Homework 4 Handed out: Wednesday, September 28, 2022
Due: Wednesday, October 5, 2022 by 11:59pm on Gradescope

Material covered: Outcomes 3.4, 3.5, 4.1, 6.3

1. **Harmonic Functions.** Check whether the following functions $u(x, y)$ are harmonic, and if so find the harmonic conjugate $v(x, y)$ so that $f(x + iy) = u(x, y) + iv(x, y)$ is an analytic function.

   a) $u(x, y) = x^3 - 3xy^2$

   b) $u(x, y) = x^2 + y^2$

   c) $u(x, y) = \cosh(y)\,(\sin(x) + \cos(x))$

2. **Visualizing conformality.**

   a) Sketch a grid of lines in the complex plane corresponding to

   $$\mathrm{Re}(z) = -1, \quad \mathrm{Re}(z) = 1, \quad \mathrm{Im}(z) = -1, \quad \mathrm{Im}(z) = 1$$

   Now sketch the images of these lines under the transformation e^z.

   b) Sketch the following curves in $\mathbb{C}$ and their images under $\mathrm{Log}(z)$.

   $$z = te^{\pi i/4}, \quad z = te^{-\pi i/4}, \quad t \in (0, \infty)$$

   $$z = e^{it}, \quad z = 2e^{it}, \quad t \in (-\pi, \pi)$$

   c) Visually, do these two maps appear conformal? Why or why not?

3. **Möbius Transformations.** The *extended complex plane* $\hat{\mathbb{C}}$ is defined as the complex plane plus an extra point "at infinity," which has infinite magnitude and corresponds to values such as $\lim_{z\to 0} \frac{1}{z}$.

   A *Möbius transformation*

   $$f(z) = \frac{az + b}{cz + d}$$

maps a point in $\hat{\mathbb{C}}$ to another point in $\hat{\mathbb{C}}$. Here, the values a, b, c, and d are constant complex numbers and $ad - bc \neq 0$.

For a visual introduction to Möbius transformations, watch the following video: https://youtu.be/0z1fIsUNhO4

a) What values of a, b, c, and d correspond to a Möbius transformation that rotates the complex plane by $\pi/4$ counter-clockwise?

b) What values of a, b, c, and d corrrespond to a Möbius transformation that leaves $z = i$ and $z = -i$ unchanged, but sends $z = 1$ to ∞?

c) Pick some nonzero values for a, b, c, and d so that $ad - bc = 0$. Sketch the image of the unit square (where $z = x + iy$, $x \in [0, 1]$ and $y \in [0, 1]$) under this Möbius transform. What happens to the square in this case?

4. **Numerical Image Transformations.**

a) Open the IPython notebook here and follow the instructions.

b) From the notebook above, upload a screenshot of your favorite custom image here. Include the transform code that produced it. (We will vote on an image for the AM104 T-shirt in a future class!)

c) What is your shirt size?

5. **Contour Integration.** For each of the following functions $f(z)$, compute the two integrals:

$$\int_{C_1} f(z)\,\mathrm{d}z \qquad \text{and} \qquad \int_{C_2} f(z)\,\mathrm{d}z$$

where C_1 is the contour from -1 to 1 along the lower half of the unit circle and C_2 is the contour along the real line from -1 to 1.

a) $f(z) = 2z^4$ b) $f(z) = |z|^2$