Spring 2022 – Cooperative Communications and Networks

Assignment 1

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Numerical Questions

Q1) [SNR, 10%] A receiver measured the received SNR equal to 5dB and the thermal noise power equal to $N_0 = 10^{-14}$ Watts. What is the received power strength in Watts?

$10^{-14} \times 10^{5/10} \approx 3.163e^{-14}$ Watts. For such a small value, it is commonly represented in dBm:

$$x = 10\log_{10}\frac{P}{1\text{ mW}}.$$

So $3.163e^{-14}$ Watts corresponds to -105 dBm. Typical cell phone's output power is about 10 17 dBm or 10 50 mW.

Q2) [Shadow fading, 20%] The path loss due to distance is deterministic but it becomes non-deterministic when shadowing comes into play. Determine the probability that the path loss due to both distance and log-normal shadowing with zero mean and variance 5^2 dB at distance of 500 m is below 118 dB, given the path-loss exponent $\nu = 4.35$ and the reference distance $d_0 = 1$ m.

With shadowing, the relative path loss at distance d is given by

$$\Delta L_p(d) = 10\nu\log_{10}\frac{d}{d_0} + X_{(dB)}. \tag{1}$$

For the shadowing variable $X \sim N(0, \sigma^2)$, the probability that the path loss at distance d is below the threshold γ dB is

$$\begin{aligned}\Pr[\Delta L_p(d) \le \gamma] &= \Pr[X \le \gamma - 10\nu\log_{10}(d/d_0)] &(2)\\ &= 1 - Q\left(\frac{\gamma - 10\nu\log_{10}(d/d_0)}{\sigma}\right) &(3)\end{aligned}$$

where $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp(-t^2/2)dt$. Plugging in the given parameter values, we obtain

$$\Pr[\Delta L_p(d) \le \gamma] = \boxed{0.5473}.$$

Q3) [Distribution, 40%] The following question helps you to recall some basics learned from Probability. It also builds the foundation of Monte Carlo Simulation (will be discussed next week), an important tool to study the performance of wireless communications systems using simulations.

Suppose X is an indicator variable that characterizes whether a transmitter has sent a bit successfully.

$$X = \begin{cases} 1, & \text{if a bit is sent in error} \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

a) Let P_e denote the probability of transmission error. Derive the mean and variance of X in terms of P_e.

The transmission process is a Bernoulli trial, which has the mean $\mathbb{E}[X] = P_e$ and variance $\mathbb{V}[X] = P_e(1 - P_e)$.

b) Suppose the transmitter sends n bits among which the number of erroneous bits is denoted as Y. Derive the mean and variance of Y.

The number of erroneous bits follows the binomial distribution which has the mean $\mathbb{E}[X] = n \cdot P_e$ and variance $\mathbb{V}[X] = nP_e(1 - P_e)$.

c) The average rate of transmission errors can be found as $\hat{P}_e = Y/n$ (which is also known as the *sample mean* or the *empirical mean*). Derive the mean and variance of $\hat{P}_e$.

Since Y is binomial distributed, the ratio Y/n is also binomial distributed with mean

$$\mathbb{E}\left[\hat{P}_e\right] = \mathbb{E}\left[\frac{Y}{n}\right] = \frac{1}{n}\mathbb{E}\left[Y\right] = P_e, \tag{5}$$

and variance

$$\mathbb{V}\left[\hat{P}_e\right] = \mathbb{V}\left[\frac{Y}{n}\right] = \frac{1}{n^2}\mathbb{V}\left[Y\right] = \frac{P_e(1 - P_e)}{n}. \tag{6}$$

d) When n goes large, $\hat{P}_e$ can be approximately normal. Derive the 95% confidence interval for $\hat{P}_e$.

Hint: Think of X, Y, and $\hat{P}_e$ as random variables and try to figure our their statistical distributions.

Suppose we have n samples for an experiment: $X_1, \cdots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. It is convenient to normalize this samples such that the mean is zero and variance is one. Suppose $\bar{X}$ is the sample mean (i.e., $\bar{X} = \hat{P}_e$).

$$\text{standardization}: \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \triangleq Z \sim \mathcal{N}(0, 1). \tag{7}$$

Confidence interval: $\mathbb{P}[-z \leq Z \leq z] = 1 - \alpha$.

$$F_Z(z) - F_Z(-z) = \Phi(z) - (1 - \Phi(z)) = 1 - \alpha \tag{8}$$

$$\Rightarrow \Phi(z) = \frac{2 - \alpha}{2} = 1 - \frac{\alpha}{2} \tag{9}$$

$$\Rightarrow z = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right). \tag{10}$$

For $\alpha = 0.05, z = 1.96$.

$$\mathbb{P}[-1.96 \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq 1.96] \tag{11}$$

$$\Rightarrow \mathbb{P}[\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}], \tag{12}$$

where $\sigma = \sqrt{P_e(1-P_e)}$, according to Q3(b). The confidence interval is

$$\left(\bar{X} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}\right). \tag{13}$$

How many samples we need to have the 95% confidence interval? The confidence interval can be also expressed as

$$|\hat{P}_e - P_e| < 1.96\frac{\sigma}{\sqrt{n}} = 1.96\frac{\sqrt{P_e(1-P_e)}}{\sqrt{n}}. \tag{14}$$

Given a predefined confidence interval, say I, we can find the number of simulations required, which is equal to

$$n = P_e(1-P_e)\left(\frac{1.96}{I}\right)^2. \tag{15}$$

For example, we want to simulate a transmission system whose theoretical error probability is $P_e = 10^{-3}$. How many number of simulation runs should be performed to ensure the 95% confidence interval with $I = 10^{-4}$?

When simulating communication systems, the error rate is often very small. Then it is desirable to specify the confidence interval as a fraction of the error rate,

$$I = \beta P_e, \tag{16}$$

for a $\beta\%$ acceptable estimation error. Hence Eq. (15) can be modified as

$$n \cdot P_e = (1-P_e)\left(\frac{1.96}{\beta}\right)^2 \approx \left(\frac{1.96}{\beta}\right)^2. \tag{17}$$

Since $n \cdot P_e$ is the expected number of errors, Eq. (17) provides a rule of thumb that we can stop the simulation when the number of errors reaches $(1.96/\beta)$!

Simulation

Q4) [Random Generator, 40%] Write a computer program to generate the normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$.

a) Verify the accuracy of your random generator by plotting the probability mass function (PMF) of empirical results and comparing it with the theoretical one. In your figure, clearly indicate the simulation and theoretical curves.

b) Describe the programming software you use (name and version) and what commands/functions are used to generate random samples. Also, show the number of samples you generated to obtain the statistics.

The generation of random samples is used in nearly every simulation for communications networks, e.g., signal transmission at the physical layer, channel access and routing protocols at the MAC layer, and packet delivery at the link layer.

See Fig. 1.

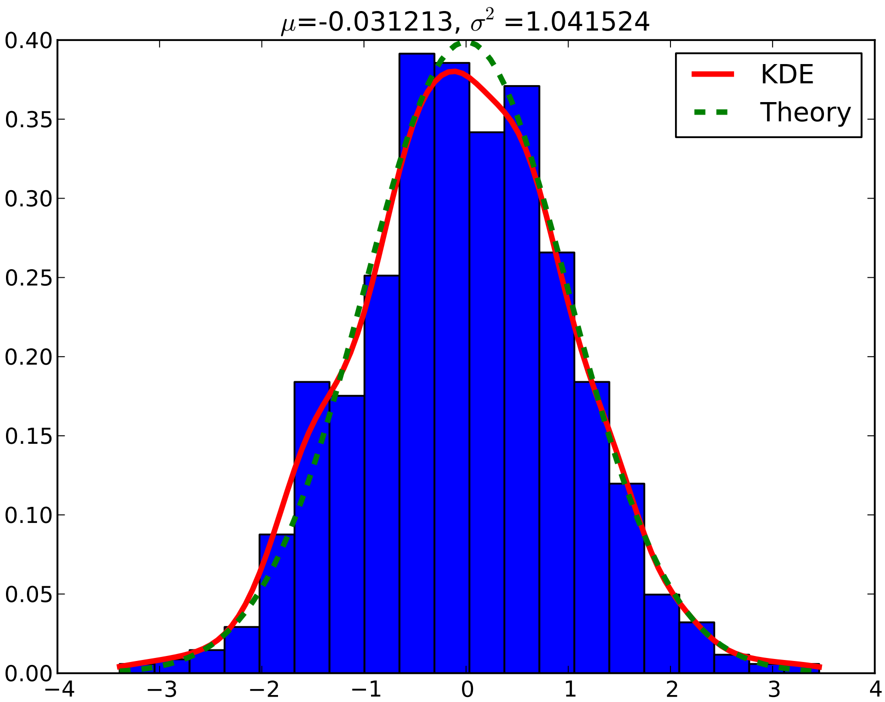

Figure 1: Empirical density function of the normal distribution from $10,000$ samples with $\mu = 0$ and $\sigma^2 = 1$. The blue bars show the histogram, the red line shows the kernel density estimation, and the green dashed line shows the theoretical density function.