Section 10 Solutions Friday, November 18, 2022

1. **Convolutions.** Recall that the convolution of two functions f and g is the integral

$$(f * g)(y) = \int_{-\infty}^{\infty} f(y-x)g(x)\,\mathrm{d}x$$

Consider the functions

$$f(x) = \begin{cases} 1-x & (0 \le x \le 1) \\ 0 & (x < 0 \text{ or } x > 1) \end{cases}, \qquad g(x) = \begin{cases} 1 & (0 \le x \le 1) \\ 0 & (x < 0 \text{ or } x > 1) \end{cases}$$

a) Plot $f(x)$ and $g(x)$.

b) On the same graph, plot $f(y-x)$ and $g(x)$ as functions of x for $y = 0$. Repeat for $y = 1/2$, $y = 1$, and $y = 3/2$.

c) Using the above plots, what is the value of $(f * g)(y)$ for $y = \{0, \frac{1}{2}, 1, \frac{3}{2}\}$?

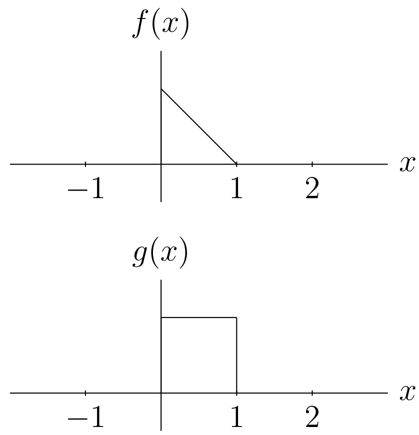
d) Write down expressions the result of the convolution, $(f * g)(y)$ for each of the following regions:

$$y \le 0, \quad 0 < y \le 1, \quad 1 < y \le 2, \quad y > 2$$

e) Conceptually: when convolving a function g in time with a *filter* function f, why is the plot of the filter flipped "backward" in time?

Solution.

a)



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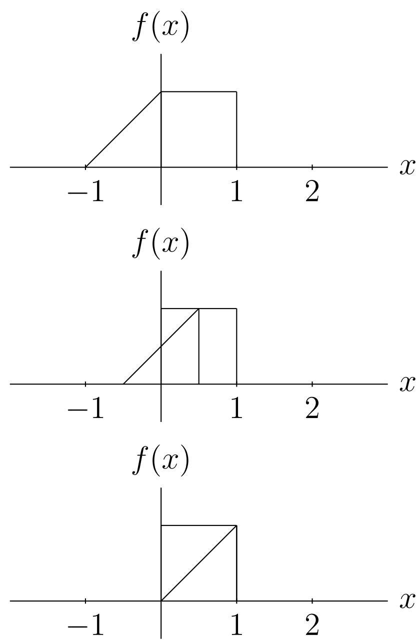
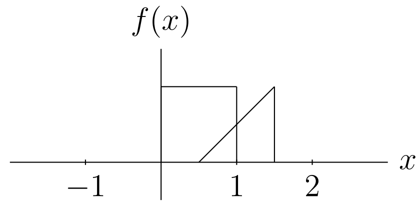
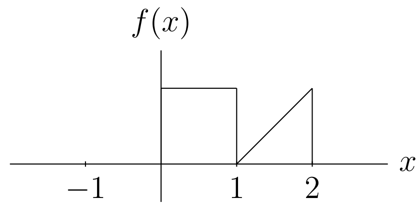






c) For $y = 0$, there is no overlap between the two functions in the convolution integral, so the value of $(f * g)(0) = 0$. When $y = 1/2$, the overlap region is a trapezoid with area $3/8$, so $(f * g)(1/2) = 3/8$. When $y = 1$, the overlap region is a right triangle with area $1/2$, so $(f*g)(1) = 1/2$. Finally, when $y = 3/2$, the overlap region is a smaller right triangle with area $1/8$, so $(f * g)(3/2) = 1/8$.

d) When $y \leq 0$, the convolution integral vanishes completely. When $0 < y \leq 1$, the integral consists of a trapezoid with area

$$\int_0^y (x + 1 - y)\,\mathrm{d}x = y - \frac{1}{2}y^2.$$

When $1 < y \leq 2$, the convolution integral consists of a triangular region with area

$$\int_{y-1}^1 (x + 1 - y)\,\mathrm{d}x = \frac{1}{2}y^2 - 2y + 2.$$

Finally, when $y > 2$, the convolution integral vanishes and the convolution has value 0.

e) Consider the general convolution integral with a filter f and signal g:

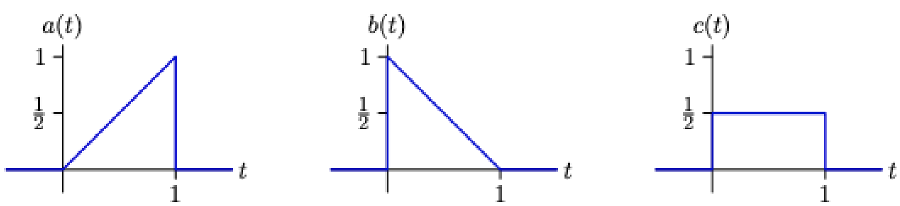
$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,\mathrm{d}\tau$$

where $g(\tau)$ represents the signal at time τ and $f(t-\tau)$ represents the contribution from the signal from time τ to the present time t. (At time t, it has been $t-\tau$ amount of time since time τ, so that's the argument we need to use for the filter function f.)

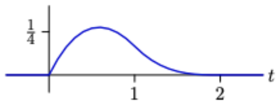
In representing the filter function itself, we consider the horizontal axis to be the "time since the signal at time 0", whereas in the convolution integral we consider a range of signals from previous times. These contrasting definitions require us to flip (and shift) the filter function before we integrate.

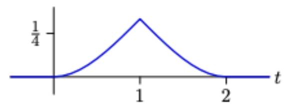
There are several other explanations, including one here: https://dsp.stackexchange.com/questions/43286/why-filter-kernel-is-flipped-during-convolution-process

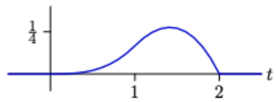
2. **Visually identifying convolutions.** Consider the functions $a(t)$, $b(t)$, and $c(t)$:

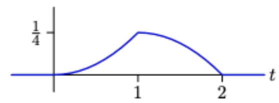


For each plot below, indicate which pair of functions (of a, b, and c) that, when convolved, would result in a function with this graph.

a) 

b) 

c) 

d) 

Solution.

a) $b * b$
This function has the most rapid rise at $t = 0$, so the height of the overlap between the function and the filter must be greatest at $t = 0$. This occurs when b is convolved with itself.

b) $a * b$ or $b * a$
As this result is symmetric, the functions whose convolution is this function must be flipped versions of each other.

c) $a * a$
Note that a and c are flipped in time with each other, suggesting that their constitutent functions are the same, but both are flipped in time.

d) $a * c$ or $c * a$
This is similar to the example worked in lecture for a decaying filter and a box function.

3. **Gaussian filter.** Consider the Gaussian function f and a sinusoidal function with amplitude 1 and frequency k:

$$f(x) = \sqrt{\frac{a}{\pi}} e^{-ax^2}, \quad g(x) = e^{ikx}$$

a is a positive real number, and k is real.

The convolution of f and g is

$$\begin{aligned}(f*g)(y) &= \int_{-\infty}^{\infty} \sqrt{\frac{a}{\pi}} e^{-ax^2} \cdot e^{ik(y-x)}\,\mathrm{d}x \\ &= \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} e^{-ax^2-ikx+iky}\,\mathrm{d}x \\ &= \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} e^{-a(x-ik/2)^2+a(-ik/2)^2+iky}\,\mathrm{d}x \\ &= \sqrt{\frac{a}{\pi}} e^{-ak^2/4+iky} \int_{-\infty}^{\infty} e^{-a(x-ik/2)^2}\,\mathrm{d}x\end{aligned}$$

where all of the steps above are only algebraic manipulations of the exponent. We can now use the fact that

$$\sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} e^{-a(x-x_0)^2} = 1$$

for any positive real a and complex x_0 to obtain

$$(f*g)(y) = e^{-ak^2/4+iky} = e^{-ak^2/4} e^{iky}.$$

Additionally, if we represent the sine and cosine functions as $c_k(x) = \cos(kx)$ and $s_k(x) = \sin(kx)$

$$\left(f*(c_k+is_k)\right)(y) = e^{-ak^2/4}\left(\cos(kx) + i\sin(kx)\right)$$

$$(f*c_k)(y) = e^{-ak^2/4}\cos(ky), \quad (f*s_k)(y) = e^{-ak^2/4}\sin(ky)$$

From this, we see that the convolution of a centered Gaussian function with a sinusoid is another sinusoid of the same frequency, but the amplitude is multiplied by a factor of $e^{-ak^2/4}$.

The amplitude of the sinusoid will be diminished greatly for high frequencies (large values of k) and/or wide Gaussian filters (small values of a).

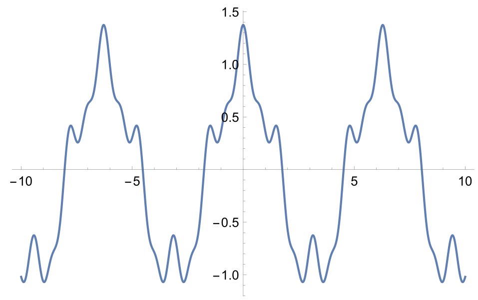


Figure 1: A plot of the function $G(x) = \cos(x) + \frac{1}{4}\cos(4x) + \frac{1}{8}\cos(8x)$.

a) Consider the Gaussian filter F and the periodic function G (plotted in Figure 1):

$$F(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}, \quad G(x) = \cos(x) + \frac{1}{4}\cos(4x) + \frac{1}{8}\cos(8x).$$

Using the results derived above and the linearity property of convolutions, compute the convolution $H(y) = (F * G)(y)$.

b) Sketch the convolution of G with a Gaussian filter, $H(y)$ for $y \in [-10, 10]$. What do you notice about the filter's effect?

c) What kind of filter does the Gaussian correspond to (e.g., low-pass, high-pass, band-pass, etc.)? Does this agree visually with the result of your computation? Compare this with the interpretation of the Gaussian filter in the Fourier domain. Is it consistent with your results? *(Hint: recall the Fourier properties of convolutions, as well as the Fourier transform of a Gaussian function.)*

Solution.

a) We can take the convolution of F with the different terms in G separately, due to the linearity of convolution. Let's define:

$$G_1(x) = \cos(x), \quad G_2(x) = \frac{1}{4}\cos(4x), \quad G_3(x) = \frac{1}{8}\cos(8x)$$

so that

$$F * G = F * G_1 + F * G_2 + F * G_3.$$

Using the relationship

$$(f * c_k)(y) = e^{-ak^2/4}\cos(ky)$$

derived above, we take $a = 1$ and plug in the different terms of G to obtain

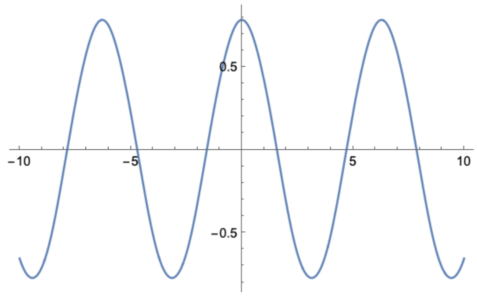
$$\begin{aligned}(F * G_1)(y) &= e^{-1/4}\cos(y)\\(F * G_2)(y) &= \frac{1}{4}e^{-4}\cos(4y)\\(F * G_3)(y) &= \frac{1}{8}e^{-16}\cos(8y)\end{aligned}$$

so

$$(F * G)(y) = e^{-1/4}\cos(y) + \frac{1}{4}e^{-4}\cos(4y) + \frac{1}{8}e^{-16}\cos(8y).$$

b) The filter drastically reduces the high-frequency (short wavelength) features in the original function.

c) The Gaussian filter corresponds to a low-pass filter (when centered around 0). This agrees visually with the result in part (b), where high frequency features were almost eliminated.
In the Fourier domain, convolution corresponds to multiplication of the two respective functions. Since a Gaussian centered at zero will have a Fourier transform that is



also a Gaussian centered at zero, multiplying this with the Fourier transform of the signal function will result in high frequency parts of the signal function being multiplied by very small numbers.

4. **Heaviside Step Function.** Let $\Theta(x)$ denote the Heaviside step function, and let

$$f(x) = \Theta(x)e^{-ax} = \begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0. \end{cases}$$

where $a > 0$.

Use the fact that

$$\lim_{a \to 0^+} f(x) = \Theta(x)$$

to show that

$$\hat{\Theta}(k) = \lim_{a \to 0^+} \hat{f}(k) = \frac{1}{ik} + \pi\delta(k).$$

Hint: Write $\hat{f}(k)$ in the form $\hat{f}(k) = \operatorname{Re}\hat{f}(k) + i\operatorname{Im}\hat{f}(k)$ and consider the cases $k = 0$ and $k \neq 0$ separately when you take the limit.

Solution. We saw in problem 1 that

$$\hat{f}(k) = \frac{a}{a^2 + k^2} - \frac{ik}{a^2 + k^2}.$$

We take the limit $a \to 0^+$ of the real and imaginary parts separately.

For the real part, we obtain for $k \neq 0$,

$$\lim_{a\to 0^+} \operatorname{Re} \hat{f}(k) = \lim_{a\to 0^+} \frac{a}{a^2 + k^2} = 0$$

while

$$\lim_{a\to 0^+} \operatorname{Re} \hat{f}(0) = \lim_{a\to 0^+} \frac{a}{a^2} = +\infty.$$

Thus $\operatorname{Re} \hat{\theta}(k) = C\delta(k)$ for some constant C. We determine C by integration:

$$\int_{-\infty}^{\infty} \frac{a}{a^2 + k^2}\, \mathrm{d}k = \int_{-\infty}^{\infty} \frac{1}{1 + u^2}\, \mathrm{d}u = \pi.$$

Therefore, if φ is any smooth function,

$$\lim_{a\to 0^+} \int_{-\infty}^{\infty} \frac{a}{a^2 + k^2} \varphi(k)\, \mathrm{d}k = \pi\varphi(0)$$

and we conclude that

$$\operatorname{Re} \hat{\theta}(k) = \lim_{a\to 0^+} \operatorname{Re} \hat{f}(k) = \pi\delta(k).$$

The limit of the imaginary part is relatively straightforward:

$$\lim_{a\to 0^+} \operatorname{Im} \hat{f}(k) = \lim_{a\to 0^+} \frac{-k}{a^2 + k^2} = -\frac{k}{k^2} = -\frac{1}{k}.$$

for $k \neq 0$ and

$$\lim_{a\to 0^+} \operatorname{Im} \hat{f}(0) = \lim_{a\to 0^+} 0 = 0.$$

To summarize,

$$\hat{\theta}(k) = \pi\delta(k) + \frac{1}{ik}.$$

5. **Audio processing.** Recall that the Fourier series of a square wave indicates that a large range of frequencies are present. This is due to the discontinuity at each step up or down in the square wave signal requiring contributions from an infinite number of sinusoidal basis functions.

   a) *White noise* (the sound of "static") is defined to be a function with a roughly uniform distribution of frequencies. It does not sound like it has a specific pitch or musical note, because one frequency (or a small number of them) cannot be identified as the dominant pitch of the sound; this is unlike square waves, which still have a peak in their Fourier transform that corresponds to the regular periodicity.

      To emulate percussive sounds like claps or snare drum hits, white noise is often sampled for a brief period of time. Why use static noise rather than a more periodic waveform?

   b) When audio is suddenly turned on or off, you will often hear a "click" or a "pop", even if this sound was not present in the original signal. (You might also hear this in headphones or amplifiers when changing an input channel.) Why do you think this occurs?

   c) Watch the following videos to explore some of the auditory differences between sounds made of a pure sine wave at 100Hz vs. a square wave at 100Hz. What are the auditory differences between the two waveforms? How would you be able to tell which sound corresponded to which waveform, simply by looking at the waveforms?

   d) When someone speaks into a microphone too loudly, you may notice that their voice begins to sound distorted, and more "static-y" than usual. This also happens when speaking into a cheap microphone or loudspeaker. This is primarily due to *saturation*, or *clipping*, of the sound wave in

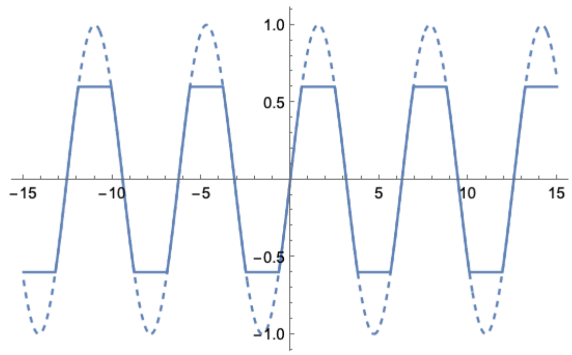


Figure 2: A sine wave (dashed) and the same wave clipped to a range of $[-0.6, 0.6]$ (solid).

the microphone or amplifier (Figure 2). This often occurs when the sound signal is digitized at some point. Considering what you know about Fourier representations, why do we hear the sound as being distorted in this way?

Solution.

a) Since static noise does not contain any predominant pitch (frequency) information, it is a good choice for sampling percussive sounds. A more periodic waveform might result in a sound like a beep.

b) By cutting off the sound signal suddenly, the sound signal experiences a discontinuity. Recalling that a discontinuity introduces a large range of frequencies in the Fourier transform, we interpret the sudden change in the sound signal as a sudden noise with no specific pitch, which is perceived as a percussive sound, typically a click or a pop.

c) Due to its discontinuities, square wave contains a wide range of high frequencies while a sine wave only has one frequency. As a result, we hear far more high-frequency noise on top of the pure 100Hz tone. The large amount of high-frequency content is characteristic of harsh or sharp sounds.

d) Clipping causes a sound wave to develop sharp features, leading to a sharp increase in the amount of high-frequency noise atop the original signal. We interpret the wide range of frequencies as sounding noisier, harsher, or simply more akin to static noise than usual.