Machine Learning 2

Exercise Sheet 3

Exercise 1: Maximum Entropy Distributions

(a)

(b)

Exercise 2: Independent Components in Two Dimensions

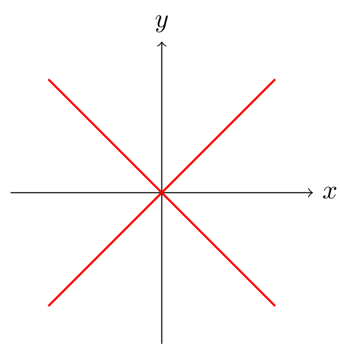
(a)

$$\omega = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\|\omega\|^2 = \cos^2\theta + \sin^2\theta = 1.$$

$$\int \delta(x)\mathrm{d}x = 1$$

$$\int y\delta(x-y)\mathrm{d}x = x$$



(b) We have

$$\begin{aligned}
\mathrm{E}\left[z\right] &= \int\int z(x,y)\cdot p(x,y)\,\mathrm{d}x\mathrm{d}y \\
&= \int\int (x\cos\theta + y\cos\theta)\cdot p(x)\left(\frac{1}{2}\delta(y-x) + \frac{1}{2}\delta(y+x)\right)\mathrm{d}x\mathrm{d}y \\
&= \int \left(\frac{1}{2}\left(x\cos\theta + x\sin\theta\right) + \frac{1}{2}\left(x\cos\theta - x\sin\theta\right)\right) p(x)\,\mathrm{d}x \\
&= \left(\frac{\cos\theta+\sin\theta}{2} + \frac{\cos\theta-\sin\theta}{2}\right)\underbrace{\int x\,p(x)\,\mathrm{d}x}_{\mathrm{E}[x]=0} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\left[z^2\right] &= \int \left(\frac{1}{2}\left(x\cos\theta + x\sin\theta\right)^2 + \frac{1}{2}\left(x\cos\theta - x\sin\theta\right)^2\right) \, p(x)\,\mathrm{d}x \\
&= \left(\frac{(\cos\theta+\sin\theta)^2}{2} + \frac{(\cos\theta-\sin\theta)^2}{2}\right) \underbrace{\int x^2 \, p(x)\,\mathrm{d}x}_{\mathrm{E}[x^2]=\mathrm{Var}[x]=1} \\
&= \cos^2\theta + \sin^2\theta \\
&= 1.
\end{aligned}$$

Since the variance is the same in any direction, there are no principal components.

(c)

$$\begin{aligned}
\mathrm{E}\left[z^4\right] - 3 &= \left(\frac{(\cos\theta+\sin\theta)^4}{2} + \frac{(\cos\theta+\sin\theta)^4}{2}\right) \underbrace{\int x^4 \, p(x)\,\mathrm{d}x}_{\mathrm{E}[x^4]=3} -3 \\
&= \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta + 4\cos^2\theta\sin^2\theta \\
&= \left(\cos^2\theta + \sin^2\theta\right)^2 + 4\left(\cos\theta\sin\theta\right)^2 \\
&= 1 + \left(\sin(2\theta)\right)^2 \\
&= \frac{1-\cos(4\theta)}{2}
\end{aligned}$$

Exercise 3: Deriving a Special Case of FastICA

(a)

(b)

(c)

(d)