Homework #03

1

1a)

1a) so $x(t) = e^{At} x_0$ first

taking derivative

$$\frac{d}{dt}(e^{At} x_0) = \frac{d}{dt}(e^{At}) x_0$$

from exp definition

$$= \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots\right) x_0$$

$$= \left(0 + A + A^2t + \frac{A^3t^2}{2!} + \dots\right) x_0$$

$$= A\left(I + At + \frac{A^2t^2}{2!} + \dots\right) x_0$$

exp def. again

$$= A(e^{At}) x_0 = A e^{At} x_0$$

and $x(t) = e^{At} x_0$ we said so

$$\frac{dx}{dt} = A x(t)$$

& for $x(t) = \sum_i a_i e_i e^{\lambda_i t}$

one solution $x(t) = e_i e^{\lambda_i t}$

$$\frac{d}{dt}(e_i e^{\lambda_i t}) = \lambda e_i e^{\lambda_i t}$$

using $\lambda e_i = A e_i$ bc eigenvector & eigenvalues

$$= A e_i e^{\lambda_i t}$$

and $x(t) = e_i e^{\lambda_i t}$ so

$$= A x$$

$\sum_i a_i e_i e^{\lambda_i t}$ is sum of all these sols

so it's the general solution for $\frac{dx}{dt} = Ax$ too

1b)

```
A=[-0.7108,-2.995;1.336,-0.7892];
x0 = [-5; 2];

% finding eigenvalues
a=A(1,1);
b=A(1,2);
c=A(2,1);
d=A(2,2);
disp('eigenvalues:')
```

```
eigenvalues:
```

```
lambda = [(a+d)/2+( ((a+d)/2).^2 - ((a.*d) - (b.*c)) )^(1/2), 0; ...
    0, (a+d)/2-( ((a+d)/2).^2 - ((a.*d) - (b.*c)) )^(1/2)];
```

```
lambda_vec = [lambda(1,1) lambda(2,2)]
```

```
lambda_vec = 1×2 complex
  -0.7500 + 1.9999i  -0.7500 - 1.9999i
```

```
% finding eigenvectors
e_vecs = zeros(length(lambda_vec), 2);
for i=1:1:length(lambda_vec)
    e_space = A-lambda_vec(1,i)*eye(2);
    e_vecs(:, i) = null(e_space);
end
disp('eigenvectors:')
```

```
eigenvectors:
```

```
disp(e_vecs)
```

```
   0.8316 + 0.0000i   0.8316 + 0.0000i
   0.0109 - 0.5553i   0.0109 + 0.5553i
```

```
% finding the a constant based on x0
disp('the a constant:')
```

```
the a constant:
```

```
a_const = e_vecs^(-1)*x0;
a_const=a_const.'
```

```
a_const = 1×2 complex
  -3.0063 + 1.8598i  -3.0063 - 1.8598i
```

1c)

```
t=2.0;

% diag turns elements of vector into diagonal elements
expm_A2=e_vecs*diag(exp(lambda_vec*t))*e_vecs^(-1)
```

```
expm_A2 = 2×2 complex
  -0.1492 + 0.0000i   0.2529 + 0.0000i
  -0.1128 - 0.0000i  -0.1426 + 0.0000i
```

```
disp('from Matlab expm() function:')
```

```
from Matlab expm() function:
```

```
disp(expm(A*t))
```

```
   -0.1492    0.2529
   -0.1128   -0.1426
```

```

```

The calculated sum is indeed equal to Matlab's expm output

```
disp('solution at t=2:')
```

```
solution at t=2:
```

```
disp(expm_A2*x0)
```

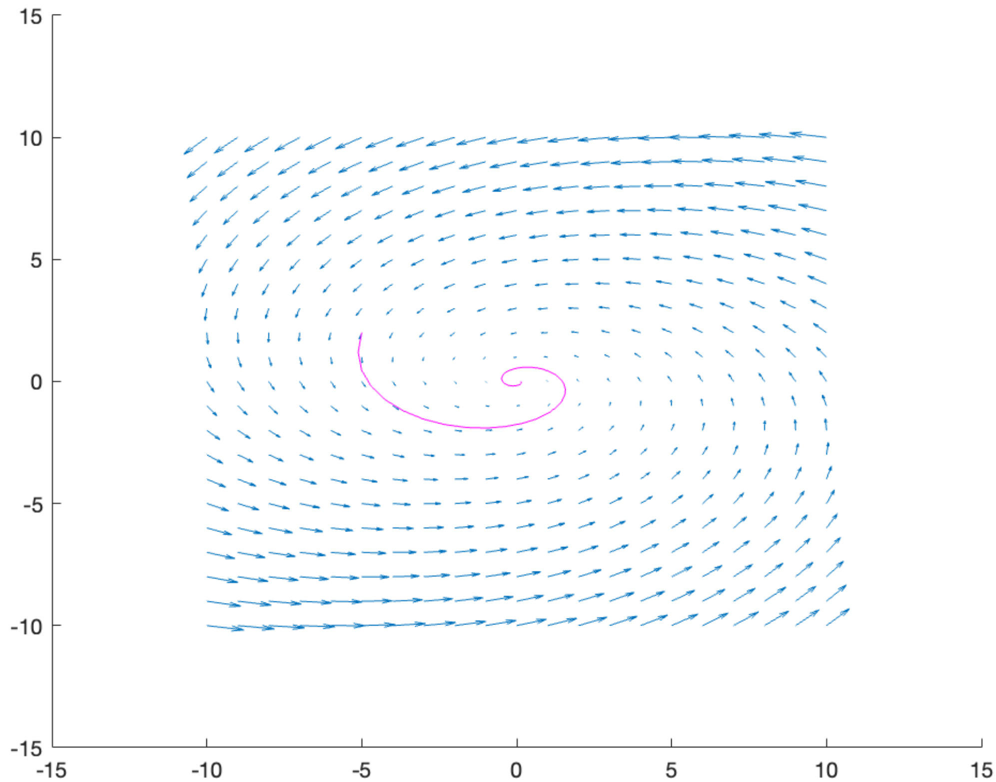
```
   1.2516 - 0.0000i
   0.2789 + 0.0000i
```

1d)

```
t=0:0.1:5;
x_traj=zeros(2,15);
for i=1:1:length(t)
    x_traj(:,i)=e_vecs*diag(exp(lambda_vec*t(1,i)))*e_vecs^(-1)*x0;
end

figure
hold on
[X,Y] = meshgrid(-10:1:10,-10:1:10);
X=X(:).';
Y=Y(:).';
gradient = A*[X;Y];
U = gradient(1,:);
V = gradient(2,:);

q = quiver(X,Y,U,V);

plot(x_traj(1,:), x_traj(2,:), 'm')
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored
```

```
hold off
```



1e) So $\frac{dx}{dt} = ax + by$

then $\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt}$

and if $\frac{dy}{dt} = cx + dy$

then $\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b(cx + dy)$

$= (aa + bc)x + (ab + bd)y$

```
disp(['x''''=' num2str((a*a) + (b*c)) 'x+' num2str((a*b)+(b*d)) 'y'])
```

```
x''=-3.4961x+4.4925y
```

1fi)

```
t=2.0;
x2_sad=[-2; -2];
```

```
disp('initial conditions based on x2=[-2;-2]:')
```

```
initial conditions based on x2=[-2;-2]:
```

```
x0_sad=expm_A2^(-1)*x2_sad
```

```
x0_sad = 2×1 complex
  15.8843 - 0.0000i
   1.4615 - 0.0000i
```

1fii)

For t=2, it is possible to optimize a starting condition point out on the trajectory that would lead to both positive values specifically at t=2.

However, long-term no, the eigenvalues have negative real parts:

```
disp(lambda_vec)
```

```
  -0.7500 + 1.9999i  -0.7500 - 1.9999i
```

So they will always eventually end up at 0, 0

1gi)

```
t=0:0.1:15;
x_sad=zeros(2,15);
for i=1:1:length(t)
    x_sad(:,i)=e_vecs*diag(exp(lambda_vec*t(1,i)))*e_vecs^(-1)*x0_sad;
end

figure
hold on
plot(t, x_sad(1,:), 'r')
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored
```

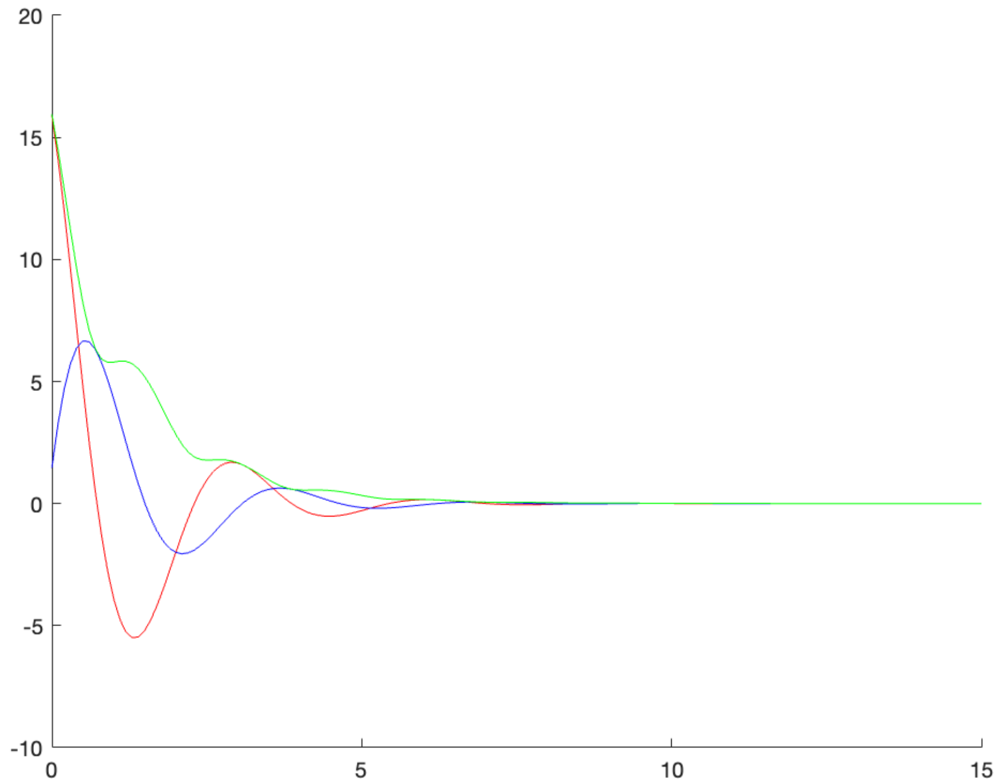
```
plot(t, x_sad(2,:), 'b')
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored
```

```
plot(t, sqrt(x_sad(1,:).^2+x_sad(2,:).^2), 'g')
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored
```

```
hold off
```



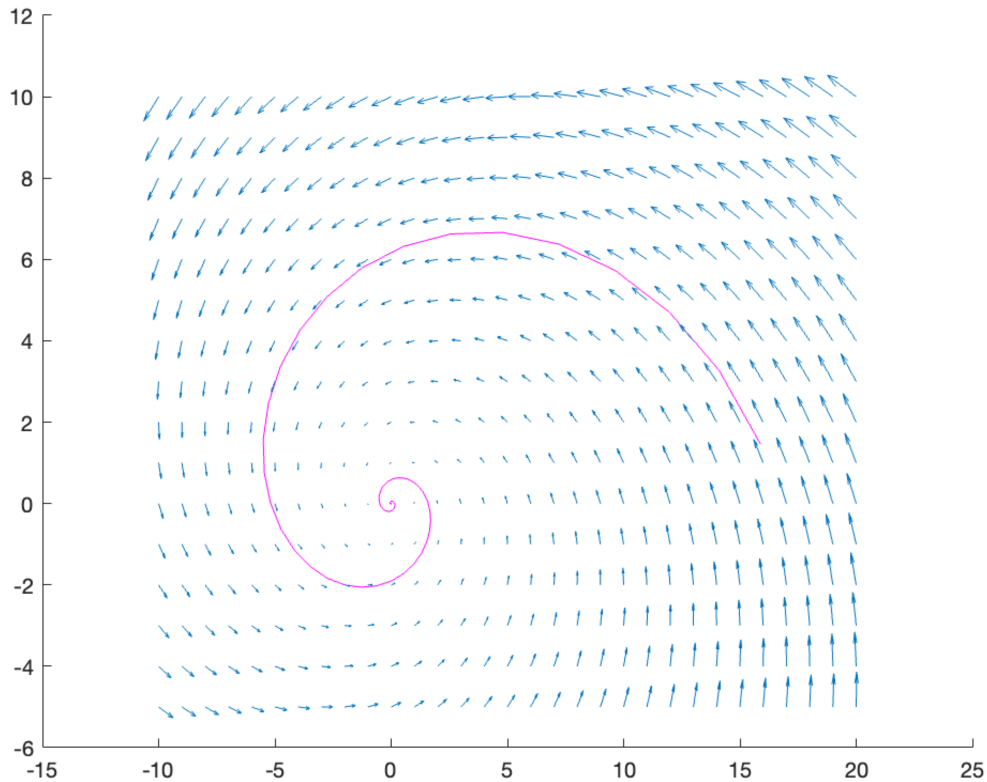
1gii)

```
figure
hold on
plot(x_sad(1,:), x_sad(2,:), 'm')
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored
```

1giii)

```
[X,Y] = meshgrid(-10:1:20,-5:1:10);
X=X(:).';
Y=Y(:).';
gradient = A*[X;Y];
U = gradient(1,:);
V = gradient(2,:);

q = quiver(X,Y,U,V);
hold off
```



2

2a)

```
A=[ -64.0859, -22.6535;
177.346, 62.6859];
```

TF during OH said I can use the eig() function directly here

```
[e_vecs, D]=eig(A);

disp('eigenvalues:')
```

```
eigenvalues:
```

```
lambda_vec = [D(1,1) D(2,2)]
```

```
lambda_vec = 1×2
   -1.2145   -0.1855
```

```
disp('eigenvectors:')
```

```
eigenvectors:
```

```
disp(e_vecs)
```

```
   -0.3390    0.3341
    0.9408   -0.9425
```

Since $Ax = \sum_{i=1}^{n} c_i e_i e^{\lambda t}$ and the eigenvalues are negative, $e^{\lambda t}$ and Ax overall will go towards 0. However, although they are both negative, one is much greater than the other.

Moreover, the two eigenvectors are also very close to each other and are therefore clearly linearly dependent on each other. Thus, there will also be some oscillation before it goes towards 0.

2b)

```
t=1.5;

% diag turns elements of vector into diagonal elements
B=e_vecs*diag(exp(lambda_vec*t))*e_vecs^(-1);
BT_B=B.'*B;

% eigenvector with max eigenvalue
v=BT_B^(50)*[1;1];
disp('initial condition for max amplitude:')
```

```
initial condition for max amplitude:
```

```
x0=v/norm(v)
```

```
x0 = 2×1
    0.9403
    0.3403
```

```
disp('corresponding eigenvalue from B^T*B:')
```

```
corresponding eigenvalue from B^T*B:
```

```
amp_lambda = mean(BT_B*x0./x0)
```

```
amp_lambda = 1.3391e+04
```

Amplitude eigenvalue for maximizing sqrt(eigenvalue) is positive

2c)

```
% finding the a constant based on x0
disp('the a constant:')
```

```
the a constant:
```

```
a_const = inv(e_vecs)*x0;
a_const=a_const.'
```

```
a_const = 1×2
 -194.3623 -194.3663
```

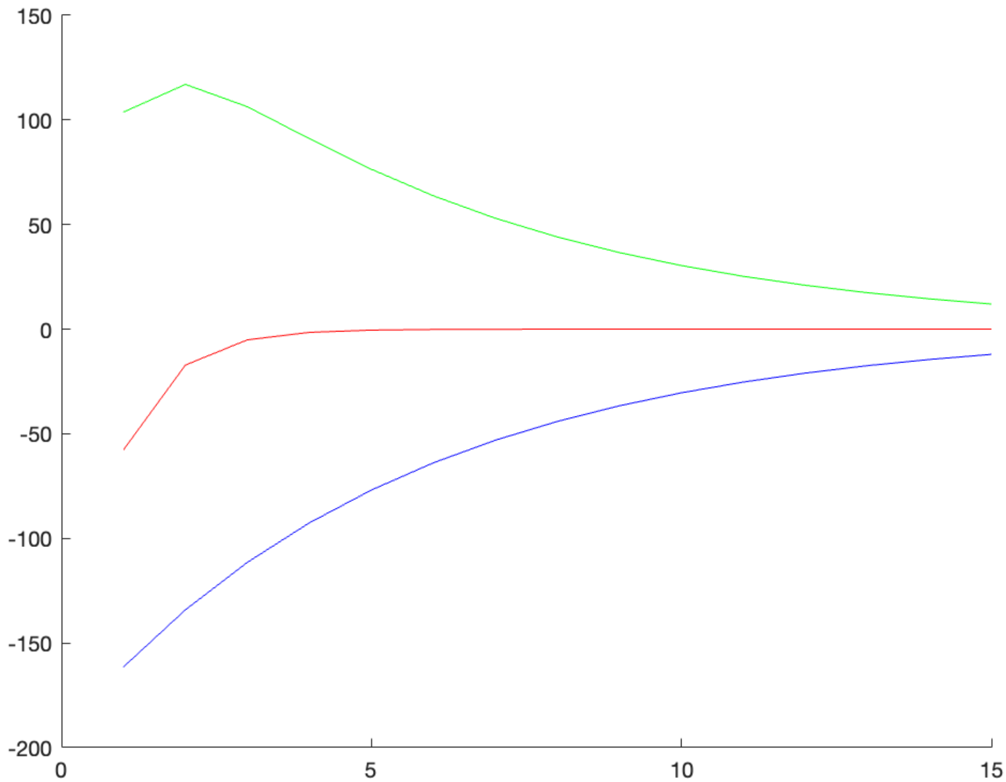
so writing it out explicitly:

2c) $\sum_i a_i e_i e^{\lambda_i t} =$

$$-194.362 \begin{bmatrix} -0.3390 \\ 0.9408 \end{bmatrix} e^{-1.2145t} - 194.3663 \begin{bmatrix} 0.3341 \\ -0.9425 \end{bmatrix} e^{-0.1855}$$

2d)

```
xsols=zeros(2, 15);
xsols_norm=zeros(1, 15);
coeff1=zeros(1, 15);
coeff2=zeros(1, 15);
for i=1:1:15
    coeffs=exp(lambda_vec*i).*a_const;
    coeff1(1,i)=coeffs(1,1);
    coeff2(1,i)=coeffs(1,2);
    % sum(,2) sums each of the rows
    xsols(:, i) = sum([coeffs;coeffs].*e_vecs, 2).';
    % typo in pset according to OH since norm is sqrt
    xsols_norm(1, i)=sqrt(xsols(1, i).^2 + (xsols(2, i).^2));
end

figure
hold on
t=1:1:15;
plot(t, xsols_norm, 'g')
plot(t, coeff1, 'r')
plot(t, coeff2, 'b')
hold off
```



Because the eigenvectors are close to parallel, there is some initial oscillation and the green goes up for a bit, but eventually goes towards 0 since the eigenvalues are still negative.

3

3a)

```
A=[5, -1, -1;
    -10, 5, 3;
    16, -2, -1];
```

TF during OH said I can use the eig() function directly here too

```
[V, D] = eig(A)
```

```
V = 3×3 complex
  -0.1374 - 0.0000i  -0.1374 + 0.0000i  -0.1374 + 0.0000i
   0.5494 + 0.0000i   0.5494 - 0.0000i   0.5494 + 0.0000i
  -0.8242 + 0.0000i  -0.8242 + 0.0000i  -0.8242 + 0.0000i
D = 3×3 complex
   3.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   3.0000 - 0.0000i   0.0000 + 0.0000i
```

```
    0.0000 + 0.0000i   0.0000 + 0.0000i   3.0000 + 0.0000i
```

```
disp('det(V):')
```

```
det(V):
```

```
disp(det(V))
```

```
  -9.5745e-23 - 1.1290e-16i
```

No, we can't directly diagonalize A. Literally all of our eigenvalues are 3/our eigenvectors are the same, so we don't have enough distinct eigenvectors for a basis. This is reflected by the determinant which is zero here so V isn't invertible.

```
disp('M and J from A:')
```

```
M and J from A:
```

```
[M,J]=jordan(A)
```

```
M = 3×3
    -2     2     1
     8   -10     0
   -12    16     0
J = 3×3
     3     1     0
     0     3     1
     0     0     3
```

3b)

```
A1=A+[0 0 0.0000001; 0 0 0; 0 0 0];

disp('M and J from A1:')
```

```
M and J from A1:
```

```
[V, D] = eig(A1)
```

```
V = 3×3 complex
    0.1369 + 0.0000i  -0.1376 - 0.0004i  -0.1376 + 0.0004i
   -0.5491 + 0.0000i   0.5496 + 0.0004i   0.5496 - 0.0004i
    0.8244 + 0.0000i  -0.8240 + 0.0000i  -0.8240 + 0.0000i
D = 3×3 complex
    2.9893 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
    0.0000 + 0.0000i   3.0053 + 0.0092i   0.0000 + 0.0000i
    0.0000 + 0.0000i   0.0000 + 0.0000i   3.0053 - 0.0092i
```

The Jordan form is very sensitive to noise. As we can see, when adding noise, the new eigenvectors and values will change and can quickly start to differ from each other, allowing a matrix to become diagonalizable.

3c)

So we start with eigenvalue 3

```
lambda=3;
I = eye(3);
B = A - (lambda*I);
B(abs(B)<1.e-13)=0;

B3 = B^3;
B3(abs(B3)<1.e-13)=0;
B3null=null(B3)
```

```
B3null = 3×3
     1     0     0
     0     1     0
     0     0     1
```

```
v3=B3null(:, 1);

v2=B*v3
```

```
v2 = 3×1
     2
   -10
    16
```

```
v1=B*v2
```

```
v1 = 3×1
    -2
     8
   -12
```

```
disp('M matrix found using generalized eigenvectors:')
```

```
M matrix found using generalized eigenvectors:
```

```
M=[v1 v2 v3]
```

```
M = 3×3
    -2     2     1
     8   -10     0
   -12    16     0
```

3d)

```
disp('J matrix:')
```

```
J matrix:
```

```
J=inv(M)*A*M
```

```
J = 3×3
```

```
    3.0000    1.0000         0
    0.0000    3.0000    1.0000
    0.0000   -0.0000    3.0000
```

```
disp('Matlab calculated M and J from jordan():')
```

```
Matlab calculated M and J from jordan():
```

```
[M,J]=jordan(A)
```

```
M = 3×3
    -2     2     1
     8   -10     0
   -12    16     0
J = 3×3
     3     1     0
     0     3     1
     0     0     3
```

The M and J matrices found using generalized eigenvectors are indeed equal to the ones calculated from Matlab's jordan() function.