Communication theory

Homework #4 Solution

Due:

1. (1) $\mathrm{H(X)} = -\sum_{k=1}^{\infty} p(1-p)^{k-1}\log_2\left(p(1-p)^{k-1}\right)$

$$= -\mathrm{p}\log_2 p \sum_{k=1}^{\infty}(1-p)^{k-1} - p\log_2(1-p)\sum_{k=1}^{\infty}(k-1)(1-p)^{k-1}$$

$$= -\mathrm{p}\log_2 p\frac{1}{1-(1-p)} - p\log_2(1-p)\frac{1-p}{\left(1-(1-p)\right)^2}$$

$$= -\log_2 p - \frac{1-p}{p}\log_2(1-p)$$

(2) For $\mathrm{k} \leq \mathrm{K}$, $\mathrm{P(X = k|X > K) = 0}$

For k > K, $\mathrm{P(X = k|X > K)} = \frac{P(X=k,X>K)}{P(X>K)} = \frac{p(1-p)^{1-k}}{P(X>K)}$ ,

$$\mathrm{P(X > K)} = \sum_{k=K+1}^{\infty} p(1-p)^{k-1} = p\frac{(1-p)^K}{1-(1-p)} = (1-p)^K$$

so that,

$$\mathrm{P(X = k|X > K)} = \frac{p(1-p)^{k-1}}{(1-p)^k} = p(1-p)^{l-1}, \mathrm{l = k - K}$$

The conditional entropy is,

$$\mathrm{H(X|X > K)} = -\sum \mathrm{P(X = k|X > K)}\log_2 \mathrm{P(X = k|X > K)}$$

$$= -\sum_{l=1}^{\infty} p(1-p)^{l-1}\log_2\left(p(1-p)^{l-1}\right)$$

$$= -\log_2 p\sum_{l=1}^{\infty} p(1-p)^{l-1} - p\sum_{l=1}^{\infty}(1-p)^{l-1}\log_2(1-p)^{l-1}$$

$$= -\mathrm{p}\log_2 p\frac{1}{1-(1-p)} - p\log_2(1-p)\frac{1-p}{(1-(1-p))^2}$$

$$= -\log_2 p - \frac{1-p}{p}\log_2(1-p)$$

2. $\mathrm{P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1)} = \frac{5}{14}$

$$\mathrm{P(X = 1) = P(X = 1, Y = 0) + P(X = 1, Y = 1)} = \frac{9}{14}$$

$$\mathrm{P(Y = 0) = P(X = 0, Y = 0) + P(X = 1, Y = 0)} = \frac{2}{7}$$

$$\mathrm{P(Y = 1) = P(X = 0, Y = 1) + P(X = 1, Y = 1)} = \frac{5}{7}$$

So that,

$$\mathrm{H(X)} = -\sum_k P(X = k)\log_2 P(X = k) = -\frac{5}{14}\log_2\frac{5}{14} - \frac{9}{14}\log_2\frac{9}{14} = 0.9402$$

$$\mathrm{H(Y)} = -\sum_k P(Y=k)\log_2 P(Y=k) = -\frac{2}{7}\log_2\frac{2}{7} - \frac{5}{7}\log_2\frac{5}{7} = 0.8631$$

$$\mathrm{H(X,Y)} = -\sum_k\sum_j P(X=j,Y=k)\log_2 P(X=j,Y=k)$$

$$= -\frac{1}{14}\log_2\frac{1}{14} - \frac{2}{7}\log_2\frac{2}{7} - \frac{3}{14}\log_2\frac{3}{14} - \frac{3}{7}\log_2\frac{3}{7} = 1.786$$

$$\mathrm{H(X|Y)} = \mathrm{H(X,Y)} - \mathrm{H(Y)} = 1.786 - 0.8631 = 0.9229$$

$$\mathrm{H}(\mathrm{Y}|X) = \mathrm{H(X,Y)} - \mathrm{H}(X) = 1.786 - 0.9402 = 0.8458$$

3. (1) $\mathrm{H(X)} = -\int_0^\infty \frac{1}{\lambda}e^{-\frac{x}{\lambda}}\ln\left(\frac{1}{\lambda}e^{-\frac{x}{\lambda}}\right)dx$

$$= -\ln\frac{1}{\lambda}\int_0^\infty \frac{1}{\lambda}e^{-\frac{x}{\lambda}}dx + \int_0^\infty \frac{1}{\lambda}e^{-\frac{x}{\lambda}}\frac{x}{\lambda}dx$$

$$= \ln\lambda + \frac{1}{\lambda}\int_0^\infty \frac{1}{\lambda}e^{-\frac{x}{\lambda}}x\,dx$$

$$= \ln\lambda + \frac{1}{\lambda}\lambda = 1 + \ln\lambda$$

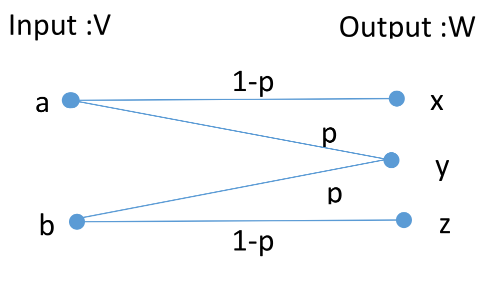
(2) $\mathrm{H(X)} = -\int_{-\infty}^\infty \frac{1}{2\lambda}e^{-\frac{|x|}{\lambda}}\ln\left(\frac{1}{2\lambda}e^{-\frac{|x|}{\lambda}}\right)dx$

$$= -\ln\frac{1}{2\lambda}\int_{-\infty}^\infty \frac{1}{2\lambda}e^{-\frac{|x|}{\lambda}}dx + \frac{1}{\lambda}\int_{-\infty}^\infty |x|\frac{1}{2\lambda}e^{-\frac{|x|}{\lambda}}dx$$

$$= \ln(2\lambda) + \frac{1}{\lambda}\left[\int_{-\infty}^0 -x\frac{1}{2\lambda}e^{\frac{x}{\lambda}}dx + \int_0^\infty x\frac{1}{2\lambda}e^{-\frac{x}{\lambda}}dx\right]$$

$$= \ln(2\lambda) + \frac{1}{2\lambda}\lambda + \frac{1}{2\lambda}\lambda = 1 + \ln(2\lambda)$$

4. (1) $\mathrm{I} = \mathrm{I(V;W)} = \mathrm{H(W)} - \mathrm{H(W|V)}$




Assume $\mathrm{P(V)} = [P(a), P(b)] = [r, 1-r]$

$$\mathrm{P(W|V)} = \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix}$$

$$\mathrm{P}(W,V) = \begin{bmatrix} r & 0 \\ 0 & 1-r \end{bmatrix}\begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix}$$

$$= \begin{bmatrix} r(1-p) & rp & 0 \\ 0 & (1-r)p & (1-r)(1-p) \end{bmatrix}$$

$$\mathrm{H(W|V)} = -r(1-p)\log_2(1-p) - rp\log_2 p$$
$$-(1-r)\log_2 p - (1-r)(1-p)\log_2(1-p)$$
$$= -\mathrm{p}\log_2 p - (1-p)\log_2(1-p)$$

$$\mathrm{P(W)} = \mathrm{P(V)P(W|V)} = [r \quad 1-r]\begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix}$$

$$= [r(1-p) \quad p \quad (1-r)(1-p)]$$

$$\mathrm{H(W)} = -r(1-p)\log_2 r(1-p) - p\log_2 p$$
$$-(1-r)(1-p)\log_2(1-r)(1-p)$$
$$\mathrm{I(V;W)} = \mathrm{H(W)} - \mathrm{H(W|V)} = (1-\mathrm{p})[-\mathrm{r}\log_2 r - (1-r)\log_2(1-r)]$$

(2) $\mathrm{C} = \max\{\mathrm{I(V;W)}\}$ occurs at $\mathrm{r} = \frac{1}{2}$

So that, $\mathrm{C} = (1-\mathrm{p})$

5. (1) $\mathrm{P(correct\ codeword)} = (1-p)^R$

(2) $\mathrm{P(at\ least\ one\ bit\ error\ in\ the\ codeword)} = 1 - \mathrm{P(correct\ codeword)}$
$= 1-(1-p)^R$

(3) $\mathrm{P}(n_e \text{ or less errors in R bits}) = \sum_{i=1}^{n_e}\binom{R}{i}p^i(1-p)^{R-i}$

(4) For R = 5, p = 0.1, $n_e = 5$

$(1-p)^R = 0.5905$

$1-(1-p)^R = 0.409$

$\sum_{i=1}^{n_e}\binom{R}{i}p^i(1-p)^{R-i} = 1-(1-p)^R = 0.409$