1. For as set of N parallel Gaussian channels, the capacity is given

$$Y_i = X_i + N_i$$

where $i = 0,1,2,\ldots,N-1$ and N_i are white gaussian noises obey the distribution $\mathcal{N}(0,\ i^2)$. There is a power constraint $\sum_{i=0}^{N-1} X_i^2 \le 5$

A. Find the channel capacity for $N = 2$

B. Find the channel capacity for $N = 4$

C. Find the channel capacity for $N \to \infty$

For a set of N parallel Gaussian channels, the capacity is given by

$$C = \sum_{n=1}^{N} \frac{1}{2} \log\left(1 + \frac{P_n}{n^2}\right)$$

where for $n = 1, 2, \ldots, N$,

$$P_n = \begin{cases} \nu - n^2, & \text{if } n^2 < \nu \\ 0, & \text{if } n^2 \ge \nu \end{cases}$$

and ν is chosen to satisfy $\sum_{n=1}^{N} P_n = 5$.

i. For $N = 2$, let $\nu = 5$, and setting $P_1 = 4$ and $P_2 = 1$ achieves the capacity. Hence we have

$$C = \frac{1}{2}\log\left(1 + \frac{4}{1^2}\right) + \frac{1}{2}\log\left(1 + \frac{1}{2^2}\right) = \log 5 - 1 \approx 1.3219 \text{ bits.}$$

ii. For $N = 4$, let $\nu = 5$, and setting $P_1 = 4$, $P_2 = 1$, and $P_3 = P_4 = 0$ achieves the capacity. Hence we have

$$C = \frac{1}{2}\log\left(1 + \frac{4}{1^2}\right) + \frac{1}{2}\log\left(1 + \frac{1}{2^2}\right) + 0 + 0 = \log 5 - 1 \approx 1.3219 \text{ bits.}$$

iii. For $N = \infty$, similarly, setting $P_1 = 4$, $P_2 = 1$, and $P_i = 0$ for $i \ge 3$ achieves the capacity. We have

$$C = \frac{1}{2}\log\left(1 + \frac{4}{1^2}\right) + \frac{1}{2}\log\left(1 + \frac{1}{2^2}\right) + 0 + \cdots + 0 = \log 5 - 1 \approx 1.3219 \text{ bits.}$$

2. Following the previous question, if the power constraint is changed to

$\sum_{i=0}^{N-1} \frac{X_i^2}{i} \le 5$

A. Find the channel capacity for $N = 2$

B. Find the channel capacity for $N = 4$

C. Find the channel capacity for $N \to \infty$

The constraint can be written as

$$\sum_{n=1}^{N} \frac{P_n}{n} = \sum_{n=1}^{N} \overline{P}_n \le 5$$

where $\overline{P}_n = P_n/n$, for $n = 1, 2, \ldots, N$. The capacity is then given by

$$C = \sum_{n=1}^{N} \frac{1}{2} \log\left(1 + \frac{P_n}{n^2}\right) = \sum_{n=1}^{N} \frac{1}{2} \log\left(1 + \frac{\overline{P}_n}{n}\right)$$

where for $n = 1, 2, \ldots, N$,

$$\overline{P}_n = \begin{cases} \nu - n, & \text{if } n < \nu \\ 0, & \text{if } n \ge \nu \end{cases}$$

and ν is chosen to satisfy $\sum_{n=1}^{N} \overline{P}_n = 5$.

i. For $N = 2$, let $\nu = 4$, and setting $\overline{P}_1 = 3$ and $\overline{P}_2 = 2$, i.e., $P_1 = 3 \cdot 1 = 3$ and $P_2 = 2 \cdot 2 = 4$, achieves the capacity. We have

$$C = \frac{1}{2} \log\left(1 + \frac{3}{1}\right) + \frac{1}{2} \log\left(1 + \frac{2}{2}\right) = \frac{1}{2} \log 8 = \frac{3}{2} \text{ bits.}$$

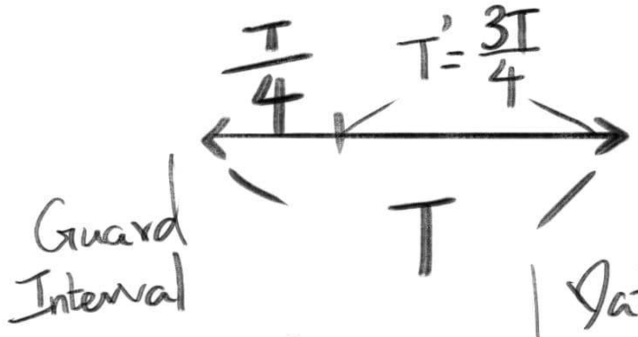
ii. For $N = 4$, let $\nu = 11/3$, and setting $\overline{P}_1 = 8/3$, $\overline{P}_2 = 5/3$, $\overline{P}_3 = 2/3$, and $\overline{P}_4 = 0$, i.e., $P_1 = (8/3) \cdot 1 = 8/3$, $P_2 = (5/3) \cdot 2 = 10/3$, $P_3 = (2/3) \cdot 3 = 2$, and $P_4 = 0$, achieves the capacity. We have

$$\begin{aligned} C &= \frac{1}{2} \log\left(1 + \frac{8/3}{1}\right) + \frac{1}{2} \log\left(1 + \frac{5/3}{2}\right) + \frac{1}{2} \log\left(1 + \frac{2/3}{3}\right) + 0 \\ &= \frac{3}{2} \log 11 - 2 \log 3 - \frac{1}{2} \approx 1.5192 \text{ bits.} \end{aligned}$$

iii. For $N = \infty$, let $\nu = 11/3$, and setting $\overline{P}_1 = 8/3$, $\overline{P}_2 = 5/3$, $\overline{P}_3 = 2/3$, and $\overline{P}_i = 0$, for $i \ge 4$, i.e., $P_1 = 8/3$, $P_2 = 10/3$, $P_3 = 2$, and $P_i = 0$, for $i \ge 4$, achieves the capacity. We have

$$\begin{aligned} C &= \frac{1}{2} \log\left(1 + \frac{8/3}{1}\right) + \frac{1}{2} \log\left(1 + \frac{5/3}{2}\right) + \frac{1}{2} \log\left(1 + \frac{2/3}{3}\right) + 0 + \cdots + 0 \\ &= \frac{3}{2} \log 11 - 2 \log 3 - \frac{1}{2} \approx 1.5192 \text{ bits.} \end{aligned}$$

3. Consider an OFDM system with 16 QAM, 0.5 code rate, 64 subcarriers which 48 of them are used to transfer data and 16 of them are used as channel estimation or protection. Each subcarrier is spaced 312.5 kHz and use $\frac{1}{4}$ frame as guard interval. Find the data rate.



$T' = \frac{1}{f} = 3.2\mu S$

$T = \frac{5}{4}T' = 4\mu S$

$16QAM \rightarrow k=4$ (per symbol)

Data rate

$\frac{1}{T} \times k \times 48 \times \frac{1}{2} = 24$ (Mbps) #

4. W is the DFT matrix in OFDM. Show that $\frac{1}{N}\mathrm{ww}^{\mathrm{H}} = I_N$

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5. Consider a linear channel with bandwidth W. The channel is equally divided into three sub-channels which has squared magnitude response $|\mathrm{H(f)}|^2$ ଶ in the piecewise-linear form with $|\mathrm{H(f)}|^2 = 1, \frac{1}{3}, \frac{1}{5}$ for sub-channels i =1,2 and 3 respectively. Assume the system transmits data at the rate equal to the Shannon's channel capacity and the noise variance $\sigma_i^2 = 1, \frac{1}{2}, \frac{1}{3}$ for subchannels i = 1,2 and 3 respectively.

A. Let the total transmit power be constrained such that $P = p_1 + p_2 + p_3$ and P is constant. Please derive the formulas for the optimum powers p_1, p_2, and p_3 allocated to the three sub-channels of frequency bands such that the overall channel capacity of the entire system can be maximized

(a) The total capacity is $C = \sum_{i=1}^{3} \Delta f_i \log_2[1 + \frac{P(f)|H(f_i)|^2}{\sigma_i^2}]$, where $P(f_i)$ is the PSD if the subchannel i, $i = 1,2,3$ and $P_i = P(f_i)\Delta f_i$

The problem is

$$\max C \text{ subject to } P = \sum_{i=1}^{3} \Delta f_i P(f_i) \text{ with Lagrange multiplier } \lambda$$

The Largrange constrain could be express as:

$$J = C + \lambda(\mathrm{P} - \sum_{i}^{3} P_i)$$

$$= \sum_{i=1}^{3} \Delta f_i \log_2[1 + \frac{P(f)|H(f_i)|^2}{\sigma_i^2}] + \lambda[\mathrm{P} - \sum_{i}^{3} \Delta f_i P(f_i)]$$

Take derivative with respect to $P(f_i) \to \frac{\partial J}{\partial P(f_i)} = 0.$

$$\Delta f_i \frac{\log_2 e}{P(f_i) + \frac{\sigma_i^2}{|H(f_i)|^2}} - \lambda \Delta f_i = 0$$

$$\Rightarrow \lambda = \frac{\log_2 e}{P(f_i) + \frac{\sigma_i^2}{|H(f_i)|^2}}$$

$$\Rightarrow \begin{cases} P(f_i) = \left(\kappa - \frac{\sigma_i^2}{|H(f_i)|^2}\right)^+, i = 1,2,3 \\ P = \sum_{i=1}^{3} \Delta f_i P(f_i) \end{cases}, \text{ where } \kappa = \frac{\log_2 e}{\lambda}$$

B. Given the total transmit power $P = 2$, and subchannel bandwidth Δf= 2, please calculate the corresponding values of p_1, p_2, and p_3

(b)

Because the channel is equally divided in to 3 subchannels, $\Delta f_i = \Delta f$ for $i = 1,2,3$.

$$P = \sum_{i=1}^{3} P_i = \sum_{i=1}^{3} \Delta f_i P(f_i) = 2$$

$$\Rightarrow \Delta f \left(k - \frac{1}{1}\right) + \Delta f \left(k - \frac{1/2}{1/3}\right) + \Delta f \left(k - \frac{1/3}{1/5}\right) = 2$$

$$\Rightarrow k = \frac{2}{3\Delta f} + \frac{25}{18}$$

$$P(f_1) = k - 1 = \frac{2}{3\Delta f} + \frac{7}{18} = \frac{13}{18} \text{ , } P_1 = \frac{13}{9}$$

$$P(f_2) = k - \frac{3}{2} = \frac{2}{3\Delta f} - \frac{2}{18} = \frac{4}{18} \text{ , } P_2 = \frac{4}{9}$$

$$P(f_1) = k - \frac{5}{3} = \frac{2}{3\Delta f} - \frac{5}{18} = \frac{1}{18} \text{ , } P_3 = \frac{1}{9}$$