End-to-End Performance of Transmission Systems With Relays Over Rayleigh-Fading Channels

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***Abstract*—End-to-end performance of two-hops wireless communication systems with nonregenerative relays over flat Rayleigh-fading channels is presented. This is accomplished by deriving and applying some new closed-form expressions for the statistics of the harmonic mean of two independent exponential variates. It is shown that the presented results can either be exact or tight lower bounds on the performance of these systems depending on the choice of the relay gain. More specifically, average bit-error rate expressions for binary differential phase-shift keying, as well as outage probability formulas for noise limited systems are derived. Finally, comparisons between regenerative and nonregenerative systems are presented. Numerical results show that the former systems clearly outperform the latter ones for low average signal-to-noise-ratio (SNR). They also show that the two systems have similar performance at high average SNR.**

***Index Terms*—Bit-error rate (BER), collaborative/cooperative diversity, harmonic mean, outage probability, Rayleigh fading, transmission with relays.**

I. Introduction

CONSIDER a communication system in which two terminals are communicating via a third terminal that acts as a relay. This scenario was encountered originally in bent-pipe satellites where the primary function of the spacecraft is to relay the uplink carrier into a downlink [1]. It is also common in various fixed microwave links by enabling greater coverage without the need of large power at the transmitter. More recently, this concept has gained new actuality in collaborative/cooperative wireless communication systems [2]–[8]. In this case, the key idea is that a mobile terminal relays a signal between the base station and a nearby mobile terminal when the direct link between the base station and the original mobile terminal is in deep fade. As a result, similar to the scenarios described above, the signal from the source to the destination propagates through two hops/links in series.

In this paper, we focus on these two-hops systems and study their end-to-end performance over independent, not necessarily identical, Rayleigh-fading channels. One of the important features of this type of fading, in addition to be the fading model used for non line-of-sight scenarios, is that the power of a signal that is perturbed by Rayleigh type of fading is exponentially distributed. Thus, many statistical attributes of the exponential distribution can be exploited when analyzing the performance of wireless communication systems over this type of fading channels. For our problem and in the case of nonregenerative systems (i. e., systems in which relays just amplify and retransmit the information), we need to find the statistics of the harmonic mean of two exponential random variables (RVs) since the end-to-end signal-to-noise ratio (SNR) of these systems is related to the harmonic mean of the SNR of the two links (as we will explain in more details in Sections II–IV). To the best of our knowledge, this problem has not been investigated even in the specialized statistical literature. The main contribution of this paper is to derive closed form expressions for the statistics [i.e., probability density function (PDF), cumulative distribution function (CDF), and moment generating function (MGF)] of the harmonic mean of two independent exponential RVs, then to use these new results to study the end-to-end performance of wireless communication systems with relays over Rayleigh-fading channels.

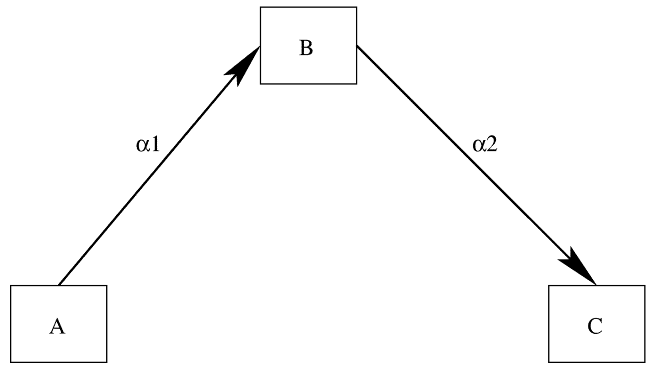

Fig. 1. A wireless communication system where terminal B is relaying the signal from terminal A to terminal C.

The remainder of this paper is organized as follows. Section II introduces the system and channel models under consideration. Section III presents the statistics of the harmonic mean of two independent exponential RVs. These results are applied in Section IV to evaluate the end-to-end performance of wireless communication systems with nonregenerative relays. Finally, Section V summarizes the main results of the paper.

II. System and Channel Models

Consider the wireless communication system shown in Fig. 1. Here, terminal A is communicating with terminal C through terminal B which acts as a relay. Assume that terminal A is transmitting a signal $s(t)$, which has an average power normalized to one. The received signal at terminal B can be written as

$$r_b(t) = \alpha_1 s(t) + n_1(t) \tag{1}$$

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where α_1 is the fading amplitude of the channel between terminals A and B, and $n_1(t)$ is an additive white Gaussian noise (AWGN) signal with one sided power spectral density (PSD) N_0. The received signal is then multiplied by the gain of the relay at terminal B, G, and then retransmitted to terminal C. The received signal at terminal C can be written as

$$r_c(t) = \alpha_2 G(\alpha_1 s(t) + n_1(t)) + n_2(t) \tag{2}$$

where α_2 is the fading amplitude of the channel between terminals B and C, and $n_2(t)$ is an AWGN signal with one sided PSD N_0. The overall SNR at the receiving end can then be written as

$$\gamma_{\text{eq}} = \frac{[\alpha_2 G \alpha_1]^2}{[(\alpha_2 G)^2 + 1]\, N_0} = \frac{\frac{\alpha_1^2}{N_0}\frac{\alpha_2^2}{N_0}}{\frac{\alpha_2^2}{N_0} + \frac{1}{G^2 N_0}}. \tag{3}$$

It is clear from (3) that the choice of the relay gain defines the equivalent SNR of the two channels. One choice for the gain was given in [3] to be

$$G^2 = \frac{1}{\alpha_1^2 + N_0}. \tag{4}$$

In this case, substituting (4) in (3) leads to γ_{eq_1} given by

$$\gamma_{\text{eq}_1} = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 + 1} \tag{5}$$

where $\gamma_i = \alpha_i^2/N_0$, $i = 1, 2$ is the per-hop SNR. The choice of the above gain aims to limit the output power of the relay if the fading parameter of the first channel, α_1, is low. The form of the equivalent SNR in (5) is not easily tractable with our approach due to the complexity in finding the statistics (i.e., the PDF, CDF, and MGF) associated with it. Fortunately, this form can be tightly bounded by

$$\gamma_{\text{eq}_2} = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2}. \tag{6}$$

The form of γ_{eq_2} [given in (6)] has the advantage of mathematical tractability over that in (5) in addition to be a tight upper bound for the form in (5), specially at high average SNR, as the numerical examples will illustrate later on. This will later translate to a tight lower bound when talking about average bit-error rate (BER) and outage probability performance criteria. The form of the equivalent SNR in (6) can be thought of as if it is arising from an ideal/hypothetical relay that is able to invert the channel regardless of its magnitude and with a gain given by

$$G^2 = \frac{1}{\alpha_1}^{2}. \tag{7}$$

Consequently, the performance of the relay in (7) can serve as a benchmark for all practical relays. As the form of equivalent SNR given in (6) is related to the harmonic mean of the instantaneous SNRs of the two links, we derive in Section III the statistics of the harmonic mean of two exponential variates, related to the Rayleigh type of fading, as mentioned earlier.

III. Harmonic Mean of Exponential Variates

In this section, we first recall the definitions of the harmonic mean and of an exponential variate and then present the key results on the statistics of the harmonic mean of two independent exponential RVs.

A. Definitions

Definition 1 (Harmonic Mean): Given two numbers X_1 and X_2, the harmonic mean of X_1 and X_2, $\mu_H(X_1, X_2)$, is defined as the reciprocal of the arithmetic mean of the reciprocals of X_1 and X_2 [9, Sec. 9.1.3], that is

$$\mu_H(X_1, X_2) = \frac{2}{\frac{1}{X_1} + \frac{1}{X_2}} = \frac{2\, X_1\, X_2}{X_1 + X_2}. \tag{8}$$

It is clear that the harmonic mean of two numbers is equal to the square of their geometric mean divided by their arithmetic mean.

Definition 2 (Exponential RV): X follows an exponential distribution with parameter $\beta > 0$ if the PDF of X is given by [10, Sec. 4.3]

$$p_X(x) = \beta e^{-\beta x} U(x) \tag{9}$$

where $U(\cdot)$ is the unit step function. In what follows, as a short hand notation, we will use the notation $X \sim \mathcal{E}(\beta)$ to denote that X is exponentially distributed with parameter β.

B. Main Results

Lemma 1 (PDF of 1/X): Given a RV $X \sim \mathcal{E}(\beta)$, the PDF of $Y = 1/X$ can be evaluated with the help of [10, Sec. 5.2], to yield

$$p_Y(y) = \frac{\beta}{y^2} e^{-\beta/y} U(y). \tag{10}$$

Lemma 2 (MGF of 1/X): Given a RV $X \sim \mathcal{E}(\beta)$, the MGF of $Y = 1/X$, $\mathcal{M}_Y(s) = \mathbf{E}_Y\left(e^{-sy}\right)$ can be evaluated with the help of [11, eq. (3.471.9)] and using the symmetry property of the modified Bessel function (i.e., $K_v(x) = K_{-v}(x)$) given in [11, eq. (8.486.16)] to yield

$$\mathcal{M}_Y(s) = 2\sqrt{\beta s} K_1\left(2\sqrt{\beta s}\right) \tag{11}$$

where $K_1(\cdot)$ is the first order modified Bessel function of the second kind defined in [12, eq. 9.6.22)].

Theorem 1 (CDF of the Harmonic Mean of Two Exponential RVs): Let X_1 and X_2 be two independent exponential RVs with parameters β_1 and β_2, respectively, [i.e., $X_i \sim \mathcal{E}(\beta_i)$, $i = 1, 2$]. Then, the CDF of $X = \mu_H(X_1, X_2)$, $P_X(x)$, is given by

$$P_X(x) = 1 - x\sqrt{\beta_1 \beta_2} e^{-x/2(\beta_1 + \beta_2)} K_1\left(x\sqrt{\beta_1 \beta_2}\right). \tag{12}$$

Proof: See Appendix A.

Corollary 1 (PDF of the Harmonic Mean of Two Exponential RVs): Let X_1 and X_2 be two independent exponential RVs with parameters β_1 and β_2, respectively, [i.e., $X_i \sim \mathcal{E}(\beta_i)$, $i = 1, 2$]. Then, the PDF of $X = \mu_H(X_1, X_2)$, $p_X(x)$, is given by

$$\begin{aligned} p_X(x) = \frac{1}{2}\beta_1 \beta_2 x e^{-x/2(\beta_1 + \beta_2)} \Bigg[\left(\frac{\beta_1 + \beta_2}{\sqrt{\beta_1 \beta_2}}\right) & \\ \times K_1\left(x\sqrt{\beta_1 \beta_2}\right) + 2K_0\left(x\sqrt{\beta_1 \beta_2}\right) \Bigg] U(x) & \end{aligned} \tag{13}$$

where $K_0(\cdot)$ is the zeroth-order modified Bessel function of the second kind defined in [12, eq. 9.6.21)].

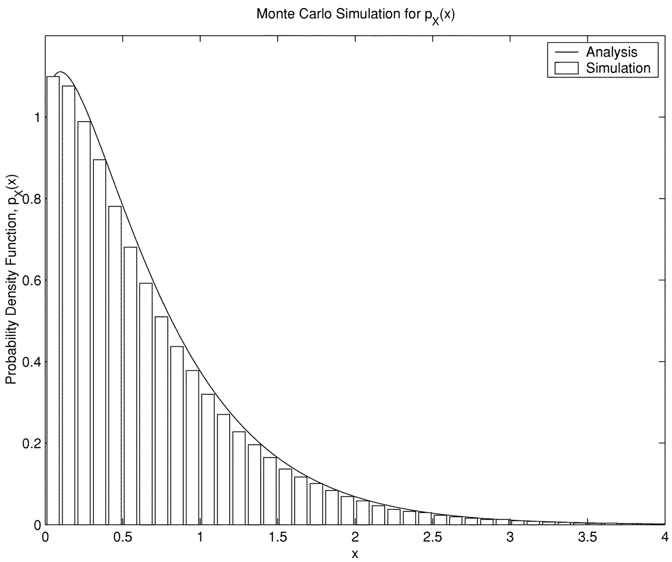

Fig. 2. Comparison between the analytical results for the PDF (13) and the Monte Carlo simulation ($\beta_1 = 1$, $\beta_2 = 1$, and 200 000 iterations).

Proof: Taking the derivative of (12) with respect to x and using the expression for the derivative of the modified Bessel function, given in [11, eq. (8.486.12)] as

$$z\frac{d}{dz}K_v(z) + vK_v(z) = -zK_{v-1}(z) \tag{14}$$

yields (13).

As a double check, the PDF in (13) was validated by Monte Carlo simulation, as illustrated in Fig. 2. It is clear that there is an excellent match between the analytical result in (13) and the Monte Carlo simulation.

Corollary 2 (MGF of the Harmonic Mean of Two Exponential RVs): Let X_1 and X_2 be two independent exponential RVs with parameters β_1 and β_2, respectively, [i.e., $X_i \sim \mathcal{E}(\beta_i)$, $i = 1, 2$]. Then, the MGF of $X = \mu_H(X_1, X_2)$, $\mathcal{M}_X(s) = \mathbf{E}_X(e^{-sx})$ is given by

$$\begin{aligned}\mathcal{M}_X(s) = &\frac{16\beta_1\beta_2}{3\left(\beta_1+\beta_2+2\sqrt{\beta_1\beta_2}+s\right)^2}\\ &\times\left[\frac{4(\beta_1+\beta_2)}{\left(\beta_1+\beta_2+2\sqrt{\beta_1\beta_2}+s\right)}\right.\\ &\times {}_2F_1\left(3,\frac{3}{2};\frac{5}{2};\frac{\beta_1+\beta_2-2\sqrt{\beta_1\beta_2}+s}{\beta_1+\beta_2+2\sqrt{\beta_1\beta_2}+s}\right)\\ &\left.+\,{}_2F_1\left(2,\frac{1}{2};\frac{5}{2};\frac{\beta_1+\beta_2-2\sqrt{\beta_1\beta_2}+s}{\beta_1+\beta_2+2\sqrt{\beta_1\beta_2}+s}\right)\right]\end{aligned} \tag{15}$$

where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the Gauss' hypergeometric function defined in [12, eq. (15.1.1)]. If $\beta_1 = \beta_2 = \beta$, it can be shown that (15) greatly simplifies to the compact form

$$\mathcal{M}_X(s) = {}_2F_1\left(1,2;\frac{3}{2};-\frac{\beta}{2}s\right). \tag{16}$$

Proof: The integral involved in the derivation of the MGF can be evaluated with the help of [11, eq. (6.621.3)], which once used with some extra manipulations leads to the desired result in (15).

IV. Application to the End-to-End Performance of Communication Systems With Relays

In light of the definition of the harmonic mean of RVs presented in (8), we notice that γ_{eq_2} in (6) can be written as $\gamma_{\text{eq}_2} = \mu_H(\gamma_1, \gamma_2)/2$. Assume now that the two links are subject to Rayleigh type of fading. The instantaneous SNR of the two links are $\gamma_1 \sim \mathcal{E}(1/\overline{\gamma}_1)$ and $\gamma_2 \sim \mathcal{E}(1/\overline{\gamma}_2)$, where $\overline{\gamma}_1$ and $\overline{\gamma}_2$ are the average SNRs of the two links, respectively. To get the PDF of γ_{eq_2} we use corollary 1 with two modifications. First, we set

$$\beta_i = \frac{1}{\overline{\gamma}_i}. \tag{17}$$

Next, we use the following transformation of variables

$$p_\Gamma(\gamma) = 2\,p_X(2\gamma). \tag{18}$$

As a result, substituting (17) and (13) in (18) leads to the PDF of γ_{eq_2}, $p_\Gamma(\gamma)$, as

$$\begin{aligned}p_\Gamma(\gamma) = \frac{2\gamma e^{-\gamma(1/\overline{\gamma}_1+1/\overline{\gamma}_2)}}{\overline{\gamma}_1\overline{\gamma}_2}&\left[\left(\frac{\overline{\gamma}_1+\overline{\gamma}_2}{\sqrt{\overline{\gamma}_1\overline{\gamma}_2}}\right)K_1\left(\frac{2\gamma}{\sqrt{\overline{\gamma}_1\overline{\gamma}_2}}\right)\right.\\ &\left.+2K_0\left(\frac{2\gamma}{\sqrt{\overline{\gamma}_1\overline{\gamma}_2}}\right)\right]U(\gamma).\end{aligned} \tag{19}$$

A. Average BER

1) Nonregenerative Systems: Based on (17) and (15) of Corollary 2, the MGF of γ_{eq_2}, $\mathcal{M}_\Gamma(s)$, can be shown to be given by

$$\begin{aligned}\mathcal{M}_\Gamma(s) = &\frac{16}{3\overline{\gamma}_1\overline{\gamma}_2\left(\frac{1}{\overline{\gamma}_1}+\frac{1}{\overline{\gamma}_2}+\frac{2}{\sqrt{\overline{\gamma}_1\overline{\gamma}_2}}+s\right)^2}\\ &\times\left[\frac{4\left(\frac{1}{\overline{\gamma}_1}+\frac{1}{\overline{\gamma}_2}\right)}{\left(\frac{1}{\overline{\gamma}_1}+\frac{1}{\overline{\gamma}_2}+\frac{2}{\sqrt{\overline{\gamma}_1\overline{\gamma}_2}}+s\right)}\right.\\ &\times {}_2F_1\left(3,\frac{3}{2};\frac{5}{2};\frac{\frac{1}{\overline{\gamma}_1}+\frac{1}{\overline{\gamma}_2}-\frac{2}{\sqrt{\overline{\gamma}_1\overline{\gamma}_2}}+s}{\frac{1}{\overline{\gamma}_1}+\frac{1}{\overline{\gamma}_2}+\frac{2}{\sqrt{\overline{\gamma}_1\overline{\gamma}_2}}+s}\right)\\ &\left.+\,{}_2F_1\left(2,\frac{1}{2};\frac{5}{2};\frac{\frac{1}{\overline{\gamma}_1}+\frac{1}{\overline{\gamma}_2}-\frac{2}{\sqrt{\overline{\gamma}_1\overline{\gamma}_2}}+s}{\frac{1}{\overline{\gamma}_1}+\frac{1}{\overline{\gamma}_2}+\frac{2}{\sqrt{\overline{\gamma}_1\overline{\gamma}_2}}+s}\right)\right].\end{aligned} \tag{20}$$

If the two links are identical, i.e., $\overline{\gamma}_1 = \overline{\gamma}_2 = \overline{\gamma}$, then based on (16), the MGF in (20) reduces to

$$\mathcal{M}_\Gamma(s) = {}_2F_1\left(1,2;\frac{3}{2};-\frac{\overline{\gamma}}{4}s\right). \tag{21}$$

which can be written in terms of the more common inverse hyperbolic sin function as

$$\mathcal{M}_\Gamma(s) = \frac{\sqrt{\frac{\overline{\gamma}}{4}s\left(\frac{\overline{\gamma}}{4}s+1\right)}+\operatorname{arcsinh}\left(\sqrt{\frac{\overline{\gamma}}{4}s}\right)}{2\sqrt{\frac{\overline{\gamma}}{4}s}\left(\frac{\overline{\gamma}}{4}s+1\right)^{3/2}}. \tag{22}$$

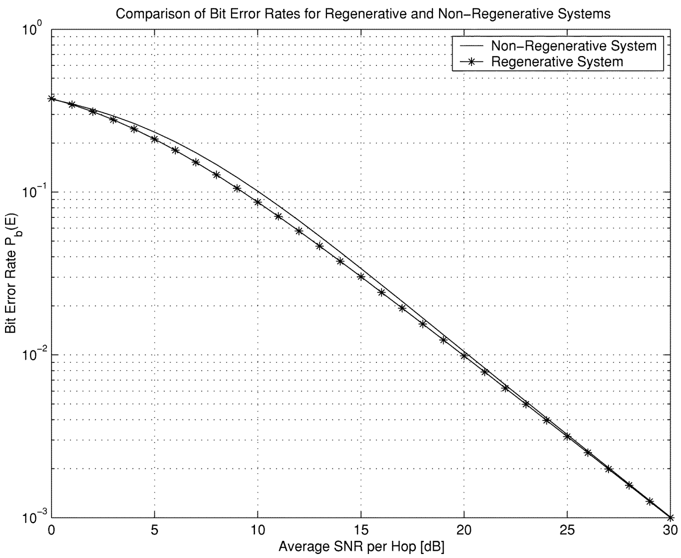

Fig. 3. Comparison between the BER of nonregenerative and regenerative systems (balanced links, $\overline{\gamma}_1 = \overline{\gamma}_2 = \overline{\gamma}$).

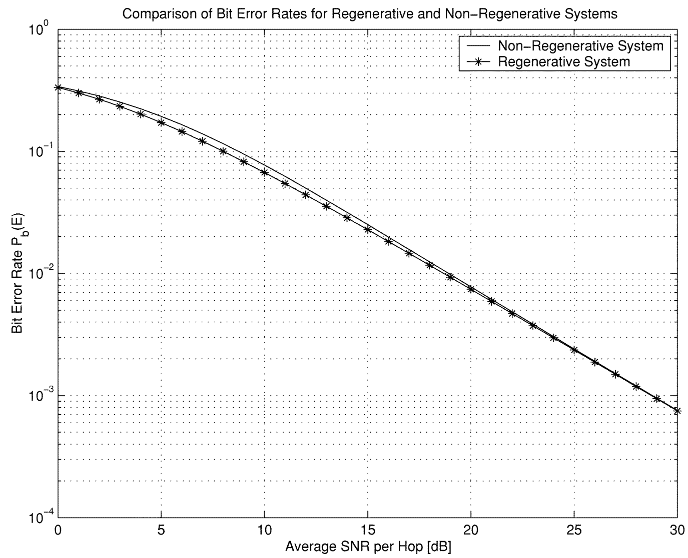

Fig. 4. Comparison between the BER of nonregenerative and regenerative systems (unbalanced links, $\overline{\gamma}_1 = 2\overline{\gamma}_2$).

Having the MGF of γ_{eq_2} in closed form as in (20), and using the MGF-based approach for the performance evaluation of digital modulations over fading channels [13] allows to obtain the average symbol-error rate (SER) for a wide variety of M-ary modulations [such as M-ary phase-shift keying (M-PSK), M-ary differential phase-shift keying (M-DPSK), and M-ary quadrature amplitude modulation (M-QAM)]. For example, the average BER of binary DPSK is given by $P_b(E) = 1/2\, \mathcal{M}_\gamma(1)$.

2) Regenerative Systems: In regenerative systems, the relaying node decodes the signal and then transmits the detected version to the destination node. This means that the transmitted signal undergoes two states of decoding in cascade, and the overall probability of error is given by [1, eq. (11.4.12)]

$$P_b(E/\gamma_1, \gamma_2) = P_b(E/\gamma_1) + P_b(E/\gamma_2) - 2P_b(E/\gamma_1)\, P_b(E/\gamma_2) \quad (23)$$

which when averaged over the two independent RVs γ_1 and γ_2 reduces to

$$P_b(E) = P_b(E_1) + P_b(E_2) - 2P_b(E_1)P_b(E_2) \quad (24)$$

where $P_b(E_i)$ is the average BER of link i, $i = 1, 2$. For DPSK over independent Rayleigh-fading channels, (24) can be rewritten as

$$\begin{aligned} P_b(E) =& \frac{1}{2}\left[\left(\frac{1}{1+\overline{\gamma}_1}\right) + \left(\frac{1}{1+\overline{\gamma}_2}\right) - \left(\frac{1}{1+\overline{\gamma}_1}\right)\left(\frac{1}{1+\overline{\gamma}_2}\right)\right] \\ =& \frac{1+\overline{\gamma}_1+\overline{\gamma}_2}{2\,(1+\overline{\gamma}_1)\,(1+\overline{\gamma}_2)}. \end{aligned} \quad (25)$$

3) Comparison: Fig. 3 compares the BER performance of DPSK with regenerative systems (25) and with the nonregenerative systems (20). It is clear from the figure that: 1) regenerative systems perform better at low average SNR and 2) at high average SNR, the two systems are equivalent BER wise. These results hold also when the two links are unbalanced as shown in Fig. 4. Note that this is in contrast to the AWGN scenario in which there is always (over the whole SNR range) a 3-dB performance gain of regenerative systems over nonregenerative systems when the two hops are balanced (in terms of SNR).

B. Outage Probability

1) Nonregenerative Systems: In noise limited systems, outage probability is defined as the probability that the instantaneous SNR γ falls below a predetermined protection ratio γ_{th}, namely

$$P_{\mathrm{out}} = \mathrm{P}[\gamma \le \gamma_{\mathrm{th}}] = \int_0^{\gamma_{\mathrm{th}}} p_\Gamma(\gamma)d\gamma = P_\Gamma(\gamma_{\mathrm{th}}). \quad (26)$$

In (26), the predetermined protection ratio γ_{th} is a threshold SNR above which the quality of service is satisfactory and which essentially depends on the type of modulation employed and the type of application supported. Using (12) of Theorem 1 along with (17), the outage probability is equal to $P_X(2\gamma_{\mathrm{th}})$ and can be shown to be given by

$$P_{\mathrm{out}} = 1 - \frac{2\gamma_{\mathrm{th}}}{\sqrt{\overline{\gamma}_1\overline{\gamma}_2}} K_1\left(\frac{2\gamma_{\mathrm{th}}}{\sqrt{\overline{\gamma}_1\overline{\gamma}_2}}\right) e^{-\gamma_{\mathrm{th}}(1/\overline{\gamma}_1 + 1/\overline{\gamma}_2)}. \quad (27)$$

This expression for the outage probability is, in addition to be an exact formula for the choice of relay gain in (7), is also a tight lower bound for a nonregenerative system employing a relay as in (4). To validate this claim, a Monte Carlo simulation was run, and the results are shown in Fig. 5. It is clear from the figure that for all practical purposes the two curves are essentially indistinguishable.

2) Regenerative Systems: For comparison, we consider also a regenerative system for which an outage occurs if either one of the links is in outage. Equivalently, it is the complement event of having both links operating above the threshold, γ_{th}. Hence, outage probability is given by

$$\begin{aligned} P_{\mathrm{out}} =& 1 - \left(\int_{\gamma_{\mathrm{th}}}^{\infty} \frac{1}{\overline{\gamma}_1} e^{-\gamma/\overline{\gamma}_1} d\gamma\right)\left(\int_{\gamma_{\mathrm{th}}}^{\infty} \frac{1}{\overline{\gamma}_2} e^{-\gamma/\overline{\gamma}_2} d\gamma\right) \\ =& 1 - e^{-\gamma_{\mathrm{th}}(1/\overline{\gamma}_1 + 1/\overline{\gamma}_2)}. \end{aligned} \quad (28)$$

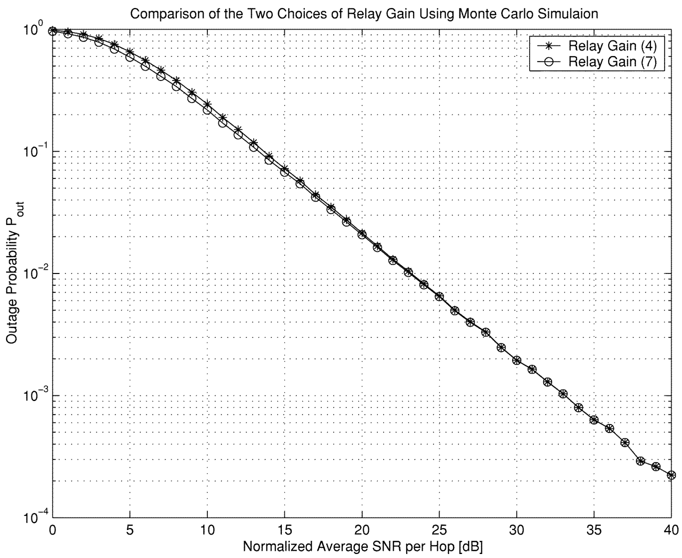

Fig. 5. Monte Carlo simulations for the types of relay gains in (4) and (7).

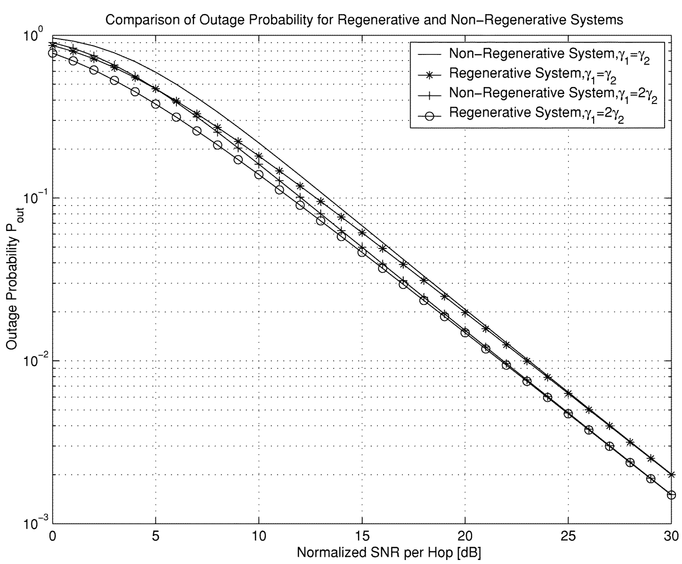

Fig. 6. Comparison of the outage probability of regenerative and nonregenerative systems.

3) Comparison: Fig. 6 compares the performance of nonregenerative systems with that of regenerative systems. Similar to the average BER results, it is clear that regeneration improves the performance at low average SNR in exchange of an increased complexity. At high average SNR, the two systems are equivalent outage probability wise. Note that the results in this case can be proven analytically by comparing the second terms in (27) and (28), which differ only by the scaling factor $[2\gamma_{\mathrm{th}}/\sqrt{\overline{\gamma}_1\overline{\gamma}_2}K_1\left(2\gamma_{\mathrm{th}}/\sqrt{\overline{\gamma}_1\overline{\gamma}_2}\right)]$ in front of the exponential function. At high average SNR, this scaling factor converges to one since the $K_1(x)$ function converges to $1/x$ when x approaches zero [12, eq. 9.6.9)]. Also, note that this behavior holds for both balanced or unbalanced links as shown in the same figure.

V. Conclusion

In this paper, end-to-end performance of wireless communication systems with nonregenerative relays is presented by applying some new statistical results on the harmonic mean of two independent exponential variates. Numerical results show that systems with regenerative relays outperform those with nonregenerative relays at low average SNR. However, at high average SNR, these same results show that the two systems are essentially equivalent in terms of average BER and outage probability. Outage capacity was also investigated for both regenerative and nonregenerative systems and the results are omitted here due to space limitations but are presented in [14].

Appendix
Proof of Theorem 1

Let X_1 and X_2 be two independent RVs distributed as $X_i \sim \mathcal{E}(\beta_i)$, $i = 1, 2$. To find the CDF of the harmonic mean of X_1 and X_2, we define a new RV Z as

$$Z = \frac{1}{2}\left(\frac{1}{X_1} + \frac{1}{X_2}\right) \tag{29}$$

which is the reciprocal of $\mu_H(X_1, X_2)$. Under the independence assumption between X_1 and X_2, the MGF of Z can be written as half the product of the MGF of $Y_1 = 1/X_1$ and $Y_2 = 1/X_2$. Hence, using lemma 1 the MGF of Z can be shown to be given by

$$\mathcal{M}_Z(s) = 2\sqrt{\beta_1\beta_2}sK_1\left(2\sqrt{\beta_1 s}\right)K_1\left(2\sqrt{\beta_2 s}\right). \tag{30}$$

The CDF of $X = \mu_H(X_1, X_2)$, $P_X(x)$, is given by

$$\begin{aligned} P_X(x) =& \Pr\left(X < x\right) \\ =& \Pr\left(\frac{1}{X} > \frac{1}{x}\right) = \Pr\left(Z > \frac{1}{x}\right) \\ =& 1 - \Pr\left(Z < \frac{1}{x}\right) = 1 - P_Z\left(\frac{1}{x}\right) \end{aligned} \tag{31}$$

where $P_Z(\cdot)$ is the CDF of Z. Using the differentiation property of Laplace transform, $P_Z(z)$ can be written as

$$P_Z(z) = \mathcal{L}^{-1}\left(\frac{\mathcal{M}_Z(s)}{s}\right) \tag{32}$$

where $\mathcal{L}^{-1}(\cdot)$ denotes the inverse Laplace transform. Substituting (30) in (32) which in turn when substituted in (31) leads to $P_X(x)$ as

$$\begin{aligned} P_X(x) = 1 - \mathcal{L}^{-1}\Bigg(&2\sqrt{\beta_1\beta_2}K_1\left(2\sqrt{\beta_1 s}\right) \\ &\times K_1\left(2\sqrt{\beta_2 s}\right)\Bigg)\Big|_{z=1/x} \end{aligned} \tag{33}$$

which can be evaluated with the help of [15, eq. (13.2.20)] to yield the desired result (12).

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References

[1] R. M. Gagliardi, *Introduction to Communications Engineering*. New York: Wiley, 1988.

[2] A. Sendonaris, E. Erkip, and B. Aazhang, "Increasing uplink capacity via user cooperation diversity," in *Proc. IEEE Int. Symp. Information Theory (ISIT'98)*, Cambridge, MA, Aug. 1998, p. 156.
[3] J. N. Laneman and G. W. Wornell, "Energy efficient antenna sharing and relaying for wireless networks," in *Proc. IEEE Wireless Communication and Networking Conf. (WCNC'00)*, Chicago, IL, Oct. 2000, pp. 7–12.
[4] ——, "Exploiting distributed spatial diversity in wireless networks," in *Proc. 40th Allerton Conf. Communication, Control, and Computing (Allerton'00)*, Allerton Park, IL, Sept. 2000, pp. 775–785.
[5] M. Grossglauser and D. Tse, "Mobility increases the capacity of ad-hoc wireless networks," in *Proc. IEEE Conf. Computer Communications (INFOCOM'01)*, Anchorage, AK, April 2001, pp. 1360–1369.
[6] J. N. Laneman, G. W. Wornell, and D. N. C. Tse, "An efficient protocol for realizing cooperative diversity in wireless networks," in *Proc. IEEE Int. Symp. Inform. Theory (ISIT'01)*, Washington, DC, June 2001, p. 294.
[7] V. Emamian and M. Kaveh, "Combating shadowing effects for systems with transmitter diversity by using collaboration among mobile users," in *Proc. Int. Symp. Communications, (ISC'01)*, Taiwan, R.O.C., Nov. 2001, pp. 105.1–105.4.
[8] H. Yanikomeroglu, "Fixed and mobile relaying technologies for cellular networks," in *Proc. 2nd Workshop on Applications and Services in Wireless Networks (ASWN'02)*, Paris, France, July 2002, pp. 75–81.
[9] M. D. Springer, *The Algebra of Random Variables*. New York: Wiley, 1979.
[10] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, 3rd ed. New York: McGraw-Hill, 1991.
[11] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. San Diego, CA: Acdemic, 1994.
[12] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, 9th ed. New York: Dover, 1970.
[13] M. K. Simon and M.-S. Alouini, *Digital Communication Over Fading Channels*. New York: Wiley, 2000.
[14] M. O. Hasna and M.-S. Alouini, "Performance analysis of two-hop Relayed transmission over Rayleigh-fading channels," in *Proc. IEEE Vehicular Technology Conf. (VTC'02)*, Vancouver, BC, Canada, Sept. 2002, pp. 1992–1996.
[15] G. E. Roberts and H. Kaufman, *Table of Laplace Transform*. Philadelphia, PA: Saunders, 1966.