~~Exc~~ Sheet 04 ML2 Ex ~~01~~

(a) $$\sum_{ij} c_i c_j k(x_i, x_j) \geq 0$$

$$\sum_{ij} c_i c_j (x_i * x_j)_t (x_j * x_i)_t$$

$$\boxed{\begin{aligned}&(x_i * x_j) = (x_j * x_i)\\ &\Rightarrow (x_i * x_j)^2 = (x_i * x_j)(x_j * x_i)\end{aligned}}$$

$$= \sum_{ij} c_{ij} \sum_{t\tau\tau'} x_i(\tau) x_j(t-\tau)\ x_j(\tau')\ x_i(t-\tau')$$

mit $s = t - \tau - \tau' \Leftrightarrow t = s + \tau + \tau'$

folgt:

$$\sum_{ij} c_i c_j \sum_{s\tau\tau'} x_i(\tau)\ x_j(s+\tau')\ x_j(\tau')\ x_i(t+\tau)$$

$$\sum_s \underbrace{\left(\sum_{i\tau} x_i(\tau)\ x_i(t+\tau)\right)}_{(1)} \underbrace{\left(\sum_{j\tau'} x_j(\tau')\ x_j(t+\tau')\right)}_{(2)}$$

Since (1) and (2) are exactly the same except for different variable names, $(1)\cdot(2)$ can be replaced by $(1)^2$

$$\sum_s \underbrace{\left(\sum_{i\tau} x_i(\tau)\ x_j(t+\tau)\right)^2}_{\geq 0} \geq 0 \quad \square$$

NL2

Sheet 04 Ex 1

(b) $k(x,x') = \sum_s \underbrace{\sum_\tau x(\tau)\, x(s+\tau)}\, \sum_{\tau'} x'(\tau')\, x'(t-\tau')$

$= \sum_s (x \star x)_s \quad (x' \star x')_s$

$= \langle \underbrace{x \star x}_{\phi(x)}, x' \star x' \rangle$

Ex 2

(a) $\sum_{ij} c_i c_j k(x_i, x_j) \geq 0$

$\sum_{ij} c_i c_j \sum_m \beta_m \sum_\ell I(u_{\ell m}(x_i) = u_{\ell m}(x_j)) \cdot \underbrace{\left(\sum_{s \in A^m} I(u_{\ell m}(x_i) = s)\right)}_{=1}$

rearange:

$= \sum_{ij} c_i c_j \sum_{m\ell} \beta_m \sum_{s \in A^m} I(u_{\ell m}(x_i) = s)\, I(s = u_{\ell m}(x_j))$

$u_{\ell m}(x_i)$ can be replaced by s since the first indicator function ~~is~~ always zero for $u_{\ell m}(x_i) \neq s$

$= \sum_{m\ell s} \beta_m \underbrace{\left(\sum_i c_i I(u_{\ell m}(x_i = s))\right)^2}_{\geq 0} \cdot \left(\sum_j c_j I(u_{\ell m}(x_j = s))\right)$

Same procedure as in (a)

and with $\beta_m \geq 0\ \forall m$ the entire expression is ≥ 0 and therefore pos. semi-definite.

ML2 ~~E~~ ESheet04
Ex2(b)

Use representation ~~der of~~ $k(x,x')$ derived intermediary ~~in~~ in (a) and simplify for special case $M = 1$

$$k(x,x') = \sum_{l} \beta \sum_{s\in A} I(u_l(x)=s)\, I(u_l(x')=s)$$

$$= \sum_{l s \in A} \sqrt{\beta}\, I(u_l(x)=s)\ \sqrt{\beta}\,(u_l(x')=s)$$

$$= \langle \underbrace{\sqrt{\beta}\left(I(u_l(x)=s)\right)_{ls}}_{\phi(x)\in \mathbb{R}^{L\times q}},\ \sqrt{\beta}\left(I(u_l(x')=s)\right)_{ls} \rangle$$

$$\begin{pmatrix} |l| = L \\ |A| = q \end{pmatrix}$$

$(\ \ )_{ls}$ is a vector with one summand of the $\sum_{ls}$ sum at each entry. ~~since there are~~

Ex3 (a)

$$\log p_\mu(x) = cst. - \frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)$$

$$G_x = \frac{\partial}{\partial \mu} = \Sigma^{-1}(x-\mu)$$

$$k(x,x') = G_x^T\, E_z[G_z G_z^T]^{-1} G_{x'}$$

$$= (x-\mu)^T \Sigma^{-1}\left(E_z\left[\Sigma^{-1}(z-\mu)(z-\mu)^T\Sigma^{-1}\right]\right)^{-1}\Sigma^{-1}(x'-\mu)$$

$$\underbrace{\left(\Sigma^{-1}\underbrace{E_z[(z-\mu)(z-\mu)^T]}_{\Sigma}\Sigma^{-1}\right)^{-1}}_{\Sigma}$$

$$= (x-\mu)^T \Sigma^{-1}(x'-\mu)$$

Ex3 (b)

$$k(x,x') = (x-\mu)^T L L^T (x'-\mu)$$
$$= \left(L^T(x-\mu)\right)^T \cdot \left(L^T(x'-\mu)\right)$$
$$= \langle \underbrace{L^T(x-\mu)}_{\phi(x)}, L^T(x'-\mu) \rangle$$