Homework 4 Handed out: Wednesday, September 28, 2022
Due: Wednesday, October 5, 2022 by 11:59pm on Gradescope

Material covered: Outcomes 3.4, 3.5, 4.1, 6.3

Solutions

1. **Harmonic Functions.** Check whether the following functions $u(x, y)$ are harmonic, and if so find the harmonic conjugate $v(x, y)$ so that $f(x + iy) = u(x, y) + iv(x, y)$ is an analytic function.

   a) $u(x, y) = x^3 - 3xy^2$
   b) $u(x, y) = x^2 + y^2$
   c) $u(x, y) = \cosh(y) (\sin(x) + \cos(x))$

   Solution.

   Recall that a function $u(x, y)$ is *harmonic* if it satisfies *Laplace's equation*: $\nabla^2 u = \partial_x^2 u + \partial_y^2 u = 0$. We can then reconstruct the harmonic conjugate $v(x, y)$ by plugging into the Cauchy-Riemann (CR) equations.

   (a)

$$\begin{aligned} \nabla^2 u(x, y) &= \partial_x \left(3x^2 - 3y^2\right) + \partial_y (-6xy) \\ &= 6x - 6x \\ &= 0 \end{aligned}$$

   The function is harmonic.

   Plugging into the first CR equation and integrating over y:

$$\begin{aligned} \partial_x u(x, y) &= \partial_y v(x, y) \\ 3x^2 - 3y^2 &= \partial_y v(x, y) \\ 3x^2 y - y^3 &= v(x, y) + C_1(x) \end{aligned}$$

Plugging into the second CR equation and integrating over x:

$$\begin{aligned}\partial_y u(x,y) &= -\partial_x v(x,y)\\ -6xy &= -\partial_x v(x,y)\\ -3x^2y &= -v(x,y) + C_2(y)\end{aligned}$$

So we can deduce that $v(x,y) = 3x^2y - y^3$.

The full function is $u(x,y) + iv(x,y) = x^3 - 3xy^2 + 3ix^2y - iy^3 = (x+iy)^3 = z^3$, which is patently analytic.

(b) Plugging into Laplace's equation, we get that

$$\nabla^2 u(x,y) = 4$$

so the function is not harmonic. We can confirm this by inspecting the function, which clearly has a global minimum at the origin.

(c) Plug into Laplace's equation to get

$$\begin{aligned}\nabla^2 u(x,y) &= \nabla^2\left[\cosh(y)\left(\sin(x)+\cos(x)\right)\right]\\ &= -\cosh(y)\left(\sin(x)+\cos(x)\right) + \cosh(y)\left(\sin(x)+\cos(x)\right)\\ &= 0\end{aligned}$$

because $\partial_y^2 \cosh(y) = \cosh(y)$, $\partial_x^2 \sin(x) = -\sin(x)$, and $\partial_x^2 \cos(x) = -\cos(x)$. The function is therefore harmonic.

Plug into the first CR equation and integrate over y:

$$\begin{aligned}\partial_x u(x,y) &= \partial_y v(x,y)\\ \cosh(y)\left(\cos(x)-\sin(x)\right) &= \partial_y v(x,y)\\ \sinh(y)\left(\cos(x)-\sin(x)\right) &= v(x,y) + C_1(x)\end{aligned}$$

Plug into the second CR equation and integrate over x:

$$\begin{aligned}\partial_y u(x,y) &= -\partial_x v(x,y)\\ \sinh(y)\left(\sin(x)+\cos(x)\right) &= -\partial_x v(x,y)\\ \sinh(y)\left(-\cos(x)+\sin(x)\right) &= -v(x,y) + C_2(y)\end{aligned}$$

We end up with

$$v(x, y) = \sinh(y)\,(\cos(x) - \sin(x))$$

The final complex function simplifies to $\sin(z) + \cos(z)$.

2. **Visualizing conformality.**
   a) Sketch a grid of lines in the complex plane corresponding to

   $$\mathrm{Re}(z) = -1, \quad \mathrm{Re}(z) = 1, \quad \mathrm{Im}(z) = -1, \quad \mathrm{Im}(z) = 1$$

   Now sketch the images of these lines under the transformation e^z.

   b) Sketch the following curves in $\mathbb{C}$ and their images under $\mathrm{Log}(z)$.

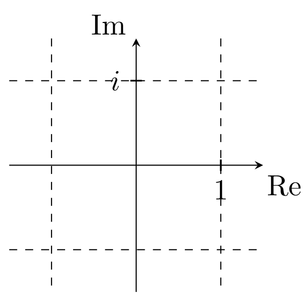
   $$z = te^{\pi i/4}, \quad z = te^{-\pi i/4}, \quad t \in [0, \infty)$$

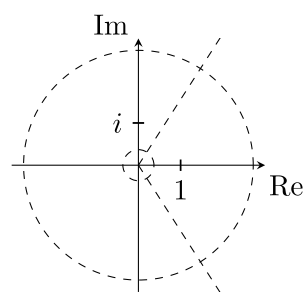
   $$z = e^{it}, \quad z = 2e^{it}, \quad t \in (-\pi, \pi)$$

   c) Visually, do these two maps appear conformal? Why or why not?

Solution.

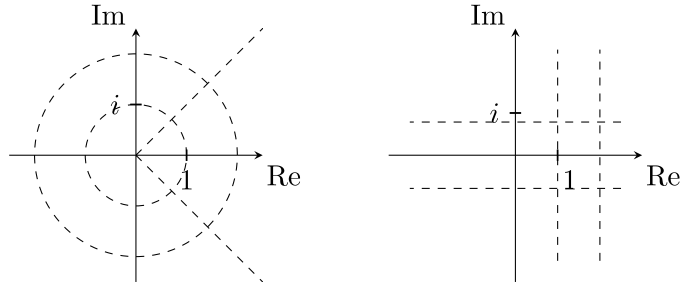
(a) The vertical lines map to circles with radius e^x; the horizontal lines map to straight lines emanating from the origin with angle y.





(b) We get a similar (but inverted) transformation here, with circles mapping to vertical lines and radial lines mapping to horizontal lines. Recall the form of the Log function to see why this should be the case:

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$$



(c) The defining property of conformal functions is that they preserve angles. In particular, curves that intersect at right angles should map to curves that continue to intersect at right angles. This is indeed the case with the above pictures, which suggests that these maps are conformal.

3. **Möbius Transformations.** The *extended complex plane* $\hat{\mathbb{C}}$ is defined as the complex plane plus an extra point "at infinity," which has infinite magnitude and corresponds to values such as $\lim_{z\to 0}\frac{1}{z}$.

   A *Möbius transformation*

   $$f(z) = \frac{az+b}{cz+d}$$

   maps a point in $\hat{\mathbb{C}}$ to another point in $\hat{\mathbb{C}}$. Here, the values a, b, c, and d are constant complex numbers and $ad - bc \neq 0$.

   For a visual introduction to Möbius transformations, watch the following video: https://youtu.be/0z1fIsUNhO4

   a) What values of a, b, c, and d correspond to a Möbius transformation that rotates the complex plane by $\pi/4$ counter-clockwise?

b) What values of a, b, c, and d corrrespond to a Möbius transformation that leaves $z = i$ and $z = -i$ unchanged, but sends $z = 1$ to ∞?

c) Pick some nonzero values for a, b, c, and d so that $ad - bc = 0$. Sketch the image of the unit square (where $z = x + iy$, $x \in [0, 1]$ and $y \in [0, 1]$) under this Möbius transform. What happens to the square in this case?

Solution.

(a) Recall that multiplying two complex numbers results in the product of their magnitudes and a total rotation equal to the sum of each number's argument. Here we need to multiply by a number with magnitude 1 and angle $\pi/4$, so the transformation should be $f(z) = e^{i\pi/4}z$ here. Therefore we need

$$a = e^{i\pi/4}, \quad b = 0, \quad c = 0, \quad d = 1$$

(b) To send $z = 1$ to ∞, we need the denominator to vanish when $z = 1$, so $d = -c$. To fit the first two conditions, we plug in:

$$\begin{aligned} i &= \frac{ai + b}{c(i - 1)} \\ -c(1 + i) &= ai + b \\ -i &= \frac{-ai + b}{c(i - 1)} \\ c(-1 + i) &= -ai + b \end{aligned}$$

Adding these equations, we get

$$-2c = 2b \implies b = -c$$

Subtracting these equations, we get

$$-2ic = 2ia \implies a = -c$$

Plugging in to the final form:

$$f(z) = \frac{-cz + -c}{cz - c}$$

We can divide through by c to get

$$f(z) = \frac{-z - 1}{z - 1}$$

(c) We can pick

$$a = 1, \;\; b = i, \;\; c = -i, \;\; d = 1$$

This leads to the transform

$$f(z) = \frac{z + i}{-iz + 1} = \frac{z + i}{-i(z + i)} = i$$

This tells us that every single point in the complex plane will map to the single point i.

In fact, if we use the condition $ad - bc = 0$ to solve for $d = bc/a$ and plug back into the original formulation:

$$f(z) = \frac{az + b}{cz + bc/a} = \frac{az + b}{(c/a)(az + b)} = \frac{a}{c}$$

So the unit square (or any region in C) will map to the single point a/c.

4. **Numerical Image Transformations.**
   a) Open the IPython notebook here and follow the instructions.
   b) From the notebook above, upload a screenshot of your favorite custom image here. Include the transform code that produced it. (We will vote on an image for the AM104 T-shirt in a future class!)
   c) What is your shirt size?

Solution. (n/a)

5. **Contour Integration.** For each of the following functions $f(z)$, compute the two integrals:

$$\int_{C_1} f(z)\,\mathrm{d}z \qquad \text{and} \qquad \int_{C_2} f(z)\,\mathrm{d}z$$

where C_1 is the contour from -1 to 1 along the lower half of the unit circle and C_2 is the contour along the real line from -1 to 1.

a) $f(z) = 2z^4$ b) $f(z) = |z|^2$

Solution.

(a) Using the parameterization $\gamma_1(t) = e^{it},\ t \in [-\pi, 0]$:

$$\begin{aligned}
\int_{C_1} 2z^4\,\mathrm{d}z &= \int_{-\pi}^{0} 2(e^{it})^4 \left(ie^{it}\right) \mathrm{d}t \\
&= 2i \int_{-\pi}^{0} e^{5it}\,\mathrm{d}t \\
&= \frac{2i}{5i}\left[e^{5it}\right]_{t=-\pi}^{0} \\
&= \frac{2}{5}\left[1-(-1)\right] \\
&= \frac{4}{5}
\end{aligned}$$

Using the parameterization $\gamma_2(t) = t,\ t \in [-1, 1]$:

$$\begin{aligned}
\int_{C_2} 2z^4\,\mathrm{d}z &= \int_{-1}^{1} 2t^4\,\mathrm{d}t \\
&= \frac{2}{5}\left[t^5\right]_{t=-1}^{1} \quad = \frac{2}{5}\left[1-(-1)\right] \\
&= \frac{4}{5}
\end{aligned}$$

(b) Using the parameterization $\gamma_1(t) = e^{it},\ t \in [-\pi, 0]$:

$$\begin{aligned}
\int_{C_1} |z|^2 \, \mathrm{d}z &= \int_{-\pi}^{0} e^{-it} e^{it} \left(i e^{it} \right) \, \mathrm{d}t \\
&= \int_{-\pi}^{0} i e^{it} \mathrm{d}t \\
&= \left[e^{it} \right]_{t=-\pi}^{0} \\
&= 1 - (-1) \\
&= 2
\end{aligned}$$

Using the parameterization $\gamma_2(t) = t,\ t \in [-1, 1]$:

$$\begin{aligned}
\int_{C_2} |z|^2 \, \mathrm{d}z &= \int_{1}^{1} t^2 \, \mathrm{d}t \\
&= \left[\frac{1}{3} t^3 \right]_{t=-1}^{1} \\
&= \frac{1}{3} \left[1 - (-1) \right] \\
&= \frac{2}{3}
\end{aligned}$$