COM 5120 Communication Theory
Homework #2
Reference solution

1.

(a) We first observe that $s_1(t)$, $s_2(t)$ and $s_3(t)$ are linearly independent.

The energy of $s_1(t)$ is

$$E_1 = \int_0^1 (2)^2 dt = 4$$

The first basis function is therefore

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwsie} \end{cases}$$

Define

$$s_{21} = \int_0^T s_2(t)\ \phi_1(t)dt$$

$$= \int_0^1 (-4)(1)dt = -4$$

$$g_2(t) = s_2(t) - s_{21}\phi_1(t) = \begin{cases} -4, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the second basis function is

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}}$$

$$= \begin{cases} -1, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Define

$$s_{31} = \int_0^T s_3(t)\ \phi_1(t)dt$$

$$= \int_0^1 (3)(1)dt = 3$$

$$s_{32} = \int_T^{2T} s_3(t)\ \phi_2(t)dt$$

$$= \int_1^2 (3)(-1)dt = -3$$

$$g_3(t) = s_3(t) - s_{31}\ \phi_1(t) - s_{32}\ \phi_2(t)$$

$$= \begin{cases} 3, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the thrid basis function is

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t)dt}}$$

$$= \begin{cases} 1, & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(b) $s_1(t) = 2\phi_1(t)$

$s_2(t) = -4\phi_1(t) + 4\phi_2(t)$

$s_3(t) = 3\phi_1(t) - 3\phi_2(t) + 3\phi_3(t)$

2.

(a). Consider the QAM constellation. Using the Pythagorean Theorem we can find the radius of the inner circle as:

$$a^2 + a^2 = A^2 \Longrightarrow a = \frac{1}{\sqrt{2}}A$$

The radius of the outer circle can be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between then equal to

$\theta = 105^\circ$, we obtain $b^2 = a^2 + A^2 - 2aA\cos 105^\circ \Rightarrow b = \dfrac{1+\sqrt{3}}{2}A$

(b). If we denote by r the radius of the circle, then using the cosine theorem we obtain:

$$A^2 = r^2 + r^2 - 2r\cos 45^o \Longrightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}$$

(c). The average transmitted power of the PSK constellation is:

$$P_{\text{PSK}} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2-\sqrt{2}}}\right)^2 \Longrightarrow P_{\text{PSK}} = \frac{A^2}{2-\sqrt{2}}$$

The average transmitted power of the QAM constellation:

$$P_{\text{QAM}} = \frac{1}{8}\left(4\frac{A^2}{2} + 4\frac{(1+\sqrt{3})^2}{4}A^2\right) \Longrightarrow P_{\text{QAM}} = \left[\frac{2+(1+\sqrt{3})^2}{8}\right]A^2$$

The relative power advantage of the PSK constellation over the QAM constellation is:

$$\text{gain} = \frac{P_{\text{PSK}}}{P_{\text{QAM}}} = \frac{8}{(2+(1+\sqrt{3})^2)(2-\sqrt{2})} = 1.5927 \text{ dB}$$

3.

(a)

$$\theta(t, I_n) = 4\pi f_d T \int_{-\infty}^{t} d(\tau)d\tau$$

$$= 4\pi f_d T \int_{-\infty}^{t} (\textstyle\sum_n I_n g(\tau - nT))d\tau$$

$$= 4\pi f_d T \textstyle\sum_{k=-\infty}^{n-1} I_k \int_{-\infty}^{nT} g(\tau - kT)d\tau + 4\pi f_d T I_n \int_{nT}^{t} g(\tau - nT)d\tau$$

$$= 2\pi f_d T \textstyle\sum_{k=-\infty}^{n-1} I_k + 4\pi f_d T I_n q(t - nT)$$
$$= \theta(nT) + 4\pi f_d T I_n q(t - nT)\ , \qquad \mathrm{nT} \le \mathrm{t} \le (\mathrm{n}+1)\mathrm{T}$$

(b)

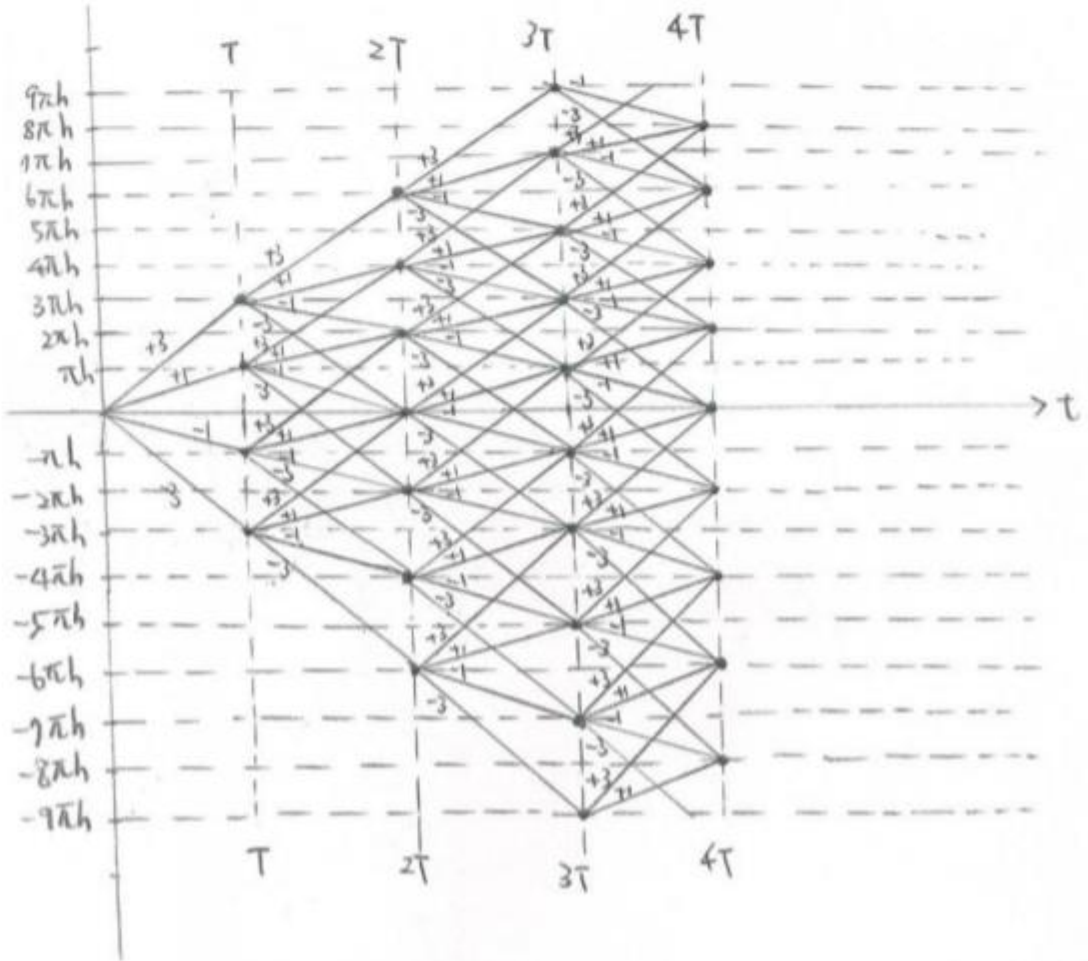
h = $\frac{3}{5}$, where 3 and 5 are mutually prime integers, we have 2x5 = 10 states. (3 is odd)

$$\rightarrow \left\{0, \frac{3}{5}\pi, \frac{6}{5}\pi, \frac{9}{5}\pi, \frac{12}{5}\pi, \frac{15}{5}\pi, \frac{18}{5}\pi, \frac{21}{5}\pi, \frac{24}{5}\pi, \frac{27}{5}\pi\right\}$$

$$\rightarrow \left\{0, \frac{1}{5}\pi, \frac{2}{5}\pi, \frac{3}{5}\pi, \frac{4}{5}\pi, \pi, \frac{6}{5}\pi, \frac{7}{5}\pi, \frac{8}{5}\pi, \frac{9}{5}\pi\right\}$$

(c)

$M = 4, \quad I_n \in \{\pm 1, \pm 3\}$



4.

We have that

$$S_{dd}(f) = \frac{1}{T} S_{II}(f) \,|\, G(f) \,|^2$$

And

$$E[I_n] = \frac{1}{4}[\frac{1}{2}(1+j) + \frac{1}{2}(1-j) + \frac{1}{2}(-1+j) + \frac{1}{2}(-1-j)] = 0$$

$$R_{II}(k) = E[I_n I_{n+k}] = \begin{cases} E[I_n]E[I_{n+k}] & , \ k \neq 0 \\ E[I_n^2] & , \ k = 0 \end{cases}$$

$$= \begin{cases} 0 & , \ k \neq 0 \\ \frac{1}{4}[\frac{1}{4}|1+j|^2 + \frac{1}{4}|1-j|^2 + \frac{1}{4}|-1+j|^2 + \frac{1}{4}|-1-j|^2] & , \ k = 0 \end{cases}$$

$$= \begin{cases} 0 & , \ k \neq 0 \\ \frac{1}{2} & , \ k = 0 \end{cases}$$

$$= \frac{1}{2}\delta(\mathrm{k})$$

$$\Rightarrow S_{II}(f) = \sum_k R_{II}(k)\, e^{-\mathrm{j}2\pi \mathrm{f\, k}} = \frac{1}{2} \qquad \text{(a)}$$

(1) $g(t) = rect(\frac{t - \frac{T}{2}}{T}) \Rightarrow G(f) = \int_0^T 1 \cdot e^{-\mathrm{j}2\pi \mathrm{f\, t}} dt = T \sin c(\mathrm{Tf}) e^{-\mathrm{j}2\pi \mathrm{f} \frac{\mathrm{T}}{2}}$

$$\Rightarrow |G(f)|^2 = T^2 \sin c^2(\mathrm{Tf}) \qquad \text{(b)}$$

Using (a) and (b), we can get

$$S_{dd}(f) = \frac{1}{T}\frac{1}{2}T^2 \sin c(\mathrm{T}f)^2 = \frac{T}{2}\sin c^2(\mathrm{T}f)$$

(2) $G(f) = \int_0^T \sin(\frac{\pi \mathrm{t}}{\mathrm{T}}) \cdot e^{-\mathrm{j}2\pi \mathrm{f\, t}} dt = \frac{2T}{\pi}\frac{\cos(\pi \mathrm{f T})}{1-4f^2T^2} e^{-\mathrm{j}2\pi \mathrm{f}\frac{\mathrm{T}}{2}}$

$$\Rightarrow |G(f)|^2 = \frac{4T^2}{\pi^2}\frac{\cos^2(\pi \mathrm{f T})}{(1-4f^2T^2)^2}$$

Using (a) and (c), we can get

$$S_{dd}(f) = \frac{1}{T}\frac{1}{2}\frac{4T^2}{\pi^2}\frac{\cos^2(\pi \mathrm{f T})}{(1-4f^2T^2)^2} = \frac{2T}{\pi^2}\frac{\cos^2(\pi \mathrm{f T})}{(1-4f^2T^2)^2}$$

5.

(a)

$R_b(m) = E[b_{n+m}b_n] = E[(a_{n+m} - a_{n+m-2})(a_n - a_{n-2})]$

(i) m = 0, $R_b(0) = E[a_n^2] - 2E[a_n a_{n-2}] + E[a_{n-2}^2]$

$$= \left(\frac{1}{2} + \frac{1}{2}\right) - 0 + \left(\frac{1}{2} + \frac{1}{2}\right) = 2$$

(ii) m = 2, $R_b(2) = E[a_{n+2}a_n] - E[a_n^2] - E[a_{n+2}a_{n-2}] + E[a_n a_{n-2}]$

$$= 0 - \left(\frac{1}{2} + \frac{1}{2}\right) - 0 + 0 = -1$$

$$R_b(m) = \begin{cases} 2, & m = 0 \\ -1, & m = \pm 2 \\ 0, & else \end{cases}$$

(b)

$$R_s(\tau) = \frac{1}{T} \sum_{m=-\infty}^{\infty} R_b(m) R_h(\tau - mT)$$

$$= \frac{1}{T}\{2R_h(\tau) - R_h(\tau + 2T) - R_h(\tau - 2T)\}$$

$$S_s(f) = F\{R_s(\tau)\} = \frac{1}{T}\left\{2|H(f)|^2 - |H(f)|^2 e^{j4\pi fT} - |H(f)|^2 e^{-j4\pi fT}\right\}$$

$$= 4\mathrm{T} sinc^2(Tf) sin^2(2\pi fT)$$