

# **COM 5120**

## **Communications Theory**

# **Chapter 3**

## **Digital Modulation**

**Jen-Ming Wu**  
**Inst. of Communications Engineering**  
**Dept. of Electrical Engineering**  
**National Tsing Hua University**

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# Outline

- **Signal Space Representation of Waveform**
- **Signaling Schemes with Modulation**
  - 1. Modulation without Memory**
    - M-PAM, M-PSK, M-QAM
    - M-ary Orthogonal Signaling (M-FSK)
    - Bi-orthogonal Signaling
    - Simplex Signaling
  - 2. Modulation with Memory**
    - Linear Modulation with Memory
    - Non-linear Modulation with Memory
- **Power Spectral Density of Digital Modulation Signal**

# Signal Space Representation of Waveform

Given signal waveforms :  $\{S_1(t), S_2(t), \dots, S_M(t)\}$

Q: What is the necessary signal space to represent the waveforms?

With the basis functions :

$$\{\varphi_1(t), \varphi_2(t), \dots, \varphi_N(t)\} \quad N < M$$

Then the transmission signal

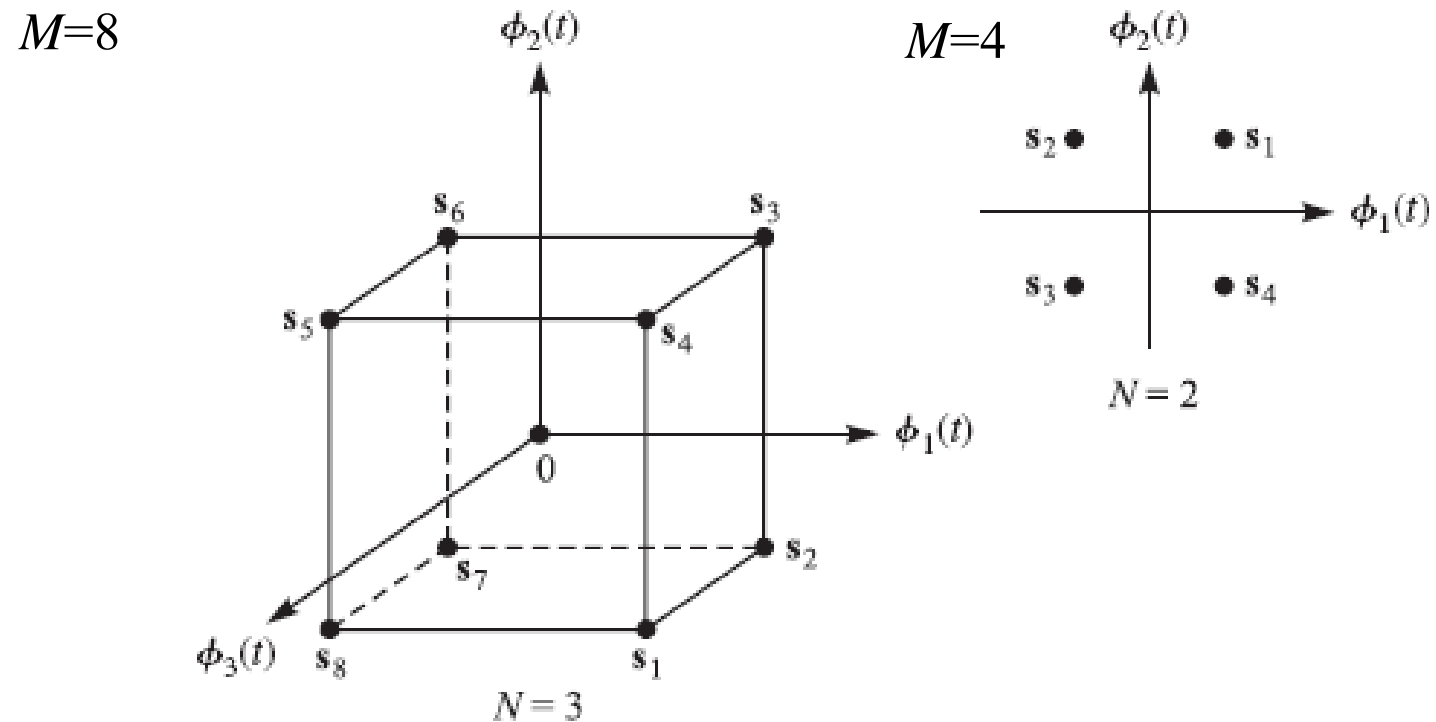
$$S_i(t) = \sum_{j=1}^N S_{ij} \varphi_j(t), \quad i = 1, 2, \dots, M \quad 0 \leq t \leq T$$

where  $\{\phi_j(t)\}$  are orthonormal, i.e.,

$$\int_0^T \phi_i(t) \phi_j(t) dt = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \quad S_{ij} = \int_0^T S_i(t) \phi_j(t) dt$$

$\Rightarrow \{\phi_j(t)\}$  spans the signal space of  $S_i(t)$

# Signal Space Representation of Waveform

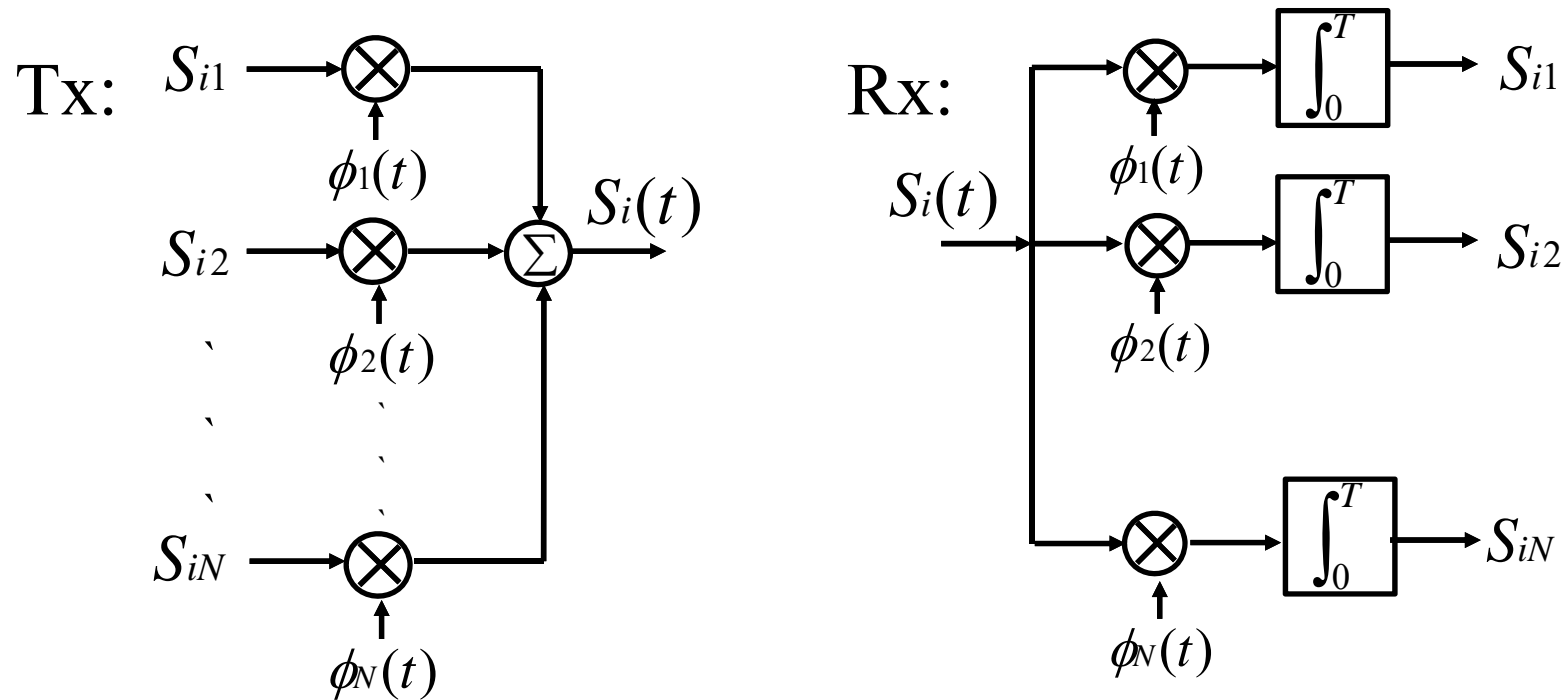


**FIGURE 3.2–10**

Signal space diagrams for signals generated from binary codes.

# How does the signal space affect the system?

- Signal space defines the Tx/Rx architecture



The continuous  $S_i(t), i = 1, \dots, M$  can be *fully represented* by the *signal vector*.

$$\underline{S}_i = [S_{i1}, S_{i2}, \dots, S_{iN}]^T, i = 1, 2, \dots, M$$

# Signal Vectorization

- Signal Energy

$$\|\underline{S_i}\|^2 = \underline{S_i}^T \underline{S_i} = \sum_{j=1}^N S_{ij}^2 = \int_0^T S_i(t)^2 dt \text{ (Signal energy)}$$

- Signal Inner Product

$$\int_0^T S_i(t) S_k(t) dt = \int_0^T \left( \sum_j S_{ij} \phi_j(t) \right) \left( \sum_{j=1}^N S_{kj}(t) \phi_j(t) \right) dt = \underline{S_i}^T \underline{S_k}$$

$$\text{when } k = i, E_i = \int_0^T S_i^2(t) dt = \|\underline{S_i}\|^2$$

- Signal Distance

$$\int_0^T (S_i(t) - S_k(t))^2 dt = \|\underline{S_i} - \underline{S_k}\|^2 = d_{ik}^2$$

- The continuous time domain signal operation can be replaced by the vector operation in linear algebra.

Q: How to find the basis  $\{\phi_j(t)\}$  from the waveforms  $\{S_i(t)\}$ ?

# Orthogonal Expansion of Signals

- Given a set of signals  $\{S_1(t), S_2(t), \dots, S_M(t)\}$ , find the set of orthogonal basis.

✓ **Gram-Schmidt Orthogonalization Procedure :**

$$\phi_1(t) = \frac{S_1(t)}{\sqrt{E_1}} \Rightarrow S_1(t) = \sqrt{E_1} \phi_1(t)$$

$$r_2(t) = S_2(t) - C_{21} \phi_1(t), \text{ where } C_{21} = \int_0^T S_2(t) \phi_1(t) dt$$

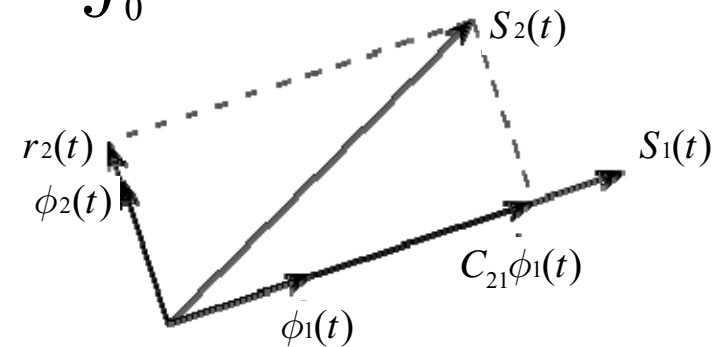
$$\phi_2(t) = \frac{r_2(t)}{\sqrt{E_2}}, E_2 = \int_0^T r_2^2(t) dt$$

• • •

$$r_i(t) = S_i(t) - \sum_{j=1}^{i-1} C_{ij} \phi_j(t), C_{ij} = \int_0^T S_i(t) \phi_j(t) dt$$

$$\phi_i(t) = \frac{r_i(t)}{\sqrt{E_i}}, \quad i = 1, \dots, M$$

$\Rightarrow$  The set of orthogonal basis is  $\{\phi_1(t), \phi_2(t), \dots, \phi_N(t)\}$   $N \leq M$



# Signaling Schemes with Modulation

## 1. Modulation without Memory

- M-PAM, M-PSK, M-QAM
- M-ary Orthogonal Signaling (M-FSK)
- Bi-orthogonal Signaling
- Simplex Signaling

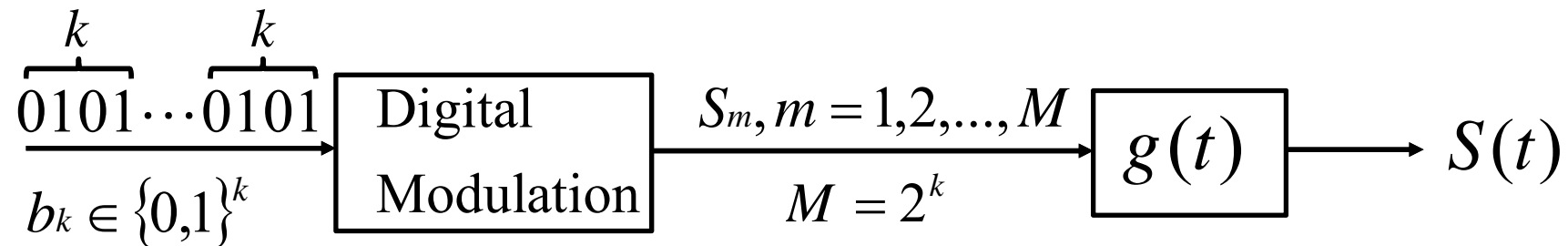
## 2. Modulation with Memory

- Linear Modulation with Memory
- Non-linear Modulation with Memory



# Introduction to Digital Modulation

- The mapping of digital sequence into a set of waveforms.



$$\text{Bit rate} = \frac{1}{T_b}$$

$$S(t) = \sum_i s_m g(t - iT), \quad s_m \in \{s_1, s_2, \dots, s_M\} \\ 0 \leq t \leq T$$

where  $g(t)$  is the pulse shaping function.

- The modulation process is to embed the message info into **amplitude/phase/frequency** of the symbol  $S_m$ . The signal waveform bears the information of the bit stream.

$$R_s = \text{symbol rate} = \frac{1}{kT_b} = \frac{1}{T}$$

# The modulation schemes can be classified

## (1) By memory

- With memory :

The mapping of  $b_n \rightarrow S_m$  depends on the past symbols of  $S_{m-i}$

Ex: DPSK, MSK, CPM, ...,etc

- Without memory(Memoryless):

The mapping of  $b_n \rightarrow S_m$  depends on the current set of  $b_n$  and  $s_m$  only.

Ex: PAM, BPSK, QAM, MPSK

(2) By linearity

- Linear modulation ( Ex: PAM)

Mapping satisfy the superposition principle.

If  $a_1 \rightarrow b_1$  and  $a_2 \rightarrow b_2$ , then  $a_1 + a_2 \rightarrow b_1 + b_2$

- Non-linear modulation (Ex: M-PSK, M-FSK)

(3) By coherence

- Coherent modulation ( Ex: M-PSK, M-QAM)

- Non-coherent modulation (Ex: DPSK)

## Memoryless Modulation

- Pulse-Amplitude Modulation(PAM)

The M-ary PAM signal.

$$S_m(t) = A_m g(t) \cos(2\pi f_c t) = \operatorname{Re} \left\{ A_m g(t) e^{j2\pi f_c t} \right\}, \quad \begin{matrix} m = 1, 2, \dots, M \\ 0 \leq t \leq T \end{matrix}$$

where  $g(t)$  is the pulse shaping function,

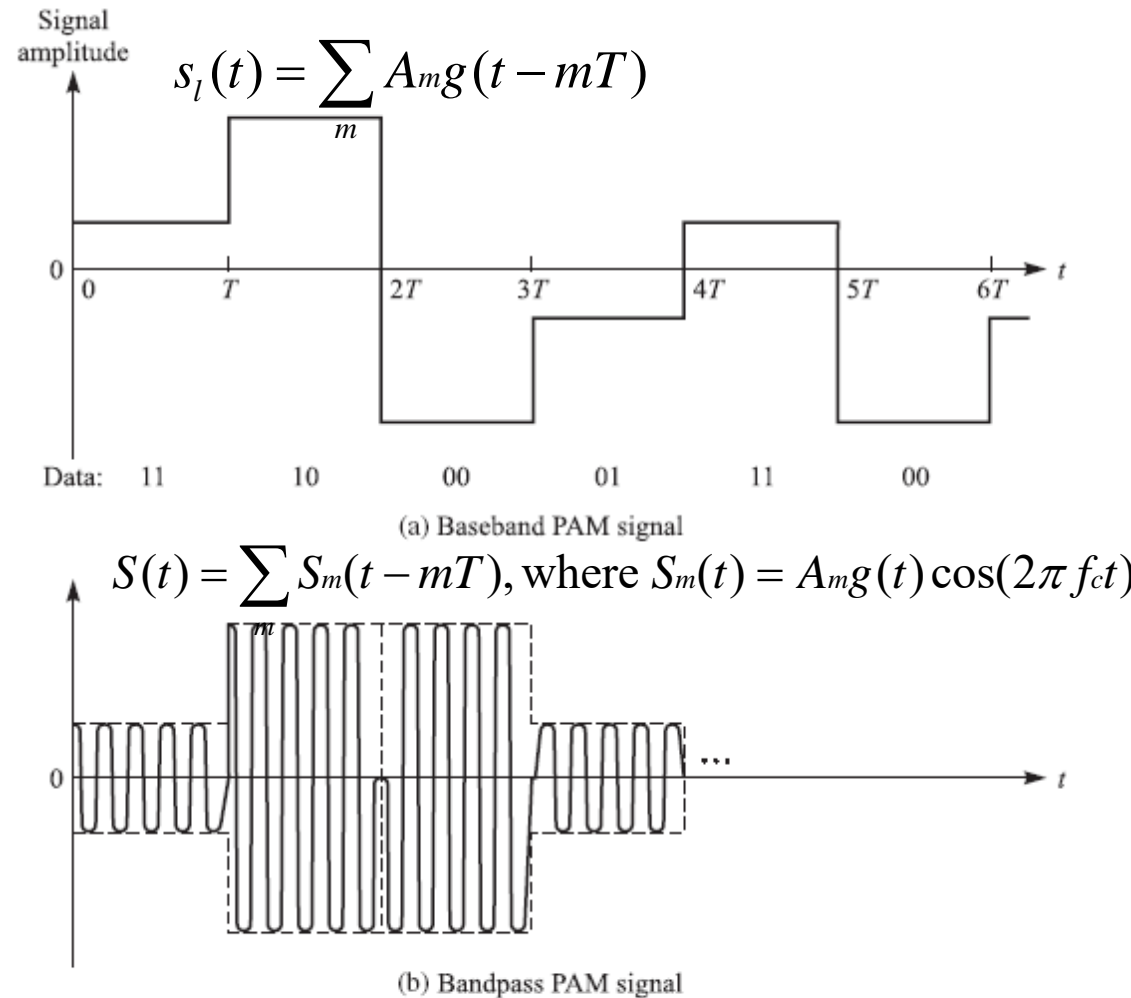
$$A_m = (2m - 1 - M), m = 1, 2, \dots, M = \{\pm 1, \pm 3, \dots, \pm(M - 1)\}$$

Example :

$$k = 1, M = 2, A_m \in \{\pm 1\} \Rightarrow 2 - PAM$$

$$k = 2, M = 4, A_m \in \{\pm 1, \pm 3\} \Rightarrow 4 - PAM$$

# Signaling of M-PAM



**FIGURE 3.2-2**

Example of (a) baseband and (b) carrier-modulated PAM signals.

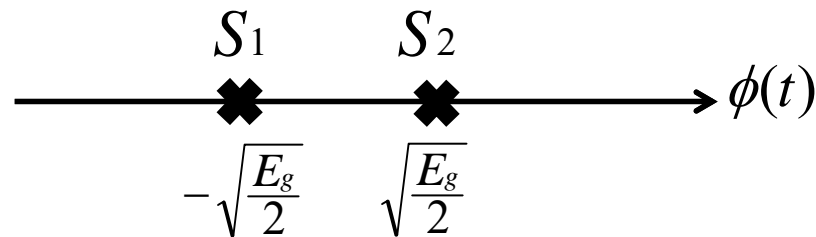
$S_m(t)$  is a one-dimension signal with

$$S_m(t) = A_m g(t) \cos(2\pi f_c t) = S_m \phi(t)$$

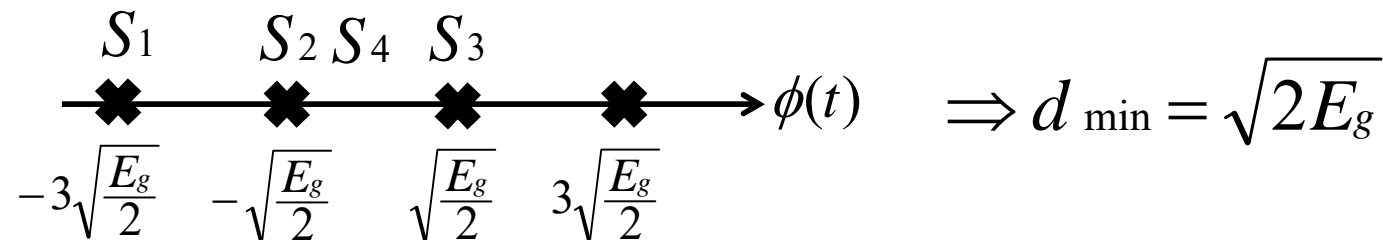
Let  $E_g = \int_0^T |g(t)|^2 dt \equiv \|g(t)\|^2$ ,  $\phi(t) = \frac{S_m(t)}{|S_m(t)|} = \sqrt{\frac{2}{E_g}} g(t) \cos(2\pi f_c t)$ ,

$$\|\phi(t)\|^2 = \int_0^T \phi^2(t) dt = 1, S_m = A_m \sqrt{\frac{E_g}{2}}, m = 1, 2, \dots, M$$

Example :  $M = 2, A_m = \pm 1$

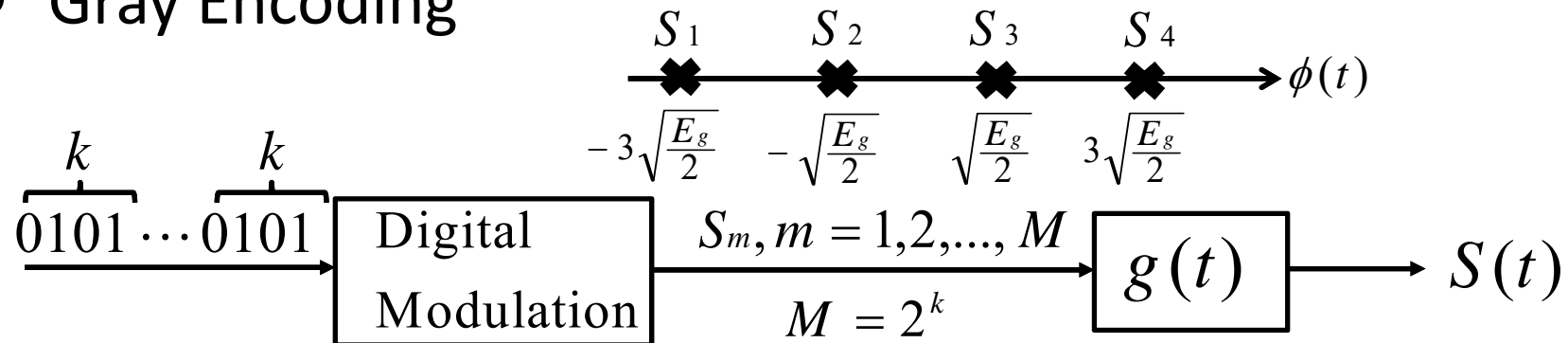


$M = 4, A_m = \pm 1, \pm 3$



# Mapping between bits and symbols

## ● Gray Encoding



Encoding A : 00      01      10      11

Encoding B : 00      01      11      10

Which encoding is better?

- Rule : Adjacent signal symbol point differ by only 1 bit.
- Reason : When error occurs in a symbol, it most likely migrates to nearby points.
- With Gray Encoding, we could minimize the error bits when symbol error occurs.
- The symbol error rate depends on the  $d_{\min}$ .

# Phase Modulation (or Phase Shift Keying, PSK)

The M-ary PSK signal

$$\begin{aligned} S_m(t) &= g(t) \cos(2\pi f_c t + \theta_m) \\ &= g(t) \cos(\theta_m) \cos(2\pi f_c t) - g(t) \sin(\theta_m) \sin(2\pi f_c t) \end{aligned}$$

where  $\theta_m = \frac{m-1}{M} 2\pi, m = 1, 2, \dots, M$

Equal signal energy :  $E_m = \int_0^T S_m^2(t) dt = \frac{1}{2} \int_0^T g^2(t) dt = \frac{E_g}{2}$

The orthogonal basis are :

$$\phi_1(t) = \sqrt{\frac{2}{E_g}} g(t) \cos(2\pi f_c t)$$

$$\phi_2(t) = -\sqrt{\frac{2}{E_g}} g(t) \sin(2\pi f_c t)$$

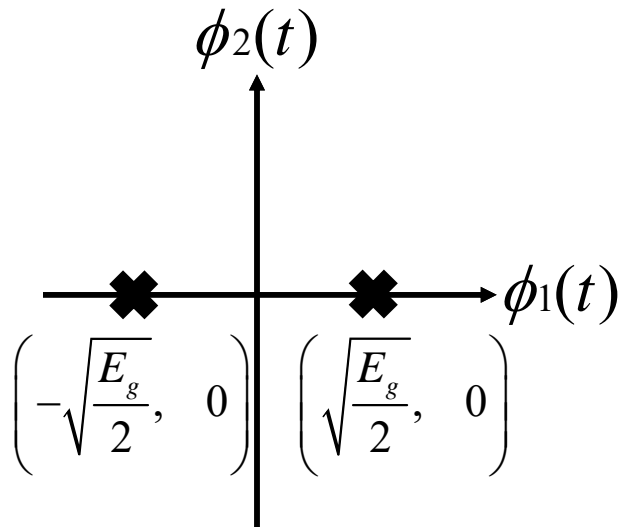
$$\Rightarrow S_m(t) = \sqrt{\frac{E_g}{2}} \cos(\theta_m) \phi_1(t) + \sqrt{\frac{E_g}{2}} \sin(\theta_m) \phi_2(t)$$

$$\underline{S}_m = \left[ \sqrt{\frac{E_g}{2}} \cos(\theta_m), \sqrt{\frac{E_g}{2}} \sin(\theta_m) \right]^T, m = 1, 2, \dots, M, N = 2$$

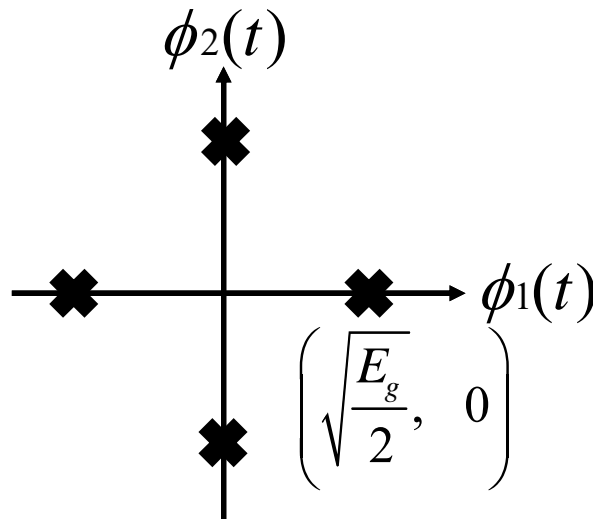


- Example :

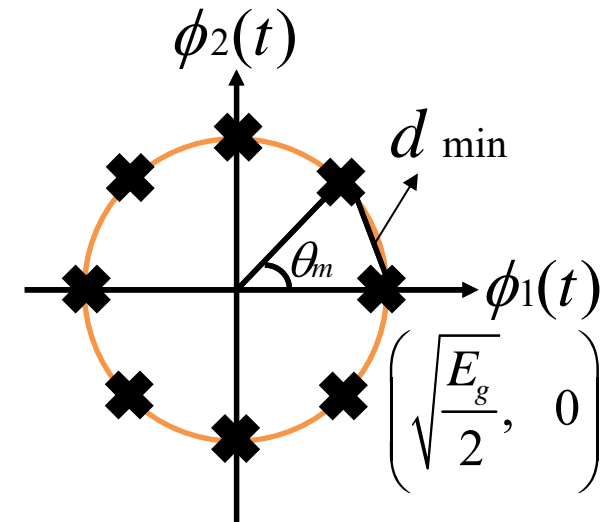
M=2, BPSK



M=4, QPSK



M=8-PSK



$$\Rightarrow d_{\min} = 2\sqrt{\frac{E_g}{2}} \sin\left(\frac{\theta_m}{2}\right) = \sqrt{2E_g \sin^2\left(\frac{\pi}{M}\right)} = \sqrt{E_g \left(1 - \cos \frac{2\pi}{M}\right)}$$

- M-ary Quadrature Amplitude Modulation(QAM)

Consider a 2-D PAM with orthogonal basis.

$$\phi_1(t) = \sqrt{\frac{2}{E_g}} g(t) \cos(2\pi f_c t), 0 \leq t \leq T \Rightarrow \text{In-phase}$$

$$\phi_2(t) = -\sqrt{\frac{2}{E_g}} g(t) \sin(2\pi f_c t), 0 \leq t \leq T \Rightarrow \text{Quadrature-phase}$$

$$S_m(t) = A_{mI}g(t) \cos(2\pi f_c t) - A_{mQ}g(t) \sin(2\pi f_c t)$$

$$= V_m g(t) \cos(2\pi f_c t + \theta_m), \text{ where } A_{mI}, A_{mQ} \in \{\pm 1, \pm 3, \dots, \pm(M-1)\}$$

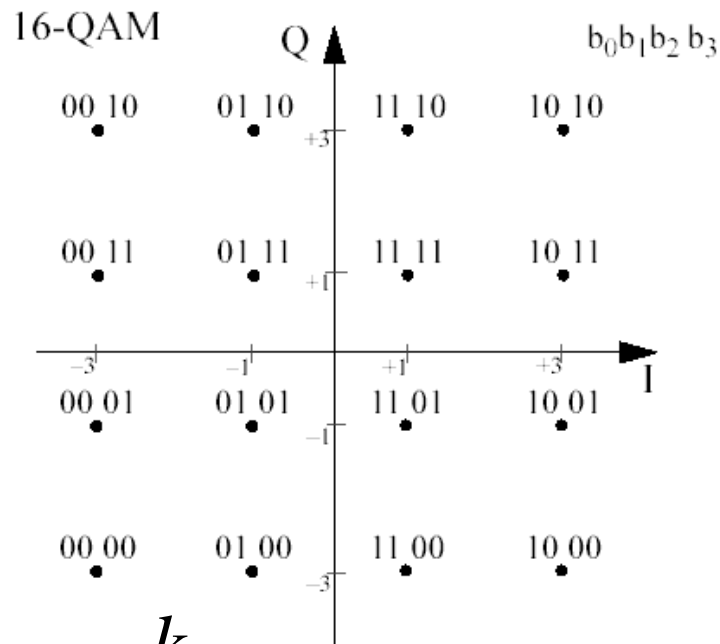
$$V_m = (A_{mI}^2 + A_{mQ}^2)^{\frac{1}{2}}, \quad \theta_m = \tan^{-1}\left(\frac{A_{mQ}}{A_{mI}}\right)$$

On the basis :

$$S_m(t) = S_{mI}\phi_1(t) + S_{mQ}\phi_2(t)$$

$$\text{with } \underline{S}_m = \begin{bmatrix} S_{mI} & S_{mQ} \end{bmatrix}^T = \begin{bmatrix} \sqrt{\frac{E_g}{2}} A_{mI} & \sqrt{\frac{E_g}{2}} A_{mQ} \end{bmatrix}^T, m = 1, 2, \dots, M$$

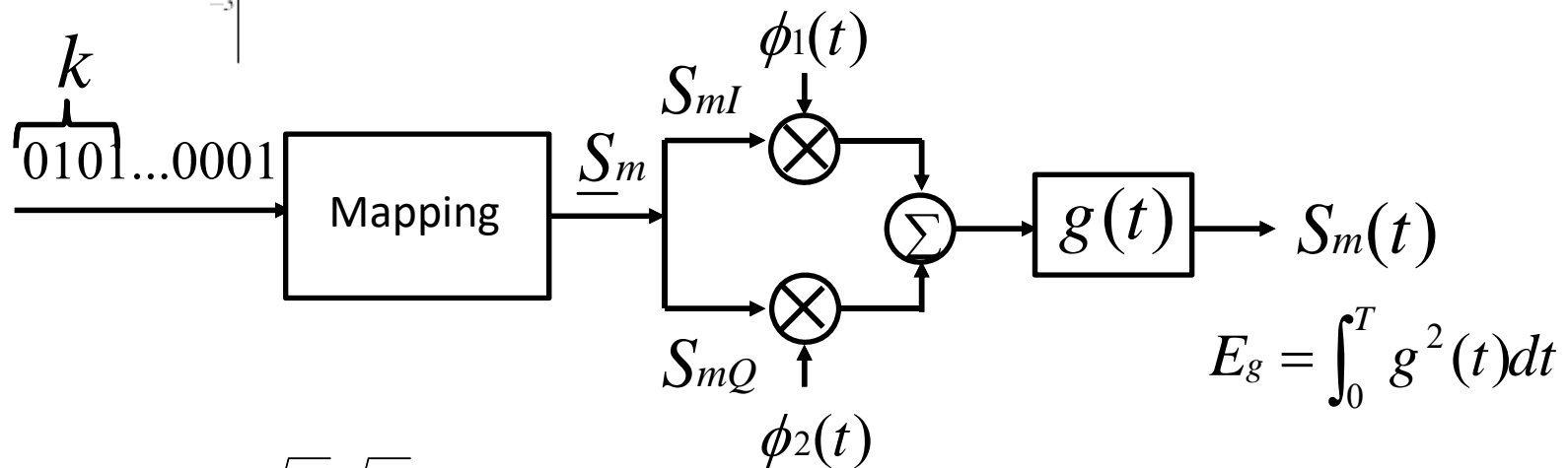
## Example M=16, 16-QAM



$$\underline{S}_m = \left[ \sqrt{\frac{E_g}{2}} A_{mI}, \sqrt{\frac{E_g}{2}} A_{mQ} \right]^T, m = 1, 2, \dots, M$$

where  $A_{mI}, A_{mQ} \in \{\pm 1, \pm 3, \dots, \pm(M-1)\}$

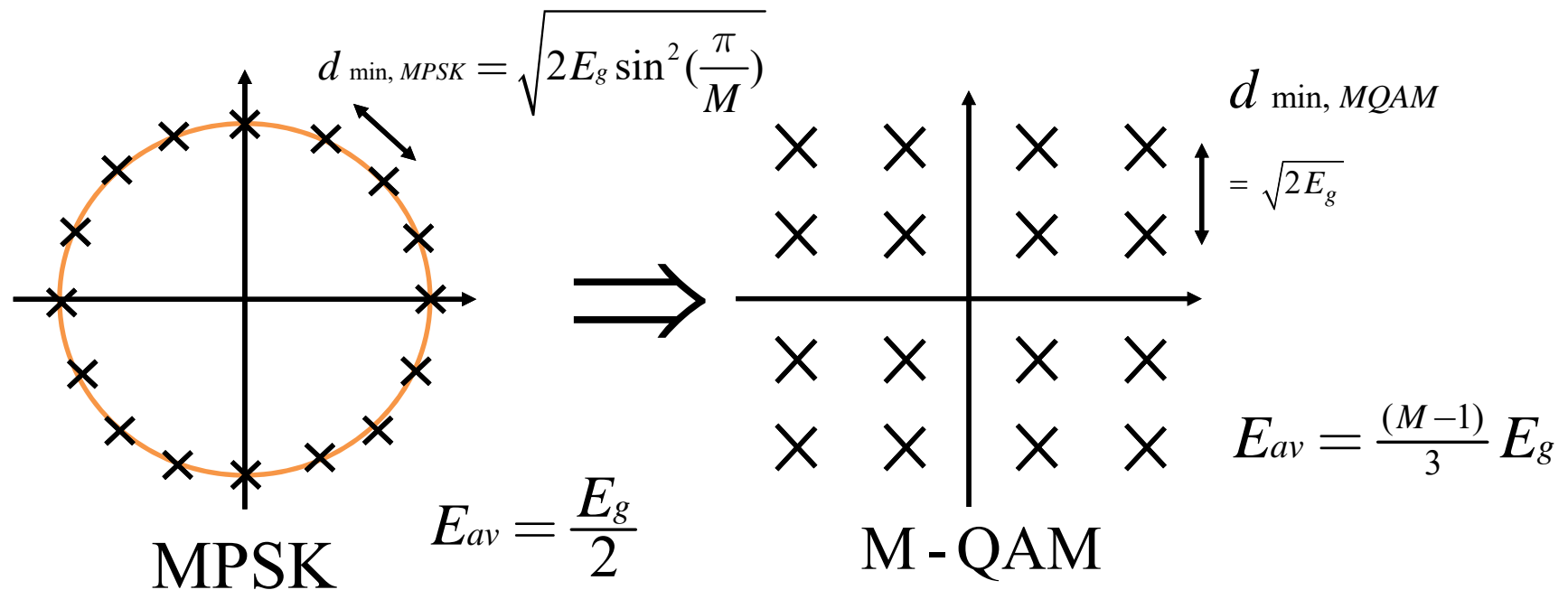
$$d_{\min} = \sqrt{2E_g}$$



$$E_{av} = \frac{1}{M} \frac{E_g}{2} \sum_{m=1}^{\sqrt{M}} \sum_{m=1}^{\sqrt{M}} (A_{mI}^2 + A_{mQ}^2) = \frac{E_g}{2M} \frac{2M(M-1)}{3} = \frac{(M-1)}{3} E_g$$

## Notes on QAM v.s. PAM and MPSK

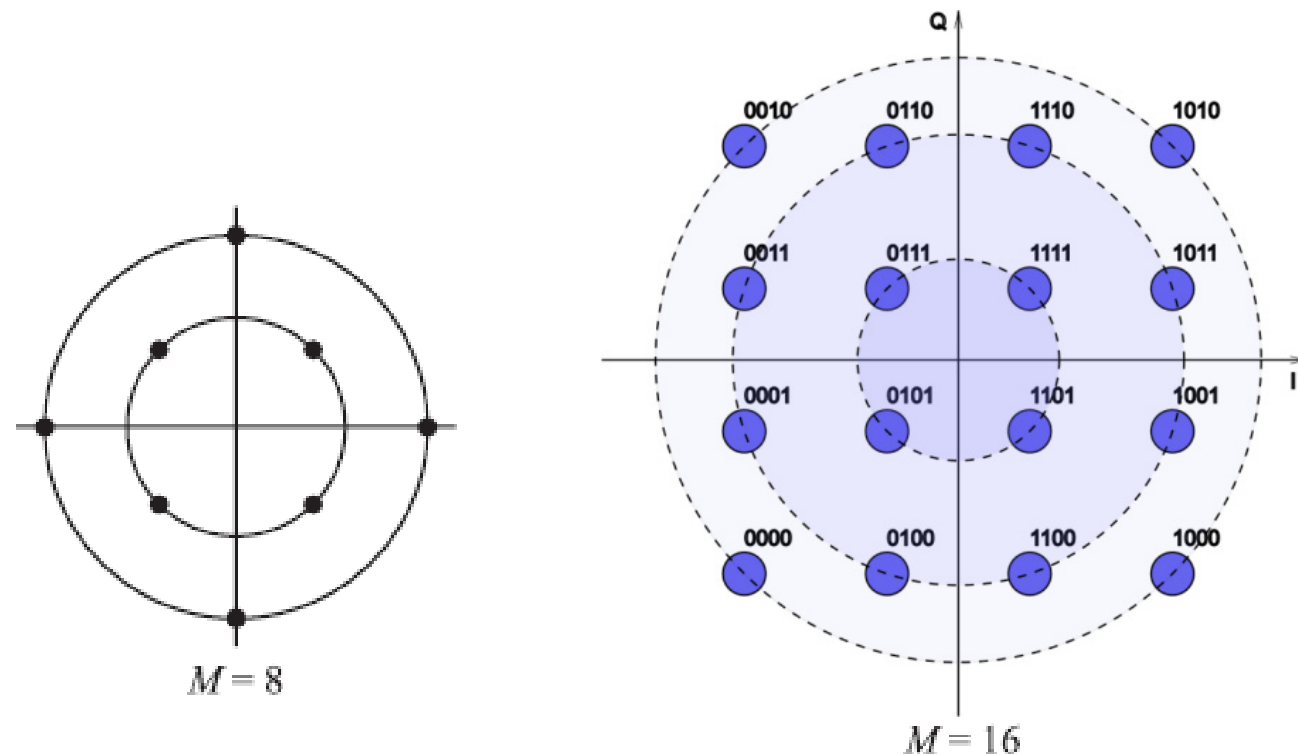
- (1) QAM improves the BW efficiency of PAM.
- (2) Given the same  $E_g$ , QAM gives better utilization (larger  $d_{\min}$ ) in the constellation space than MPSK.



$\Rightarrow$  QAM has larger  $d_{\min}$  and hence better error performance.

Q: Given the same  $E_{av}$ , please compare the  $d_{\min}$  of MPSK and M-QAM?

# QAM can be viewed as combined PAM-PSK Modulation



**FIGURE 3.2–4**  
Examples of combined PAM-PSK constellations.

# Multi-dimensional Signals

## ● Orthogonal multidimensional basis

Example : M-ary Frequency Shift Keying(MFSK)

$$S_m(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + 2\pi f_m t), \quad m = 1, 2, \dots, M \\ 0 \leq t \leq T$$

The orthogonal basis :

$$\phi_m(t) = \frac{S_m(t)}{\sqrt{E}} = \sqrt{\frac{2}{T}} \cos(2\pi(f_c + f_m)t), \quad m = 1, 2, \dots, M \\ 0 \leq t \leq T$$

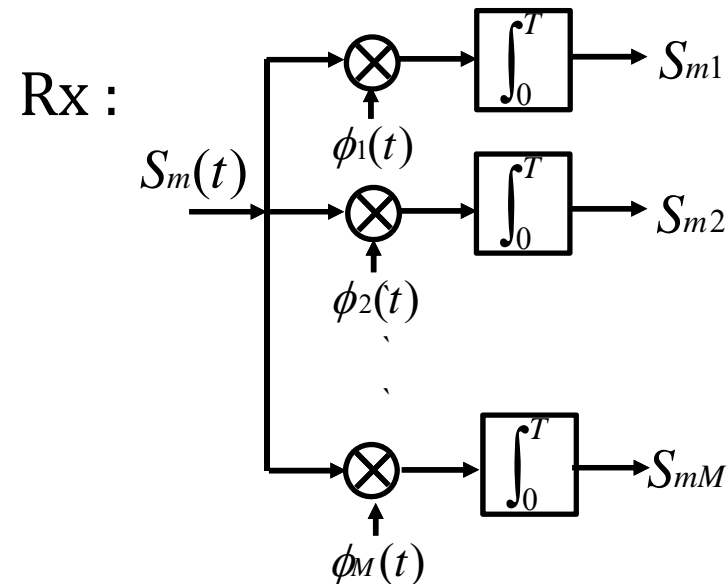
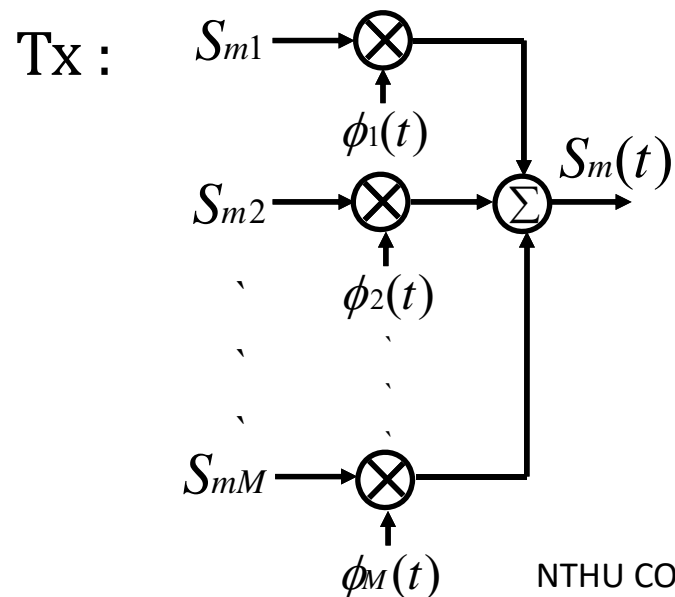
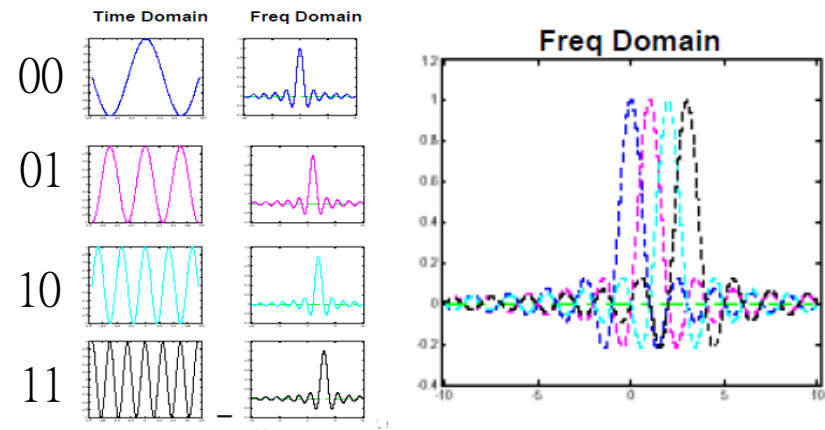
$$\begin{aligned} \int_0^T \phi_m(t) \phi_n(t) dt &= \frac{2}{T} \int_0^T \cos[2\pi(f_c + f_m)t] \cos[2\pi(f_c + f_n)t] dt \\ &= \frac{1}{T} \int_0^T \cos[2\pi(2f_c + f_m + f_n)t] dt + \frac{1}{T} \int_0^T \cos[2\pi(f_m - f_n)t] dt \\ &= \frac{A_1}{T} \sin[2\pi(2f_c + f_m + f_n)T] + \frac{A_2}{T} \sin[2\pi(f_m - f_n)T] \\ &= 0, \quad (\text{Has to be 0 to satisfy orthogonality}) \end{aligned}$$

Sufficient solution:

$$\Rightarrow \begin{cases} (1) 2f_c + f_m + f_n = \frac{n_c}{2T} , & n_c \in \text{integer} \\ (2) \Delta f = |f_m - f_n| = \frac{k}{2T} , & k \in \text{integer} , \end{cases} \Rightarrow \Delta f_{\min} = \frac{1}{2T}$$

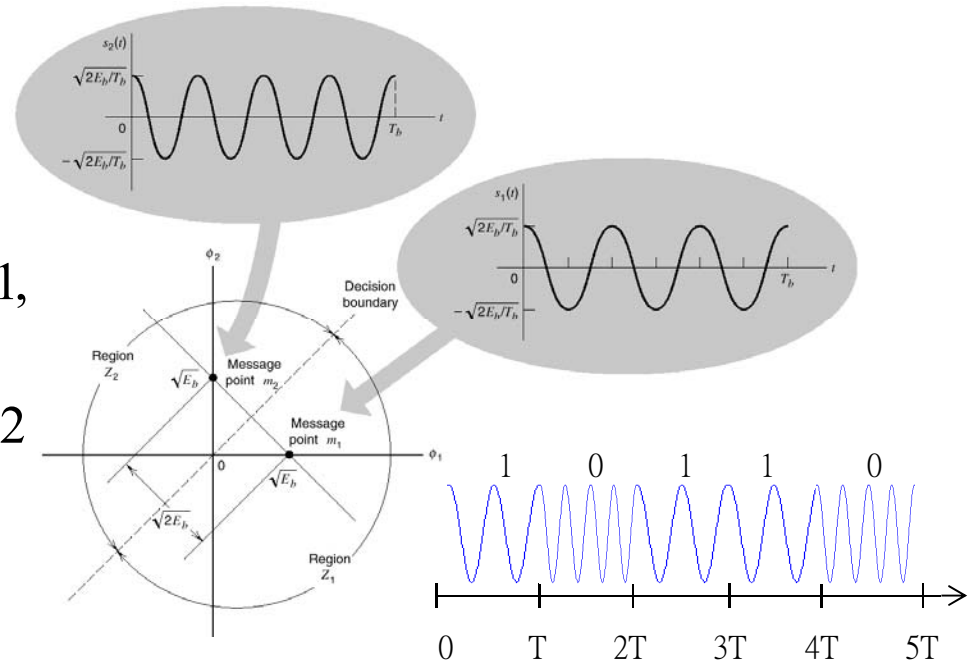
$$S_{mn} = \int_0^T S_m(t) \phi_n(t) dt , m = 1, 2, \dots, M$$

$$= \begin{cases} \sqrt{E} , & m = n \\ 0 , & m \neq n \end{cases}$$



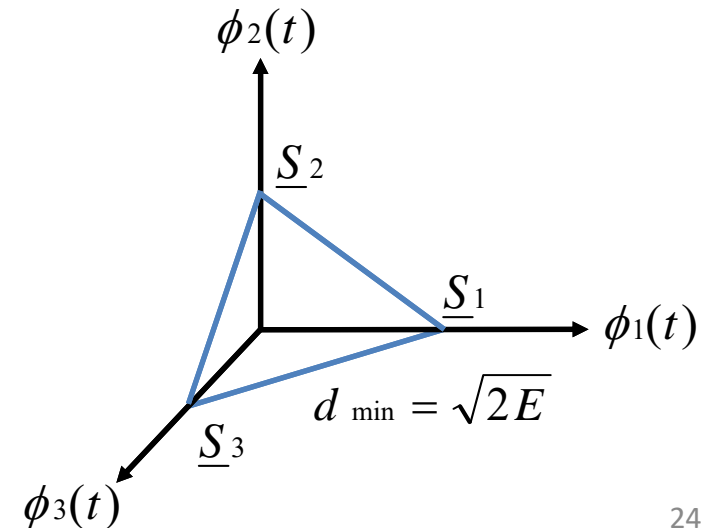
$$M = 2 \Rightarrow BFSK$$

$$S_m(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos[2\pi(f_c - \frac{\Delta f}{2})t] , & m = 1, \\ \sqrt{\frac{2E}{T}} \cos[2\pi(f_c + \frac{\Delta f}{2})t] , & m = 2 \end{cases}$$



General MFSK ( $k=\log_2 M$ ),

$$\underline{S}_1 = \begin{bmatrix} \sqrt{E} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \underline{S}_2 = \begin{bmatrix} 0 \\ \sqrt{E} \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \underline{S}_M = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sqrt{E} \end{bmatrix}$$





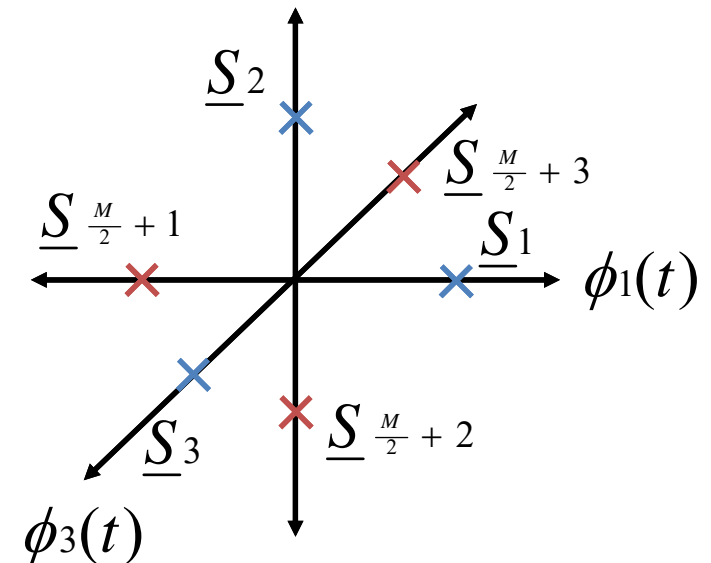
# Bi-orthogonal Signaling

Construct M-biorthogonal signals from M/2 orthogonals in pairs,  $M \in \text{even integer}$

$$S_m(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos[2\pi(f_c + f_m)t], & m = 1, 2, \dots, \frac{M}{2} \\ -\sqrt{\frac{2E}{T}} \cos[2\pi(f_c + f_m)t], & m = \frac{M}{2} + 1, \frac{M}{2} + 2, \dots, M \end{cases}$$

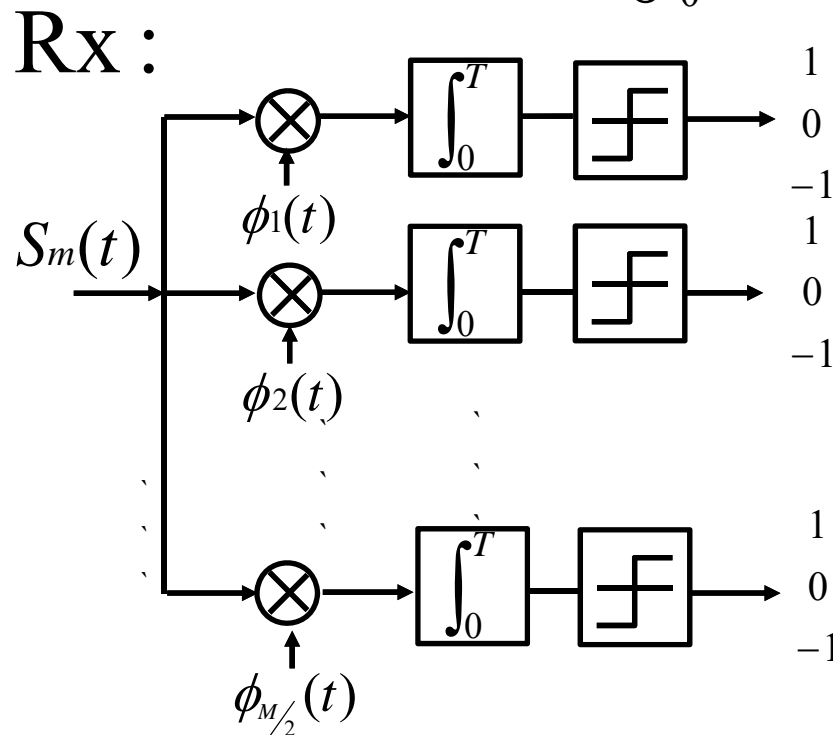
with the basis functions :

$$\phi_i(t) = \sqrt{\frac{2}{T}} \cos[2\pi(f_c + f_i)t], i = 1, 2, \dots, \frac{M}{2}$$



# Mutual correlation of Bi-orthogonal signals

$$\rho = -1, 0, +1 \quad \text{i.e., } \rho = \frac{\int_0^T S_m(t) S_n(t) dt}{\int_0^T S_m^2(t) dt} = \begin{cases} +1 & , n = m \\ 0 & , \text{otherwise} \\ -1 & , n = \frac{M}{2} + m \end{cases}$$



Q: What is the benefit of the bi-orthogonal?

✓ The system complexity saves by 50%

- What's the advantages in bi-orthogonal signal?

- (1) Complexity saving

- (2) Better spectral efficiency than MFSK or M-ary orthogonal signals.

- Can you think of any disadvantages of bi-orthogonal signal?

- ✓ Increased average energy ( $E_{av}$ ) from  $E/2$  to  $2E/3$ .

# Simplex Signaling

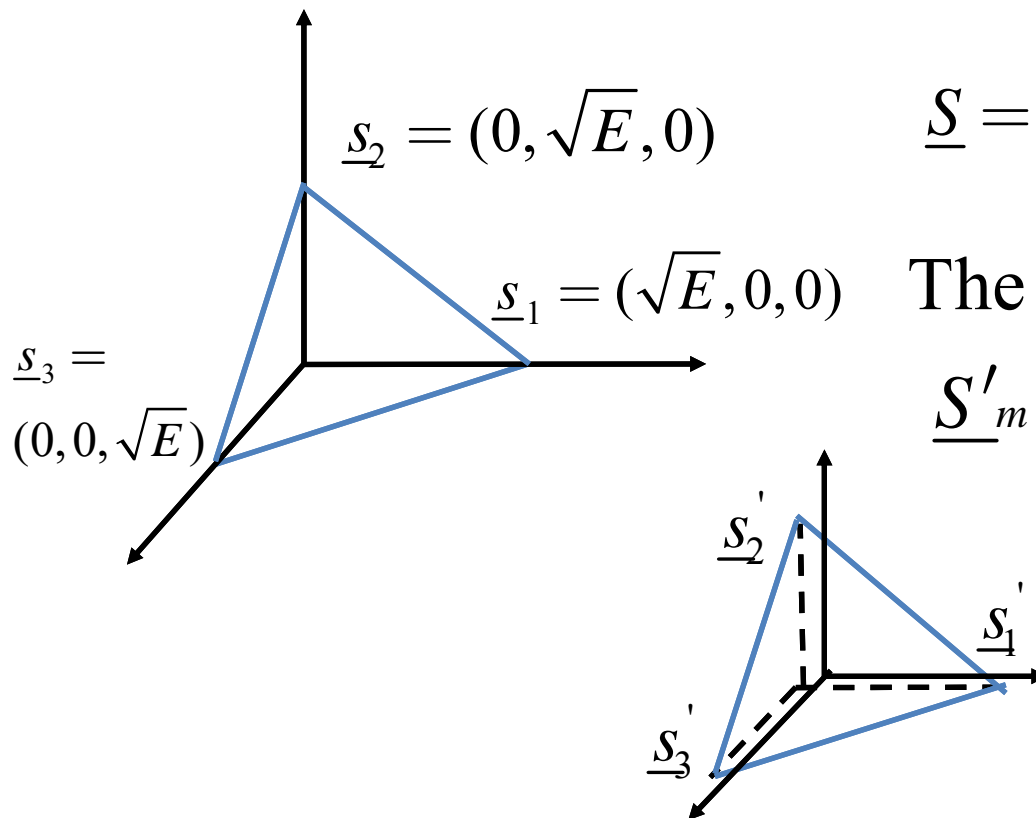
Translate a set of M-ary orthogonal signals from the origin to the centroid of the constellation.

The centroid locates at :

$$\underline{S} = \frac{1}{M} \sum_{m=1}^M \underline{S}_m, \text{ (mean value)}$$

The simplex signals are then

$$\underline{S}'_m = \underline{S}_m - \underline{S}$$



# Characteristics of Simplex Signal

(1) Reduced Energy :

$$\begin{aligned}\|\underline{S}'_m\|^2 &= \|\underline{S}_m - \underline{S}\|^2 = \|\underline{S}_m\|^2 - 2\underline{S}_m^T \underline{S} + \|\underline{S}\|^2 \\ &= E - \frac{2}{M} E + \frac{1}{M} E = (1 - \frac{1}{M})E\end{aligned}$$

(2) Equal cross-correlation :

$$\rho_{mk} = \frac{\underline{S}'_m{}^T \underline{S}'_k}{\|\underline{S}'_m\| \|\underline{S}'_k\|} = \begin{cases} 1 & , m = k \\ \frac{-1}{M-1} & , m \neq k \end{cases}$$

where  $\underline{S}'_m{}^T \underline{S}'_k = (\underline{S}_m - \underline{S})^T (\underline{S}_k - \underline{S}) = \underline{S}_m^T \underline{S}_k - \underline{S}_k^T \underline{S} - \underline{S}_m^T \underline{S} + \|\underline{S}\|^2$

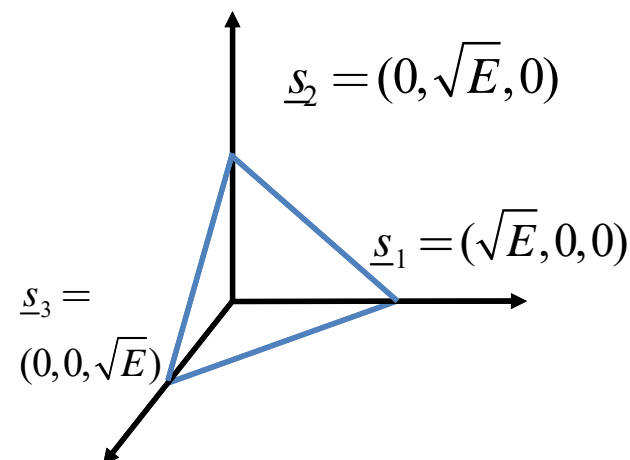
$$= -\frac{E}{M} - \frac{E}{M} + \frac{E}{M} = -\frac{1}{M} E$$

$$\rho_{mk} = \frac{-\frac{1}{M} E}{1 - \frac{1}{M}} = \frac{-1}{M-1}$$

# Example of Simplex Signal

- M-orthogonal signal

$$[\underline{s}_1 \quad \underline{s}_2 \dots \underline{s}_M] = \begin{bmatrix} \sqrt{E} & 0 & 0 \\ 0 & \sqrt{E} & \vdots \\ \vdots & 0 & \dots \vdots \\ \vdots & \vdots & \vdots \\ 0 & 0 & \sqrt{E} \end{bmatrix}$$



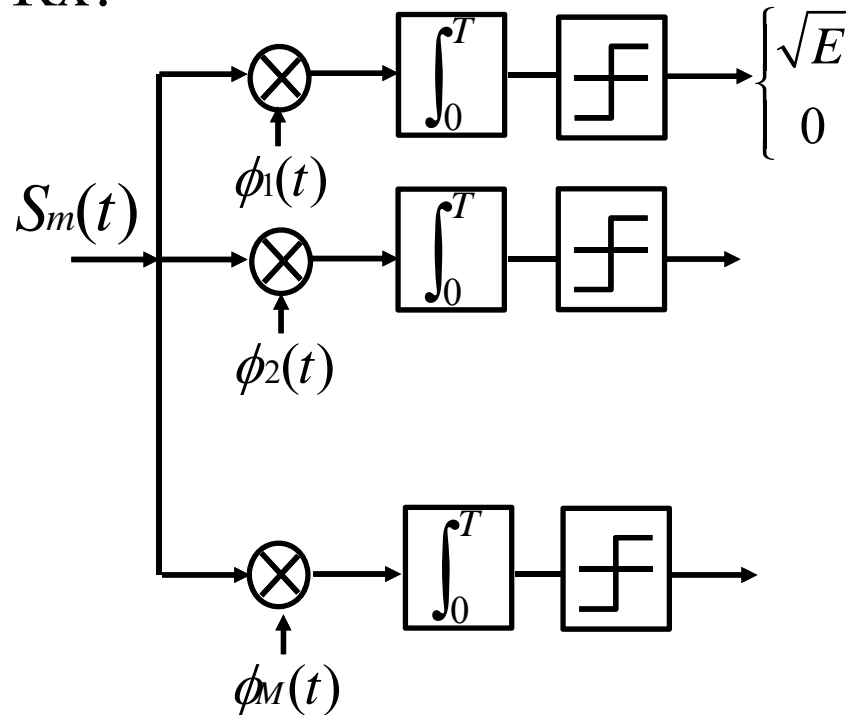
- Simplex Signal

$$[\underline{s}'_1 \quad \underline{s}'_2 \dots \underline{s}'_M] = \begin{bmatrix} \left(1 - \frac{1}{M}\right)\sqrt{E} & -\frac{1}{M}\sqrt{E} & \dots & -\frac{1}{M}\sqrt{E} \\ -\frac{1}{M}\sqrt{E} & \left(1 - \frac{1}{M}\right)\sqrt{E} & \dots & -\frac{1}{M}\sqrt{E} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{M}\sqrt{E} & -\frac{1}{M}\sqrt{E} & \dots & \left(1 - \frac{1}{M}\right)\sqrt{E} \end{bmatrix}$$

● For MFSK :

$$\int S_m(t)\phi_k(t)dt = \underline{S}_m^T \cdot \frac{\underline{S}_k}{\|\underline{S}_k\|} = \begin{cases} \sqrt{E} & , m = k \\ 0 & , m \neq k \end{cases}$$

Rx:

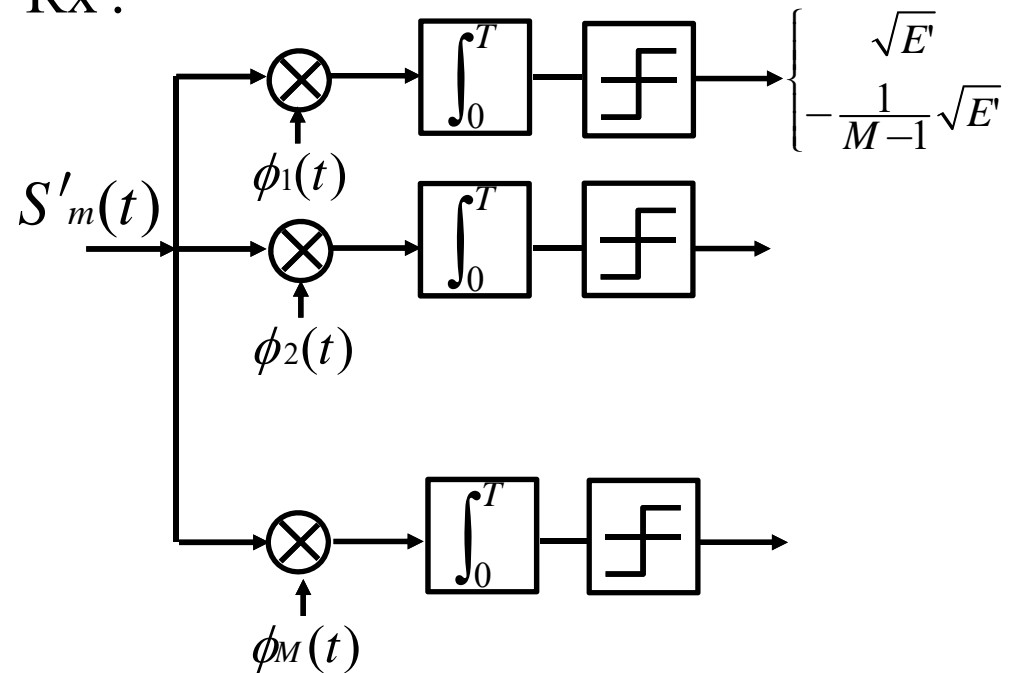


● For MFSK with simplex signaling

$$\int S'_m(t)\phi_k(t)dt = \underline{S}'_m^T \cdot \frac{\underline{S}_k}{\|\underline{S}_k\|} = \begin{cases} \sqrt{E'} & , m = k \\ -\frac{\sqrt{E'}}{M-1} & , m \neq k \end{cases}$$

where  $E' = \left(1 - \frac{1}{M}\right)E$

Rx :



# Test Your Understanding

- 1. What is digital modulation?**
- 2. How to find the signal space that represents a modulation scheme?**



# **Modulation with Memory**

- 1. Linear Modulation with Memory**
- 2. Non-linear Modulation with Memory**

# 1. Linear Modulation with Memory (Proakis 3.3)

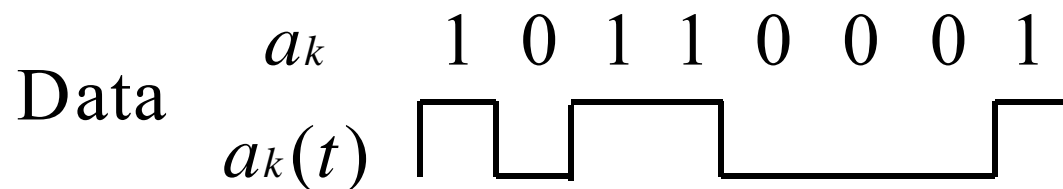
## ● Linear Correlated Encoding :

Linear modulation satisfies the superposition principle:

$$\text{If } a_1 \rightarrow b_1 \text{ and } a_2 \rightarrow b_2, \text{ then } a_1 + a_2 \rightarrow b_1 + b_2$$

### Ex: Differential encoding

Transmit signal makes a transition when a message 1 is to transmit.



Q: Can you identify the relation of  $a_k$  and  $b_k$  from the waveforms?

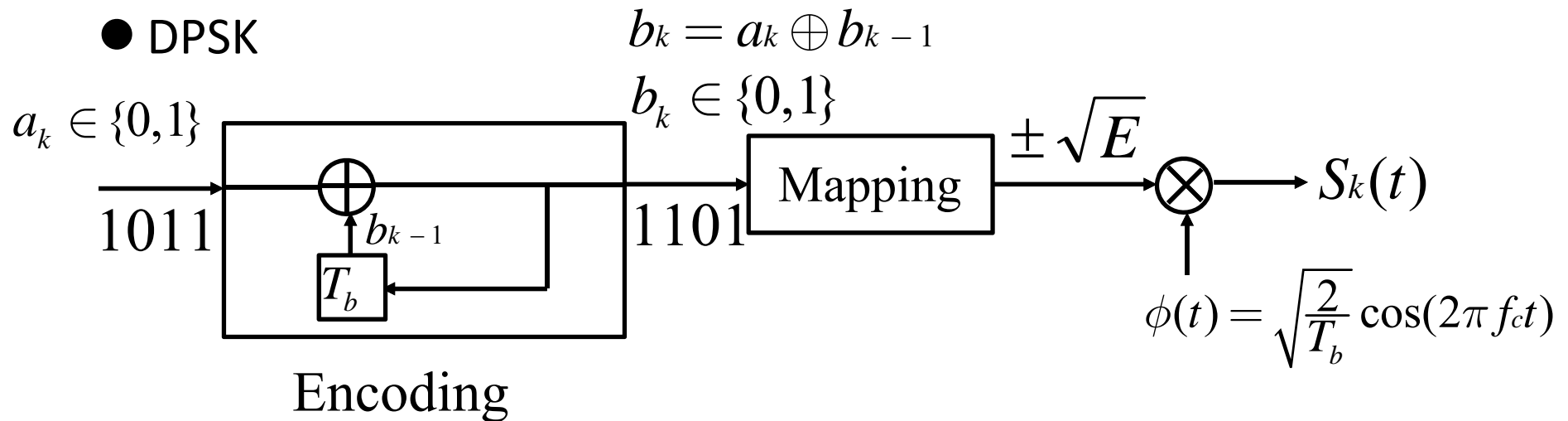
$$\Rightarrow b_k = a_k \oplus b_{k-1}$$

Q: Is this a linear modulation? Superposition principle?

$$\text{If } b_k = a_k \oplus b_{k-1} \text{ and } b'_k = a'_k \oplus b_{k-1}, \text{ then } (a_k + a'_k) \oplus b_{k-1} = b_k + b'_k$$

# Differential Encoding Tx

- DPSK

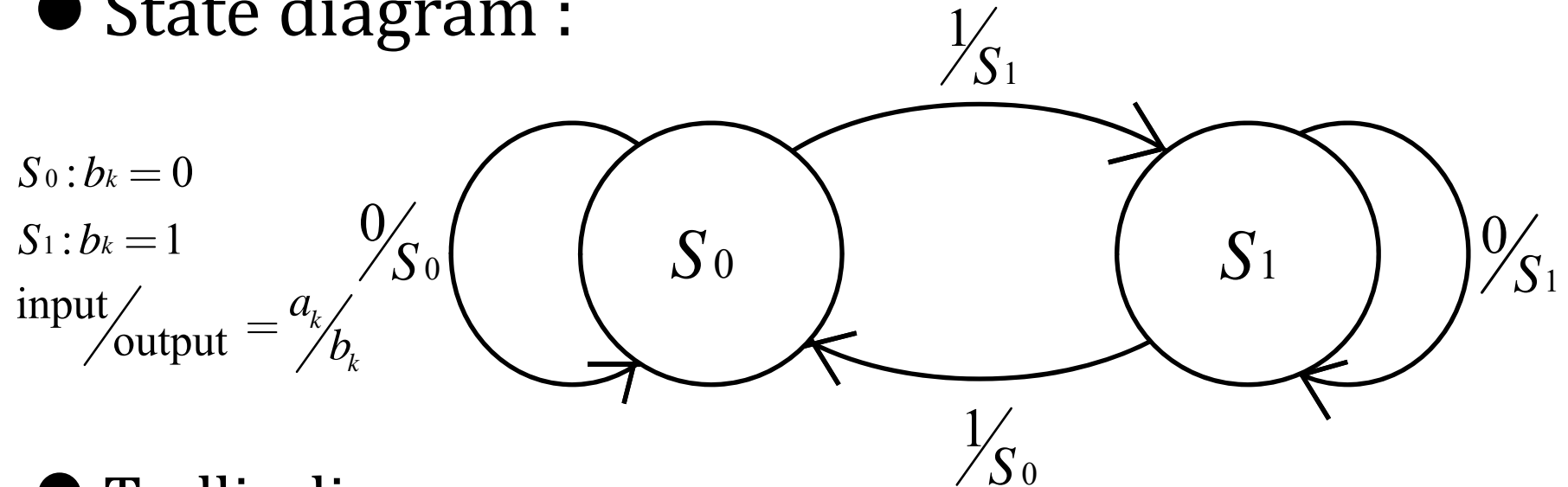


- The encoder output depends on the current input and the pervious symbol(s)

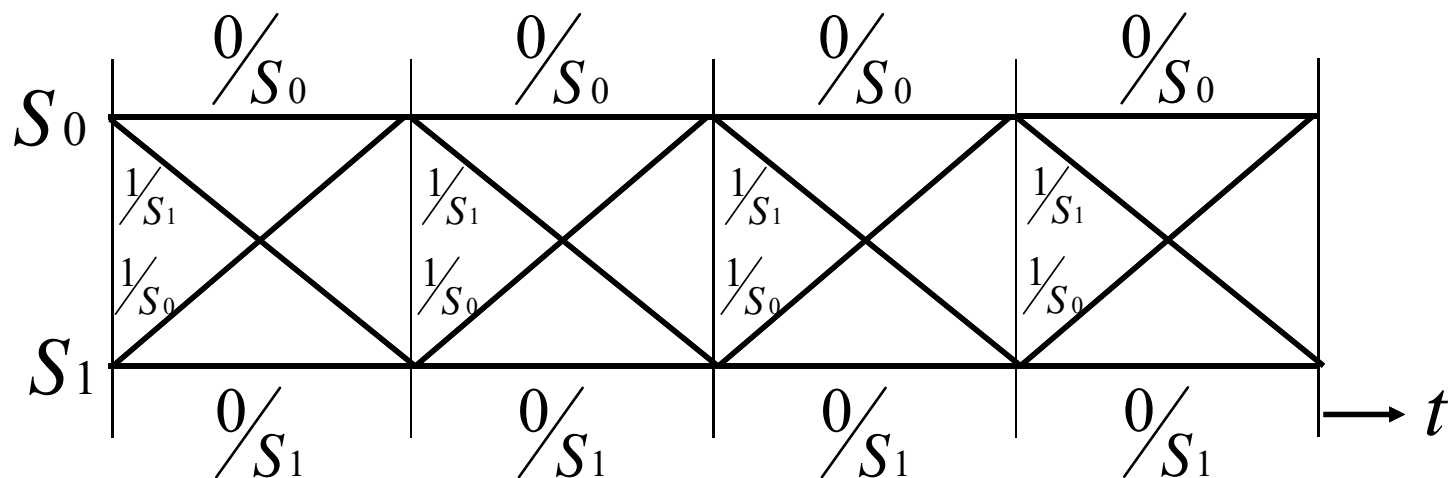
⇒ Encoding with memory.

The transition can be expressed by:

● State diagram :



● Trellis diagram :



Linear modulation with memory can be characterized by the Markov chain.

- ✓ The output  $b_k$  can be considered as the state of Markov chain with transition probability from  $s_i$  to  $s_j$  as

$$\Pi_{ij} \quad i, j \in \{0, 1\} \quad \Pi_{ij} = P(s_j | s_i)$$

Let the Markov transition matrix be:  $\underline{\Pi} = \begin{bmatrix} \Pi_{00} & \Pi_{10} \\ \Pi_{01} & \Pi_{11} \end{bmatrix}$

Let  $\underline{P}(n) = \begin{bmatrix} P_0(n) \\ P_1(n) \end{bmatrix}$  = Probability distribution of  $b_k = 0$  and  $b_k = 1$  at  $t = nT$

$$\Rightarrow \underline{P}(1) = \underline{\Pi} \cdot \underline{P}(0) = \begin{bmatrix} \Pi_{00} & \Pi_{10} \\ \Pi_{01} & \Pi_{11} \end{bmatrix} \begin{bmatrix} P_0(0) \\ P_1(0) \end{bmatrix}$$

$$\underline{P}(1) = \underline{\Pi} \cdot \underline{P}(0)$$

$$\underline{P}(2) = \underline{\Pi} \cdot \underline{P}(1)$$

$$\vdots$$

$$\underline{P}(n) = \underline{\Pi} \cdot \underline{P}(n-1)$$

$$\rightarrow \underline{P}(n) = \underline{\Pi}^n \cdot \underline{P}(0)$$

## Remarks

If  $\lim_{n \rightarrow \infty} \underline{\Pi}^n$  exists, then the Markov chain stabilizes to a stationary state, and the probability distribution  $\underline{P}$  becomes a steady state.

$\Rightarrow$  In the steady state, the probability distribution remains unchanged, *i.e.*  $\underline{P}(n) = \underline{P}(n-1) = \underline{P}$ .

$\Rightarrow$  By the Markov chain rule,  $\underline{P}(n) = \underline{\Pi} \cdot \underline{P}(n-1)$ , *i.e.*  $\underline{P} = \underline{\Pi} \cdot \underline{P}$   
*i.e.*,  $\underline{P}$  is the eigenvector of  $\underline{\Pi}$  with the eigenvalue = 1.

Ex: If the source is equiprobable and all transition probabilities = 1/2

$$\underline{\Pi} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \text{ then } \lim_{n \rightarrow \infty} \underline{\Pi}^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \lim_{n \rightarrow \infty} \underline{P}(n) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

# Modulation with Memory

## 1. Linear Modulation with Memory

## 2. Non-linear Modulation with Memory

- ✓ CP-BFSK: Minimum Shift Keying(MSK)
- ✓ CP-MFSK

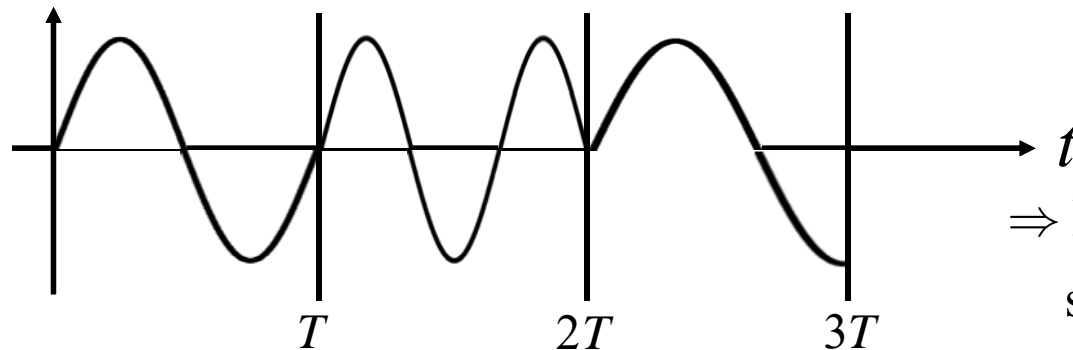
## 2. Non-linear Modulation with Memory

- Continuous Phase Modulation with FSK(CPFSK)

Motivation /Observation:

For memoryless FSK, the phase is dis-continuous.

$$S_m(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + 2\pi f_m t), \quad \begin{matrix} m = 1, 2, \dots, M \\ 0 \leq t \leq T \end{matrix}$$



⇒ Produce high frequency spectral components.

- What would happen?

(1) Abrupt transition of phases leads to large spectral side-lobes.

(2) Passing through a BPF to save the spectral BW, but leads to distorted signal.

✓ The continuous phase design helps to shape the spectrum of Tx signals for band-limited transmission.

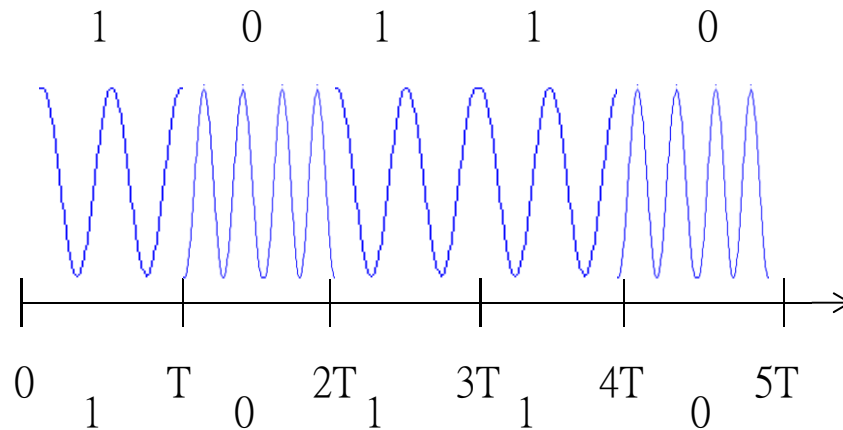


# Example of Memoryless BFSK

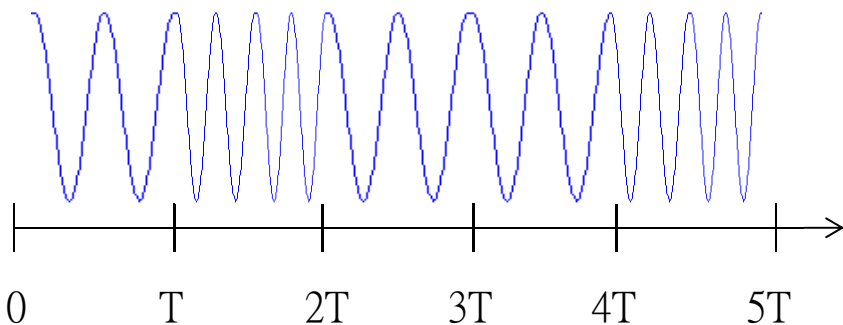
For the  $n^{\text{th}}$  symbol in time,  $nT \leq t \leq (n+1)T$

$$S_n(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos[2\pi f_1 t] & \text{for symbol 1} \\ \sqrt{\frac{2E}{T}} \cos[2\pi f_2 t] & \text{for symbol 0} \end{cases},$$

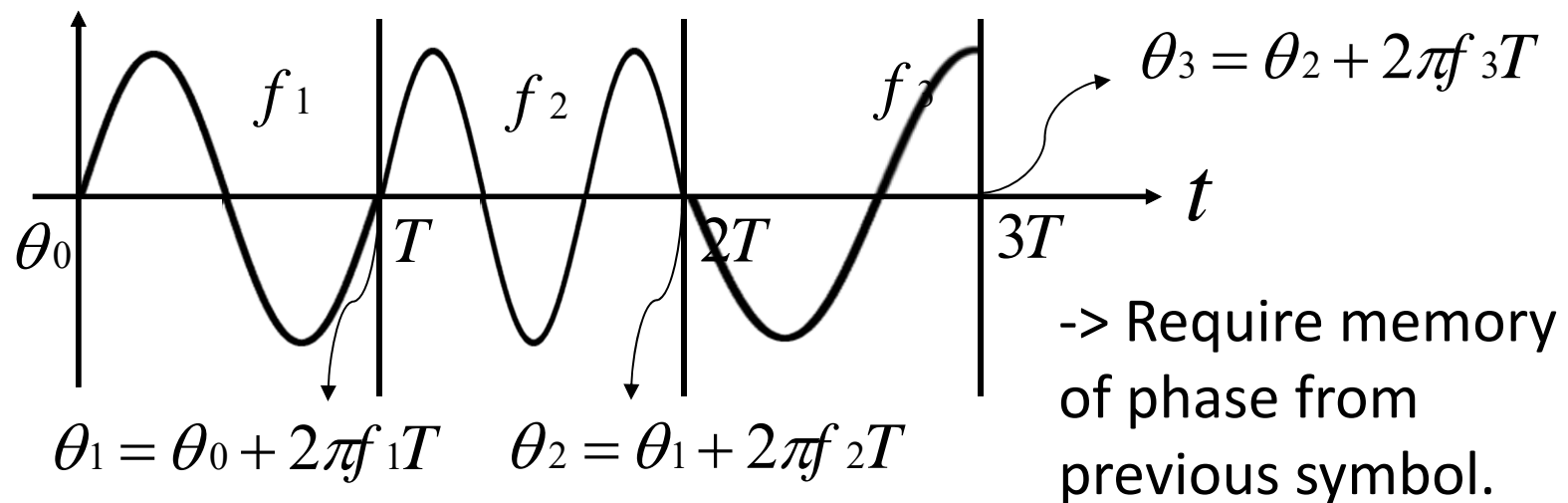
## ● Dis-continuous-Phase BFSK



## ● Continuous-Phase BFSK



## ● Continuous-Phase FSK



## ● Minimum Shift Keying(MSK)

✓ MSK is a special form of CP-BFSK

For the  $n^{\text{th}}$  symbol in time,  $nT \leq t \leq (n+1)T$

$$S_n(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos[2\pi f_1(t - nT) + \theta(nT)] & \text{for symbol 1} \\ \sqrt{\frac{2E}{T}} \cos[2\pi f_2(t - nT) + \theta(nT)] & \text{for symbol 0} \end{cases},$$

where  $\theta(nT)$  denotes the phase value at end of the previous symbol.

$$\text{where } \begin{cases} f_1 = f_c + \frac{h}{2T} = f_c + h \cdot \Delta f_{\min}, \text{ symbol 1} \\ f_2 = f_c - \frac{h}{2T} = f_c - h \cdot \Delta f_{\min}, \text{ symbol 0} \end{cases} \quad \text{and } \Delta f_{\min} = \frac{1}{2T}$$

$$h = T \cdot \Delta f \triangleq \text{modulation index}$$

$$\Rightarrow h_{\min} = \frac{1}{2} (= T \cdot \Delta f_{\min}) \text{ (Minimum Shift)}$$

$$\begin{aligned} \text{The MSK signal } S_n(t) &= \sqrt{\frac{2E}{T}} \cos[2\pi f_c(t - nT) \pm \underbrace{2\pi h \Delta f_{\min}(t - nT)}_{\theta(t)} + \theta(nT)] \\ &= \sqrt{\frac{2E}{T}} \cos[2\pi f_c(t - nT) + \theta(t)] \end{aligned}$$

$$\text{where } \theta(t) = \theta(nT) \pm \frac{\pi h}{T}(t - nT), \quad nT \leq t \leq (n+1)T$$

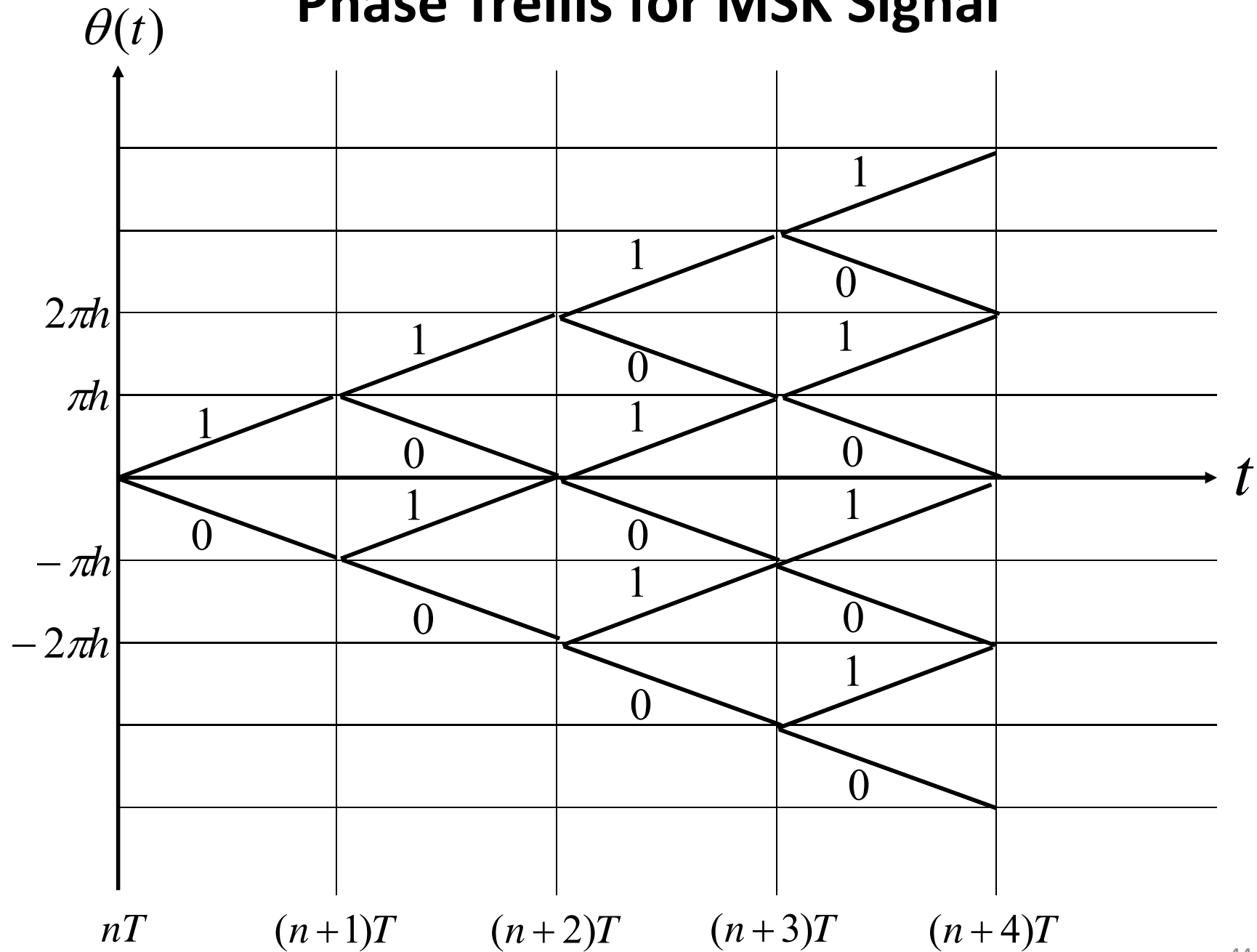
$$\text{for the } n\text{th symbol and } \begin{cases} +: \text{symbol 1} \\ -: \text{symbol 0} \end{cases}$$

The phase transition becomes **continuous** with phase trellis.

$$\theta[(n+1)T] - \theta[nT] = \begin{cases} \pi h, & \text{for symbol 1} \\ -\pi h, & \text{for symbol 0} \end{cases}$$

✓ Symbol detection is changed from frequencies to phase transitions.

# Phase Trellis for MSK Signal



# Signal space diagram of MSK

$$\begin{aligned} S(t) &= \sqrt{\frac{2E}{T}} \cos[2\pi f_c t + \theta(t)] \\ &= \sqrt{\frac{2E}{T}} \cos[\theta(t)] \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \sin[\theta(t)] \sin(2\pi f_c t) \end{aligned}$$

Let the basis function of MSK be :

$$\begin{aligned} \phi_1(t) &= \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \quad , \quad nT \leq t \leq (n+1)T \\ \phi_2(t) &= -\sqrt{\frac{2}{T}} \sin(2\pi f_c t) \quad , \quad nT \leq t \leq (n+1)T \end{aligned}$$

Then the projection on  $\phi_1(t)$  ,  $\phi_2(t)$  are

$$\underline{S} = \left[ \sqrt{E} \cos[\theta(t)] , \sqrt{E} \sin[\theta(t)] \right]^T = [S_I(t) , S_Q(t)]^T$$

$$\theta(t) = \theta(nT) \pm \frac{\pi}{2T} (t - nT), \quad nT \leq t \leq (n+1)T$$

$$\text{for MSK with } \Delta f_{\min} = \frac{1}{2T}, \quad h = \frac{1}{2}$$

Without losing generality, assume the starting phase  $\theta(nT) = 0$ , or  $\pi$

$$S_I(t) = \sqrt{E} \cos(\theta(t)), \quad S_Q(t) = \sqrt{E} \sin(\theta(t))$$

$$\theta(t) = \theta(nT) \pm \frac{\pi}{2T} (t - nT) \quad \text{for } nT \leq t \leq (n+1)T$$

$$\text{At } t = nT \Rightarrow S_I(t) = S_I(nT) = \sqrt{E} \cos[\theta(nT)]$$

$$S_Q(t) = S_Q(nT) = \sqrt{E} \sin[\theta(nT)] = 0$$

$$\text{At } t = (n+1)T \Rightarrow S_I(t) = S_I[(n+1)T] = \sqrt{E} \cos[\theta(nT) \mp \frac{\pi}{2}] = 0$$

$$S_Q(t) = S_Q[(n+1)T] = \sqrt{E} \sin[\theta(nT) \pm \frac{\pi}{2}] = \pm \sqrt{E} \cos[\theta(nT)]$$

where “+”=symbol 1, “−” = symbol 0

The phasor representation and the constellation point of the signal are defined as :

Phasor :  $\tilde{S}(t) = S_I(t) + jS_Q(t+T)$ , or

$$\text{Vector : } \underline{S} = [S_I(nT) \quad , \quad S_Q(n+1)T] \quad \text{where} \begin{cases} S_I(nT) = \sqrt{E} \cos[\theta(nT)] \\ S_Q[(n+1)T] = \pm \sqrt{E} \cos[\theta(nT)] \end{cases}$$

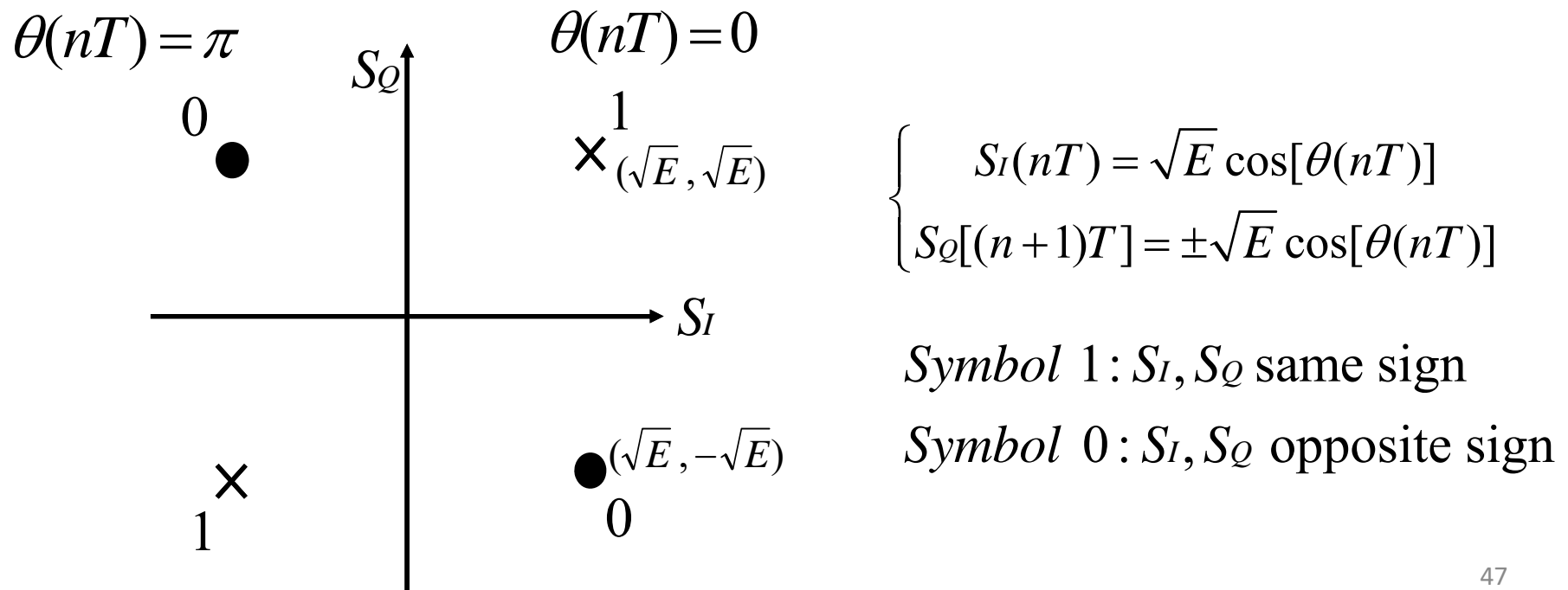
$$\stackrel{\Delta}{=} [S_I \quad , \quad S_Q]$$

Example :  $\underline{S} = [S_I(nT) \ , \ S_Q(n+1)T] \stackrel{\Delta}{=} [S_I \ , \ S_Q]$

$$\theta(nT) = 0 \Rightarrow (S_I, S_Q) = \begin{cases} (\sqrt{E} \ , \ \sqrt{E}) : & \text{symbol 1,} \\ (\sqrt{E} \ , \ -\sqrt{E}) : & \text{symbol 0,} \end{cases}$$

Note that the reference initial phase  $\theta(nT)$  is not necessarily 0, e.g.

$$\theta(nT) = \pi \Rightarrow (S_I, S_Q) = \begin{cases} (-\sqrt{E} \ , \ -\sqrt{E}) : & \text{symbol 1} \\ (-\sqrt{E} \ , \ \sqrt{E}) : & \text{symbol 0} \end{cases}$$



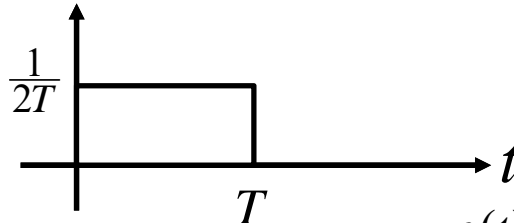
# Non-linear Modulation with Memory (M-ary CPFSK )

- Constructing M-ary CPFSK from M-ary PAM

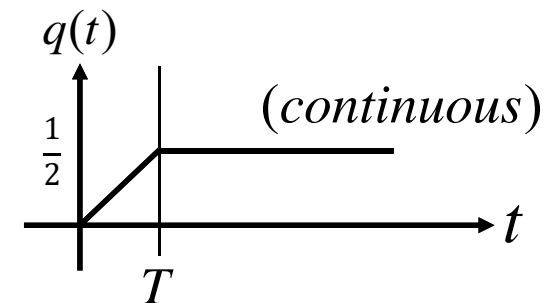
Given M-PAM  $d(t) = \sum_n I_n g(t - nT)$ ,

where  $\{I_n\} \in \{\pm 1, \pm 3, \dots, \pm(M-1)\}$  are M-PAM symbols.

Let  $g(t) = \frac{1}{2T} \text{rect}(t - \frac{T}{2})$



Define  $q(t) = \int_0^t g(\tau) d\tau = \begin{cases} 0 & , t \leq 0 \\ \frac{t}{2T} & , 0 \leq t \leq T \\ \frac{1}{2} & , t > T \end{cases}$



Each M-ary PAM symbol  $I_n$  is mapped into frequencies  $\{\pm f_d, \pm 3f_d, \dots, \pm(M-1)f_d\}$



The CPFSK signal is :

$$S(t) = \sqrt{\frac{2E}{T}} \cos[2\pi f_c t + \theta(t, d(t))], \quad nT \leq t \leq (n+1)T$$

where  $\theta(t, d(t)) = 4\pi T f_d \int_{-\infty}^t d(\tau) d\tau$

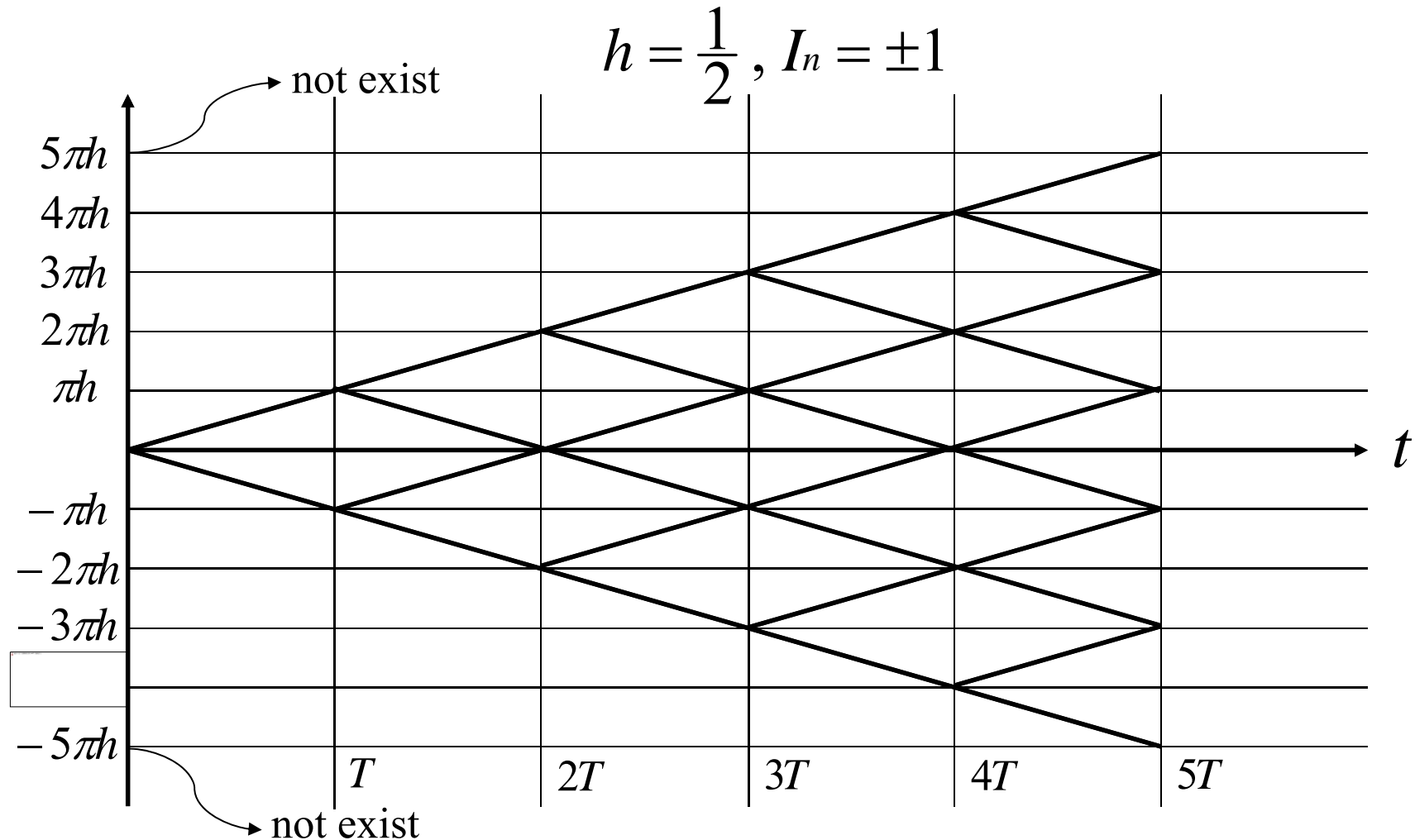
$$\begin{aligned} &= 4\pi f_d T \int_{-\infty}^t \left( \sum I_n g(\tau - nT) \right) d\tau \\ &= 4\pi f_d T \sum_{k=-\infty}^{n-1} I_k \underbrace{\int_{-\infty}^{nT} g(\tau - kT) d\tau}_{=1/2} + 4\pi f_d T I_n \int_{nT}^t g(\tau - nT) d\tau \\ &= 2\pi f_d T \sum_{k=-\infty}^{n-1} I_k + 4\pi f_d T I_n q(t - nT) \\ &= \theta(nT) + 2\pi h I_n q(t - nT), \quad nT \leq t \leq (n+1)T \end{aligned}$$

Eq. (3.3-10)

Note :  $h = T \cdot \Delta f = 2 f_d T$

$$\begin{cases} t = nT, & \theta(t) = \theta(nT) \\ t = (n+1)T, & \theta(t) = \theta(nT) + \pi h I_n = \theta[(n+1)T] \Rightarrow \text{phase trellis} \end{cases}$$

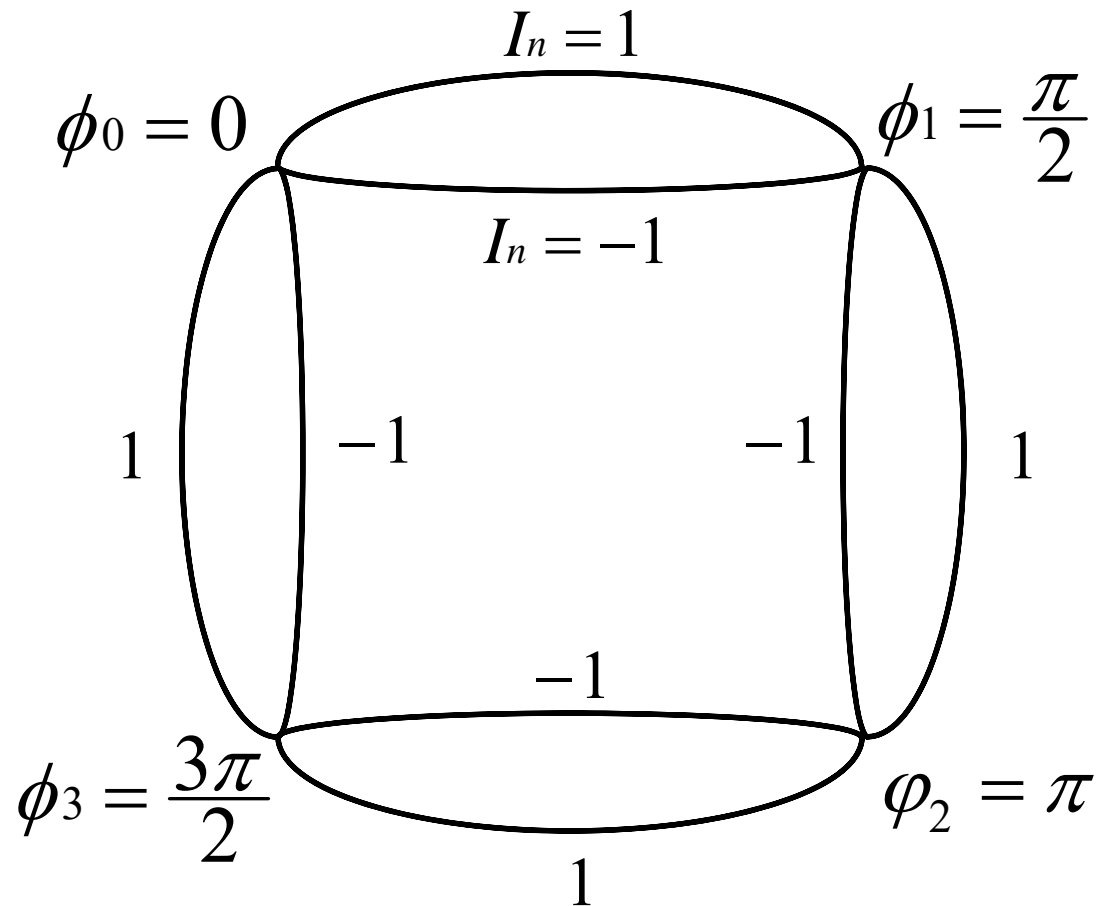
# Phase Trellis for M-ary CPFSK Signal, M=2



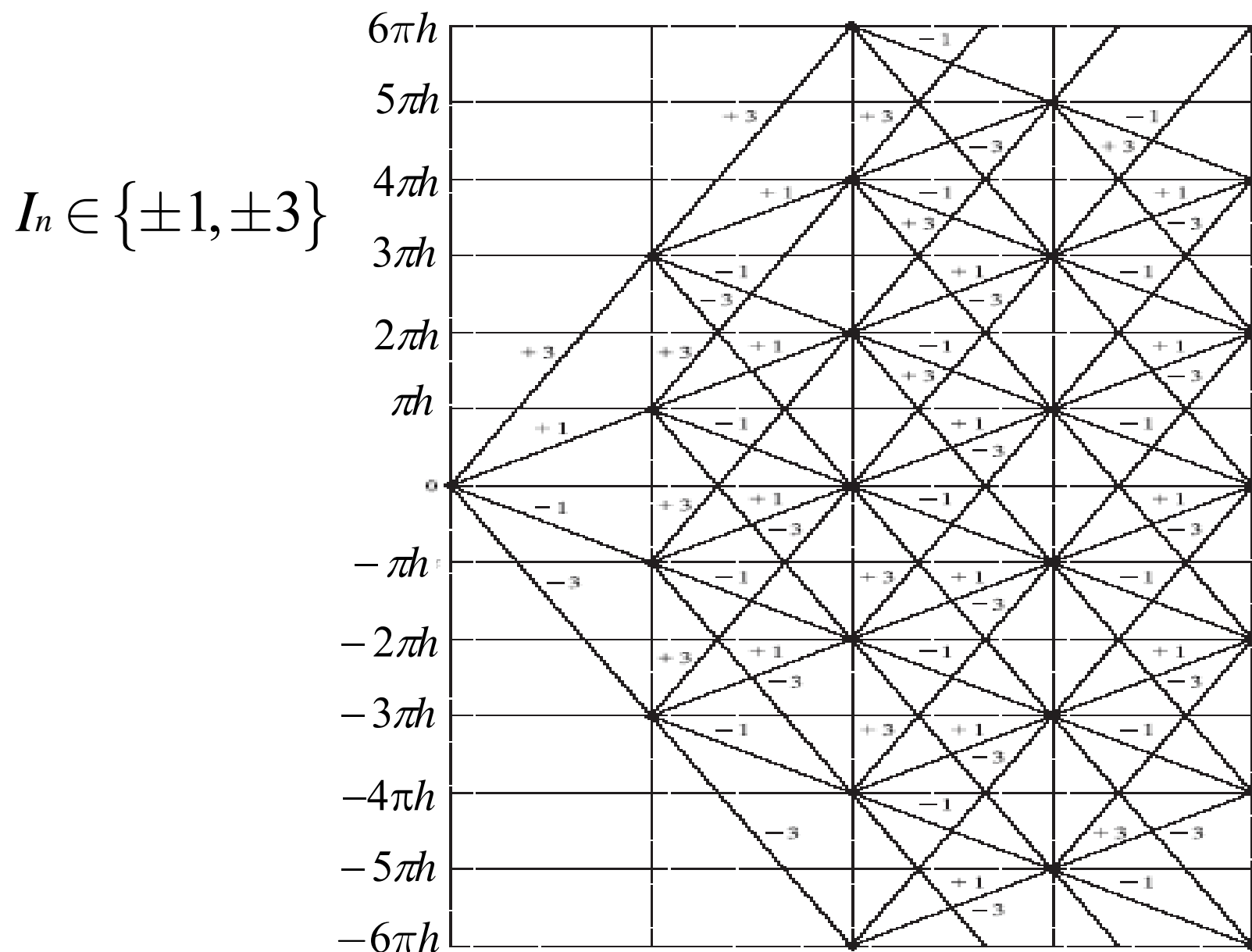
In fact, in this example, there are only four states.

$$(\pi h I_n = \frac{\pi}{2} \Rightarrow \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi)$$

- State diagram (Binary CFSK,  $h = \frac{1}{2}$ ,  $I_n = \pm 1$ )



# Phase Trellis for M-ary CPFSK Signal (M=4)



Thus CPM can be decoded by Viterbi trellis decoding.

- The number of possible phase states is determined by  $h$ .

For  $h = \frac{m}{p}$ , where  $m$  and  $p$  are mutually prime integers, we have :

$$\begin{cases} p \text{ states, i.e., } \left\{ 0, \frac{\pi}{p} m, \dots, \frac{(p-1)\pi}{p} m \right\} & \text{when } m \text{ is even} \\ 2p \text{ states, i.e., } \left\{ 0, \frac{\pi}{p} m, \dots, \frac{(2p-1)\pi}{p} m \right\} & \text{when } m \text{ is odd} \end{cases}$$

i.e. there are  $\frac{2\pi}{\pi h} = \frac{2}{h}$  states.

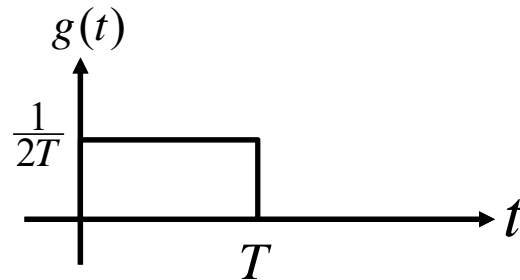
Example :

$$h = \frac{1}{3} \Rightarrow \frac{2}{h} = 6 \text{ states } (= 2p)$$

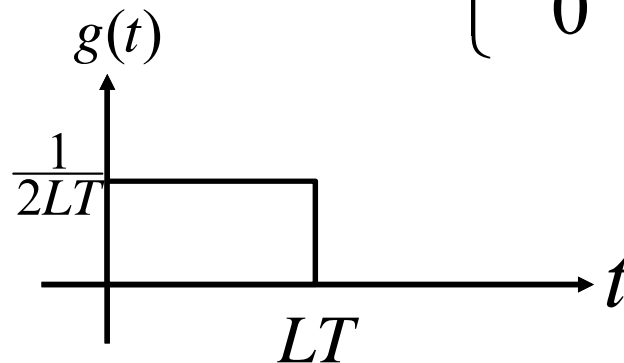
$$h = \frac{2}{3} \Rightarrow \frac{2}{h} = 3 \text{ states } (= p)$$

In general , CPM has many variations by choosing different combination of  $h$  ,  $M$  , and  $g(t)$ .

For CPFSK :  $g(t) = \frac{1}{2T} \text{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$



For L - Rect CPFSK :  $g(t) = \begin{cases} \frac{1}{2LT} & , 0 \leq t \leq LT \\ 0 & , otherwise \end{cases}$

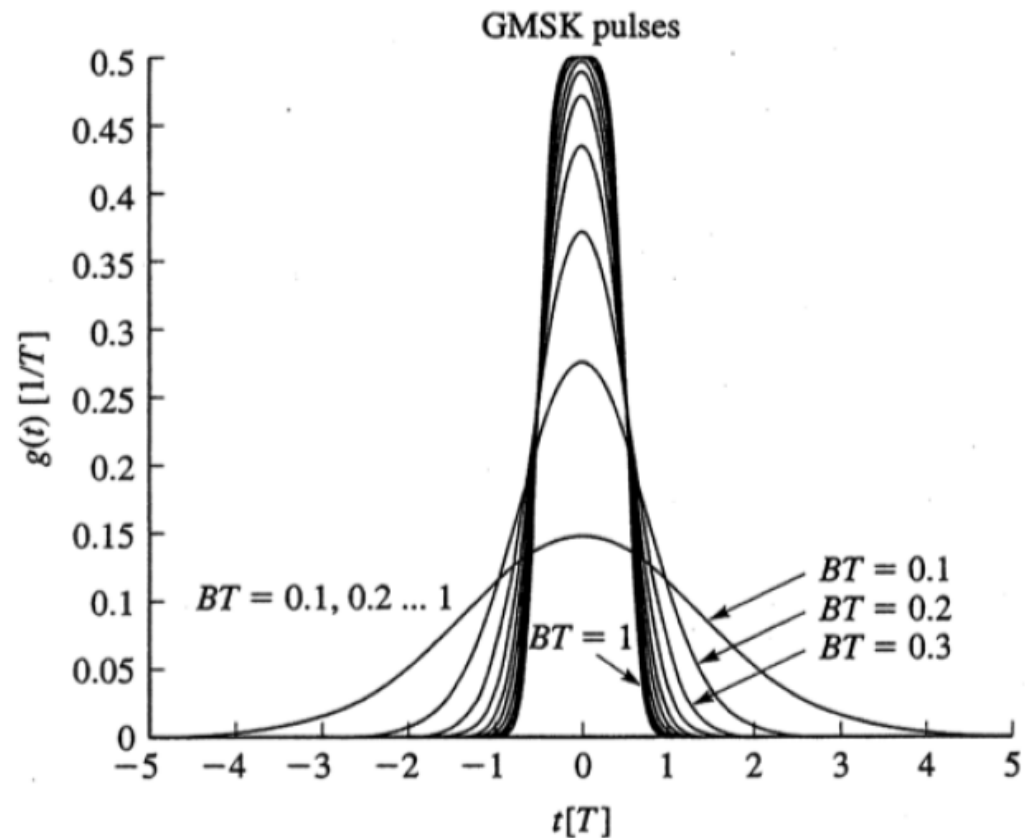


- Gaussian pulse shaping function for MSK(GMSK)

$$g(t) = \frac{1}{\sqrt{\ln 2}} \{Q[2\pi B(t - \frac{T}{2})] - Q[2\pi B(t + \frac{T}{2})]\}$$

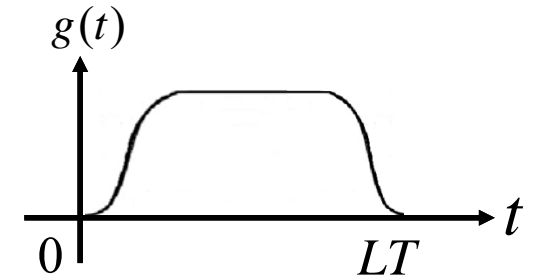
Note:

GMSK with  $BT = 0.3$  is used in the European digital cellular communication system, called GSM (2G).



- LRC(L - Raised Cosine)

$$g(t) = \begin{cases} \frac{1}{2LT} [1 - \cos(\frac{2\pi t}{LT})] & , 0 \leq t \leq LT \\ 0 & , otherwise \end{cases}$$



In general, the  $g(t)$  can be classified into two types:

Full - response CPM :  $g(t) = 0$  , for  $t \geq T$

Partial - response CPM :  $g(t) \neq 0$  , for  $t \geq T$



# Power Spectral Density of Digital Modulation Signal

Why studying power spectral characteristics?

- Defines the bandwidth requirement for a modulation scheme in the transmission channel.

# Power Spectral Density of Digital Modulation Signal

Given a low-pass (i.e. baseband) signal  $v(t)$  that is WSS random proc.,  
the passband signal :  $s(t) = v(t) \cos(2\pi f_c t) = \text{Re} \left\{ v(t) e^{j2\pi f_c t} \right\}$

It can be shown that the power spectral density :

$$S_s(f) = \frac{1}{4} [S_v(f - f_c) + S_v(f + f_c)]$$

*Proof.*  $R_s(\tau) = E[v(t + \tau) \cos(2\pi f_c(t + \tau)) v(t) \cos(2\pi f_c t)]$

$$\begin{aligned} &= E \left[ v(t + \tau) v(t) \frac{1}{2} [\cos(2\pi f_c(2t + \tau)) + \cos(2\pi f_c \tau)] \right] \\ &= \frac{1}{2} E[v(t + \tau) v(t)] \cos(2\pi f_c \tau) \quad (\text{assume ergodic r.p.}) \\ &= \frac{1}{2} R_{vv}(\tau) \cos(2\pi f_c \tau) \end{aligned}$$

# Power Spectral Density of Digital Modulation Signal

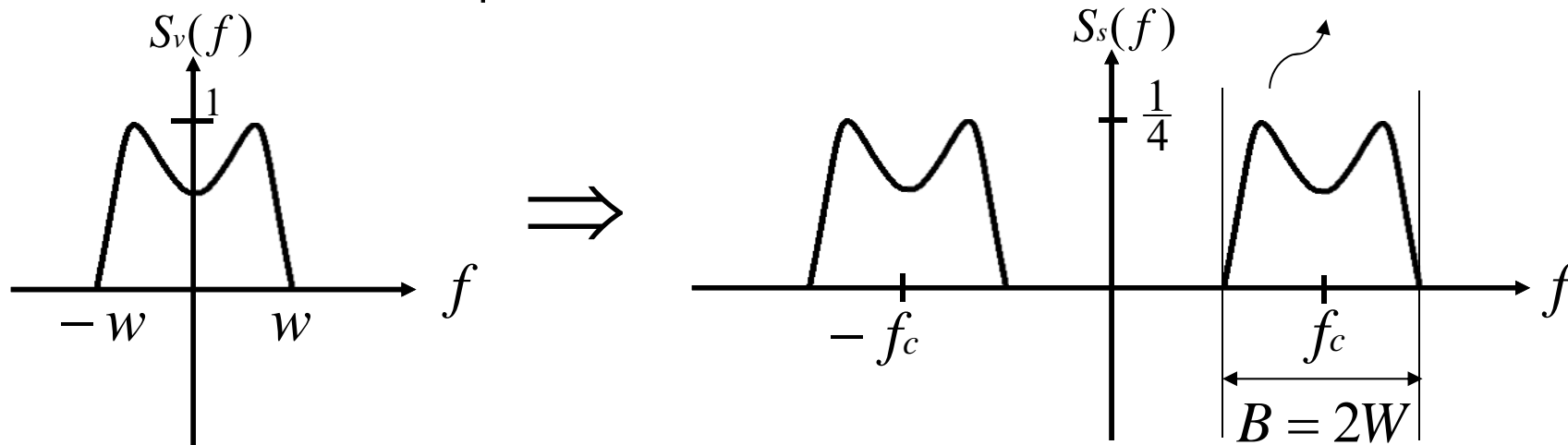
The PSD of  $s(t)$  is the F.T. of  $R_s(\tau)$ .

$$S_s(f) = F\{R_s(\tau)\} = F\left\{\frac{1}{2}R_{vv}(\tau)\cos(2\pi f_c\tau)\right\}$$

$$= \frac{1}{2}S_v(f) * \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

$$= \frac{1}{4}[S_v(f - f_c) + S_v(f + f_c)]$$

Transmission BW:  $B = 2W$



Therefore, we can focus on baseband  $S_v(f)$  and the passband PSD  $S_s(f)$  can be obtained accordingly.

# Power Spectral Density of Cyclo-Stationary Signals

Let the low-pass (i.e. baseband) signal  $v(t)$  be

$$v(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT),$$

where

$T$  is the symbol period,

$\{I_n\}$  is W.S.S. random sequence with  $E[I_n] = \mu_n$

and autocorrelation  $R_{II}[k] = E\left[I_n I_{n+k}^*\right]$ .

**Lemma :**  $v(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT)$  is cyclo-stationary random process

Proof :  $E[v(t)] = \mu_I \cdot \sum_n g(t - nT) \Rightarrow$  periodic in  $t$  with period of  $T$ .

$$R_{vv}(t, \tau) = E[v(t + \tau)v^*(t)] = \sum_n \sum_m E[I_n I_m^*] \cdot g(t + \tau - nT) \cdot g^*(t - mT)$$

Let  $k = n - m$ , then  $E[I_n I_m^*] = R_{II}[n - m] \equiv R_{II}[k] \quad \because \{I_n\}$  is W.S.S.

$$g(t + \tau - nT) = g(t - (m + k)T + \tau)$$

$$R_{vv}(t, \tau) \stackrel{n=k+m}{=} \sum_k R_{II}[k] \cdot \sum_m g(t - (m + k)T + \tau) g^*(t - mT)$$

$\Rightarrow$  Periodic in  $t$  with period  $T$

$\Rightarrow v(t)$  is cyclo-stationary.

*Note :* As  $v(t)$  is cyclo-stationary, the averaged auto-correlation can be obtained by taking time-average over one period  $T$ .

The time averaged auto-correlation function is

$$\begin{aligned}\overline{R_{vv}}(\tau) &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} R_{vv}(t, \tau) dt \\ &= \sum_k R_{\Pi}[k] \sum_m \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t + \tau - kT - mT) g^*(t - mT) dt\end{aligned}$$

$$\begin{aligned}\text{let } t' = t - mT & \\ &= \sum_k R_{\Pi}[k] \cdot \sum_m \frac{1}{T} \int_{-\frac{T}{2}-mT}^{\frac{T}{2}-mT} g(t' + \tau - kT) g^*(t') dt' \\ &= \sum_k R_{\Pi}[k] \cdot \frac{1}{T} \int_{-\infty}^{\infty} g(t' + \tau - kT) g^*(t') dt' \\ &= \frac{1}{T} \sum_k R_{\Pi}[k] R_{gg}(\tau - kT)\end{aligned}$$

$$\text{where } R_{gg}(\tau) \triangleq \int_{-\infty}^{\infty} g(t + \tau) g^*(t) dt = g(\tau) * g^*(-\tau)$$

$$\Rightarrow R_{gg}(\tau - kT) = \int_{-\infty}^{\infty} g(t + \tau - kT) g^*(t) dt$$

The power spectral density :

$$S_{vv}(f) = \mathbb{E} \left\{ \overline{R_{vv}}(\tau) \right\} = \mathbb{E} \left\{ \frac{1}{T} \sum_k R_{II}[k] R_{gg}(\tau - kT) \right\}$$

Recall from Fourier transform properties,

$$\begin{aligned} g(\tau) &\leftrightarrow G(f) \Rightarrow g^*(\tau) \leftrightarrow G^*(-f) \\ g^*(-\tau) &\leftrightarrow G^*(f) \end{aligned}$$

$$\begin{aligned} \Rightarrow S_{vv}(f) &= \frac{1}{T} \sum_k R_{II}[k] \mathbb{E} \left\{ R_{gg}(\tau - kT) \right\} \\ &= \frac{1}{T} \sum_k R_{II}[k] \cdot G(f) \cdot G^*(f) \cdot e^{-j2\pi f k T} \\ &= \frac{1}{T} S_{II}(f) |G(f)|^2 \end{aligned}$$

$$\text{where } S_{II}(f) = \sum_{k=-\infty}^{\infty} R_{II}[k] e^{-j2\pi f k T}$$

represents the PSD of the discrete random process  $\{I_n\}$

$$\Rightarrow \boxed{S_{vv}(f) = \frac{1}{T} S_{II}(f) |G(f)|^2}$$

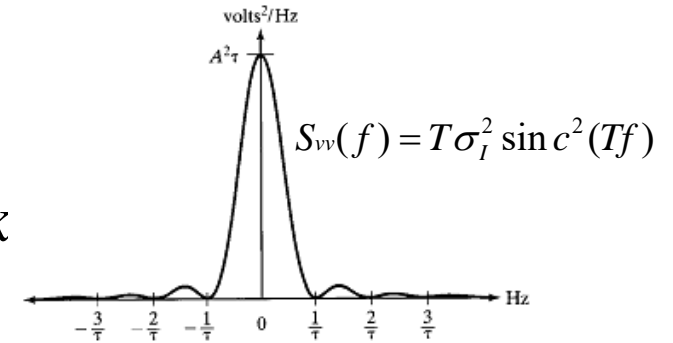
- **Example : Power Spectral Density of M-PAM**

$$I_n \in \{\pm 1, \pm 3, \dots, \pm(M-1)\} \Rightarrow \text{M-PAM}$$

$$\mu_I = 0, \quad R_{II}[k] = \begin{cases} \sigma_I^2 & , k = 0 \\ 0 & , k \neq 0 \end{cases} = \sigma_I^2 \delta[k]$$

$$\Rightarrow S_{II}(f) = \sigma_I^2$$

$$\Rightarrow S_{vv}(f) = \frac{1}{T} \sigma_I^2 \cdot |G(f)|^2$$



Required transmission  
Bandwidth  $B=2/T$

$$\text{If } g(t) = \begin{cases} 1 & , 0 \leq t \leq T \\ 0 & , \text{otherwise} \end{cases}, \quad S_{vv}(f) = \frac{1}{T} \cdot \sigma_I^2 \cdot |T \sin c(Tf)|^2 = T \sigma_I^2 \sin^2(Tf)$$

In general, for memoryless modulation with non-zero mean value,

$$R_{II}[k] = \begin{cases} \sigma_I^2 + \mu_I^2 & , k = 0 \\ \mu_I^2 & , k \neq 0 \end{cases} = \sigma_I^2 \delta[k] + \sum_{n \neq 0} \mu_I^2 \delta[k - n]$$

$$\Rightarrow S_{II}(f) = \sigma_I^2 + \mu_I^2 \sum_k e^{-j2\pi k f T} = \sigma_I^2 + \frac{\mu_I^2}{T} \sum_k \delta(f - \frac{k}{T})$$

$$\Rightarrow S_{vv}(f) = \frac{1}{T} S_{II}(f) |G(f)|^2 \quad (= \sigma_I^2, \text{ if } \mu_I = 0)$$



- Example : Power Spectral Density of QPSK modulation**

$$S_i(t) = \sqrt{\frac{2E}{T}} \cos[(2i-1)\frac{\pi}{4}]g(t) \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \sin[(2i-1)\frac{\pi}{4}]g(t) \sin(2\pi f_c t)$$

$$= \text{Re} \left\{ \tilde{S}_i e^{j2\pi f_c t} g(t) \right\}, 0 \leq t \leq T$$

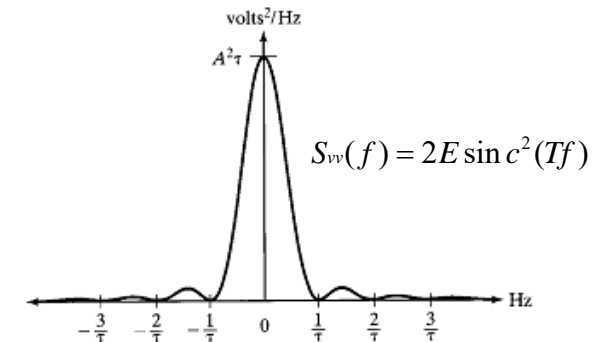
where  $g(t) = \begin{cases} 1 & , 0 \leq t \leq T \\ 0 & , \text{otherwise} \end{cases}$  ,  $\tilde{S}_i = \pm\sqrt{\frac{E}{T}} \pm j\sqrt{\frac{E}{T}} = S_I + jS_Q$

Let the baseband waveform be  $v(t) = \sum_n \tilde{I}_n g(t - nT)$

where  $\tilde{I}_n = S_I + jS_Q$   $S_I, S_Q \in \left\{ \pm\sqrt{\frac{E}{T}} \right\}$

$$R_{II}[k] = \begin{cases} E[|\tilde{S}_i|^2] = \frac{2E}{T} & , k = 0 \\ 0 & , k \neq 0 \end{cases}$$

$$\bar{R}_{vv}(\tau) = \frac{1}{T} \sum_k R_{II}[k] R_{gg}(\tau - kT) = \frac{1}{T} E[|\tilde{S}_i|^2] \cdot R_{gg}(\tau)$$



$$S_{vv}(f) = \frac{1}{T} \cdot E[|\tilde{S}_i|^2] \cdot |G(f)|^2 = \frac{1}{T} \cdot \frac{2E}{T} \cdot |T \sin c(Tf)|^2 = 2E \sin^2(Tf)$$

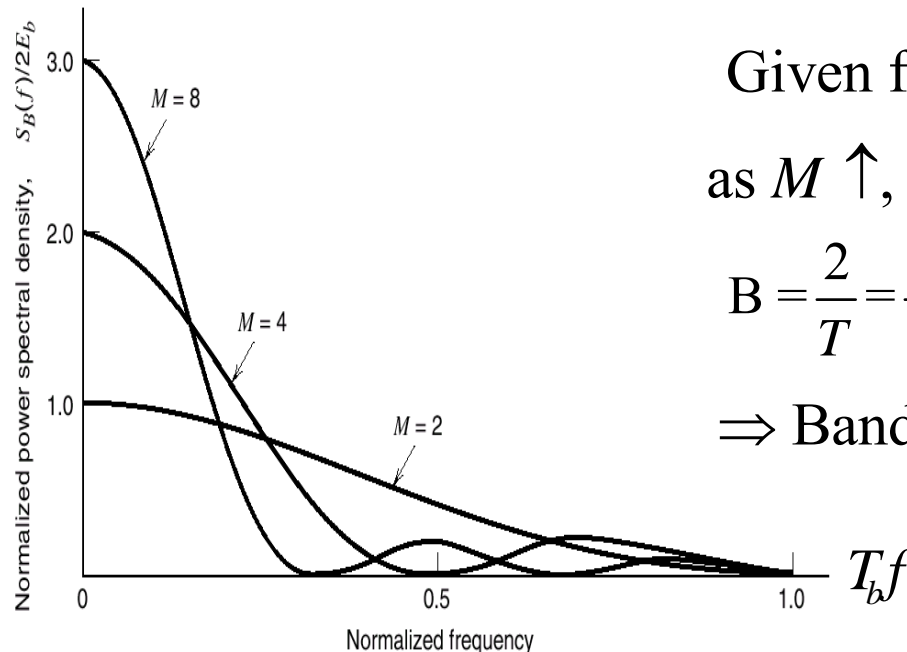
# Power Spectral Density of MPSK modulation

$$S_i(t) = \sqrt{\frac{2E}{T}} \cos[(2i-1)\frac{\pi}{M}] \cos(2\pi fct) - \sqrt{\frac{2E}{T}} \sin[(2i-1)\frac{\pi}{M}] \sin(2\pi fct), 0 \leq t \leq T$$

$$= \text{Re} \{ \tilde{S}_i e^{j2\pi fct} g(t) \}$$

$$S_{vv}(f) = \frac{1}{T} \cdot E[|\tilde{S}_i|^2] \cdot |G(f)|^2 = \frac{1}{T} \cdot \frac{2E}{T} \cdot |T \text{sinc}(Tf)|^2 = 2E \text{sinc}^2(Tf)$$

$$= 2E \text{sinc}^2(fT_b \log_2 M)$$



Given fixed  $T_b$ , i.e. fixed bit rate,

as  $M \uparrow$ , the required transmission bandwidth

$$B = \frac{2}{T} = \frac{2}{T_b \log_2 M} \downarrow$$

$$\Rightarrow \text{Bandwidth efficiency } \rho = \frac{R_b}{B} \uparrow$$

What has to pay for increased  $\rho$ ?

$$\Rightarrow P_e \uparrow$$

# Power Spectral Density of MSK modulation

- Consider a single transmission of a MSK signal,

$$\begin{aligned}
 s(t) &= \sqrt{\frac{2E}{T}} \cos[2\pi f_c t + \theta(t)] \\
 &= \underbrace{\sqrt{\frac{2E}{T}} \cos[\theta(t)]}_{s_I(t)} \cos(2\pi f_c t) - \underbrace{\sqrt{\frac{2E}{T}} \sin[\theta(t)]}_{s_Q(t)} \sin(2\pi f_c t), \quad 0 \leq t \leq T
 \end{aligned}$$

where  $\theta(t) = \theta(0) \pm \frac{\pi}{2T}t$ , (i.e.  $h = \frac{1}{2}$ )

The baseband PSD is the combination of  $s_I(t)$  and  $s_Q(t)$ ,

- For  $s_I(t)$ , assume  $\theta(0) = 0$  or  $\pi$ , i.e.  $S_B(f) = S_I(f) + S_Q(f)$

$$\begin{aligned}
 s_I(t) &= \sqrt{\frac{2E}{T}} \cos\left(\theta(0) \pm \frac{\pi t}{2T}\right) = \sqrt{\frac{2E}{T}} \left[ \cos \theta(0) \cos\left(\pm \frac{\pi t}{2T}\right) - \sin \theta(0) \sin\left(\pm \frac{\pi t}{2T}\right) \right] \\
 &= \sqrt{\frac{2E}{T}} \cos \theta(0) \cos\left(\pm \frac{\pi t}{2T}\right) = \cos \theta(0) g_I(t) \quad \text{where } g_I(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left(\frac{\pi t}{2T}\right), & -T \leq t \leq T \\ 0, & \text{other} \end{cases}
 \end{aligned}$$

$$\Rightarrow \text{At } t = 0, s_I(t) = \sqrt{\frac{2E}{T}} \cos \theta(0)$$

$$\text{The PSD of } s_I(t) \text{ is } S_I(f) = \frac{1}{2T} |G_I(f)|^2 = \frac{1}{2T} \frac{32ET}{\pi^2} \left[ \frac{\cos(2\pi T f)}{16T^2 f^2 - 1} \right]^2 = \frac{16E}{\pi^2} \left[ \frac{\cos(2\pi T f)}{16T^2 f^2 - 1} \right]^2$$

# Power Spectral Density of MSK modulation

- For  $s_Q(t)$ , assume  $\theta(0) = 0$  or  $\pi$ ,

$$s_Q(t) = \sqrt{\frac{2E}{T}} \sin\left(\theta(0) \pm \frac{\pi t}{2T}\right) = \sqrt{\frac{2E}{T}} \left[ \sin \theta(0) \cos\left(\pm \frac{\pi t}{2T}\right) + \cos \theta(0) \sin\left(\pm \frac{\pi t}{2T}\right) \right]$$

$$= \pm \sqrt{\frac{2E}{T}} \cos \theta(0) \sin\left(\pm \frac{\pi t}{2T}\right) = \pm \cos \theta(0) g_Q(t)$$

where  $g_Q(t) = \begin{cases} \sqrt{\frac{2E}{T}} \sin\left(\frac{\pi t}{2T}\right), & 0 \leq t \leq 2T \\ 0 & , \text{ other} \end{cases}$

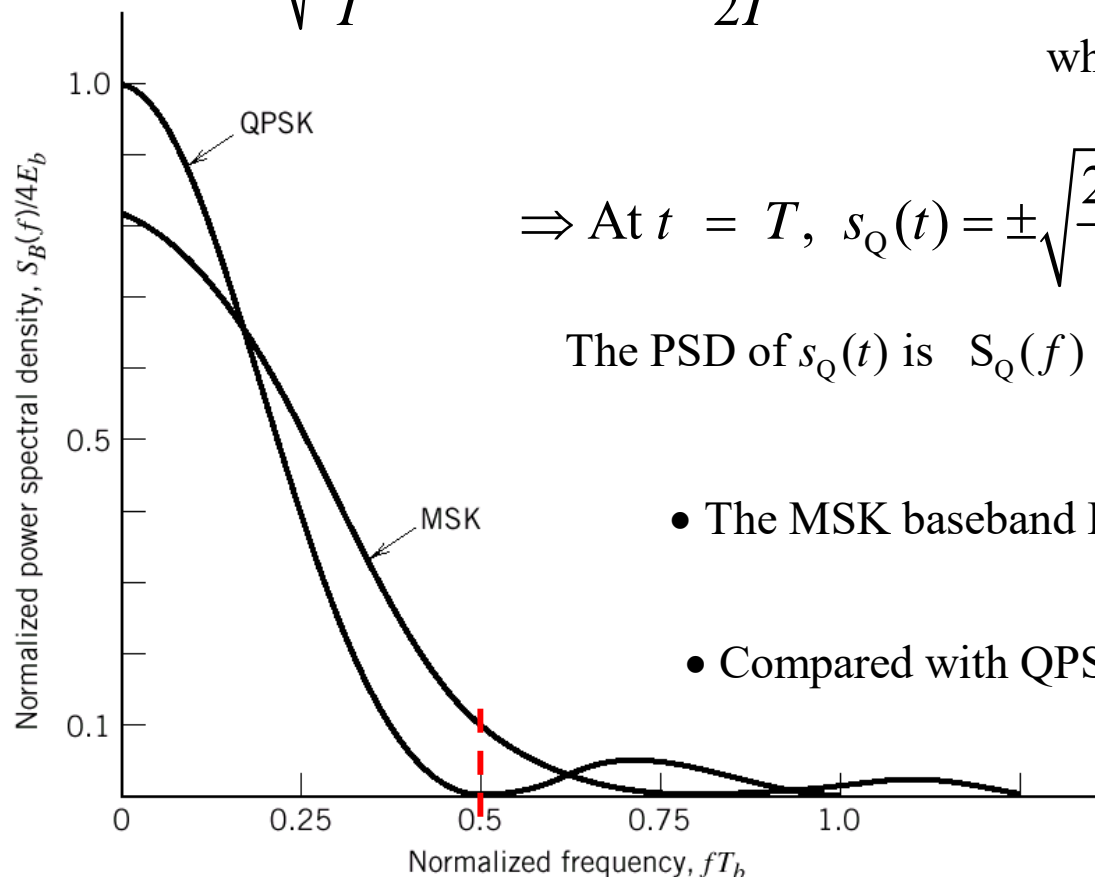
$$\Rightarrow \text{At } t = T, s_Q(t) = \pm \sqrt{\frac{2E}{T}} \cos \theta(0)$$

The PSD of  $s_Q(t)$  is  $S_Q(f) = \frac{1}{2T} |G_Q(f)|^2 = \frac{16E}{\pi^2} \left[ \frac{\cos(2\pi T f)}{16T^2 f^2 - 1} \right]^2$

- The MSK baseband PSD is  $S_B(f) = \frac{32E}{\pi^2} \left[ \frac{\cos(2\pi T f)}{16T^2 f^2 - 1} \right]^2$

- Compared with QPSK, the MSK achieves the same BER,

*but* MSK produces less interference outside the channel bandwidth.



# HW#2

## Due: TBD

# Midterm I

- Time: TBD
- Place: Delta 216
- Coverage: Ch2 and Ch3
- Remark: Closed book. Open 1 sheet of A4 size note (two sides). Non-programmable calculator is allowed.