Homework 7 Handed out: Wednesday, October 26, 2022
Due: Wednesday, November 2, 2022 by 11:59pm

Material covered: Outcomes 5.4–5.5.

1. In this problem we compute the integral

$$I(a) = \int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\, \mathrm{d}x$$

where $a \in \mathbb{R}$.

a) Let

$$f(z) = \frac{e^{iaz}}{1+z^2}$$

so that

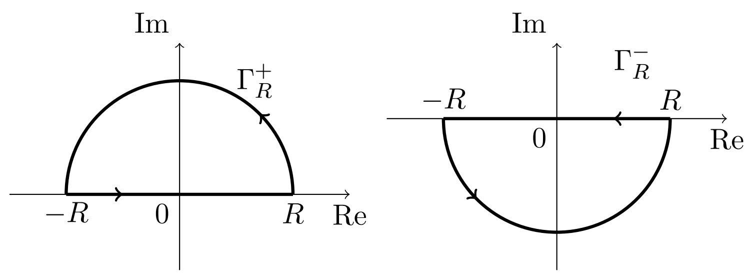
$$I(a) = \int_{-\infty}^{\infty} \mathrm{Re}(f(x))\, \mathrm{d}x.$$

Determine the poles of f and compute the residue at each pole.

b) Compute the contour integrals

$$\int_{\Gamma_R^+} f(z)\, \mathrm{d}z, \quad \int_{\Gamma_{R-}} f(z)\, \mathrm{d}z$$

where $R > 1$ and Γ_R^+, Γ_R^- are the following contours:



c) Does the integral of f along the two semicircles converge to 0 as $R \to \infty$? Does the answer depend on a?

d) Use the results of parts (a)–(c) to compute the integrals:

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\,\mathrm{d}x,$$
$$I_2 = \int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\,\mathrm{d}x$$

2. Compute the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(1+x+x^2)^2}.$$

3. Compute the integral

$$\int_{-\infty}^{\infty} \frac{xe^{-ikx}}{1+x^2}\,\mathrm{d}x.$$

where (a) $k > 0$, (b) $k < 0$.

4. In this problem, we compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,\mathrm{d}x.$$

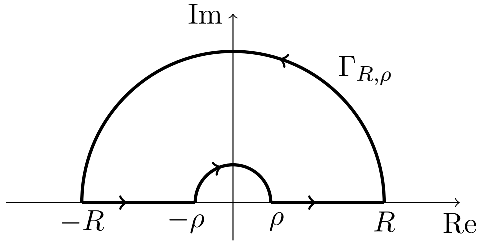
using the residue theorem. We cannot use the same method that we used in problem 1, because e^{iz}/z has a pole on the real line. We modify the method by integrating around the singularity, as shown in the figure below.

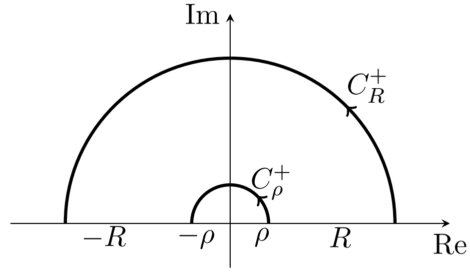
We start by proving the following result:

a) Let C_ρ^+ be the semicircle of radius $\rho > 0$ in the upper half-plane, traversed counter-clockwise. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an analytic function with a simple pole at $z = 0$. Show that

$$\lim_{\rho\to 0} \int_{C_\rho^+} f(z)\,\mathrm{d}z = i\pi\,\mathrm{Res}(f, 0).$$

Hint: Expand f in a Laurent series around 0 and integrate term by term.





Left: The contour $\Gamma_{R,\rho}$. Right: The semicircles C_R^+ and C_ρ^+.

b) Let $f(z) = e^{iz}/z$ and let $\Gamma_{R,\rho}$ be the contour shown above with $0 < \rho < R$, so that

$$\int_{\Gamma_{R,\rho}} f(z)\,\mathrm{d}z = \int_\rho^R f(x)\,\mathrm{d}x + \int_{C_R^+} f(z)\,\mathrm{d}z + \int_{-R}^{-\rho} f(x)\,\mathrm{d}x - \int_{C_\rho^+} f(z)\,\mathrm{d}z.$$

and thus, after rearranging,

$$\int_\rho^R \frac{e^{ix} - e^{-ix}}{x}\,\mathrm{d}x = \int_{\Gamma_{R,\rho}} f(z)\,\mathrm{d}z + \int_{C_\rho^+} f(z)\,\mathrm{d}z - \int_{C_R^+} f(z)\,\mathrm{d}z.$$

Evaluate each term on the right-hand side in the limit $R \to \infty$, $\rho \to 0$. Use this to compute

$$I_1 = \int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x$$

and

$$I_2 = \int_{-\infty}^\infty \frac{\sin(x)}{x}\,\mathrm{d}x.$$