Sheet 3 Ex 1

a)

$$\mathcal{L}(s(x), \lambda_1, \lambda_2, \lambda_3) = -\sum_{x\in G} \exp(s(x)) \log(\exp(s(x)))\, \delta$$
$$+ \lambda_1 \left(\sum_{x\in G} \exp(s(x))\, \delta - 1 \right)$$
$$+ \lambda_2 \left(\sum_{x\in G} x \exp(s(x))\, \delta - 0 \right)$$
$$+ \lambda_3 \left(\sum_{x\in G} x^2 \exp(s(x))\, \delta - \sigma^2 \right)$$

b)

$$\frac{\partial \mathcal{L}}{\partial s(x)} = -\cancel{\exp(s(x))} (1 + s(x)) \cancel{\delta}$$
$$+ \lambda_1 \cancel{\exp(s(x))\, \delta}$$
$$+ \lambda_2\, x \cancel{\exp(s(x))\, \delta}$$
$$+ \lambda_3\, x^2 \cancel{\exp(s(x))\, \delta}$$

$$= -1 - s(x) + \lambda_1 + \lambda_2 x + \lambda_3 x^2 \overset{!}{=} 0$$

$$\Leftrightarrow \quad s(x) = \lambda_1 + \lambda_2 x + \lambda_3 x^2 - 1$$

$p(x) = \exp(s(x))$

$$\Rightarrow \quad p(x) = \exp\left(\lambda_3 x^2 + \lambda_2 x + \lambda_1 - 1 \right)$$

Gaussian:

$$p(x) = \exp\left(-\frac{1}{2\sigma^2} x^2 + \frac{1}{2\sigma^2} x \mu - \frac{1}{2\sigma^2} \mu^2 + C \right)$$

Sheet 3

Ex 2

a) See graphs in programmatic solution

b) $$E_{xy}[z] = E[x\cos(\theta) + y\sin(\theta)]$$

$$= \cos\theta \underbrace{E[x]}_{=0} + \sin(\theta) E[y]$$

$$= \sin(\theta) E_x[\underbrace{E_{y|x}[y]}_{0}]$$

$$= 0$$

$$Var(z) = E[(z - \underbrace{E[z]}_{=0})^2] = E[z^2]$$

$$= E[(x\cos(\theta) + y\sin(\theta))^2]$$

$$= E[x^2\cos^2(\theta) + 2x\cos(\theta)y\sin(\theta) + y^2\sin^2(\theta)]$$

(1) $$= \cos^2(\theta)\underbrace{E[x^2]}_{=1} + 2\cos(\theta)\sin(\theta)E[xy] + \sin^2(\theta)E[y^2]$$

$$E_{xy}[xy] = E_x[E_{y|x}[xy]] = E_x[x \underbrace{E_{y|x}[y]}_{=0}] = 0 \quad (2)$$

$$E[y^2] = \underbrace{E_x[\underbrace{E_{y|x}[y^2]}_{=x^2}]}_{=1} \quad (3)$$

(2), (3) in (1)

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Variance is always 1, independent of θ.

Sheet 3 Ex 2 c)

c) $\mathrm{kurt}(z) = E[z^4] - 3$

$$= E\Big[x^4 \cos^4\theta + 4x^3 y \cos^3\theta \sin(\theta) + 6x^2y^2\cos^2\theta\sin^2\theta + 4xy^3\cos(\theta)\sin^3\theta + y^4\sin^4(\theta)\Big] - 3$$

$$= \cos^4\theta \; E[x^4]$$
$$+ 4\cos^3\theta\sin\theta \; E_x\Big[x^3 \underbrace{E_{y|x}[y]}_{=0}\Big]$$
$$+ 6\cos^2\theta\sin^2\theta \; \underbrace{E_x\Big[x^2 \underbrace{E_{y|x}[y^2]}\Big]}_{1}$$
$$+ 4\cos(\theta)\sin(\theta) \; \underbrace{E_x\Big[x \, E_{y|x}[y^3]\Big]}_{=0}$$
$$+ \sin^4(\theta) \underbrace{E[y^4]}_{E[x^4]}$$

$$= E[x^4]\cdot\big(\cos^4(\theta) + \underbrace{6}_{2+4}\cos^2\theta\sin^2\theta + \sin^4(\theta)\big) - 3$$

$\cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta = (\cos^2\theta + \sin^2\theta)^2 = 1$

$$= E[x^4]\cdot\big(1 + 4\cos^2\theta\sin^2\theta\big) - 3 = E[x^4]\big(1 + (2\cos\theta\sin\theta)^2\big) - 3$$

$$= E[x^4]\big(1 + \sin^2(2\theta)\big) - 3 = E[x^4]\Big(1 + \frac{1-\cos(4\theta)}{2}\Big) - 3$$

$$= \frac{1}{2} E[x^4]\big(3 - \cos(4\theta)\big) - 3$$

Sheet 3 Ex 3

(a) $$E[z] = E[w^T x] = w^T \underbrace{E[x]}_{=0} = 0$$

$$Var(z) = E[z^2] = E[(w^T x)^2] = E[w^T x x^T w]$$

$$= w^T \underbrace{E[x x^T]}_{I} w = w^T \cancel{I} w = \underbrace{\|w\|^2}_{=1} = 1$$

(b) $$\cancel{\mathcal{L} = \frac{E[(w^T x - E[w^T x])^4]}{\underbrace{Var(w^T x)^2}_{=0\,1}} - 3 = E[(w^T x)^4] - 3}$$

$$\mathcal{L} = \frac{E[(w^T x - E[w^T x])^4]}{\underbrace{Var(w^T x)^2}_{=1}} + \lambda \cdot 2 \cdot (1 - \|w\|^2) = E[(w^T x)^4] + \lambda \cdot 2 \cdot (1 - \|w\|^2)$$

$$\frac{\partial \mathcal{L}}{\partial w} = E[4 (w^T x)^3 x] - \lambda 2 \cdot 2 \cdot w \overset{!}{=} 0$$

$$\Leftrightarrow \cancel{4} E[x (w^T x)^3] = \lambda w \cancel{4}$$

(c) $$F(w) = E[x (w^T x)^3] - \lambda w = 0$$

$$J(w) = E[3 x x^T (w^T x)^2] - \lambda I$$

$$w^+ = w - (J(w))^{-1} \cdot F(w)$$

(d) $$E[3 x x^T (w^T x)^2] - \lambda I = 3 \underbrace{E[x x^T]}_{=I} \cdot \underbrace{E[(w^T x)^2]}_{=1} - \lambda I = (3 - \lambda) I$$

$$w^+ = w - \frac{1}{3-\lambda} I \cdot (E[x (w^T x)^3] - \lambda w) \Leftrightarrow (3 - \lambda) w^+ = 3w - \cancel{\lambda w} - E[x (w^T x)^3] \text{ \cancel{thw}}$$

$$\Leftrightarrow -(3 - \lambda) w^+ = E[x (w^T x)^3] - 3w$$

$$w^+ = \underbrace{-\frac{1}{3-\lambda}}_{\gamma} \cdot E[x (w^T x)^3] - 3w$$