COM 5120
Communications Theory

Chapter 2
Deterministic and Random Signal Analysis

Jen-Ming Wu
Inst. Of Communications Engineering
Dept. of Electrical Engineering
National Tsing Hua University
Email: jmwu@ee.nthu.edu.tw

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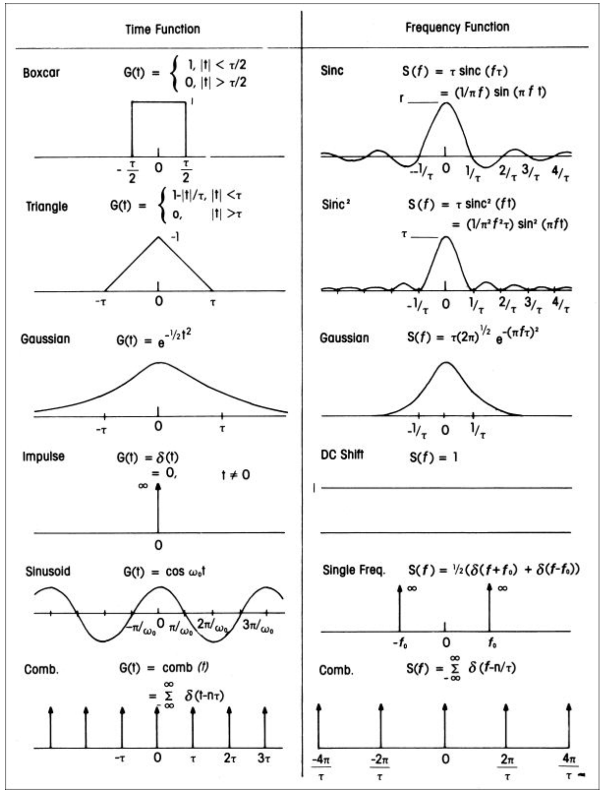
Outline

- Deterministic Signal Analysis
  - Fourier and Hilbert analysis
  - Bandpass and lowpass signal representation

- Random Signal Analysis
  - Random variables
  - Bounds on tail probabilities
  - Random processes
  - Series expansion of random processes

Fourier Analysis



- Fourier Transform
  - ✓ For a non-periodic signal $x(t)$, $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df$$

- Fourier Series
  - ✓ For a periodic signal $x(t)$ with period T_0, $\int_{-\infty}^{\infty} |x_0(t)|^2 dt < \infty$

$$x(t) = \sum_{n=-\infty}^{\infty} x_0(t-nT_0) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi nf_o t)$$

$$\text{where } c_n = \frac{1}{T_0}\int_{-\infty}^{\infty} x_0(t)\exp(-j2\pi nf_o t)dt, \qquad n = 0, \pm1, \pm2, \ldots$$

- Please check the book for the F.T. pairs and properties.

Fourier Transform of Periodic Signals

- Given a periodic signal $x(t)$ with period T_0, $\int_{-\infty}^{\infty} |x_0(t)|^2 dt < \infty$

$$x(t) = \sum_{n=-\infty}^{\infty} x_0(t - nT_0) = x_0(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

- From Fourier transform pairs, it can be shown that

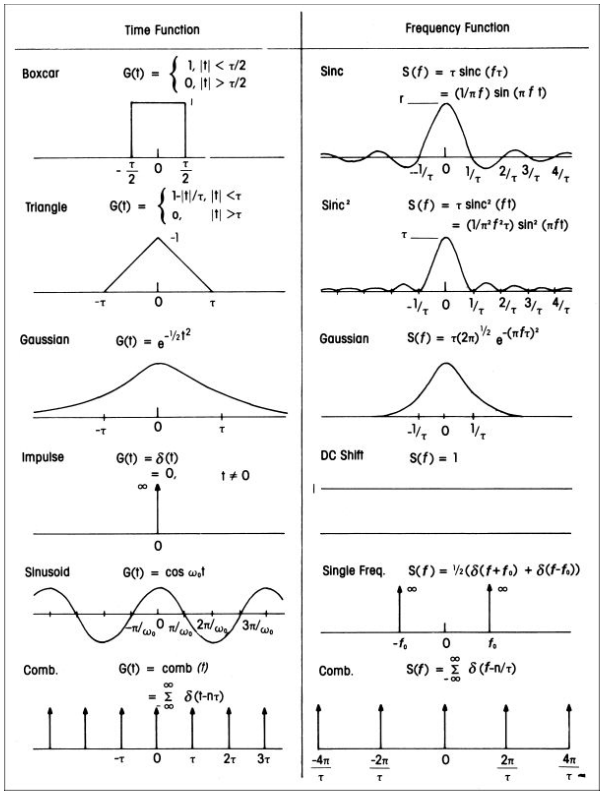
$$\sum_{n=-\infty}^{\infty} \delta(t - nT_0) \leftrightarrow f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

then $X(f) = X_0(f) F\left\{ \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \right\}$

$$= X_0(f) f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

$$= f_0 \sum_{n=-\infty}^{\infty} X_0(nf_0) \delta(f - nf_0)$$

where $X_0(f) = \int_{-\infty}^{\infty} x_0(t) \exp(-j2\pi ft) dt$



Fourier Transform of Periodic Signals

- For the periodic signal x(t) with period T_0, $\int_{-\infty}^{\infty} |x_0(t)|^2 dt < \infty$

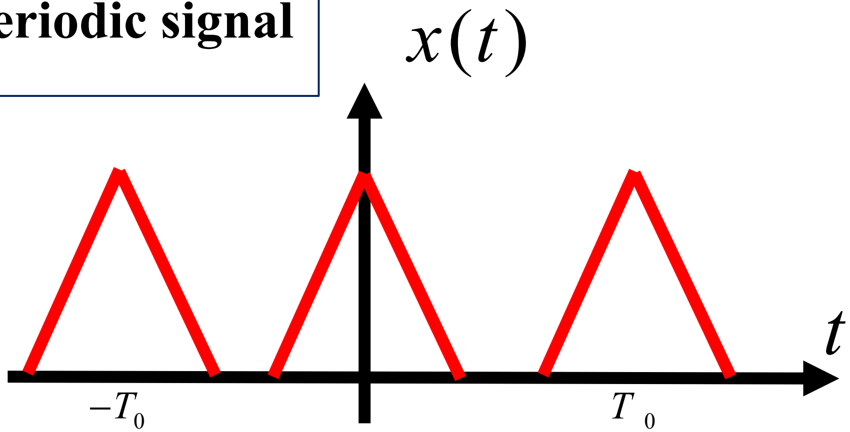
$$x(t) = \sum_{n=-\infty}^{\infty} x_0(t - nT_0) = x_0(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

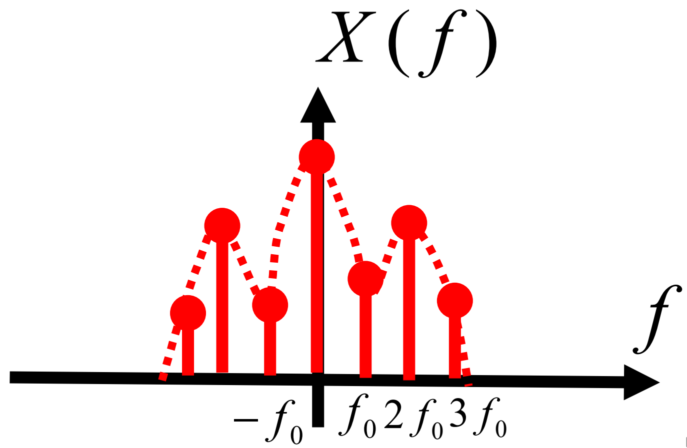
$$X(f) = X_0(f) f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

$$= f_0 \sum_{n=-\infty}^{\infty} X_0(nf_0)\delta(f - nf_0)$$

✓ $X(f)$ is a discrete signal

***x*(*t*) is a periodic signal**





Test Your Understanding

not discrete
not periodic

$$\xrightarrow{\mathcal{F}}$$

not periodic
not discrete

Hilbert Transform

- Hilbert transform

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau = x(t) * \frac{1}{\pi t}$$

- Inverse Hilbert transform

$$x(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{x}(\tau)}{t-\tau} d\tau = -\hat{x}(t) * \frac{1}{\pi t}$$

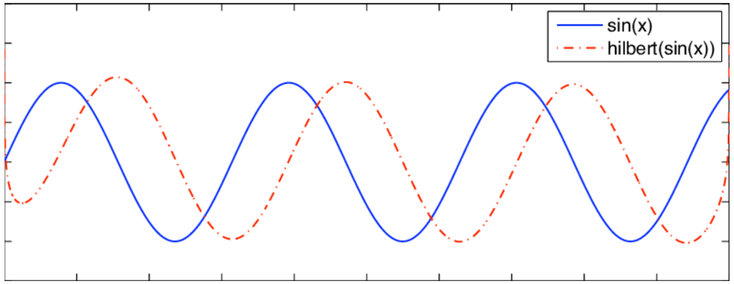
- F.T. versus H.T. in communication systems?
  - The F.T. is used for evaluating the frequency content that provides the math basis for analyzing frequency response.
  - The H.T. is used to shift the phase of a signal. In the communication system, the phase shift between signals can be utilized to separate signals in communication systems.

Hilbert Transform

- The Fourier Transform of H.T. pairs

$$x(t) \leftrightarrow X(f)$$

$$\hat{x}(t) \leftrightarrow \hat{X}(f)$$



- What is the relation between $X(f)$, and $\hat{X}(f)$?

Since $\hat{x}(t) = x(t) * \dfrac{1}{\pi t} \Rightarrow \hat{X}(f) = -j\,\mathrm{sgn}(f) X(f)$

Note from F.T. pair: $\dfrac{1}{\pi t} \overset{F.T.}{\leftrightarrow} -j\,\mathrm{sgn}(f)$

- The Hilbert transform is called a quadrature filter that shift the phase of $x(t)$ by $\pi/2$ in time domain.

$Ex:\ x(t) = \cos(2\pi f_c t) \leftrightarrow \hat{x}(t) = \sin(2\pi f_c t)$

Hilbert Transform Pairs

- $x(t) = \cos(2\pi f_c t) \leftrightarrow \hat{x}(t) = \sin(2\pi f_c t)$

  $\therefore X(f) = \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right],$

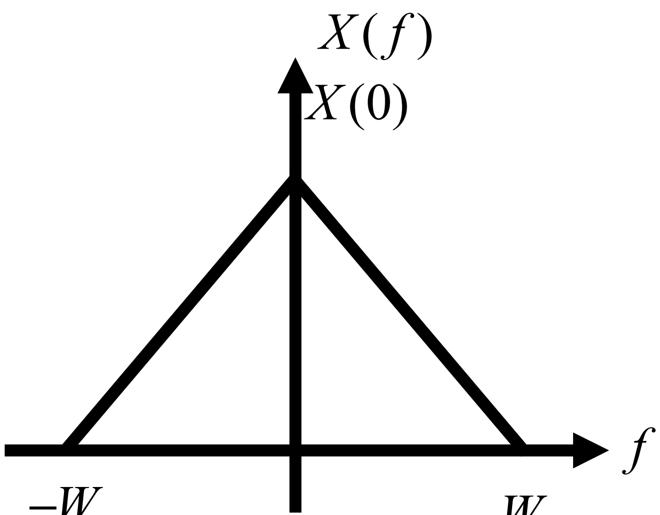
  $\hat{X}(f) = -j\,\mathrm{sgn}(f) X(f) = \frac{1}{2j}\left[\delta(f - f_c) - \delta(f + f_c)\right]$

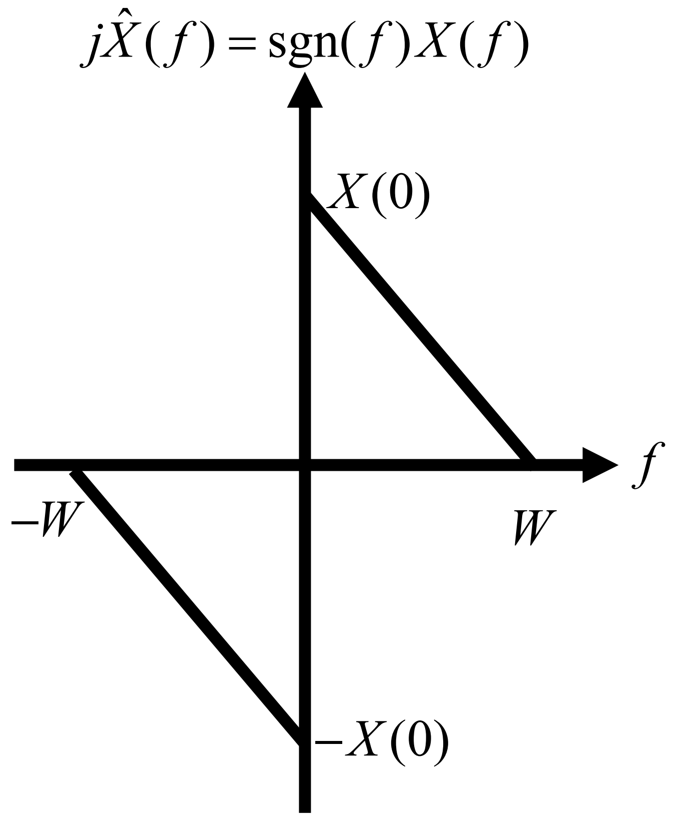
- $\sin(2\pi f_c t) \leftrightarrow -\cos(2\pi f_c t)$
- $\frac{\sin(t)}{t} \leftrightarrow \frac{1 - \cos(t)}{t}$
- $\delta(t) \leftrightarrow \frac{1}{\pi t}$
- $\frac{1}{t} \leftrightarrow -\pi\delta(t)$

Hilbert Transform of Low Pass Signal

- Given the low-pass signal $x(t)$ with H.T. $\hat{x}(t)$, then

$$\hat{X}(f) = \frac{1}{j}\operatorname{sgn}(f)X(f)$$





Pre-envelopes

- Motivation

  Q1: How to represent positive/negative frequency parts of a signal in time domain?

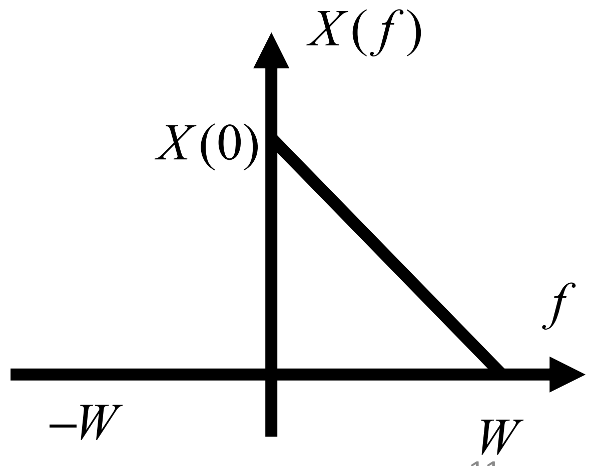
  Q2: How can we modify the frequency content of a real-valued signal *x*(*t*) ? For example, elimination of the negative frequency components of a signal in time domain?

- Pre-envelope for positive frequency:

$$x_{+}(t) = \frac{1}{2}x(t) + \frac{j}{2}\hat{x}(t)$$

$$\Rightarrow X_{+}(f) = \frac{1}{2}X(f) + \frac{1}{2}\operatorname{sgn}(f)X(f)$$

$$= \begin{cases} X(f), & f > 0 \\ \frac{1}{2}X(0), & f = 0 \\ 0, & f < 0 \end{cases}$$



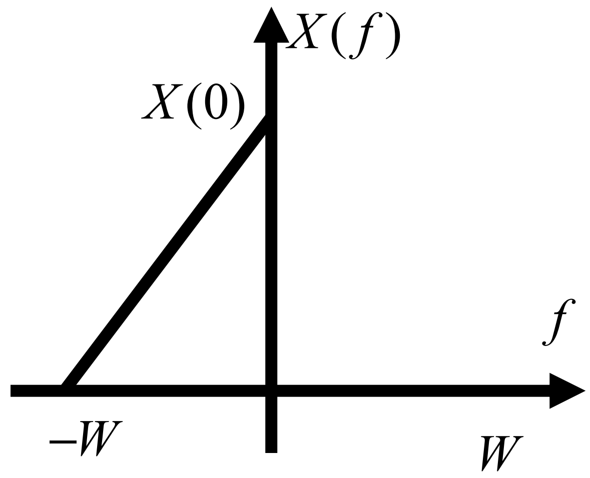
Pre-envelopes

- Pre-envelope for negative frequency:

$$x_{-}(t) = \frac{1}{2}x(t) - \frac{j}{2}\hat{x}(t)$$

$$\Rightarrow X_{-}(f) = \frac{1}{2}X(f) - \frac{1}{2}\mathrm{sgn}(f)X(f)$$

$$= \begin{cases} 0, & f > 0 \\ \frac{1}{2}X(0), & f = 0 \\ X(f), & f < 0 \end{cases}$$



- The pre-envelope signal represents the positive or negative frequencies of a signal and is usually complex in time domain.

Pre-envelopes

- Properties of pre-envelope signal

(1) $x_+(t) \leftrightarrow X_+(f)$: positive frequency part of $X(f)$

$x_-(t) \leftrightarrow X_-(f)$: negative frequency part of $X(f)$

(2) $x(t) = x_+(t) + x_-(t)$

$$X(f) = X_+(f) + X_-(f)$$

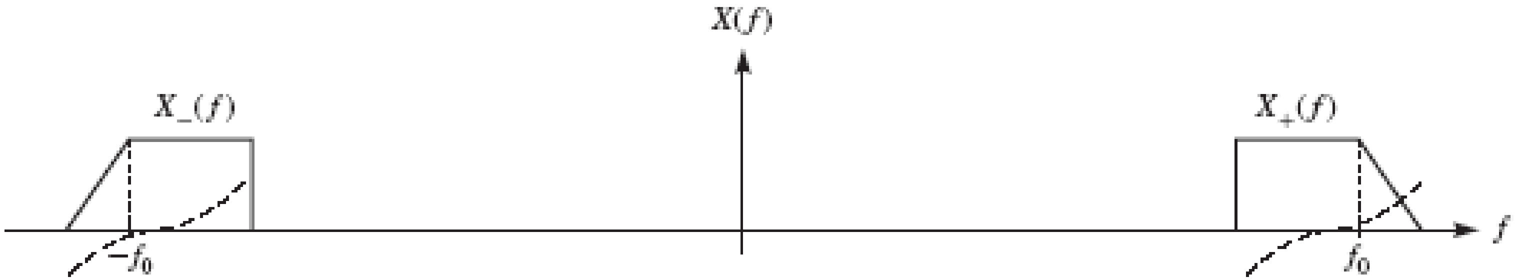
(3) The positive and negative pre-envelopes are conjugate to each other,

$$x_-(t) = x_+^*(t) \qquad \rightarrow X_-(f) = X_+^*(-f)$$

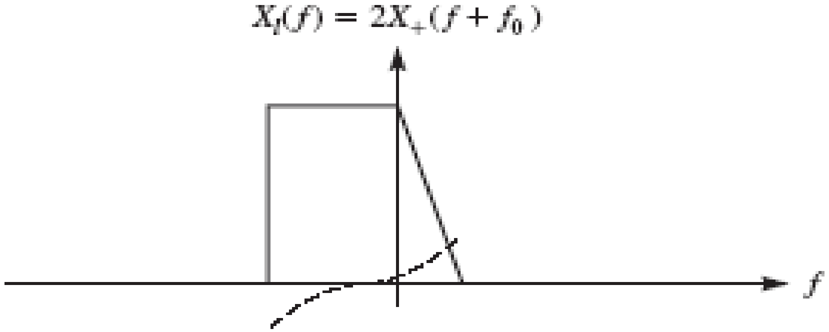
Complex Envelope and Band-pass Signal

- Given the real-valued symmetric band-pass signal $x(t)$ with

$$X(f) = X_+(f) + X_-(f) = X_+(f) + X_+^*(-f)$$



- Define the low-pass signal of $x(t)$ be $x_l(t) = x_I(t) + jx_Q(t)$ so that



$$X_l(f) = 2X_+(f + f_0)$$

$$\rightarrow \quad x_l(t) = x_I(t) + jx_Q(t) \text{ is called}$$

the complex envelope of $x(t)$.

Q: What is the physical meaning of $x_I(t)$ and $x_Q(t)$ in the system?

- The passband signal can be represented

by the lowpass signal $X(f) = \dfrac{1}{2}\left[X_l(f - f_0) + X_l^*(-f - f_0)\right]$

Complex Envelope and Band-pass Signal

Q: What is the relation of bandpass and lowpass signal in time domain? i.e. $x(t)$ and $x_l(t)$ in the system?

- The positive pre-envelope of band-pass signal is

$$x_+(t) = \frac{1}{2} x_l(t)\exp(j2\pi f_0 t) \quad i.e. \quad X_+(f) = \begin{cases} X(f), & f > 0 \\ 0, & f \le 0 \end{cases}$$

$$x_l(t) = 2x_+(t)\exp(-j2\pi f_0 t) \qquad (\because X_l(f) = 2X_+(f + f_0))$$

$$= \left(x(t) + j\hat{x}(t)\right)\exp(-j2\pi f_0 t)$$

$$\Rightarrow x_l(t)\exp\left(j2\pi f_0 t\right) = x(t) + \mathrm{j}\hat{x}(t)$$

- As result, the relation of bandpass and lowpass signals is

$$x(t) = \mathrm{Re}\left\{x_l(t)\exp(j2\pi f_0 t)\right\}$$

$$= \underbrace{x_I(t)}_{\substack{\text{In-phase} \\ \text{component} \\ \text{of } x(t)}} \cos(2\pi f_o t) - \underbrace{x_Q(t)}_{\substack{\text{Quadrature phase} \\ \text{component} \\ \text{of } x(t)}} \sin(2\pi f_o t)$$

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- Random Signal Analysis
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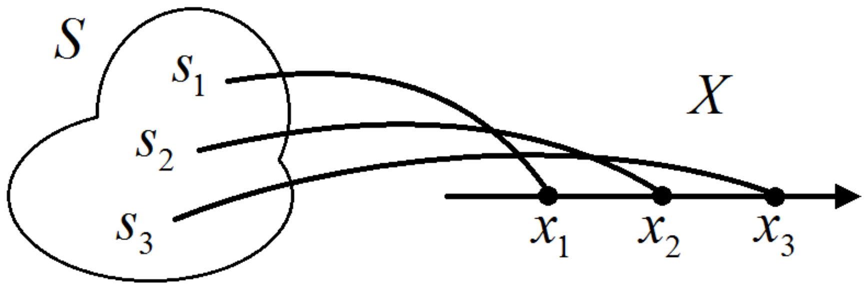
Random Variable

- A random variable (r.v.) X is a mapping from the sample space S to a real number space R , i.e. $X : S \rightarrow R$

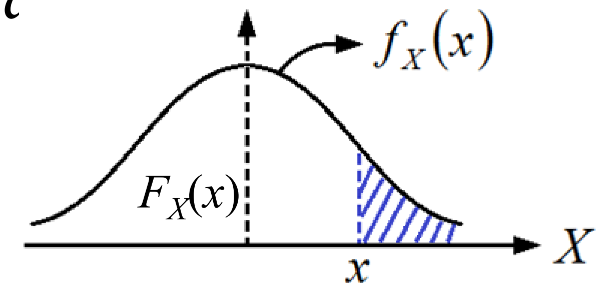


- A random variable can be defined by the distribution functions

$$F_X(x) = \mathrm{P}_X[X \le x]$$

or probability density function $f_X(x) = \frac{d}{dx} F_X(x)$

- Discrete r.v. with discrete p.d.f.
  Continuous r.v. with continuous p.d.f.



Properties

1. Let $g(X)$ be a function of *r.v.* X. The expectation of $g(X)$ is

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(X) f_X(x) dx$$

$$\text{or } \sum_X g(X) \mathrm{P}_X(x)$$

When $g(X) = X$, then $\mathrm{E}[g(X)] = \mathrm{E}[X] = \mu_X$ (mean)

When $g(X) = (X - \mu_X)^2$,

then $\mathrm{E}[g(X)] = \mathrm{E}\left[(X - \mu_X)^2\right] = \sigma_X^2$ (variance)

Properties

2. For two random variables X and Y, we say X and Y are：

(1) independent if $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

(2) uncorrelated if $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$

Properties

3. Moment generating function $\psi_X(s)$

$$g(X) = e^{sX}, \forall s$$

$$\mathrm{E}[g(X)] = \mathrm{E}\left[e^{sX}\right] \equiv \psi_X(s)$$

Moment generating function

$$\frac{d}{ds}\psi_X(s)\Bigg|_{s=0} = \mathrm{E}[X] = \mu_X$$

First moment

$$\frac{d^2}{ds^2}\psi_X(s)\Bigg|_{s=0} = \mathrm{E}\left[X^2\right] = \sigma_X^2 + \mu_X^2$$

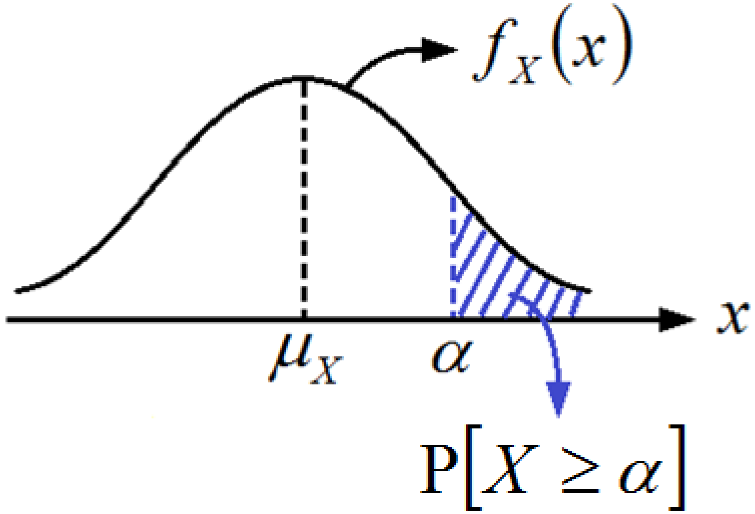
Second moment

Bounds on tail probability

(1) Markov Inequality

Given a non-negative r.v. X, and $\alpha > 0, \alpha \in \mathrm{R}^+$

$$\mathrm{P}[X \ge \alpha] \le \frac{\mathrm{E}[X]}{\alpha}$$

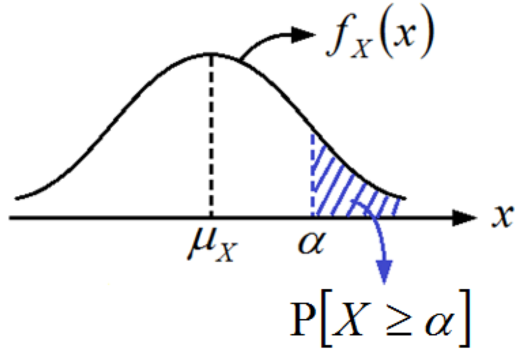


proof

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx \ge \int_{\alpha}^{\infty} x f_X(x)\, dx$$

$$\ge \alpha \int_{\alpha}^{\infty} f_X(x)\, dx = \alpha\ \mathrm{P}[X \ge \alpha]$$

Bounds on tail probability



(2) Chernov Bound

$$\mathrm{P}[X \geq \alpha] \leq e^{-s\alpha}\psi_X(s), \forall\, \alpha > \mathrm{E}[X]$$

$$s > 0$$

Proof

$$\mathrm{P}\left[X \geq \alpha\right] = \mathrm{P}\left[e^{sX} \geq e^{s\alpha}\right]$$

By Markov inequality

$$\mathrm{P}\left[e^{sX} \geq e^{s\alpha}\right] \leq \frac{\mathrm{E}\left[e^{sX}\right]}{e^{s\alpha}} = e^{-s\alpha}\psi_X\left(s\right) \ ,\forall \alpha > \mathrm{E}\left[X\right] \ ,\forall s > 0$$

✓ Note:

- The Chernov bound is a function of s.
- There exists an optimal value of s that minimize the Chernov bound which is subject to the distribution function of X.

Some useful random variables

(1) **Bernoulli random variable** X

$$\mathrm{P}[X=1]=p\,,\ \mathrm{P}[X=0]=1-p$$

$$\mu_X=\mathrm{E}[X]=p$$

$$\sigma_X^2=\mathrm{E}\left[(X-\mu_X)^2\right]=p(1-p)$$

(2) **Binomial random variable** X

$X =$ Sum of n indep. Bernoulli r.v.'s $=\sum_{i=1}^{n}X_i$

$$\mathrm{P}[X=k]=\binom{n}{k}p^k(1-p)^{n-k},\ k=0,1,\ldots,n$$

$$\mu_X=np\quad;\quad\sigma_X^2=np(1-p)$$

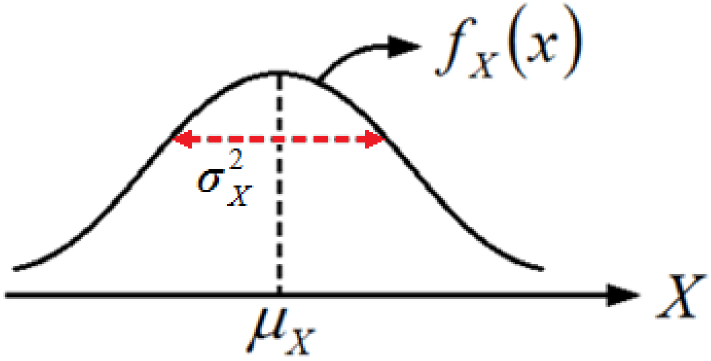
Some useful random variables

(3) Gaussian random variable X

X is a Gaussian r.v. if

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{\frac{-(X-\mu_X)^2}{2\sigma_X^2}},$$

denoted by $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$



$\rightarrow$ completely defined by μ_X and σ_X^2

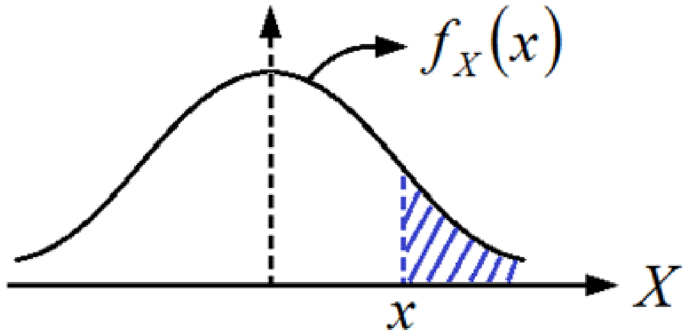
(degrees of freedom = 2)

Let $X \sim \mathcal{N}(0,\ 1)$, define Q-function

$$Q(x) \equiv 1 - F_X(x) = \mathrm{P}[X \geq x]$$

$$= \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt \ , \ x > \mu_X$$

$=$ The tail probability of a Gaussian function



∗ Note：

$$\operatorname{erfc}(x) \equiv \int_x^{\infty} \frac{1}{\sqrt{\pi}} e^{-t^2} dt \ \Rightarrow Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}} \right)$$

$$\boxed{\begin{array}{l} \text{Let } X \sim \mathcal{N}(0,\ \sigma_X^2) \text{ , then} \\ \mathrm{P}[X \geq x] = \int_x^{\infty} \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{-t^2}{2\sigma_X^2}} dt = Q\left(\frac{x}{\sigma_X} \right) \end{array}}$$

Chernov bound of Gaussian random variable

$$Q(x) \le \exp\left(\frac{-x^2}{2}\right)$$

proof

By Chernov bound

$$Q(x) \le e^{-sx}\psi_X(s) = e^{-sx}\mathrm{E}\left[e^{sX}\right], \ \forall s$$

The tightest bound (i.e. min.) of the Chernov bound occurs when

$$\frac{\partial}{\partial s}\left\{ e^{-sx}\mathrm{E}\left[e^{sX}\right] \right\} = 0$$

$$\Rightarrow \cancel{e^{-sx}}\mathrm{E}\left[Xe^{sX}\right] - x\cancel{e^{-sx}}\mathrm{E}\left[e^{sX}\right] = 0$$

$$\Rightarrow \mathrm{E}\left[Xe^{sX}\right] = x\mathrm{E}\left[e^{sX}\right]$$

where

$$\text{(i) } \mathrm{E}\left[e^{sX}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{st} e^{\frac{-t^2}{2}} dt$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{s^2}{2} - st + \frac{t^2}{2}\right)\right] dt}_{=\ 1} \cdot e^{\frac{s^2}{2}} = e^{\frac{s^2}{2}}$$

$$\text{(ii) } \mathrm{E}\left[Xe^{sX}\right] = \frac{d}{ds}\mathrm{E}\left[e^{sX}\right] = se^{\frac{s^2}{2}}$$

$$\mathrm{E}\left[Xe^{sX}\right] - x\mathrm{E}\left[e^{sX}\right] = 0 \Rightarrow se^{\frac{s^2}{2}} - xe^{\frac{s^2}{2}} = 0 \Rightarrow s = x$$

The min Chernov bound at $s = x$ is then

$$Q(x) \le e^{-sx}\mathrm{E}\left[e^{sX}\right] = e^{-sx}e^{\frac{s^2}{2}} = e^{-x^2}e^{\frac{x^2}{2}} = e^{\frac{-x^2}{2}}, \quad \text{Q.E.D}$$

Tighter than the Chernov bound?

- It can be shown that $Q(x) \le \dfrac{1}{2}\exp\left(\dfrac{-x^2}{2}\right)$

Proof: Given $X \sim \mathcal{N}(0,1)$, then

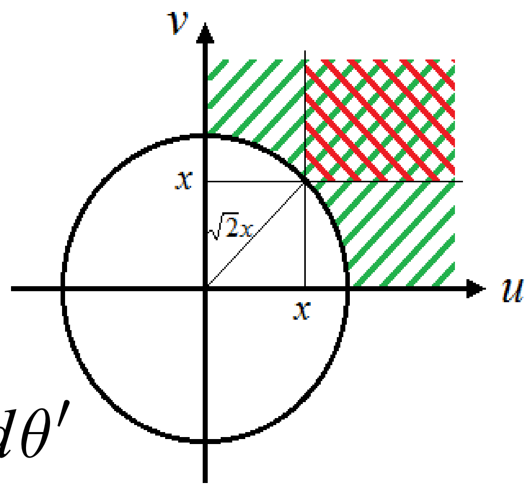
$$(Q(x))^2 = \left(\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}du\right)\left(\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{v^2}{2}}dv\right) = \frac{1}{2\pi}\int_x^\infty\int_x^\infty e^{-\frac{u^2+v^2}{2}}dudv$$

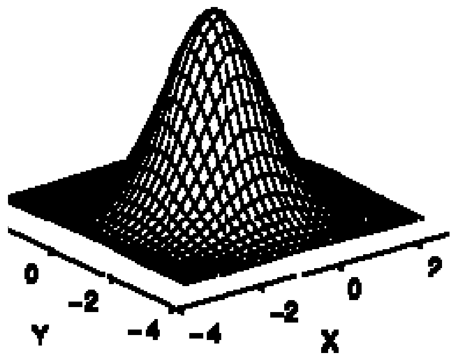
Let $r^2 = u^2 + v^2,\ \theta = \tan^{-1}\left(\dfrac{v}{u}\right)$

$$\Rightarrow (Q(x))^2 = \frac{1}{2\pi}\int_r\int_\theta e^{-\frac{r^2}{2}}rdrd\theta$$

$$\le \frac{1}{2\pi}\int_{\sqrt{2}x}^\infty re^{-\frac{r^2}{2}}dr\int_0^{\frac{\pi}{2}}d\theta'$$

$$= \frac{1}{2\pi}\left[-e^{-\frac{r^2}{2}}\right]\Bigg|_{\sqrt{2}x}^{\infty}\cdot\frac{\pi}{2} = \frac{1}{4}e^{-x^2},\quad \therefore \mathrm{Q}(x) \le \frac{1}{2}e^{-\frac{x^2}{2}}$$





Some useful random variables

(4) **Complex random variable**

$$X = X_1 + j\,X_2$$

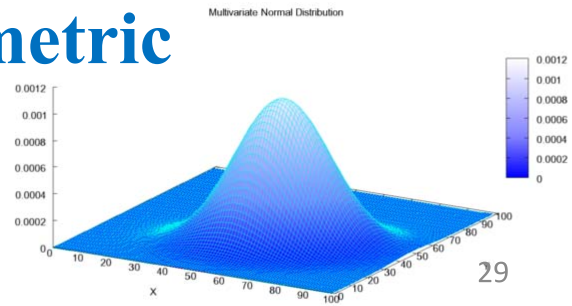
Let X_1 and X_2 be jointly Gaussian r.v. with joint prob. density $f(X_1, X_2)$.

$$\mu_X = \mu_{X_1} + j\mu_{X_2}\ ;\ \sigma_X^2 = \mathrm{E}\left[(X-\mu_X)(X-\mu_X)^*\right]$$

If X_1 and X_2 are *i.i.d.* (independent and identically distributed) Gaussian r.v.'s with $\sigma_{X_1}^2 = \sigma_{X_2}^2$

$$\Rightarrow \sigma_X^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2$$

In this case, X is called **circularly symmetric** complex Gaussian or $X \sim \mathcal{CN}\left(\mu_X, \sigma_X^2\right)$



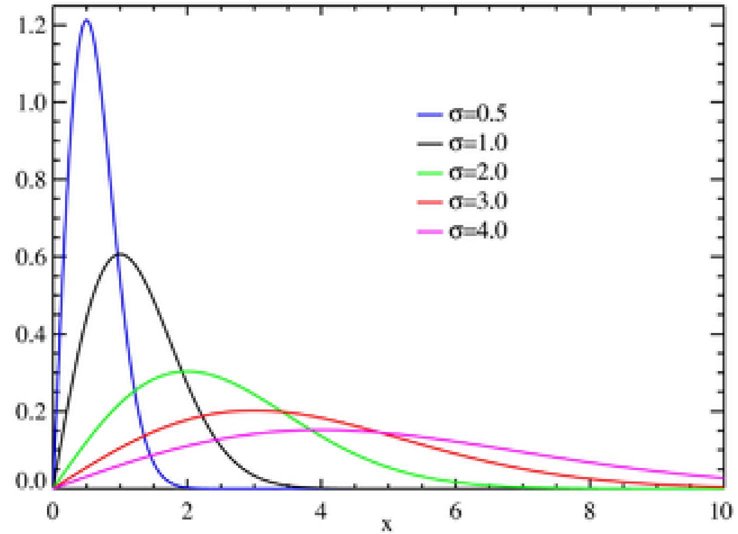
Some useful random variables

(5) **Rayleigh distributed random variable**

If X_1 and X_2 are i.i.d. Gaussian r.v.'s with $X_1, X_2 \sim \mathcal{N}(0,\sigma^2)$, then $X = \sqrt{X_1^2 + X_2^2}$ is a Rayleigh r.v. with PDF：

$$f_X(x) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}, x > 0$$

$$\mu_X = \sigma\sqrt{\frac{\pi}{2}} \ ; \sigma_X^2 = \left(2 - \frac{\pi}{2}\right)\sigma^2$$



Example: Let Y= X_1 +j X_2, which is a complex Gaussian random variable.

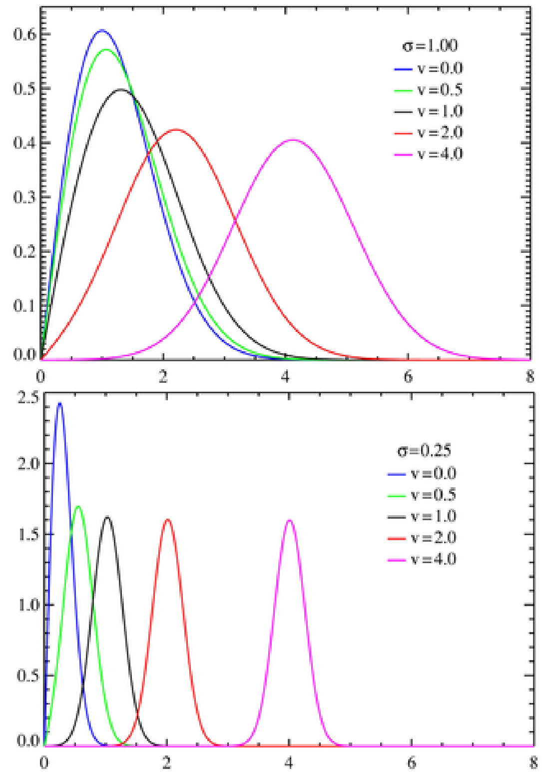
If X = |Y|, then X = $\sqrt{X_1^2 + X_2^2}$ is a Rayleigh distributed r.v.

Some useful random variables

(6) **Rician distributed random variable**

If X_1 and X_2 are i.i.d. Gaussian r.v.'s with X_1 , $X_2 \sim \mathcal{N}\left(\mu,\sigma^2\right)$

then $X = \sqrt{X_1^2 + X_2^2}$ is a **Rician r.v.**



$$f_X\left(x\right) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2+u^2}{2\sigma^2}\right)I_o(\frac{u\ x}{\sigma^2}),\ x>0$$

$$\mu_X = \sigma\sqrt{\frac{\pi}{2}}\ \mathrm{e}^{-\frac{K}{2}}\left[(1+K)I_0(\frac{K}{2})+KI_1(\frac{K}{2})\right];$$

$$\mathrm{E}[\mathrm{X}^2]\text{=}2\ \sigma^2+2\mu,\ \text{ where } K=\frac{\mu^2}{\sigma^2}$$

Example: Let Y= X_1 +j X_2, which is a complex Gaussian random variable.

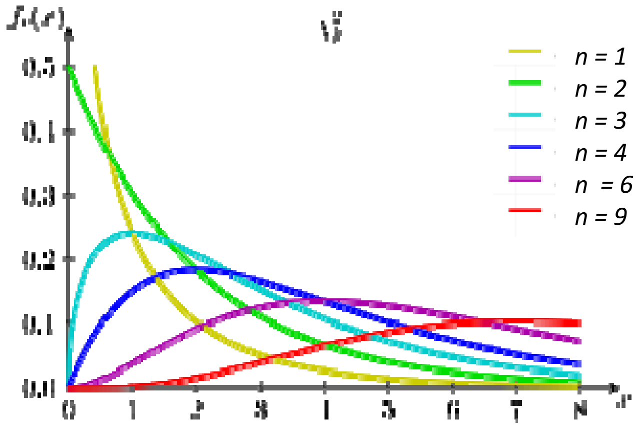
If $\mathrm{X} = \left|\mathrm{Y}\right|$, then $\mathrm{X} = \sqrt{\mathrm{X}_1^2+\mathrm{X}_2^2}$ is a Rician distributed r.v.

Some useful random variables

(7) **Chi-squared** (χ^2) **distributed random variable**

If X_i, $i=1,2,\ldots,n$ are i.i.d. Gaussian r.v. with $X_i \sim \mathcal{N}(0,\sigma^2)$, then $X = X_1^2 + X_2^2 \ldots + X_n^2$ is a Chi-square r.v. with n-degrees of freedom with pdf

$$f_X(x) = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)\sigma^n} x^{\frac{n}{2}-1} e^{\frac{-x}{2\sigma^2}}, x > 0$$



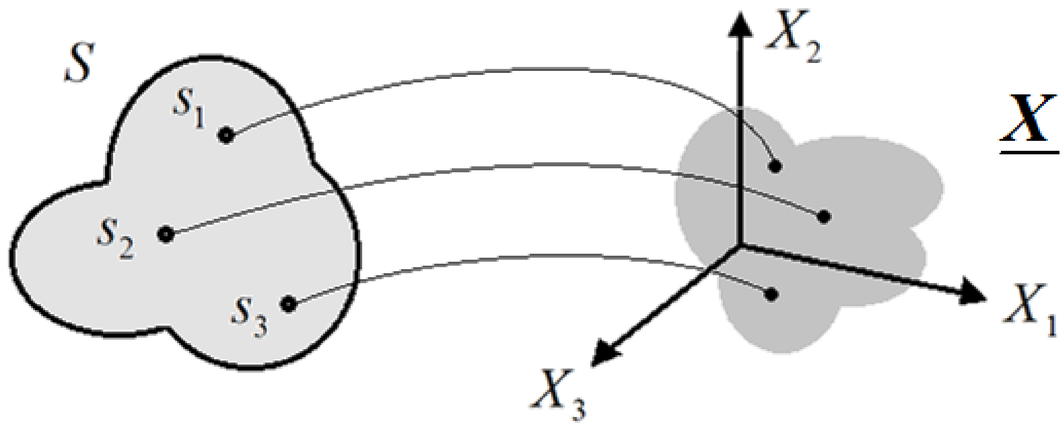
where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the Gamma function

Special case：$n = 2, X = X_1^2 + X_2^2$ is the power of complex Gaussian r.v. $Y = X_1 + jX_2$

Random Vector

$\underline{\boldsymbol{X}} = [X_1, X_2, \ldots, X_N]^{\mathrm{T}}$ is a finite collection of r.v.'s mapping from S to R^n

e.g. R^3



where X_i are real or complex random variables.

Q: Could you identify examples of application scenario using random vector in communication systems?

Gaussian random vector $\underline{\boldsymbol{X}}$

$\underline{\boldsymbol{X}}$ is joint Gaussian and is characterized by its joint distribution function with mean vector

$$\underline{\mu}_{\underline{\boldsymbol{X}}} = [\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_N}]^{\mathrm{T}}$$

and covariance matrix

$$\boldsymbol{C}_{\underline{\boldsymbol{X}}} = \mathrm{E}[(\underline{\boldsymbol{X}} - \underline{\mu}_{\underline{\boldsymbol{X}}})(\underline{\boldsymbol{X}} - \underline{\mu}_{\underline{\boldsymbol{X}}})^{*\mathrm{T}}] \in \mathbf{C}^{N \times N}$$

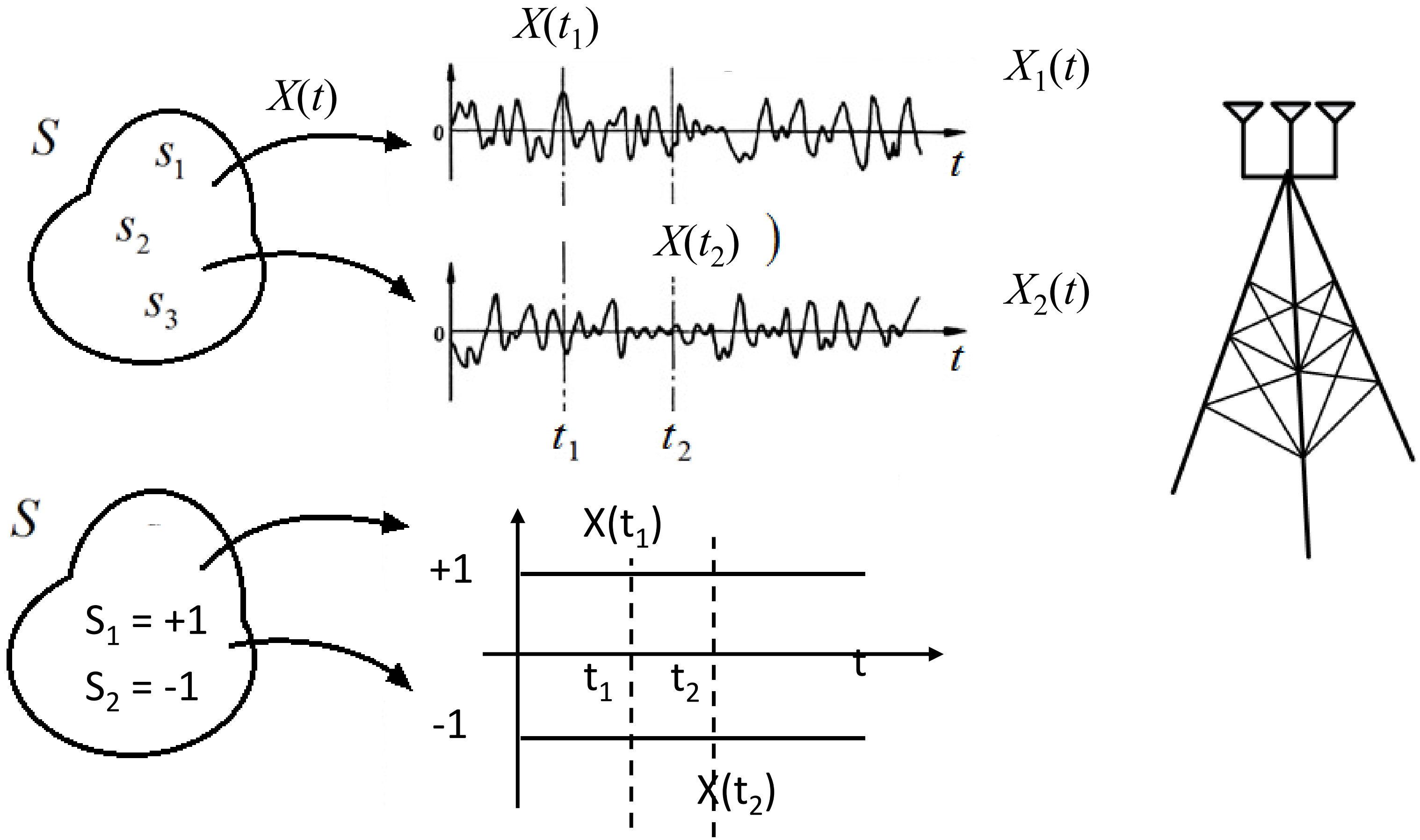
The joint distribution function

$$f_{\underline{\boldsymbol{X}}}(\underline{\boldsymbol{X}}) = \frac{1}{(2\pi)^{N/2} \cdot \left|\boldsymbol{C}_{\underline{\boldsymbol{X}}}\right|^{1/2}} \exp\left[-\frac{1}{2}(\underline{\boldsymbol{X}} - \underline{\mu}_{\underline{\boldsymbol{X}}})^{\mathrm{T}} \boldsymbol{C}_{\underline{\boldsymbol{X}}}^{-1} (\underline{\boldsymbol{X}} - \underline{\mu}_{\underline{\boldsymbol{X}}})\right]$$

where $\left|\boldsymbol{C}_{\underline{\boldsymbol{X}}}\right|$ = determinant of $\boldsymbol{C}_{\underline{\boldsymbol{X}}}$

Random Process (r. p.)

A r.p. $\{X(t)\}$ is an infinite collections of r.v.'s with mapping onto a functional space



For fixed t_0, $X(t) \equiv X(t_0)$ is a random variable

For fixed s_i , $X(t) \equiv X_i(t)$ is a sample function

A r.p. $X(t)$ is completely characterized by all finite order distributions

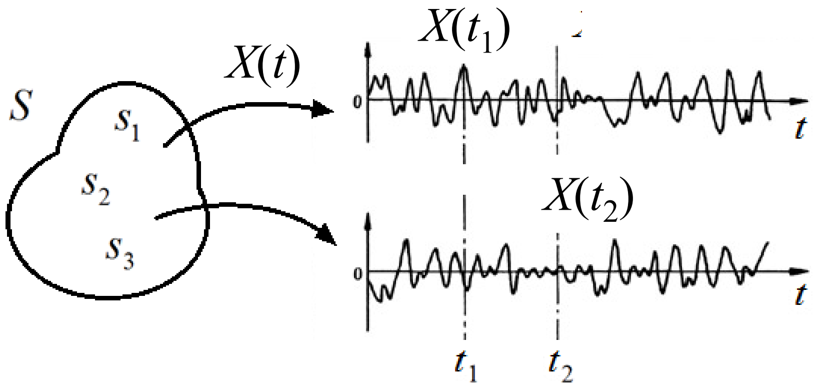
$$F_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n), \forall t_1, \ldots t_n, \forall n$$

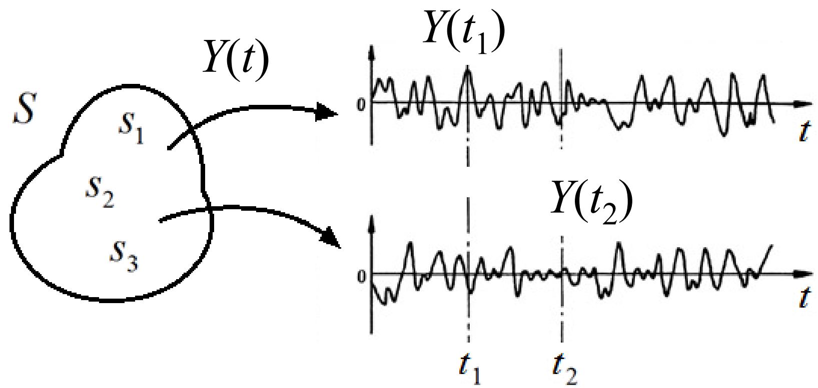
Statistical Properties

(1) Mean：$\mu_X(t) = E\left[X(t) \right] \Rightarrow$ a time function

(2) Autocorrelation： $R_X(t_1, t_2) = E\left[X(t_1)X^*(t_2)\right]$

(3) Cross-correlation： $R_{XY}(t_1, t_2) = \mathrm{E}\left[X(t_1)Y^*(t_2)\right]$





Remarks

We say two random processes $X(t)$ and $Y(t)$ are：

(1) Uncorrelated if

$$\mathrm{E}\left[X(t_1)Y(t_2)\right] = \mathrm{E}\left[X(t_1)\right]\mathrm{E}\left[Y(t_2)\right] = \mu_X(t_1)\mu_Y(t_2)\ ,\ \forall t_1, t_2$$

(2) Orthogonal if $\mathrm{E}\left[X(t_1)Y(t_2)\right] = 0, \forall t_1, t_2$

(3) Independent if

$$F_{XY}\left(x_1 \ldots x_n, y_1 \ldots y_n\right) = F_X\left(x_1 \ldots x_n\right) F_Y\left(y_1 \ldots y_n\right)$$

or $f_{XY}\left(x_1 \ldots x_n, y_1 \ldots y_n\right) = f_X\left(x_1 \ldots x_n\right) f_Y\left(y_1 \ldots y_n\right), \forall n$

Stationary random process

(1) **Strict Sense Stationary (SSS)**

For $\forall n$, the r.v.'s $\{X(t_1),\ldots,X(t_n)\}$ and $\{X(t_1+\tau),\ldots,X(t_n+\tau)\}$ are identically distributed, i.e.

$$f_{X(t_1)\ldots X(t_n)}(x_1\ldots x_n) = f_{X(t_1+\tau)\ldots X(t_n+\tau)}(x_1\ldots x_n), \forall n$$

✓ Q: What is the physical meaning of stationary?

A: A stationary process means that the observation of $X(t)$ statistics is shift-invariant.

✓ SSS concerns any high order of statistics and is generally difficult to prove.

Stationary random process

(2) **Wide Sense Stationary (WSS)**

A r.p. is said to be WSS if

(i) Mean: $\mu_X(t) = \text{Const.}$

(ii) Autocorrelation: a function of time difference $t_1 - t_2$ only.

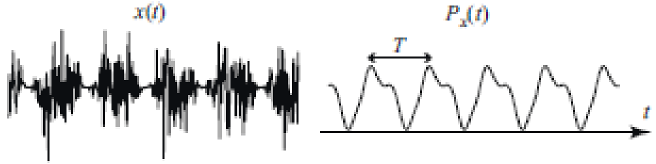
$$
\begin{aligned}
R_X(t_1, t_2) &= \mathrm{E}\left[X(t_1)X^*(t_2)\right] \\
&= \mathrm{E}\left[X(t_1+\tau)X^*(t_2+\tau)\right] \\
&= R_X(t_1 - t_2)
\end{aligned}
$$

✓ WSS concerns only the 1^{st} and the 2^{nd} orders of statistics.

✓ Q: Why should we care about WSS?

A: The signal amplitude and power are what we can observe typically.

Stationary random process



(3) **Cyclo-Stationary (Cyclo-S)**

$X(t)$ is cyclo-stationary with period T if

$$f_{X(t_1)\ldots X(t_n)}(x_1 \ldots x_n) = f_{X(t_1+T)\ldots X(t_n+T)}(x_1 \ldots x_n),$$

$$\forall n, \forall (t_1, \ldots, t_n)$$

✓ The statistical properties are periodic.

(4) **Wide-Sense Cyclo-Stationary (WSCS)**

A r.p. $X(t)$ is WSCS if the mean and autocorrelation functions are periodic.

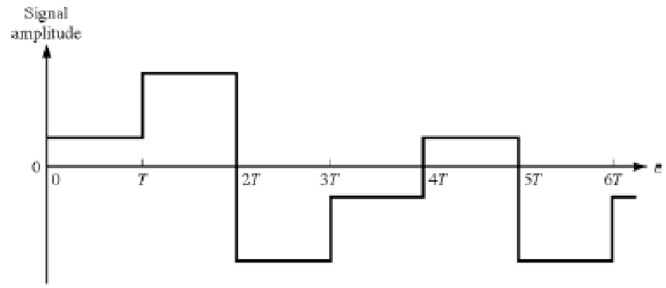
(i) $\mathrm{E}\left[X(t)\right] = \mathrm{E}\left[X(t+kT)\right]$, $\forall k \in \mathbf{Z}$

(ii) $R_X(t_1,t_2) = R_X(t_1 + kT, t_2 + kT), \forall k \in \mathbf{Z}, \forall t_1, t_2$

Ex. Let $s(t)$ be a finite energy signal pulse and the discrete r.p. b_k be WSS with mean μ_b and the autocorrelation depends on n only, $R_b(n) = E\left[b_{k+n} b_k^*\right]$

Then the transmitted waveform with period T is

$$X(t) = \sum_{k=-\infty}^{\infty} b_k s(t - kT)$$



which is WS cyclo-stationary.

Proof (i) The 1st order statistics

$$\mu_X(t) = \mathrm{E}\left[X(t)\right] = \sum_k \mathrm{E}\left[b_k\right] s(t - kT)$$

$$= \mu_b \sum_k s(t - kT) = \mu_b \sum_k s(t + T - kT)$$

$$= \mu_X(t + T)$$

proof

(ii) The 2nd order statistics

$$R_X(t_1,t_2) = \mathrm{E}\left[X(t_1)X^*(t_2)\right], \qquad \forall t_1,t_2$$

$$= \sum_k \sum_l \mathrm{E}\left[b_k b_l^*\right] s(t-kT)s^*(t-lT)$$

$$= \sum_k \sum_l R_b(k-l)\; s(t-kT)s^*(t-lT)$$

$$= \sum_k \sum_l R_b\left[(k-1)-(l-1)\right] s(t+T-kT)\; s^*(t+T-lT)$$

$$= R_X(t_1+T, t_2+T)$$

From (i) and (ii), $X(t)$ is wide sense cyclo-stationary.

Gaussian Random Process

Def. $X(t)$ is a real Gaussian r.p. if $\forall n, \forall(t_1,\ldots,t_n)$ $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly Gaussian random variables with joint PDF

$$f_X(x_1,\ldots x_n) = \frac{1}{(2\pi)^{n/2} \cdot |\Lambda|^{1/2}} \exp\left[-\frac{1}{2}(\underline{\boldsymbol{X}} - \underline{\boldsymbol{\mu}}_X)^{\mathrm{T}} \Lambda^{-1} (\underline{\boldsymbol{X}} - \underline{\boldsymbol{\mu}}_X)\right]$$

where

$$\underline{\boldsymbol{\mu}}_X = \mathrm{E}[\underline{\boldsymbol{X}}], \quad \boldsymbol{\Lambda} = \mathrm{E}\left[(\underline{\boldsymbol{X}} - \underline{\boldsymbol{\mu}}_X)(\underline{\boldsymbol{X}} - \underline{\boldsymbol{\mu}}_X)^{\mathrm{T}}\right] \in \mathrm{R}^{N\times N}$$

$$= E\left\{\begin{bmatrix} (X_1 - \mu_{X_1}) \\ \vdots \\ (X_N - \mu_{X_N}) \end{bmatrix} \left[(X_1 - \mu_{X_1}), \ldots, (X_N - \mu_{X_N})\right]\right\}$$

$\Rightarrow f_X(x_1,\ldots x_n)$ is characterized by $\underline{\boldsymbol{\mu}}_X$ and Λ, which is determined by the 2nd order statistics $\mu_X(t)$ and $R_X(t_1,t_2)$, $\forall t_1, t_2$

Gaussian Random Process

Def. $X(t)$ is a complex Gaussian r.p. if $\forall n$, $\forall t_1,\ldots,t_n$, $X(t_1),\ldots,X(t_n)$ are jointly Gaussian complex random variables with joint PDF

$$f_X(x_1,\ldots x_n)=\frac{1}{\pi^n\cdot|\Lambda|}\exp\left[-\frac{1}{2}(\underline{\boldsymbol{X}}-\underline{\mu}_X)^{\mathrm{H}}\Lambda^{-1}(\underline{\boldsymbol{X}}-\underline{\mu}_X)\right]$$

$\Rightarrow f_X(x_1,\ldots x_n)$ is characterized by

$$\mu_X(t)\ \text{and}\ R_{XX^*}(t_1,t_2)=\mathrm{E}\left[X(t_1)X^*(t_2)\right]$$

∗ Note：A WSS real Gaussian r.p. $X(t)$

i.e. $\mu_X(t)=\text{Const.}$ and $R_X(t_1,t_2)=R_X(t_1-t_2)$

then $X(t)$ is also a SSS

Ergodicity

Def. $X(t)$ is ergodic in the mean if

$$\mathrm{E}[X(t)] = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} X(t)\,dt = \mu_X$$

$\downarrow$ statistical average $\qquad$ $\downarrow$ time average

Def. $X(t)$ is ergodic in the autocorrelation if

$$\mathrm{E}[X(t+\tau)X^*(t)] = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \underbrace{X(t+\tau)X^*(t)\,dt}_{\text{time-average autocorrelation}}$$

∗ Note： With the assumption of ergodic r.p. , we can obtain the $\mu_X(t)$ and $R_X(t_1,t_2)$ and hence the PDF of a Gaussian r.p.

Transmission of a r.p. through a Linear Time-Invariant (LTI) system

- Passing a deterministic signal $x(t)$ through a LTI system $h(t)$

LTI

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

- Passing a r.p. X(t) through a LTI system $h(t)$

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

$$Y(t) = \int h(t-\tau)\; X(\tau)\, d\tau$$

✓The convolution relationship still holds.

Passing a WSS r.p. X(t) through a LTI system $h(t)$

Mean： $\mu_Y(t) = \mathrm{E}\left[Y(t)\right] = \mathrm{E}[\int h(\tau)\ X(t-\tau)\,d\tau]$

$$= \int h(\tau)\ \mathrm{E}\left[X(t-\tau)\right]d\tau$$

$$= \mu_X \int h(\tau)\ d\tau = \mu_X H(0) = const$$

Autocorrelation：

$$R_Y(t_1,t_2) = \mathrm{E}\left[Y(t_1)Y^*(t_2)\right]$$

$$= \mathrm{E}[\int h(\tau_1)X(t_1-\tau_1)\,d\tau_1 \cdot \int h^*(\tau_2)X^*(t_2-\tau_2)\,d\tau_2]$$

$$= \int_{-\infty}^{\infty} d\tau_1 h(\tau_1) \int_{-\infty}^{\infty} d\tau_2 h^*(\tau_2)\,\mathrm{E}\left[X(t_1-\tau_1)X^*(t_2-\tau_2)\right]$$

$$= \int_{-\infty}^{\infty} d\tau_1 h(\tau_1) \int_{-\infty}^{\infty} d\tau_2 h^*(\tau_2) R_X(t_1-t_2-\tau_1+\tau_2)$$

Passing a WSS r.p. $X(t)$ through a LTI system $h(t)$

Let $\tau = t_1 - t_2$

$$R_Y(t_1,t_2) = \int_{-\infty}^{\infty} d\tau_1 h(\tau_1) \int_{-\infty}^{\infty} d\tau_2 h^*(\tau_2) R_X(\tau - \tau_1 + \tau_2)$$

$$= R_X(\tau) * \underset{\substack{\downarrow \\ H(f)}}{h(\tau)} * \underset{\substack{\downarrow \\ H^*(f)}}{h^*(-\tau)} \Rightarrow R_Y(t_1,t_2) \text{ is } f_X \text{ of } \tau$$

$$= R_Y(\tau)$$

✓ The LTI output $Y(t)$ is still WSS !

Note If we define $S_X(f) = \mathscr{F}\{R_X(\tau)\}$

$$S_Y(f) = \mathscr{F}\{R_Y(\tau)\}$$

then $S_Y(f) = S_X(f)\left|H(f)\right|^2$

$S_X(f)$ is called the power spectral density of $X(t)$.

Spectral Analysis

Define：The power spectral density of $X(t)$

is $S(f)=\int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau$

(known as Wiener-Khinchin Theorem)

Properties of $S(f)$

(1) $S_X(0)=\int_{-\infty}^{\infty} R_X(\tau)d\tau \Rightarrow$ d.c. power of $X(t)$

(2) Average power $= \mathrm{E}\left[X^2(t)\right]= R_X(0)$

$$=\int_{-\infty}^{\infty} S_X(f)\,df \quad \rightarrow \text{Area of PSD}$$

where $R_X(\tau)=\int_{-\infty}^{\infty} S_X(f)e^{j2\pi f\tau}df$

Properties of $S(f)$

(3) Non-negative $\quad S_X(f) \geq 0,\ \forall f$

(4) If $X(t)$ is real valued r.p., then $S(f) = S(-f)$

proof

Since $R_X(\tau) = \mathrm{E}\left[X(t_1)X(t_2)\right]$ (where $t_1 - t_2 = \tau$ by definition)

$$R_X(-\tau) = \mathrm{E}\left[X(t_2)X(t_1)\right] \quad (t_2 - t_1 = -\tau)$$

$$R_X(\tau) = R_X(-\tau),$$

so $S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau) e^{j2\pi f\tau} d\tau = \int_{-\infty}^{\infty} R_X(-\tau) e^{j2\pi f\tau} d\tau$

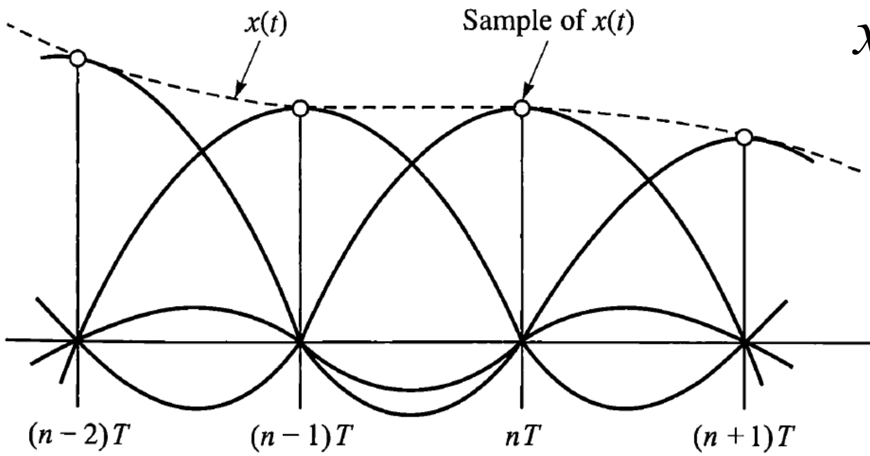
Let $\quad \tau' = -\tau \Rightarrow S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau')\ e^{-j2\pi f\tau'} d\tau' = S_X(f)$

Serial Expansion of Random Process

(1) Sampling Theorem (for deterministic signals)

Let the deterministic real signal $x(t)$ be bandlimited with bandwidth W then $x(t)$ can be represented by the discrete samples $x[n] = x(nT)$

at **Nyquist rate** $\frac{1}{T} \geq 2W$, i.e.



$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\left\{ (t - nT)/T \right\}$$

$$= \sum_{n} x\left(\frac{n}{2W}\right) \mathrm{sinc}\left\{ \left(t - \frac{n}{2W}\right) 2W \right\}$$

$$= \sum_{n} x[n]\,\phi_n(t)$$

Q：Do we have an equivalent sampling theorem for the random process?

- ✓ Can we represent the r.p. with a set of random variables?

For a bandlimited r.p. X(t) with $S_X(f) = 0, |f| \geq W$

$$\underset{\downarrow \atop \text{r.p.}}{X(t)} = \sum_n \underbrace{X(nT)}_{\text{Seq. of r.v.}} \underbrace{\operatorname{sinc}\left(2W\left(t - \frac{n}{2W}\right)\right)}_{\phi_n(t)}$$

$$= \sum_n X_n \phi_n(t)$$

$\Rightarrow$ There are different ways to expand $X(t)$

Serial Expansion of Random Process

(2) Karhunen-Loève (K-L) Expansion (1955)

If $\{\phi_n(t)\}$ are the eigen functions of COV function of $X(t)$, then we have $\{X_n\}$ to be mutually uncorrelated

$\Rightarrow$ Use the least number of r.v. $\{X_n\}$ to represent $X(t)$

$$\text{K - L expansion : } X(t) = \sum_{n=1}^{\infty} X_n \phi_n(t),\ a < t < b$$

$$\text{where } X_n = \langle X(t), \varphi_n(t) \rangle = \int_a^b X(\mathrm{t}) \varphi_n^*(t)\, dt$$

$$\int_a^b |\varphi_n(t)|^2\, dt = 1\ ,\ \int_a^b \varphi_n(t) \varphi_m^*(t)\, dt = \begin{cases} 1\ ,\ n = m \\ 0,\ n \neq m \end{cases}$$

The basis $\varphi_n(t)$ are eigen functions of the cov function,

$$\int_a^b C_X(t_1,t_2)\ \varphi_n(t_2)\,dt_2 = \lambda_n \varphi_n(t)\ ,\ a < t < b$$

and $C_X(t_1,t_2) = \mathrm{E}\left\{ \left[X(t_1) - \mu_X(t_1)\right] \left[X(t_2) - \mu_X(t_2)\right]^* \right\}$

$$= R_X(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2)$$

Note: It can be shown that $\{X_n\}$ are uncorrelated,

i.e. $\mathrm{COV}[X_n, X_m] = E\left[\left(X_n - \mu_{X_n}\right)\left(X_m - \mu_{X_m}\right)^*\right] = \begin{cases} \lambda_n\ , if\ n = m \\ 0\ , if\ n \neq m \end{cases}$

It can be also shown that the $C_X(t_1, t_2)$ can be decomposed as

$$C_X(t_1,t_2) = \sum_{n=1}^{\infty} \lambda_n \phi_n(t_1)\phi_n^*(t_2),\ a < t_1, t_2 < b$$

This is called Mercer's Theorem

HW#1
(available on eLearn)

Due: 10/14/2021