Spring 2022– Cooperative Communications and Networks

Solutions for Assignment 3

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Q1) [Selection DF, 40%] Derive the outage probability of Selection Decode-and-Forward (SDF) when the average SNR of the first hop is identical to that of the direct channel, i.e., $\bar{\gamma}_0 = \bar{\gamma}_1$. Show the diversity gain of SDF is equal to two. In your derivations, use $\gamma_i = P_i a_i^2 / N_0$ to denote the received SNR of the i-th hop, where $i = 0$ refers to the source-destination channel, $i = 1$ corresponds to the source-relay channel, and $i = 2$ represents the relay-destination channel.

The achievable end-to-end rate is

$$C^{\mathsf{SDF}} = \begin{cases} \frac{1}{2}\log_2\left(1 + \gamma_0^{(1)} + \gamma_2\right), & \text{if } \gamma_1 \geq \mathtt{T}, \\ \frac{1}{2}\log_2\left(1 + \gamma_0^{(1)} + \gamma_0^{(2)}\right), & \text{if } \gamma_1 < \mathtt{T}. \end{cases}$$

By definition, the outage probability can be obtained as

$$\begin{aligned} P_{out}^{\mathsf{SDF}} &= \mathbb{P}[\gamma_1 < \mathtt{T}]\mathbb{P}\left[\log_2(1 + \gamma_0^{(1)} + \gamma_0^{(2)}) < \mathtt{R}\right] \\ &\quad + \mathbb{P}[\gamma_1 \geq \mathtt{T}]\mathbb{P}\left[\log_2(1 + \gamma_0^{(1)} + \gamma_2) < \mathtt{R}\right] \\ &= \underbrace{\mathbb{P}[\gamma_1 < \mathtt{T}]\mathbb{P}\left[\gamma_0^{(1)} + \gamma_0^{(2)} < \mathtt{T}\right]}_{①} \\ &\quad + \underbrace{\mathbb{P}[\gamma_1 \geq \mathtt{T}]\mathbb{P}\left[\gamma_0^{(1)} + \gamma_2 < \mathtt{T}\right]}_{②}. \end{aligned}$$

1) The first term accounts for the case when R can't help and thus S retransmits in phase II. For Rayleigh fading,

$$\begin{aligned} &\mathbb{P}\left[\gamma_0^{(1)} + \gamma_0^{(2)} < \mathtt{T}\right] = 1 - e^{-\frac{\mathtt{T}}{\bar{\gamma}_0^{(1)} + \bar{\gamma}_0^{(2)}}} \\ \Rightarrow &\mathbb{P}[\gamma_1 < \mathtt{T}]\mathbb{P}\left[\gamma_0^{(1)} + \gamma_0^{(2)} < \mathtt{T}\right] = \left(1 - e^{-\frac{\mathtt{T}}{\bar{\gamma}_1}}\right)\left(1 - e^{-\frac{\mathtt{T}}{\bar{\gamma}_0^{(1)} + \bar{\gamma}_0^{(2)}}}\right) \end{aligned}$$

2) The second term involves the summation of two independent SNRs, $\gamma_0^{(1)}$ and γ_2, whose CDF can be

found in Eq. (18) in the lecture notes. Combining the above results, we have

$$\begin{aligned} P_{out} =& \left(1 - e^{-\frac{\mathtt{T}}{\bar{\gamma}_1}}\right)\left(1 - e^{-\frac{\mathtt{T}}{\bar{\gamma}_0^{(1)}+\bar{\gamma}_0^{(2)}}}\right) \\ &+ \begin{cases} e^{-\frac{\mathtt{T}}{\bar{\gamma}_1}}\left[1 - \left(\frac{\bar{\gamma}_0^{(1)}}{\bar{\gamma}_0^{(1)}-\bar{\gamma}_2} e^{-\frac{\mathtt{T}}{\bar{\gamma}_0^{(1)}}} + \frac{\bar{\gamma}_2}{\bar{\gamma}_2-\bar{\gamma}_0^{(1)}} e^{-\frac{\mathtt{T}}{\bar{\gamma}_2}}\right)\right], & \text{for } \bar{\gamma}_0^{(1)} \neq \bar{\gamma}_2 \\ e^{-\frac{\mathtt{T}}{\bar{\gamma}_1}}\left[1 - \left(1 + \frac{\mathtt{T}}{\bar{\gamma}_1}\right) e^{-\frac{\mathtt{T}}{\bar{\gamma}_1}}\right], & \text{for } \bar{\gamma}_0^{(1)} = \bar{\gamma}_2 \end{cases} \end{aligned}$$

At high SNR, using the first-order Taylor approximation yields

$$P_{out} \approx \begin{cases} \frac{\mathtt{T}^2}{\bar{\gamma}_1(\bar{\gamma}_0^{(1)}+\bar{\gamma}_0^{(2)})}, & \text{for } \bar{\gamma}_0^{(1)} \neq \bar{\gamma}_2 \\ \left(\frac{\mathtt{T}}{\bar{\gamma}_0^{(1)}}\right)^2 + \frac{\mathtt{T}^2}{(\bar{\gamma}_0^{(1)}+\bar{\gamma}_0^{(2)})\bar{\gamma}_1} - \left(\frac{\mathtt{T}}{\bar{\gamma}_0^{(1)}}\right)^2 \frac{\mathtt{T}}{\bar{\gamma}_1}, & \text{for } \bar{\gamma}_0^{(1)} = \bar{\gamma}_2. \end{cases}$$

To see the diversity gain, take the case $\bar{\gamma}_0^{(1)} = \bar{\gamma}_2$ for example. Recall the definition of average SNR, $\bar{\gamma}_i = P_i/N_0\bar{\alpha}_i^2$. Let $P_i = P$ and $\mathtt{SNR} = P/N_0$, the outage probability can be expressed as

$$P_{out} \approx \frac{\mathtt{T}^2}{\mathtt{SNR}^2} \frac{1}{\mathbb{E}[\alpha_1^2](\mathbb{E}[(\alpha_0^{(1)})^2] + \mathbb{E}[(\alpha_0^{(2)})^2]}.$$

Since P_{out} of SDF is proportional to $\mathtt{SNR}^{-2}$, diversity gain is two.

Q2) $[10\%]$ The SNR mentioned above is the ratio between the transmission power P and the noise power N_0. Suppose $P = 1$. For a given SNR in decibel, show the formula you used to determine the noise power.

The formula is

$$10^{-\mathsf{SNR}_{\mathsf{dB}}/20} \tag{1}$$

Q3) $[30\%]$ Plot both the simulated and theoretical outage probabilities as a function of SNR in one figure.

- SNR range: $0 \ \sim 30$ dB.
- Clearly label each curve (simulated or analytical).
- Set y-axis as logarithm scale.
- Colors are not differentiable when printed in black and white. Use different line styles and symbols to represent different curves.

See Fig. 1.

Q4) $[20\%]$ Plot the average squared gain versus SNR (dB). Explain how the average squared gain varies with the noise power. With the trend you observed, does the noise amplification problem get worse at higher SNR?

See Fig. 2. Since the transmission power is fixed, a larger SNR means a smaller noise power. When the noise power is small, the relay can use a larger amplification gain to strengthen the signal power without enlarged noise amplification.

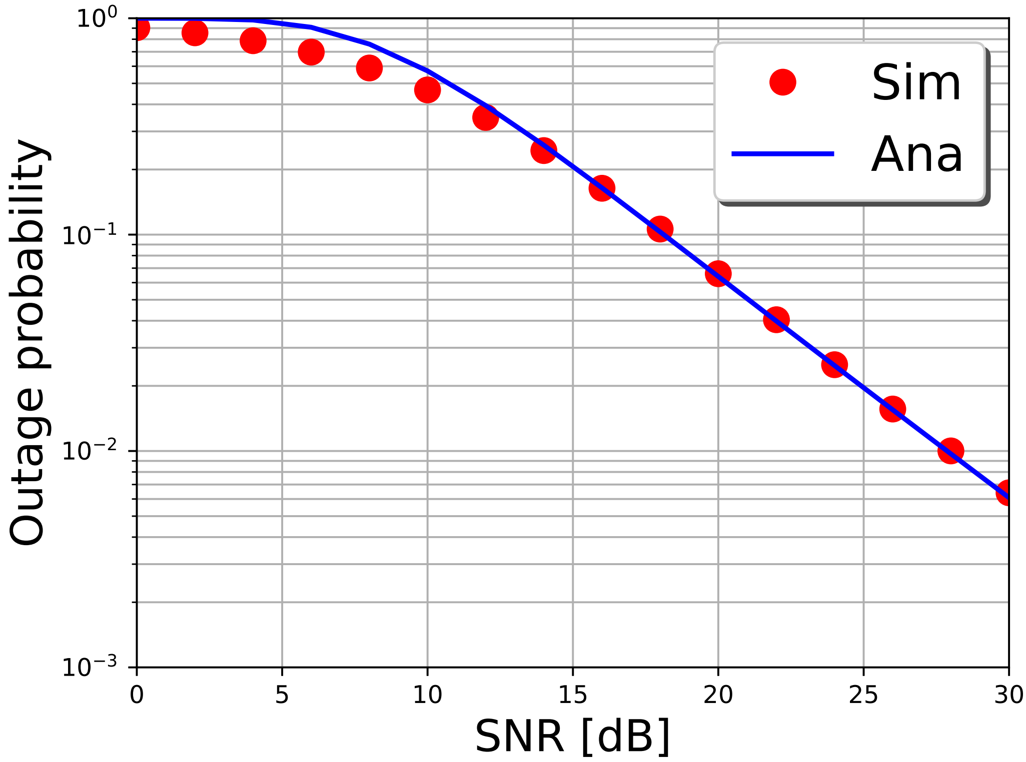

Figure 1: Outage probability of VAF.

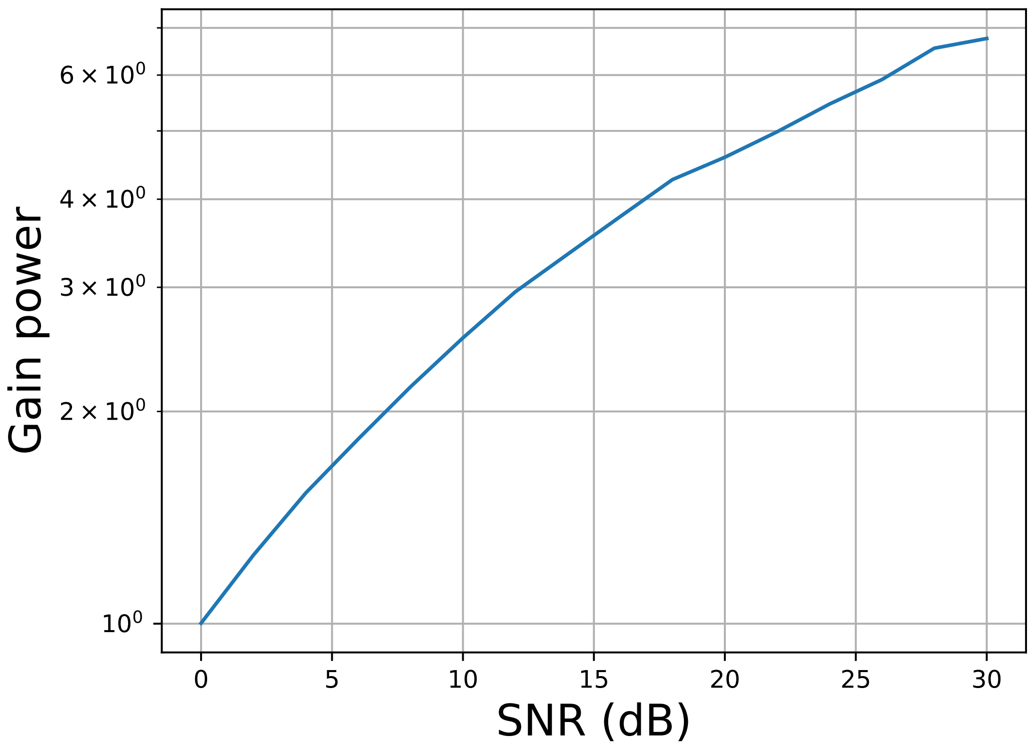

Figure 2: Average squared gain of VAF.