# Chapter 5 Cooperative Communications with Multiple Relays

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# Scenario

$N$ relays use the two-phase protocol to cooperate.
**Phase I** : $S$ broadcasts signal $s(t)$.
The received signal at $R_n$ is

$$r_n(t) = a_{1,n}s(t) + w_{1,n}(t)$$

**Phase II** : A set of *cooperating relays* $\mathcal{D}$ forward. The received signal at $D$ from relay $R_n$ is

$$r_{D,n}(t) = a_{2,n}s(t) + w_{2,n}(t)$$

- Key to enable diversity gain: relays transmit over orthogonal channels.
- Which set of relays should be $\mathcal{D}$?

Start from the AF case in the following.

# Orthogonal Cooperation with AF Relays

Setting

- All relays forward using VAF at different time slots.
- $D$ combines signals received in $N+1$ slots using MRC.

Effective SNR at the output of MRC

$$\gamma_c = \gamma_0 + \sum_{n=1}^{N} \frac{\gamma_{1,n} \cdot \gamma_{2,n}}{\gamma_{1,n} + \gamma_{2,n} + 1}$$

where $\gamma_0 = P_S a_{S,D}^2 / N_0$, $\gamma_{1,n} = P_S a_{1,n}^2 / N_0$, and $\gamma_{2,n} = P_{R,n} a_{2,n}^2 / N_0$.

E2e capacity: $C^{\mathsf{VAF}} = \frac{1}{N+1} \log_2 (1 + \gamma_c)$.

# Outage Probability

Given the required transmission rate $\mathtt{R}$, the outage probability is

$$\mathbb{P}_{\mathsf{out}} = \mathbb{P}[C^{\mathsf{VAF}} < \mathtt{R}] = \mathbb{P}\left[\frac{1}{N+1}\log_2\left(1+\gamma_c\right) < \mathtt{R}\right]$$
$$= \mathbb{P}\left[\gamma_0 + \sum_{n=1}^{N}\frac{\gamma_{1,n}\cdot\gamma_{2,n}}{\gamma_{1,n}+\gamma_{2,n}+1} < 2^{\mathtt{R}(N+1)} - 1\right]. \qquad (1)$$

(1) involves $N+1$-fold integral!

How to simplify? $\Longrightarrow$ replace the summation by the maximum.

Let $\mathtt{T} = 2^{\mathtt{R}(N+1)} - 1$,

$$\mathbb{P}_{\text{out}} \le \mathbb{P}[\gamma_0 + \gamma_{\max} < \mathtt{T}]. \tag{2}$$

This is equivalent to the case with a single AF relay and the direct link (Ch. 4, p. 19)

$$\begin{aligned}\mathbb{P}_{\text{out}} &\le \int_0^{\mathtt{T}} \mathbb{P}[\gamma_{\max} < \mathtt{T} - x] f_{\gamma_0}(x) dx \\ &= \int_0^{\mathtt{T}} \prod_{n=1}^{N} \mathbb{P}\left[\frac{\gamma_{1,n} \cdot \gamma_{2,n}}{\gamma_{1,n} + \gamma_{2,n} + 1} < \mathtt{T} - x\right] \frac{1}{\bar{\gamma}_0} e^{-x/\bar{\gamma}_0} dx.\end{aligned}$$

Using the high-SNR approximation (Ch. 4, p. 23), one can obtain

$$\mathbb{P}_{\text{out}} \le \frac{\mathtt{T}^{N+1}}{N+1} \frac{1}{\bar{\gamma}_0} \prod_{n=1}^{N} \left(\frac{1}{\bar{\gamma}_{1,n}} + \frac{1}{\bar{\gamma}_{2,n}}\right) \tag{3}$$

# Asymptotic Analysis

Formally, diversity order $d$ is defined as

### Diversity Order

$$d = -\lim_{\mathsf{SNR}\to\infty} \frac{\log \mathbb{P}_{\mathsf{out}}}{\log \mathsf{SNR}}$$

Assuming $P_S = P_{R_n} = P$, (3) can be bounded as

$$\mathbb{P}_{\mathsf{out}} \leq \frac{1}{N+1}\left(\frac{\mathtt{T}}{\mathsf{SNR}}\right)^{N+1} \frac{1}{\sigma_0^2} \prod_{n=1}^{N} \left(\frac{1}{\sigma_{1,n}^2} + \frac{1}{\sigma_{2,n}^2}\right) \tag{4}$$

### Conclusion

With $N$ relays and the existence of direct link, VAF achieves diversity order of $N+1$.

# Orthogonal Cooperation with DF Relays

Scenarios

- Among $N$ relays, only those successfully decoding relays forward using DF at different time slots.
- Suppose $N+1$ slots are used to complete a cooperation round,

$$\mathcal{D} = \left\{ R_i : \ \frac{1}{N+1} \log_2(1+\gamma_{1,n}) \geq \mathtt{R}. \right\} \tag{5}$$

- $D$ combines signals received from $S$ and relays in $\mathcal{D}$ using MRC.

Effective SNR at the output of MRC

$$\gamma_c = \gamma_0 + \sum_{R_i \in \mathcal{D}} \gamma_{2,i}$$

where $\gamma_0 = P_S a_{S,D}^2 / N_0$ and $\gamma_{2,i} = P_{R,i} a_{2,i}^2 / N_0$.

E2e capacity: $C^{\mathsf{DF}} = \frac{1}{N+1} \log_2 (1+\gamma_c)$.

# Power Allocation

Given the total relay power constraint $\sum_{R_i \in \mathcal{D}} P_{R_i} \leq P$, optimizing the received power at $D$.

$$\begin{aligned} \max_{P_{R_i}} \quad & \sum_{i \in \mathcal{D}} P_{R_i} \frac{a_{2,i}^2}{N_0} \\ \text{subject to} \quad & \sum_{i \in \mathcal{D}} P_{R_i} \leq P \text{ and } P_{R_i} \geq 0, \forall i \in \mathcal{D}. \end{aligned}$$

The optimal solution:

$$P_{R_n} = \begin{cases} P, & a_{2,n}^2 \geq a_{2,n'}^2, \ \forall n', \\ 0, & \text{otherwise.} \end{cases}$$

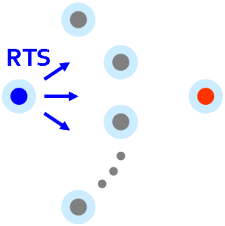
- Using a single relay with the best relay-destination channel to forward.
- Only needing two slots $\Rightarrow$ more bandwidth efficient.

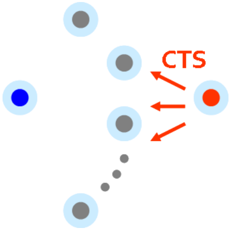
Properly choosing cooperative relays can avoid waste of bandwidth.

- Since only one relay is used, spectral efficiency is greatly improved.
- Need a certain "communication" between nodes.

Opportunistic relaying (OR): Select the "best" as the one with the strongest end-to-end path between $S$ and $D$ [1].

Step 1: Before data transmission, $S$ sends a ready-to-send (RTS) packet followed by a clear-to-send (CTS) packet sent by $D$.



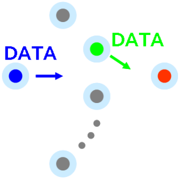
Suppose there are $N$ relays available to help.

Step 2: $R_i$ estimates $\gamma_{1,i}$ based on RTS and $\gamma_{2,i}$ based on CTS.

Step 3: Each relay maintains a backoff timer $T_i = \frac{1}{\Gamma_i}$, where

$$\Gamma_i = \min\{\gamma_{1,i}, \gamma_{2,i}\}. \tag{6}$$

Each relay counts down the timer $T_i$ and transmits.




Step 4: Whenever a relay transmits, other relays sense the medium being busy and stop to count down.

Assumptions

- No direct link.
- Relays perform DF.
- I.I.D. Rayleigh fading channels.

**Cooperating set**: $\mathcal{D} = \{R_i : \frac{1}{2}\log_2(1+\gamma_{1,R_i}) \geq \mathtt{R}\}$

If $R^*$ is the best relay. The end-to-end capacity of using this relay is

$$\mathtt{C} = \frac{1}{2}\log_2(1+\Gamma_{R^*})$$

The outage probability can be computed as

$$\mathbb{P}_{\text{out}} = \Pr[\mathtt{C} < \mathtt{R}] = \Pr[\Gamma_{R^*} < \mathtt{T}]$$

where $\mathtt{T} = 2^{2\mathtt{R}} - 1$.

# Analysis of OR (cont'd)

$$\mathbb{P}_{\text{out}} = \Pr[\max_{R_i \in \mathcal{D}} \Gamma_i < \mathtt{T}] \quad \text{(Law of Total Probability)}$$

$$= \sum_{n=1}^{N} \Pr[\max_{i \in \mathcal{D}} \Gamma_i < \mathtt{T} \Big| |\mathcal{D}| = n] \Pr[|\mathcal{D}| = n] + \Pr[\mathcal{D} = \emptyset]$$

For $|\mathcal{D}| = n > 0$,

$$\Pr[\max_{R_i \in \mathcal{D}, |\mathcal{D}|=n} \min(\gamma_{1,i}, \gamma_{2,i}) < \mathtt{T} \Big| |\mathcal{D}| = n] = \prod_{i=1}^{n} \Pr[\min(\gamma_{1,i}, \gamma_{2,i}) < \mathtt{T}]$$

$$= \prod_{i=1}^{n} \Big(1 - \Pr[\min(\gamma_{1,i}, \gamma_{2,i}) \geq \mathtt{T}]\Big)$$

$$= \prod_{i=1}^{n} \Big(1 - \mathbb{P}[\gamma_{1,i} \geq \mathtt{T}] \Pr[\gamma_{2,i} \geq \mathtt{T}]\Big) = \prod_{i=1}^{n} \Big(1 - e^{-(\frac{1}{\bar{\gamma}_{1,i}} + \frac{1}{\bar{\gamma}_{2,i}})\mathtt{T}}\Big)$$

# Analysis of OR (cont'd)

At high SNR, $|\mathcal{D}| \to N$ and using the first-order Taylor expansion (Ch. 4, p. 32) yields

$$\mathbb{P}_{\text{out}} \approx \prod_{i=1}^{N} \left[ \left( \frac{1}{\bar{\gamma}_{1,i}} + \frac{1}{\bar{\gamma}_{2,i}} \right) \mathtt{T} \right] = \left( \frac{\mathtt{T}}{\mathtt{SNR}} \right)^{N} \prod_{i=1}^{N} \left( \frac{1}{\sigma_{1,i}^2} + \frac{1}{\sigma_{2,i}^2} \right). \tag{7}$$

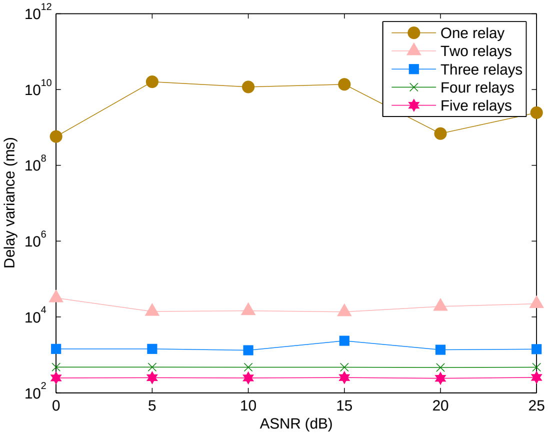
Assume i.i.d. fading for both links, i.e., $\sigma_{1,i}^2 = \sigma_{2,i}^2 = \sigma^2$,

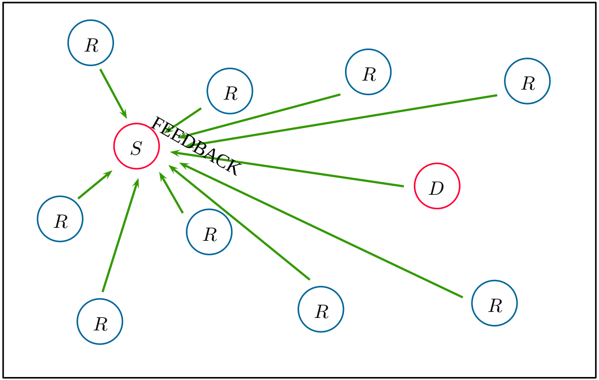
$$\mathbb{P}_{\text{out}} \approx \left( \frac{2\mathtt{T}}{\mathtt{SNR} \cdot \sigma^2} \right)^{N} \tag{8}$$

**Conclusion**

With $N$ DF relays, OR achieves diversity order of $N$.

- Similar protocol as WiFi and thus can be used with the off-the-shelf hardware.
- Does not need each relay to report CSI.
- CSI-based timer results in large delay variances.






Relay selection based on CSI may incur large feedback overhead.

# Partial Relay Selection

Idea: reduce the channel estimation and signaling overheads for relay selection by utilizing the CSI at the $S$-$R$ link only [2].

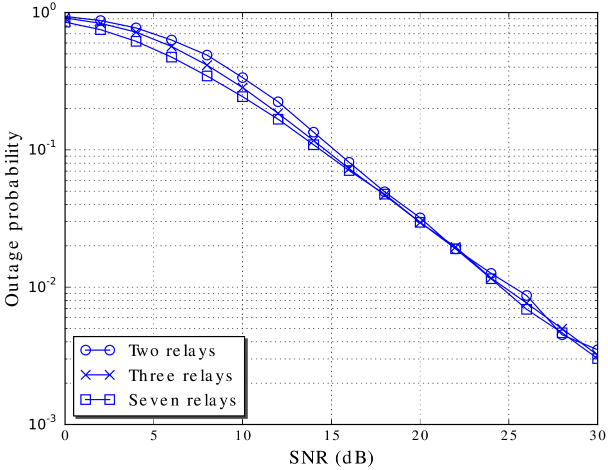
$S$ monitors the quality of its connectivity with the relays via the transmission of local (one-hop) feedback. This scheme is basically a centralized one.

In combination with AF relaying, partial relay selection is very attractive to IoT demanding low complexity and low power consumption.

**Selection rule**

$$R^* = \max_{R_i} \gamma_{1,R_i}$$

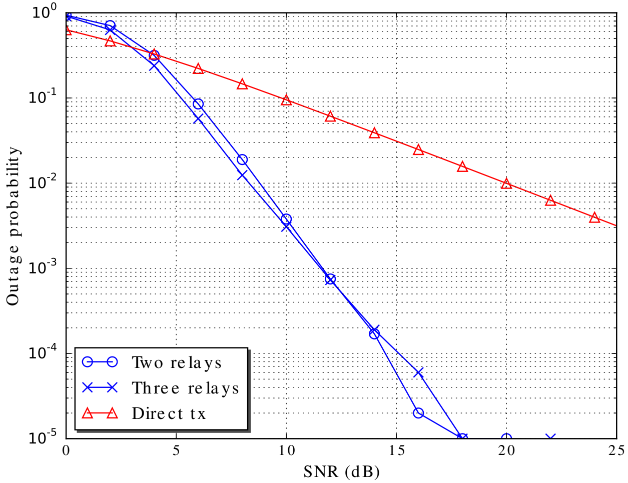
# Outage Probability




$\bar{\gamma}_{1,i} = \bar{\gamma}_{2,i} = \text{SNR}$

- No diversity gain.

# Outage Probability (cont'd)




$(\bar{\gamma}_0, \bar{\gamma}_{1,i}, \bar{\gamma}_{2,i}) = (1, 2, 3) \times \mathsf{SNR}$

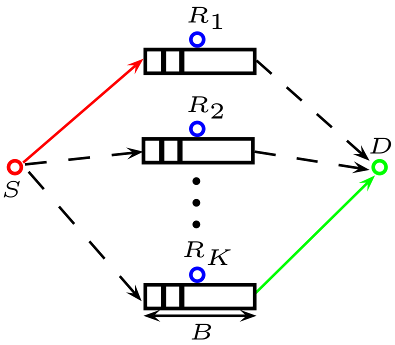
- Significant improvement over direct transmission.

# Buffer-Aided Relay Selection

Recall the key to achieve diversity gain: relays transmit over orthogonal channel $\Rightarrow$ more relays, lower spectral efficiency.

- With relay selection, only two slots are used $\Rightarrow$ higher spectral efficiency than using all relays.
- Channel mismatch: Due to independent fading, the first-hop and the second-hop channels of the selected relay may not be the best at the same time.
- Using the "same" relay for reception and transmission is not optimal.
- When a relay receives the source signal but its relay-destination channel falls in a deep fade, the relay should wait until the channel quality becomes good.

# Max-Max Relay Selection




- Nonbuffer-aided: $R^* = \arg\ \max_{R_i} \{\min\{\gamma_{1,R_i}, \gamma_{2,R_i}\}\}$
- Max-Max Relay Selection (MMRS) [3]:

$$(R_r, R_t) = \arg\Big(\max_{R_i}\ \{\gamma_{1,R_i}\}, \max_{R_i}\ \{\gamma_{2,R_i}\}\Big)$$

For MMRS with DF relays, the analysis of the outage probability is similar to the case of a single DF relay (this relay is selected according to MMRS). Thus the outage probability (refer to Ch. 4, p. 31):

$$\begin{aligned}\mathbb{P}_{\text{out}}^{\text{MMRS}} &= \Pr\left[\tfrac{1}{2}\min\{\log_2(1+\gamma_{R_r}),\log_2(1+\gamma_{R_t})\} < \mathtt{R}\right] \\ &= 1 - \Pr\left[\gamma_{R_r} \geq \mathtt{T}\right]\cdot\Pr\left[\gamma_{R_t} \geq \mathtt{T}\right] \\ &= 1 - \left[1 - \prod_{i=1}^{N}\Big(1-\exp\big(-\tfrac{\mathtt{T}}{\bar{\gamma}_{1,i}}\big)\Big)\right]\cdot\left[1 - \prod_{i=1}^{N}\Big(1-\exp\big(-\tfrac{\mathtt{T}}{\bar{\gamma}_{2,i}}\big)\Big)\right]\end{aligned} \tag{9}$$

where $\mathtt{T} = 2^{2\mathtt{R}} - 1$, $\gamma_{R_r} = \max\limits_{i=1,\cdots,N}\{\gamma_{1,i}\}$ and $\gamma_{R_t} = \max\limits_{i=1,\cdots,N}\{\gamma_{2,i}\}$

# Outage Probability of MMRS (cont'd)

At high SNR, $1 - e^{-x} \approx x$ for $x \to 0$,

$$\mathbb{P}_{\text{out}}^{\text{MMRS}} \approx \left( \frac{1}{\prod_{i=1}^{N} \bar{\gamma}_{1,i}} + \frac{1}{\prod_{i=1}^{N} \bar{\gamma}_{2,i}} \right) \mathtt{T}^N.$$

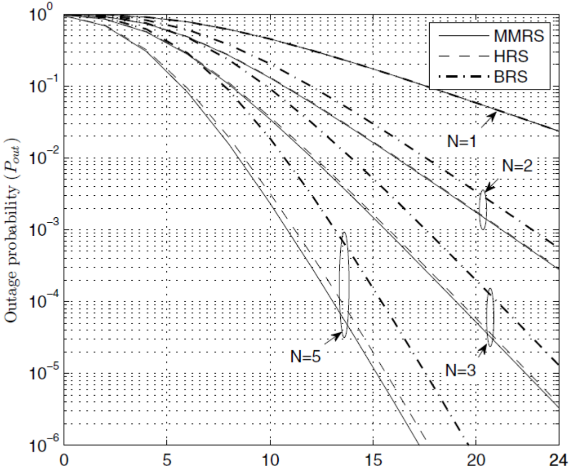
Assume i.i.d. fading for both links, $\bar{\gamma}_{1,i} = \bar{\gamma}_{2,i} = P\sigma^2/N_0$

$$\begin{aligned}
\mathbb{P}_{\text{out}}^{\text{MMRS}} &\approx \left(\tfrac{\mathtt{T}}{\mathsf{SNR}}\right)^N \left(\tfrac{1}{\sigma^{2N}} + \tfrac{1}{\sigma^{2N}}\right) \\
&= \left( \frac{2^{\frac{1}{N}}\mathtt{T}}{\mathtt{SNR} \cdot \sigma^2} \right)^N.
\end{aligned}$$

The outage probability of MMRS scales with $2^{\frac{1}{N}}$. Comparing with the outage probability of OR (see (8)), which scales with $2^1$, MMRS achieves the SNR gain of

$$10 \log_{10}(\mathbb{P}_{\text{out}}^{\text{OR}} / \mathbb{P}_{\text{out}}^{\text{MMRS}}) = 3(1 - 1/N) \text{ dB}$$

# Outage Probability of MMRS (cont'd)




- Both MMRS and OR achieve the diversity gain of $N$.
- The gap between MMRS and OR (SNR gain) increases with $N$.

# Max-Link

With $N$ relays, MMRS selects the reception/transmission relay from $N$ channels $\rightarrow$ diversity gain of $N$.

Actually, there are $2N$ channels available to select, if we don't confine the order of reception/transmission $\rightarrow$ each slot can be used by a relay either for reception or transmission.

Max-Link [4]: choose the relay with the strongest quality among all $2N$ channels.

$$R^* = \arg \max_{R_i} \left( \bigcup_{R_i : B_i \neq L} \{\gamma_{1,R_i}\} \bigcup_{R_i : B_i \neq 0} \{\gamma_{2,R_i}\} \right)$$

where $B_i$ is the buffer length of $R_i$ at the selection time and $L$ is the buffer size.

**Conclusion**

With infinite buffer size, Max-Link achieves diversity order of $2N$.

# Hybrid Relay Selection

Issues with relay buffers

- Inevitable overflow or underflow problem with finite buffer $B < \infty$.
- Need to monitor buffer status in addition to CSI for relay selection

  **If** selected relay sees buffer full or empty
  use the same relay to rx and tx

Let $L_i$ denote the buffer length of relay $R_i$ at a particular slot

**Hybrid Relay Selection (HRS)**

Tx relay: $R_t^* = \begin{cases} R^* & \text{if } L_{R_r^*} = B - 1 \text{ or } L_{B_t^*} = 0 \\ R_t & \text{otherwise} \end{cases}$

Rx relay: $R_r^* = \begin{cases} R^* & \text{if } L_{R^*} = B - 1 \text{ or } L_{B_t^*} = 0 \\ R_r & \text{otherwise} \end{cases}$

Note: $R^*$ is the relay selected according to OR.

# Performance of HRS

HRS switches between MMRS and OR. From the total probability law,

$$\boxed{\mathbb{P}_{\text{out}}^{\text{HRS}} = P_{\text{MMRS}}\mathbb{P}_{\text{out}}^{\text{MMRS}} + P_{\text{OR}}\mathbb{P}_{\text{out}}^{\text{OR}}}$$

where $\mathbb{P}_{\text{out}}^{\text{MMRS}}$ and $\mathbb{P}_{\text{out}}^{\text{OR}}$ have been derived in (9) and (7), respectively, and $P_{\text{MMRS}} = 1 - P_{\text{OR}}$.

From p. 29, whether MMRS and OR is used depends on the buffer status. Let's model the buffer status by a finite-state Markov chain with $N_s$ states.

**State**: $S_i = X_1 X_2 \cdots X_N$ where $X_j$: number of packets in the $j$th buffer. Then

$$P_{\text{OR}} = \sum_{i=1}^{N_s} P_{\text{OR},i} \cdot \pi_i$$

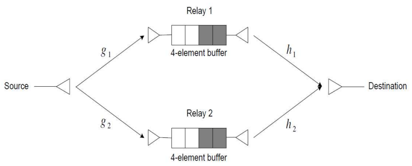
where $P_{\text{OR},i}$: probability of using OR in state $i$ and $\pi_i$: probability being in state $i$.

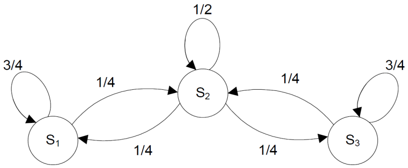
Constraint of each state

- $\sum_{i=1}^{N} X_i = N_e$, where $N_e$: total number of packets of all buffers.
- $0 \le X_i \le B - 1$: leave one space of buffer empty to ensure each reception relay can receive.
- With i.i.d. channels, each state transition has the probability of $1/N^2$.

**Example**: $N = 2$, $B = 4$, each buffer is half full ($N_e = 4$).

$$X_1 + X_2 = 4, X_1 \le 3, X_2 \le 3, \quad \xRightarrow{\text{possible states}} \quad \{13, 22, 31\}$$



# HRS: Example (cont'd)

**Transition probability matrix**: $\mathbf{P} = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/4 & 3/4 \end{bmatrix}$

**Definition (Doubly stochastic matrix)**

*A doubly stochastic matrix is a square matrix for which the sum of the elements in each of its rows and columns is 1.*

**Theorem**

*For a doubly stochastic transition matrix, the stationary distribution is uniform, i.e., all the states are equally likely. For an $N_s$-state Markov chain, the probability of being in state $S_i$, $i = 1, \cdots, N_s$, is $\pi_i = 1/N_s$, regardless of the initial state.*

According to Theorem 1, the steady-state probability $\pi_i = 1/3$ and thus $P_{\mathsf{OR}} = \frac{1}{3}\sum_{i=1}^{N_s} P_{\mathsf{OR},i}$.

To represent the selection result, define an indicator matrix $\mathbf{D} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ where

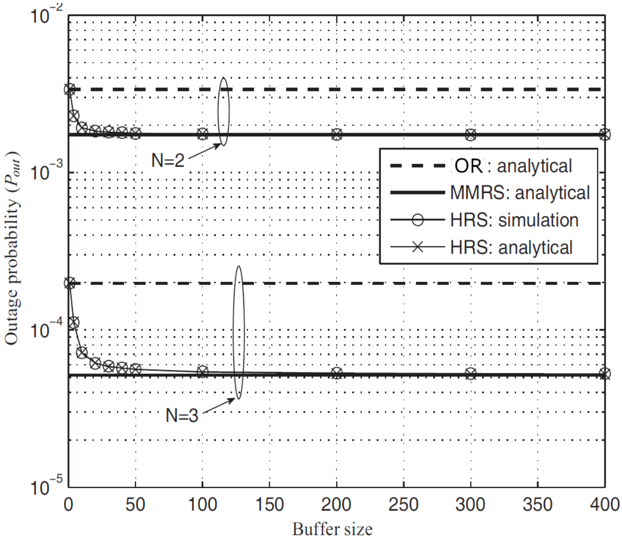
$$a_i = \begin{cases} 1 & R_i \text{ selected for reception} \\ 0 & \text{otherwise} \end{cases} \text{ and } b_i = \begin{cases} 1 & R_i \text{ selected for transmission} \\ 0 & \text{otherwise} \end{cases}$$

Since only one relay is selected at a time, only one element in the column of $\mathbf{D}$ is one. Possible combinations (each happens with probability $1/4$)

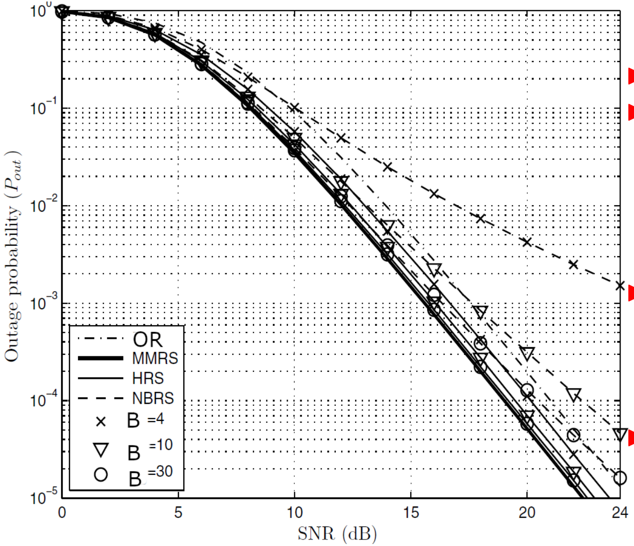
$$\mathbf{D}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{D}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \mathbf{D}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{D}_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- State $S_1 = (1, 3)$: For $\mathbf{D}_1$, $R_1$ selected for reception has a non-full buffer, so MMRS is used. Similarly, $\mathbf{D}_3$ implies MMRS is used. For $\mathbf{D}_2$ and $\mathbf{D}_4$, OR is used with probability $1/4$. In total, $P_{\text{OR},1} = 1/2$.
- State $S_2 = (2, 2)$: Since buffers are not full or empty, MMRS is used. So $P_{\text{OR},2} = 0$.
- State $S_3 = (3, 1)$: Symmetric to $S_1$. So $P_{\text{OR},3} = 1/2$.

Finally, $P_{\text{OR}} = \frac{1}{3}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{3}$.




- SNR=20 dB
- $B = 1$: HRS behaves like OR.
- HRS converges to MMRS as $B \uparrow$.




- $N = 3$ relays
- NBRS: if the buffer of rx (tx) relay is full (empty), select the next best relay with non-full (non-empty) buffer
- Higher gain of HRS when $B$ is small: buffer gets full or empty more often
- At high SNR, OR is better than NBRS even when $B$ is large

Further improvement?

- HRS still suffers buffer overflow and underflow.
- Key: avoid selected relay for tx/rx seeing empty/full buffer
- Idea: *equalize* all buffers.
  - The relay with a long/short buffer should transmit/receive.
  - No need to check every buffer: only those qualified.

**Shortest-In Longest-Out (SILO)**

Define $\mathcal{T} = \{R_i : \gamma_{1,i} \geq \mathtt{T}\}$ and $\mathcal{R} = \{R_i : \gamma_{2,i} \geq \mathtt{T}\}$.

Selection rule:

$$(R_r^*, R_t^*) = \arg\left(\min_{R_i \in \mathcal{R}} \{L_i\}, \max_{R_i \in \mathcal{T}} \{L_i\}\right)$$

where $\mathtt{T} = 2^{2\mathrm{R}} - 1$.

With finite buffers of size $B$, SILO encounter buffer overflow and underflow.

**Solution**: combine SILO with OR.

### Combined Relay Selection (CRS)

$$(\hat{R}_r^*, \hat{R}_t^*) = \begin{cases} (R_r^*, R_t^*), & \text{if } (\mathcal{R} \neq \emptyset) \cap (\mathcal{T} \neq \emptyset) \\ & \quad \cap (L_{R_r^*} < B) \cap (L_{R_t^*} > 0) \\ (R^*, R^*), & \text{otherwise,} \end{cases}$$

# Modeling for CRS

Causes of outage

- CRS switches between SISO and OR. When SILO is used, no outage occurs.
- OR causes outage and it is used when
  - Empty event: $\mathcal{R}$ or $\mathcal{T}$ is empty with probability $p_{\text{empty}}$.
  - Null event: Both $\mathcal{R}$ and $\mathcal{T}$ are not empty but the buffer of $R_r^*$ is full or that of $R_t^*$ is empty (using SILO).

According to the law of total probability,

$$\mathbb{P}_{\text{out}}^{\text{CRS}} = p_{\text{empty}} \mathbb{P}_{\text{out, empty}} + p_{\text{null}} \mathbb{P}_{\text{out, null}}.$$

Model the buffer status as a Markov chain with state $S_i = X_1 X_2 \cdots X_N$ where $X_j$ number of packets in the $j$th buffer.

$$\begin{cases} 0 \leq X_j \leq B-1, \ \forall\, j = 1, 2, \ldots, N \\ \sum_{i=1}^{N} X_i = \sum_{i=1}^{N} L_{0,j} \triangleq L \end{cases}$$

# CRS: Example

Consider $N = 2$ relays, buffer size $B = 6$, initial buffer length $L_0 = 2$.

$$X_1 + X_2 = 4, X_1 \leq 5, X_2 \leq 5, \quad \overset{\text{possible states}}{\Longrightarrow} \quad \{40, 31, 22, 13, 04\}$$

| | | | | |
|---|---|---|---|---|
| Case 1.1 | $R_r^* \neq R_t^*$ | $X_1 = X_2$ $(S_3)$ | $(\mathcal{R} \notin \emptyset) \cap (\mathcal{T} \notin \emptyset)$ | SILO |
| Case 1.2 | | $X_1 \neq X_2$ $(S_1, S_2, S_4, S_5)$ | | |
| Case 2.1 | $R_r^* = R_t^*$ | $X_1 = X_2$ $(S_3)$ | $(\mathcal{R} \notin \emptyset) \cap (\mathcal{T} \notin \emptyset)$ | SILO |
| | | | $(\mathcal{R} \in \emptyset) \cup (\mathcal{T} \in \emptyset)$ | OR |
| Case 2.2 | | $X_1 \neq X_2 \neq 0$ $(S_2, S_4)$ | $(\mathcal{R} \notin \emptyset) \cap (\mathcal{T} \notin \emptyset)$ | SILO |
| | | | $(\mathcal{R} \in \emptyset) \cup (\mathcal{T} \in \emptyset)$ | OR |
| Case 2.3 | | $X_1 \neq X_2$, $X_1 = 0$ $(S_1)$ or $X_2 = 0$ $(S_5)$ | $(\mathcal{R} \notin \emptyset) \cap (\mathcal{T} \notin \emptyset)$ | SILO |
| | | | $(\mathcal{R} \in \emptyset) \cup (\mathcal{T} \in \emptyset)$ | OR |

Notations: $\alpha_c = e^{-\mathtt{T}/\bar{\gamma}_c}$ and $\tilde{\alpha}_c = 1 - e^{-\mathtt{T}/\bar{\gamma}_c}$ for $c \in \{1,2\}$.

**Case 1.1** $\boxed{j = \{2,4\}}$ when $(\mathcal{R} \notin \emptyset) \cap (\mathcal{T} \notin \emptyset)$

$$\begin{aligned}
\Pr[s_3 \to s_j] &= \Pr[R_r^* \neq R_t^* | s_3] \\
&= \Big(\Pr[|\mathcal{R}| = 2] \cdot \frac{1}{2} + \Pr[|\mathcal{R}| = 1] \cdot 1\Big)\Big(\Pr[|\mathcal{T}| = 2] \cdot \frac{1}{2} + \Pr[|\mathcal{T}| = 1] \cdot 1\Big) \\
&= \big(\alpha_1 - \frac{1}{2}\alpha_1^2\big) \cdot \big(\alpha_2 - \frac{1}{2}\alpha_2^2\big).
\end{aligned}$$

---

**Case 1.2**

$\boxed{(i,j) = (1,2),(2,3),(4,3),(5,4)}$ when $(\mathcal{R} \notin \emptyset) \cap (\mathcal{T} \notin \emptyset)$

$$\begin{aligned}
\Pr[s_i \to s_j] &= \Pr[R_r^* \neq R_t^* | s_i] \\
&= \Pr[R_r^* \in \mathcal{R}, R_t^* \in \mathcal{T}] \\
&= \alpha_1 \alpha_2.
\end{aligned}$$

$\boxed{(i,j) = (2,1),(4,5)}$ only when $|\mathcal{R}| = |\mathcal{T}| = 1$

$$\begin{aligned}
&\Pr[s_i \to s_j] \\
&\quad = \Pr[\mathcal{R} = \{R_r^*\}, \mathcal{T} = \{R_t^*\} | s_i] \\
&\quad = \alpha_1 \tilde{\alpha}_1 \alpha_2 \tilde{\alpha}_2.
\end{aligned}$$

**Case 2.1** $R_r^* = R_t^*, X_1 = X_2 = 2 \Rightarrow \boxed{(i,j) = (3,3)}$

If $(\mathcal{R} \notin \emptyset) \cap (\mathcal{T} \notin \emptyset)$

$$\begin{aligned}
\Pr[s_3 \to s_3] &= \Pr[R_r^* = R_t^* | s_3, (\mathcal{R} \notin \emptyset) \cap (\mathcal{T} \notin \emptyset)] \\
&= \Big(\Pr[|\mathcal{R}| = 2] \cdot \frac{1}{2} + \Pr[|\mathcal{R} = 1] \cdot 1\Big) \cdot \Big(\Pr[|\mathcal{T}| = 2] \cdot \frac{1}{2} + \Pr[|\mathcal{T}| = 1] \cdot 1\Big) \\
&= (\alpha_1 - \frac{1}{2}\alpha_1^2) \cdot (\alpha_2 - \frac{1}{2}\alpha_2^2).
\end{aligned}$$

If $(\mathcal{R} \in \emptyset) \cup (\mathcal{T} \in \emptyset)$

$$\begin{aligned}
\Pr[s_3 \to s_3] &= \Pr[R_r^* = R_t^* | s_3] \\
&= \Pr[(\mathcal{R} \in \emptyset) \cup (\mathcal{T} \in \emptyset)] \\
&= 1 - \Pr[(\mathcal{R} \notin \emptyset) \cap (\mathcal{T} \notin \emptyset)] \\
&= 1 - \Big(1 - \Pr[\mathcal{R} \in \emptyset]\Big)\Big(1 - \Pr[\mathcal{T} \in \emptyset]\Big) \\
&= 1 - (1 - \tilde{\alpha}_1^2)(1 - \tilde{\alpha}_2^2). \qquad (10)
\end{aligned}$$

# CRS: Example (cont'd)

**Case 2.2** $R_r^* = R_t^* = R^*, X_1 \neq X_2 \neq 0 \Rightarrow \boxed{i = \{2, 4\}}$

$\underline{\text{If } (\mathcal{R} \notin \emptyset) \cap (\mathcal{T} \notin \emptyset) \Rightarrow (\mathcal{R} = \{R^*\}) \cap (R^* \in \mathcal{T})}$

or $\underline{(\mathcal{T} = \{R^*\}) \cap (R^* \in \mathcal{R})}$

$$
\begin{aligned}
\Pr[s_i \to s_i] &= \Pr[(\mathcal{R} = \{R^*\}) \cap (R^* \in \mathcal{T})] + \Pr[(\mathcal{T} = \{R^*\}) \cap (R^* \in \mathcal{R})] \\
&= \alpha_1 \tilde{\alpha}_1 \alpha_2 + \alpha_1 \alpha_2 \tilde{\alpha}_2.
\end{aligned}
$$

$\underline{\text{If } (\mathcal{R} \in \emptyset) \cup (\mathcal{T} \in \emptyset)}$: $\Pr[s_i \to s_i]$ is same as (10).

**Case 2.3** $R_r^* = R_t^* = \hat{R}^*, X_1 = 0$ or $X_2 = 0 \Rightarrow \boxed{i = \{1, 5\}}$

This case is similar to Case 2.2, except that one of the buffers is empty (will be exercised in the next assignment).

# CRS: Example (cont'd)

## Conditional probability

1) Empty event: $p_{\text{empty}} = \Pr[(\mathcal{R} \in \emptyset) \cup (\mathcal{T} \in \emptyset)] =$ as given in (10).

2) Null event: It happens when

- At state $S_1 = (4, 0)$, $R_2$ is selected for transmission $\Rightarrow$ $\mathcal{T} = \{R_2\}, \mathcal{R} \notin \emptyset$.
- At state $S_5 = (0, 4)$, $R_1$ is selected for transmission $\Rightarrow$ $\mathcal{T} = \{R_1\}, \mathcal{R} \notin \emptyset$.

$$p_{\text{null},i} = \begin{cases} (1 - \tilde{\alpha}_i^2)\alpha_2\tilde{\alpha}_2, & i = 1, 5 \\ 0, & i = 2, 3, 4. \end{cases}$$

Remark: $p_{\text{null},i}$ is state dependent.

**Conditional outage probability**: $\mathbb{P}_{\text{out,a}} = \Pr[\min(\gamma_1^*, \gamma_2^*) < \mathtt{T}]$ assuming AF.

1) Empty event: In this case, $\mathcal{R} \in \emptyset$ or $\mathcal{T} \in \emptyset$, which occurs if $\gamma_{1,i} < \mathtt{T}$ or $\gamma_{2,i} < \mathtt{T}$ for all relays. This implies $\min(\gamma_1^*, \gamma_2^*) < \mathtt{T}$ and so $\mathbb{P}_{\text{out,empty}} = 1$.

## High SNR approximation

When $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are sufficiently large, $p_{\text{null}} \to 0$ and thus $\mathbb{P}_{\text{out}}^{\text{CRS}}$ is dominated by the empty event.

$$\begin{aligned}
\mathbb{P}_{\text{empty}} &= \Pr[(\mathcal{R} \in \emptyset) \cup (\mathcal{T} \in \emptyset)] \\
&= 1 - \left(1 - \tilde{\alpha}_1^N\right)\left(1 - \tilde{\alpha}_2^N\right) \\
&\overset{\tilde{\alpha}_c \approx \frac{\mathrm{T}}{\bar{\gamma}_c}}{=} \left(\frac{\mathrm{T}}{\bar{\gamma}_1}\right)^N + \left(\frac{\mathrm{T}}{\bar{\gamma}_2}\right)^N - \left(\frac{\mathrm{T}^2}{\bar{\gamma}_1\bar{\gamma}_2}\right)^N .
\end{aligned}$$

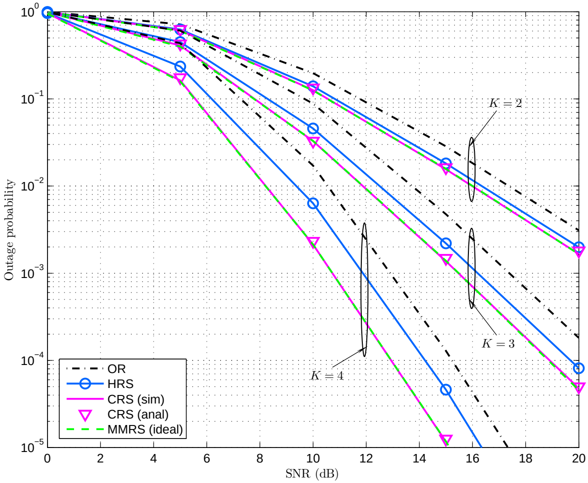
Together with $\mathbb{P}_{\text{out, empty}} = 1$ and $\bar{\gamma}_1 = \bar{\gamma}_2 = \gamma = P\sigma^2/N_0$

$$\mathbb{P}_{\text{out}}^{\text{CRS}} \approx \left(\frac{2^{\frac{1}{N}}\mathrm{T}}{\mathsf{SNR} \cdot \sigma^2}\right)^N$$

### Diversity gain of CRS

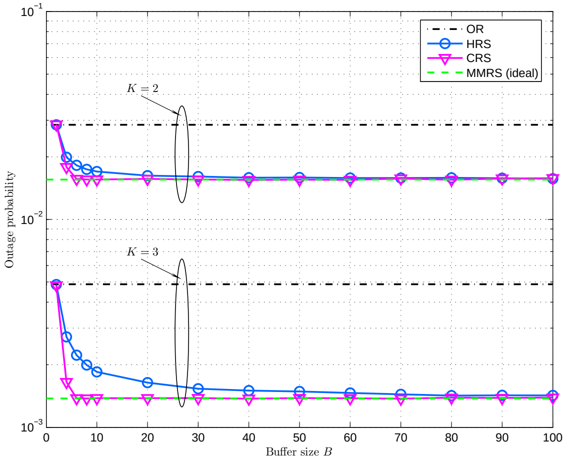
CRS achieves diversity order of $N$.

# Outage Probability ($B = 6$)



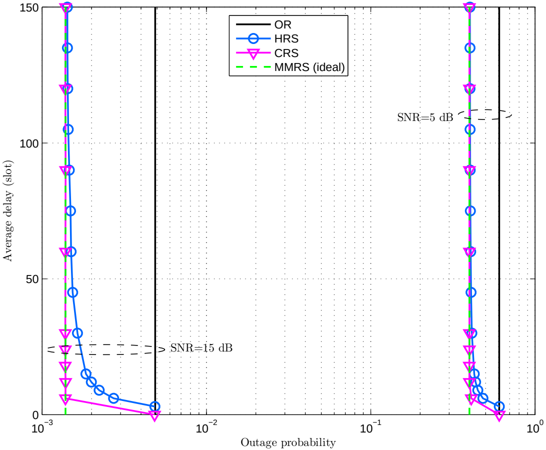

- CRS is as good as the ideal MMRS with $B \to \infty$.
- The gain increases with the num. of relays.

# Impact of Buffer Size (SNR = 15 dB)




- $\mathbb{P}_{\mathbf{out}}$ of ideal MMRS is flat because ideal MMRS assumes $B \to \infty$.
- Minimal $\mathbb{P}_{\mathbf{out}}$:
  - $K = 2$: $B = 6$ for CRS, $B = 30$ for HRS
  - $K = 3$: $B = 6$ for CRS, $B = 100$ for HRS

# Delay vs. $\mathbb{P}_{\text{out}}$ ($K = 3$ relays)




- $y$-axis: delay cost for maintaining $\mathbb{P}_{\text{out}}$.
- Delay exponentially increases with decreasing $\mathbb{P}_{\text{out}}$.

Delay cost:

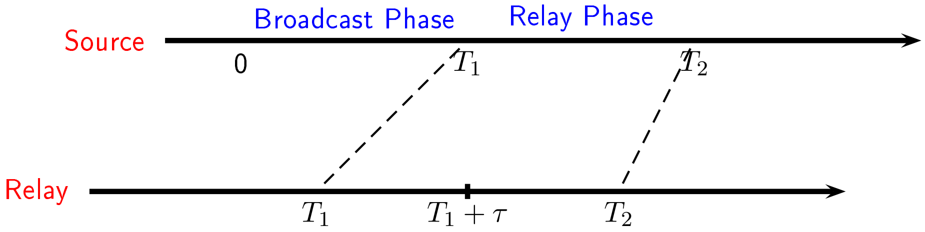
- SNR $= 5$ dB: 6 slots for CRS, 60 for HRS
- SNR $= 15$: 6 slots for CRS, 120 slots for HRS

# Effect of Imperfect CSI

Relay selection is promising in achieving diversity gain. However, its achievable gain strictly replies on perfect CSI.

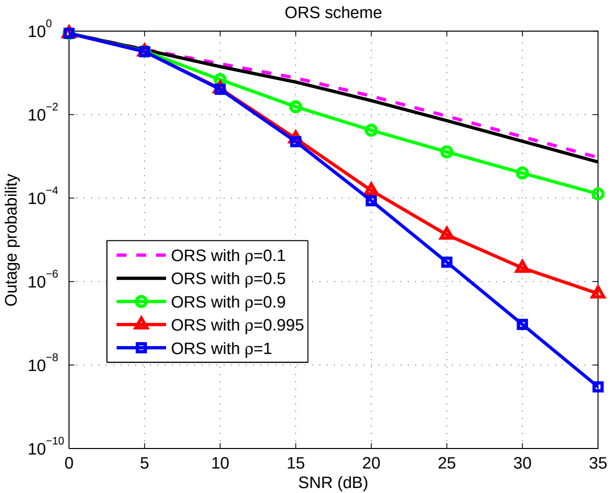
In fact, there exists a delay $\tau$ between the instant ($T_1$) that CSI is obtained and that ($T_1 + \tau$) when the selected relay starts to transmit.

Therefore, the selected relay might not be the optimal one if the selection delay $\tau$ is larger than the channel coherence time, which primarily depends on the Doppler frequency.



# OR with Outdated CSI (cont'd)

Let $a$ denote the actual channel coefficient and $\hat{a}$ the delayed copy. Their correlation can be modeled by $\rho = J_0(2\pi f_d \tau)$ where $f_d$ is the Doppler frequency and $J_0(\cdot)$ is the zero-order Bessel function of the first kind.



Overall, the diversity order of OR with outdated CSI can only take two possible values

$$d = \begin{cases} 1 & \text{if } \rho < 1 \\ K & \text{if } \rho = 1. \end{cases}$$

Remarks

- Accurate CSI is generally not available due to feedback delay and node mobility.
- Achieving full diversity is only possible if update frequency is fast enough to capture the channel variation.
- CSI-based relay selection also suffers from channel estimation error.
- Location-based relay selection is less sensitive to the change of small-scale fading statistics [5].

# Effect of Co-Channel Interference

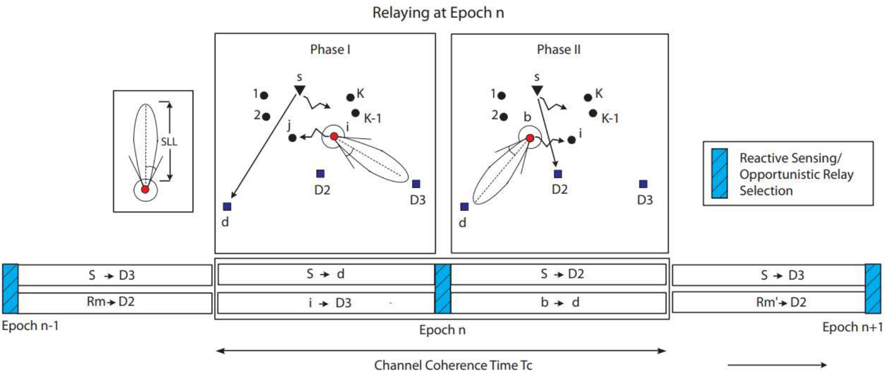
In wireless networks, co-channel interference (CCI) may arise when two different transmitters use the same frequency.

Since frequency spectrum is a precious resource, cellular systems reply on frequency reuse to improve resource utilization. However, CCI may occur when two cells using the same frequency are not sufficiently apart from each other.

In cooperative networks, relays can transmit simultaneously with the source to enhance spectral efficiency, if CCI is well controlled [6].

In the following, we discuss the effect of CCI to OR, considering DF.

# OR with Temporal Reuse




Suppose $r_i$ transmits to $d$ in phase I of epoch $n$ while another relay is selected following the rule

$$r_b = \max \arg_{r_k} \{\gamma_{r_k,d}\}, r_k \in \mathcal{D} \backslash r_i$$

During phase II, $r_b$ transmits the (old) signal received in phase I while $S$ transmits a (new) one.

The received signal at $d$ can be expressed as

$$\mathbf{y} = \underbrace{\begin{bmatrix} a_{s,d} \\ a_{b,d} \end{bmatrix}}_{\mathbf{a}} x_0 + \underbrace{\begin{bmatrix} a_{i,d} \\ 0 \end{bmatrix}}_{\mathbf{c_1}} x_1 + \underbrace{\begin{bmatrix} 0 \\ a_{s,d} \end{bmatrix}}_{\mathbf{c_2}} x_2 + \mathbf{n}$$

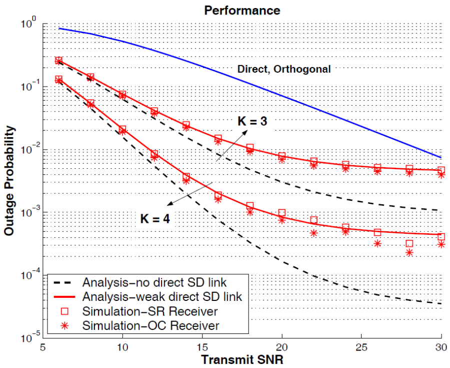
$$= \mathbf{a} x_0 + \underbrace{\sum_{i=1}^{2} \mathbf{c}_i x_i + \mathbf{n}}_{\mathbf{v}}$$

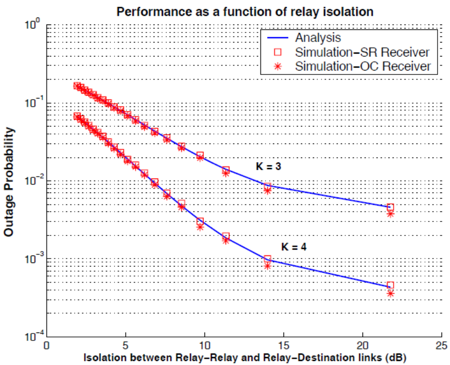
where $x_0$ is the signal of interest, $x_1$ and $x_2$ are interfering signals.

The compound channel $\mathbf{a}$ of $x_0$ and the compound channel $\mathbf{c_2}$ of $x_2$ are correlated because of $a_{s,d}$.

Node $D$ linearly processes the received vector with $\mathbf{w}^*\mathbf{y}$, where $\mathbf{w}$ depends on the specified diversity combiner.

- Optimal combining (OC): the weighting vector $\mathbf{w}$ that maximizes the signal-to-interference-and-noise ratio (SINR) is proportional to $\mathbf{R_v}^{-1}\mathbf{a}$, where
$$\mathbf{R_v} = \mathbb{E}[\mathbf{v}^*\mathbf{v}^T].$$
Here, CSI regarding the interfering node $(a_{i,d})$ is required.
- Selection diversity (SR): the receiver processes the two, time-separated signal independently and succeeds if either reception, during phase I or II succeeds.




- Left: OC slightly outperforms SC but both fail to achieve full diversity gain. The absence of direct link has a lower outage probability than the case with direct link. Why?
- Right: Isolation between relays in terms of $\mathbb{E}[\gamma_{jd}]/\mathbb{E}[\gamma_{ji}]$ helps to combat CCI. How such isolation can be achieved?

# Conclusion

- Relay selection achieves good tradeoff between achievable diversity gain and bandwidth cost.
- CSI-based selection rules
  - OR is optimal if perfect CSI is available.
  - Partial selection is appealing for its reduced feedback overhead.
  - CCI is a killer to diversity gain (but may not be that bad from energy harvesting perspective)
- Relay buffers can be used to best utilize the two-hop channels.

# Markov Chain

Imagine that there were two possible states for weather: sunny or cloudy. You can always directly observe the current weather state, and it is guaranteed to always be one of the two aforementioned states.

Now, you decide you want to be able to predict what the weather will be like tomorrow. Intuitively, you assume that there is an inherent **transition** in this process, in that the current weather has some bearing on what the next day's weather will be. So, being the dedicated person that you are, you collect weather data over several years, and calculate that the chance of a sunny day occurring after a cloudy day is 0.25. You also note that, by extension, the chance of a cloudy day occurring after a cloudy day must be 0.75, since there are only two possible states.

You can now use this **distribution** to predict weather for days to come, based on what the current weather state is at the time.

# Markov Chain (cont'd)

A random process $X(t)$ is a **Markov process** if the future process given the present is independent of the past.

- Memoryless: For arbitrary times $t_1 < t_2 < \cdots < t_k < t_{k+1}$,

$$\Pr[X(t_{k+1} = x_{k+1}|X(t_k) = x_k, \dots, X(t_1) = x_1] \\ = \Pr[X(t_{k+1}) = x_{k+1}|X(t_k) = x_k]$$

- An integer-valued Markov random process is called a **Markov chain**.
- Let $X_n$ be a discrete-time Markov chain starting at $n = 0$ with probability mass function (PMF) $p_j(0) \triangleq P[X_0 = j]$, $j = 0, 1, 2, \dots$.
- Joint PMF for the first $n+1$ values of the process:

$$P[X_n = i_n, \dots, X_0 = i_0] = P[X_n = i_n|X_{n-1} = i_{n-1}] \cdots \\ \times P[X_1 = i_1|X_0 = i_0]P[X_0 = i_0].$$

- **Homogeneous**: One step transition probability does not change with time, i.e., $P[X_{n+1} = j|X_n = i] = p_{ij}, \forall\ n$.

$$P[X_n = i_n, \dots, X_0 = i_0] = p_{i_{n-1},i_n} \cdots p_{i_0,i_1} p_{i_0}(0).$$

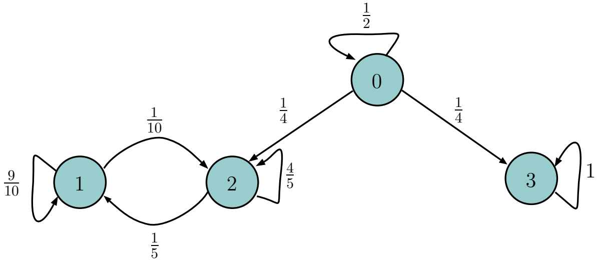
A *homogeneous* Markov chain $X_n$ is completely specified by $p_i(0)$ and the one-step **transition matrix**

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \cdot & \cdot & \cdot & \\ p_{i0} & p_{i1} & \cdots & \\ \cdot & \cdot & \cdots & \end{bmatrix}$$

whose row sum must be equal to one because

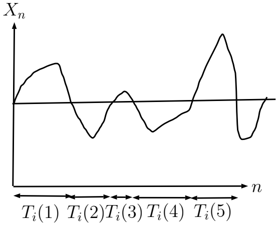
$$\sum_j p_{ij} = \sum_j P[X_{n+1} = j | X_n = i].$$

# Properties of States




The long-term behavior of the Markov chain depends on the state's characteristics.

- **Accessible**: state $j$ is accessible from state $i$ if $p_{ij}(n) > 0$ for some $n \geq 0$.
- States $i$ and $j$ **communicate** if they are accessible to each other; they belong to the same **class** if they communicate with each other.
- State $i$ is **recurrent**: $f_i = P[\text{ever returning to state } i] = 1$.
- State $i$ is **transit**: $f_i < 1$.
- State $i$ has **period** of $d$ if it needs $d$ steps to reoccur.



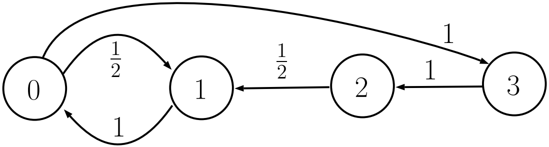

Suppose we start a Markov chain in a recurrent state $i$ at time $n = 0$. Let $T_i(1)$, $T_i(1) + T_i(2)$, ... be the time when the process returns to state $i$.

The proportion of time spent in state $i$ after $k$ returns to $i$ is $\frac{k}{T_i(1)+T_i(2)+\cdots T_i(k)} \to \frac{1}{E[T_i]} = \pi_i$, which is called the **steady-state probability** of state $i$ satisfying

$$\left.\begin{array}{l} \pi_j = \sum_i \pi_i P_{ij}, \quad \forall\ j \\ 1 = \sum_i \pi_i \end{array}\right\} \xRightarrow{\boldsymbol{\pi}=[\pi_1\ \pi_2\ \cdots]} \begin{cases} \boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi} \\ \boldsymbol{\pi}\mathbf{B} = \boldsymbol{b} \end{cases} \tag{11}$$

where $\boldsymbol{b} = [1\ 1\ \cdots\ 1]$ and $\mathbf{B}_{ij} = 1, \forall i, j$.

# Properties of Markov chain




- **Irreducible**: a Markov chain with a single class.
- States $i$ and $j$ **communicate** if they are accessible to each other; they belong to the same class if they communicate with each other.
- An irreducible Markov chain is **aperiodic** if the states in its single class have period of one.
- An aperiodic Markov chain has a unique steady-state probability vector.

# Modeling the CSI Uncertainty

Since $\alpha$ and $\hat{\alpha}$ are both complex Gaussian distributed, the PDF of $\alpha$ conditioned on $\hat{\alpha}$ can be obtained by applying Bayes' Theorem

$$\begin{aligned} f_{\alpha|\hat{\alpha}}(\alpha|\hat{\alpha}) &= \frac{f_{\alpha,\hat{\alpha}}(\alpha,\hat{\alpha})}{f_{\hat{\alpha}}(\hat{\alpha})} \\ &= \frac{1}{\pi^2 \det(\mathbf{R})} e^{-(\alpha-\rho\hat{\alpha})^H \mathbf{R}^{-1}(\alpha-\rho\hat{\alpha})} \sim \mathcal{CN}(\rho\hat{\alpha}, (1-\rho^2)N_0) \end{aligned}$$

where $\mathbf{R} = (1-\rho)\mathbf{I}_2$ is the covariance matrix, which models the degree of CSI uncertainty. When $\alpha = \hat{\alpha}$, $\mathbf{R} = 0$.

The actual SNR, $\gamma = \frac{P|\alpha|^2}{N_0}$, conditioned on its estimate, $\hat{\gamma} = \frac{P|\hat{\alpha}|^2}{N_0}$, follows a non-central chi-square distribution with 2 degrees of freedom [7]

$$f_{\gamma|\hat{\gamma}}(\gamma|\hat{\gamma}) = \frac{1}{\bar{\gamma}(1-\rho^2)} e^{-\frac{\gamma+\hat{\gamma}\rho^2}{\bar{\gamma}(1-\rho^2)}} I_0\left(\frac{2\sqrt{\gamma\hat{\gamma}\rho^2}}{\bar{\gamma}(1-\rho^2)}\right) \tag{12}$$

where $I_0(\cdot)$ is the zero-order modified Bessel function of the first kind.

Consider OR as an example.

$$P_{out} = \sum_{k=0}^{K} P_{out}\Big|_{|\mathcal{D}|=k} \Pr[|\mathcal{D}| = k]$$

where $\mathcal{D}$ denotes the decoding set.

1) Probability of decoding set

$$\Pr[|\mathcal{D}| = k] = \binom{K}{k} (1 - e^{-\frac{\mathtt{T}}{\bar{\gamma}_1}})^{K-k} e^{\frac{k\mathtt{T}}{\bar{\gamma}_1}} \tag{13}$$

where $\mathtt{T} = 2^{2\mathrm{R}} - 1$. This probability is independent from the correlation coefficient $\rho$. Here I.I.D. channels are assumed.

2) Conditional outage probability

$$F_{\gamma^*}(\mathtt{T}) = \sum_{i=1}^{k} \Pr[\gamma_i < \mathtt{T}; i = \max \arg_i\{\hat{\gamma}_i\}]$$
$$= \sum_{i=1}^{k} \int_{y=0}^{\mathtt{T}} \int_{x=0}^{\infty} f_{\gamma_i|\hat{\gamma}_i}(y|x) f_{\hat{\gamma}_i}(x) \prod_{j \neq i} F_{\hat{\gamma}_j}(x) dx dy.$$

With I.I.D. assumption, we can drop the subscript such that

$$\prod_{j \neq i} F_{\hat{\gamma}_j}(x) = (1 - e^{-\frac{x}{\bar{\gamma}_2}})^{k-1} = \sum_{j=1}^{k-1} \binom{k-1}{j} (-1)^j e^{-\frac{jx}{\bar{\gamma}_2}}.$$

Further using (12) and after some algebraic manipulations [8], we reach

$$F_{\gamma^*}(\mathtt{T}) = k \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{j+1} \left(1 - e^{-\frac{j+1}{\bar{\gamma}_2(1+j(1-\rho^2))}\mathtt{T}}\right) \propto \frac{1}{\rho^2} \qquad (14)$$

# OR with Outdated CSI - Asymptotic Analysis

In (14), the exponential term can be expanded using Taylor series

$$\left(1 - e^{-\frac{j+1}{\bar{\gamma}_2(1+j(1-\rho^2))}\mathtt{T}}\right) = \sum_{s=1}^{K}\left(\frac{j+1}{\bar{\gamma}_2(1+j(1-\rho^2))}\right)^s \frac{\mathtt{T}^s(-1)^{s-1}}{s!} + \mathcal{O}\left((\frac{1}{\bar{\gamma}_2})^{K+$$

Then (14) can be expressed as

$$F_{\gamma^*}(\mathtt{T}) \approx k\sum_{s=1}^{K}\frac{\mathtt{T}^s(-1)^{s-1}}{s!}\underbrace{\left(\frac{1}{\bar{\gamma}_2}\right)^s\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{j+1}\left(\frac{j+1}{1+j(1-\rho^2)}\right)^s}_{\mathcal{I}(s,j)}$$

1) High $\rho$ region ($\rho \approx 1^-$): $s = 1$ dominates $\mathcal{I}(s,j)$

Let $x = 1-\rho^2 \to 0$. Expand $\frac{1}{(1+jx)^s}$ at $x = 0$ leading to

$$\frac{1}{(1+jx)^s} = 1 - sjx + \frac{s(s+1)j^2x^2}{2!} - \frac{s(s+1)(s+2)j^3x^3}{3!} + \cdots \quad (15)$$

Notice that the Taylor series for $(1+x)^{-1}$ around $x = 0$ is a geometric series $1 - x + x^2 - x^3 + \cdots$.

Using (15), $\mathcal{I}(s,j)$ can be expressed as

$$\mathcal{I}(s,j) = \left(\frac{1}{\bar{\gamma}_2}\right)^s \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j (j+1)^{s-1}$$
$$\times \left(1 - sj(1-\rho^2) + \frac{s(s+1)j^2(1-\rho^2)^2}{2!} - \frac{s(s+1)(s+2)j^3(1-\rho^2)^3}{3!} \cdots \right)$$

Since $1/\bar{\gamma}_2 \to 0$, we can focus on $\mathcal{I}(s=1,j)$ given by [8]

$$\mathcal{I}(1,j) = \left(\frac{1}{\bar{\gamma}_2}\right) \left[(k-1)!(1-\rho^2)^{k-1} + \mathcal{O}((1-\rho^2)^{k-1})\right] \qquad (16)$$

Plugging (16) into (14) and combining (13) (with Taylor series expansion),

$$P_{out} \approx \underbrace{\left(\frac{\mathrm{T}}{\bar{\gamma}}\right)^K}_{\mathcal{D}=\emptyset} + \underbrace{\sum_{k=1}^{K} \binom{K}{k} \left(\frac{\mathrm{T}}{\bar{\gamma}}\right)^{K-k+1} k!(1-\rho^2)^{k-1} \underbrace{\left(1-\frac{\mathrm{T}}{\bar{\gamma}}\right)^k}_{\to 1}}_{\text{dominate by } k=K}$$

For $\rho \to 1^-$,

$$P_{out} \approx \frac{K!\mathtt{T}}{\bar{\gamma}} \left(1 - \rho^2\right)^{K-1} + \mathcal{O}(\frac{1}{\bar{\gamma}})$$

Outage probability decays exponentially as a function of the number of relays.

2) Low $\rho$ region ($\rho \approx 0^+$)

Performing the similar analysis as above, we have

$$P_{out} \approx \frac{\mathtt{T}}{\bar{\gamma}}\left(1 + \rho^2 (1 - \sum_{k=1}^{K} \frac{1}{k})\right) + \mathcal{O}(\frac{1}{\bar{\gamma}}).$$

Using more relays reduces the outage probability following the selection gain term $\sum_{k=1}^{K} \frac{1}{k}$.

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