COM5120 Communication theory

Homework #4

Due: 12/14/2021 (Tuesday)

1. (20%) Random variable X over four symbols {a, b, c, d} is the input source to a communication channel, and the output from this channel is a random variable Y over the same four symbols {a, b, c, d}. Following is the joint distribution of these two random variables, please find:

	$x = a$	$x = b$	$x = c$	$x = d$
$y = a$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$
$y = b$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$	0
$y = c$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{16}$	0
$y = d$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{16}$	0

(1) Marginal entropy of X, H(X) and Marginal entropy of Y, H(Y) in bits.
(2) Joint entropy H(X,Y) of the two random variables in bits.
(3) Conditional entropy H(X|Y) and H(Y|X) in bits.
(4) Mutual information I(X;Y) between the two random variables in bits.
(5) Channel capacity for this channel in bits.

2. (20%) Find the differential entropy of the continuous random variable X in the following case:

   (1) X is an exponential random variable with $\lambda > 0$

$$p(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda}, & x > 0 \\ 0, & otherwise \end{cases}$$

   (2) X is an Laplacian random variable with $\lambda > 0$

$$p(x) = \frac{1}{2\lambda} e^{-|x|/\lambda}$$

3. (20%) Consider an additive white Gaussian noise channel with the output $Y = X + N$, where X is a white Gaussian input with E(X) =0 and E(X^2) = σ_x^2. Given the pdf of noise N with

$$p(n) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-n^2/2\sigma_n^2}$$

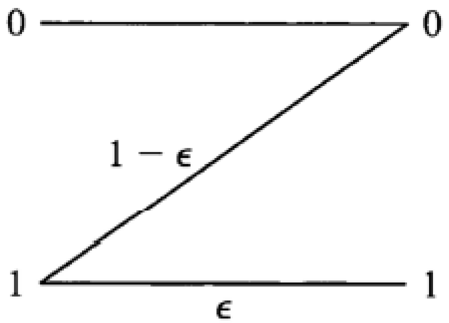
   (1) Determine the conditional differential entropy H(X|N).

   (2) Compute the mutual information I(X;Y).

4. (20%) Let X, Y and Z be the joint random variables. Prove the following inequalities and find conditions for equality.
   (Hint: One can use Venn's diagram to help you understand)

   (1) H(X, Y | Z) $\geq$ H(X|Z).
   (2) I(X, Y; Z) $\geq$ I(X; Z| Y).
   (3) H(X, Y, Z) − H(X| Y, Z) $\geq$ H(Z| X, Y).
   (4) I(X; Z | Y) $\geq$ I(Z; Y | X) − I(Z; Y) − I(X; Z).

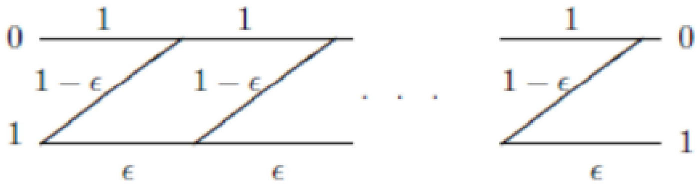
5. (20%) The Z channel is shown below:



(1) Find the input probability distribution p(0) and p(1), that maximize the channel capacity.

(2) What is the input distribution and capacity for the special case $\epsilon = 0$, $\epsilon = 1$, and $\epsilon = 0.57$?

(3) Show that if n such channels are cascaded as following figure, the resulting channel will be equivalent to a Z channel with $\epsilon_1 = \epsilon^n$.



(4) Following from (3), what is the capacity of the equivalent Z channel when n $\rightarrow$ ∞ ?