COM 5120
Communications Theory

Chapter 9

Digital Communication through Band-Limited Channels

Prof. Jen-Ming Wu

Inst. of Communications Engineering

Dept. of Electrical Engineering

National Tsing Hua University

Email: jmwu@ee.nthu.edu.tw

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Outline

Part 1: Signal Design for Bandlimited Channels

- **Lecture 1: System Model of band-limited channel**
  - ✓ ISI problem due to band-limited channel

- **Lecture 2: Signal design with controlled ISI (duo-binary precoding)**
  - ✓ Duo-binary precoding
  - ✓ Modified duo-binary precoding
  - ✓ M-ary duo-binary precoding

- **Lecture 3: Error performance analysis of controlled ISI precoding**

Outline

Part II -Optimal Receiver for Channels with ISI

- **Lecture 4: Optimal Receiver for Channels with ISI**
  - ✓ Maximum Likelihood Sequential Detection
  - ✓ Whitening filter design

- **Lecture 5: Linear equalizer for low complexity Rx with ISI**
  - ✓ Zero forcing linear equalizer
  - ✓ MMSE linear equalizer

Signal Model

- Let the modulated signal at baseband be

$$v(t) = \sum_{n=-\infty}^{\infty} I_n g_T(t - nT)$$

where $g_T(t)$ is the band-limited pulse shaping function and

$$G(f) = \mathscr{F}\{g_T(t)\} = 0 \text{ for } |f| \geq W$$

- The band-limited channel can be modeled by the linear filter $c(t)$, $C(f) = \mathscr{F}\{c(t)\} = 0$ for $|f| \geq W$

- The received signal is

$$r_l(t) = c(t) * v(t) + z(t) = \sum_{n=-\infty}^{\infty} I_n[c(t) * g_T(t - nT)] + z(t)$$

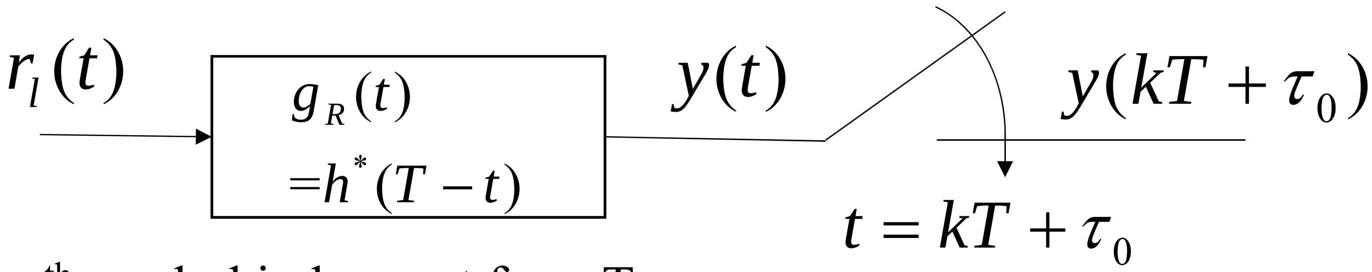
$$= \sum_{n=-\infty}^{\infty} I_n h(t - nT) + z(t)$$

- The equivalent pulse shape filter is

$$h(t)=\int_{-\infty}^{\infty} g_T(\tau)c(t-\tau)d\tau = c(t)*g_T(t)$$

$$\Rightarrow h(t-nT)=c(t)*g_T(t-nT)$$

- The received signal goes into the match filter in Rx



n = the n^{th} symbol index sent from Tx
k = the k^{th} sample index received by Rx

$$y(t)=r_l(t)*h^*(T-t)=\sum_{n=-\infty}^{\infty} I_n x(t-nT)+v(t)$$

where $x(t)=h(t)*h^*(T-t) \;\; = g_T(t)*c(t)*g_R(t)$

$$v(t)=z(t)*h^*(T-t)$$

- Sampling at $t = kT + \tau_0$ (τ_0 = transmission delay time)

n = nth symbol sent from Tx
k = kth sample received by Rx

$$\underbrace{y(kT+\tau_0)}_{y_k} = r_l(kT+\tau_0) * h^*(T-kT-\tau_0)$$

$$= \sum_{n=-\infty}^{\infty} I_n \underbrace{x(kT-nT+\tau_0)}_{x_{k-n}} + \underbrace{v(kT+\tau_0)}_{v_k}$$

$$\Rightarrow y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + v_k = \underbrace{x_0 I_k}_{\substack{\text{desired symbol} \\ n=k}} + \underbrace{\sum_{\substack{n=-\infty \\ n\neq k}}^{\infty} I_n x_{k-n}}_{\text{Intersymbol interference (ISI)}, n\neq k} + v_k$$

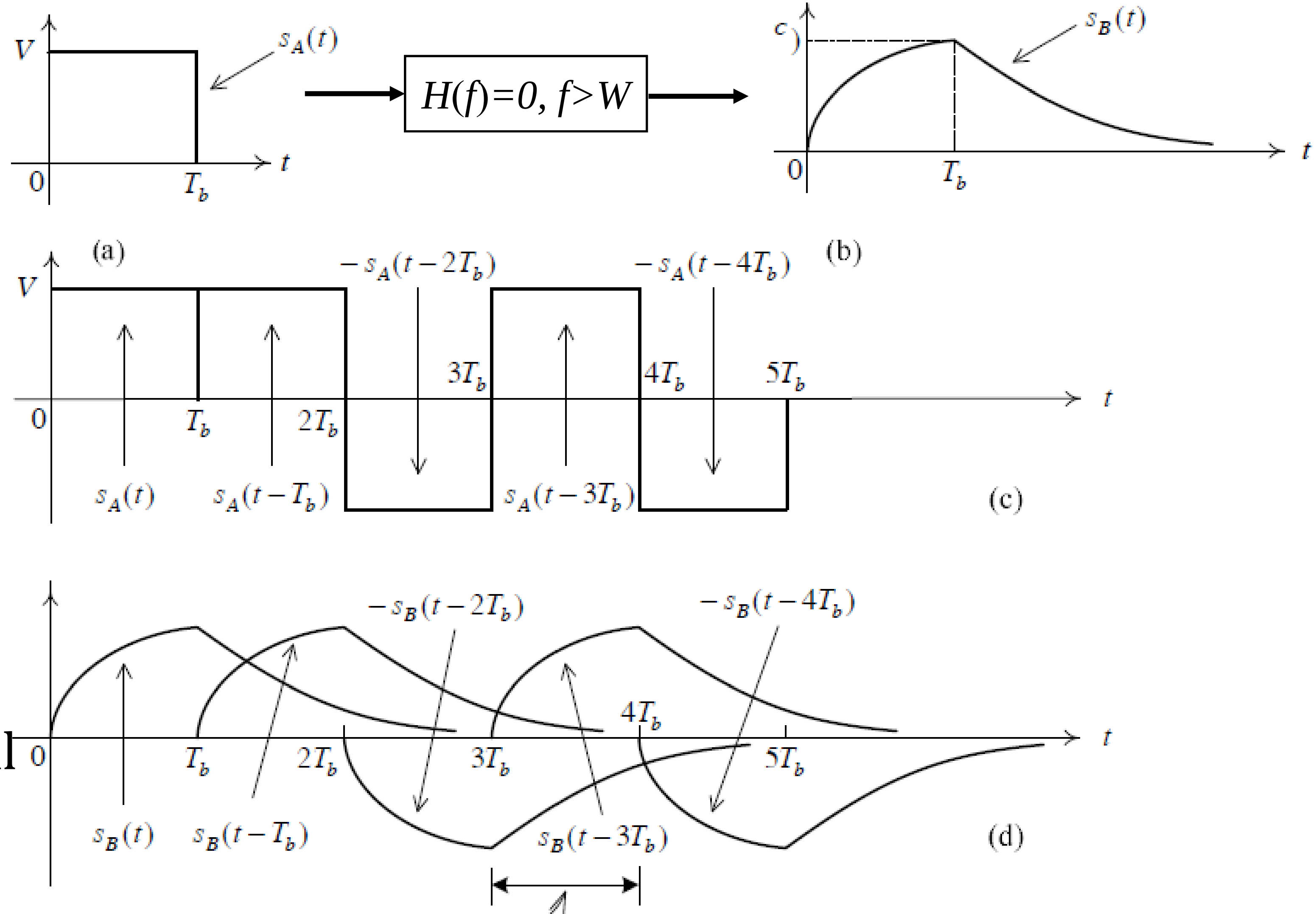
- Every sampling at $t = kT$ is affected by all symbols I_n
- For simplicity, we ignore the effect of τ_o and set (or normalize) $x_o = 1$

$$\Rightarrow y_k = I_k + \sum_{n\neq k} I_n x_{k-n} + v_k$$

$n = k$ desired symbol
$n \neq k$ ISI

where $x_k = x(kT) = x(t)\cdot \sum_{k=-\infty}^{\infty} \delta(t-kT)$

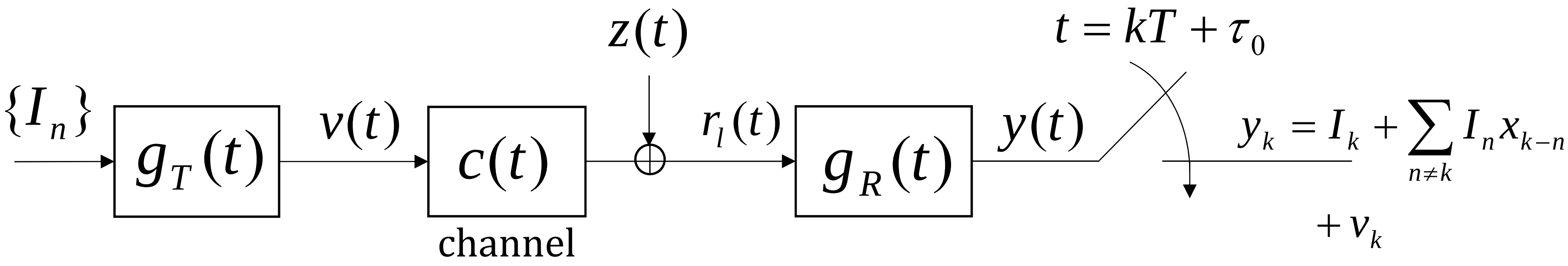
Illustration of ISI due to Band-limited Channel



- Tx signal
- Rx signal

During this interval, $\mathbf{r}(t)=\mathbf{b}_0 s_B(t)+\mathbf{b}_1 s_B(t-T_b)+\mathbf{b}_2 s_B(t-2T_b)+\mathbf{b}_3 s_B(t-3T_b)+\mathbf{w}(t)$

How to Design the Signaling for NO ISI?



- Suppose $C(f) = \begin{cases} 1 & \text{for } |f| \le W \\ 0 & \text{otherwise} \end{cases}$

$$X(f) = G_T(f)C(f)G_R(f)$$

$$x_k = x(kT) = x(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$\Rightarrow$ For no ISI $\Rightarrow x_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$ i.e. only $x_0 = 1$ (A)

(A) holds iff $\mathscr{F}\{x(t) \cdot \sum_{m=-\infty}^{\infty} \delta(t - kT)\} = \sum_{m=-\infty}^{\infty} X(f + \frac{m}{T}) = T$

i.e. The aggregate channel effect is flat over the spectrum.
This is called Nyquist Pulse Shaping Criterion

Proof of Nyquist pulse shaping criterion:

$$x(t)=\int_{-\infty}^{\infty}X(f)e^{j2\pi ft}df$$

$$\Rightarrow x_k=x(kT)=\int_{-\infty}^{\infty}X(f)e^{j2\pi fkT}df$$

$$=\sum_{m=-\infty}^{\infty}\int_{m/T-1/2T}^{m/T+1/2T}X(f)e^{j2\pi fkT}df$$

$$(Let\ f'=f-\tfrac{m}{T})$$

$$=\sum_{m=-\infty}^{\infty}\int_{-1/2T}^{1/2T}X(f'+\tfrac{m}{T})e^{j2\pi f'kT}df'e^{j2\pi\frac{m}{T}kT}$$

$$=\int_{-1/2T}^{1/2T}[\underbrace{\sum_{m=-\infty}^{\infty}X(f'+\tfrac{m}{T})}]e^{j2\pi f'kT}df'$$

For simplicity, use f (instead of f') as the dummy variable.

$$=\int_{-1/2T}^{1/2T}B(f)e^{j2\pi fkT}df,\quad \text{where } B(f)=\sum_{m=-\infty}^{\infty}X(f+\tfrac{m}{T})\quad \cdots①$$

- $B(f) = \sum_{m=-\infty}^{\infty} X(f + \frac{m}{T})$ is a periodic function with period 1/T

$$\Rightarrow B(f) = \sum_{n=-\infty}^{\infty} b_n e^{j2\pi nfT} \quad \text{(Fourier Series Expansion)}$$

where the coefficient $b_n = T\int_{-\frac{1}{2T}}^{\frac{1}{2T}} B(f) e^{-j2\pi nfT} df$

- From ①,

$$b_n = T\int_{-\frac{1}{2T}}^{\frac{1}{2T}} \left[\sum_{m=-\infty}^{\infty} X(f + \tfrac{m}{T}) \right] e^{-j2\pi nfT} df = T\sum_{m=-\infty}^{\infty} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X(f + \tfrac{m}{T}) e^{-j2\pi fnT} df$$

$$= T\int_{-\infty}^{\infty} X(f) e^{-j2\pi fnT} df = T\int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \,\Big|_{t=-nT} = Tx(-nT)$$

- To have no ISI, $x_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \Rightarrow b_n = Tx(-nT) = \begin{cases} T \text{ if } n = 0 \\ 0 \text{ if } n \neq 0 \end{cases}$

$$\Rightarrow B(f) = \sum_{n=-\infty}^{\infty} b_n e^{j2\pi nfT} = b_0 = T \qquad \Rightarrow \sum_m X(f + \frac{m}{T}) = T_{\#}$$

How to satisfy the Nyquist Pulse Shaping Criterion?

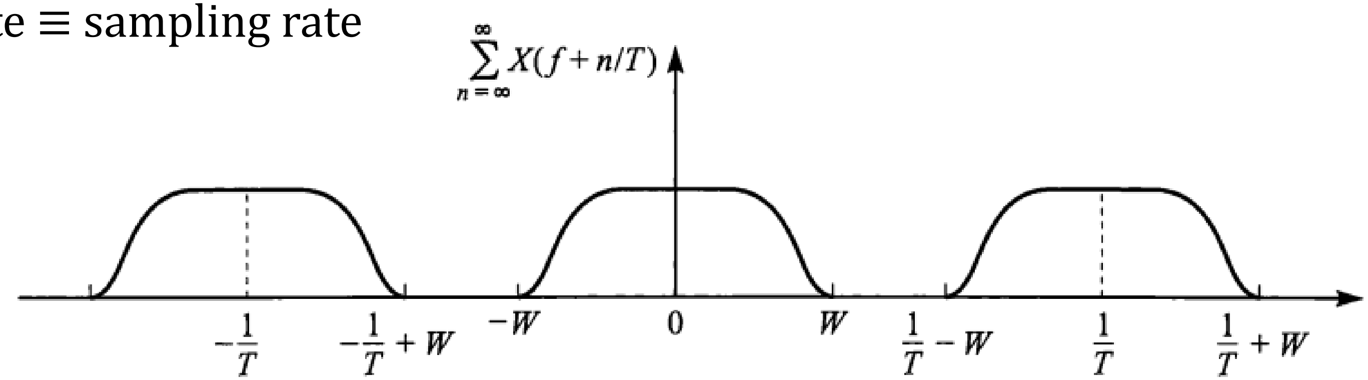
Assume band-limited transmission, $X(f) = 0 \ , \ \ |f| > W$

(limited bandwidth), and let $B(f) = \sum_{m=-\infty}^{\infty} X(f + \frac{m}{T})$.

(1) $\frac{1}{T} > 2W \quad \Rightarrow$ Higher than the transmission bandwidth

Channel bandwidth

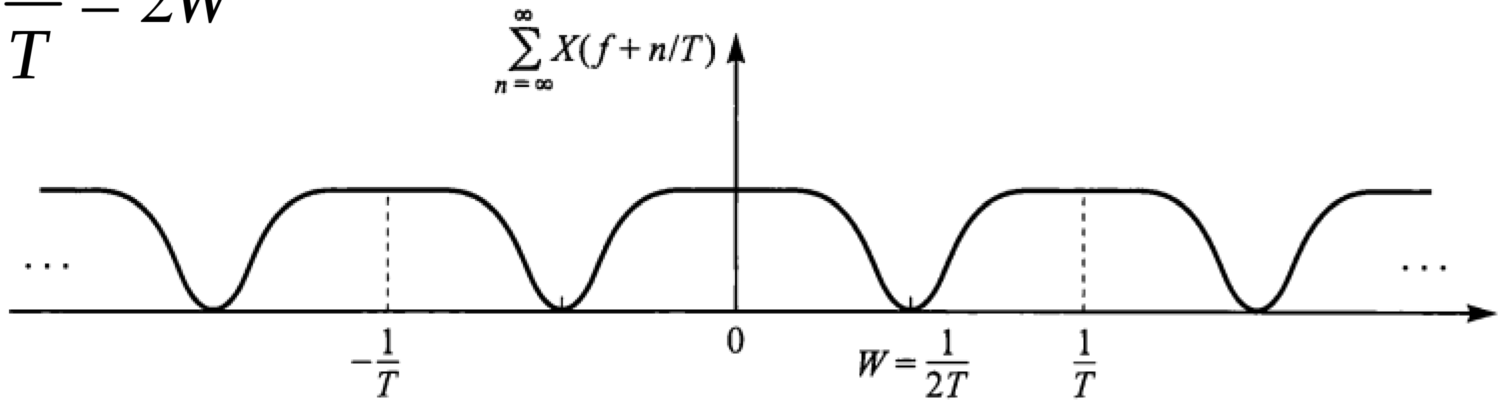
symbol rate ≡ sampling rate



$\Rightarrow$ Can not satisfy the Nyquist pulse shaping criterion $\Rightarrow \sum_{m} X(f + \frac{m}{T}) = T$

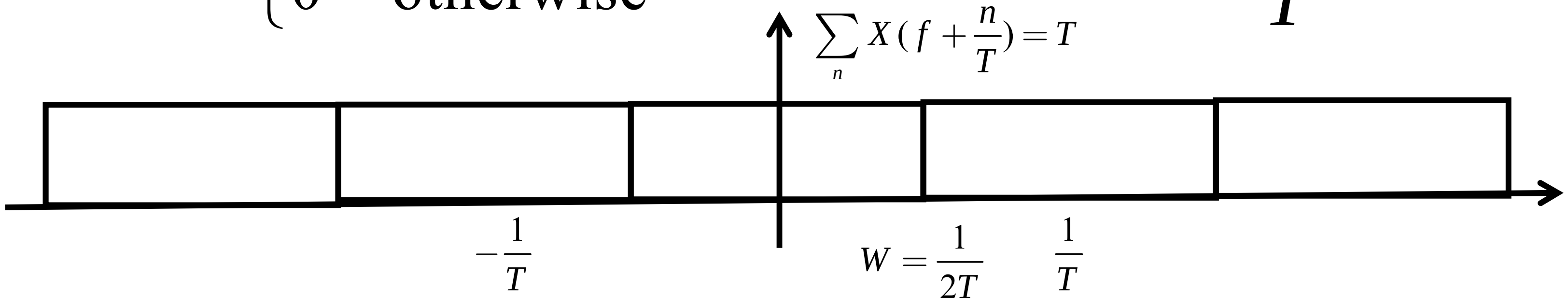
How to satisfy the Nyquist Pulse Shaping Criterion?

$$(2)\ \frac{1}{T} = 2W$$



The only pulse shape that satisfy the criterion is the rectangular

$$X(f) = \begin{cases} T & |f| < W \\ 0 & \text{otherwise} \end{cases} \qquad \Rightarrow x(t) = \mathrm{sinc}(\frac{t}{T})$$

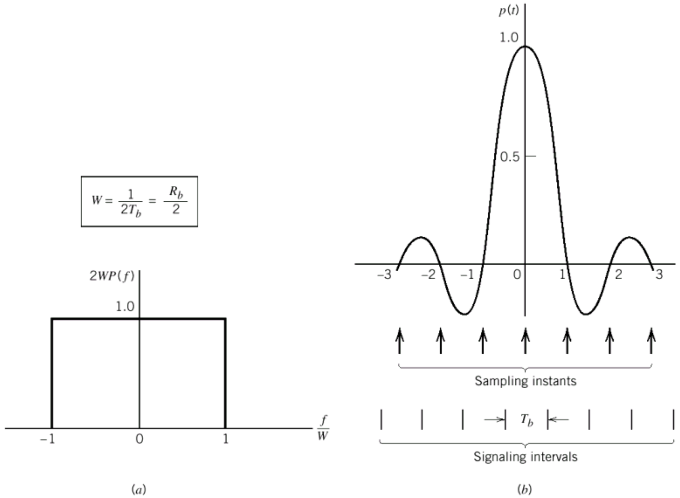


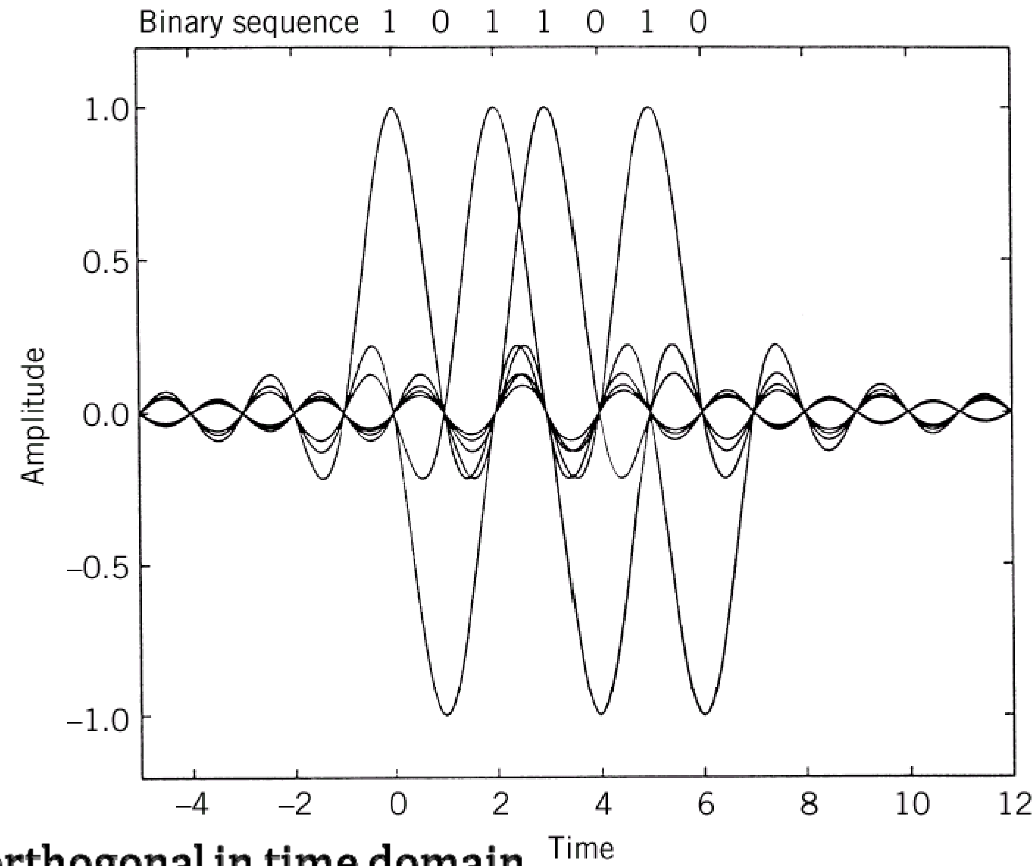
Nyquist Pulse Shaping with 1/T = 2W

$$X(f) = \begin{cases} T & |f| < W \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = r_l(t) * h^*(T-t) = \sum_{n=-\infty}^{\infty} I_n x(t-nT) + v(t)$$

$$\Rightarrow x(t) = \frac{1}{T}\operatorname{sinc}\left(\frac{t}{T}\right)$$

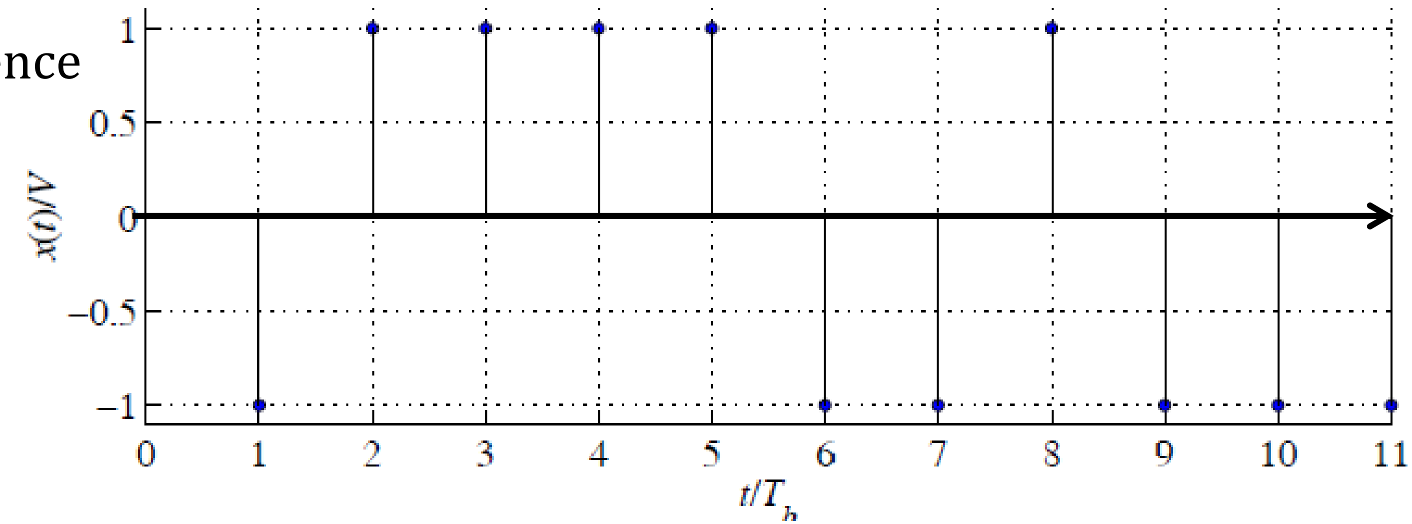




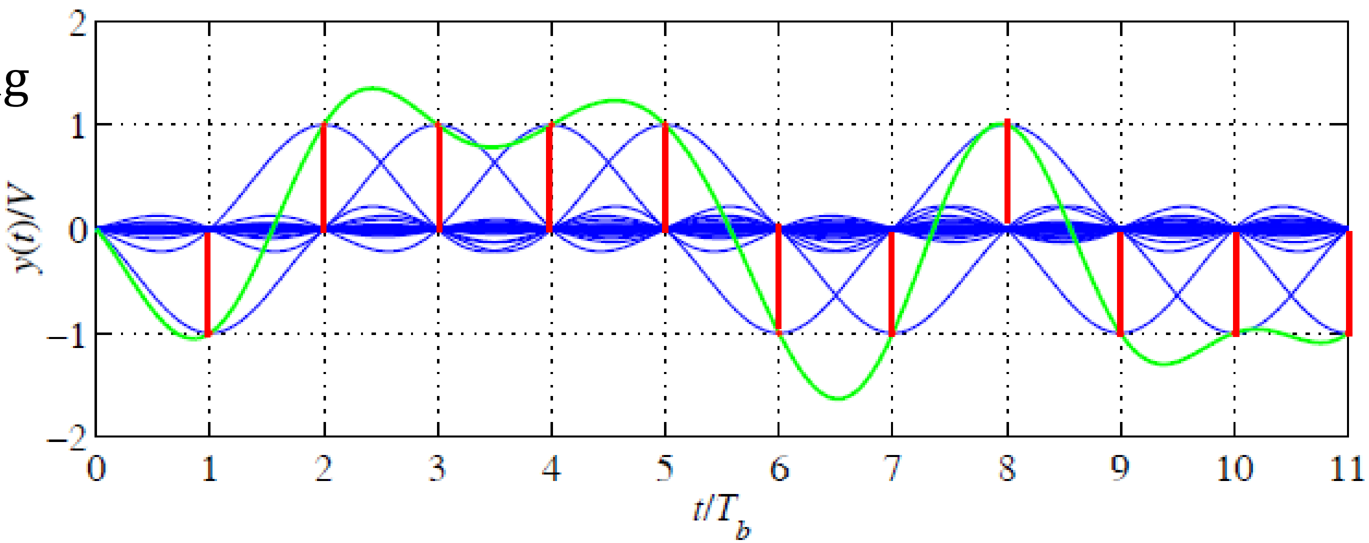
These pulses $x(t-nT)$ are orthogonal in time domain
i.e. $x(t) = 0$ at $t = \pm T, \pm 2T, \pm 3T, \ldots$
In practice, this pulse is not causal!

Signaling with Nyquist Pulse Shaping with 1/T = 2W

- Tx sequence



- Rx signaling

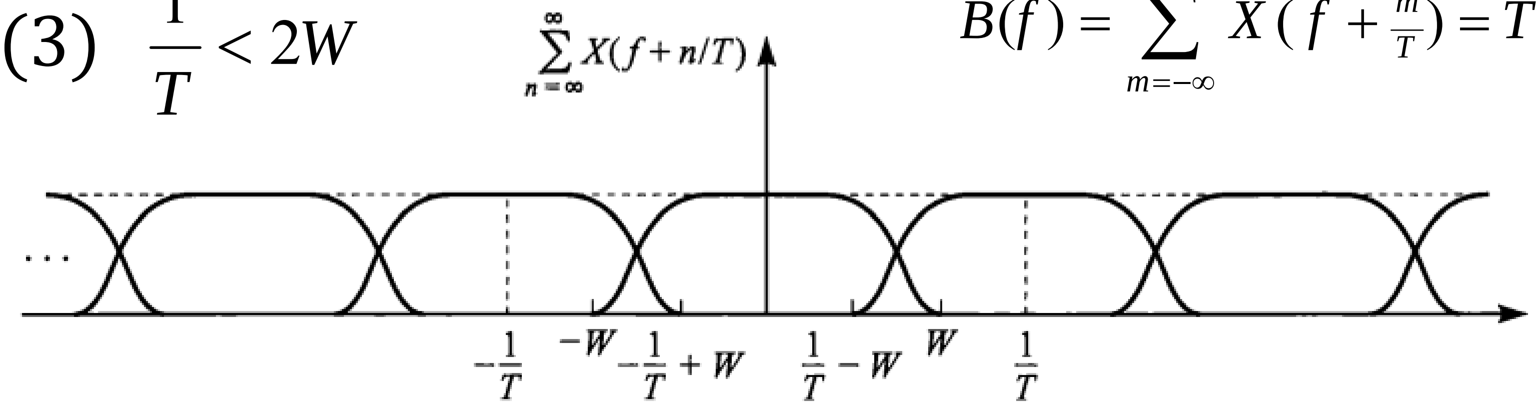


➢ NO ISI at each sampling!

How to satisfy the Nyquist Pulse Shaping Criterion?

(3) $\dfrac{1}{T} < 2W$

$$B(f) = \sum_{m=-\infty}^{\infty} X\left(f + \tfrac{m}{T}\right) = T$$



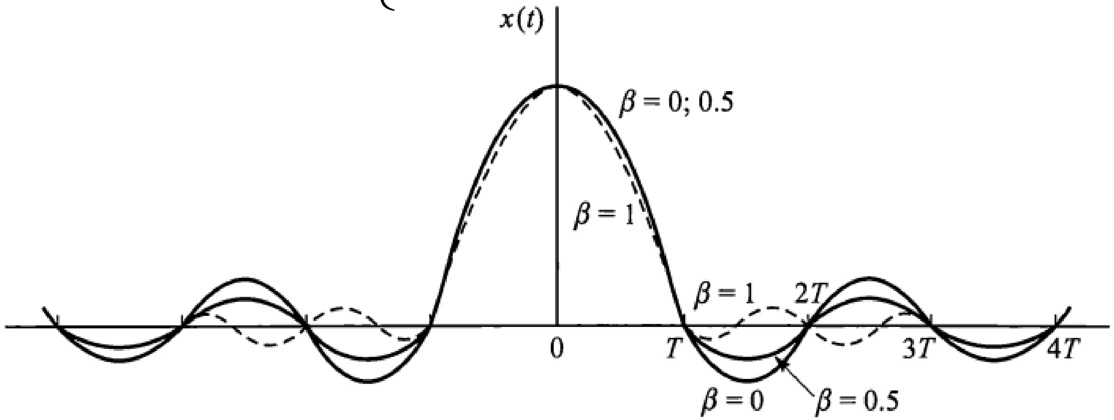
- *Pros*: It is then possible to design $X(f)$ carefully such that the Nyquist criterion can be satisfied. ISI free!
- *Cons:* Reduced symbol rate. It is a waste of channel bandwidth when the symbol rate $< 2W$.

Note: In communication standards, typically,

$$\text{BW/Rb} \approx 110\% \quad (g_T(t) = \text{raised cosine})$$

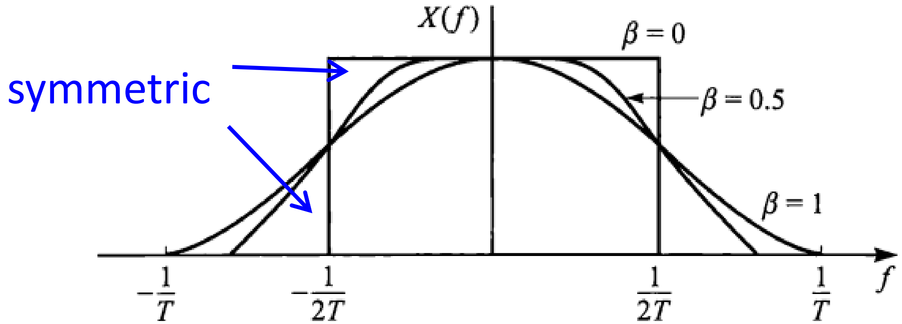
- Raised Cosine Pulse Shaping for $\frac{1}{T} < 2W$

$$X_{rc}(f) = \begin{cases} T & 0 \le |f| < \frac{1-\beta}{2T} \\ \frac{T}{2}\{1+\cos[\frac{\pi T}{\beta}(|f| - \frac{1-\beta}{2T})]\} & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0 & |f| > \frac{1+\beta}{2T} \end{cases}$$



(a)

where β is the roll-off factor $\beta \in [0,1]$. β represents the portion of BW beyond Nyquist frequency.

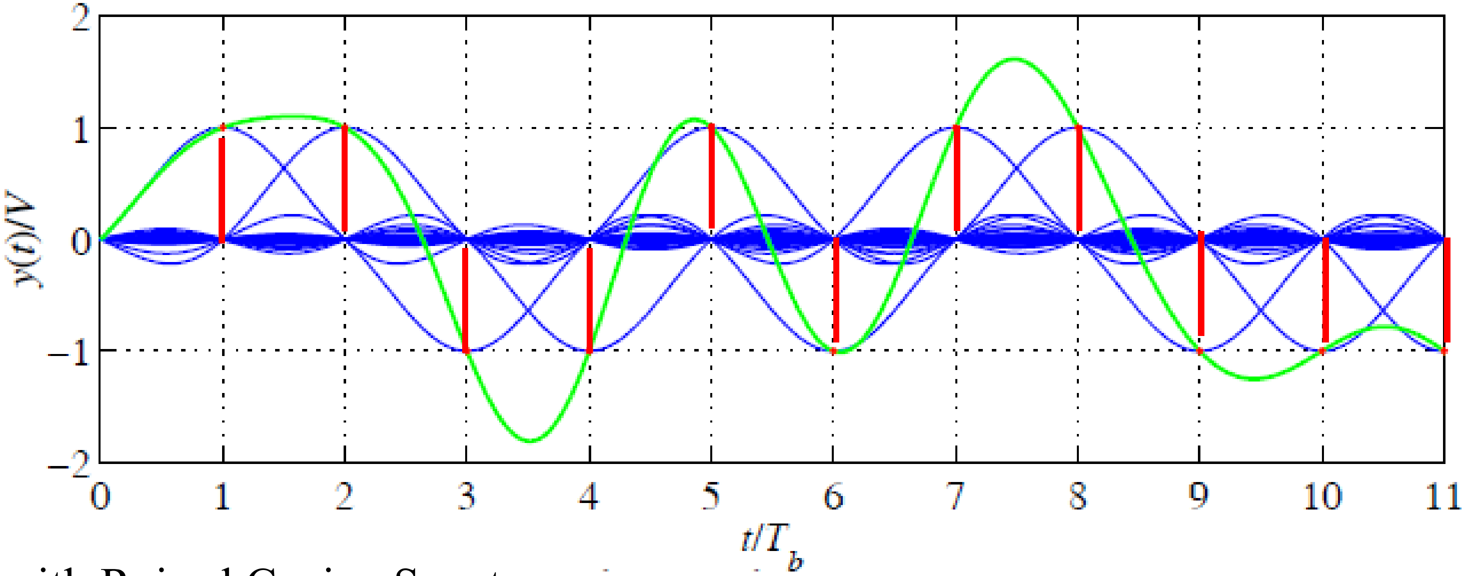


(b)

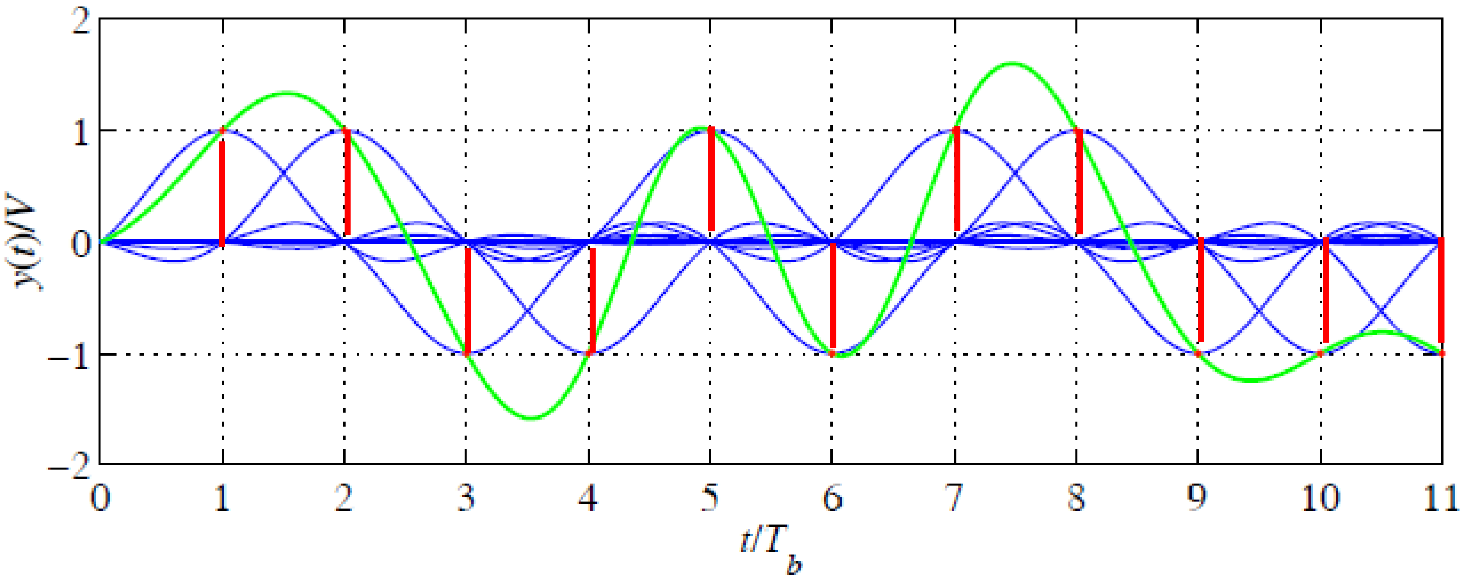
$$\begin{cases} \beta = 0 & \frac{1}{T} = 2W \\ \beta = \frac{1}{2} & \text{excess bandwidth is } 50\% \\ \beta = 1 & \text{excess bandwidth is } 100\% \end{cases}$$

Signaling with Nyquist Pulse Shaping

- $\dfrac{1}{T} = 2W$ with Rectangular Spectrum



- $\dfrac{1}{T} < 2W$ with Raised Cosine Spectrum



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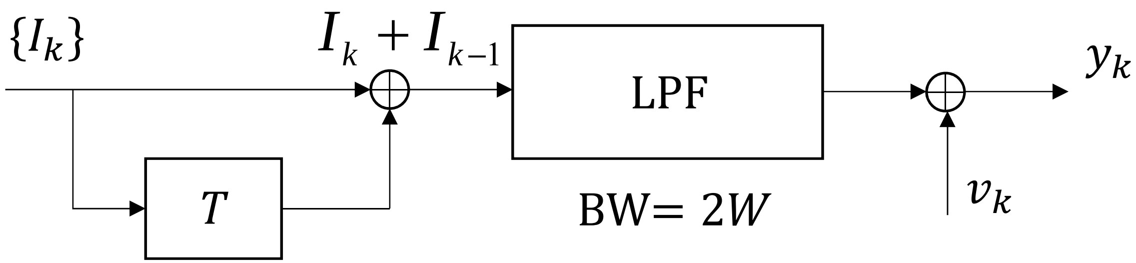
Signal Design with Controlled ISI

Motivation: How to avoid ISI, keep 1/T = 2W and have smooth rising/falling filter of *G*(*f*)?

- Duo-binary signal pulse $$y_k = x_0 I_k + \sum_{\substack{n=-\infty \\ n \neq k}}^{\infty} I_n x_{k-n} + v_k$$

$$x(nT) = \begin{cases} 1 & n = 0,1 \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow y_k = I_k + I_{k-1} + v_k \quad \Rightarrow \hat{I}_k = y_k - \hat{I}_{k-1}$$

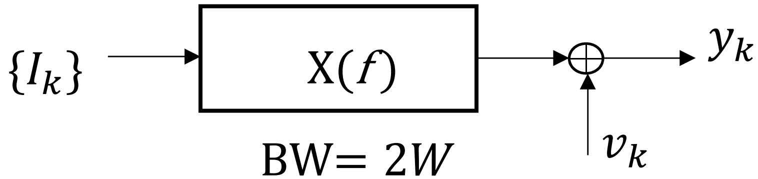


$$\overset{\text{no ISI}}{\Rightarrow} b_n = Tx(-nT) = \begin{cases} T & (n = 0,-1) \Rightarrow b_0 = b_{-1} = T \\ 0 & \text{other} \end{cases}$$

$$B(f) = \sum_{m=-\infty}^{\infty} X\left(f + \tfrac{m}{T}\right) = \sum_{n=-\infty}^{\infty} b_n e^{j2\pi nfT} = T + Te^{-j2\pi fT}$$

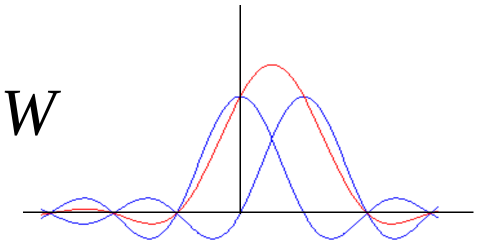
● Duo-binary signal pulse (cont'd)

For $\dfrac{1}{T} = 2W$



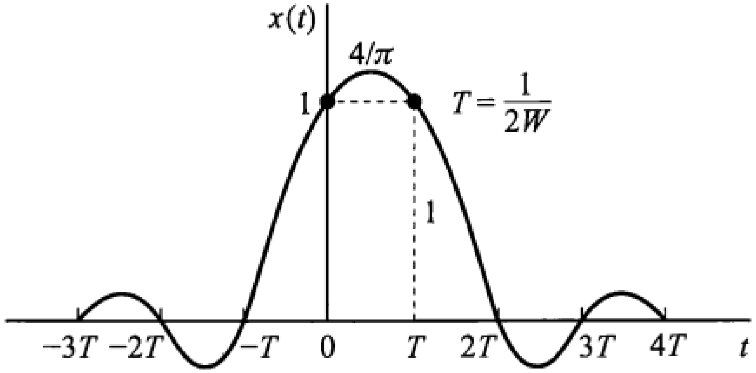
$$X(f) = T + Te^{-j2\pi fT}, \qquad |f| \le W$$

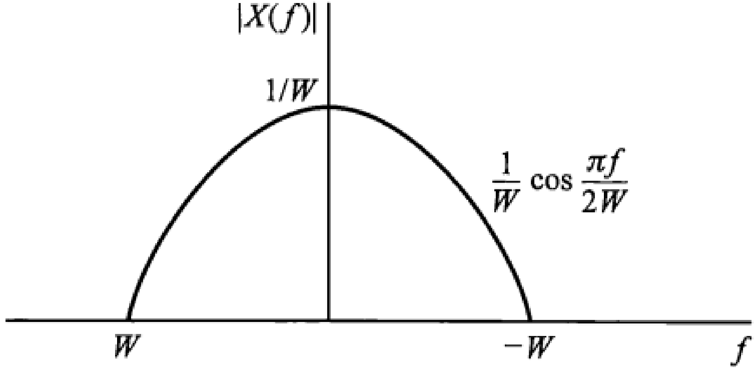
$$\Rightarrow x(t) = \mathrm{sinc}(2\pi Wt) + \mathrm{sinc}[2\pi W(t - \tfrac{1}{2W})]$$



Also $$X(f) = \frac{1}{2W} e^{-j\pi f \frac{1}{2W}} (e^{j\pi f \frac{1}{2W}} + e^{-j\pi f \frac{1}{2W}}), \quad |f| \le W$$

$$= \begin{cases} \frac{1}{W} e^{-j\pi f \frac{1}{2W}} \cos(\frac{\pi f}{2W}), & |f| \le W \\ 0, & \text{other} \end{cases}$$

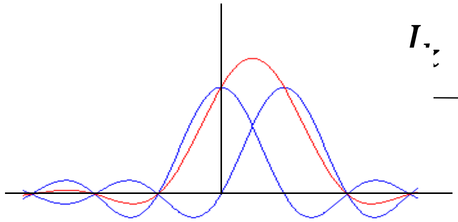


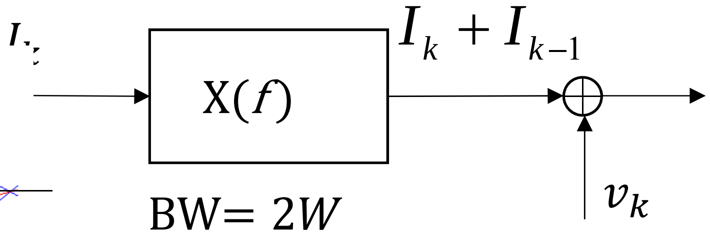


- ➢ $x(t)$ has value only at $t = 0, t = T$, other sampling time $x(nT) = 0$
- ➢ The ISI is controlled and can be resolved recursively.

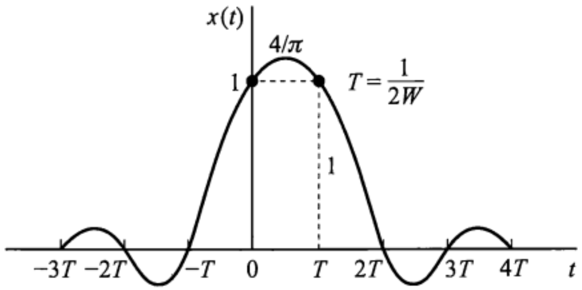
Duo-binary signal pulse (cont'd)

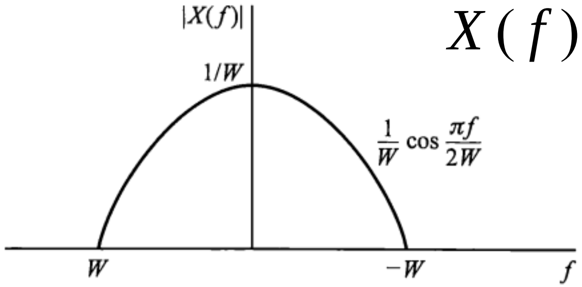
$$y(t) = \sum_{n=-\infty}^{\infty} I_n x(t - nT) + v(t)$$





$$y_k = I_k + I_{k-1} + v_k$$





$$X(f) = G_T(f)C(f)G_R(f)$$

- The Duo-binary signal precoding achieves three objectives

✓ Increased symbol rate 1/T to 2W

✓ Satisfying Nyquist pulse shaping criterion between pre-coded symbol $\{I_k+I_{k-1}\}$ and $\{y_k\}$.

✓ Smooth rising and falling of the pulse filters $G_T(f)$ and $G_R(f)$

- Modified duo-binary signal pulse with 1/T = 2W

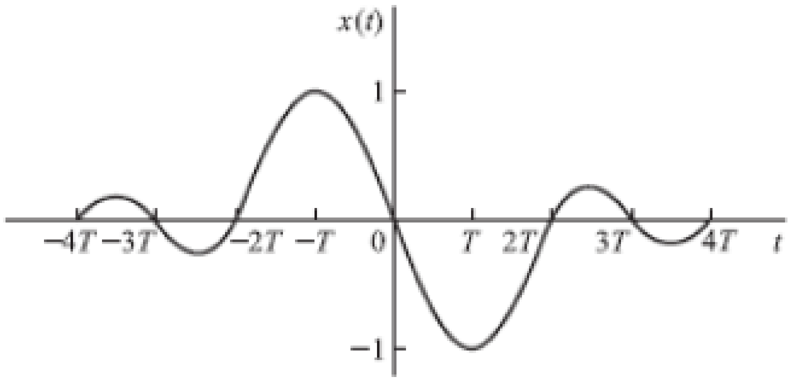
$$x(nT) = \begin{cases} 1, & n = -1 \\ -1, & n = 1 \\ 0, & \text{otherwise} \end{cases} \quad \begin{aligned} &\Rightarrow y_k = I_{k-1} - I_{k+1} + v_k \\ &\Rightarrow \hat{I}_{k+1} = y_k - \hat{I}_{k-1} \end{aligned}$$

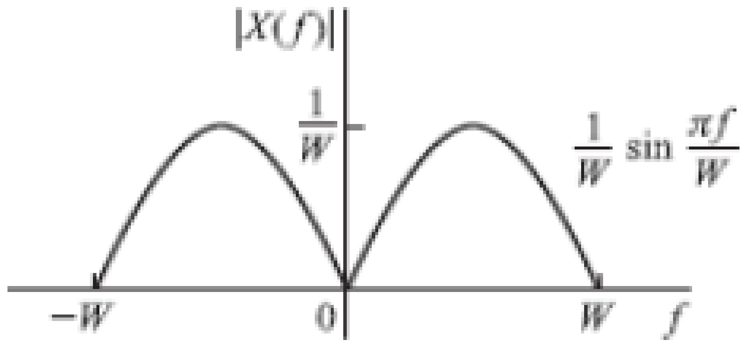
$$x(t) = \operatorname{sinc}[2\pi W(t + \tfrac{1}{2W})] - \operatorname{sinc}[2\pi W(t - \tfrac{1}{2W})] = \operatorname{sinc}\frac{\pi(t+T)}{T} - \operatorname{sinc}\frac{\pi(t-T)}{T}$$

$$B(f) = \sum_{m=-\infty}^{\infty} X(f + \tfrac{m}{T}) = \sum_{n=-\infty}^{\infty} b_n e^{j2\pi nfT}$$

$$X(f) = \tfrac{1}{2W}(e^{j2\pi fT} - e^{-j2\pi fT}), \quad |f| \le W$$

$$= \tfrac{1}{2W}(e^{j\pi f/W} - e^{-j\pi f/W}), \quad |f| \le W = \tfrac{j}{W}\sin(\frac{\pi f}{W}), \quad |f| \le W$$





- There are many other ways to design the precoding (or controlled ISI), $x(t)$.
- In general, $x(t)=\sum_k x(kT)\operatorname{sinc}[2\pi W(t-kT)]$ $\Rightarrow$ convolution of $\{x_k\}$ with $\operatorname{sinc}(2\pi Wt)$

$$X(f)=\frac{1}{2W}\sum_k x(\tfrac{k}{2W})e^{-jk\pi f/w}, |f|\le W$$

$$y(t)=\sum_n I_n x(t-nT)+v(t)=\sum_n I_n\sum_k x(kT)\operatorname{sinc}[2\pi W(t-nT-kT)]+v(t)$$

$$=\sum_n I_n\sum_k x_k\operatorname{sinc}[2\pi W(t-\tfrac{n+k}{2W})]+v(t)$$

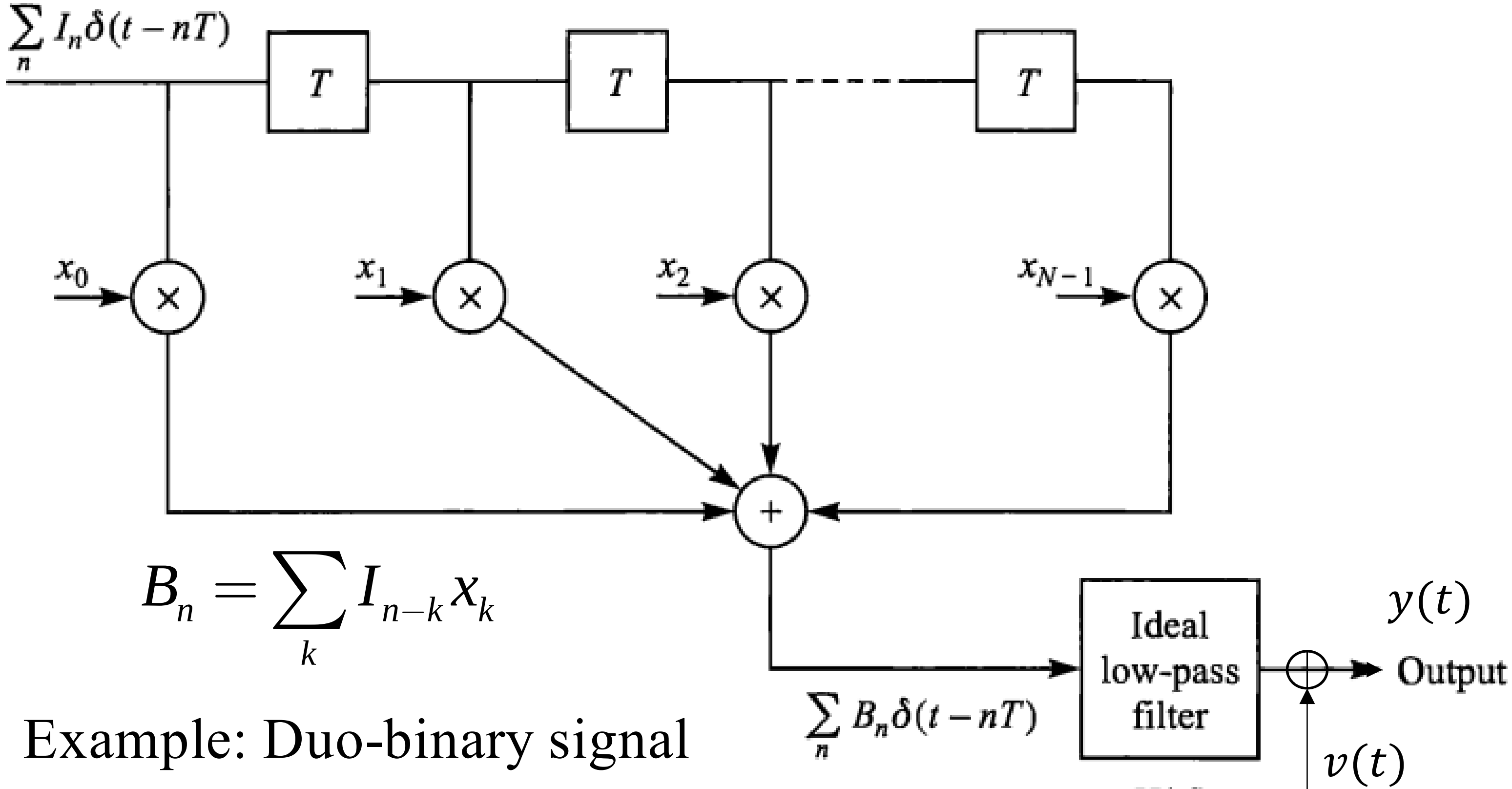
Let $n'=n+k$

$$y(t)=[\sum_{n'}\underbrace{\sum_k I_{n'-k}x_k}\delta(t-n'T)*\operatorname{sinc}(2\pi Wt)]+v(t)$$

$B_n=\sum_k I_{n-k}x_k$ (Linear combination of $\{I_n\}$)

$$=\sum_{n'}B_{n'}\delta(t-n'T)*\operatorname{sinc}(2\pi Wt)+v(t)$$

The equivalent model $y(t) = \sum_n B_n \delta(t - nT) * \mathrm{sinc}(2\pi W t) + v(t)$

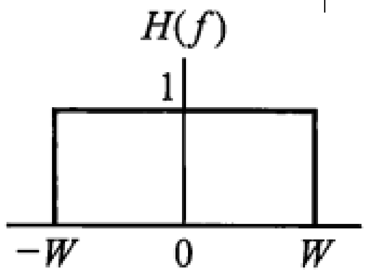


$$B_n = \sum_k I_{n-k} x_k$$

Example: Duo-binary signal

$$y_k = I_k + I_{k-1} + v_k$$
$$= x_0 I_k + x_1 I_{k-1} + v_k$$

$$\Rightarrow x(t) = \mathrm{sinc}(2\pi W t) + \mathrm{sinc}[2\pi W(t - \tfrac{1}{2W})]$$



The precoded system is equivalent to transmit a correlated symbol sequence $\{B_n\}$

$$y(t) = [\sum_n B_n \delta(t - nT)] * \mathrm{sinc}(2\pi Wt) + v(t)$$

where $B_n = \sum_{k=0}^{L} x_k I_{n-k}$ *and* L is the precoding length.

Ex: For duo-binary signal ($L = 1$), $B_n = I_n + I_{n-1}$

- The autocorrelation of B_n is

$$R_{BB}[m] = \mathrm{E}[B_{n+m} B_n] = \mathrm{E}[(\sum_k x_k I_{n-k})(\sum_l x_l I_{n+m-l})]$$

- The PSD of precoded signal is

$$S_{BB}(f) = \sum_m R_{BB}[m] e^{-j2\pi f m T}$$

- The power spectrum at the Tx output:

$$S_{VV}(f) = \frac{1}{T} S_{BB}(f) \left|G(f)\right|^2$$

Symbol-by-Symbol Detection for Controlled ISI

- Suppose the detection of I_{k-1} is available , then we can detect I_k and following symbols successively.

$$\left.\begin{aligned} I_k &= B_k - I_{k-1} \\ I_{k+1} &= B_{k+1} - I_k \\ I_{k+2} &= B_{k+2} - I_{k+1} \end{aligned}\right\rangle$$

The controlled ISI can be resolved.

Q: What would be the problem here ?

$\rightarrow$ When I_{k-1} is detected incorrectly, error propagates!

Symbol-by-Symbol Detection for Controlled ISI

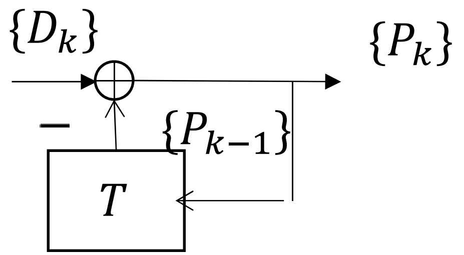
- Duo-binary signal

To solve the error propagation problem, additional precoding can be applied.

- **Precoding at Tx**

Suppose $\{D_k\}$ is the binary data sequence and $\{P_k\}$ is the precoded sequence, where $P_k = D_k - P_{k-1} \mod 2$

$$\Rightarrow D_k = P_k + P_{k-1} \mod 2$$

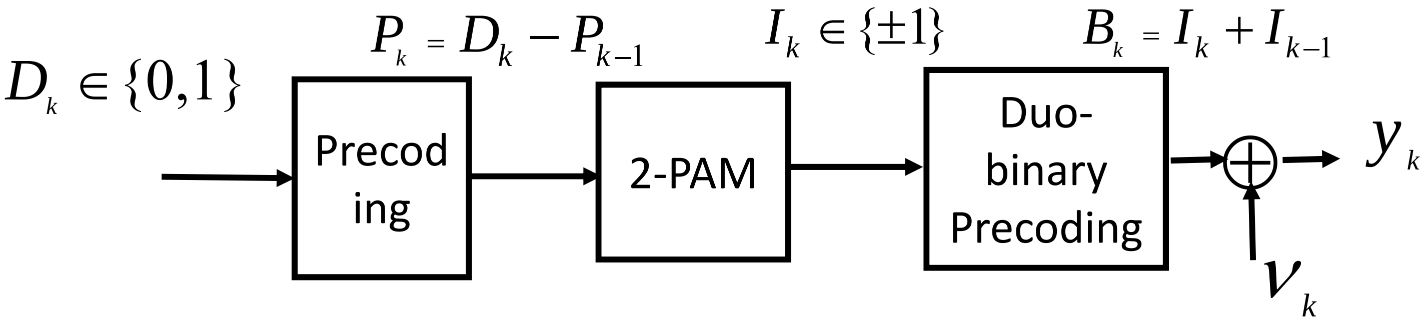


$$= \begin{cases} 0 \text{ if } & P_k = P_{k-1} \\ 1 \text{ if } & P_k \neq P_{k-1} \end{cases}$$

Symbol-by-Symbol Detection for Controlled ISI

- Duo-binary signal

$$\text{Transmit} \quad I_k = \begin{cases} 1 & \text{if } P_k = 1 \\ -1 & \text{if } P_k = 0 \end{cases} = 2P_k - 1$$



$$\text{At the receiver,} \quad y_k = B_k + \nu_k = I_k + I_{k-1} + \nu_k$$

$$\begin{aligned} B_k &= I_k + I_{k-1} \\ &= (2P_k - 1) + (2P_{k-1} - 1) \\ &= 2(P_k + P_{k-1} - 1) \end{aligned}$$

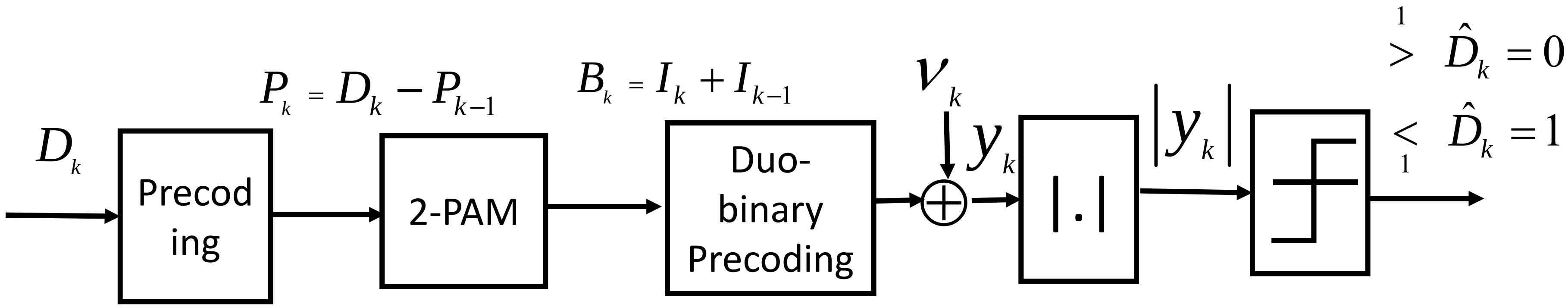
$$\Rightarrow D_k = P_k + P_{k-1} = \frac{1}{2} B_k + 1 \mod 2$$

Symbol-by-Symbol Detection for Controlled ISI

- Duo-binary signaling

$$D_k = P_k + P_{k-1} \mod 2$$
$$= (\frac{1}{2} B_k + 1) \mod 2$$
$$= \begin{cases} 0, \text{ if } B_k = \pm 2, \\ 1, \text{ if } B_k = 0, \end{cases} \overset{i.e.}{=} \begin{cases} 0, \text{ if } |y_k| \geq 1, & \text{with prob } 1/2 \\ 1, \text{ if } |y_k| < 1, & \text{with prob } 1/2 \end{cases}$$

Note: $B_k = I_k + I_{k-1} \Rightarrow B_k \in \{-2, 0, +2\}$ with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.



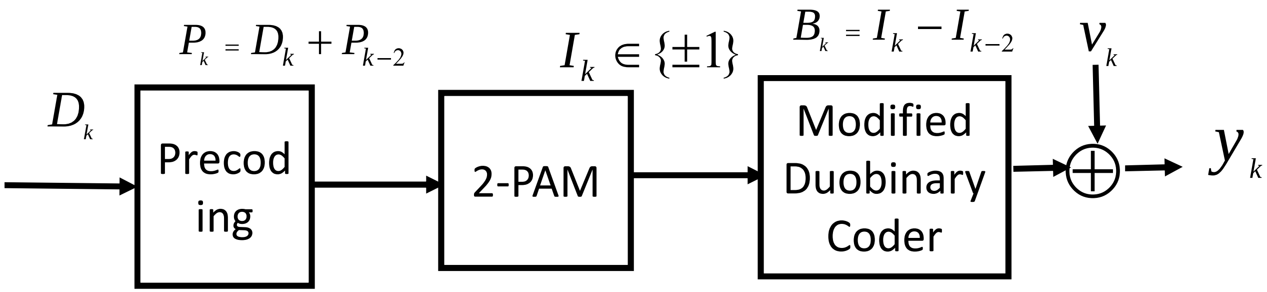
- Determine $\hat{D}_k$ only based on B_k (*i.e.* symbol by symbol detection)

$\Rightarrow$ No error propagation

Symbol-by-Symbol Detection for Controlled ISI

● Modified Duo-binary signal with precoder

Design the precoder $P_k = D_k + P_{k-2} \mod 2$



$$\text{Transmit } I_k = \begin{cases} 1 & \text{if } P_k = 1 \\ -1 & \text{if } P_k = 0 \end{cases} = 2P_k - 1$$

At the receiver, $y_k = B_k + v_k = I_k - I_{k-2} + v_k$

$$\begin{aligned} B_k &= I_k - I_{k-2} = (2P_k - 1) - (2P_{k-2} - 1) \\ &= 2(P_k - P_{k-2}) \end{aligned}$$

Note that $D_k = P_k - P_{k-2} \mod 2$

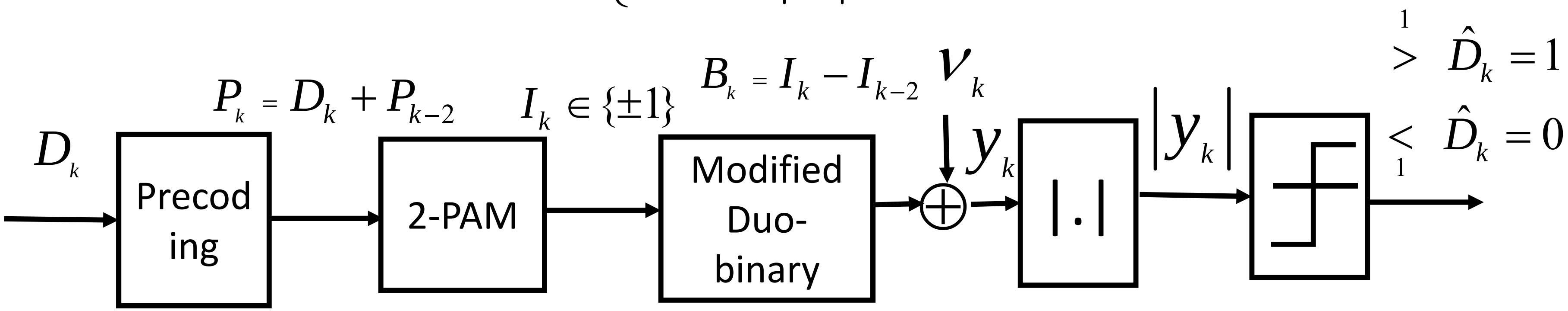
Symbol-by-Symbol Detection for Controlled ISI

- Modified Duo-binary signal with precoder

$$D_k = P_k - P_{k-2} \mod 2$$
$$= \frac{1}{2} B_k \mod 2 = \begin{cases} 1, \text{ if } B_k = \pm 2, \\ 0, \text{ if } B_k = 0, \end{cases}$$

Note: $B_k = I_k - I_{k-2} \Rightarrow B_k \in \{-2, 0, +2\}$ with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.

$$y_k = B_k + \nu_k \Rightarrow \hat{D}_k = \begin{cases} 1, \text{ if } |y_k| \geq 1 \\ 0, \text{ if } |y_k| < 1 \end{cases}$$



- Determine $\hat{D}_k$ only based on B_k (*i.e.* symbol by symbol detection)

$\Rightarrow$ No error propagation

M-ary Duo-Binary Signaling

- M-ary PAM over bandlimited channel suffers from ISI as well.
- Use symbol-by-symbol detection for controlled ISI with M-ary duo-binary signal

  - Extend from the binary duo-binary signal.

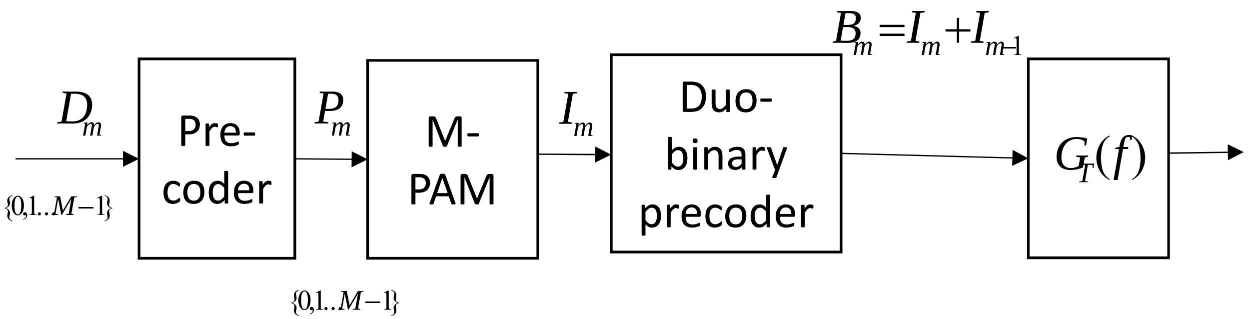
- Given $D_m \in \{0, \ldots, M-1\}$

  Design the precoder $P_m = D_m - P_{m-1} \mod M$

  $\rightarrow P_m \in \{0, \ldots, M-1\}$

  For M-ary PAM, $I_m \in \{\pm d, \pm 3d, \ldots \pm (M-1)d,\}$

  $I_m = [2P_m - (M-1)]d$



M-ary Duo-Binary Signaling

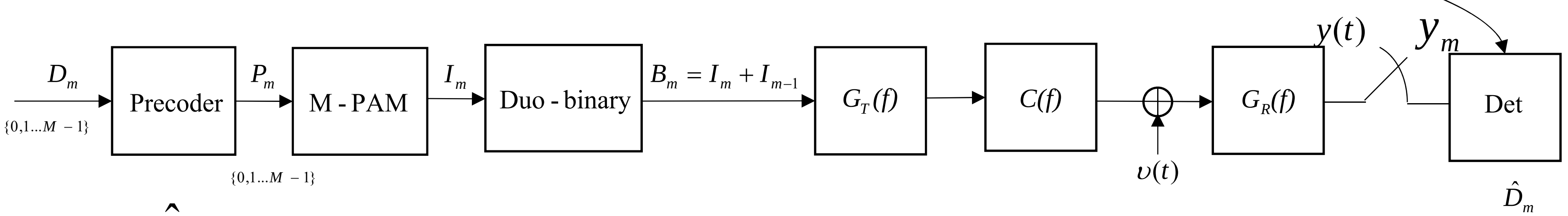
- $B_m = I_m + I_{m-1} = 2[\underbrace{P_m + P_{m-1}}_{=D_m} - (M-1)]d$

$$\Rightarrow B_m \in \{0, \pm 2d, \pm 4d, \ldots, \pm 2(M-1)d\}$$

$$\Rightarrow D_m = P_m + P_{m-1} = \frac{1}{2d} B_m + (M-1) \mod M$$

- $y_m = I_m + I_{m-1} + \upsilon_m = B_m + \upsilon_m$

$$\boxed{\Rightarrow \hat{D}_m = \frac{y_m}{2d} + (M-1) \mod M}$$



- $\hat{D}_m$ only depends on the value of y_m (symbol by symbol detection)

$\Rightarrow$ No error propagation

Outline

Part 1: Signal Design for Bandlimited Channels

- **Lecture 1: System Model of band-limited channel**
  - ✓ ISI problem due to band-limited channel

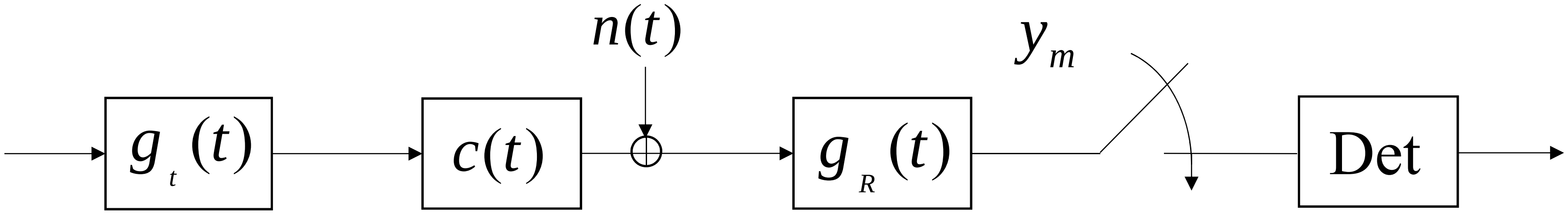
- **Lecture 2: Signal design with controlled ISI (duo-binary precoding)**
  - ✓ Duo-binary precoding
  - ✓ Modified duo-binary precoding
  - ✓ M-ary duo-binary precoding

- **Lecture 3: Error performance analysis of controlled ISI precoding**

Error Probability for Bandlimited Channel

(with Controlled ISI Precoder)

- To have a complete picture of the controlled ISI precoder design, the error performance evaluation is needed.
- NO ISI - satisfying Nyquist pulse shaping criterion



Let $g_R(t) = g_t^*(-t)$ matched filter (non-causal).

For $X(f) = G_T(f)G_R(f) = \left|G_T(f)\right|^2$

- Recall the case of M-PAM w/o ISI (unlimited BW), $y_m = x_0 I_m + \upsilon_m$

$$\text{where } x_0 = \int_{-\infty}^{\infty} \left|g_t(t)\right|^2 dt = \int_{-W}^{W} \left|g_T(f)\right|^2 df = E_g$$

Error Probability for Bandlimited Channel

The noise term $\upsilon(t) = \int_{-\infty}^{\infty} g_R(\tau) n(t-\tau) d\tau$

$$\Rightarrow S_{vv}(f) = S_{nn}(f) \cdot \left| G_T(f) \right|^2$$

$$\Rightarrow \sigma_\upsilon^{\ 2} = \int_{-\infty}^{\infty} S_{vv}(f) df = \int_{-\infty}^{\infty} S_{nn}(f) \left| G_T(f) \right|^2 df = \frac{N_0}{2} E_g$$

From Ch4 (or Eq. 4.3-4), for $I_m \in \{\pm d, \pm 3d, \ldots \pm (M-1)d\}$ the union error for M-PAM is

$$P_e = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2d^2 E_g}{N_0}} \right)$$

Error Probability for Bandlimited Channel

- M-ary duo-binary signal

$$D_m = \frac{1}{2d} B_m + (M-1) \mod M$$

where $B_m = I_m + I_{m-1} \Rightarrow B_m \in \{0, \pm 2d, \pm 4d, \ldots \pm 2(M-1)d\}$

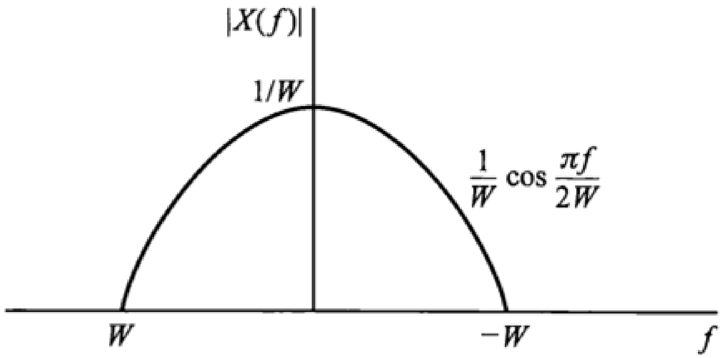
$$y_m = B_m + \upsilon_m$$

$\Rightarrow$ Equivalent to M-ary PAM transmission and detection.

For noise power,

$$\sigma_\upsilon^{\ 2} = \frac{1}{2} N_0 \int_{-w}^{w} |G_T(f)|^2 \, df$$

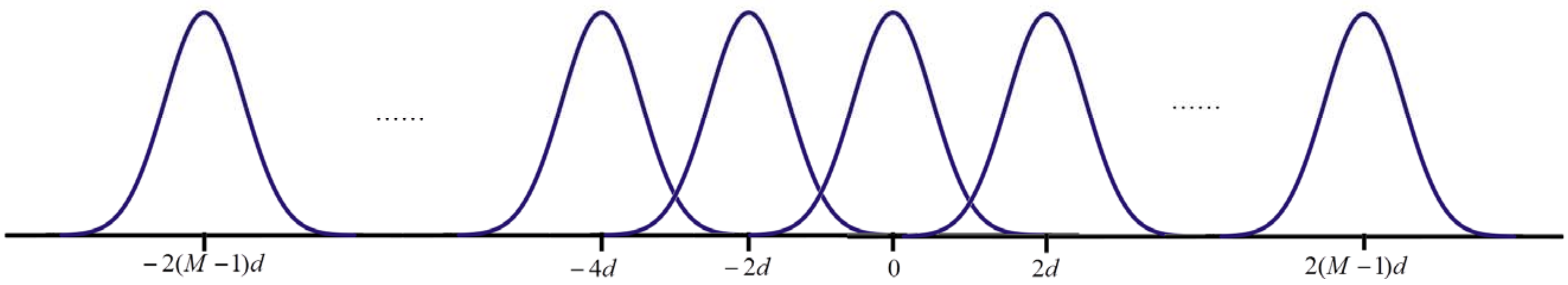
$$= \frac{1}{2} N_0 \int_{-\infty}^{\infty} |X(f)| \, df$$



$$= \frac{1}{2} N_0 \int_{-W}^{W} \frac{1}{W} \cos(\frac{\pi f}{2W}) df \ = \frac{N_0}{2} \frac{2}{\pi} \sin(\frac{\pi f}{2W}) \Big|_{-W}^{W} \ = \frac{N_0}{2} \frac{4}{\pi} = \frac{2N_0}{\pi}$$

- Error Probability for Bandlimited Channel

$$y_m = B_m + \upsilon_m \quad \text{where } B_m \in \{0, \pm 2d, \pm 4d, \ldots \pm 2(M-1)d\}$$



y_m is Gaussian distributed with mean = $\mathrm{B_m}$, and var $= 2N_0/\pi$

We the error probability of M-PAM here.

$$d_{\min} = 2d, \;\; \text{var} = 2N_0/\pi$$

$$P_e = \sum_{m=-(M-1)}^{M-1} P(B = B_m) P\left(\hat{D}_m \neq B_m / B_m\right)$$

However, $P(B_m)$ is not uniformly distributed here.

- Error Probability for Bandlimited Channel

If I_m is equally probable, i.e.$P\{I_m\} = \frac{1}{M}, \forall m$

$$I_m \in \{-(M-1)d, \cdots\cdots, -3d, -d, d, +3d, \cdots\cdots +(M-1)d\}$$

$$I_{m-1} \in \{-(M-1)d, \cdots\cdots, -3d, -d, d, +3d, \cdots\cdots +(M-1)d\}$$

$\Rightarrow$ For $B_m = 2md$, there are M - $|m|$ out of M^2 cases.

e.g. $m = 0 : \Pr(B_m = 0) = \frac{M}{M^2} = \frac{1}{M}$

$m = 1:\ \Pr(B_m = 2d) = \frac{M-1}{M^2},\ \Pr(B_m = -2d) = \frac{M-1}{M^2}$

$$\Rightarrow \Pr\{B_m = 2md\} = \frac{M-|m|}{M^2} \text{ for } m = 0, \pm 1, \ldots \pm (M-1)$$

Ex: M=2, $\Pr\{B_m\} = \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ for $m = -1, 0, +1$

- Error Probability for Bandlimited Channel

$$P_e = \sum_{m=-(M-1)}^{M-1} \mathrm{P}(B_m) P\left(\hat{D}_m \neq B_m / B_m\right)$$

where $B_m \in \{0, \pm 2d, \pm 4d, \ldots \pm 2(M-1)d\}$

$$\Rightarrow P_e = \left\{ \sum_{m=-(M-2)}^{(M-2)} \Pr(|\, y - 2md \,| > d) / B_m = 2md) \Pr(B_m = 2md) \right\}$$

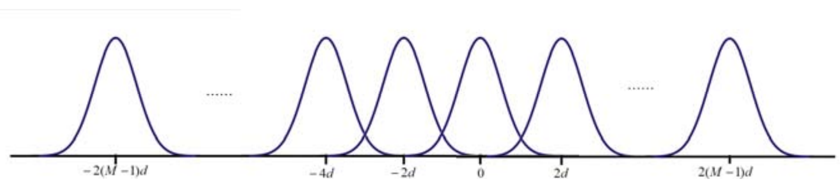
$$+ 2\Pr(y + 2(M-1)d > d) / B_m = -2(M-1)d) \Pr(B_m = -2(M-1)d)$$

$$= \underbrace{\Pr\{|y| > d \mid B_m = 0\}}_{\rightarrow = Q\left(\frac{\frac{d_{\min}}{2}}{\sigma_\nu}\right)} \cdot \{2 \sum_{m=0}^{M-1} \Pr(B_m = 2md) - \Pr(B_m = 0) - \Pr(B_m = -2(M-1)d\}$$

where $2\sum_{m=0}^{M-1} \Pr(B_m = 2md) = 2\sum_{m=0}^{M-1} \frac{M-m}{M^2} = \frac{M+1}{M}$,

$$\Pr(B_m = 0) = \frac{M}{M^2} = \frac{1}{M},$$

$$\Pr\{B_m = -2(M-1)d\} = \frac{1}{M^2}$$



- Error Probability for Bandlimited Channel

$$\Rightarrow P_e = \left(\frac{M+1}{M} - \frac{1}{M} - \frac{1}{M^2} \right) \Pr\{|\upsilon_m| > d\} = (1 - \frac{1}{M^2}) \Pr\{|\upsilon_m| > d\}$$

$$= Q(\frac{\frac{d_{\min}}{2}}{\sigma_\nu})$$

$$= (1 - \frac{1}{M^2}) \cdot 2Q(\frac{\frac{d_{\min}}{2}}{\sqrt{2N_0 / \pi}}) = (1 - \frac{1}{M^2}) \cdot 2Q(\sqrt{\frac{\pi d^2}{2N_0}})$$

Probability of error for controlled ISI with M-ary PAM precoding

$$P_M \overset{\Delta}{=} P_e = (1 - \frac{1}{M^2}) \cdot 2Q(\sqrt{\frac{\pi d^2}{2N_0}})$$

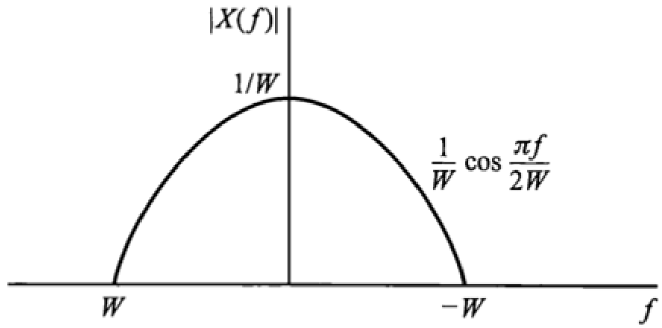
To express P_M with SNR: $E_{av} = P_{av} \cdot T$

- Error Probability for Bandlimited Channel

$$P_{av} = \int_{-w}^{w} S_{vv}(f)df$$

$$= \frac{1}{T}\int_{-w}^{w} S_{II}(f)\left|G_T(f)\right|^2 df$$

$$= \frac{1}{T}E[I_m{}^2]\int_{-w}^{w} X(f)df$$



$$\int_{-w}^{w} X(f)df = \frac{4}{\pi}$$

where $I_m \in \{\pm d, \pm 3d, \ldots \pm (M-1)d\}$ with equal prob,

$$E[I_m{}^2] = \frac{1}{3}d^2(M^2-1)$$

$$\Rightarrow P_{av} = \frac{1}{T}\cdot\frac{1}{3}d^2(M^2-1)\cdot\frac{4}{\pi}$$

$$= \frac{4}{\pi T}\frac{1}{3}d^2(M^2-1)$$

$$\text{Recall}: 1^2 + 2^2 + \ldots n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$1^2 + 3^2 + (M-1)^2 = \frac{M}{6}(M^2-1)$$

• Error Probability for Bandlimited Channel

With $E_{av} = P_{av} \cdot T$, then $d^2 = \dfrac{3\pi E_{av}}{4(M^2-1)}$

The error prob for M-ary duo-binary signaling becomes

$$\Rightarrow P_M = 2\left(1-\frac{1}{M^2}\right)Q\left(\sqrt{\left(\frac{\pi}{4}\right)^2 \frac{6}{M^2-1}\frac{E_{av}}{N_0}}\right)$$

Comparing with the P_e of M-PAM without ISI (from Ch4),

$$P_e = \frac{2(M-1)}{M}Q\left(\sqrt{\frac{6E_{av}}{(M^2-1)N_0}}\right) \quad < P_M$$

Symbol-by-symbol detection with duo-binary design have a loss

of $\left(\dfrac{\pi}{4}\right)^2$ (i.e. 2.1dB) in energy.

The loss can be recovered with MLSD at cost of complexity.

Summary for Part 1

- ISI problem due to band-limited channel
- Nyquist pulse shaping criterion
- Signal design with controlled ISI (duo-binary precoding)
  - ✓ Duo-binary precoding
  - ✓ Modified duo-binary precoding
  - ✓ M-ary duo-binary precoding
- Detection with controlled ISI precoding
  - ✓ Symbol-by-symbol detection with precoding
- Error performance analysis of controlled ISI precoding

HW #5
Due: 12/30/2021 (Thur)

COM 5120
Communications Theory

Chapter 9: Digital Communication through Band-Limited Channels

–Part II: Optimal Receiver for Channels with ISI

Prof. Jen-Ming Wu

jmwu@ee.nthu.edu.tw

Inst. of Communications Engineering

Dept. of Electrical Engineering

National Tsing Hua University

Fall, 2021

Outline

Part II -Optimal Receiver for Channels with ISI

- **Lecture 4: Optimal Receiver for Channels with ISI**
  - ✓ Maximum Likelihood Sequential Detection
  - ✓ Whitening filter design

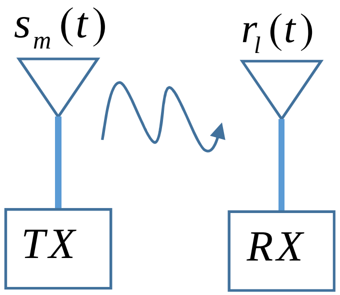
- **Lecture 5: Linear equalizer for low complexity Rx with ISI**
  - ✓ Zero forcing linear equalizer
  - ✓ MMSE linear equalizer

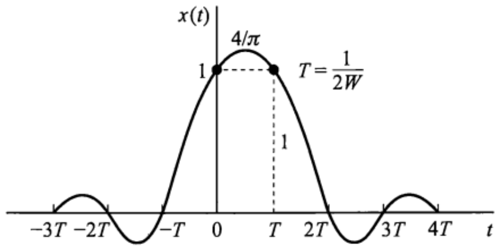
Optimal Receiver for Channels with ISI
(Proakis 9.3)

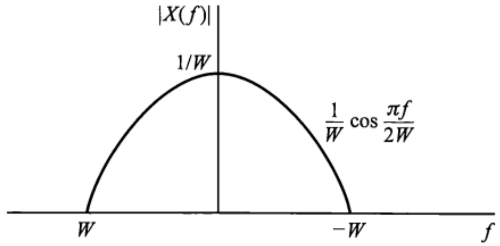
- Problem:

The channel state info (CSI) is usually not known *a priori* at Tx.

But, the controlled ISI approach requires CSI at Tx







- Approach: Instead of Tx precoding, design the optimal receiver.

  (1) Max Likelihood Sequence Detection (MLSD)

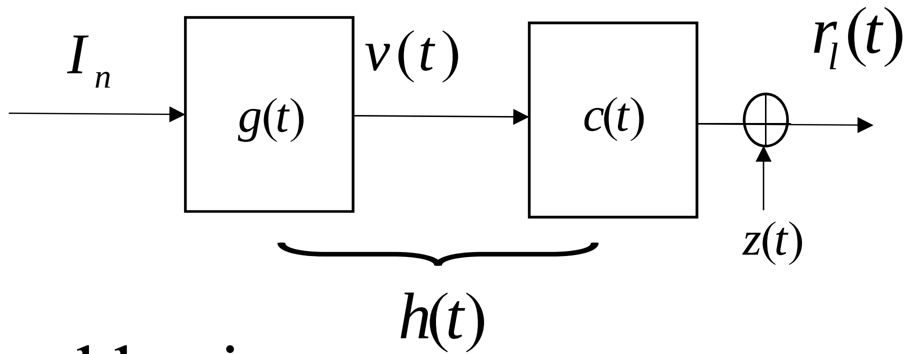
  (2) Linear equalizer (filter) with adjustable coefficients

Optimal Receiver for Channels with ISI- MLSD

The Tx signal $\quad v(t) = \sum_{n=1}^{\infty} I_n g(t - nT)$

The received signal $\quad r_l(t) = \sum_n I_n h(t - nT) + z(t)$

where $h(t) = c(t) * g(t)$



With Karhunen-Loeve Expansion,

$$r_l(t) = \sum_{k=1}^{N} r_k \phi_k(t)$$

where $\{\phi_k(t)\}$ is the orthonormal basis .

$$r_k = \int r_l(t)\phi_k(t)dt \quad \rightarrow \text{projection on } \phi_k(t)$$

$$= \sum_n I_n \int h(t - nT)\phi_k(t)dt + \int z(t)\phi_k(t)dt = \sum_n I_n h_{kn} + z_k, \quad k = 1, \ldots, N$$

$E[z_k^* z_m] = 2N_0 \delta_{km}$ (See Prob 2.55 of Proakis or slide in CH4)

Optimal Receiver for Channels with ISI- MLSD

Suppose $r_l(t)$ is the signal containing the information $\underline{I}_P = [I_1, I_2, \ldots I_p]^T$ and $r_l(t)$ is represented by $\underline{r}_N = [r_1, r_2, \ldots r_N]^T$

- The likelihood function

$$f(\underline{r}_N \mid \underline{I}_P) = \left(\frac{1}{2\pi N_0}\right)^N \exp\left\{\frac{-1}{N_0}\sum_{k=1}^{N}\left|r_k - \underbrace{\sum_n I_n h_{kn}}\right|^2\right\}$$

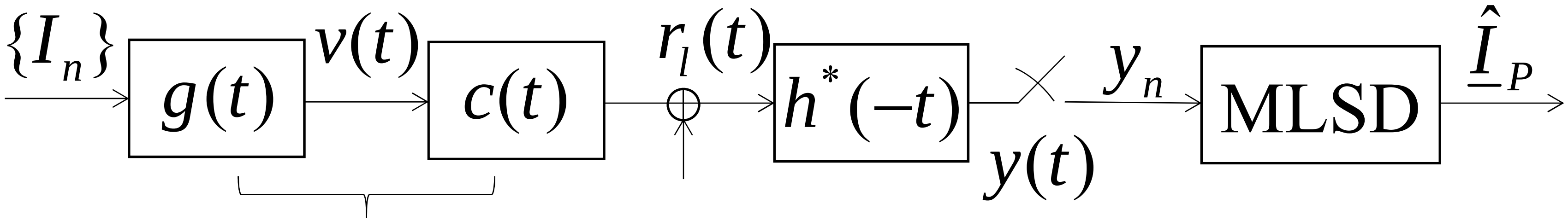
$$\boxed{B_k \overset{\Delta}{=} \sum_n I_n h_{kn}}$$

Path Metric: PM($\underline{I}_P$)= ln $\left\{ f(\underline{r}_N \mid \underline{I}_P)\right\}$

$$\equiv -\left\|\underline{r} - \underline{B}\right\|^2 = -\left|r_l(t) - \sum I_n h(t - nT)\right|^2$$

Optimal Receiver for Channels with ISI- MLSD

- MLSD Detector
$$\hat{\underline{I}}_P = \arg\max_{\underline{I}} PM(\underline{I}_P)$$
$$= \arg\max_{\underline{I}} \left\{ -\int_{-\infty}^{\infty} \left| r_l(t) - \sum_n I_n h(t-nT) \right|^2 dt \right\}$$
$$= \arg\max_{\underline{I}} \left\{ -\int_{-\infty}^{\infty} \left| p(t) - q(t) \right|^2 dt \right\}$$



$$h(t) = g(t) * c(t)$$

Optimal Receiver for Channels with ISI- MLSD

Note: Given $p(t), q(t) \in \mathbb{C}$, then

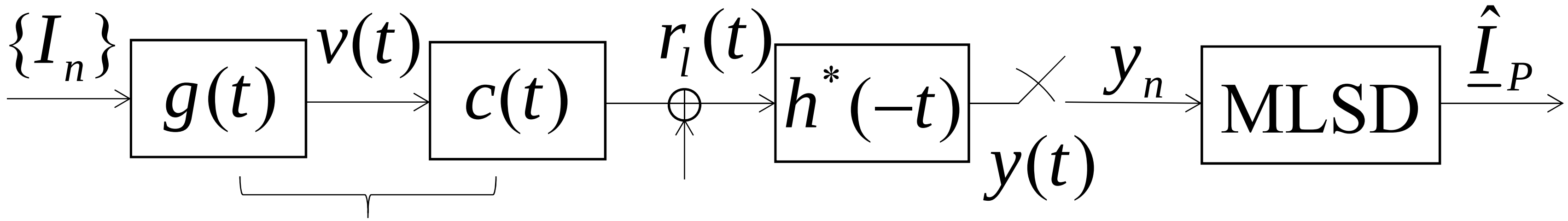
$$\left|p(t)-q(t)\right|^2 = (p(t)-q(t))(p(t)-q(t))^*$$

$$= \left|p(t)\right|^2 - (p(t)q^*(t) + p^*(t)q(t)) + \left|q(t)\right|^2$$

$$= \left|p(t)\right|^2 - 2\underset{2}{\mathrm{Re}}\{p(t)q^*(t)\} + \left|q(t)\right|^2$$

$$\underline{\hat{I}}_P = \arg\max_{\underline{I}} \{-\int_{-\infty}^{\infty} \left|r_l(t)\right| + 2\,\mathrm{Re}\{\sum_n [I_n^* \underbrace{\int_{-\infty}^{\infty} r_l(t)h^*(t-nT)dt}_{y_n}]\}$$

$$-\sum_n \sum_m I_n^* I_m \underbrace{\int_{-\infty}^{\infty} h^*(t-nT)h(t-mT)dt\}}_{x_{n-m}}$$



$$h(t) = g(t) * c(t)$$

Optimal Receiver for Channels with ISI- MLSD

- As $y(t) = r_l(t) * h^*(-t) = \int_{-\infty}^{\infty} r_l(\tau) h^*(\tau - t) d\tau$

$$\Rightarrow y_n \equiv y(nT) = \int_{-\infty}^{\infty} r_l(\tau) h^*(\tau - nT) d\tau$$

- As $x(t) \equiv g_T(t) * c(t) * g_R(t)$

$$= h(t) * h^*(-t) = \int_{-\infty}^{\infty} h^*(\tau) h(\tau + t) d\tau$$

$$\Rightarrow x_n \equiv x(nT) = \int_{-\infty}^{\infty} h^*(\tau) h(\tau + nT) d\tau$$

$$\Rightarrow x_{n-m} = \int_{-\infty}^{\infty} h^*(\tau - nT) h(\tau - mT) d\tau$$

Optimal Receiver for Channels with ISI- MLSD

- The ML Detector becomes

$$\hat{\underline{I}}_P = \arg\max_{\underline{I}_P} 2\,\mathrm{Re}\Big(\sum_n I_n^* \underbrace{y_n}_{\text{from Rx filter output}}\Big) - \sum_n \sum_m I_n^* I_m \underbrace{x_{n-m}}_{\text{from } h(t),\, h^*(-t)}$$

- Note that the Rx filter output is a linear combination of $\{I_n\}$

$$y_k = \int_{-\infty}^{\infty} r_l(t) h^*(t-kT)\,dt$$

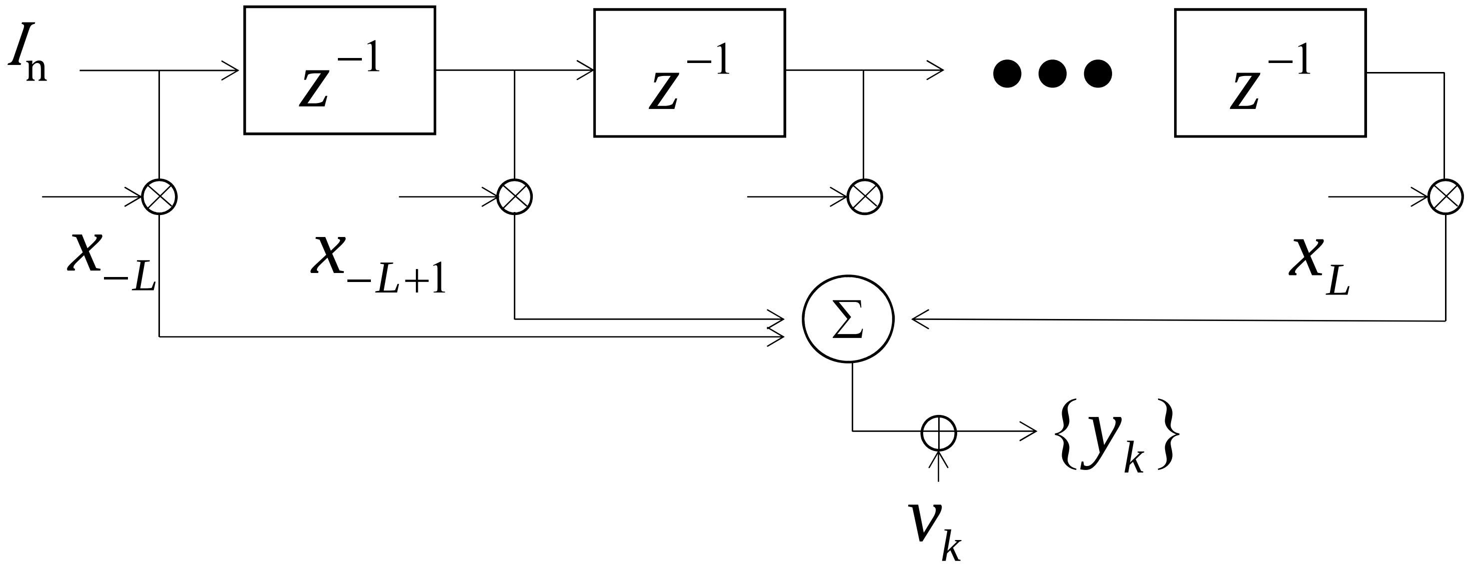
$$= \sum_n I_n \int h(t-nT) h^*(t-kT)\,dt + \int z(\tau) h^*(\tau - kT)\,d\tau$$

$$= \sum_n I_n x_{k-n} + v_k$$

Optimal Receiver for Channels with ISI- **MLSD**

- Suppose ISI affects a finite number of L symbols in $h(t)$,

$$y_k = \sum_{n=-L}^{L} x_n I_{k-n} + v_k$$



The MLSD can be realized by Viterbi algorithm with the discrete-time model.

Optimal Receiver for Channels with ISI- MLSD

The noise term

$$E[v_j v_k^*] = E[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overbrace{z(t)z^*(\tau)}^{2N_0\delta(t-\tau)} h(t-jT)h^*(\tau-kT)dtd\tau]$$

$$= 2N_0 \int_{-\infty}^{\infty} h(t-jT)h^*(t-kT)dt$$

$$= 2N_0 x_{k-j} = \begin{cases} 2N_0 x_{k-j} \ , \ |k-j| \le L \\ 0 \ , \ \text{other} \end{cases}$$

$\Rightarrow v_k$ is a non-white noise, since $x_{k\text{-}j}$ is non-flat!

$Q:$ So what is the problem here?

Optimal Receiver for Channels with ISI- MLSD

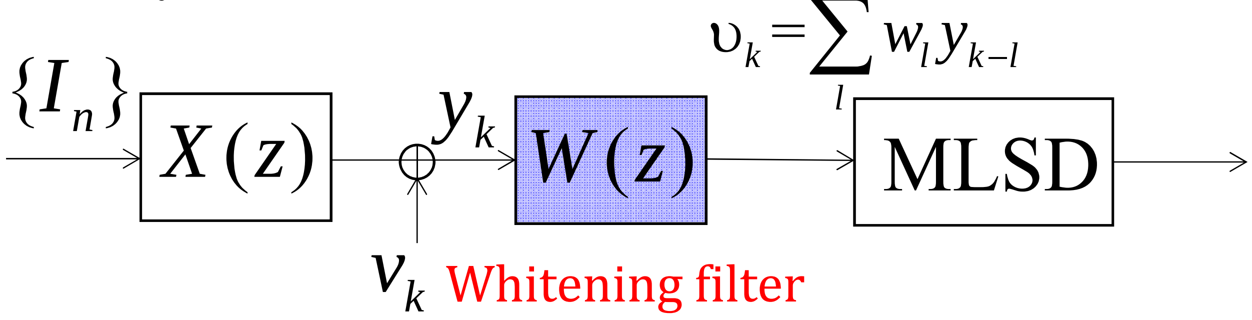
For non-white noise, we can not apply distance metric, i.e.

$\Rightarrow$ Applying distance metric $\left\| \underline{r}_N - \underline{B} \right\|^2$ in MLSD requires white noise condition, and is invalid with non-white noise !!

Optimal Receiver for Channels with ISI- MLSD

Given the z-transform $X(z) = \sum_{k=-L}^{L} x_k z^{-k}$, as $x_k = 0$ for $|k| > L$

$\Rightarrow S_{vv}(z) = 2N_0 X(z)$



➢ Use the Whitening Filter, W(z), after the receive filter.

Need to design W(z) s.t. the noise η_k can be white.

where $\eta_k = \sum_l w_l v_{k-l}$ (digital convolution)

The noise auto-correlation after W(z) is

$$\Rightarrow R_\eta[j] = E[\eta_{k+j}\eta_k^*] = \sum_l \sum_r w_l w_r^* E[v_{k+j-l} v_{k-r}^*] \quad (\star)$$

Optimal Receiver for Channels with ISI - MLSD

- Take z-transform on both sides of Eq. ☆

$$\text{LHS}: S_{\eta\eta}(z) = \sum_j R_\eta[j] z^{-j}$$

$$\text{RHS:} = \sum_j \left(\sum_l \sum_r w_l w_r^* R_{vv}[j-l+r] \right) z^{-j}$$

$$= \sum_l \sum_r w_l w_r^* \underbrace{\sum_j R_{vv}[j-l+r] z^{-j+l-r}}_{S_{vv}(z)} z^{-l+r}$$

$$= \sum_l w_l z^{-l} \sum_r w_r^* z^r \sum_j R_{vv}[j-l+r] z^{-j+l-r}$$

$$= W(z) W^*(\frac{1}{z^*}) S_{vv}(z)$$

- As $\text{LHS} = \text{RHS} \quad \Rightarrow S_{\eta\eta}(z) = W(z) W^*(\frac{1}{z^*}) S_{vv}(z) = 2N_0 W(z) W^*(\frac{1}{z^*}) X(z)$

- Can we design $W(z)$, s.t. $S_{\eta\eta}(z) = 2N_0$, and $\eta(t)$ becoems white?

Optimal Receiver for Channels with ISI- MLSD

Note that (from p.51 in slide) $x(t)=\int_{-\infty}^{\infty} h(\tau+t)h^{*}(\tau)d\tau$

$$x_k \equiv \int h(\tau+kT)h^{*}(\tau)d\tau = [\int h(\tau)h^{*}(\tau+kT)d\tau]^{*}$$

$$x_{-k} = \int_{-\infty}^{\infty} h(\tau-kT)h^{*}(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)h^{*}(\tau+kT)d\tau$$

$\Rightarrow x_k = x_{-k}^{*}$ (conjugate symmetric)

$$\Rightarrow X(z)=\sum_{k=-L}^{L} x_k z^{-k} = \sum_{k=-L}^{L} x_{-k}^{*} z^{-k} \overset{k'=-k}{=} \left[\sum_{k'=-L}^{L} x_{k'} (z^{k'})^{*}\right]^{*}$$

$$=\left[\sum x_{k'} (\frac{1}{z^{*}})^{-k'}\right]^{*} = X^{*}(\frac{1}{z^{*}})$$

Optimal Receiver for Channels with ISI- MLSD

If $X(\rho)=0$, then $X(\frac{1}{\rho^*})=X^*(\rho)=0$ $\therefore \frac{1}{\rho^*}$ is a root of $X(z)$.

$\Rightarrow$ If ρ is a root of X(z) then $\frac{1}{\rho^*}$ is also a root of X(z)

$\Rightarrow$ The 2L roots of X(z) are conjugate symmetric.

$\Rightarrow$ $X(z)$ can be factorized as $X(z)=F(z)F^*(\frac{1}{z^*})$.

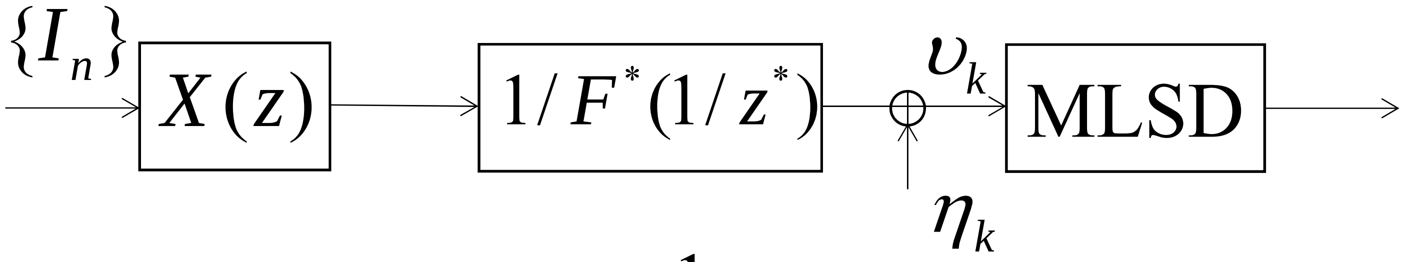
$F(z)$ is a polynomial of degree L with roots $\rho_1,..,\rho_L$

$F^*(1/z^*)$ is a polynomial of degree L with roots $1/\rho_1^*,..,1/\rho_L^*$

$\Rightarrow$ Choose the causal solution and $W(z)=\dfrac{1}{F^*(\frac{1}{z^*})}$

Optimal Receiver for Channels with ISI- MLSD

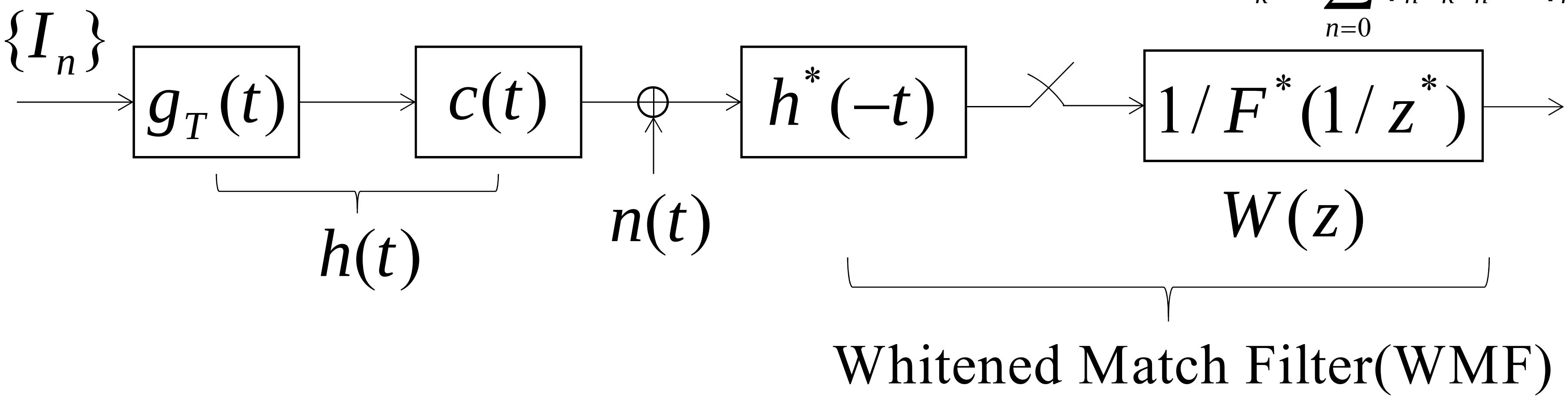
Then



$$\Rightarrow S_{\eta\eta}(z) = 2N_0 W(z) W^*\left(\frac{1}{z^*}\right) X(z)$$

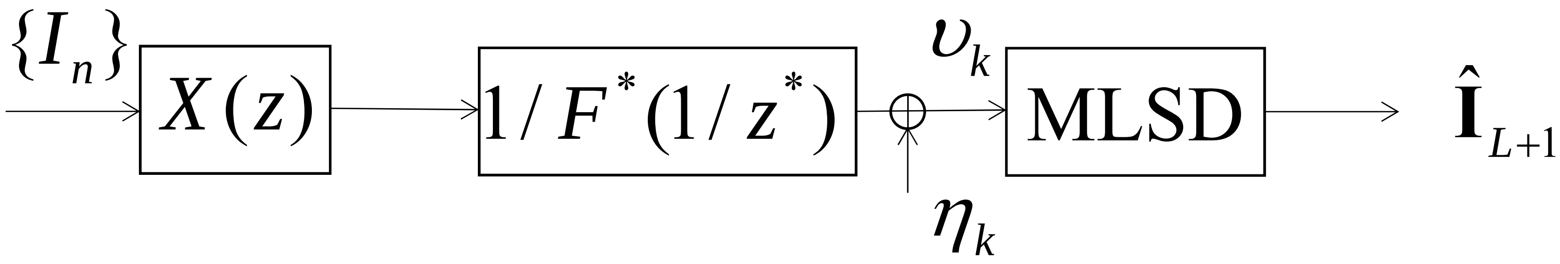
$$= 2N_0 \quad \Rightarrow \text{White !!}$$

$$\upsilon_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k$$

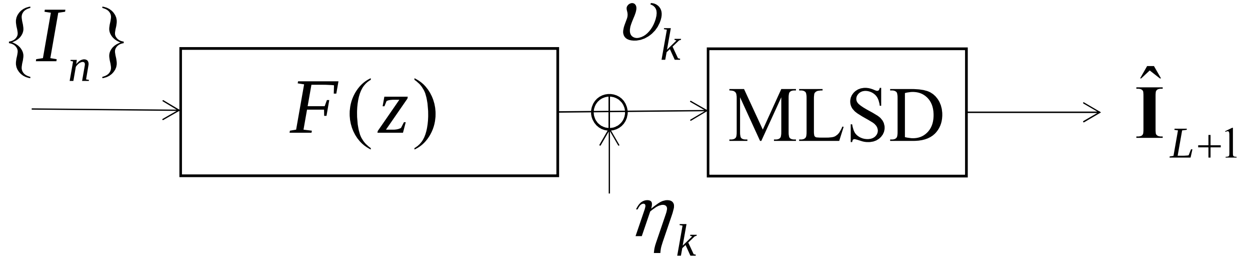


Optimal Receiver for Channels with ISI- MLS

$$X(z) = F(z)F^*(1/z^*)$$



$$\Rightarrow$$



$$\upsilon_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k = f_0 I_k + \sum_{n=1}^{L} f_n I_{k-n} + \eta_k$$

$$ML : \hat{\mathbf{I}}_{L+1} = \arg\min_{\mathbf{I}_{L+1}} \left| \upsilon_k - \sum_{n=0}^{L} f_n I_{k-n} \right|$$

Optimal Receiver for Channels with ISI- MLSD

Example:

Suppose $g(t)$ has duration T, and $\int_0^T g^2(t)dt = 1$

The channel $c(t) = \delta(t) + a\delta(t-T)$

The received channel $h(t) = g(t) + ag(t-T)$

$$\Rightarrow x_k = \int h^*(t)h(t+kT)dt$$

$$= \int [g^*(t) + a^* g^*(t-T)][g(t+kT) + ag(t+kT-T)]dt$$

$$= \begin{cases} a & k = 1 \\ 1 + |a|^2 & k = 0 \\ a^* & k = -1 \end{cases}$$

Optimal Receiver for Channels with ISI- MLSD

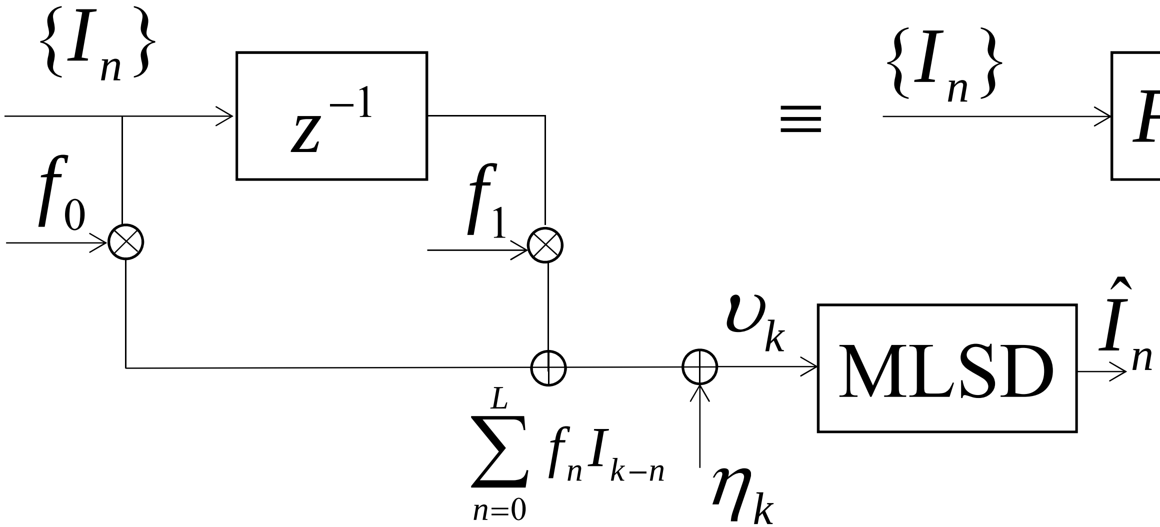
$$X(z)=\sum_{k=-1}^{1} x_k z^{-k} = a^* z + (1+|a|^2) + a z^{-1}$$

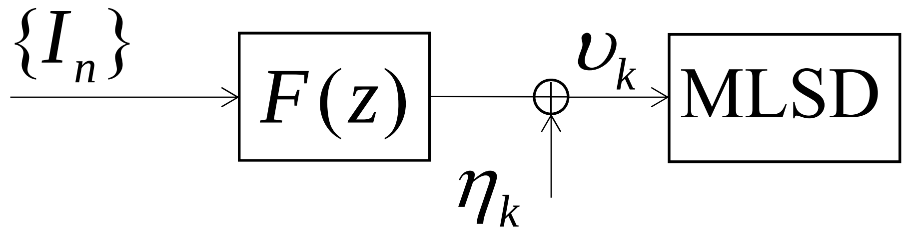
$$= (az^{-1}+1)(a^* z + 1)$$

Assume $|a|<1$ and choose the causal filter as $F(z)$

$$F(z) = f_1 z^{-1} + f_0 = az^{-1} + 1$$

$$\Rightarrow f_0 = 1,\ f_1 = a$$





MLSD :

$$\hat{\mathbf{I}}_{L+1} = \arg\min_{\mathbf{I}_{L+1}} \left| \upsilon_k - \sum_{n=0}^{L} f_n I_{k-n} \right|$$

Outline

Part II -Optimal Receiver for Channels with ISI

- **Lecture 4: Optimal Receiver for Channels with ISI**
  - ✓ Maximum Likelihood Sequential Detection
  - ✓ Whitening filter design

- **Lecture 5: Linear equalizer for low complexity Rx with ISI**
  - ✓ Zero forcing linear equalizer
  - ✓ MMSE linear equalizer

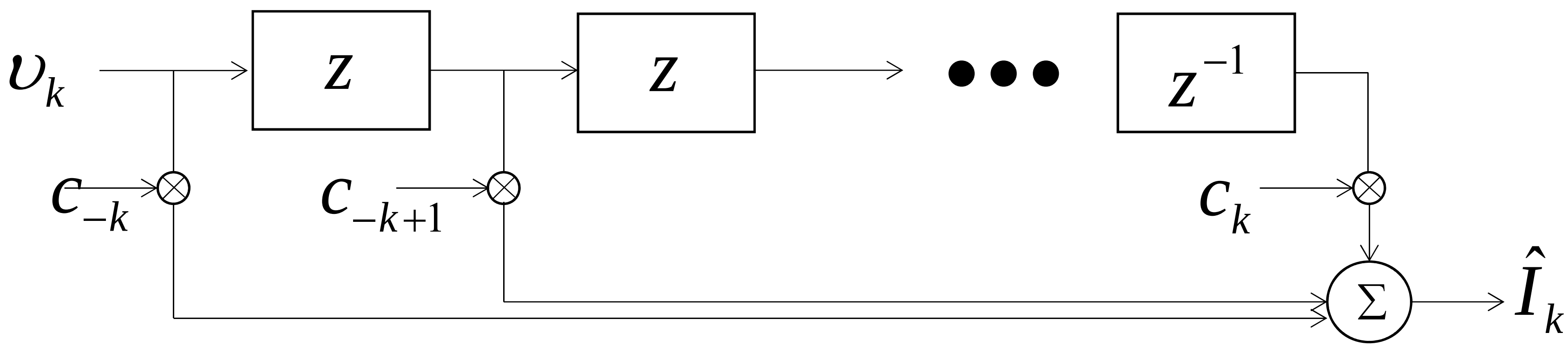
Optimal Receiver for Channels w/ ISI - Linear Equalizer

- Motivation: The MLSD has complexity M^{L+1} for each received symbol.

$$ML : \hat{\mathbf{I}}_{L+1} = \arg\min_{\mathbf{I}_{L+1}} \left| \upsilon_k - \sum_{n=0}^{L} f_n I_{k-n} \right|$$

$\rightarrow$ Need linear complexity with suboptimal solution.

- From the output of W(z)



✓ Can we obtain $\hat{I}_k$ from the linear combination of received symbol?

Optimal Receiver for Channels with ISI - Linear Equalizer

$$\hat{I}_k = \sum_{j=-k}^{k} c_j \upsilon_{k-j}$$

How do we find ?

- Peak Distortion Criterion

  (i.e. Zero-Forcing)

- Mean Square Error (MSE) Criterion

  How do design filter with minimum MSE (MMSE)?

(1) Peak Distortion Criterion

$$\upsilon_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k$$

Suppose we have infinite-tap equalizer

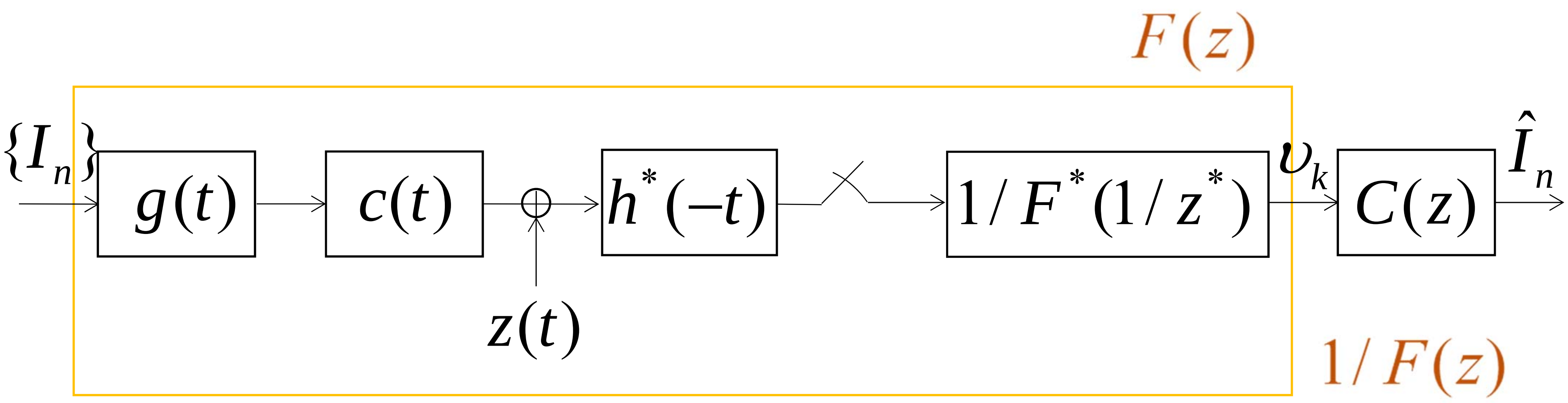
$$\Rightarrow \hat{I}_k = \sum_{j=-\infty}^{\infty} c_j \upsilon_{k-j}$$

$$= \sum_j c_j \left(\sum_n f_n I_{k-j-n}\right) + \sum_j c_j \eta_{k-j}$$

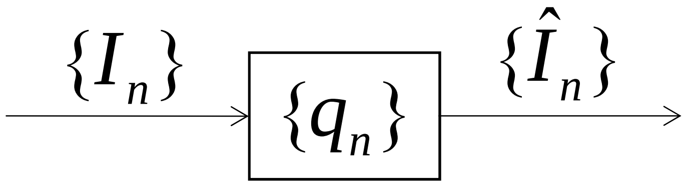
$$\overset{n' = j+n}{=} \sum_{n'} \left(\sum_j c_j f_{n'-j}\right) I_{k-n'} + \sum_j c_j \eta_{k-j}$$

$$\overset{(\sum_j c_j f_{n'-j}) = q_{n'}}{=} \sum_{n'} q_{n'} I_{k-n'} + \sum_j c_j \eta_{k-j} \qquad \text{where} \quad q_n = \sum_j c_j f_{n-j}$$

(1) Peak Distortion Criterion



$\equiv$



$$Q(z) = F(z)C(z)$$

$$\Rightarrow \hat{I}_k = q_0 I_k + \underbrace{\sum_{n\neq 0} I_n q_{k-n}}_{\text{ISI}} + \sum_j c_j \eta_{n-j}$$

(1) Peak Distortion Criterion

Normalize $q_0 = 1$

Assume $\{I_n\} \in \{\pm 1\}$

Peak Distortion Criterion

$$D(\underline{c}) = \sum_{n \neq 0} |q_n| = \sum_{n \neq 0} \left| \sum_j c_j f_{n-j} \right|$$

We want to choose $\{c_k\}$ s.t. $D(\underline{c}) = 0$

i.e. $q_n = \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

(1) Peak Distortion Criterion

$$\Rightarrow Q(z) = 1 = C(z)F(z)$$

$$\Rightarrow C(z) = \frac{1}{F(z)} : \text{zero} - \text{forcing} \quad \text{filter}$$

$$\Rightarrow c_k = \frac{1}{2\pi j} \oint C(z) z^{k-1} dz \quad (\text{Inversve} \quad \text{z} - \text{transform})$$

$$\Rightarrow \hat{I}_k = I_k + \underbrace{\sum_j c_j \eta_{k-j}}_{n_k}$$

The power spectral density of the noise after the ZF equalizer is

$$S_{nn}(z) = C(z)C^*(1/z^*)N_0 = \frac{N_0}{X(z)} \qquad \text{where } S_{\eta\eta}(z) = N_0$$

$\Rightarrow$ Non – white noise!

(1) Peak Distortion Criterion

The PSD of noise can be obtained with $z = e^{jwT}$

$$S_{nn}(e^{jwT}) = \frac{N_0}{X(e^{jwT})}$$

The noise power

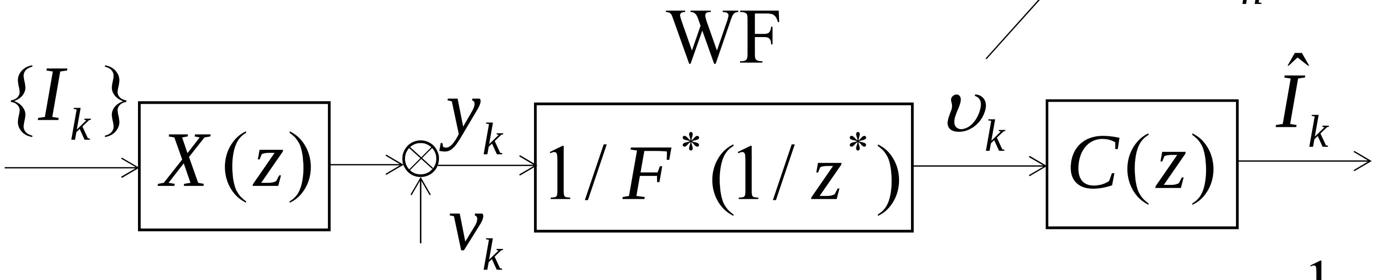
$$\sigma_n{}^2 = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} S_{nn}(e^{jwT}) dw = \frac{TN_0}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{1}{X(e^{jwT})} dw$$

$$If \int \frac{1}{X(e^{jwT})} dw > \frac{2\pi}{T} \quad \text{then } \sigma_n{}^2 > S_{\eta\eta}(z) = N_0$$

=> Noise enhancement occurs!

- Summary of ZF filter

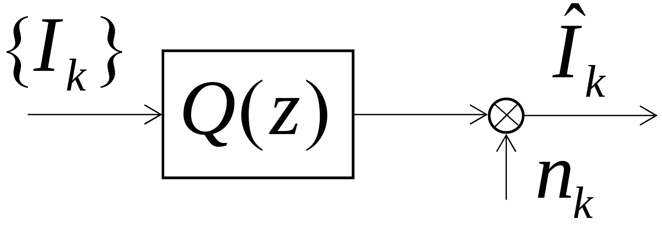
$$\upsilon_k = \sum_n f_n I_{k-n} + \eta_k$$



$$\text{ZF}: C(z) = \frac{1}{F(z)}$$

$$X(z) = F(z)F^*(1/z^*)$$

$\equiv$



$$SNR = \frac{q_0^2}{\sigma_n^2} = \frac{1}{\sigma_n^2}$$

$$= \left[\frac{TN_0}{\pi} \int_{-\pi/T}^{\pi/T} \frac{1}{X(e^{jwT})} dw \right]^{-1}$$

$$Q(z) = 1 = C(z)F(z)$$

$$i.\text{e. } q_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

If $X(e^{jwT})$ posses any zeros,

the σ_n^2 will be infinite.

$\Rightarrow \text{SNR} \to 0$

● Finite Length ZF-Equalizer

For finite length filter, suppose $c_j = 0$ for $|j| > K$.

$$q_n = \sum_{j=-K}^{K} c_j f_{n-j}$$

Then we have a set of linear equations

$$\underline{q} = \begin{bmatrix} & & \\ & f_{n-j} & \\ & & \end{bmatrix} \begin{bmatrix} c_{-K} \\ \vdots \\ c_0 \\ \vdots \\ c_K \end{bmatrix} = \begin{bmatrix} q_{-K} \\ \vdots \\ q_0 \\ \vdots \\ q_K \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

By solving the linear equations, we can find $\underline{c} = F^{-1}\underline{q}$.

$\Rightarrow$ The residual distortion is $D(\underline{c}) = \sum_{|n|>K} |q_n|$

- Finite Length Causal ZF-Equalizer

For *causal* linear filter, i.e. $c_j = 0 \quad j < 0$,

the c_j only have values at $c_0, c_1, \ldots c_K$.

Ex. Three taps causal ZF-filter (K=2).

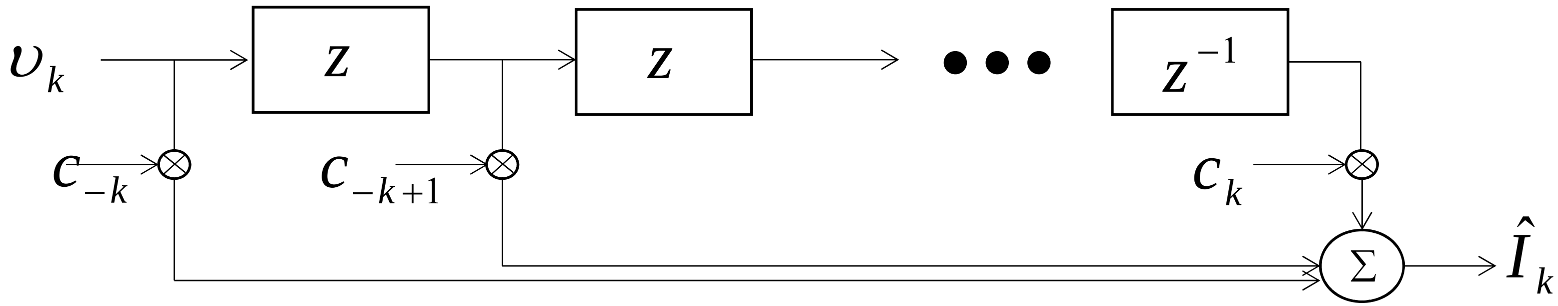
The linear equalizer becomes

$$\begin{bmatrix} f_0 & f_{-1} & f_{-2} \\ f_1 & f_0 & f_{-1} \\ f_2 & f_1 & f_0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{c} = F^{-1}\underline{q}$$

$\Rightarrow$ The residual distortion is $D(\underline{c}) = \sum_{n \neq 0} |q_n|$

Mean-Square Error Criterion

- Given the equalizer input υ_k, the output can be expressed as the linear combination of υ_k



- Let $\varepsilon_k = I_k - \hat{I}_k$ be the MMSE estimation rror

$\Rightarrow$ minimize the MSE $J(\underline{c}) = E[|\varepsilon_k|^2] = E[|I_k - \hat{I}_k|^2]$

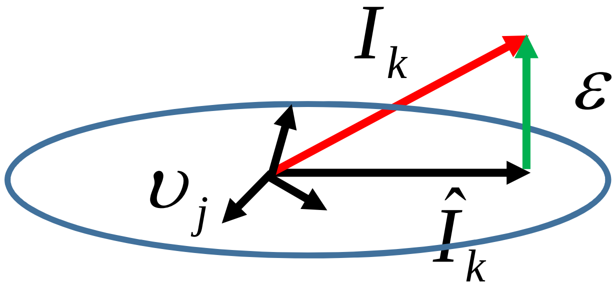
Suppose we have infinite taps. i.e.

$$\hat{I}_k = \sum_{j=-\infty}^{\infty} c_j \upsilon_{k-j} \quad \Rightarrow \{c_k\} = \arg\min_{\{c_k\}} E[\left| I_k - \sum_j c_j \upsilon_{k-j} \right|^2]$$

Mean-Square Error Criterion

- Based on orthogonality principle, the min error occurs when the error random variable is orthogonal to the equalizer output.

$$\because \hat{I}_k \text{ is linear combination on } \{\upsilon_k\}, \quad \hat{I}_k = \sum_{l=-\infty}^{\infty} c_j \upsilon_{k-l}$$



$$\varepsilon_k = I_k - \hat{I}_k$$

$$\Rightarrow \varepsilon_k \text{ is minimized when } E[\varepsilon_k \hat{I}_k^*] = 0$$

$$\Rightarrow E[\varepsilon_k \upsilon_{k-l}^*] = 0, \quad \forall l$$

$$\Rightarrow E[(I_k - \sum_j c_j \upsilon_{k-j})\upsilon^*_{k-l}] = 0, \forall l$$

$$\Rightarrow \sum_j \underbrace{c_j E[v_{k-j} \upsilon_{k-l}^*]}_{\mathrm{R}_{\upsilon\upsilon}[l-j]} = E[I_k \upsilon_{k-l}^*] \cdots (\star)$$

Mean-Square Error Criterion

υ_k is the output of WF:

$$\upsilon_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k$$

LHS: $R_{\upsilon\upsilon}[l-j]$

$$= E[\upsilon_{k-j}\upsilon^*_{k-l}]$$

$E[I_k \eta_j] = 0, \forall k, j$

$$= E[(\sum_n f_n I_{k-j-n} + \eta_{k-j})(\sum_m f_m^* I^*_{k-l-m} + \eta^*_{k-l})]$$

Assume

$$E[I_k I_j] = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

$$= \sum_n \sum_m f_n f^*_m \underbrace{\boxed{E[I_{k-j-n} I^*_{k-l-m}]}}_{\delta_{l+m-j-n}} + \underbrace{\boxed{E[\eta_{k-j}\eta^*_{k-l}]}}_{N_0 \delta_{l-j}}$$

$$= \sum_m f_m^* f_{m+l-j} + N_0 \delta_{l-j}$$

$$= \begin{cases} x_{l-j} + N_0 \delta_{l-j}, & |l-j| \leq L \\ 0, & \text{other} \end{cases} \quad \text{— ①}$$

Mean-Square Error Criterion

υ_k is the output of WF:

$$\upsilon_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k$$

RHS $\quad E[I_k \upsilon^*_{k-l}]$

$$= \sum_n f_n^{\ *} E[I_k I^*_{\ k-l-n}] + E[I_k \eta^*_{k-l}]$$

$$= \sum_n f_n^{\ *} \delta_{l+n}$$

$$= f_{-l}^{\ *}, \qquad l = 0, 1, \ldots, L \qquad \text{— } ②$$

From ①=②, $\sum_j c_j R_{\upsilon\upsilon}[l-j] = f_{-l}^{\ *}$ (MMSE equation)

Mean-Square Error (MSE) Criterion

Note that $Z\{f_n\} = F(z) = \sum_n f_n z^{-n}$

$$Z\{f_{-n}\} = \sum_n f_{-n} z^{-n} = \sum_{n'} f_{n'} z^{n'} = \sum_{n'} f_{n'} (\frac{1}{z})^{-n'} = F(\frac{1}{z})$$

$$Z\{f^*_{-n}\} = \sum_{n'} f^*_{n'} (\frac{1}{z})^{-n'} = (\sum_{n'} f_{n'} (\frac{1}{z^*})^{-n'})^* = F^*(\frac{1}{z^*})$$

- Take z-transform on the MMSE equation:

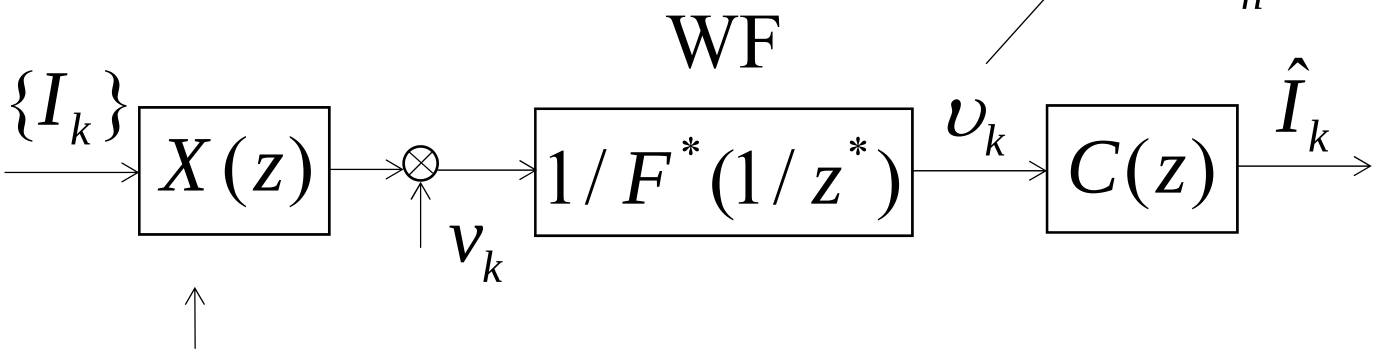
$$C(z)S_{vv}(z) = F^*(1/z^*)$$

$$\Rightarrow C(z)[F(z)F^*(1/z^*) + N_0] = F^*(1/z^*)$$

$$\Rightarrow C(z) = \boxed{\frac{F^*(1/z^*)}{F(z)F^*(1/z^*) + N_0}}$$ MMSE Equalizer

MMSE Equalizer

$$\upsilon_k = \sum_n f_n I_{k-n} + \eta_k$$



$$X(z) = F(z)F^*(1/z^*)$$

ZF: $C(z) = 1/F(z)$

$$\text{MMSE}: C(z)S_{\upsilon\upsilon}(z) = F^*(1/z^*) \text{ i.e. } \sum_{j=-\infty}^{\infty} c_j R_{\upsilon\upsilon}[l-j] = f_{-l}^{\ *}$$

$$\text{C(z)} = \frac{F^*(1/z^*)}{F(z)F^*(1/z^*) + N_0}$$

MMSE Equalizer

- Remarks: $C(z) = \dfrac{F^*(1/z^*)}{F(z)F^*(1/z^*) + N_0}$

When $N_0 \to 0$ (i.e. SNR→∞)
MMSE $C(z) \cong 1/F(z) \equiv \mathrm{ZF}$

When $N_0 > 0$, there is residual ISI, i.e. $q_n \neq \delta_n$

MMSE Equalizer

Let the residual distortion be

$$J_{\min} = E[|\varepsilon_k|^2] = E[\varepsilon_k \varepsilon_k^*] \qquad \text{where} \quad \varepsilon_k = I_k - \hat{I}_k$$

$$= E[\varepsilon_k (I_k^* - \hat{I}_k^*)]$$

$$= E[\varepsilon_k I_k^*] \quad \text{(Why?)}$$

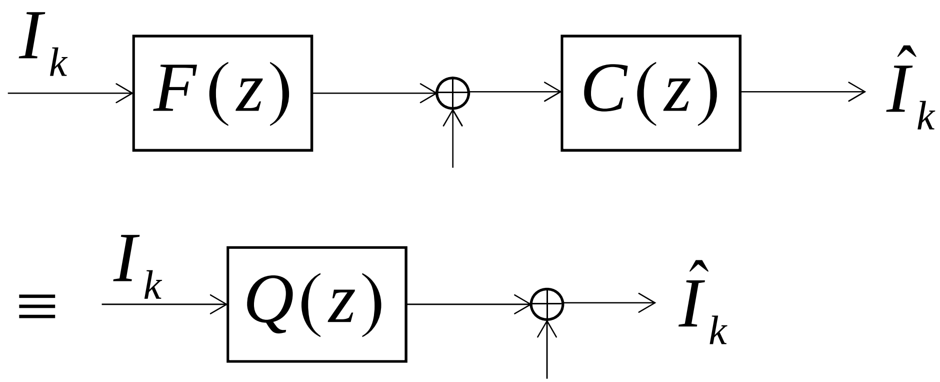
$(\varepsilon_k = I_k - \sum_j c_j \upsilon_{k-j})$

$$= E[|I_k|^2] - \sum_j c_j E[\upsilon_{k-j} I_k^*]$$

From ②, $E[I_k \upsilon_{k-l}^*] = f_{-l}^*$

$$\Rightarrow J_{\min} = 1 - \sum_{j=-\infty}^{\infty} c_j f_{-j}$$

$$=> J_{\min} = 1 - q_0 \quad \text{(Why?)}$$



where $q_n = \sum_j c_j f_{n-j}$ digital convolution

MMSE Equalizer

$$Q(z) = C(z)F(z)$$

$$= \frac{F^*(1/z^*)}{F(z)F^*(1/z^*) + N_0} F(z) = \frac{X(z)}{X(z) + N_0}$$

From inverse z-transform: $q_n = \frac{1}{2\pi j} \oint Q(z) z^{n-1} dz$

$$\Rightarrow q_0 = \frac{1}{2\pi j} \oint \frac{Q(z)}{z} dz \qquad z = e^{jwT}$$

$$= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{Q(e^{jwT})}{e^{jwT}} e^{jwT} dw \qquad dz = jTe^{jwT} dw$$

$$= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{X(e^{jwT})}{X(e^{jwT}) + N_0} dw$$

MMSE Equalizer

$$J_{\min} = 1 - q_0$$

$$= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \left(1 - \frac{X(e^{jwT})}{X(e^{jwT}) + N_0}\right) dw = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N_0}{X(e^{jwT}) + N_0} dw$$

- If no ISI, i.e. $X(e^{jwT}) = 1$ $\Rightarrow J_{\min} = \dfrac{N_0}{1 + N_0}$

- The SINR of MMSE equalizer with infinite taps

$$\hat{I}_k = q_0 I_k + \sum_{n \neq 0} q_n I_{k-n} + \sum_j c_j \eta_{k-j}$$

$$= q_0 (\underbrace{I_k}_{\text{Desired Signal}} + \underbrace{\frac{1}{q_0} \sum_{n \neq 0} q_n I_{k-n} + \frac{1}{q_0} \sum_j c_j \eta_{k-j}}_{\text{Interference + noise}})$$

MMSE Equalizer

- The unbiased error variance (interference + noise power)

$$\sigma_n^{\,2} = E\left[\left|\frac{1}{q_0}\hat{I}_k - I_k\right|^2\right]$$

$$= E\left[\left|\left(\frac{1}{q_0}\hat{I}_k - \hat{I}_k\right) + \underbrace{(\hat{I}_k - I_k)}_{\varepsilon_k}\right|^2\right]$$

$\hat{I}_k \perp \varepsilon_k \Rightarrow$ no cross term

$$= \frac{(1-q_0)^2}{q_0^{\,2}} E\left[\left|\hat{I}_k\right|^2\right] + \underbrace{E\left[\left|\hat{I}_k - I_k\right|^2\right]}_{J_{\min}}$$

MMSE Equalizer

$$\Rightarrow \sigma_n^2 = \frac{(1-q_0)^2}{q_0^{\ 2}} E\left[\left| q_0 I_k + \underbrace{\left(\sum_{n\neq 0} q_n I_{k-n} + \sum_j c_j \eta_{k-j}\right)}_{\sigma_n^2 = E\left[\frac{1}{q_0^2}\left(\sum_{n\neq 0} q_n I_{k-n} + \sum_j c_j \eta_{k-j}\right)^2\right]} \right|^2\right] + J_{\min}$$

$$= \frac{(1-q_0)^2}{q_0^{\ 2}} q_0^{\ 2}\left(E\left[\left|I_k\right|^2\right] + \sigma_n^{\ 2}\right) + J_{\min}$$

$J_{\min} = 1 - q_0$

$$E[I_k I_{k-n}] = 0,$$

$$E[I_k \eta_{k-j}] = 0$$

$$= J_{\min}^{\ \ 2}(1 + \sigma_n^{\ 2}) + J_{\min}$$

$$\Rightarrow \sigma_n^{\ 2} = \frac{J_{\min}}{1 - J_{\min}}$$

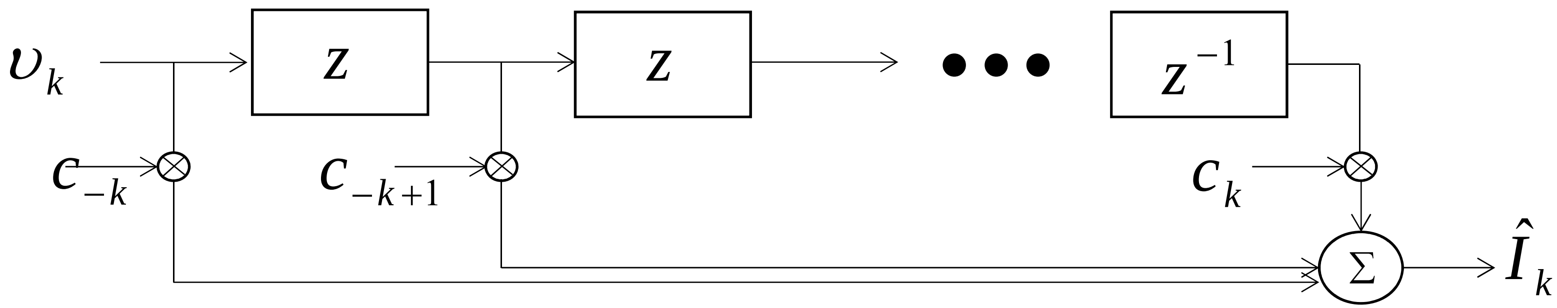
$\Rightarrow$ SINR with infinite taps: $r_\infty = \dfrac{E[\left|I_k\right|^2]}{\sigma_n^{\ 2}} = \dfrac{1}{\sigma_n^{\ 2}} = \dfrac{1 - J_{\min}}{J_{\min}}$

$J_{\min} \downarrow, \gamma_\infty \uparrow$

- If no ISI,i.e. $X(e^{jwT}) = 1$, *then* $J_{\min} = \dfrac{N_0}{1+N_0}. \Rightarrow r_\infty = \dfrac{1-J_{\min}}{J_{\min}} = \dfrac{1}{N_0}$

$\Rightarrow$ The best $J_{\min}$ and γ that MMSE could achieve.

Finite-Length MMSE Equalizer



- Assume finite length of 2K+1 taps equalizer, $c_j = 0 \quad for \ \left|j\right| > K$

$$\Rightarrow \hat{I}_k = \sum_{j=-K}^{K} c_j v_{k-j} \quad (2K+1 \text{ taps})$$

where υ_k is the output of WF, and $\upsilon_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k$

L represents the length of ISI at equalizer input.

- Note that the number of taps has to be greater than the length of ISI, i.e. $K \geq L$

$$\Rightarrow J(\underline{c}) = E[\left|I_k - \hat{I}_k\right|^2] = E[\left|I_k - \sum_{j=-K}^{K} c_j \upsilon_{k-j}\right|^2]$$

Finite-Length MMSE Equalizer

From the orthogonality principle, the coefficient of

$$\text{MMSE} \quad \sum_{j=-K}^{K} c_j R_{\upsilon\upsilon}[l-j] = f_{-l}^{\;*} = \xi_l, \qquad l = -K,\ldots,0,\ldots,K$$

From ① in p.80 $\quad R_{\upsilon\upsilon}[l-j] = \begin{cases} x_{l-j} + N_0 \delta_{l-j}, & |l-j| \le L \\ 0, & \text{other} \end{cases}$

$$\xi_l \triangleq \begin{cases} f_{-l}^{*}, & l = -L,\ldots,0 \\ 0, & \text{other} \end{cases}$$

$$i.e. \begin{bmatrix} \\ \\ \left[R_{\upsilon\upsilon}[l-j]\right] \\ \\ \\ \end{bmatrix} \begin{bmatrix} c_K \\ \vdots \\ \vdots \\ c_{-K} \end{bmatrix} = \begin{bmatrix} \xi_K \\ \vdots \\ \xi_0^{\;*} \\ \vdots \\ \xi_{-K} \end{bmatrix} \Rightarrow \mathbf{R}_{vv}\underline{\mathbf{c}} = \boldsymbol{\xi}$$

where $\mathbf{R}_{\upsilon\upsilon}$ is $(2K+1)\times(2K+1)$

Finite-Length MMSE Equalizer

$$\begin{bmatrix} x_0+N_0 & x_1 & \cdots & x_{2K-1} & x_{2K} \\ x_{-1} & x_0+N_0 & \ddots & \ddots & x_{2K-1} \\ \vdots & x_{-1} & x_{l-j}+N_0\delta_{l-j} & \ddots & \vdots \\ x_{-2K+1} & \ddots & \ddots & x_0+N_0 & x_1 \\ x_{-2K} & x_{-2K+1} & \cdots & x_{-1} & x_0+N_0 \end{bmatrix} \begin{bmatrix} c_K \\ \vdots \\ \vdots \\ c_{-K} \end{bmatrix} = \begin{bmatrix} f_{-K}{}^* \\ \vdots \\ \vdots \\ f_K{}^* \end{bmatrix}$$

$$\Rightarrow \underline{\mathbf{c}} = \mathbf{R}_{vv}^{-1}\boldsymbol{\xi}$$

For example, the 3-tap MMSE equalizer could be obtained by

$$\begin{bmatrix} x_0+N_0 & x_1 & x_2 \\ x_{-1} & x_0+N_0 & x_1 \\ x_{-2} & x_{-1} & x_0+N_0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \\ c_{-1} \end{bmatrix} = \begin{bmatrix} f_{-1}^* \\ f_0^* \\ f_1^* \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} c_1 \\ c_0 \\ c_{-1} \end{bmatrix} = \mathbf{R}_{vv}^{-1}\boldsymbol{\xi}$$

$$J_{\min} = 1 - q_0 = 1 - \sum_{j=-K}^{K} c_j f_{-j} \qquad \Rightarrow \text{SINR: } r_K = \frac{1-J_{\min}}{J_{\min}}$$

Finite-Length MMSE Equalizer

Example: Consider an equivalent discrete-time channel model F(z) with two delay components f_0 and f_1 , i.e.

$F(z) = f_0 + f_1 z^{-1}$ The f_0 and f_1 are normalized to $|f_0|^2 + |f_1|^2 = 1$

$$\Rightarrow \upsilon_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k = f_0 I_k + f_1 I_{k-1} + \eta_k$$

$$F(1/z) = f_0 + f_1 z \rightarrow F^*(1/z^*) = f_0^* + f_1^* z$$

$$\Rightarrow X(z) = F(z)F^*(1/z^*) = 1 + f_0 f_1^* z + f_0^* f_1 z^{-1} = x_0 + x_1 z + x_{-1} z^{-1}$$

For example, the 3-tap MMSE equalizer could be obtained by

$$\begin{bmatrix} x_0 + N_0 & x_1 & x_2 \\ x_{-1} & x_0 + N_0 & x_1 \\ x_{-2} & x_{-1} & x_0 + N_0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_0 \\ c_{-1} \end{bmatrix} = \begin{bmatrix} f_{-1}^* \\ f_0^* \\ f_1^* \end{bmatrix} \quad \Rightarrow \begin{bmatrix} c_1 \\ c_0 \\ c_{-1} \end{bmatrix} = \mathbf{R}_{vv}^{-1} \boldsymbol{\xi}$$

Finite-Length MMSE Equalizer

$$\text{Let} \quad z = e^{jwT}$$

$$\Rightarrow X(e^{j\omega T}) = f_0 f_1^* e^{j\omega T} + 1 + f_0^* f_1 e^{-j\omega T}$$

$$= 1 + 2\left|f_0\right|\left|f_1\right|\cos\left(\omega T + \theta\right) \quad \text{where} \ f_0 f_1^* = \left|f_0\right|\left|f_1\right| e^{j\theta}$$

- If no ISI, i.e. $f_0 = 1,\ f_1 = 0 \quad \rightarrow X(e^{jwT}) = 1$

- When $f_0 = f_1 = \frac{1}{\sqrt{2}}, (i.e.\ \theta = 0)$, and $\omega = \frac{\pi}{\mathrm{T}}$.

$$\Rightarrow \cos(\varpi T + \theta) = -1, \quad \text{then} \ X(e^{j\omega T}) = 0$$

Finite-Length MMSE Equalizer

Example – cont'd .

From the residual distortion of MMSE equalizer (p.84),

$$J_{\min} = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\frac{N_0}{X(e^{j\omega T})+N_0}d\omega = \frac{N_0 T}{2\pi}\int_{-\pi/T}^{\pi/T}\frac{1}{(N_0+1)+2|f_0||f_1|\cos(\omega T+\theta)}d\omega$$

$$= \frac{N_0}{2\pi}\frac{4}{\sqrt{(N_0+1)^2-4|f_0|^2|f_1|^2}}\tan^{-1}\left(\frac{\sqrt{(N_0+1)^2-4|f_0|^2|f_1|^2}\tan\left(\frac{u}{2}\right)}{(N_0+1)+2|f_0||f_1|}\right)\Bigg|_0^{\pi}$$

$Note:$

$$\int_{-\pi}^{\pi}\frac{1}{a+b\cos u}du = 2\int_0^{\pi}\frac{1}{a+b\cos u}du = \frac{4}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{\sqrt{a^2-b^2}\tan\left(\frac{u}{2}\right)}{a+b}\right)\Bigg|_0^{\pi}$$

$$= \frac{4}{\sqrt{a^2-b^2}}\left(\tan^{-1}(\infty)-\tan^{-1}(0)\right) = \frac{4}{\sqrt{a^2-b^2}}\left(\frac{\pi}{2}-0\right)$$

Finite-Length MMSE Equalizer

Example – cont'd

$$\Rightarrow J_{\min} = \frac{2N_0/\pi}{\sqrt{(N_0+1)^2 - 4|f_0|^2|f_1|^2}}\left(\frac{\pi}{2} - 0\right) = \frac{N_0}{\sqrt{(N_0+1)^2 - 4|f_0|^2|f_1|^2}}$$

- If no ISI, i.e. $X(e^{jwT}) = 1$ $\quad \Rightarrow J_{\min} = \dfrac{N_0}{1+N_0}$
- In case of $X(e^{jwT}) = 0$ (i.e. $f_0 = f_1 = \dfrac{1}{\sqrt{2}}$), $\quad \Rightarrow J_{\min} = \dfrac{N_0}{\sqrt{N_0^2 + 2N_0}}$

$$\Rightarrow \text{SINR } \gamma = \frac{1\text{-}J_{\min}}{J_{\min}} = \sqrt{1+\frac{2}{N_0}} - 1 \quad \Rightarrow \quad \gamma \approx \sqrt{\frac{2}{N_0}} \text{ for } N_0 << 1$$

Note: Compare with that of no ISI $\gamma = \dfrac{1}{N_0}$.(the best SINR that infinite taps MMSE)

The loss in SINR is due to ISI in this finite length example.

Summary of Chap 9 Part II

- MLSD for optimal Rx with ISI
  - Whitening filter design
- Linear equalizer for low complexity Rx with ISI
  - Zero forcing linear equalizer
    - ✓ Infinite taps ZF equalizer
    - ✓ Finite taps ZF equalizer

$$\text{ZF: } C(z) = \frac{1}{F(z)}$$

$$q_n = \sum_{j=-K}^{K} c_j f_{n-j}, n = -K, .., K$$

  - MMSE linear equalizer
    - ✓ Infinite taps MMSE equalizer
    - ✓ Finite taps MMSE equalizer

$$\text{C(z)} = \frac{F^*(1/z^*)}{F(z)F^*(1/z^*) + N_0}$$

$$\sum_{j=-K}^{K} c_j R_{\upsilon\upsilon}[l-j] = f_{-l}^{\ *}, \qquad l = -K, ..., 0, ..., K$$

- ☐ Further Study. Various more sophisticated equalizers (e.g. Decision Feedback Equalizer in Ch 9.5) can be applied.