General and MC stuff

We consider the one-dimensional integral of an arbitrary function $g(x)$ for $x \in [0,1]$ with $0 < g(x) \leq b, \forall x$.

$$I = \int_0^1 g(x)dx.$$

To estimate the integral value, you are asked to use the following two Monte Carlo estimators and a fixed number of samples N in each estimator.

$$I_1 = \frac{1}{N}\sum_{i=1}^{N} g(x_i).$$

Here, the x_i are independent samples from a uniform distribution in $[0,1]$.

$$I_2 = \frac{1}{N}\sum_{i=1}^{N} b\, h(x_i, y_i).$$

Here, the x_i are again independent samples from a uniform distribution in $[0,1]$, whereas the y_i are independent samples from a uniform distribution in $[0,b]$. Moreover,

$$h(x,y) = \begin{cases} 1 & y < g(x), \\ 0 & \text{otherwise.} \end{cases}$$

Hint:

$$\int_0^b h(x,y)dy = \int_0^{g(x)} dy.$$

19 a) **[4 points]** Show that $\mathbb{E}[I_1] = I$

b) Show that $\mathrm{E}[I_2] = I$

c) **[8 points]** Show that $\mathrm{Var}[I_2] \geq \mathrm{Var}[I_1]$. Which Monte Carlo estimator would you choose to obtain on average the smaller error for a fixed N?

c) **[8 points]** Show that $\mathrm{Var}[I_2] \geq \mathrm{Var}[I_1]$. Which Monte Carlo estimator would you choose to obtain on average the smaller error for a fixed N?

a) $E[I_1] = \frac{1}{N}\sum^{N} E[g(x)]$

$= \frac{1}{N}\sum^{N} \int_0^1 g(x)\frac{1}{1}\,dx$

$= \frac{1}{N}\sum^{N} I = I$

b) $E[I_2] = \frac{b}{N}\sum^{N} E[h(x)]$

$= \frac{b}{N}\sum^{N} \int_0^1\int_0^b h(x_i, y_i)\frac{1}{1}\frac{1}{b}\,dy_i\,dx_i$

$= \frac{1}{N}\sum^{N} \int_0^1 \int_0^{g(x)} dy_i\,dx_i$

$= \frac{1}{N}\sum^{N} \int_0^1 g(x) - 0\,dx_i = I$

c) $\frac{Var[g(x)]}{N} = \frac{E[g(x)^2] - E[g(x)]^2}{N}$

$= \frac{\int_0^1 g(x)^2 \frac{1}{1}\,dx - \left(\int_0^1 g(x)\frac{1}{1}\,dx\right)^2}{N}$

$= \frac{\int_0^1 g(x)^2\,dx - I^2}{N}$

$\frac{Var[b\,h(x,y)]}{N} = \frac{E[b^2 h(x,y)^2] - E[b\,h(x,y)]^2}{N}$

$= \frac{b^2\int_0^1\int_0^b h(x,y)^2\frac{1}{1}\frac{1}{b}\,dy\,dx - I^2}{N}$

$= \frac{b\int_0^1\int_0^{g^2(x)} dy\,dx - I^2}{N}$

$= \frac{b\int_0^1 g^2(x)\,dx - I^2}{N} \geq \frac{\int_0^1 g(x)\,dx - I^2}{N}$

so $Var[I_1]$ smaller

d) MC is unbiased & var $\to 0$ as $N \to \infty$

Consider the following integral:

$$I = \int_0^1 f(x)\,dx, \quad f(x) = x.$$

a) **[4 points]** We want to solve this integral by using the following Monte Carlo estimator:

$$I_1 = \frac{1}{N}\sum_{i=1}^{N} f(x_i).$$

The x_i are the samples from a uniform distribution in $[0, 1]$ and N is the total number of
20 samples used. What is the variance of the Monte Carlo estimator I_1?

b) **[8 points]** Consider the following transformation of the above integral

$$I = \frac{1}{2}\int_0^1 (f(x) + f(1-x))\,dx$$

and the corresponding estimator

$$I_2 = \frac{1}{2N}\sum_{i=1}^{N}(f(x_i) + f(1-x_i)),$$

where x_i are samples from a uniform distribution in $[0,1]$ and N is the number of samples. What is the variance of I_2?

c) **[8 points]** Explain why the variance of I_2 is lower than that of I_1.

a) $\frac{Var[f(x)]}{N} = \frac{Var[x]}{N} \quad x \sim U(0,1) \quad = \frac{1^2}{12}\,\frac{1}{N} = \frac{1}{12N}$

b) $\frac{Var[f(x) + f(1-x)]}{2N} = \frac{Var[x + (1-x)]}{2N} = \frac{0}{2N} = 0$

c) because with $f(x) + f(1-x)$ constant, it's no longer stochastic & gives a var of 0

In this exercise we fix the notation we will use during this course and refresh our memory on basic properties of random variables. Present your answers *in detail*.

a) A random variable with normal (or Gaussian) distribution $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ has probability density function (pdf) given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} . \tag{1}$$

Show that the mean and the variance of X are given by $\mathbb{E}\left[X\right] = \mu$ and $\mathbb{E}\left[(X-\mu)^2\right] = \sigma^2$, respectively.

b) The probability that a random variable X with pdf f_X is less or equal than any $x \in \mathbb{R}$ is given by,

$$P(X \leq x) \;=\; F_X(x) \;=\; \int_{-\infty}^{x} f_X(z)\,\mathrm{d}z\,. \tag{2}$$

The function F_X is called the cumulative distribution function (cdf).
The Laplace distribution with parameters μ and β has pdf,

$$f(x) = \frac{1}{2\beta}\,\exp\left(-\frac{|x-\mu|}{\beta}\right). \tag{3}$$

i) Find the cdf of the Laplace distribution.

ii) Use the cdf to find the median of the Laplace distribution.

c) The pdf of the quotient $Q = X/Y$ of two random variables X, Y is given by,

$$f_Q(q) \;=\; \int_{-\infty}^{\infty} |x|\, f_{X,Y}(qx, x)\,\mathrm{d}x\,, \tag{4}$$

where $f_{X,Y}$ is the joint pdf of X and Y.
Assume that X and Y are independent random variables with pdfs $f_X(x) = \mathcal{N}\left(x|0,\sigma_X^2\right)$ and $f_Y(y) = \mathcal{N}\left(y|0,\sigma_Y^2\right)$.

1 i) Find the joint pdf of X and Y.

ii) Show that $Q = X/Y$ follows a Cauchy distribution with zero location parameter and scale $\gamma = \sigma_X/\sigma_Y$. The pdf of a Cauchy distribution with location parameter x_0 and scale γ is given by,

$$f(x) \;=\; \frac{1}{\pi}\frac{\gamma}{(x-x_0)^2+\gamma^2}\,. \tag{5}$$

$$E[x] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) dx$$

$a = (x-\mu)$

$da = dx$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int (a+\mu) \exp\left(\frac{-a^2}{2\sigma^2}\right) da$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left[\int a \exp\left(\frac{-a^2}{2\sigma^2}\right) da + \mu \int \exp\left(\frac{-a^2}{2\sigma^2}\right) da\right]$$

$b = a^2$

$db = 2a\,da$

$\frac{1}{2} db = a\,da$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left[\frac{1}{2} \int_{-\infty}^{\infty} \exp\left(\frac{-b}{2\sigma^2}\right) db + \mu \int_{-\infty}^{\infty} \exp\left(\frac{-a^2}{2\sigma^2}\right) da\right]$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left[\frac{1}{2} (-2\sigma^2) \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) + \mu \int_{-\infty}^{\infty} \exp\left(\frac{-a^2}{2\sigma^2}\right) da\right]$$

$$\int_{-\infty}^{\infty} e^{-\# x^2} = \frac{1}{2}\sqrt{\frac{\pi}{\#}}\, \mathrm{erf}(\sqrt{\#}\, x)$$

$\# = \frac{1}{2\sigma^2}$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left[-\sigma^2 \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) + \mu \frac{1}{2} \sqrt{\frac{\pi}{1/2\sigma^2}}\, \mathrm{erf}\left(\sqrt{\frac{1}{2\sigma^2}}(x-\mu)\right)\right]_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left[-\sigma^2(\exp(-\infty) - \exp(-\infty)) + \mu \frac{1}{2} \sqrt{\pi 2\sigma^2}\, (1-(-1))\right]$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left[0 + \mu\sqrt{2\pi\sigma^2}\right] = \mu$$

$$E[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) dx$$

$a = (x-\mu)$

$da = dx$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} a^2 \exp\left(\frac{-a^2}{2\sigma^2}\right) da$$

$$\Gamma(\#) = \int_0^{\infty} \exp(-x)\, x^{\#-1} dx$$

$b = \frac{a^2}{2\sigma^2}$ $\quad a = \sqrt{2b\sigma^2}$

$db = 2a\,da\,\frac{1}{2\sigma^2}$

even bc a^2 & $\exp(-a^2)$

$\sigma^2 db = a\,da$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\, 2 \int_0^{\infty} \sqrt{2b\sigma^2}\, \exp(-b)\, \sigma^2 db$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\, 2\sqrt{2\sigma^2}\, \sigma^2 \int_0^{\infty} b^{\frac{1}{2}} \exp(-b)\, db$$

$$= \frac{2}{\sqrt{\pi}} \sigma^2\, \Gamma(3/2) = \frac{2}{\sqrt{\pi}} \sigma^2 \left(\frac{\sqrt{\pi}}{2}\right) = \sigma^2$$

$$cdf = \int_{-\infty}^{x} \frac{1}{2\beta} \exp\left(\frac{-|x-\mu|}{\beta}\right) dx$$

abs val so 2 cases

(1) $x-\mu \le 0$ @ $x \le \mu$

$$\frac{1}{2\beta}\int_{-\infty}^{x} \exp\left(\frac{+(x-\mu)}{\beta}\right) dx \qquad \begin{array}{l} a = x-\mu \\ da = dx \end{array}$$

$$= \frac{1}{2\beta}\int_{-\infty}^{x} \exp\left(\frac{a}{\beta}\right) da$$

$$= \frac{1}{2\beta}\,\beta\left[\exp\left(\frac{x-\mu}{\beta}\right)\right]_{-\infty}^{x}$$

$$= \frac{1}{2}\exp\left(\frac{x-\mu}{\beta}\right)$$

(2) $x-\mu > 0$ @ $x > \mu$

$$\int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{x} f(x)\,dx$$

$$= \frac{1}{2}\exp\left(\frac{\mu-\mu}{\beta}\right) + \int_{\mu}^{x}\frac{1}{2\beta}\exp\left(\frac{\mu-x}{\beta}\right)dx \qquad \begin{array}{l} a = \mu - x \\ da = (-1)dx \end{array}$$

$$= \frac{1}{2} + \frac{1}{2\beta}(-1)\,\beta\left[\exp\left(\frac{\mu-x}{\beta}\right)\right]_{\mu}^{x}$$

$$= \frac{1}{2} - \frac{1}{2}\left[\exp\left(\frac{\mu-x}{\beta}\right) - 1\right]$$

$$= 1 - \frac{1}{2}\exp\left(\frac{\mu-x}{\beta}\right)$$

median $\frac{1}{2}\exp\left(\frac{x-\mu}{\beta}\right) = \frac{1}{2}$ @ $x = \mu$

i) X & Y ind so multiply

$$f(X=x, Y=y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(\frac{-x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right)$$

ii) $$f_Q(q) = \int_{-\infty}^{\infty} |x| \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(\frac{-q^2x^2}{2\sigma_x^2} - \frac{x^2}{2\sigma_y^2}\right) dx$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} |x| \exp\left(-\frac{x^2}{2}\left(\frac{q^2}{\sigma_x^2} + \frac{1}{\sigma_y^2}\right)\right) dx$$

even bc $|x|$ & $-x^2$

$$= \frac{1}{\pi\sigma_x\sigma_y} \int_0^{\infty} x \exp\left(-\frac{x^2}{2}\left(\frac{q^2}{\sigma_x^2} + \frac{1}{\sigma_y^2}\right)\right) dx$$

$a = \# x^2$

$da = 2x\,dx\,\#$

$\frac{1}{2\#} da = x\,dx$

$$= \frac{1}{\pi\sigma_x\sigma_y} \frac{1}{2} \left[-\frac{1}{2}\left(\frac{q^2}{\sigma_x^2} + \frac{1}{\sigma_y^2}\right)\right]^{-1} \left[\exp\left(-\frac{x^2}{2}\left(\frac{q^2}{\sigma_x^2} + \frac{1}{\sigma_y^2}\right)\right)\right]_0^{\infty}$$

$$= \frac{-1}{\pi\sigma_x\sigma_y} \left(\frac{\sigma_x^2\sigma_y^2}{q^2\sigma_y^2 + \sigma_x^2}\right) [0-1]$$

$$= \frac{+1}{\pi} \frac{\sigma_x\sigma_y}{q^2\sigma_y^2 + \sigma_x^2}$$

$$\text{Cauchy}(x) = \frac{1}{\pi} \frac{\gamma}{(x-x_0)^2 + \gamma^2}$$

$$= \frac{1}{\pi} \frac{\sigma_x}{\sigma_y} \frac{1}{q^2 + \sigma_x^2/\sigma_y^2} \quad \checkmark$$

Let's say that a genetic disorder occurs in the United States population at a rate of 2% (i.e., if D is the event that an individual sampled from this population has the genetic disorder, then $P(D) = \frac{1}{50}$, $P(D^C) = 1 - \frac{1}{50} = \frac{49}{50}$).

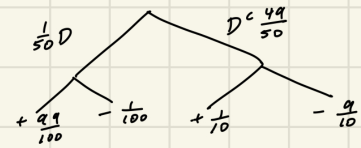
A company that markets genetic testing services has produced a test for detecting whether an individual has this genetic disorder. According to the company, their test can accurately detect positive cases at a rate of 99% and can accurately detect negative cases at a rate of 90%:

$$P(+|D) = \frac{99}{100}, \quad P(-|D^C) = \frac{9}{10}$$

a) From the above information, let's say you take this test and receive a positive result, what is the probability that you *actually* have this disorder? In math, what is $P(D|+) =$?

b) From the company's perspective, they have no control over the rates of the disorder in the general population ($P(D), P(D^C)$). However, they do have some control over the false positive rate of the test they've produced, $P(+|D^C)$. What is the maximum false positive rate their test can produce if the company wishes for a positive test result to indicate a true positive 90% of the time (i.e., what is the maximum $P(+|D^C)$ if the company wishes for
2 $P(D|+) = \frac{9}{10}$)? What about 99% of the time?

$P(+|D) = \frac{99}{100}$ $\quad P(D) = \frac{1}{50}$



Bayes

$$P(D|+) = \frac{P(+|D)\ P(D)}{P(+)}$$

$$P(+) = \frac{1}{50}\frac{99}{100} + \frac{49}{50}\frac{1}{10}$$

$$P(D|+) = \frac{\frac{99}{100}\frac{1}{50}}{\frac{1}{50}\frac{99}{100} + \frac{49}{50}\frac{1}{10}} \qquad (1)$$

from (1)

$$P(D|+) = \frac{9}{10} = \frac{y\frac{1}{50}}{\frac{1}{50}y + \frac{49}{50}x}$$

$$\frac{9}{10} = \frac{y}{y+49x}$$

$$9y + 9(49)x = 10y$$

$$9(49)x = y \quad \& \quad y \le 1$$

$$x \propto y$$

$$\max \quad x = \frac{1}{9(49)}$$

$$P(D|+) = \frac{99}{100}$$

$$\max \quad x = \frac{1}{99(49)}$$

a) Find the tangential and normal lines to the curve $y = x^2$ at the point $(1, 1)$. The *normal line* is the line that is perpendicular to the tangent line and that passes through the point of tangency.

b) Scaling t and $y(t)$ can reduce the equation

$$y_{tt} + ay_t + by = 0$$

to the form

$$u_{\tau\tau} + \epsilon u_\tau + u = 0.$$

What is ϵ in terms of a and b (a and $b > 0$ are constants)?

c) Show that $u(t, x) = \frac{a+x}{b+t}$ is a solution of the equation $u_t + u\, u_x = 0$. What initial condition ($t = 0$) does the solution correspond to (a and $b > 0$ are constants)?

d) Consider the boundary value problem

$$-\, u_{xx} = 1, \quad u(0) = u(1) = 0$$

and its finite difference approximation

$$-\frac{v_{j-1} - 2v_j + v_{j+1}}{h^2} = 1, \quad j = 1, \ldots n, \quad v_0 = v_{n+1} = 0$$

where $n \geq 1$ is an integer, v_j is the finite difference approximation of $u(jh)$ and $h = 1/(n+1)$. Show that

$$v_j = (jh)(1 - jh)/2$$

is the finite difference solution. Compare it to the exact solution at the grid points $x_j = jh$.

e) The solution of the initial value problem

$$y_t = y^2, \quad y(0) = 2$$

3 "blows up" to infinity in a finite time. Find that time.

$y = x^2$ @ $(1,1)$

$\frac{dy}{dx} = 2x \quad = 2$ @ $(1,1)$

tangent = $y - 1 = 2(x-1)$

normal = $y - 1 = \frac{-1}{2}(x-1)$

$$y_{tt} + a y_t + b y = 0 \longrightarrow u_{TT} + \epsilon u_T + u = 0$$

try $\quad u = by \qquad \frac{du}{dt} = b\frac{dy}{dt} \qquad \frac{d^2u}{dt^2} = b\frac{d^2y}{dt^2}$

$$= \frac{1}{b} u_{tt} + \frac{a}{b}\frac{du}{dt} + u = 0$$

$$\frac{d^2u}{dT^2} = \frac{1}{b}\frac{d^2u}{dt^2} \qquad T^2 = bt^2 \qquad t = \frac{T}{\sqrt{b}} \qquad \frac{1}{dt} = \frac{\sqrt{b}}{dT}$$

$$= u_{TT} + \frac{a}{b}\sqrt{b}\frac{du}{dT} + u_T = 0$$

$$\epsilon = \frac{a}{\sqrt{b}}$$

$$\frac{du}{dt} + u\frac{du}{dx} = 0$$

$$u(t,x) = \frac{a+x}{b+t}$$

$$\frac{du}{dt} = (a+x)\frac{-1}{(b+t)^2}$$

$$\frac{du}{dx} = \frac{1}{b+t}$$

$$(a+x)\frac{-1}{(b+t)^2} + \left(\frac{a+x}{b+t}\right)\frac{1}{b+t} = 0 \checkmark$$

@ $t = 0 \qquad \frac{a+x}{b+0} = u$

$$\frac{d^2u}{dx^2} = -1 \qquad u(0) = 0 \qquad u(1) = 0$$

$$\frac{du}{dx} = -x + C_1$$

$$u = -\frac{1}{2}x^2 + C_1 x + C_2$$

$$u = -\frac{1}{2}x^2 + \frac{1}{2}x + 0$$

$$\frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} \approx \frac{d^2u}{dx^2}$$

$$\frac{v_{j-1} - 2v_j + v_{j+1}}{h^2} \approx -1 = u(jh)$$

$$v_0 = 0 \qquad v_{n+1} = 0 \qquad h = \frac{1}{n+1}$$

$$v_j = \frac{-1}{2}h^2 + \frac{1}{2}v_{j-1} + \frac{1}{2}v_{j+1}$$

$$v_j = \frac{(jh)(1-jh)}{2}$$

$$\frac{(jh)(1-jh)}{2} = \frac{1}{2}h^2 + \frac{1}{2}v_{j-1} + \frac{1}{2}v_{j+1}$$

$$jh(1-jh) = h^2 + v_{j-1} + v_{j+1}$$

$$v_{j+1} = \frac{1}{2}(j+1)h(1-(j+1)h) = \frac{1}{2}(jh+h)(1-jh-h)$$

$$v_{j-1} = \frac{1}{2}(j-1)h(1-(j-1)h) = \frac{1}{2}(jh-h)(1-jh+h)$$

$$jh(1-jh) = h^2 + (jh+h)(1-jh-h) + (jh-h)(1-jh+h)$$

$$jh - j^2h^2 = h^2 + \frac{1}{2}\left[jh - j^2h^2 \cancel{-jh^2} \cancel{+h} \cancel{-jh^2} - h^2\right] + \frac{1}{2}\left[jh - j^2h^2 \cancel{+jh^2} \cancel{-h} \cancel{+jh^2} - h^2\right]$$

$$jh - j^2h^2 = h^2 + \left[jh - j^2h^2 - h^2\right]$$

$$jh - j^2h^2 = jh - j^2h^2$$

$$\frac{dy}{dt} = y^2 \qquad y(0) = 2$$

$$dy\, y^{-2} = dt$$

$$-y^{-1} = t + C$$

$$y = \frac{-1}{t+C}$$

$$2 = \frac{-1}{C} \qquad C = -\frac{1}{2}$$

$$y = \frac{1}{-t+\frac{1}{2}}$$

@ $t = \frac{1}{2}$ $\quad y = \infty$

a) The concentration $c(\mathbf{x}, t)$ of a pollutant in a body of water diffuses according to

$$\frac{\partial c}{\partial t} = \nu \nabla^2 c,$$

with initial conditions $c(\mathbf{x}, 0) = c_0(\mathbf{x})$.
Show that the total concentration remains constant.

b) Show that

$$c_g(\mathbf{x}, t) = \frac{1}{\sqrt{t}} e^{-x^2/4\nu t}$$

is a solution of the diffusion equation.

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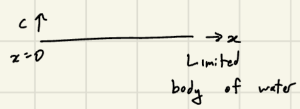
total concentration $= \iiint c(x,t)\, dV$

$c(x,0) = c_0(x)$



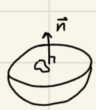
$$\frac{d}{dt} \iiint c(x,t)\, dV = 0\,?$$

$$\iiint \frac{\partial c}{\partial t}\, dV = \iiint_0^L \nu \nabla (\nabla c)\, dV$$

$$\frac{d}{dt} \iiint c\, dV = \nu \iiint_0^L \nabla\, \nabla c(x,t)\, dV$$

Divergence Theorem

$$= \nu \iint_S \nabla c(x,t) \cdot \vec{n}\, dS$$



$$= 0$$

$$\frac{dc}{dt} = \frac{d}{dt} t^{-\frac{1}{2}} exp(-x^2/4vt)$$

$$= t^{-\frac{1}{2}} \left(\frac{+x^2}{4v}\right) t^{-2} exp\left(\frac{-x^2}{4vt}\right) + exp\left(\frac{-x^2}{4vt}\right)\left(-\frac{1}{2}\right) t^{-3/2}$$

$$= t^{\frac{-3}{2}} exp\left(\frac{-x^2}{4vt}\right)\left[\frac{x^2}{4vt} + \frac{-1}{2}\right]$$

$$\nabla c = t^{-\frac{1}{2}} \frac{-2x}{4vt} exp\left(\frac{-x^2}{4vt}\right) = t^{-\frac{1}{2}} \frac{-x}{2vt} exp\left(\frac{-x^2}{4vt}\right)$$

$$\nabla^2 c = t^{-\frac{1}{2}}\left[\frac{-1}{2vt} exp\left(\frac{-x^2}{4vt}\right) + \frac{+x^2}{4v^2t^2} exp\left(\frac{-x^2}{4vt}\right)\right]$$

$$= t^{\frac{-3}{2}} exp\left(\frac{-x^2}{4vt}\right)\left[-\frac{1}{2v} + \frac{x^2}{4v^2t}\right]$$

$$\nabla^2 c \times v = t^{\frac{-3}{2}} exp\left(\frac{-x^2}{4vt}\right)\left[-\frac{1}{2} + \frac{x^2}{4vt}\right] \checkmark$$

Let $P(E)$ be the probability of an event $E \subset \Omega$ and $P(\Omega) = 1$. We denote the sample space as Ω. The three axioms of probabilities are:

- $P(E) \geq 0$ for all $E \subset \Omega$
- $P(\Omega) = 1$
- If $E_1, E_2, \cdots \subset \Omega$ and they are pairwise disjoint, i.e. $E_i \bigcap E_j = \varnothing$ when $i \neq j$, then $P(E_1 \cup E_2 \cup \ldots) = P(E_1) + P(E_2) + \cdots$

Given the three axioms, answer the following questions **[5 points each]**:

1. If $\bar{E}$ is the complement of E (i.e. $\bar{E}$ contains all the events that are not in E), show that $P(\bar{E}) = 1 - P(E)$
2. Assume, two events E_1 and E_2 are given with $E_1 \subset E_2$. Show that $P(E_2) = P(E_1) + P(E_2 \cap \bar{E_1}) \geq P(E_1)$
3. Show that $P(\cup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} P(E_i)$
4. Two event A and B are given with $0 < P(A) < 1$ and $0 < P(B) < 1$. Given A and B are disjoint, can the two events be independent ?
5. Two events A and B are given with $0 < P(A) < 1$ and $0 < P(B) < 1$. Given $A \subset B$, can the two events be independent ?

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1) if $\bar{E}$ is complement, then pairwise disjoint

$$P(E \cap \bar{E}) = \emptyset$$

3rd axiom applies

$$P(E \cup \bar{E}) = P(E) + P(\bar{E})$$

& since only E or $\bar{E}$, 2nd axiom

$$P(E \cup \bar{E}) = 1 = P(E) + P(\bar{E})$$

$$\therefore P(\bar{E}) = 1 - P(E)$$

2) since only E_1 or $\bar{E}_1$, 2nd axiom

$$P(E_1 \cup \bar{E}_1) = 1 = P(\Omega)$$

E_2 is in sample space Ω

$$P(E_2) = P(E_2 \cap \Omega)$$

$$P(E_2) = P[E_2 \cap (E_1 \cup \bar{E}_1)]$$

$$= P[(E_2 \cap E_1) \cup (E_2 \cap \bar{E}_1)]$$

E_1 & $\bar{E}_1$ disjoint, 3rd axiom

$$P(E_2) = P(E_2 \cap E_1) + P(E_2 \cap \bar{E}_1)$$

$E_1 \subset E_2$ so all E_1 events also had E_2 occur

$$P(E_1) = P(E_2 \cap E_1)$$

$$\therefore P(E_2) = P(E_1) + P(E_2 \cap \bar{E}_1) \qquad (2.1)$$

1st axiom

$$P(E_2 \cap \bar{E}_1) \geq 0$$

$$\therefore P(E_1) + P(E_2 \cap \bar{E}_1) \geq P(E_1) \qquad (2.2)$$

3) first proving case $n=2$:

$A \cup B$ is the same as all of A or the parts of B not including A (A^c)

$$P(A \cup B) = P(A \cup (B \cap A^c))$$

A & $B \cap A^c$ are therefore disjoint, 3rd axiom

$$P(A \cup B) = P(A) + P(B \cap A^c)$$

$B \cap A^c$ is a subset of all B

$$B \cap A^c \subset B$$

from (2.1) & (2.2) of previous problem

$$P(B \cap A^c) \leq P(B)$$

$$\therefore P(A \cup B) = P(A) + P(B \cap A^c) \leq P(A) + P(B) \qquad (3.1)$$

case $n=3$:

$$P(A \cup B \cup C) = P((A \cup B) \cup C)$$

from (3.1) with $A \cup B$ as first event & C as second event

$$P((A \cup B) \cup C) \leq P(A \cup B) + P(C)$$

from (3.1) again

$$P(A \cup B) \leq P(A) + P(B)$$

$$P(A \cup B) + P(C) \leq P(A) + P(B) + P(C)$$

$$\therefore P(A \cup B \cup C) \leq P(A \cup B) + P(C) \leq P(A) + P(B) + P(C)$$

case $n+1$:

assume case n works

$$P(E_1 \cup E_2 \cup \ldots E_n) \leq P(E_1) + P(E_2) + \ldots P(E_n)$$

now look at case $n+1$

$$P(E_1 \cup E_2 \cup \ldots E_n \cup E_{n+1}) = P\left[((E_1 \cup E_2) \ldots \cup E_n) \cup E_{n+1} \right]$$

from 3.1 with $(E_1 \cup E_2) \ldots \cup E_n$ as first event & E_{n+1} as second event

$$P\left[((E_1 \cup E_2) \ldots \cup E_n) \ldots \cup E_{n+1} \right] \leq P((E_1 \cup E_2) \ldots \cup E_n) + P(E_{n+1})$$

since we assumed case n works

$$P((E_1 \cup E_2) \ldots \cup E_n) + P(E_{n+1}) \leq P(E_1) + P(E_2) + \ldots P(E_n) + P(E_{n+1})$$

$$\therefore P\left[((E_1 \cup E_2) \ldots \cup E_n) \ldots \cup E_{n+1} \right] \leq P((E_1 \cup E_2) \ldots \cup E_n) + P(E_{n+1}) \leq P(E_1) + P(E_2) + \ldots P(E_n) + P(E_{n+1})$$

complete proof by induction

4) independence def: independent iff $P(A \cap B) = P(A)\,P(B)$

disjoint so

$P(A \cap B) = 0$

$P(A) > 0$ & $P(B) > 0$

$P(A)\,P(B) \neq 0$

$\therefore$ $P(A \cap B) \neq P(A)P(B)$ not independent

5) independence def: independent iff $P(A \cap B) = P(A)\,P(B)$

$A \subset B$ so all A events also had B occur

$P(A) = P(A \cap B)$

$P(A) > 0$ & $P(B) > 0$

$P(A)\,P(B) \neq P(A)$

$\therefore$ $P(A \cap B) \neq P(A)P(B)$ not independent

function:

$$f(\mathbf{x}) = 2^{-d} \sum_{i=1}^{d} x_i^2 \,,$$

where $\mathbf{x}$ is a vector $\mathbf{x} = (x_1, \ldots, x_d)$.

a) **[2 points]** Calculate the integral analytically

$$I = \int_V f(\mathbf{x}) \, \mathrm{d}\mathbf{x} \,,$$

6 where V is a d-dimensional cube with bounds $[-1, 1]^d$.

$$= 2^{-d} \int_V \vec{x}^T \vec{x} \, d\vec{x}$$

$$= 2^{-d} \int_V \sum_i^d x_i^2 \, d\vec{x}$$

$$= 2^{-d} \sum_i^d \left(\int_{-1}^1\right)^{d-1} \int_{-1}^1 x_i^2 \, dx_i \, (dx)^{d-1}$$

$$= 2^{-d} \sum_i^d \left(\int_{-1}^1\right)^{d-1} \frac{1}{3}[1-(-1)](dx)^{d-1}$$

$$= 2^{-d} \sum_i^d \left(\int_{-1}^1\right)^{d-2} \frac{2}{3}[2](dx)^{d-2}$$

$$= 2^{-d} \sum_i^d \frac{2}{3} \, 2^{d-1}$$

$$= 2^{-d} 2^d d \frac{1}{3} = d/3$$

You are in charge of a factory that produces mechanical springs. A spring is produced successfully if its stiffness K is larger than 10 $[N/m]$. The stiffnesses of all produced springs are independent from each other and follow a known distribution f_K. You want to find the unknown success rate q of the production process using Monte Carlo.

a) Find a suitable Monte Carlo estimator q_N for the production sucess rate given N independent random samples from the distribution f_K.

b) Find the expected value as well as the variance of the Monte Carlo estimator q_N.

c) We define the relative error of the Monte Carlo estimator as:

$$\alpha_N = \frac{q_N - q}{q} \tag{1}$$

Find the expected value of the squared relative error. Your result should depend on both N and q.

d) Assume the unknown true success rate is $q = \frac{1}{10001}$. How many sample N are necessary
9 such that the squared relative error is smaller than 0.01 ?

a) $I_q = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}(k_i > 10) \qquad k_i \sim f_k$

b) $E[q_N] = q$ average prob of passing

then $(k_i > 10) \sim \text{Bern}(q)$

$\text{Var}[q_N] = \frac{q(1-q)}{N}$

c) $E[\alpha^2] = \text{Var}[\alpha] + E[\alpha]^2$

$= \frac{1}{q^2} \text{Var}[q_N] + 0 + \frac{1}{q} \left[E[q_N] - q \right]$

$= \frac{1}{q^2} \frac{q(1-q)}{N} + \frac{1}{q} \left[q - q \right]$

$= \frac{(1-q)}{qN}$

d) $0.01 > \frac{1-q}{qN}$

$N > \frac{1-q}{q(0.01)}$