COM 5120 Communication Theory

Midterm II 2021

Reference Solution

1.(a)

$$\mathrm{A: d_{min}^2 = d^2 = E_0, E_{avg} = \frac{1}{6}(d^2 \times 2 + ((d/2)^2 + d^2) \times 4) = \frac{7}{6}d^2 = \frac{7}{6}E_0}$$

$$\mathrm{CFM_A = \frac{d_{min}^2}{E_{avg}} = \frac{E_0}{\frac{7}{6}E_0} = \frac{6}{7}}$$

$$\mathrm{B: d_{min}^2 = d^2 = E_0, E_{avg} = \frac{1}{6}(d^2 \times 6) = d^2 = E_0}$$

$$\mathrm{CFM_B = \frac{d_{min}^2}{E_{avg}} = \frac{E_0}{E_0} = 1}$$

1.(b)

題目沒有定義清楚，因此有寫出比較都算對，沒寫出比較扣一分

解法一:

$$\mathrm{d_{min,A} = d_{min,B} = d}$$

$$\mathrm{P_{e,A} = P_{e,B} = Q(\frac{d}{\sqrt{2N_0}})}$$

解法二:

$$\mathrm{Given \quad E_{avg}, d_{min,A}^2 = \frac{6}{7}E_{avg}, d_{min,B}^2 = E_{avg}}$$

$$\mathrm{P_{e,A} = Q(\sqrt{\frac{\frac{6}{7}E_{avg}}{2N_0}}), P_{e,B} = Q((\sqrt{\frac{E_{avg}}{2N_0}}) \rightarrow P_{e,A} > P_{e,B}}$$

解法三:

$$\mathrm{P_{e,A} = \frac{4}{6}[Q(\sqrt{\frac{d^2}{2N_0}}) + Q(\sqrt{\frac{\frac{5}{4}d^2}{2N_0}})] + \frac{2}{6}[Q(\sqrt{\frac{\frac{5}{4}d^2}{2N_0}}) \times 2] = \frac{1}{6}[4Q(\sqrt{\frac{d^2}{2N_0}}) + 4Q(\sqrt{\frac{\frac{5}{4}d^2}{2N_0}})]}$$

$$\mathrm{P_{e,B} = \frac{6}{6}[Q(\sqrt{\frac{d^2}{2N_0}}) \times 2] = \frac{1}{6}[12Q(\sqrt{\frac{d^2}{2N_0}})]}$$

$$\mathrm{\rightarrow P_{e,A} < P_{e,B}(\because 8Q(\sqrt{\frac{d^2}{2N_0}}) > 4Q(\sqrt{\frac{\frac{5}{4}d^2}{2N_0}}))}$$

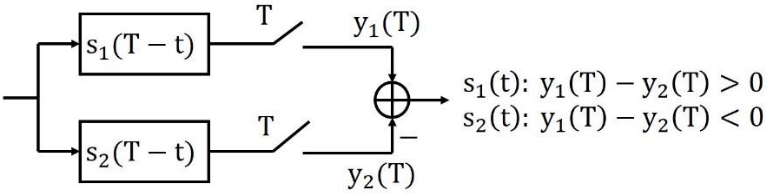
1.(c)

Noise variance 使用 $\mathrm{N_0}$ 或是 $\frac{\mathrm{N_0}}{2}$ 計算都給對，bits 計算錯誤或是沒有考慮 bits 扣三分

$$\mathrm{SNR_A = \frac{E_{avg,A}/log_2 6}{N_0} = \frac{7E_0}{6N_0 log_2 6}(= \frac{7E_0}{18N_0}, \lceil \log_2 6 \rceil = 3bits)}$$

$$\mathrm{SNR_B = \frac{E_{avg,B}/log_2 6}{N_0} = \frac{E_0}{N_0 log_2 6}(= \frac{E_0}{3N_0}, \lceil \log_2 6 \rceil = 3bits)}$$

2.(a)



$s_1(T-t) = x(T-t)(= x(t-T/2)), s_2(T-t) = x(T-t-T/2) = x(T/2-t)(= x(t))$

If transmit $s_1(t)$,

$y_1(t) = s_1(t) * s_1(T-t) = \int_0^t s_1(\tau)s_1(T-(t-\tau))d\tau \to y_1(T) = \int_0^T s_1(\tau)^2 d\tau = \frac{A^2T}{2}$

$y_2(t) = s_1(t) * s_2(T-t) = \int_0^t s_1(\tau)s_2(T-(t-\tau))d\tau \to y_2(T) = \int_0^T s_1(\tau)s_2(\tau)d\tau = 0$

$y_1(T) - y_2(T) = \frac{A^2T}{2} - 0 = \frac{A^2T}{2}$

Similarly, if transmit $s_2(t)$

$y_1(T) - y_2(T) = 0 - \frac{A^2T}{2} = -\frac{A^2T}{2}$

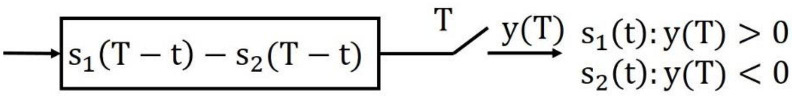
Thus, decision boundary is 0

2.(b)

$d_{min}^2 = \int_{-\infty}^{\infty}(s_1(t) - s_2(t))^2 dt = \int_0^T A^2 dt = A^2T$

$P_e = Q(\sqrt{\frac{d_{min}^2}{2N_0}}) = Q(\sqrt{\frac{A^2T}{2N_0}})$

2(c)



If transmit $s_1(t)$,

$y(t) = s_1(t) * (s_1(T-t) - s_2(T-t)) = \int_0^t s_1(\tau)(s_1(T-(t-\tau)) - s_2(T-(t-\tau)))d\tau$

$\to y(T) = \int_0^T s_1(\tau)^2 - s_1(\tau)s_2(\tau)d\tau = \frac{A^2T}{2}$

The results is similar to (a)

3.(a)

$$\mathrm{Using \quad ML \quad rule, detect \quad X = A}$$

$$\mathrm{p(Y|X=A) > p(Y|X=-A)}$$

$$\rightarrow \frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} e^{-\frac{(Y-0.5A)^2}{2N_0/2}} > \frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} e^{-\frac{(Y+0.5A)^2}{2N_0/2}}$$

$$\rightarrow Y^2 - AY + 0.25A^2 < Y^2 + AY + 0.25A^2$$

$$\rightarrow Y > 0 \quad \mathrm{(Decision \quad boundary \quad is \quad 0)}$$

$$X = \begin{cases} A, Y > 0 \\ -A, Y \leq 0 \end{cases}$$

$$P_e = \frac{1}{2} Q\left(\frac{0.5A-0}{\sqrt{N_0/2}}\right) + \frac{1}{2} Q\left(\frac{-0.5A-0}{\sqrt{N_0/2}}\right) = Q\left(\frac{0.5A}{\sqrt{N_0/2}}\right)$$

$$\left(Q\left(\frac{-0.5A}{\sqrt{N_0/2}}\right) = Q\left(\frac{0.5A}{\sqrt{N_0/2}}\right), \because \mathrm{symmetry}\right)$$

3.(b)

$$\mathrm{Using \quad ML \quad rule, detect \quad X = A}$$

$$\mathrm{p(Y|X=A) > p(Y|X=-A)}$$

$$\frac{1}{2}p(Y|X=A,\alpha=1) + \frac{1}{2}p(Y|X=A,\alpha=-1) > \frac{1}{2}p(Y|X=-A,\alpha=1) + \frac{1}{2}p(Y|X=-A,\alpha=-1)$$

$$\rightarrow \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} e^{-\frac{(Y-A)^2}{2N_0/2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} e^{-\frac{(Y+A)^2}{2N_0/2}} > \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} e^{-\frac{(Y+A)^2}{2N_0/2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} e^{-\frac{(Y-A)^2}{2N_0/2}}$$

$$\mathrm{Decision \quad rule \quad is \quad random \quad guess.}$$

$$P_e = \frac{1}{2}$$

3.(c)

$$\mathrm{Using \quad ML \quad rule, detect \quad X = A}$$

$$\mathrm{p(Y|X=A) > p(Y|X=-A)}$$

$$\rightarrow \frac{1}{2}p(Y|X=A,\alpha=1) + \frac{1}{2}p(Y|X=A,\alpha=0) > \frac{1}{2}p(Y|X=-A,\alpha=1) + \frac{1}{2}p(Y|X=-A,\alpha=0)$$

$$\rightarrow \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} e^{-\frac{(Y-A)^2}{2N_0/2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} e^{-\frac{(Y-0)^2}{2N_0/2}} > \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} e^{-\frac{(Y+A)^2}{2N_0/2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{N_0/2}} e^{-\frac{(Y-0)^2}{2N_0/2}}$$

$$\rightarrow Y^2 - 2AY + A^2 < Y^2 + 2AY + A^2$$

$$\rightarrow Y > 0 \quad \mathrm{(Decision \quad boundary \quad is \quad 0)}$$

$$X = \begin{cases} A, Y > 0 \\ -A, Y \leq 0 \end{cases}$$

$$P_e = p(\alpha=0)P_{e,\alpha=0} + p(\alpha=1)P_{e,\alpha=1} = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}Q\left(\frac{A-0}{\sqrt{N_0/2}}\right) + \frac{1}{2}Q\left(\frac{-A-0}{\sqrt{N_0/2}}\right)\right) = \frac{1}{4} + \frac{1}{2}Q\left(\frac{A}{\sqrt{N_0/2}}\right)$$

$$\left(Q\left(\frac{-A}{\sqrt{N_0/2}}\right) = Q\left(\frac{A}{\sqrt{N_0/2}}\right), \because \mathrm{symmetry}\right)$$

3.(d)

$$\mathrm{Using \quad ML \quad rule, detect \quad X = A}$$

$$\mathrm{p(Y|X=A) > p(Y|X=0)}$$

$$\rightarrow \tfrac{1}{2}\mathrm{p(Y|X=A,\alpha=1)} + \tfrac{1}{2}\mathrm{p(Y|X=A,\alpha=0)} > \tfrac{1}{2}\mathrm{p(Y|X=0,\alpha=1)} + \tfrac{1}{2}\mathrm{p(Y|X=0,\alpha=0)}$$

$$\rightarrow \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{\mathrm{N_0/2}}}\mathrm{e}^{-\frac{(\mathrm{Y-A})^2}{2\mathrm{N_0}/2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{\mathrm{N_0/2}}}\mathrm{e}^{-\frac{(\mathrm{Y-0})^2}{2\mathrm{N_0}/2}} > \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{\mathrm{N_0/2}}}\mathrm{e}^{-\frac{(\mathrm{Y-0})^2}{2\mathrm{N_0}/2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\sqrt{\mathrm{N_0/2}}}\mathrm{e}^{-\frac{(\mathrm{Y-0})^2}{2\mathrm{N_0}/2}}$$

$$\rightarrow \mathrm{Y^2 - 2AY + A^2 < Y^2}$$

$$\rightarrow \mathrm{Y} > \tfrac{\mathrm{A}}{2} \quad \mathrm{(Decision \quad boundary \quad is \quad \tfrac{A}{2})}$$

$$\mathrm{X} = \begin{cases} \mathrm{A, Y} > \frac{\mathrm{A}}{2} \\ \mathrm{0, Y} \le 0 \end{cases}$$

4.

(a)

$$X \sim \left(\frac{1}{8}, \quad \frac{1}{4}, \quad \frac{7}{16}, \quad \frac{3}{16} \right)$$

$$\mathrm{H(X)} = -\tfrac{1}{8} \times \log\tfrac{1}{8} - \tfrac{1}{4} \times \log\tfrac{1}{4} - \tfrac{7}{16} \times \log\tfrac{7}{16} - \tfrac{3}{16} \times \log\tfrac{3}{16} = 1.8496 \text{ (bits)}$$

$$Y \sim \left(\frac{7}{16}, \quad \frac{7}{32}, \quad \frac{11}{32} \right)$$

$$\mathrm{H(Y)} = \left(-\frac{7}{16} \times \log\frac{7}{16}\right) + \left(-\frac{7}{32} \times \log\frac{7}{32}\right) + \left(-\frac{11}{32} \times \log\frac{11}{32}\right) = 1.5310 \text{ (bits)}$$

$$\mathrm{H(X,Y)} = 1 \times \left(-\frac{1}{4} \times \log\frac{1}{4}\right) + 2 \times \left(-\frac{1}{8} \times \log\frac{1}{8}\right) + 7(-\frac{1}{16} \times \log\frac{1}{16})$$

$$+ 2\left(-\frac{1}{32} \times \log\frac{1}{32}\right) = 3.3125 \text{ (bits)}$$

(b)

$$\mathrm{H(X|Y) = H(X,Y) - H(Y)} = 1.7815 \text{ (bits)}$$

$$\mathrm{H(Y|X) = H(X,Y) - H(X)} = 1.4629 \text{ (bits)}$$

(c)

$$\mathrm{I(X,Y) = H(X) + H(Y) - H(X,Y)} = 0.0681\ (bits)$$

(d)

	Y	0	1	2
X	Z			
0		0	1	2
1		1	2	3
2		2	3	0
3		3	0	1

$$\mathrm{P}(Z=0) = \frac{1}{16} + \frac{1}{16} + \frac{1}{8} = \frac{1}{4}$$

$$\mathrm{P}(Z=1) = \frac{1}{32} + \frac{1}{16} + \frac{1}{16} = \frac{5}{32}$$

$$\mathrm{P}(Z=2) = \frac{1}{4} + \frac{1}{16} + \frac{1}{32} = \frac{11}{32}$$

$$\mathrm{P}(Z=3) = \frac{1}{16} + \frac{1}{16} + \frac{1}{8} = \frac{1}{4}$$

$$\mathrm{I(X;Y|Z)} = \sum_{z=0}^{3} \mathrm{p(z)} \sum_{x=0}^{3}\sum_{y=0}^{2} \mathrm{p(x,y|z)\log\frac{p(x,y|z)}{p(x|z)p(y|z)}} = 1.3645 \text{ (bits)}$$

Note:

$p(x,y|z) = p(x,y,z)/p(z)$; $p(x|z) = p(x,z)/p(z)$; $p(y|z) = p(y,z)/p(z)$

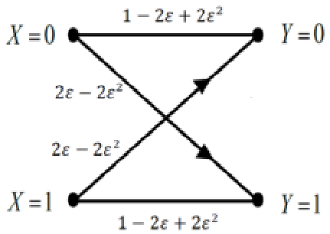
5.

(a) (8%)

First, we can compute the joint probabilities of X, Y

$$P(X=0, Y=0) = (1-\varepsilon)(1-\varepsilon) + \varepsilon \cdot \varepsilon = 1 - 2\varepsilon + 2\varepsilon^2$$
$$P(X=0, Y=1) = (1-\varepsilon)\varepsilon + \varepsilon(1-\varepsilon) = 2\varepsilon - 2\varepsilon^2$$
$$P(X=1, Y=0) = \varepsilon(1-\varepsilon) + (1-\varepsilon)\varepsilon = 2\varepsilon - 2\varepsilon^2$$
$$P(X=1, Y=1) = \varepsilon \cdot \varepsilon + (1-\varepsilon)(1-\varepsilon) = 1 - 2\varepsilon + 2\varepsilon^2$$

Therefore, the channel is equivalent to a BSC



Let $P(X=0) = p$, $P(X=1) = 1\text{-}p$

$$P_e = p(2\varepsilon - 2\varepsilon^2) + (1-p)(2\varepsilon - 2\varepsilon^2) = 2\varepsilon - 2\varepsilon^2$$

(b) (8%)

The capacity of this channel:

$C = \max\{I(X;Y)\} = \max\{H(Y) - H(Y|X)\}$

Since this channel is symmetric, the probabilities of sending 0 or 1 should be equal to maximize the capacity: $P(X=0) = P(X=1) = 0.5$

Hence,

$$P(Y=0) = \frac{1}{2}(1-2\varepsilon+2\varepsilon^2) + \frac{1}{2}(2\varepsilon - 2\varepsilon^2) = \frac{1}{2}$$
$$P(Y=1) = \frac{1}{2}(1-2\varepsilon+2\varepsilon^2) + \frac{1}{2}(2\varepsilon - 2\varepsilon^2) = \frac{1}{2}$$

(c) (8%)

$$H(Y) = -\frac{1}{2}\log_2\frac{1}{2} + -\frac{1}{2}\log_2\frac{1}{2} = 1$$

$$H(Y|X) = -\sum_y \sum_x p(x,y)\log_2 p(y|x)$$
$$= -(1-2\varepsilon+2\varepsilon^2)\log_2(1-2\varepsilon+2\varepsilon^2) - (2\varepsilon+2\varepsilon^2)\log_2(2\varepsilon+2\varepsilon^2)$$
$$= H(2\varepsilon+2\varepsilon^2)$$

Therefore, the capacity is $C = \max\{H(Y) - H(Y|X)\} = 1 - H(2\varepsilon - 2\varepsilon^2)$

6.

Capacity of the BL-AWGN channel is given as

$$\mathrm{C} = \mathrm{W}log_2\left(1 + \frac{P}{N_0 W}\right) \quad \text{,where B = 2W}$$

For a reliable channel, bit rate should be less or equal to capacity, i.e.

$$\mathrm{R} \leq \mathrm{C} = \mathrm{W}log_2\left(1 + \frac{P}{N_0 W}\right), \text{ where } \mathrm{R} = \frac{P}{E_b}$$

By defining $\gamma = \frac{\mathrm{R}}{\mathrm{W}}$ and replacing P, the inequality becomes

$$\gamma = \frac{\mathrm{R}}{W} \leq log_2\left(1 + \frac{E_b}{N_0}\gamma\right)$$

Taking into the value of $\gamma = \frac{\mathrm{R}}{\mathrm{W}} = \frac{2\times10^6}{1\times10^6/2} = 4$, and the inequality becomes

$$\frac{E_b}{N_0} \geq \frac{2^\gamma - 1}{\gamma} = \frac{15}{4}$$

Therefore, we derive the bit energy

$$E_b \geq \frac{15}{4} N_0 = 3.75 \times 10^{-6} \ (J)$$