COM 5120
Communications Theory

Chapter 11
Multi-carrier Communications

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Outline

- **Lecture 1: Single carrier vs Multi-carrier Communications**

✓ **Multicarrier capacity**

✓ **Power allocation of multicarrier communications**

- **Lecture 2: Orthogonal Frequency Division Multiplexing(OFDM)**

✓ **OFDM architecture**

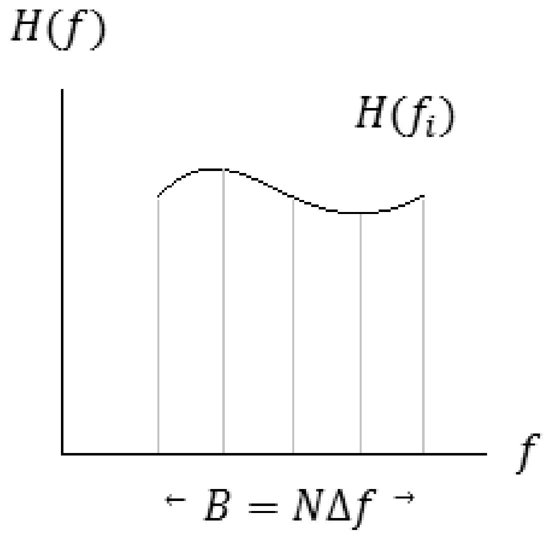
✓ **Implementation of OFDM with IDFT/DFT**

✓ **Channel effect for OFDM signaling**

✓ **PAPR problem in OFDM**

Single carrier vs Multi-carrier Communications

The multicarrier communication approach divide the available channel bandwidth into *N* subchannels, such that each subchannel has nearly flat fading.



Motivation:

Complicated equalizer could be saved or replaced with simple (i.e. short-length) equalizer.

Single carrier vs Multi-carrier Communications

Given that frequency-selective fading channel with bandwidth B.

Divide the bandwidth into N subchannels. $B = N\Delta f$

The capacity of each subchannel is

$$C_i = \Delta f \log_2[1 + \frac{\Delta f P(f_i)\left|H(f_i)\right|^2}{\Delta f S_{nn}(f_i)}]$$

where $P(f_i)$ is the power spectral density of transmit signal. i.e. $\Delta f P(f_i)$ is the power at subchannel i,

$H(f_i)$ is the channel fading gain at f_i.

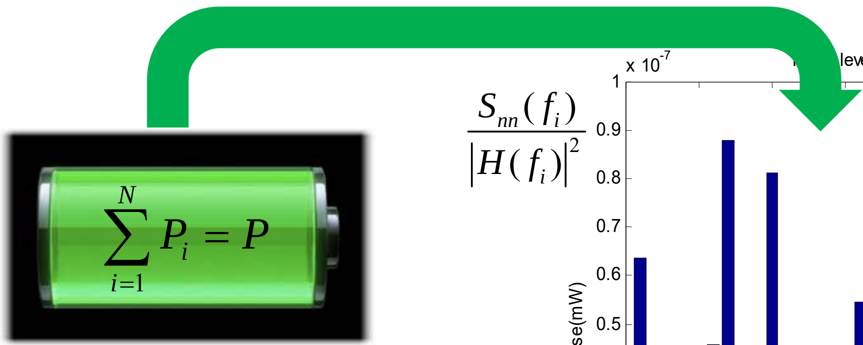
$S_{nn}(f_i)$ is the PSD of noise.

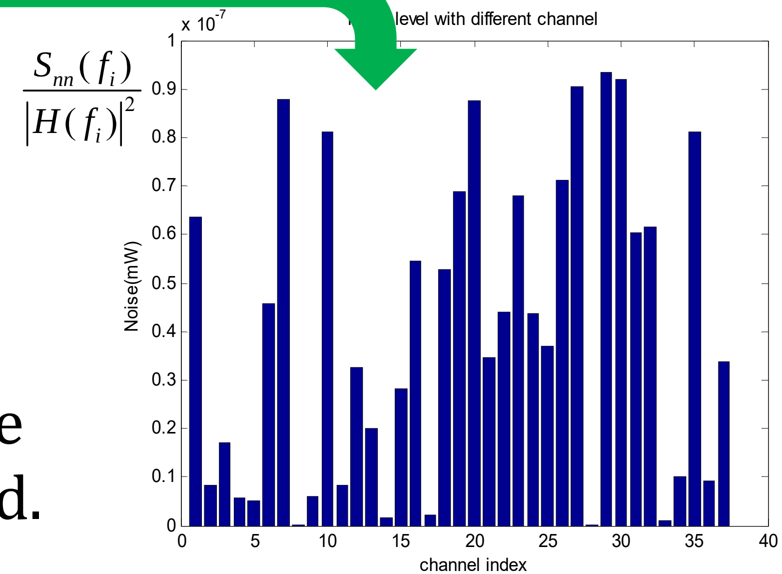
Single carrier vs Multi-carrier Communications

The total capacity is

$$C = \sum_{i=1}^{N} C_i = \Delta f \sum_{i=1}^{N} \log_2[1 + \frac{P(f_i)|H(f_i)|^2}{S_{nn}(f_i)}]$$

The transmit signal power is allocated over the subchannels subject to constraint $\sum_{i=1}^{N} P_i = P$, where $P_i = \Delta f P(f_i)$





With power allocation of P(f_i) ,the total capacity C can be maximized.

Single carrier vs Multi-carrier Communications

Q: How to find $\{P(f_i), i = 1 \sim N\}$ to maximize C while satisfying the power constraint?

- With the help of Lagrange multiplier, joint consideration of the objective function and the constraint can be formulated as,

$$J = C + \lambda(P - \sum_{i=1}^{N} P_i)$$

$$= \Delta f \sum_{i=1}^{N} \log_2[1 + \frac{P(f_i)\left|H(f_i)\right|^2}{S_{nn}(f_i)}] + \lambda[P - \Delta f \sum_{i=1}^{N} P(f_i)]$$

The J is maximized when $\dfrac{\partial J}{\partial P(f_i)} = 0, i = 1, \ldots N$

Single carrier vs Multi-carrier Communications

Note that $\dfrac{d\ln x}{dx} = \dfrac{1}{x},$

$$\frac{d\ln(1+ax)}{dx} = \frac{d\ln(1+ax)}{d(1+ax)}\frac{d(1+ax)}{dx} = \frac{1}{x+\frac{1}{a}}$$

Let $x_i = P(f_i),\ a_i = \dfrac{\left|H(f_i)\right|^2}{S_{nn}(f_i)}$

$$\Rightarrow J = \Delta f \sum_{i=1}^{N} \log_2[1 + \frac{x_i\left|H(f_i)\right|^2}{S_{nn}(f_i)}] + \lambda[P - \Delta f \sum_{i=1}^{N} x_i]$$

$$\frac{\partial J}{\partial P(f_i)} = \Delta f \cdot \frac{\log_2 e}{P(f_i) + \frac{S_{nn}(f_i)}{\left|H(f_i)\right|^2}} - \lambda \Delta f = 0, \qquad i = 1, \ldots, N$$

Single carrier vs Multi-carrier Communications

$$\Rightarrow \lambda = \frac{\log_2 e}{P(f_i) + \frac{S_{nn}(f_i)}{\left|H(f_i)\right|^2}}$$

$$\Rightarrow \begin{cases} P(f_i) = \frac{1}{\lambda \ln 2} - \frac{S_{nn}(f_i)}{\left|H(f_i)\right|^2}, \ \ i = 1, \ldots N \\ \Delta f \sum_{i=1}^{N} P(f_i) = P \end{cases}$$

$N+1$ linear equations for $N+1$ variables,

$P(f_i), i = 1, .., N,$ and λ

Single carrier vs Multi-carrier Communications

Besides, $P(f_i) \geq 0$, for all f_i

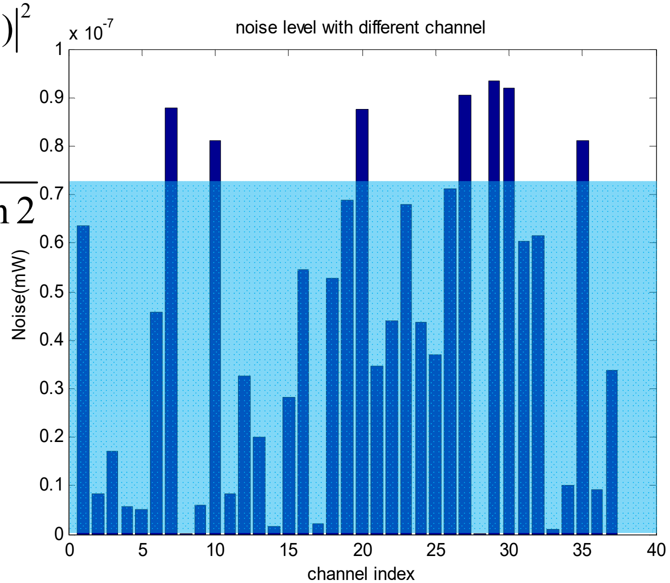
$$\Rightarrow P(f_i) = \left[\frac{1}{\lambda \ln 2} - \frac{S_{nn}(f_i)}{|H(f_i)|^2} \right]^+ , i = 1 \sim N, \text{where } [x]^+ \equiv \max\{x, 0\}$$

- Analogy to water-filling power allocation with water level= $\dfrac{1}{\lambda \ln 2}$

$$P(f_i) + \frac{S_{nn}(f_i)}{|H(f_i)|^2} = \frac{1}{\lambda \ln 2} = \text{constant}$$

$\dfrac{S_{nn}(f_i)}{|H(f_i)|^2}$

$\dfrac{1}{\lambda \ln 2}$



$\Rightarrow$ More power allocated for better subchannel,

i.e. as $\dfrac{S_{nn}(f_i)}{|H(f_i)|^2} \downarrow, P(f_i) \uparrow$

- What if the $P(f_i) < 0$ in the calculation?
  - ➢ Iterative water-filling (IWF) power allocation is needed.

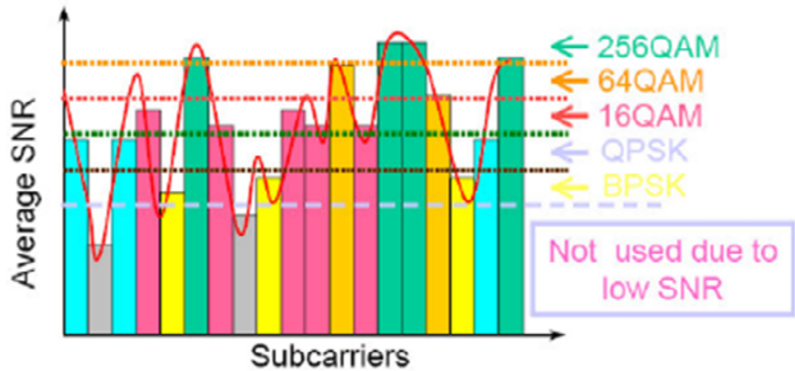
Q: What happen to the water level when Iterative water-filling is needed?

Power Allocation Strategy Applications

- The power allocation strategy can be applied to many other daily life scenario, such as time management, resource management, and investment, etc.

Bit and Power Allocation in Multicarrier Modulation

- Assume N subcarriers and M_{i} points QAM modulation symbols on each subcarrier
  $\rightarrow M_i = 2^{b_i}$, and b_i bits transmitted on each subcarrier.
- The total bit rate $R_b = \dfrac{1}{\mathrm{T}} \sum_{i=1}^{N} b_i$



•The power allocated on subcarrie

power allocated is $P = \sum_{i=1}^{N} P_i$

•The QAM error probability on subcarrier i is $\mathrm{P_e} \cong 4Q\left(\sqrt{\dfrac{3P_i|c_i|^2}{N_0(M_i - 1)}}\right)$

$\rightarrow$ Given P_{i} and $\mathrm{P_e}$ requirement$_{\mathrm{i}}$, the QAM modulation order M_i can be

determined by $Q\left(\sqrt{\dfrac{3P_i|c_i|^2}{N_0(M_i - 1)}}\right) \le \dfrac{\mathrm{P_e}}{4}$. Hence $R_b = \sum_{i=1}^{N} \dfrac{\log_2 M_i}{T}$ is determined.

Outline

- Lecture 1: Single carrier vs Multi-carrier Communications

✓ Multicarrier capacity

✓ Power allocation of multicarrier communications

- **Lecture 2: Orthogonal Frequency Division Multiplexing(OFDM)**

✓ **OFDM architecture**

✓ **Implementation of OFDM with IDFT/DFT**

✓ **Channel effect for OFDM signaling**

✓ **PAPR problem in OFDM**

Orthogonal Frequency Division Multiplexing

- The available channel bandwidth B_T is divided into N subchannels, each of bandwidth Δf, i.e, $B_T = N\Delta f$
- Assign a subcarrier signal for each subchannel. Suppose each subcarrier is modulated with M-ary QAM symbols. Then the signal on the kth subcarrier:

$$s_k(t) = \sqrt{\frac{2}{T}} A_{ki} \cos(2\pi f_k t) - \sqrt{\frac{2}{T}} A_{kq} \sin(2\pi f_k t), \; k = 0,1,\ldots N-1$$

$$= \mathrm{Re}\{\sqrt{\frac{2}{T}} X_k e^{j2\pi f_k t}\}$$

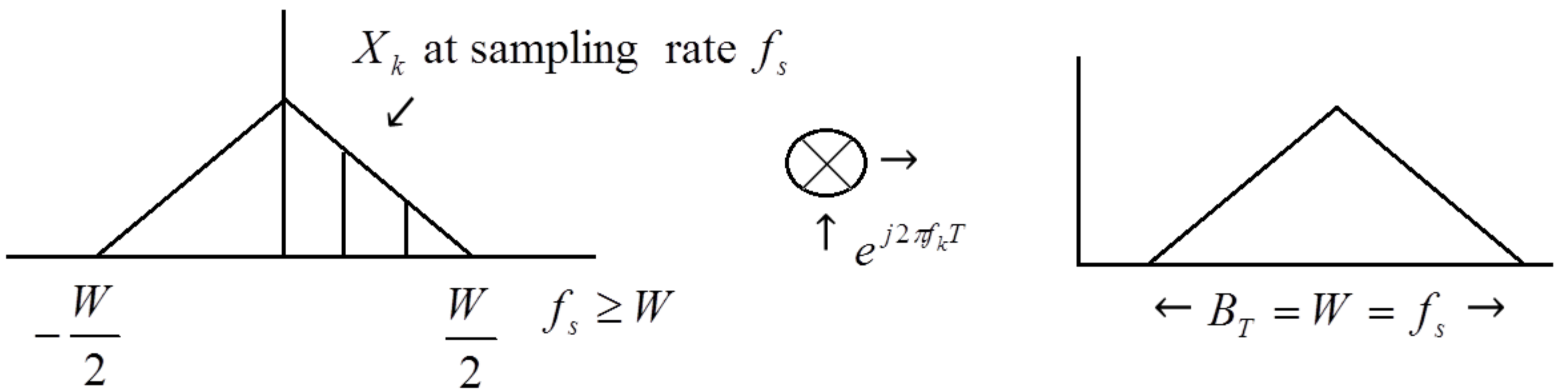
where $X_k = A_{ki} + jA_{kq}, A_{ki} \; A_{kq} \in \{\pm1, \pm3, \ldots \pm(M-1)\}$

Orthogonal Frequency Division Multiplexing

Define $\phi_k(t) = \sqrt{\frac{2}{T}} e^{j2\pi f_k t}, 0 \le t \le T$

$$\Rightarrow \int_0^T \varphi_k(t)\varphi_j^*(t)dt = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$$

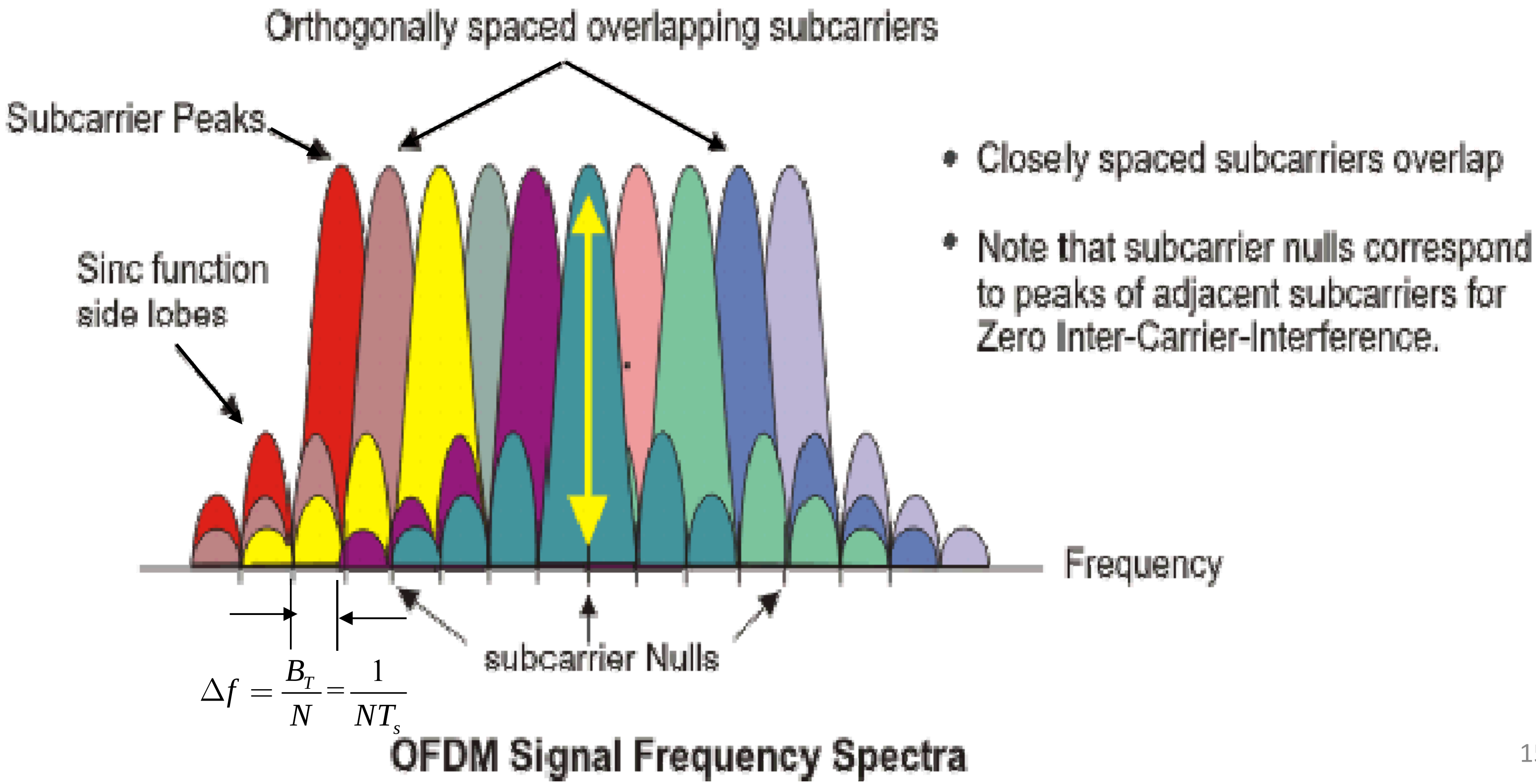
X_k represents the modulated symbols at sample rate of f_s, where $f_s = 1/T_s = B_T$ and $T = NT_s$



Orthogonal Frequency Division Multiplexing

Time domain $\varphi_k(t) = \sqrt{\frac{2}{T}} e^{j 2\pi f_k t}, \; 0 \le t \le T$

Freq domain: $\phi_k(f) = \sqrt{2T}\,\mathrm{sinc}\left[2\pi\left(f - f_k\right)T\right]$



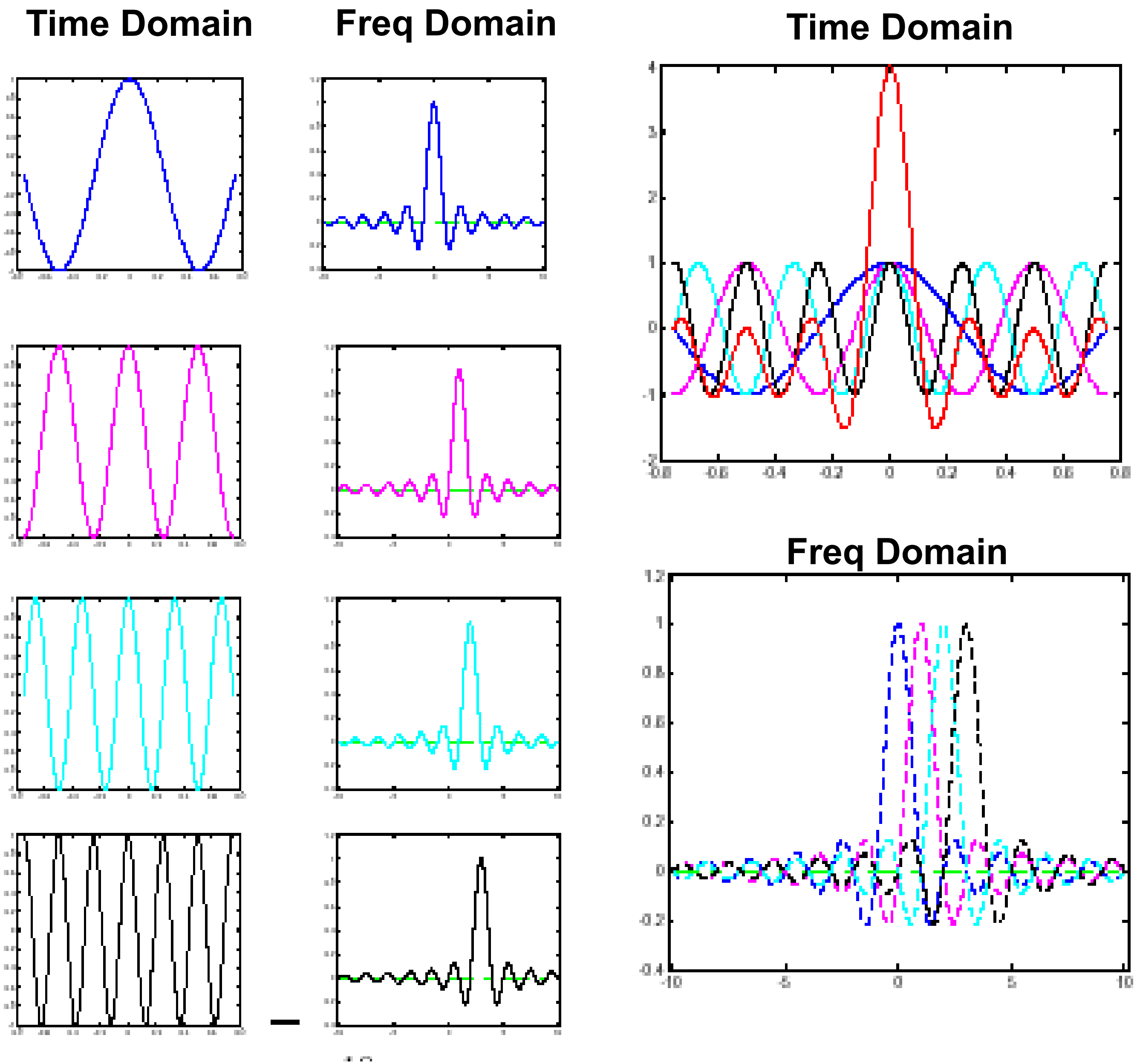
Subcarrier Orthogonality of OFDM

Subcarrier k=1

Subcarrier k=2

Subcarrier k=3

Subcarrier k=4



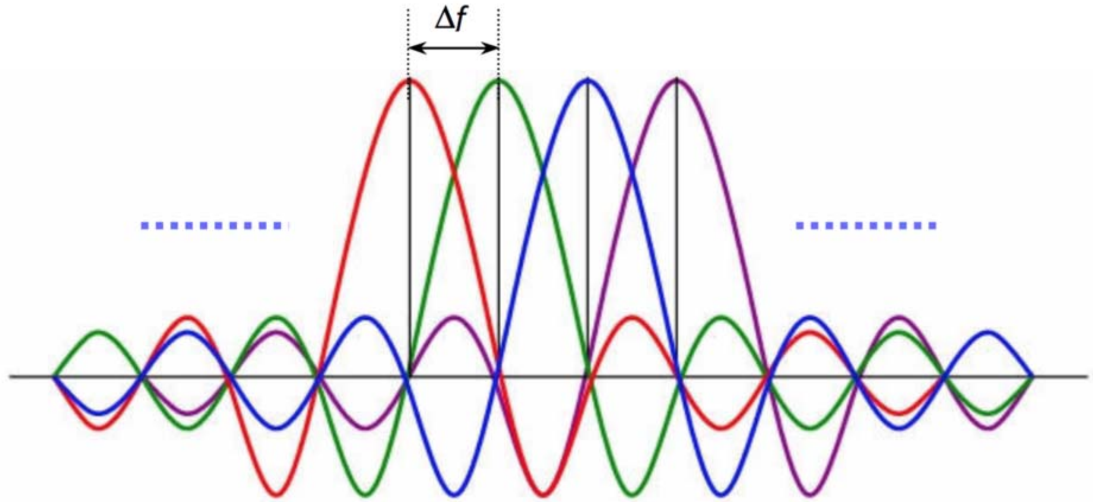
Orthogonal Frequency Division Multiplexing

The subcarrier spacing $\Delta f = \frac{B_T}{N} = \frac{1}{NT_s} \equiv \frac{1}{T}$

T_s = sampling period

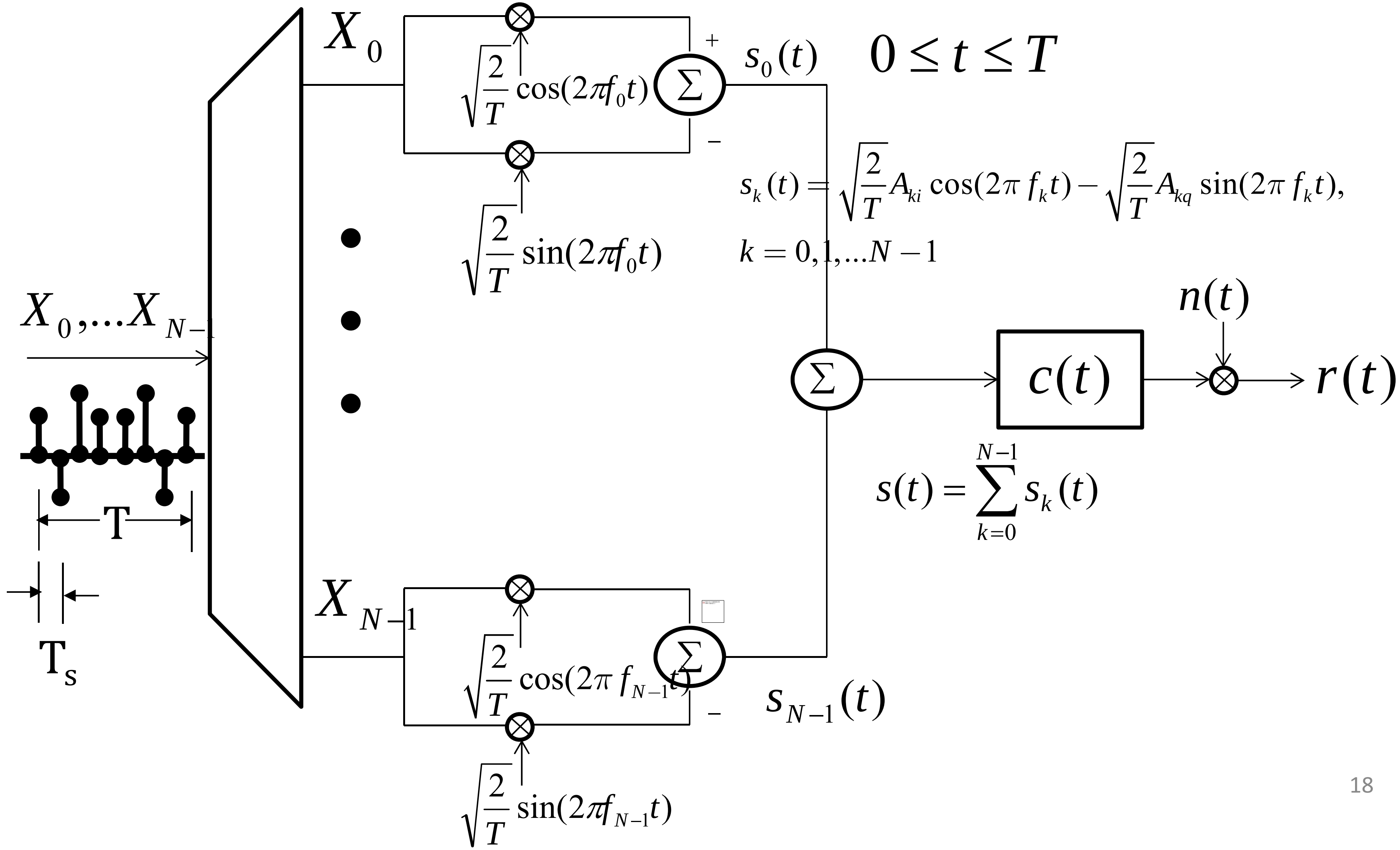
$T = NT_s$ = Period of N modulation symbols X_k

= 1 OFDM symbol time



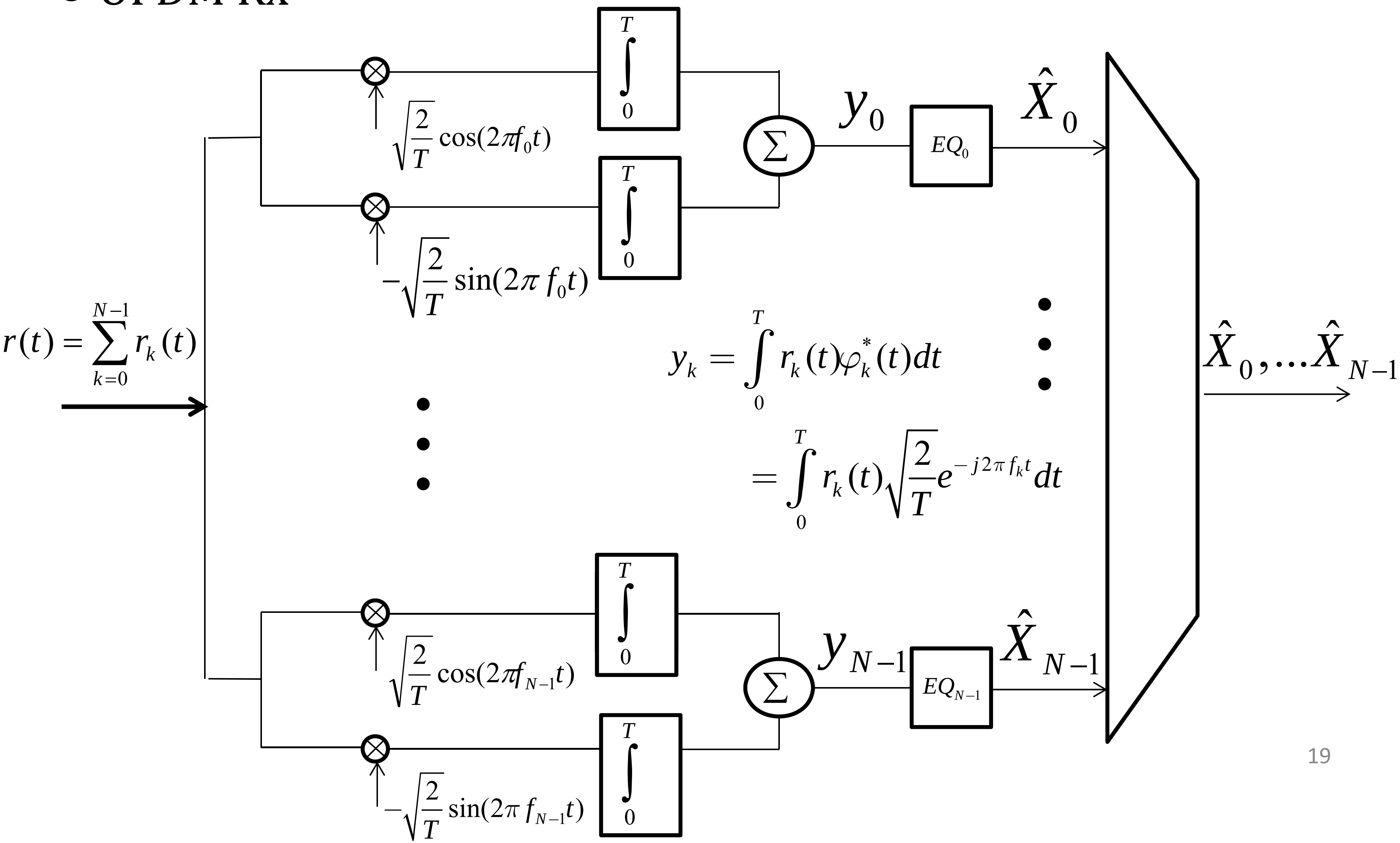
Orthogonal Frequency Division Multiplexing

- OFDM Tx



Orthogonal Frequency Division Multiplexing

- OFDM Rx



$$y_k = \int_0^T r_k(t)\varphi_k^*(t)dt$$

$$= \int_0^T r_k(t)\sqrt{\frac{2}{T}}e^{-j2\pi f_k t}dt$$

Orthogonal Frequency Division Multiplexing

- Let $c(f_k)$=c_k= complex channel frequency response at f_k ,

symbols $X_k = \mathrm{A}_k \mathrm{e}^{j\theta_k} = A_{ki} + jA_{kq}$, and $A_{ki}\ A_{kq} \in \{\pm 1, \pm 3, \ldots \pm (M-1)\}$

Each subchannel is nearly flat with $\mathrm{c}_k = \mathrm{c}_{ki} + j\mathrm{c}_{kq} = \left|\mathrm{c}_k\right| \mathrm{e}^{j\phi_k}$.

The received signal

$$r_k(t) = \sqrt{\frac{2}{T}} \left|c_k\right| A_{ki} \cos(2\pi f_k t + \phi_k) - \sqrt{\frac{2}{T}} \left|c_k\right| A_{ki} \sin(2\pi f_k t + \phi_k) + n_k(t)$$

$$= \mathrm{Re}\left\{ \sqrt{\frac{2}{T}} c_k X_k e^{j 2\pi f_k t} \right\} + n_k(t)$$

The correlation receiver basis are

$$\varphi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_k t), \quad 0 \le t \le T$$

$$\varphi_2(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_k t), \quad 0 \le t \le T$$

Orthogonal Frequency Division Multiplexing

After correlation receiver filter,

$$\Rightarrow \ y_k = \int_0^T r_k(t)\varphi_1(t)dt + j\int_0^T r_k(t)\varphi_2(t)dt$$

$$= \left(c_{ki}A_{ki} + n_{ki}\right) + j\left(c_{ki}A_{kq} + n_{kq}\right)$$

- The correlator outpu y_k can be detected by simple linear equalization.

- The detection of X_k can be realized by the Linear Equalizer.

Ex: 1-tap ZF : $\hat{\mathrm{X}}_k = \frac{\mathrm{y}_{ki}}{c_{ki}} + j\frac{\mathrm{y}_{kq}}{c_{kq}} = \mathrm{X}_k + \frac{n_k}{c_k}$

Implementation of OFDM with IDFT/DFT

- Direct implementation requires N analog RF frontends !
- ✓ The cost is too expensive and prevents the OFDM realization for 20 years since the birth of concept of OFDM.
- It can be shown that the OFDM processing is mathematically equivalent to the IDFT/DFT .
- ✓ The IDFT/DFT processing can be realized in digital baseband with low cost.

- The OFDM transmit signal is $s(t)=\sum_{k=0}^{N-1} s_k(t)$

$$s_k(t)=\sqrt{\frac{2}{T}}A_{ki}\cos(2\pi f_k t)-\sqrt{\frac{2}{T}}A_{kq}\sin(2\pi f_k t), \quad k=0,1,\ldots N-1$$

$$=\mathrm{Re}\{\sqrt{\frac{2}{T}}X_k e^{j2\pi f_k t}\}=\mathrm{Re}\{\underbrace{\sqrt{\frac{2}{T}}X_k e^{j2\pi k\Delta f t}}_{=X_k(t)} e^{j2\pi f_c t}\}$$

where $f_k = f_c + k\Delta f, \quad k=0,\ldots,N-1$

Implementation of OFDM with IDFT/DFT

Define the baseband signal in subcarrier k

$$X_k(t) = \sqrt{\frac{2}{T}} X_k e^{j2\pi k\Delta f t}, k = 0, \ldots, N-1$$

The passband signal is $s_k(t) = \mathrm{Re}\{X_k(t) e^{j2\pi f_c t}\}, k = 0, \ldots, N-1$

The baseband transmitted signal becomes

$$x(t) = \sum_{k=0}^{N-1} X_k(t) = \sum_{k=0}^{N-1} \sqrt{\frac{2}{T}} X_k e^{j2\pi k\Delta f t}$$

The passband transmitted signal is

$$s(t) = \sum_{k=0}^{N-1} s_k(t) = \mathrm{Re}\left\{\sum_{k=0}^{N-1} X_k(t) e^{j2\pi f_c t}\right\} = \mathrm{Re}\left\{x(t) e^{j2\pi f_c t}\right\}$$

Implementation of OFDM with IDFT/DFT

The discrete-time representation of $x(t)$ at $t=nT_s$ is

$$x(nT_s) \equiv x[n] = \sum_{k=0}^{N-1} \sqrt{\frac{2}{T}} X_k e^{j2\pi k\Delta f \cdot nT_s}$$

Recall: $\Delta f = \frac{1}{NT_s}$

$$= \sum_{k=0}^{N-1} \sqrt{\frac{2}{T}} X_k e^{j2\pi k(\frac{1}{NT_s})\cdot nT_s}$$

$$= \sqrt{\frac{2}{T}} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

The transmit signal $x[n]$ is IDFT of the modulated symbols, X_k

- The baseband signal relation is equivalent to DFTpairs,

$$x[n] \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftharpoons}} X_k$$

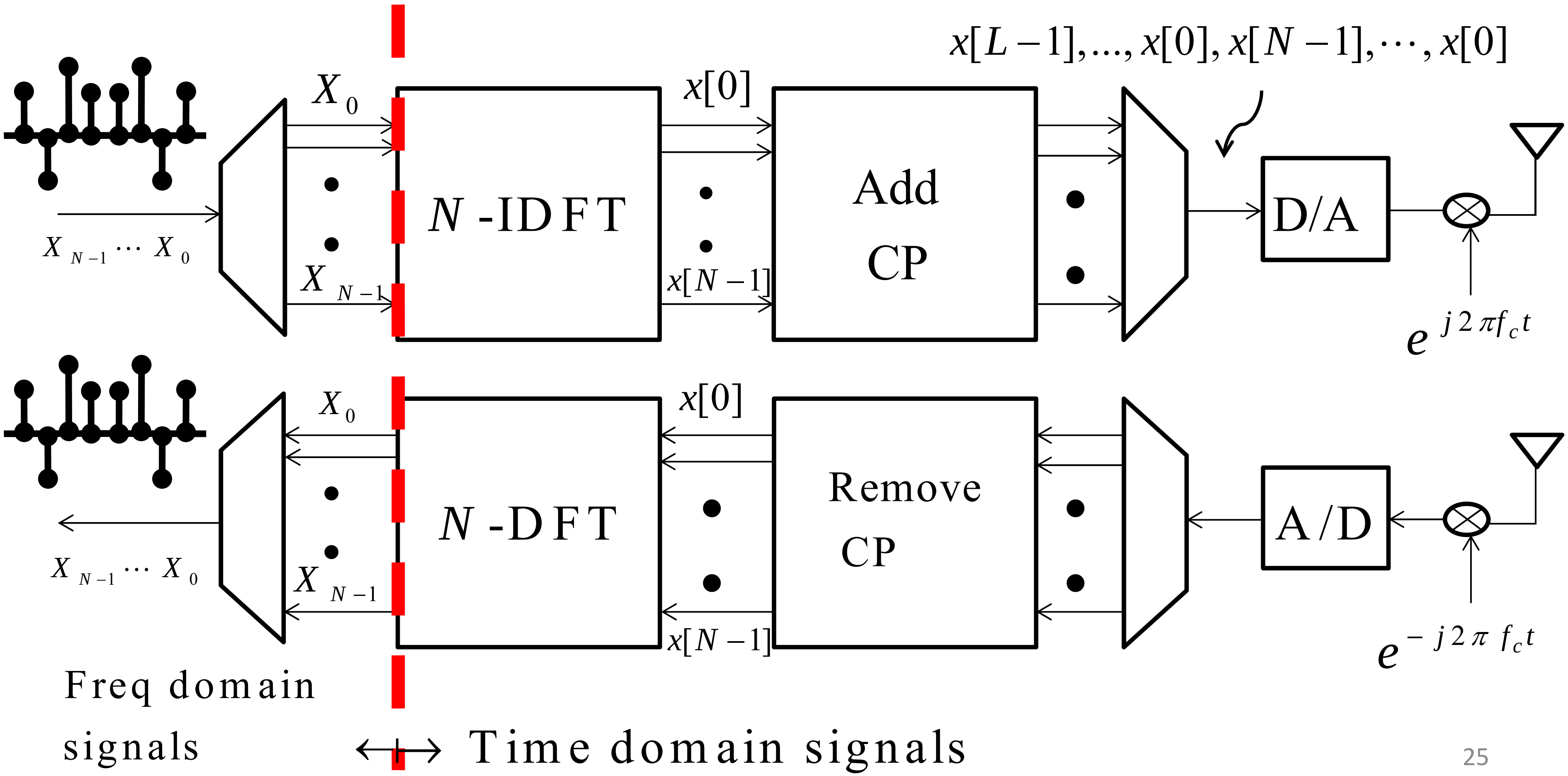
where k is index of subcarriers, and n is index of time.

Implementation of OFDM with IDFT/DFT

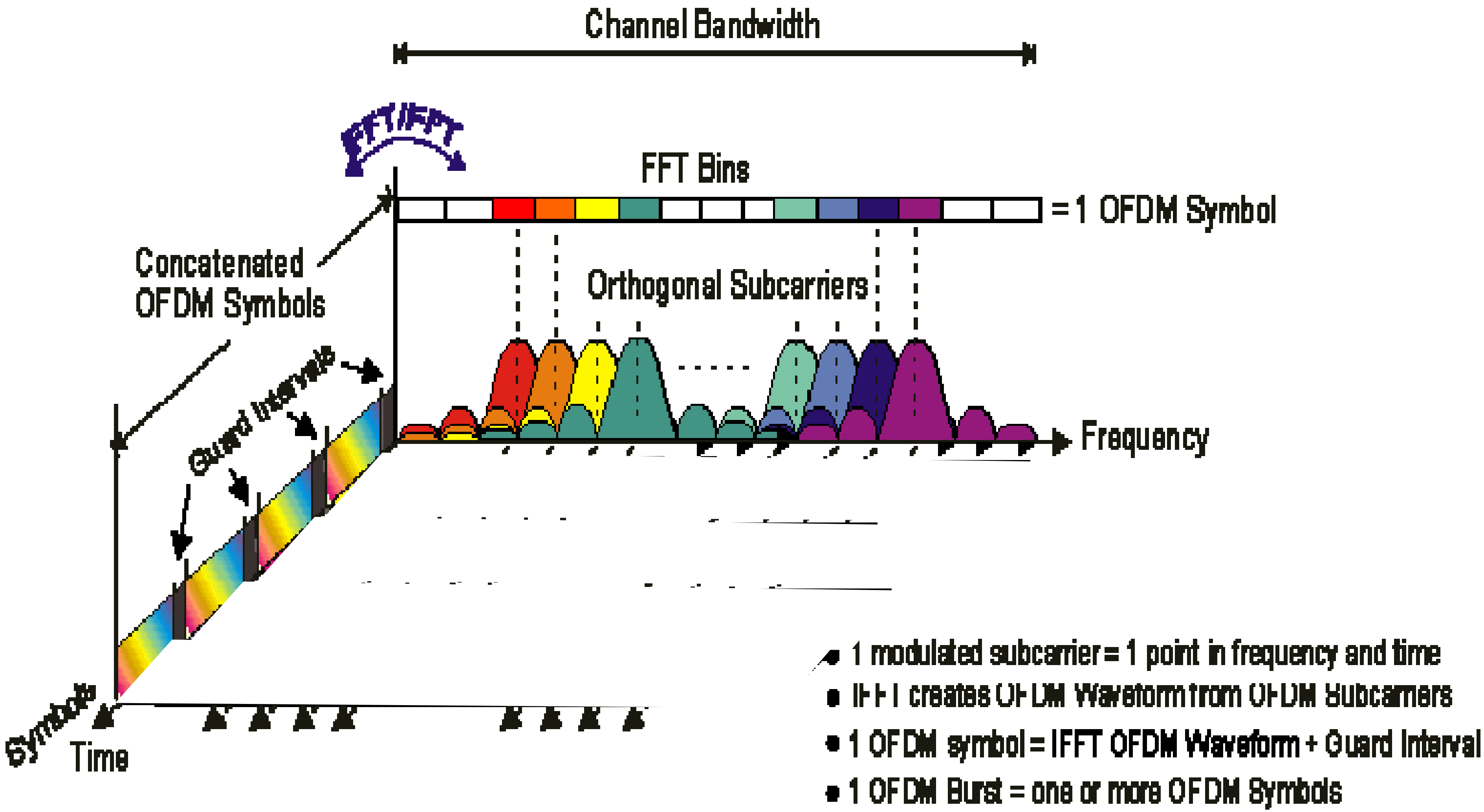
- $X_k \leftrightarrow x[n]$ becomes IDFT / DFT pairs

  The OFDM Tx can be represented by IDFT

  The OFDM Rx can be represented by DFT

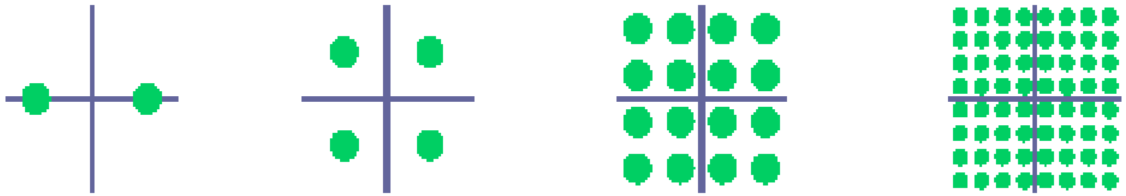


Frequency-Time Representation of OFDM Signal



OFDM Symbols

- **Each OFDM symbol has N frequency carriers**
  - D data carriers transmit information (D<=N)
  - (N-D) free carriers
- **Choose encoding scheme for carriers**
  - 1,2,4 or 6 bits/carrier -> points c_i in complex plane

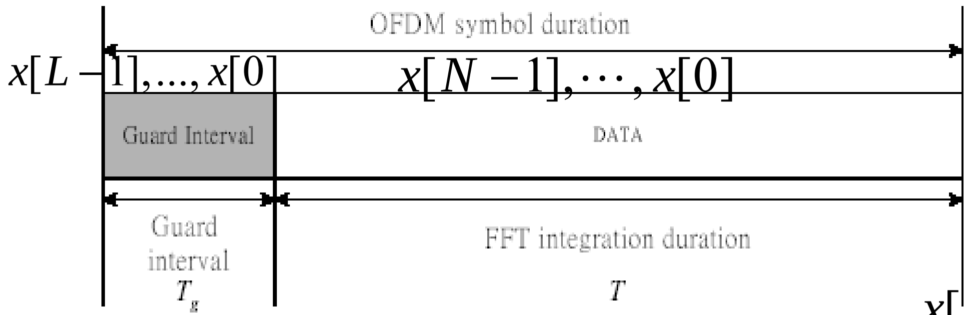


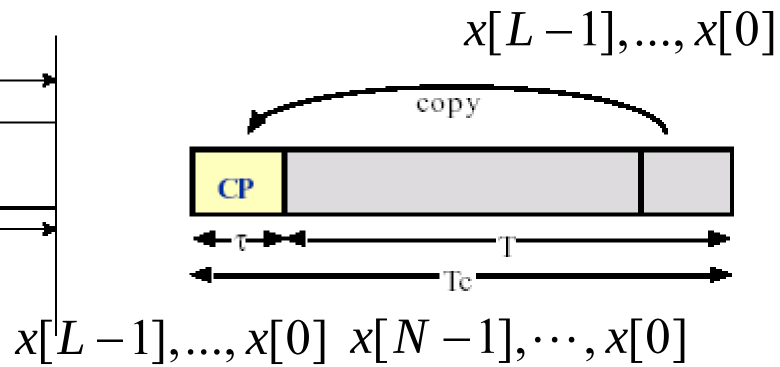
BPSK QPSK 16QAM 64QAM

- **Symbol representations**
  - Frequency-domain constellation $c=[c_1\ c_2\ \ldots\ c_N]$
  - Time-domain waveform $x = \mathrm{IFFT}(c)$

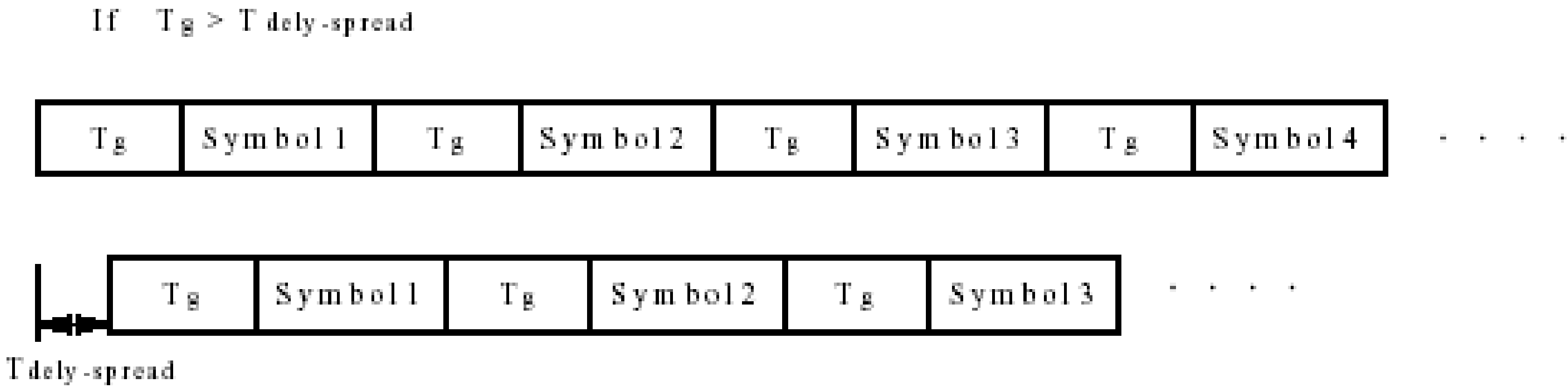
OFDM Guard Interval to remove ISI

- OFDM Symbol duration = Tg+T

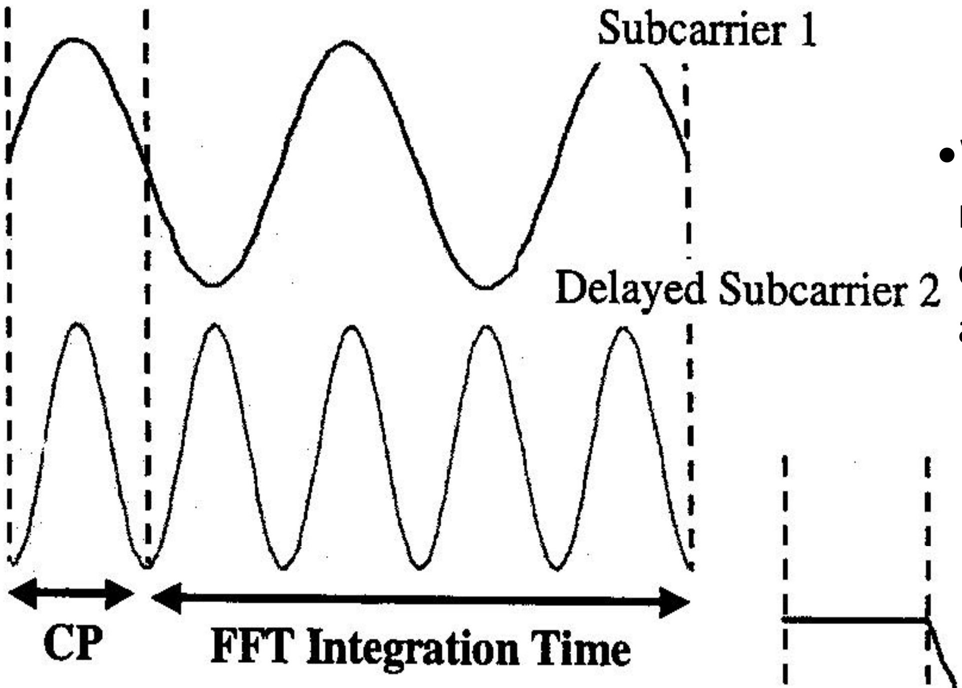




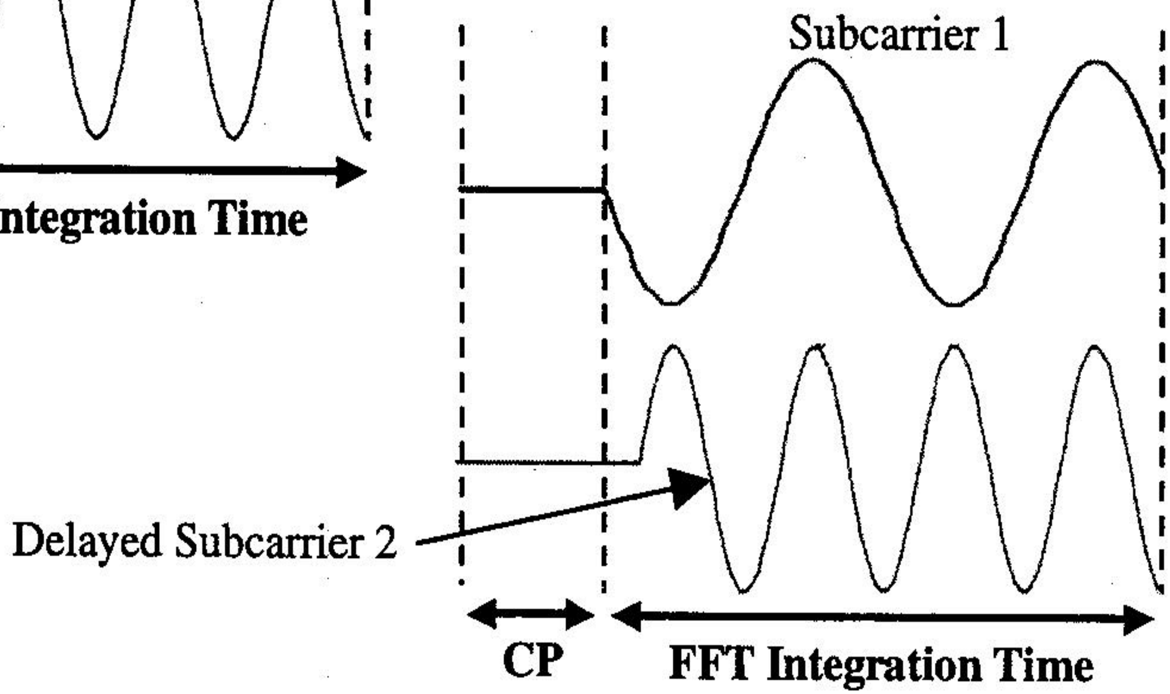
Guard interval (using cyclic prefix extension) is used in OFDM systems to combat against multipath fading, $T_g > T_{delay\_spread}$



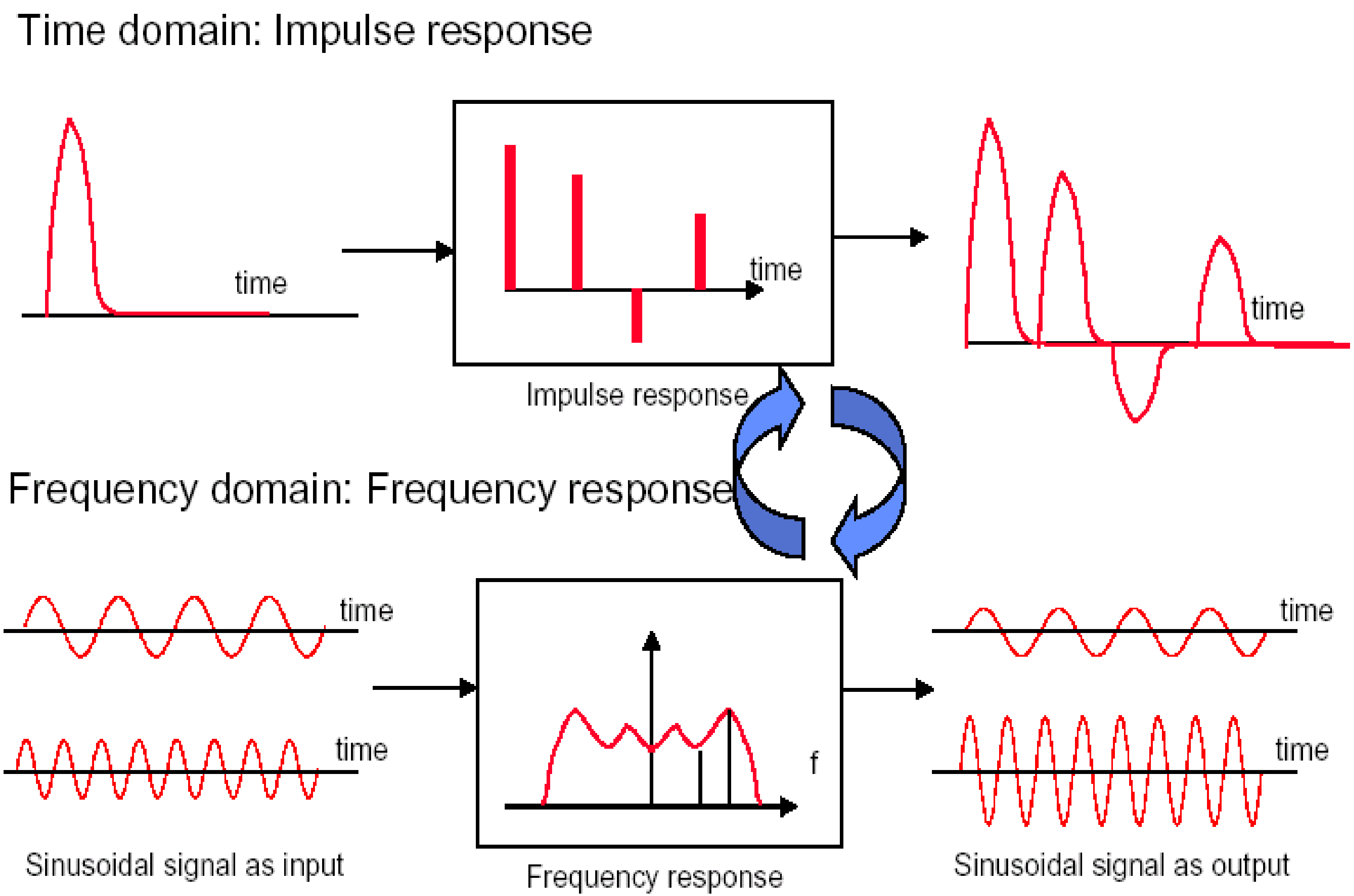
Cyclic Prefix for OFDM symbol Guard



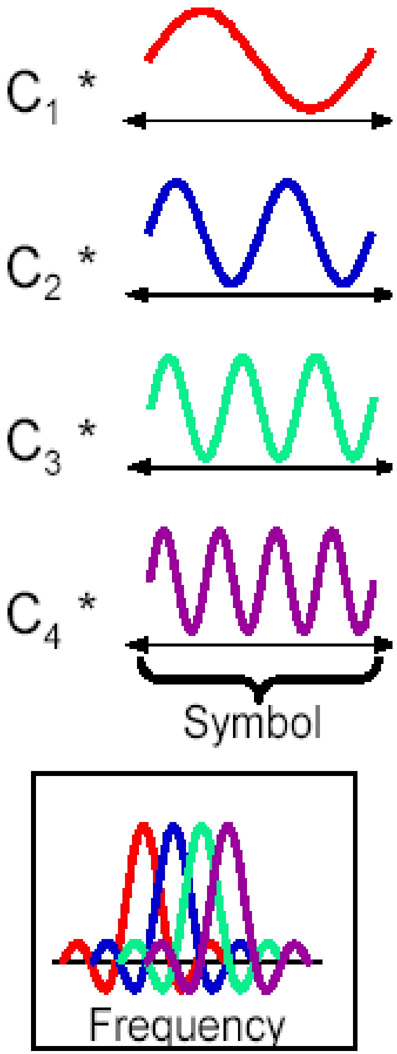
- Why Cyclic Prefix for GI?
  -> symbol timing error could happen so that FFT integration time could misalign
- With null signal in the GI, an integral number waveform cycles does not exist. Orthogonality may not preserve, and ICI will occur.



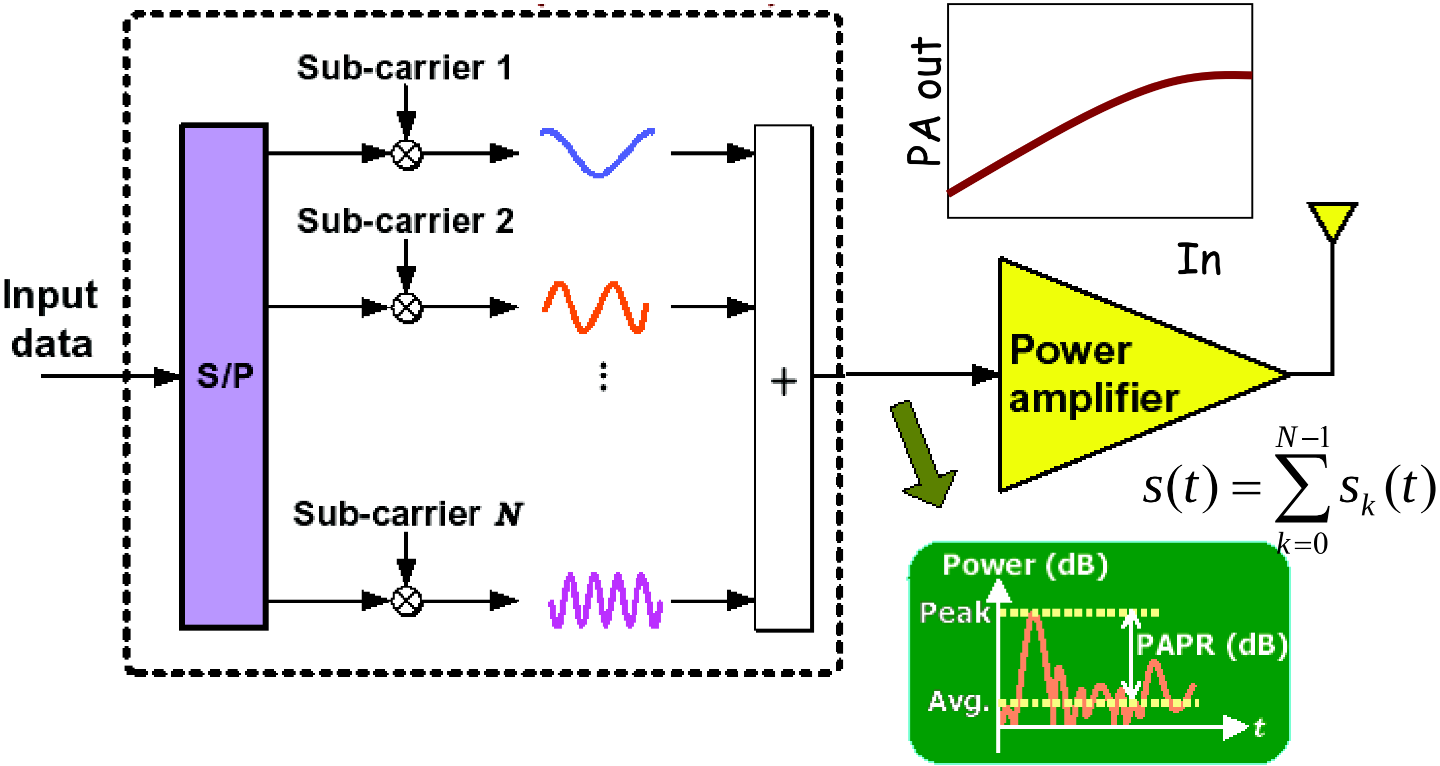
Multipath Problem in Time and Frequency Domains



OFDM Signaling over Multipath Channel



Peak to Average Power Ratio (PAPR) Problem in OFDM Transmitter



- PAPR = Peak-to-Average Power Ratio (PAPR)
- Distortion occurs when the transmitting power run into saturating region of the Power Amplifier
- PAPR gets worse as the number of OFDM subcarriers N increases.

Peak-to-Average Power Ratio (PAPR) of OFDM

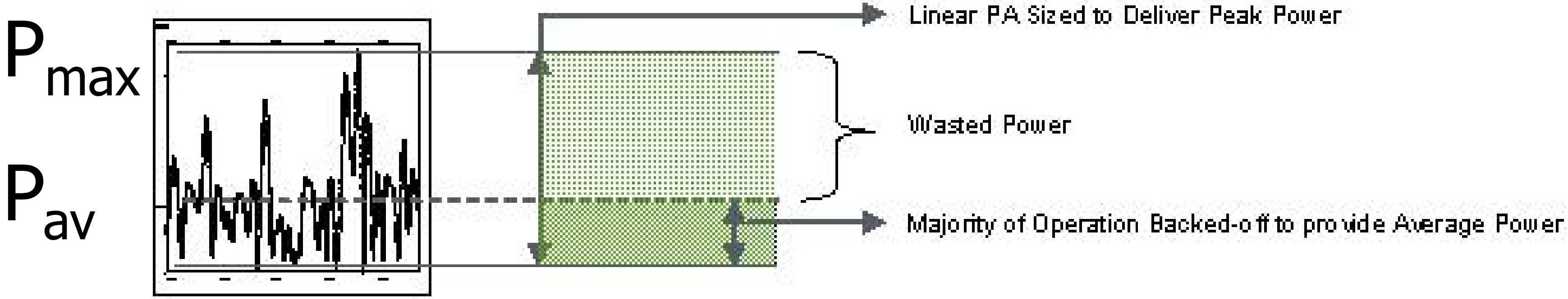
- PAPR of OFDM Signal
  - PAPR

$$\text{Peak Power: } P_{peak} = \max_n |x[n]|^2$$

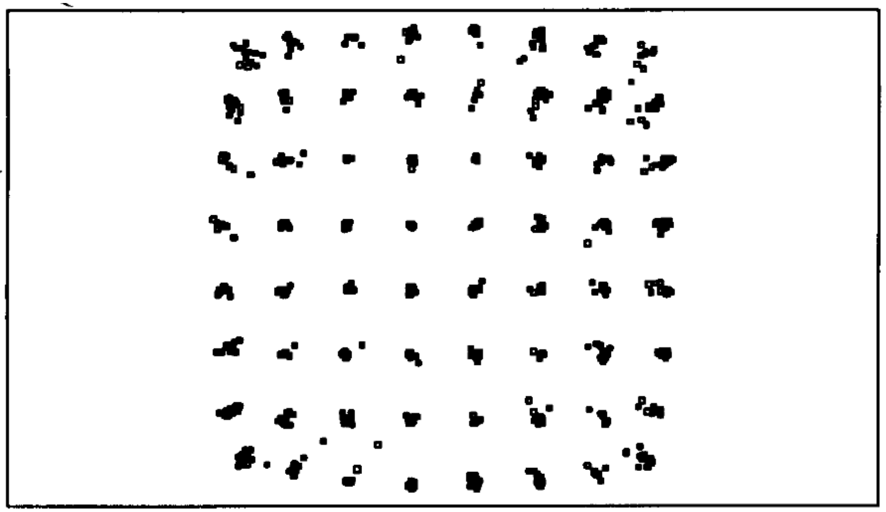
$$\text{Average Power: } P_{av} = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

$$\mathrm{PAPR} = \frac{P_{peak}}{P_{av}}$$

The Peak-to-Average Power Ratio (PAPR) Problem



- High PAPR signals are more likely to enter the saturation region.
- Need larger backoff to avoid signal distortion.



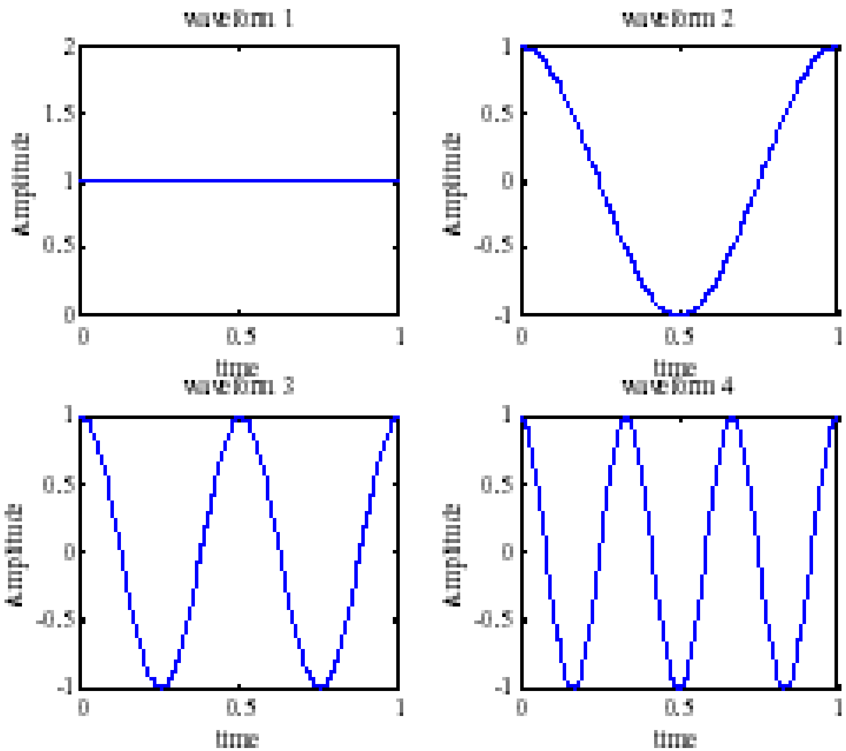
- How to alleviate the PAPR problem?

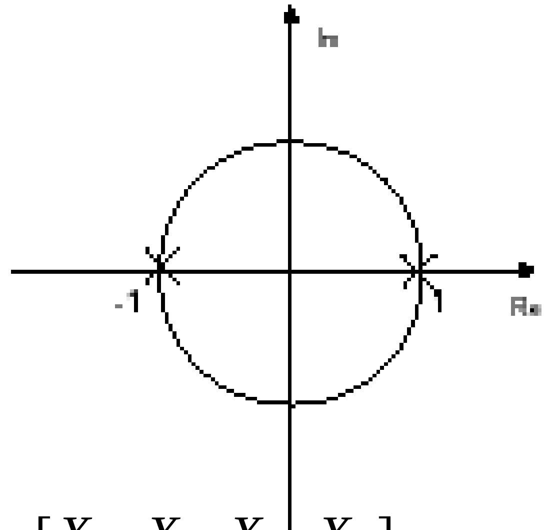
➢ If the Tx decrease P_{av} to avoid distortion, then the system may not be able to deliver enough SNR at the Rx.

➢ If the Tx increase P_{av} to to improve SNR then the system needs higher P_{sat} and larger linear region (expensive PA if available). Besides, it consumes more power.

Peak-to-Average Power Ratio (PAPR) of OFDM

- Basic waveforms of OFDM signal with 4-DFT and BPSK modulation





$$\mathbf{X} = [X_0 \quad X_1 \quad X_2 \quad X_3]$$

$$\mathbf{x} = \begin{bmatrix} x[0] & x[1] & x[2] & x[3] \end{bmatrix}$$

Maximum PAPR case

$$\mathbf{X} = [\ 1,\ 1,\ 1,\ 1] \rightarrow \mathbf{x} = [\ 4,0,0,0]$$

$$\mathbf{X} = [-1,-1,-1,-1] \rightarrow \mathbf{x} = [-4,0,0,0]$$

$$\mathbf{X} = [\ 1,-1,\ 1,-1] \rightarrow \mathbf{x} = [0,0,\ 4,0]$$

$$\mathbf{X} = [-1,\ 1,-1,\ 1] \rightarrow \mathbf{x} = [0,0,-4,0]$$

$$x[n] = \sqrt{\frac{2}{T}} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

Peak-to-Average Power Ratio (PAPR) of OFDM

- MPSK case: Let $\mathrm{X}_k \in \{\exp(j2\pi / m), \quad m = 0, \ldots, M-1\}$

$$P_{peak} = \max_n \left| x[\mathrm{n}] \right|^2 = N \quad \rightarrow \text{ occurs when } \mathrm{X}_0 = \ldots = \mathrm{X}_{N-1}$$

$$P_{av} = \frac{1}{N} \sum_{n=0}^{N-1} \left\| x[\mathrm{n}] \right\|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left| X_k \right|^2 = 1$$

$$\mathrm{PAPR}_{\max} = 10 \log_{10} \frac{P_{peak}}{P_{av}} = 10 \log_{10} N \quad \text{(dB)}$$

$$\mathrm{N{=}64,\ PAPR} \left\{ \mathrm{x[n]} \right\} \leq 18dB$$

$$N = 8192,\ \mathrm{PAPR} \left\{ \mathrm{x[n]} \right\} \leq 39dB$$

PAPR Example: QPSK on N=4 OFDM

$$x[n]=\sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \qquad \mathbf{X}=[X_0 \;\; X_1 \;\; X_2 \;\; X_3] \qquad \mathbf{x}=\left[x[0] \quad x[1] \quad x[2] \quad x[3]\right]$$

The PAPR occurs when

$$\mathrm{M}\begin{cases}\mathbf{X}=[\;\;1,\;1,\;1,\;1]\rightarrow\mathbf{x}=[\;\;4,0,0,0] \\ \mathbf{X}=[-1,-1,-1,-1]\rightarrow\mathbf{x}=[-4,0,0,0] \\ \mathbf{X}=[\;\;i,\;\;i,\;\;i,\;\;i]\rightarrow\mathbf{x}=[\;\;4i,0,0,0] \\ \mathbf{X}=[-i,-i,\;-i,-i]\rightarrow\mathbf{x}=[-4i,0,0,0]\end{cases}$$

$$\mathbf{X}=[\;1,\;\;i,-1,-i]\rightarrow\mathbf{x}=[0,4,0,0]$$
$$\mathbf{X}=[-1,-i,\;1,\;\;i]\rightarrow\mathbf{x}=[0,-4,0,0]$$
$$\mathbf{X}=[\;\;i,-1,-i,\;1]\rightarrow\mathbf{x}=[0,4i,0,0]$$
$$\mathbf{X}=[-i,\;1,\;\;i,-1]\rightarrow\mathbf{x}=[0,-4i,0,0]$$

$$\mathbf{X}=[\;1,\;-1,1,-1]\rightarrow\mathbf{x}=[0,0,4,0]$$
$$\mathbf{X}=[-1,\;\;1,-1,\;1]\rightarrow\mathbf{x}=[0,0,-4,0]$$
$$\mathbf{X}=[\;\;i,\;-i,\;\;i,-i]\rightarrow\mathbf{x}=[0,0,4i,0]$$
$$\mathbf{X}=[-i,\;\;i,-i,\;\;i]\rightarrow\mathbf{x}=[0,0,-4i,0]$$

$$\mathbf{X}=[\;1,-i,-1,\;\;i]\rightarrow\mathbf{x}=[0,0,0,4]$$
$$\mathbf{X}=[-1,\;i,\;1,\;-i]\rightarrow\mathbf{x}=[0,0,0,-4]$$
$$\mathbf{X}=[\;\;i,\;1,-i,-1]\rightarrow\mathbf{x}=[0,0,0,4i]$$
$$\mathbf{X}=[-i,\;-1,\;\;i,1]\rightarrow\mathbf{x}=[0,0,0,-4i]$$

$\rightarrow$ Total of MN cases that the PAPR occurs.

Peak-to-Average Power Ratio (PAPR) of OFDM

- How serious is the PAPR problem?

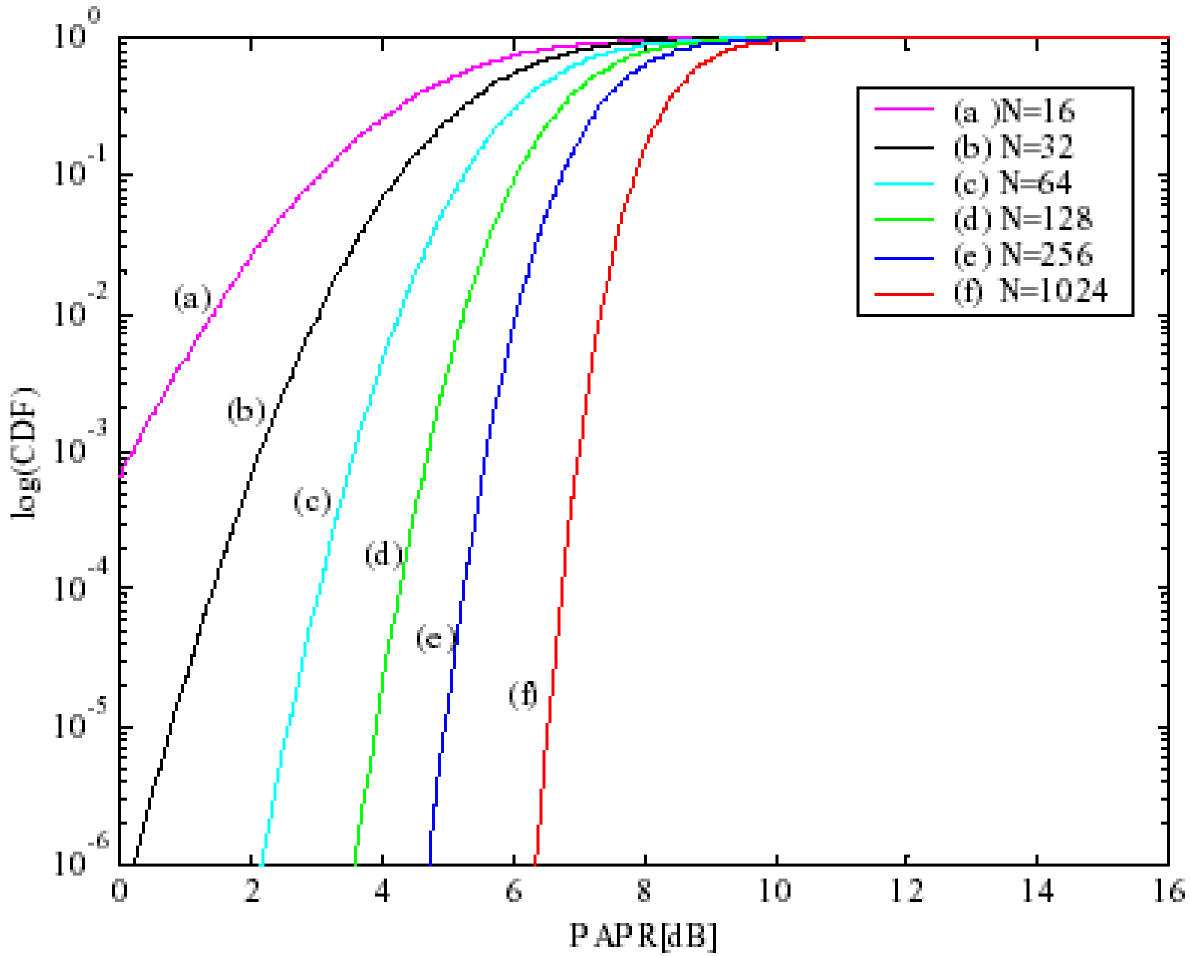
✓ The occurrence of PAPR problem.

- It can be shown that for an M-ary PSK N-point OFDM system, there are at most MN cases that yield the max PAPR (= N)
- The probability of observing the max PAPR is $\frac{MN}{M^N} = NM^{1-N}$
- *F*or N = 32 and M = 4, the probability of max PAPR is $8.7\text{x}10^{-19}$
- For OFDM with T = 100 μs, the max PAPR occurs once every $3.7\text{x}10^{6}$ years.

- What matters is the probability of signals fall into the saturation region.
- What is the PAPR for a single carrier M-ary PSK modulation system?

Cumulative Distribution of the PAPR

- PAPR distribution of Random transmission at number of subcarriers N =16/32/64/128 /256 /1024

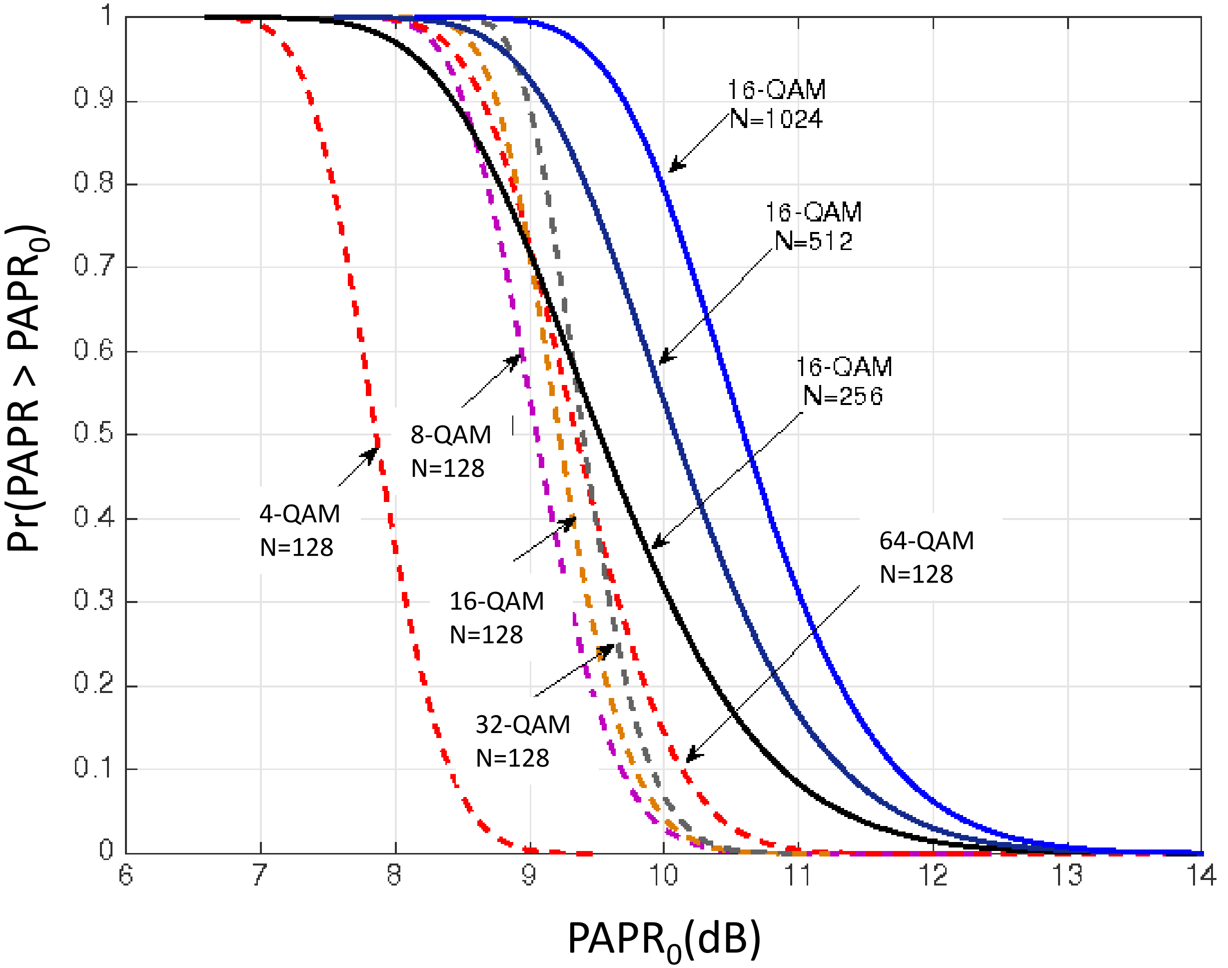


Cumulative Distribution of the PAPR

- Analytical PAPR distribution $Pr(PAPR > PAPR_0)$

✓ # subcarriers N = 256 /512/1024
✓ Modulation order M= 16-QAM

✓ # subcarriers N=128
✓ Modulation order M=4/8/16/32/64



Summary

- Single carrier vs multi-carrier communications
- Power allocation for multi-carrier communications
- OFDM architecture for multi-carrier communication
- Implementation OFDM with IDFT/DFT
- Multipath fading channel and OFDM
- PAPR problem in OFDM communications

Announcement

HW#6

Due: 1/11/2022 (Tue) 18:00 @EECS611

Final Exam

Time: Jan. 13, 2022 18:30pm – 21:00pm

Place: Delta 215 & 217

Coverage: Ch9 and Ch11