Macro - 7020: TA Session 2

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Taylor Expansion

- Taylor's theorem tells us the following

$$f(x_t) = f(x) + f'(x)(x_t - x) + \frac{f^{(2)}(x)}{2!}(x_t - x)^2 + \frac{f^{(3)}(x)}{3!}(x_t - x)^3 + \dots$$

  where the expansion is considered "at x"
  - usually we expand around the steady state (x^*) in the context of Macro
- For smooth functions, the magnitude of the terms dissipates quickly with n
- So the bumbling idiots in economics usually feel they can simply write

$$f(x_t) = f(x) + f'(x)(x_t - x) \; \underline{\textbf{and}} \; f(x_t, y_t) = f(x, y) + f_x(x, y)(x_t - x) + f_y(x, y)(y_t - y)$$

  where equality is imposed <u>but it's really an approximation</u>.

Log(-)Linearization

- The usual definition for a (log) linearized variable is $\widehat{x}_t = \frac{x_t - x}{x}$
  - Think about this as relative deviation from the steady state.
  - Value: non-relative numbers are arbitrary. Also, cycles and shocks.
- First order Taylor expansion about $\widehat{x}_t = 0$

$$\ln(1+\widehat{x}_t) \approx \ln(1) + \left(\frac{\mathrm{d}}{\mathrm{d}\widehat{x}_t} \ln(1+\widehat{x}_t) \Big|_{\widehat{x}_t=0} \right) (\widehat{x}_t - 0) = \left(\frac{1}{1+\widehat{x}_t} \Big|_{\widehat{x}_t=0} \right) \widehat{x}_t = \widehat{x}_t$$

  "valid" since we consider $\widehat{x}_t$ to be small in magnitude

- So now consider the following property, which is extremely useful

$$\widehat{x}_t \approx \ln(1+\widehat{x}_t) = \ln\left(1 + \frac{x_t - x}{x}\right) = \ln\left(\frac{x_t}{x}\right) = \ln(x_t) - \ln(x)$$

Still Log-Linearizing

- Now we have $\widehat{x}_t \approx \ln(x_t) - \ln(x)$. So for instance

$$y_t = x_t z_t \implies \widehat{y}_t \approx \ln(y_t) - \ln(y) = \big(\ln(x_t) + \ln(z_t)\big) - \big(\ln(x) - \ln(z)\big) \approx \widehat{x}_t + \widehat{z}_t$$

  This gives us the first in several rules

  - $y_t = x_t z_t \implies \widehat{y}_t = \widehat{x}_t + \widehat{z}_t$ *product rule*
  - $y_t = x_t^\alpha \implies \widehat{y}_t = \alpha \widehat{x}_t$ *power rule*
  - $y_t = f(x_t) \implies \widehat{y}_t = \left[\frac{f'(x)}{f(x)} x\right] \widehat{x}_t$ *function rule*
  - $y_t = x_t + z_t \implies y\widehat{y}_t = x\widehat{x}_t + z\widehat{z}_t$ *sum rule*
- These are: incredibly useful, all you need, and (mostly) transparent
  - Implicit, (mathematically) trivial, but important rule: linearized constant = 0
  - Rule 3 Proof Appendix

The Brutal Truth about LL..

- Taylor series expansions in econ are ***not*** about mathematical precision
  - Jensen's inequality? Never heard of her
- Write equality signs. Just do it.
  - The last slide I went from $\approx$ to $=$ when writing "rules". I'm never going back
  - For this year, don't worry. But for future: "does this matter?" is good for research
- To add onto the ingrained grainyness..
  - "black box": people say log-linearize and magically a solution appears
  - Because there's lots of messy math behind the scenes, can be hard to implement
- I have been taught a ton of different ways to do this over the years. Here are in my opinion the best two: rule based and brute force

Method 1: Rule Based (or The Method of **Big Hat™**)

- Intuitive and step by step: just apply the rules over and over
  - $y_t = x_t z_t \implies \widehat{y}_t = \widehat{x}_t + \widehat{z}_t$
  - $y_t = x_t^\alpha \implies \widehat{y}_t = \alpha \widehat{x}_t$
  - $y_t = f(x_t) \implies \widehat{y}_t = \left[\frac{f'(x)}{f(x)} x\right] \widehat{x}_t$
  - $y_t = x_t + z_t \implies y \widehat{y}_t = x \widehat{x}_t + z \widehat{z}_t$
- *Example:* Consider $k_{t+1} = (1-\delta)k_t + sA_t k_t^\alpha$. This means

$$\begin{aligned}
\widehat{k}_{t+1} &= \frac{(1-\delta)k}{k} \widehat{(1-\delta)k_t} + \frac{sAk^\alpha}{k} \widehat{sA_t k_t^\alpha} && \textbf{Rule 4} \\
&= (1-\delta)[\widehat{(1-\delta)} + \widehat{k}_t] + sAk^{\alpha-1}(\widehat{s} + \widehat{A}_t + \widehat{k_t^\alpha}) && \textbf{Rule 1} \\
&= (1-\delta)\widehat{k}_t + sAk^{\alpha-1}(\widehat{A}_t + \alpha \widehat{k}_t) && \textbf{Rule 2}
\end{aligned}$$

- key is treating entire term as one linearized variable and then "shrinking the hat"

Method 2: Brute Force

- No intuition, just compute a formula

$$\boxed{0 = f(x_t, y_t, z_t) \Longrightarrow 0 = f_x(x,y,z)x\widehat{x}_t + f_y(x,y,z)y\widehat{y}_t + f_z(x,y,z)z\widehat{z}_t}$$

- Breaking this down: set everything equal to 0. Call this expression $f(\dots)$
  - Say you have k variables $\{x_{i,t}\}_{i=1}^k$
  - Log-linearizing is

$$0 = \sum_{i=1}^k f_{x_i}(\text{ss}) \cdot x_i \widehat{x}_{i,t}$$

  - Each term: partial derivative of f w.r.t x_i evaluated at the steady states $(x_1, \dots, x_k)$ multiplied by $x_i \cdot \widehat{x}_{i,t}$ (steady state of x_i times linearized x_i)
- Returning to our $k_{t+1} = (1-\delta)k_t + sA_t k_t^\alpha$ example (so $0 = -k_{t+1} + (1-\delta)k_t + sA_t k_t^\alpha$)

$$0 = -1 \times k\widehat{k}_{t+1} + \left(1 - \delta + \alpha s A k^{\alpha-1}\right) \times k\widehat{k}_t + sk^\alpha \times A\widehat{A}_t$$

$$\Longrightarrow \widehat{k}_t = (1-\delta)\widehat{k}_t + sAk^{\alpha-1}(\widehat{A}_t + \alpha\widehat{k}_t)$$

Which method do I use?

- Try on your own and see what your brain likes best
- For most people, brute force will be best (cleaner)
- But always remember the rules! Brute force can be cumbersome in simple cases
- Say you want to linearize δx_t
  - Rules-based immediately gives you $\widehat{x}_t$ (just use product rule)
    - Remember: "hat" treats all objects equally. Can't think about constants until it's isolated
  - Brute force only really makes sense with an equation. So you have to redefine $y_t = \delta x_t$

$$\implies 0 = -y\widehat{y}_t + \delta x\widehat{x}_t$$

    and then you have to realize/recognize the steady state of y_t is δx
  - "Realizing" is often an essential simplifying step and source of struggle with LL

Practice!

- For reference, recall: $0 = \sum_{i=1}^{k} f_{x_i}(\text{ss}) \cdot x_i \widehat{x}_{i,t}$
  - $y_t = x_t z_t \implies \widehat{y}_t = \widehat{x}_t + \widehat{z}_t$
  - $y_t = x_t^{\alpha} \implies \widehat{y}_t = \alpha \widehat{x}_t$
  - $y_t = f(x_t) \implies \widehat{y}_t = \left[\frac{f'(x)}{f(x)} x\right] \widehat{x}_t$
  - $y_t = x_t + z_t \implies y\widehat{y}_t = x\widehat{x}_t + z\widehat{z}_t$

1. $y_t = -x_t$
2. $y_t = (x_t + \beta z_t)^{\alpha}$
3. $c_t + k_{t+1} - (1-\delta)k_t = A_t k_t^{\alpha} \ell_t^{1-\alpha}$
4. How does #3 simplify if you know steady states $c = \gamma k$ and $A, \ell = 1$
5. $c_{t+1} = \beta \left[c_t \left(\alpha A_t k_{t+1}^{\alpha - 1} + 1 - \delta \right) \right]$

Proof of Rule 3

▸ Back to Rules

- A simple way to see this is

$$\widehat{y}_t \approx \ln(y_t) - \ln(y) = \ln(f(x_t)) - \ln(f(x)) \approx \left[\ln(f(x)) + \frac{f'(x)}{f(x)}(x_t - x)\right] - \ln(f(x))$$
$$= \frac{f'(x)}{f(x)}(x_t - x) = \frac{xf'(x)}{f(x)}\widehat{x}_t$$

- We can also consider $f(x_t) = \frac{g(x_t)}{h(x_t)} \implies \ln(f(x_t)) = \ln(g(x_t)) - \ln(h(x_t))$. So

$$\ln(f(x_t)) = \ln(f(x)) + \frac{f'(x)}{f(x)}(x_t - x) \quad \ln(g(x_t)) = \ln(g(x)) + \frac{g'(x)}{g(x)}(x_t - x)$$
$$\ln(h(x_t)) = \ln(h(x)) + \frac{h'(x)}{h(x)}(x_t - x)$$

- Combing these taylor expansions with $\ln(f(x_t)) = \ln(g(x_t)) - \ln(h(x_t))$ yields

$$\frac{f'(x)}{f(x)}(x_t - x) = \frac{g'(x)}{g(x)}(x_t - x) - \frac{h'(x)}{h(x)}(x_t - x) \implies \frac{xf'(x)}{f(x)}\widehat{x}_t = \frac{xg'(x)}{g(x)}\widehat{x}_t - \frac{xh'(x)}{h(x)}\widehat{x}_t$$