Tutorial on Monte Carlo Simulation

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Monte Carlo Simulation

- General definition: any technique that approximates solutions to quantitative problems through statistical sampling
- Rely on the process of explicitly representing uncertainties by specifying inputs as probability distributions
- General idea (consider the error rate of a transmission system)
  - Simulate the system repeatedly,
  - for each simulation count the number of transmitted data and data errors,
  - estimate the symbol error rate as the ratio of the total
  - number of observed errors and the total number of transmitted data.
- Implementation idea: Inside a programming loop:
  - perform a system simulation, and
  - accumulate counts for the quantities of interest

Stop Criterion

- **Question**: How many times should the loop be executed?
- **Answer**: It depends
  - on the desired level of accuracy (confidence), and
  - on the actual error rate.
- **Confidence Intervals**:
  - Assume we form an estimate of the error rate $\hat{P}_e$.
  - Then, the true error rate P_e is (hopefully) close to our estimate.
  - Put differently, we would like to be reasonably sure that $|P_e - \hat{P}_e|$ is small.

Choosing the Number of Simulations

In simulations, a stop criterion can be chosen according to

- a desired confidence level $1-\alpha$
- an acceptable confidence interval I
- the error rate P_e

Example: How many samples n we need to have the 95% confidence interval?
Recall the confidence interval is given as (from Assignment 1, $z = \Phi^{-1}\left(1-\frac{\alpha}{2}\right)$)

$$\left(\hat{P}_e - z\frac{\sigma}{\sqrt{n}}, \hat{P}_e + z\frac{\sigma}{\sqrt{n}}\right) \quad z = 1.96 \text{ when } 1-\alpha = 0.95$$

or $|\hat{P}_e - P_e| < z\frac{\sigma}{\sqrt{n}} = z\frac{\sqrt{P_e(1-P_e)}}{\sqrt{n}}$. Now given a predefined confidence interval, say I, we can find the number of simulations required: ① $n =$ ________________.

A better stop criterion

- When simulating communications systems, the error rate is often very small.
- Then, it is desirable to specify the confidence interval as a fraction of the error rate.

$$I = \beta P_e.$$

For example, we accept 10% estimation error if $\beta = 0.1$

- From ①, $n \cdot P_e = (1 - P_e)\left(\frac{z}{\beta}\right)^2 \approx \left(\frac{z}{\beta}\right)^2$
- Since $n \cdot P_e$ is the expected number of errors, a rule of thumb is to stop the simulation when the number of errors reaches

② ________________

Outage probability over Rayleigh Fading Channel

The term "outage" is used in communication theory to represent the event of a transmission error.

- An error occurs when the channel capacity $\mathtt{C}$ is less than the transmission rate $\mathtt{R}$ [1].

  $$\mathbb{P}_{\text{out}} = \mathbb{P}[\mathtt{C} < \mathtt{R}]$$

  where $\mathtt{C}$ is the channel capacity and $\mathtt{R}$ is the required transmission rate.

  - For point-to-point transmission: $\mathtt{C} = \log_2(1 + \gamma)$ where $\gamma = \frac{P}{N_0}a^2$ with $a = |\alpha|$ being the fading channel gain.

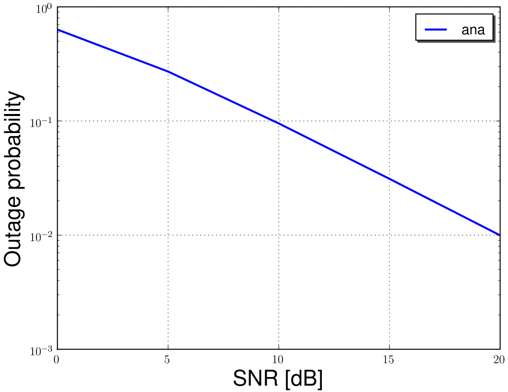
- Alternatively, an error occurs when the received SNR γ is less than a threshold γ_0

  $$\mathbb{P}_{\text{out}} = \mathbb{P}[\gamma < \gamma_0]$$

  where $\gamma_0 =$ 3 ________________.

Outage Probability vs. SNR

Want a graph like this



How to set up SNR?

Simulation Setup

Recall $\text{SNR} = P|\alpha|^2/N_0$

- Fix $P = |\alpha|^2 = 1$
- $10\log_{10}(|\alpha|^2/N_0) = \text{SNR}_{\text{dB}} \Rightarrow N_0 = 10^{-\frac{\text{SNR}_{\text{dB}}}{10}}$

Procedure

- Generate channel coefficient α (depending on fading conditions)
- Generate noise term $n = 10^{-\frac{\text{SNR [dB]}}{20}} \cdot \frac{1}{\sqrt{2}}(x + \mathrm{j}y)$ where $x, y \sim \mathcal{N}(0,1)$
- Verify $\frac{|\alpha|^2}{|n|^2} = \text{SNR}$

Then perform Monte Carlo simulation

- Determine the confidence interval
- Determine the stopping criteria

References

A. Goldsmith, *Wireless Communications*. Cambridge University Press, 2004.