Section 4 Friday, September 30, 2022

Material covered: Outcomes 6 (conformal maps)

Exercises that are slightly more difficult are labeled ***Exercise**. You may want to move on to another exercise if you get stuck.

1 Harmonic conjugates

Determine whether the following functions $u(x,y)$ are harmonic. If u is harmonic, find a harmonic conjugate, i.e. a real-valued function v such that $f = u + iv$ is holomorphic.

1. $u(x,y) = e^x \sin(y)$
2. $u(x,y) = x^3 - y^3$

Solution. 1. To determine whether u is harmonic, we compute the Laplacian of u:

$$\partial_x^2 u(x,y) = e^x \sin(y)$$
$$\partial_y^2 u(x,y) = -e^x \sin(y)$$

so

$$\nabla^2 u(x,y) = \partial_x^2 u(x,y) + \partial_y^2 u(x,y) = 0.$$

Thus u is harmonic.

Now if v is a harmonic conjugate to u, then u, v satisfy the Cauchy–Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The first equation tells us that

$$\frac{\partial v}{\partial y} = e^x \sin(y)$$

which implies that

$$v(x, y) = -e^x \cos(y) + c(x)$$

where c is some real-valued function of x only. We determine c by substituting this into the second Cauchy–Riemann equation:

$$-\frac{\partial v}{\partial x} = e^x \cos(y) + c'(x) = \frac{\partial u}{\partial y} = e^x \cos(y)$$

Thus $c'(x) = 0$ so c is a constant. This shows that

$$v(x, y) = -e^x \cos(y)$$

is a harmonic conjugate to u (the other harmonic conjugates differ by a constant).

We remark that

$$u + iv = e^x \big(\sin(y) - i \cos(y)\big) = -ie^x \big(\cos(y) + i \sin(y)\big) = -ie^{x+iy}$$

which is indeed a holomorphic function.

2. To determine whether u is harmonic, we compute the Laplacian of u:

$$\partial_x^2 u(x, y) = 6x$$
$$\partial_y^2 u(x, y) = -6y.$$

so

$$\nabla^2 u(x, y) = 6x - 6y.$$

Since $\nabla^2 u(x, y)$ does not vanish identically on $\mathbb{C}$ (or indeed on any open set), u is not harmonic.

2 Möbius transformations

The Möbius transformations are particularly simple examples of conformal maps. In this section we discuss some of their properties and helpful calculation rules. We encourage readers to try the exercises for themselves before reading the solutions.

2.1 Basic properties

A function of the form

$$f(z) = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{C},\ ad - bc \neq 0 \tag{1}$$

is called a Möbius transformation. If $c = 0$, the function is analytic on $\mathbb{C}$, but if $c \neq 0$, the function is analytic on $\mathbb{C} \setminus \{-d/c\}$, and is singular at $z = -d/c$. The derivative

$$f'(z) = \frac{ad - bc}{(cz+d)^2}$$

is nowhere zero, so f is conformal[1].

***Exercise.** Show that a Möbius transformation is injective, i.e. if z_1, z_2 are complex numbers in the domain of f and $f(z_1) = f(z_2)$, then $z_1 = z_2$.

Solution. Consider first the case $c = 0$. Then f is an affine map (i.e. a map of the form $f(z) = Az + B$ with $A, B \in \mathbb{C}$) and thus a bijection $\mathbb{C} \to \mathbb{C}$, as is easily verified: Suppose $w \in \mathbb{C}$. Then the equation $f(z) = w$ has exactly one solution, namely $z = (w - B)/A$.

In the case $c \neq 0$, let $w \in \mathbb{C}$ be given. We need to show that the equation

$$f(z) = \frac{az+b}{cz+d} = w$$

has at most one solution z. Rearranging, we obtain,

$$(a - cw)z = dw - b.$$

As long as $w \neq \frac{a}{c}$, we can divide both sides by $a - cw$ and we conclude that there is exactly one solution,

$$z = \frac{dw - b}{-cw + a} \tag{2}$$

Thus f is a bijection $\mathbb{C} \setminus \{-\frac{d}{c}\} \to \mathbb{C} \setminus \{\frac{a}{c}\}$.

[1]The case when $ad = bc$ is excluded since in that case $f'(z) = 0$ for all z and f is a constant function (thus not conformal).

The previous exercise shows that f has an inverse f^{-1} which is given by

$$f^{-1}(w) = \frac{dw - b}{-cw + a}. \tag{3}$$

Thus f^{-1} is a Möbius transformation (note that the condition $da-(-b)(-c) = ad - bc \neq 0$ is satisfied), and thus analytic on its domain.

The following simple transformations are examples of Möbius transformations:

- $f(z) = z + b$, $(a = 1, b \in \mathbb{C}, c = 0, d = 1)$ (translation)
- $f(z) = az$, $(a \in \mathbb{C} \setminus \{0\}, b = 0, c = 0, d = 1)$ (rotation-scaling)
- $f(z) = 1/z$ $(a = 0, b = 1, c = 1, d = 0)$ (inversion)

***Exercise.** Show that if f_1 and f_2 are Möbius transformations, then their composition $f_3 = f_2 \circ f_1$ is also a Möbius transformation. How are the coefficients of f_3 related to those of f_1, f_2?

Solution. Let f_1, f_2 be the Möbius transformations

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}, \qquad f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}.$$

Then

$$\begin{aligned}
f_2(f_1(z)) &= \left(a_2 \frac{a_1 z + b_1}{c_1 z + d_1} + b_2\right)\left(c_2 \frac{a_1 z + b_1}{c_1 z + d_1} + d_2\right)^{-1} \\
&= \Big(a_2(a_1 z + b_1) + b_2(c_1 z + d_1)\Big)\Big(c_2(a_1 z + b_1) + d_2(c_1 z + d_1)\Big)^{-1} \\
&= \frac{(a_2 a_1 + b_2 c_1) z + (a_2 b_1 + b_2 d_1)}{(c_2 a_1 + d_2 c_1) z + (c_2 b_1 + d_2 d_1)}
\end{aligned}$$

from which we can read the coefficients a_3, b_3, c_3, d_3 of f_3. Notice that they given by the following matrix multiplication:

$$\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}. \tag{4}$$

In fact, it can be shown that any Möbius transformation can be written as a composition of translations, rotation-scalings and inversions.

2.2 Möbius transformations on the extended complex plane

Let f be defined as in (1). If $c \neq 0$, then f is a bijective map $\mathbb{C} \setminus \{-\frac{d}{c}\} \to \mathbb{C} \setminus \{\frac{a}{c}\}$, and we notice that

$$\lim_{z \to -d/c} |f(z)| = +\infty, \qquad \lim_{|z| \to \infty} f(z) = \frac{a}{c}.$$

Möbius transformations become much easier to work with if we extend their domain of definition by adding a point, *the point at infinity* ∞, to the complex plane. We define the *extended complex plane*

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

and extend f to a bijective map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by letting

$$f(-d/c) = \infty, \qquad f(\infty) = \lim_{|z| \to \infty} f(z) = \frac{a}{c}, \qquad \text{if } c \neq 0.$$

If $c = 0$, we instead let

$$f(\infty) = \infty, \qquad \text{if } c = 0.$$

The advantage of this will be revealed in the next section.

2.3 The "three point rule"

Let z_1, z_2, z_3 be three distinct points in $\widehat{\mathbb{C}}$. Then there exist exactly one Möbius transformation f which satisfies

$$f(z_1) = 0, \quad f(z_2) = 1, \quad f(z_3) = \infty, \tag{5}$$

namely the function

$$f(z) = \frac{(z - z_1)}{(z - z_3)} \cdot \frac{(z_2 - z_3)}{(z_2 - z_1)}.$$

If one of the points z_i is ∞, then the above definition should be interpreted by taking the appropriate limit $|z_i| \to +\infty$. Namely, if $z_1 = \infty$, then

$$f(z) = \lim_{|z_1| \to \infty} \frac{(z - z_1)}{(z - z_3)} \cdot \frac{(z_2 - z_3)}{(z_2 - z_1)} = \frac{(z_2 - z_3)}{(z - z_3)}.$$

Similarly,

$$f(z) = \frac{(z - z_1)}{(z - z_3)}, \quad \text{if } z_2 = \infty$$
$$f(z) = \frac{(z - z_1)}{(z_2 - z_1)}, \quad \text{if } z_3 = \infty.$$

We can use this result to find a Möbius transformation f which maps the distinct points z_1, z_2, z_3 to any three distinct points $w_1, w_2, w_3 \in \widehat{C}$, i.e.

$$f(z_1) = w_1, \quad f(z_2) = w_2, \quad f(z_3) = w_3. \tag{6}$$

To see this, find Möbius transformations F and G which satisfy

$$F(w_1) = G(z_1) = 0, \qquad F(w_2) = G(z_2) = 1, \qquad F(w_3) = G(z_3) = \infty.$$

Then the composition $f(z) = F^{-1} \circ G(z)$ is a Möbius transformation which satisfies(6). To summarize:

Theorem 1 (Three point rule)**.** *If z_1, z_2, z_3 are three distinct points in $\widehat{C}$ and w_1, w_2, w_3 be three distinct points in $\widehat{C}$, then there exists exactly one bijective holomorphic map $f \colon \widehat{C} \to \widehat{C}$ such that $f(z_i) = w_i$, $1 \le i \le 3$. Moreover, f is the Möbius transformation $f = F^{-1} \circ G$ where*

$$F(w) = \frac{(w - w_1)}{(w - w_3)} \cdot \frac{(w_2 - w_3)}{(w_2 - w_1)}, \quad G(z) = \frac{(z - z_1)}{(z - z_3)} \cdot \frac{(z_2 - z_3)}{(z_2 - z_1)}.$$

Exercise. Find a Möbius transformation f which satisfies the conditions:

1. $f(i) = 0, \quad f(\infty) = 1, \quad f(-i) = \infty$
2. $f(1) = i, \quad f(i) = 2, \quad f(-i) = 0$

Solution. 1. In this case, we have $z_1 = i$, $z_2 = \infty$, $z_3 = -i$ and we can read f directly from the equation

$$f(z) = \frac{(z - z_1)}{(z - z_3)} \cdot \frac{(z_2 - z_3)}{(z_2 - z_1)}.$$

in the limit $|z_2| \to \infty$, that is

$$f(z) = \frac{(z - z_1)}{(z - z_3)} = \frac{(z - i)}{z + i}.$$

2. Let $z_1 = 1, z_2 = i, z_3 = -i$ and $w_1 = i, w_2 = 2, w_3 = 0$. Define

$$F(w) = \frac{(w - w_1)}{(w - w_3)} \cdot \frac{(w_2 - w_3)}{(w_2 - w_1)} = \frac{(4 + 2i)w + (2 - 4i)}{5w}$$

and

$$G(z) = \frac{(z - z_1)}{(z - z_3)} \cdot \frac{(z_2 - z_3)}{(z_2 - z_1)}. = \frac{(1 - i)z + (i - 1)}{z + i}.$$

The inverse of F is (recall (3))

$$F^{-1}(z) = \frac{-(2 - 4i)}{-5z + (4 + 2i)} = \frac{2}{(1 + 2i)z - 2i}.$$

Then $f = F^{-1} \circ G$ is the Möbius transformation whose coefficients can be found using matrix multiplication (recall (4)):

$$f(z) = \frac{(1 + i)z + (i - 1)}{(2 + i)z - i}.$$

The reader is encouraged to visualize the above functions and note especially what happens to the unit circle and the unit disk.

3 Riemann mapping theorem (optional)

Let U and V be two domains (connected open sets) in $\mathbb{C}$. Under what conditions does there exist a conformal map between U and V? This question is of interest in many areas of physics, but also in image processing where one might want to warp an image into a given shape in a way that preserves angles locally (see Homework 4).

The Riemann mapping theorem states that such a map f always exists if both U and V are simply connected, proper subsets of $\mathbb{C}$. In fact, the theorem guarantees that there exists an *invertible* conformal map $f: U \to V$ whose inverse $f^{-1}: V \to U$ is also conformal. “Simply connected” means (roughly) that they are connected sets which have no holes. “Proper subset” means that neither U nor V is the entire complex plane.

An example if shown in figure 1. The Riemann mapping theorem guarantees that there exists a conformal map from the complicated domain U to the unit disk $\mathbb{D}$.

Theorem 2 (Riemann mapping theorem)**.** *Let $U \subset \mathbb{C}$, $V \subset \mathbb{C}$ be simply connected, open, proper subsets of $\mathbb{C}$. Then*

1. *there exists a bijective holomorphic map $f: U \to V$,*

2. *f is conformal and so is its inverse.*

Perhaps surprisingly, there is no analogous theorem for non-simply connected domains: Given two domains in $\mathbb{C}$ with the same number of holes, there typically does *not* exist a bijective holomorphic map between them. An example is shown in figure 2.

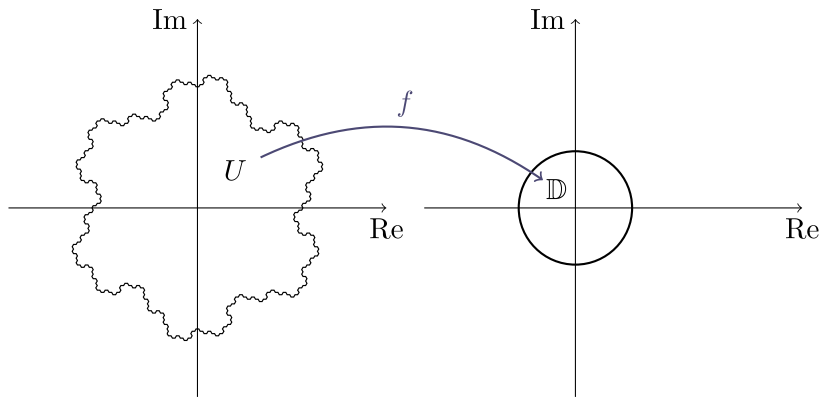

Figure 1: By the Riemann mapping theorem, there exists a conformal map f which maps the complicated domain U bijectively to the unit disk $\mathbb{D}$.

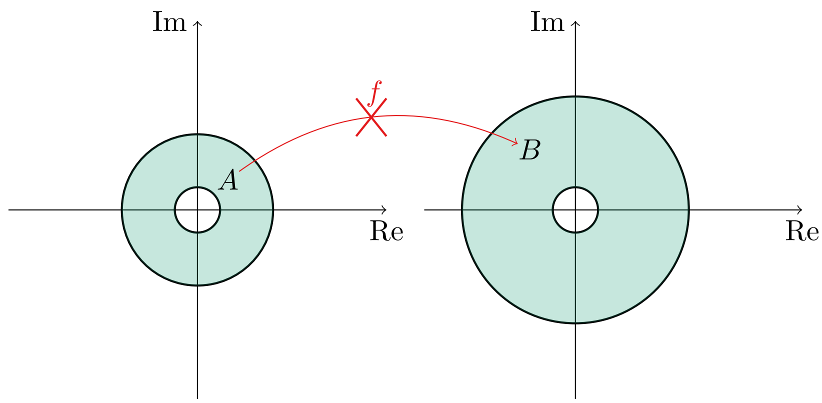

Figure 2: The annuli A and B have the same inner radius but different outer radii. No bijective conformal map $A \to B$ exists.