb{R}_-$.

   Compute the integral
   $$\int_C \mathrm{Log}(z)\,\mathrm{d}z$$
   for each of the following contours C (pictured below):

   b) C is the right half-circle with center 0 and radius 1, traversed counter-clockwise.

   c) $C = C(2,1)$ is the circle with center 2 and radius 1, traversed counter-clockwise.

   d) $C = C(0,1)$ is the circle with center 0 and radius 1, traversed counter-clockwise.



Solution.

a) It follows from the product rule for derivatives that F is differentiable on $U := \mathbb{C} \setminus \mathbb{R}_-$ with derivative

$$F'(z) = \mathrm{Log}(z) - z\,\mathrm{Log}'(z) - 1 = \mathrm{Log}(z) - z\frac{1}{z} - 1 = \mathrm{Log}(z).$$

This shows that F is an antiderivative of $\mathrm{Log}(z)$ on U.

b) Since the contour C is contained in U, we can compute the integral using the antiderivative F:

$$\int_C f(z)\,\mathrm{d}z = F(i) - F(-i) = \Big(-\frac{\pi}{2} - i\Big) - \Big(-\frac{\pi}{2} + i\Big) = -2i.$$

c) Since C is a simple closed contour in U and Log is analytic on U, it follows from the Cauchy integral theorem that

$$\int_C \mathrm{Log}(z)\,\mathrm{d}z = 0.$$

Alternatively, this follows from the fact that Log has an antiderivative F on U and the start and end points of C are the same.

d) The interior of the contour C is not entirely contained in U since it crosses the negative real axis, so Cauchy's integral theorem does not apply. We'll show two methods to evaluate this integral.

Firstly, we can compute the integral directly by parameterizing the contour:

$$\gamma\colon [-\pi, \pi] \to \mathbb{C}, \quad \gamma(t) = e^{it}.$$

We obtain

$$\int_C \mathrm{Log}(z)\,\mathrm{d}z = \int_{-\pi}^{\pi} \mathrm{Log}(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_{-\pi}^{\pi} \mathrm{Log}(e^{it}) i e^{it}\,\mathrm{d}t.$$

Now, $\mathrm{Log}(e^{it}) = it$ for $-\pi < t < \pi$. Therefore,

$$\int_C \mathrm{Log}(z)\,\mathrm{d}z = \int_{-\pi}^{\pi} (it) i e^{it}\,\mathrm{d}t = -\int_{-\pi}^{\pi} t e^{it}\,\mathrm{d}t.$$

The last integral can be evaluated using integration by parts:

$$\int_C \mathrm{Log}(z)\,\mathrm{d}z = -t\frac{e^{it}}{i}\Big|_{-\pi}^{\pi} + \frac{1}{i}\int_{-\pi}^{\pi} e^{it}\,\mathrm{d}t = -2\pi i.$$

Alternatively, we can use the antiderivative F as follows:

$$\int_C f(z)\,\mathrm{d}z = \lim_{t\nearrow\pi} F(e^{it}) - \lim_{t\searrow-\pi} F(e^{it}).$$

In the first term, we are approaching the point -1 from the upper half-plane, where

$$\lim_{t\nearrow\pi} F(e^{it}) = \lim_{t\nearrow\pi} e^{it}\,\mathrm{Log}(e^{it}) - e^{it} = -1(i\pi) - (-1) = -i\pi + 1.$$

For the second term, we are approaching -1 from the lower half-plane, so

$$\lim_{t\searrow-\pi} F(e^{it}) = \lim_{t\searrow-\pi} e^{it}\,\mathrm{Log}(e^{it}) - e^{it} = -1(-i\pi) - (-1) = i\pi + 1.$$

Combining these results, we have

$$\int_C f(z)\,\mathrm{d}z = (-i\pi + 1) - (i\pi + 1) = -2\pi i.$$

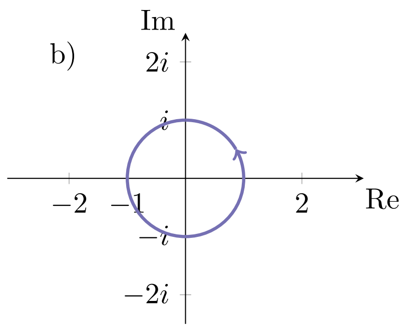
2. Let
$$f(z) = \frac{3z^2+4}{z(z^2+4)}.$$

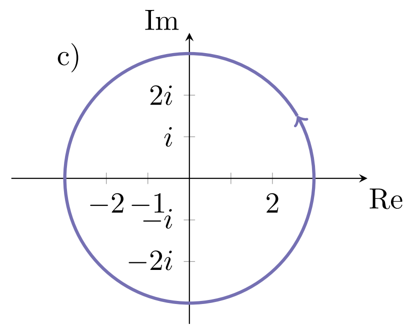
   a) Find a partial fraction decomposition for f.

   Compute
   $$\int_C f(z)\,\mathrm{d}z$$
   for the following contours (pictured below):

   b) $C = C(0,1)$, the circle with center 0 and radius 1, traversed counter-clockwise.

   c) $C = C(0,3)$, the circle with center 0 and radius 3, traversed counter-clockwise.





Solution.

a) We start by factoring the denominator:

$$z(z^2+4) = z(z+2i)(z-2i).$$

Then $f(z)$ has a partial fraction decomposition of the form

$$\frac{3z^2+4}{z(z^2+4)} = \frac{A_1}{z} + \frac{A_2}{z+2i} + \frac{A_3}{z-2i}$$

where $A_1, A_2, A_3 \in \mathbb{C}$. We find the coefficients in the usual way:[1]

$$\begin{aligned}
A_1 &= \lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{3z^2+4}{z^2+4} = 1,\\
A_2 &= \lim_{z\to 2i} (z-2i) f(z) = \lim_{z\to 2i} \frac{3z^2+4}{z(z+2i)} = 1,\\
A_3 &= \lim_{z\to -2i} (z+2i) f(z) = \lim_{z\to -2i} \frac{3z^2+4}{z(z-2i)} = 1.
\end{aligned}$$

Thus

$$f(z) = \frac{1}{z} + \frac{1}{z-2i} + \frac{1}{z+2i}.$$

b) We use the partial fraction decomposition we found in (a) to write

$$\int_C f(z)\,\mathrm{d}z = \int_C \frac{1}{z}\,\mathrm{d}z + \int_C \frac{1}{z+2i}\,\mathrm{d}z + \int_C \frac{1}{z-2i}\,\mathrm{d}z, \tag{1}$$

[1]For a detailed explanation, see the solution to problem 4 on Section 1.

and evaluate each integral in turn. It follows from the Cauchy integral formula that

$$\int_C \frac{1}{z}\,\mathrm{d}z = 1 \cdot 2\pi i = 2\pi i.$$

Since the second integrand $1/(z+2i)$ is analytic on $\mathbb{C} \setminus \{-2i\}$, which contains the contour C and its interior, Cauchy's integral theorem tells us that

$$\int_C \frac{1}{z+2i}\,\mathrm{d}z = 0.$$

Similarly, we see that the third integral is zero. Thus

$$\int_C f(z)\,\mathrm{d}z = 2\pi i.$$

c) Now the contour $C = C(0,3)$ winds around all three poles. It follows from Cauchy's integral formula that

$$\int_C \frac{1}{z}\,\mathrm{d}z = \int_C \frac{1}{z+2i}\,\mathrm{d}z = \int_C \frac{1}{z-2i}\,\mathrm{d}z = 2\pi i,$$

and thus

$$\int_C f(z)\,\mathrm{d}z = 6\pi i.$$

3. Compute the following integrals ($C(z_0, r)$ denotes a circle with center z_0 and radius r):

   a)
   $$\int_{C(i,1)} \frac{e^{z^2}}{z^2+1}\,\mathrm{d}z$$

   b)
   $$\int_{C(0,4)} \frac{\sin z}{(z-\pi)^4}\,\mathrm{d}z$$

Solution.
a) Let us write C for $C(i,1)$. The denominator $z^2+1=(z-i)(z+i)$ vanishes at $z=i$ and $z=-i$. Since only the former point is enclosed by C, we can write

$$\int_C \frac{e^{z^2}}{z^2+1}\,\mathrm{d}z = \int_C \frac{e^{z^2}/(z+i)}{z-i}\,\mathrm{d}z = \int_C \frac{f(z)}{z-i}\,\mathrm{d}z$$

where $f(z)=e^{z^2}/(z+i)$ is analytic on and inside C. Cauchy's integral formula gives

$$\int_C \frac{f(z)}{z-1}\,\mathrm{d}z = 2\pi i f(i) = 2\pi i\frac{e^{-1}}{2i} = \frac{\pi}{e}.$$

We could also evaluate the integral by expanding $1/(z^2+1)$ into partial fractions, but the above method is slightly quicker.
b) We can use Cauchy's integral formula for derivatives directly. With $f(z)=\sin(z)$,

$$\int_{C(0,4)} \frac{f(z)}{(z-\pi)^4}\,\mathrm{d}z = \frac{2\pi i}{3!}f^{(3)}(\pi) = \frac{2\pi i}{6}(-\cos(\pi)) = i\frac{\pi}{3}.$$

4. Compute the integral

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{(2+\cos\theta)^2}$$

by evaluating an appropriately chosen contour integral, as follows:

a) Show that
$$I = \int_{C(0,1)} \frac{1}{\left(2+(z+1/z)/2\right)^2}\frac{\mathrm{d}z}{iz}$$
where $C(0,1)$ is the unit circle centered at the origin.
Hint: Parameterize $C(0,1)$ by $\gamma(\theta)=e^{i\theta}$, $0\le\theta\le 2\pi$.

b) Compute the contour integral in (a) using the Cauchy integral formula.

Solution.
a) If $z = e^{i\theta}$ is a point on the unit circle, then

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}(z + 1/z),$$

and $\mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta$. Therefore, we can write I as a contour integral over the unit circle $C(0,1)$,

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{(2+\cos\theta)^2} = \int_{C(0,1)} \frac{1}{(2+(z+1/z)/2)^2}\frac{\mathrm{d}z}{iz}.$$

b) Let's rewrite the integrand:

$$\begin{aligned}\frac{1}{iz}\frac{1}{(z/2+2+1/2z)^2} &= \frac{1}{iz}\frac{1}{(2z)^{-2}(z^2+4z+1)^2}\\ &= \frac{4z/i}{(z^2+4z+1)^2} = \frac{4z/i}{(z-a)^2(z-b)^2}\end{aligned}$$

where $a = -2+\sqrt{3}$ and $b = -2-\sqrt{3}$ are the two roots of the denominator. Notice that the root a is in the interior of $C(0,1)$ and b is not. Cauchy's integral formula gives

$$I = \int_{C(0,1)} \frac{f(z)}{(z-a)^2}\,\mathrm{d}z = 2\pi i f'(a)$$

where

$$f(z) = \frac{4z/i}{(z-b)^2}, \quad f'(z) = -4\frac{z+b}{i(z-b)^3}, \quad f'(a) = \frac{2}{3\sqrt{3}i}.$$

Therefore, we obtain

$$I = 2\pi i \cdot \frac{2}{3\sqrt{3}i} = \frac{4\pi}{3\sqrt{3}}.$$