COM 5120 Communication Theory
Homework #6 Solution

1.

(a)

$$h_0 = 1, h_1 = -\frac{1}{4}$$

$$q_k = \sum_{n=1}^{1} c_n h_{k-n} \Rightarrow c = [c_{-1}\ c_0\ c_1] = [0\ 1\ \frac{1}{4}]$$

(b)

$$h_0 = 1, h_1 = -\frac{1}{4}$$

$$q_k = \sum_{n=1}^{1} c_n h_{k-n}$$

$$q_k = c_{-1}h_3 + c_0 h_2 + c_1 h_1 = -\frac{1}{16}$$

2.

(a) The total capacity is $C = \sum_{i=1}^{3} \Delta f_i \log_2[1 + \frac{P(f_i)|H(f_i)|^2}{\sigma_i^2}]$

Where $P(f_i)$ is the PSD of the subchannel $i, i = 1,2,3$ and

$$\Delta f_1 = W_1$$
$$\Delta f_2 = W_2 - W_1$$
$$\Delta f_3 = W - W_2$$
$$P_i = P(f_i)\Delta f_i, i = 1,2,3$$

This problem becomes

maximize C subject to $P = \sum_{i=1}^{3} P_i = \sum_{i=1}^{3} \Delta f_i P(f_i)$

with Lagrange multiplier λ

$$J = C + \lambda(P - \sum_{i=1}^{3} P_i)$$

$$= \sum_{i=1}^{3} \Delta f_i \log_2[1 + \frac{P(f_i)|H(f_i)|^2}{\sigma_i^2}] + \lambda[P - \sum_{i=1}^{3} \Delta f_i P(f_i)]$$

$$\frac{\partial J}{\partial P(f_i)} = 0$$

$$\Delta f_i \frac{\log_2 e}{P(f_i)+\frac{\sigma_i^2}{|H(f_i)|^2}} - \lambda \Delta f = 0$$

$$\Rightarrow \lambda = \frac{\log_2 e}{P(f_i)+\frac{\sigma_i^2}{|H(f_i)|^2}}$$

$$\begin{cases} P(f_i) = \left(\kappa - \frac{\sigma_i^2}{|H(f_i)|^2}\right)^+, i = 1 \sim 3 \quad \text{where } [x]^+ \equiv \max\{x,0\} \text{ (power } P(f_i) \geq 0 \text{ for all } f_i \text{)} \\ P = \sum_{i=1}^{3} \Delta f_i P(f_i) \end{cases}$$

where $\kappa = \frac{\log_2 e}{\lambda}$

(b)

$$P = \sum_{i=1}^{3} P_i = \sum_{i=1}^{3} \Delta f_i P(f_i) = 2$$

$$\Rightarrow \Delta f_1(\kappa - 1) + \Delta f_2(\kappa - \frac{3}{2}) + \Delta f_3(\kappa - \frac{9}{4}) = 2$$

If $\Delta f_i = \Delta f, i = 1 \sim 3$

$$\kappa = \frac{2/3}{\Delta f} + \frac{19}{12}$$

$$P(f_1) = \kappa - 1 = \frac{2}{3\Delta f} + \frac{7}{12} = \frac{5}{4}$$

$$P(f_2) = \kappa - \frac{3}{2} = \frac{2}{3\Delta f} + \frac{1}{12} = \frac{3}{4}$$

$$P(f_3) = \kappa - \frac{9}{4} = \left(\frac{2}{3\Delta f} - \frac{2}{3}\right)^+ = 0$$

3.

If $I_k = I_{k-1}$, then $y_k = I_{k-1} + I_{k-1}/4 + n_k = \frac{5}{4} I_k + n_k$

$$d = \frac{5}{4}\sqrt{E_b} \Rightarrow P_{e,I_k = I_{k-1}} = Q(\frac{5}{4}\sqrt{\frac{2E_b}{N_0}})$$

If $I_k \neq I_{k-1}$, then $y_k = I_{k-1} + I_{k-1}/4 + n_k = \frac{3}{4} I_k + n_k$

$$d = \frac{3}{4}\sqrt{E_b} \Rightarrow P_{e,I_k \neq I_{k-1}} = Q(\frac{3}{4}\sqrt{\frac{2E_b}{N_0}})$$

Therefore, $P_e = \frac{1}{2} P_{e,I_k = I_{k-1}} + \frac{1}{2} P_{e,I_k \neq I_{k-1}} = \frac{1}{2}\left[Q(\frac{5}{4}\sqrt{\frac{2E_b}{N_0}}) + Q(\frac{3}{4}\sqrt{\frac{2E_b}{N_0}}) \right]$

4. When $a=1$ is transmitted, the error probability is

$$\begin{aligned}
P_{e,1} &= P(y_m < 0 \mid a_m = 1) \\
&= P(1 + n_m + i_m < 0) \\
&= \sum_{i_m} P(n_m < -1 - i_m \mid i_m) P(i_m) \\
&= \frac{1}{4} P(n_m < -\frac{3}{2}) + \frac{1}{2} P(n_m < -1) + \frac{1}{4} P(n_m < -\frac{1}{2}) \\
&= \frac{1}{4} P(\frac{n_m}{\sigma_n} < -\frac{3}{2\sigma_n}) + \frac{1}{2} P(\frac{n_m}{\sigma_n} < -\frac{1}{\sigma_n}) + \frac{1}{4} P(\frac{n_m}{\sigma_n} < -\frac{1}{2\sigma_n}) \\
&= \frac{1}{4} Q(\frac{3}{2\sigma_n}) + \frac{1}{2} Q(\frac{1}{\sigma_n}) + \frac{1}{4} Q(\frac{1}{2\sigma_n})
\end{aligned}$$

Due to the symmetry of this scheme,

$$P_{e,1} = P_{e,-1}$$

So

$$\begin{aligned}
P_e &= \frac{1}{2}(P_{e,-1} + P_{e,1}) \\
&= \frac{1}{4} Q(\frac{3}{2\sigma_n}) + \frac{1}{2} Q(\frac{1}{\sigma_n}) + \frac{1}{4} Q(\frac{1}{2\sigma_n})
\end{aligned}$$

5.

(a) The equivalent channel taps are

$$h_0 = 0.9, h_1 = 0.3, h_{-1} = 0.3$$

We know

$$q_m = \sum_n c_n h_{m-n}$$

Given

$$q_m = \begin{cases} 1, & m = 0 \\ 0, & m = \pm 1 \end{cases}$$

We have

$$\begin{bmatrix} 0.9 & 0.3 & 0 \\ 0.3 & 0.9 & 0.3 \\ 0 & 0.3 & 0.9 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} -0.4762 \\ 1.4286 \\ -0.4762 \end{bmatrix}$$

(b) Using

$$q_m = \sum_n c_n h_{m-n}$$

We can compute

$$q_{-2} = -0.1429$$
$$q_{-3} = 0$$
$$q_2 = -0.1429$$
$$q_3 = 0$$