

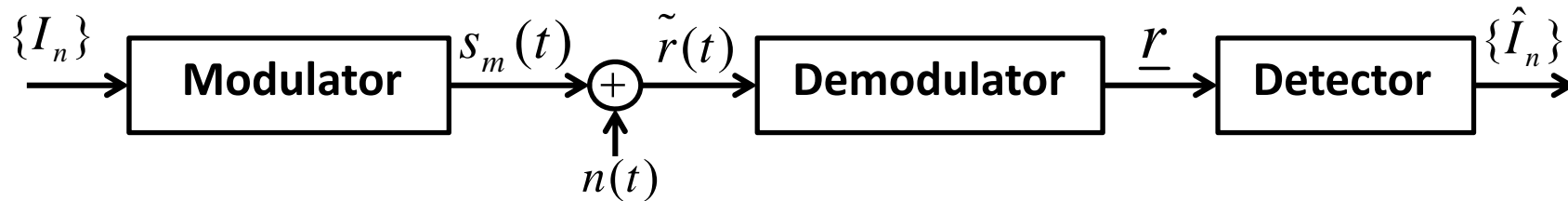
**COM 5120**  
**Communications Theory**

**Chapter 4**  
**Optimal Receiver for AWGN**

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## 4.1 Optimal Detection for a General Vector Channel



Let  $\{s_m(t), m = 1, 2, \dots, M\}$  be the  $M$  transmitted signal waveforms.

The received signal is

$$\tilde{r}(t) = s_m(t) + n(t) \quad 0 \leq t \leq T$$

$$n(t) \sim \mathcal{N}(0, \frac{N_0}{2})$$

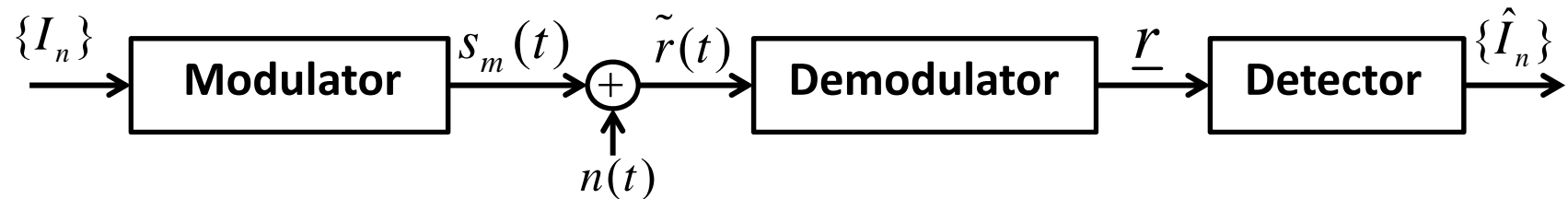
Suppose the basis set  $\{\phi_1(t), \dots, \phi_N(t)\}$  is the minimum set for spanning  $\{s_m(t)\}$

$$s_m(t) = \sum_{k=1}^N s_{mk} \phi_k(t) \quad k = 1, \dots, N$$

$$s_{mk} = \langle s_m(t), \phi_k(t) \rangle = \int_0^T s_m(t) \phi_k(t) dt$$

$$\underline{s}_m = [s_{m1}, s_{m2}, \dots, s_{mN}]^T$$

## 4.1 Optimal Detection for a General Vector Channel



Note that, in general,

$$\tilde{r}(t) = s_m(t) + n(t) \notin \text{span}\{\phi_1(t), \dots, \phi_N(t)\} \quad \text{Why?}$$

$$\because n(t) \notin \text{span}\{\phi_1(t), \dots, \phi_N(t)\}$$

Suppose that

$$\tilde{r}(t) \in \text{span}\{\phi_1(t), \dots, \phi_K(t)\} \quad K \geq N$$

$$\tilde{r}(t) = \sum_{k=1}^K r_k \phi_k(t) \quad \underline{r} = [r_1, r_2, \dots, r_K]^T$$

## 4.1 Optimal Detection for a General Vector Channel

**Objective :** Based on  $\tilde{r}(t)$  or equivalently  $\underline{\tilde{r}}$ , the receiver wants to find the optimally estimated  $\hat{\underline{s}}_m$ .

The detector is a function that maps  $\underline{\tilde{r}}$  to one of the signals  $\{\underline{s}_1, \dots, \underline{s}_M\}$ .

### Definition of Optimality:


The optimal detector is the detector that minimize  $\Pr(\text{error})$  i.e

$$D(\underline{\tilde{r}}) = \underline{s}_m \text{ if } P(\underline{s}_m | \underline{\tilde{r}}) \geq P(\underline{s}_n | \underline{\tilde{r}}) \quad \forall n \neq m$$

$$D(\underline{\tilde{r}}) = \arg \max_{\underline{s} \in \{\underline{s}_1, \dots, \underline{s}_M\}} P(\underline{s} | \underline{\tilde{r}})$$

This is called *Maximum A Posteriori* (MAP) detector

$$\hat{\underline{s}} = \arg \max_{\underline{s} \in \{\underline{s}_1, \dots, \underline{s}_M\}} P(\underline{s} | \underline{\tilde{r}})$$

  
N-dimension      K-dimension

# Theorem of Irrelevance

For MAP detector, to represent the received signal, it is sufficient to have

$$r(t) = \sum_{k=1}^N r_k \phi_k(t) \text{ where } r_k = \langle \tilde{r}(t), \phi_k(t) \rangle$$

Given the received signal  $\tilde{r}(t) = \sum_{k=1}^K r_k \phi_k(t) = s_m(t) + n(t)$

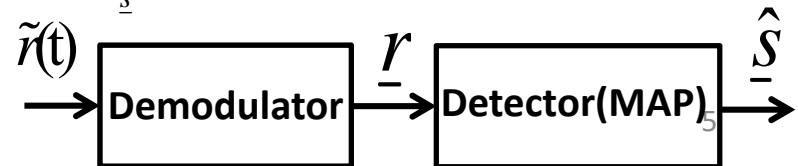
where  $r_k = \langle \tilde{r}(t), \phi_k(t) \rangle = \langle s_m(t), \phi_k(t) \rangle + \langle n(t), \phi_k(t) \rangle$

$$= \begin{cases} s_{mk} + n_k & 1 \leq k \leq N \\ n_k & k > N \end{cases}$$

The noise vector  $\underline{\tilde{n}} = [n_1, n_2, \dots, n_K]^T$  and  $\underline{\tilde{n}} \sim \mathcal{N}(0, \frac{N_0}{2} \mathbf{I}_K)$ , i.e.  $E[n_k] = 0$ , and  $E[n_k n_m] = \frac{N_0}{2} \delta_{km}$

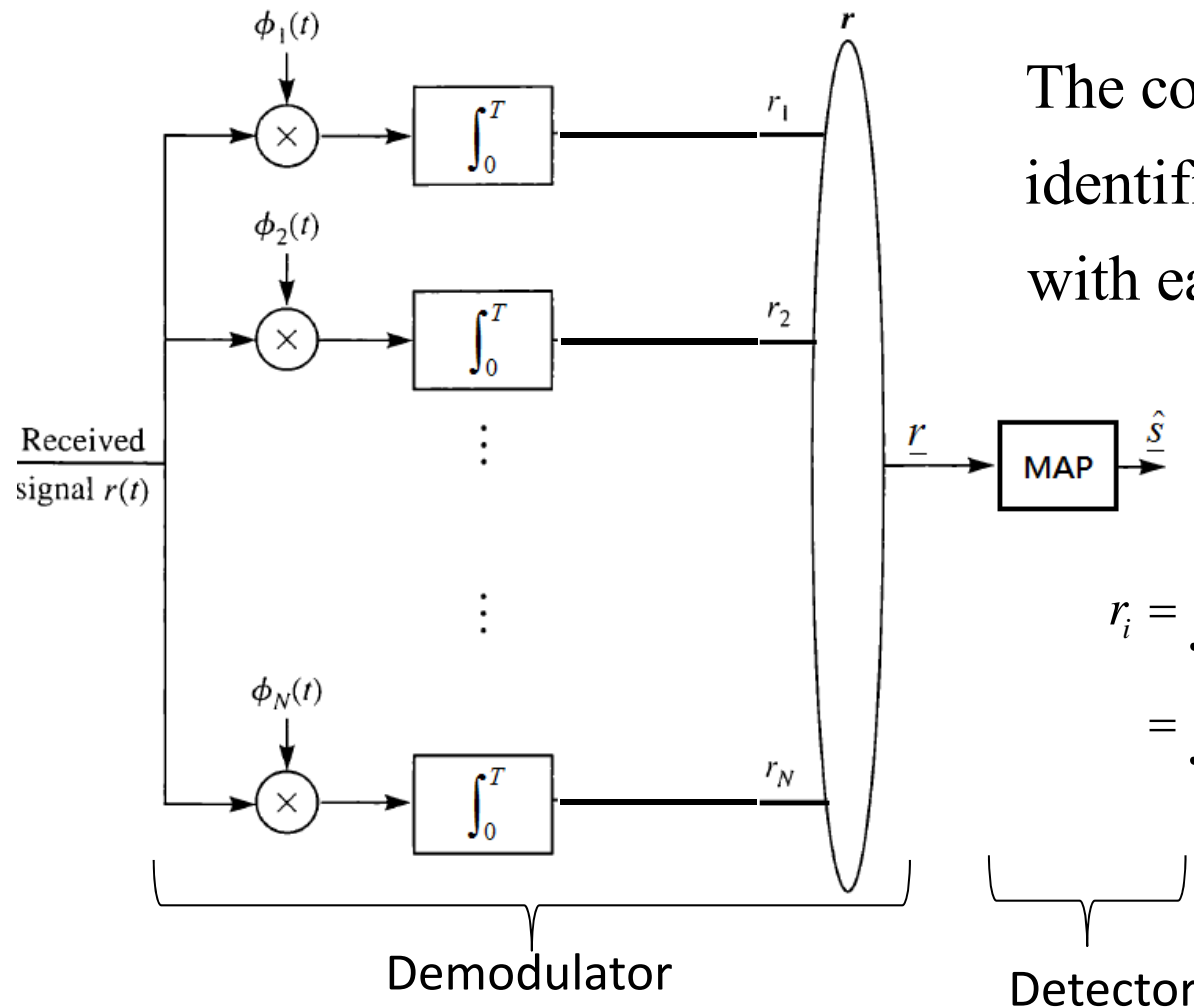
where  $\{n_k\}_{k=N+1}^K$  are mutually independent and independent of  $\underline{s}_m$

$$\begin{aligned} \text{Therefore } \hat{\underline{s}} &= \arg \max_{\underline{s}} P(\underline{s} | \underline{\tilde{r}}) = \arg \max_{\underline{s}} \frac{P(\underline{s}, \underline{\tilde{r}})}{P(\underline{\tilde{r}})} = \arg \max_{\underline{s}} \frac{P(\underline{s}) P(\underline{\tilde{r}} | \underline{s})}{P(\underline{\tilde{r}})} \\ &= \arg \max_{\underline{s}} \frac{P(\underline{s}) P(r_1, \dots, r_N, r_{N+1}, \dots, r_K | \underline{s})}{\sum_{\underline{s}} P(\underline{s}) P(r_1, \dots, r_N, r_{N+1}, \dots, r_K | \underline{s})} = \arg \max_{\underline{s}} \frac{P(\underline{s}) P(r_1, \dots, r_N | \underline{s}) P(r_{N+1}, \dots, r_K)}{\sum_{\underline{s}} P(\underline{s}) P(r_1, \dots, r_N | \underline{s}) P(r_{N+1}, \dots, r_K)} \\ &= \arg \max_{\underline{s}} \frac{P(\underline{s}) P(\underline{r} | \underline{s})}{P(\underline{r})} = \arg \max_{\underline{s}} P(\underline{s} | \underline{r}) \end{aligned}$$



## 4.2 Waveform and Vector AWGN Channel

### ● Correlation demodulator



The correlation demodulator identifies the correlation of  $r(t)$  with each basis  $\phi_k(t)$ ,  $k = 1, \dots, N$ .

$$\begin{aligned} r_i &= \int_0^T r(t) \phi_i(t) dt \\ &= \int_0^T s(t) \phi_i(t) dt + \int_0^T n(t) \phi_i(t) dt \\ &\quad i = 1, \dots, N \end{aligned}$$

- ✓ The receiver architecture generally contains two parts: demodulator, and detector.

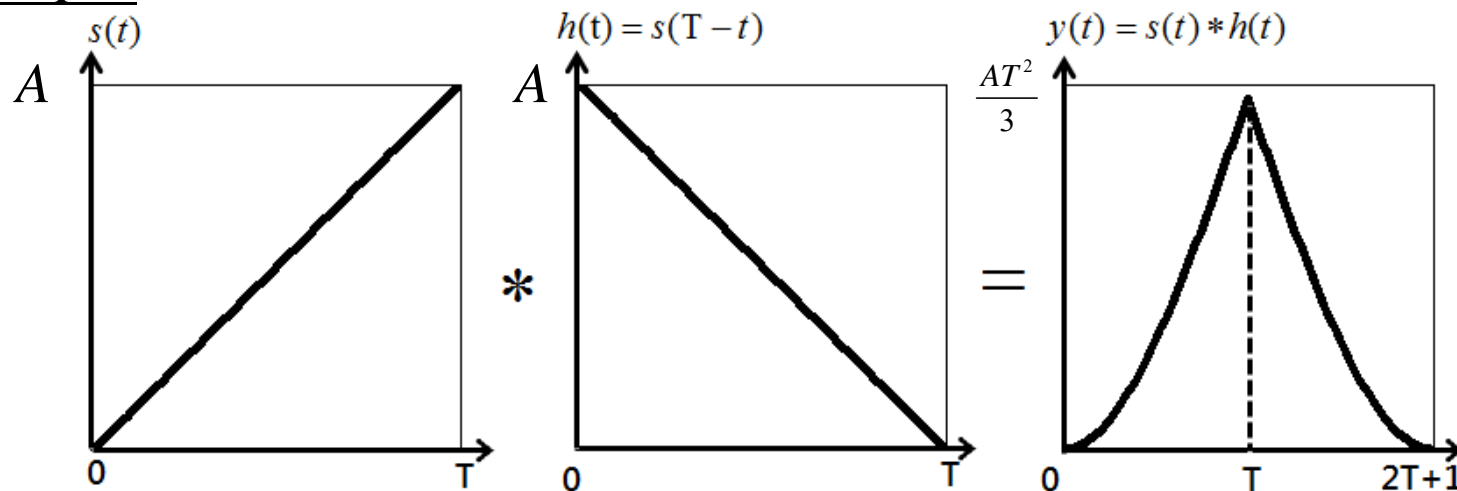
# Matched Filter Demodulator

## ● Matched Filter

### Definition

For a signal  $s(t)$  which is zero outside  $[0, T]$ , a filter with impulse response  $h(t) = s(T - t)$  is a matched filter for  $s(t)$ .

### Example

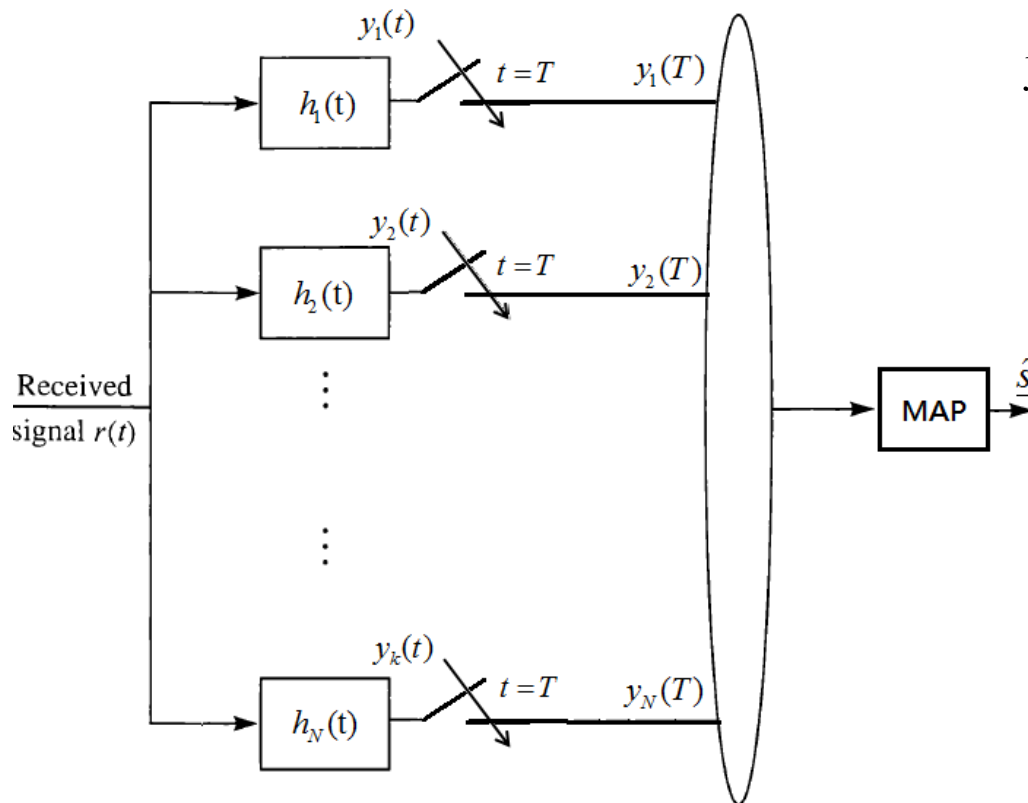


$$y(T) = \int_0^T |s(t)|^2 dt = \frac{AT^2}{3}$$

The maximum occurs at  $t=T$

# Matched Filter Demodulator

For the matched filter demodulator, the  $N$  filters  $h_k(t)$  are matched to the basis  $\{\phi_k(t)\}$ , i.e.  $h_k(t) = \phi_k(T-t)$ .



$$y_k(t) = \int_0^t r(\tau) h_k(t - \tau) d\tau$$

$$= \int_0^t r(\tau) \phi_k(T - t + \tau) d\tau$$

Q: Is the MF demodulator equivalent to correlation demodulator?

Sampling at  $t = T$

$$y_k(T) = \int_0^T r(\tau) \phi_k(\tau) d\tau$$

➤ The MF demodulator is equivalent to correlation demodulator.



# Matched Filter Demodulator

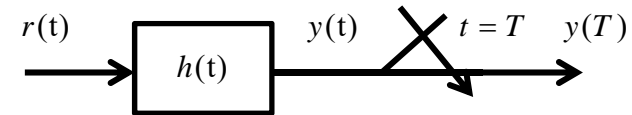
**Theorem:** The MF demodulator maximizes the SNR for demodulator output, i.e.

For  $r(t) = s(t) + n(t)$  where  $n(t) \sim \mathcal{N}(0, \frac{N_0}{2})$  and  $s(t)$  is confined within  $[0 T]$ ,

then the SNR of  $y(T)$  is maximized if  $h(t)$  matched to  $s(t)$ .

Proof

$$y(t) = \int_0^t r(\tau) h(t-\tau) d\tau$$



$$= \int_0^t s(\tau) h(t-\tau) d\tau + \int_0^t n(\tau) h(t-\tau) d\tau$$

Sampling at  $t = T$

$$y(T) = \underbrace{\int_0^T s(\tau) h(T-\tau) d\tau}_{y_s(T)} + \underbrace{\int_0^T n(\tau) h(T-\tau) d\tau}_{y_n(T)}$$

## Proof (cont.)

$$\Rightarrow SNR = \frac{y_s^2(T)}{E[y_n^2(T)]}$$

$$\begin{aligned} E[y_n^2(T)] &= \int_0^T \int_0^T \underbrace{E[n(\tau_1)n(\tau_2)]}_{=\frac{N_0}{2}\delta(\tau_1-\tau_2)} h(T-\tau_1)h(T-\tau_2) d\tau_1 d\tau_2 \\ &= \frac{N_0}{2} \int_0^T h^2(T-\tau) d\tau \end{aligned}$$

$$SNR = \frac{\left( \int_0^T s(\tau) h(T-\tau) d\tau \right)^2}{\frac{N_0}{2} \int_0^T h^2(T-\tau) d\tau} \leq \frac{\left( \int_0^T s^2(\tau) d\tau \right) \left( \int_0^T h^2(T-\tau) d\tau \right)}{\frac{N_0}{2} \int_0^T h^2(T-\tau) d\tau} = \frac{2E}{N_0}$$

Recall Cauch-Schwaz Inequality

$$\left( \int_0^T g_1(t) g_2(t) dt \right)^2 \leq \left( \int_0^T g_1^2(t) dt \right) \left( \int_0^T g_2^2(t) dt \right)$$

"=" holds when  $g_1(t) = c g_2(t)$

The "=" holds when

$$h(T-\tau) = cs(\tau) \Rightarrow h(t) = cs(T-t)$$

## MAP and Maximum Likelihood (ML) Detector

The MAP  $\hat{\underline{s}} = \arg \max_{\underline{s} \in \{\underline{s}_1, \underline{s}_2, \dots, \underline{s}_M\}} P(\underline{s} | \underline{r})$

$$= \arg \max_{\underline{s}} \frac{P(\underline{s})P(\underline{r} | \underline{s})}{P(\underline{r})}$$
$$= \arg \max_{\underline{s}} P(\underline{s})P(\underline{r} | \underline{s})$$

If  $P(\underline{s}_m) = \frac{1}{M} \quad \forall m$  (i.e. equal probability)

then  $\hat{\underline{s}} = \arg \max_{\underline{s}} P(\underline{r} | \underline{s})$

This is called maximum likelihood detector (ML Detector)

$P(\underline{r} | \underline{s})$  is called the likelihood function.

- In most cases, usually  $P(\underline{s} | \underline{r})$  is not available, but  $P(\underline{r} | \underline{s})$  is available (subject to the channel models).

# MAP and Maximum Likelihood Detector

## Example (AWGN channel case)

For  $\underline{r} = \underline{s} + \underline{n}$  where  $\underline{n} \sim \mathcal{N}(0, \frac{N_0}{2} \mathbf{I}_N)$  assume  $\underline{s}_m$  is equally probable.

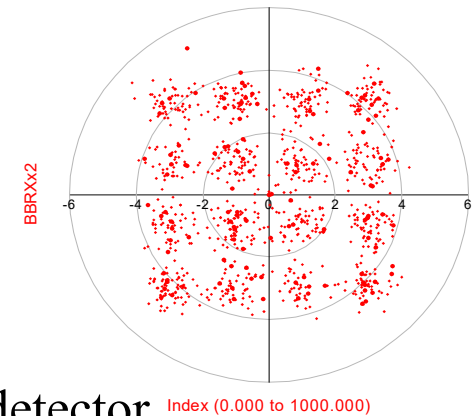
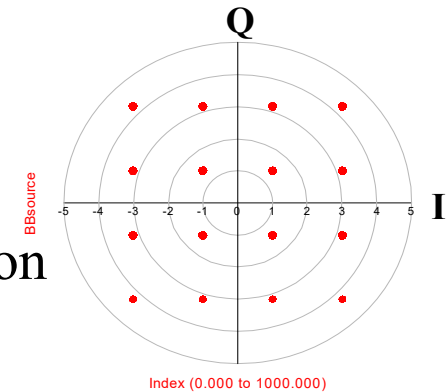
The MAP detector becomes ML detector.

$$\begin{aligned} \hat{\underline{s}} &= \underset{\underline{s}}{\operatorname{argmax}} p(\underline{r} | \underline{s}) \\ &= \underset{\underline{s}}{\operatorname{argmax}} \left( \frac{1}{\pi N_0} \right)^{\frac{N}{2}} \exp\left(-\frac{1}{N_0} \|\underline{r} - \underline{s}\|^2\right) \rightarrow \text{likelihood function} \\ &= \underset{\underline{s}}{\operatorname{argmax}} \ln(f(\underline{r} | \underline{s})) \rightarrow \text{log-likelihood function} \\ &= \underset{\underline{s}}{\operatorname{argmax}} -\frac{1}{N_0} \|\underline{r} - \underline{s}\|^2 \\ &= \underset{\underline{s}}{\operatorname{argmin}} \|\underline{r} - \underline{s}\|^2 \Rightarrow \text{Minimum distance(MD) detector} \end{aligned}$$

If  $\|\underline{s}_m\|^2 = E_s \quad \forall m$  (e.g. MPSK)

$$\hat{\underline{s}} = \underset{\underline{s}}{\operatorname{argmin}} \|\underline{r}\|^2 - 2\underline{r} \cdot \underline{s} + \|\underline{s}\|^2 = \underset{\underline{s}}{\operatorname{argmax}} \underline{r} \cdot \underline{s} \rightarrow \text{max correlation detector}$$

✓ With different conditions, the detector can be reformulated.



## 4.3 Optimal detection and error probability for band limited and power limited signals

- Binary PAM(Binary Antipodal Signaling)

$$s_1(t) = Ag(t) \cos(2\pi f_c t) \quad 0 \leq t \leq T_b$$

$$s_2(t) = -Ag(t) \cos(2\pi f_c t) \quad 0 \leq t \leq T_b$$

1-D signal with basis with  $\phi(t) = \sqrt{\frac{2}{E_g}} g(t) \cos(2\pi f_c t)$

$$s_1 = \int_0^{T_b} s_1(t) \phi(t) dt = A \sqrt{\frac{E_g}{2}} = \sqrt{E_b} \quad E_b = \|s_1(t)\|^2 = \|s_2(t)\|^2$$

$$s_2 = \int_0^{T_b} s_2(t) \phi(t) dt = -A \sqrt{\frac{E_g}{2}} = -\sqrt{E_b}$$

The received signal  $r(t) = s(t) + n(t)$

After the decorrelator  $r = s + n$

$$\text{where } s = \begin{cases} \sqrt{E_b} & \text{if } s_1 \text{ transmitted} \\ -\sqrt{E_b} & \text{if } s_2 \text{ transmitted} \end{cases} \quad n \sim \mathcal{N}(0, \frac{N_0}{2})$$

- MAP detector

$$\hat{s} = \underset{s \in \{s_1, \dots, s_M\}}{\operatorname{argmax}} p(s | r)$$

Q: How to find  $p(s | r)$ ?

$$\hat{s} = \underset{s}{\operatorname{argmax}} p(s) p(r | s)$$

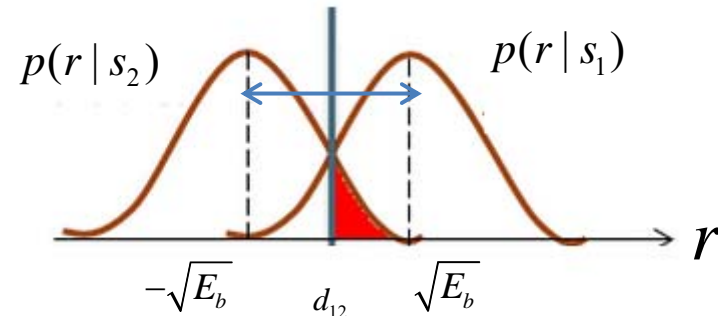
Assume equal probability of  $s$ ,

$$\begin{aligned} \hat{s} &= \underset{s}{\operatorname{argmax}} p(r | s) \\ &= \underset{s}{\operatorname{argmin}} |r - s|^2 \\ &= \underset{s}{\operatorname{argmax}} r \cdot s \end{aligned}$$

$$D_1 = \{r : r\sqrt{E_b} > -r\sqrt{E_b}\}$$

$$D_2 = \{r : -r\sqrt{E_b} > r\sqrt{E_b}\}$$

→ Given  $p(s)$  and probability model of noise,  $p(r | s)$  can be obtained based on received  $r$ .



$$p(r | s_i) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{1}{N_0} (r - s_i)^2\right), i = 1, 2$$

→ Hence  $p(s | r)$  can be obtained.

## Error probability:

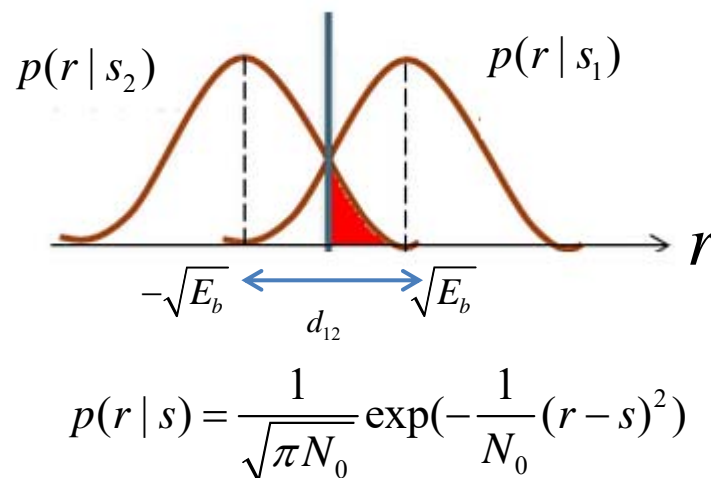
$$P(\text{error} | s_2) = P(\hat{s} = s_1 | s_2) = P(r \in D_1 | s_2)$$

$$= \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r + \sqrt{E_b})^2}{N_0}\right) dr$$

$$= \int_{\sqrt{E_b}}^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x^2}{N_0}\right) dx$$

$$= Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$\text{where } Q\left(\frac{x}{\sigma_x}\right) = \int_x^\infty \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left(-\frac{t^2}{2\sigma_x^2}\right) dt$$



- Overall  $P_e$  and Point to point distance  $d_{12}$

$$P_e = P(s_1)P(\text{error} | s_1) + P(s_2)P(\text{error} | s_2)$$

$$= \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right)$$

$$= Q\left(\frac{d_{12} / 2}{\sqrt{N_0 / 2}}\right)$$

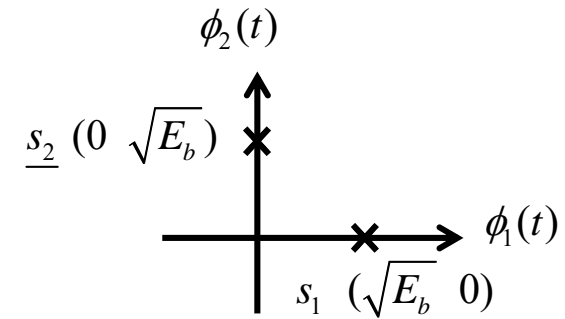
$$= Q\left(\frac{d_{12} / 2}{\sigma_n}\right)$$

As  $d_{12} \uparrow$ ,  $P_e \downarrow$

## ● Binary Orthogonal signal

Two dimension signal

$$\underline{s}_1 = [\sqrt{E_b} \ 0]^T \quad \underline{s}_2 = [0 \ \sqrt{E_b}]^T$$



with the basis  $\phi_1(t)$  and  $\phi_2(t)$

$$\underline{r} = \underline{s} + \underline{n} = \begin{bmatrix} \sqrt{E_b} + n_1 \\ n_2 \end{bmatrix} \text{ or } \begin{bmatrix} n_1 \\ \sqrt{E_b} + n_2 \end{bmatrix}$$

MAP detector  $\hat{\underline{s}} = \underset{\underline{s}}{\operatorname{argmax}} p(\underline{s} | \underline{r})$

$$= \underset{\underline{s}}{\operatorname{argmax}} p(\underline{r} | \underline{s}) \quad \left( \because \text{Equallly probable source symbols} \right)$$

$$= \underset{\underline{s}}{\operatorname{argmin}} |\underline{r} - \underline{s}|^2 \quad \left( \because \text{AWGN channel} \right)$$

$$= \underset{\underline{s}}{\operatorname{argmax}} \underline{r} \cdot \underline{s} \quad \left( \because \text{Equallly energy source symbols} \right)$$

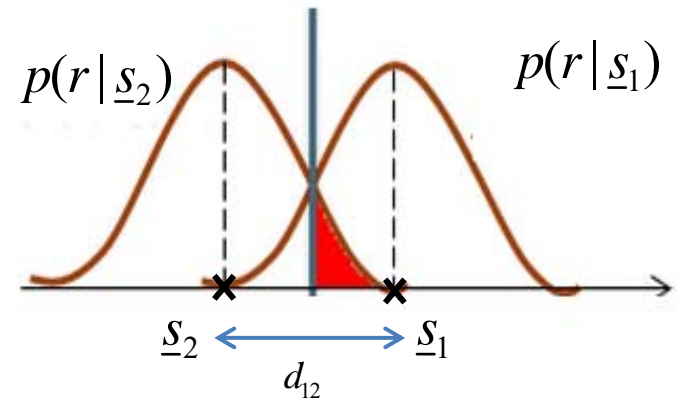
$$D_1 = \{ \underline{r} : \underline{r} \cdot \underline{s}_1 > \underline{r} \cdot \underline{s}_2 \}$$

$$D_2 = \{ \underline{r} : \underline{r} \cdot \underline{s}_1 < \underline{r} \cdot \underline{s}_2 \}$$



Error probability: For the case that  $\underline{s}_2$  was sent,

$$\begin{aligned}
 P(\text{error} \mid \underline{s}_2) &= P(\underline{r} \mid \underline{s}_2) < P(\underline{r} \mid \underline{s}_1) \\
 &= P(\underline{r} \cdot \underline{s}_2 < \underline{r} \cdot \underline{s}_1 \mid \underline{s}_2) \\
 &= P(E_b + \sqrt{E_b} n_2 < \sqrt{E_b} n_1) \\
 &= P(n_1 - n_2 > \sqrt{E_b}) \\
 &= Q\left(\sqrt{\frac{E_b}{N_0}}\right) \quad (\because n_1 - n_2 \sim \mathcal{N}(0, N_0))
 \end{aligned}$$



Point to point distance  $d_{12}$  and  $P_e$

$$\begin{aligned}
 P_e &= P(s_1)P(\text{error} \mid s_1) + P(s_2)P(\text{error} \mid s_2) \\
 &= \frac{1}{2}Q\left(\sqrt{\frac{E_b}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{E_b}{N_0}}\right) \\
 &= Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right) = Q\left(\frac{d_{12}/2}{\sqrt{N_0}/2}\right)
 \end{aligned}$$

As  $d_{12} \downarrow$ ,  $P_e \uparrow$

# Probability Error of M-ary Orthogonal Signals

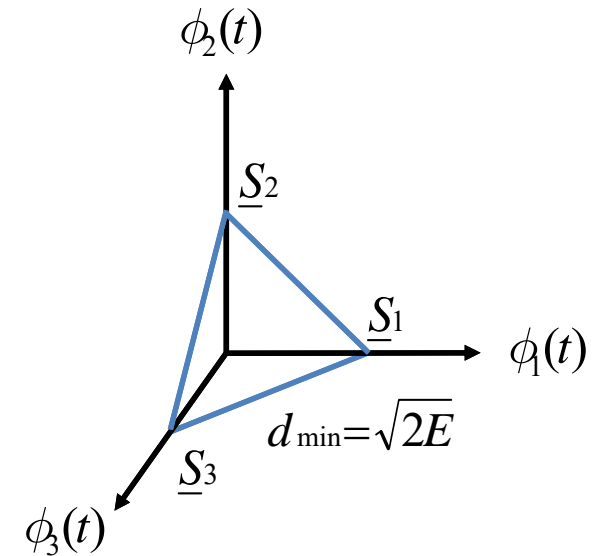
Transmitted symbols

$$\underline{s}_1 = [\sqrt{E_s} \quad 0 \quad \dots \quad 0 \quad 0]^T$$

$$\underline{s}_2 = [0 \quad \sqrt{E_s} \quad 0 \quad \dots \quad 0]^T$$

$$\vdots \quad \quad \quad \ddots$$

$$\underline{s}_M = [0 \quad 0 \quad \dots \quad 0 \quad \sqrt{E_s}]^T$$



Suppose  $\underline{s}_1$  was transmitted

$$\underline{r} = \underline{s}_1 + \underline{n} = \begin{bmatrix} \sqrt{E_s} + n_1 \\ n_2 \\ \vdots \\ n_M \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_M \end{bmatrix}$$

$$p(r_1 | \underline{s}_1) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{1}{N_0} (r_1 - \sqrt{E_s})^2\right)$$

$$p(r_k | \underline{s}_1) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{r_k^2}{N_0}\right),$$

$$k = 2, \dots, M$$

MAP detector

$$\hat{\underline{s}} = \underset{\underline{s}_k}{\operatorname{argmax}} \quad \underline{r} \cdot \underline{s}_k \quad \Rightarrow \quad \underline{r} \cdot \underline{s}_1 = r_1 \cdot \sqrt{E_s} = E_s + \sqrt{E_s} n_1$$

$$\underline{r} \cdot \underline{s}_k = r_k \cdot \sqrt{E_s} = \sqrt{E_s} n_k, \quad k = 2, \dots, M$$

Error probability for  $\underline{s}_1$

$$P(\text{error} \mid \underline{s}_1) = 1 - P(\text{correct} \mid \underline{s}_1)$$

$$P(\text{correct} \mid \underline{s}_1) = P(\underline{r} \cdot \underline{s}_1 > \underline{r} \cdot \underline{s}_k \quad \forall k \neq 1)$$

$$= P(r_1 > n_2, r_1 > n_3, \dots, r_1 > n_M)$$

$$= \int_{-\infty}^{\infty} P(r_1 > n_2, r_1 > n_3, \dots, r_1 > n_M \mid r_1) p(r_1) dr_1$$

Given  $r_1$

$$P(r_1 > n_2, r_1 > n_3, \dots, r_1 > n_M \mid r_1) = \prod_{k=2}^M P(r_1 > n_k \mid r_1)$$

where

$$P(r_1 > n_k \mid r_1) = \int_{-\infty}^{r_1} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{n_k^2}{N_0}\right) dn_k = 1 - Q\left(\frac{r_1}{\sqrt{N_0/2}}\right)$$

Therefore, the correct probability

$$P(\text{correct} | \underline{s}_1) = \int_{-\infty}^{\infty} P(r_1 > n_2, r_1 > n_3, \dots, r_1 > n_M | r_1) p(r_1) dr_1$$

$$= \int_{-\infty}^{\infty} [1 - Q(\frac{r_1}{\sqrt{N_0/2}})]^{M-1} \frac{1}{\sqrt{\pi N_0}} \exp(-\frac{1}{N_0} (r_1 - \sqrt{E_s})^2) dr_1$$

The error probability,

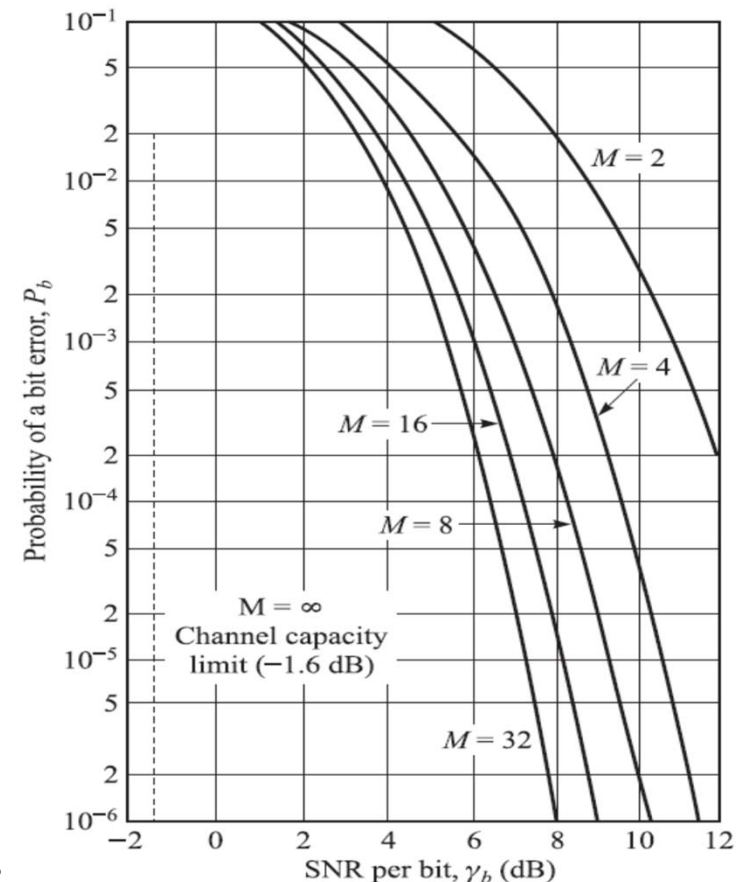
$$P(\text{error} | \underline{s}_1) = 1 - \int_{-\infty}^{\infty} [1 - Q(\frac{r_1}{\sqrt{N_0/2}})]^{M-1} p(r_1) dr_1$$

The overall error probability,

$$P_e = \sum_{i=1}^M P(\underline{s}_i) P(\text{error} | \underline{s}_i) = P(\text{error} | \underline{s}_1)$$

$$M \uparrow \quad R_b \uparrow \quad P_e \uparrow \quad P_b \downarrow$$

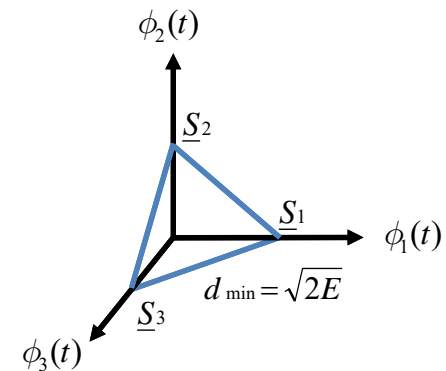
Q: What has to pay? A: More basis or BW.



# Bit error rate of M-ary orthogonal Signals

- To compare different modulation schemes with different M values , it is more fair to convert symbol error  $P_e$  to bit error rate(BER).
- In M-ary orthogonal signals, a symbol  $\underline{s}_m$  has equal probability of having error to any other symbol,  $\underline{s}_l$ .

$$P(\hat{\underline{s}} = \underline{s}_\ell \mid \underline{s} = \underline{s}_m) = \begin{cases} 1 - P_e & \text{if } \ell = m \\ \frac{P_e}{M-1} & \text{if } \ell \neq m \end{cases}$$



Suppose that symbol  $\underline{s}_m$  represents  $K$  bits message  $b_{m1} \ b_{m2} \ ... \ b_{mK}$  and symbol  $\underline{s}_l$  also represents  $K$  bits message  $b_{l1} \ b_{l2} \ ... \ b_{lK}$  ,  $K = \log_2^M$  .

- For a single bit error, there are  $2^{K-1}$  combination of cases for the same symbol error.

# Bit error rate ( $P_b$ ) of M-ary orthogonal Signals

- Example ( $M=8$   $K=3$ )

Assume  $\underline{s}_1$  is sent

$$\left. \begin{array}{l} \underline{s}_1 = 0 \quad 0 \quad 0 \\ \underline{s}_2 = 0 \quad 0 \quad 1 \\ \underline{s}_3 = 0 \quad 1 \quad 0 \\ \underline{s}_4 = 0 \quad 1 \quad 1 \\ \underline{s}_5 = 1 \quad 0 \quad 0 \\ \underline{s}_6 = 1 \quad 0 \quad 1 \\ \underline{s}_7 = 1 \quad 1 \quad 0 \\ \underline{s}_8 = 1 \quad 1 \quad 1 \end{array} \right\} P_c = 1 - P_e(M)$$

$$\left. \begin{array}{l} \underline{s}_1 = 0 \quad 0 \quad 0 \\ \underline{s}_2 = 0 \quad 0 \quad 1 \\ \underline{s}_3 = 0 \quad 1 \quad 0 \\ \underline{s}_4 = 0 \quad 1 \quad 1 \\ \underline{s}_5 = 1 \quad 0 \quad 0 \\ \underline{s}_6 = 1 \quad 0 \quad 1 \\ \underline{s}_7 = 1 \quad 1 \quad 0 \\ \underline{s}_8 = 1 \quad 1 \quad 1 \end{array} \right\} P_e = P(\hat{\underline{s}} \neq \underline{s}_1)$$

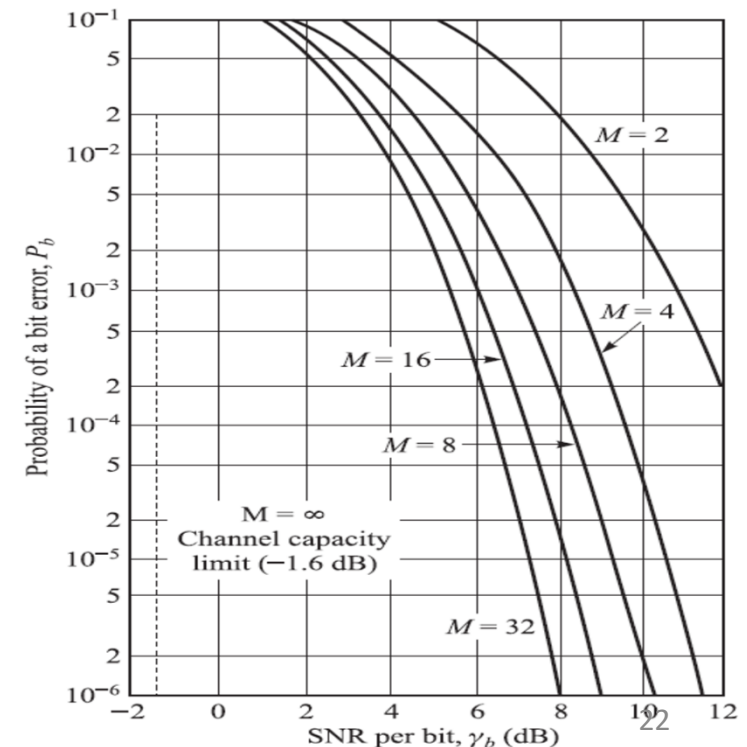
$$\Rightarrow P_b = \frac{4}{7} P_e$$

M	BER ( $P_b$ )
2	$P_b = P_e$ (M=2)
4	$P_b = \frac{2}{3} P_e$ (M=4)
8	$P_b = \frac{4}{7} P_e$ (M=8)
16	$P_b = \frac{8}{15} P_e$ (M=16)

- In general, given M,  $k=\log_2 M$ )

$$P_b = \frac{\frac{M}{2}}{M-1} P_e = \frac{2^{k-1}}{2^k - 1} P_e$$

$$M \uparrow \quad P_b \downarrow \quad \text{as } M \rightarrow \infty \quad P_b \rightarrow \frac{1}{2} P_e$$



# Union Bound of Probability of Error for M-ary Orthogonal Signals

- Union rule: Suppose we have event  $E_1, E_2, \dots, E_M$ , then  $P(\bigcup_{i=1}^M E_i) \leq \sum_{i=1}^M P(E_i)$ .
- Applying the union rule to  $P_e$

$$P_e = \sum_{i=1}^M P(\underline{s}_m) P(\bigcup_{\ell \neq m} \{\underline{r} \cdot \underline{s}_m < \underline{r} \cdot \underline{s}_\ell \mid \underline{s}_m\})$$

$$= P(\bigcup_{\ell \neq 1} \{\underline{r} \cdot \underline{s}_1 < \underline{r} \cdot \underline{s}_\ell \mid \underline{s}_1\})$$

$$\leq \sum_{\ell=2}^M P(\underline{r} \cdot \underline{s}_1 < \underline{r} \cdot \underline{s}_\ell \mid \underline{s}_1)$$

$$= (M-1)P(\underline{r} \cdot \underline{s}_1 < \underline{r} \cdot \underline{s}_2 \mid \underline{s}_1)$$

$$= (M-1)P(E_s + \sqrt{E_s} n_1 < \sqrt{E_s} n_2)$$

$$= (M-1)P(n_2 - n_1 > \sqrt{E_s})$$

$$= (M-1)Q\left(\frac{\sqrt{E_s}}{\sqrt{N_0}}\right) \text{ As } M \uparrow, P_e \uparrow \text{ or } \downarrow ?$$

→ Depends on whether  $E_s$

$$\leq \frac{(M-1)}{2} \exp\left(\frac{-E_s}{2N_0}\right) \text{ is fixed.}$$

$$= \frac{1}{2} (2^K - 1) \exp\left(\frac{-KE_b}{2N_0}\right)$$

$$= \frac{1}{2} (e^{K \ln 2} - 1) \exp\left(\frac{-KE_b}{2N_0}\right)$$

What does it suggest?

$$\leq \frac{1}{2} e^{K(\ln 2 - \frac{E_b}{2N_0})}$$

For  $\frac{E_b}{2N_0} > \ln(2)$ , i.e.  $\frac{E_b}{N_0} > 2 \ln(2) = 1.42$

as  $K \rightarrow \infty$ ,  $P_e \rightarrow 0$

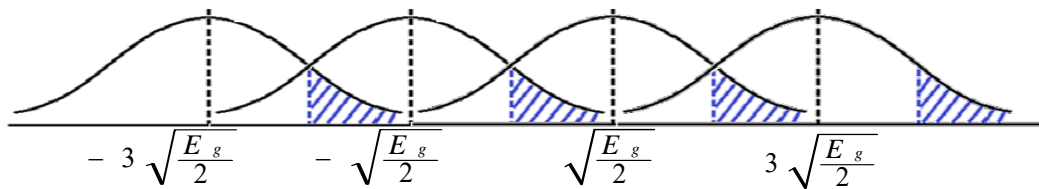
⇒ Reliable communication requirement.

# Probability of Error for M-ary PAM

One dimension signal with

$$s_m = \sqrt{\frac{E_g}{2}} A_m \quad m = 1, \dots, M$$

$$A_m = (2m - 1 - M), m = 1, 2, \dots, M = \{\pm 1, \pm 3, \dots, \pm(M - 1)\}$$



$$d_{\min} = \sqrt{2E_g}$$

MAP detector

$$\hat{s} = \underset{s_m}{\operatorname{argmax}} p(s_m | r) = \underset{s_m}{\operatorname{argmin}} |r - s_m|$$

$$D_m = \{r : |r - s_m| < |r - s_\ell| \quad \forall \ell \neq m\}$$

For inner points,  $m = 2, \dots, M-1$

$$P(\text{error} | s_m) = P(|r - s_m| > \frac{d_{\min}}{2}) \approx 2Q\left(\frac{\frac{d_{\min}}{2}}{\sqrt{\frac{N_0}{2}}}\right) = 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

For outer points,  $m = 1, M$

$$P(\text{error} | s_1) = P(\text{error} | s_M) = Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$



- Error probability (M-PAM)

$$P_e = \sum_{m=1}^M P(s_m) P(\text{error} | s_m) = \frac{1}{M} \sum_{m=2}^{M-1} 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) + \frac{2}{M} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = \frac{2(M-1)}{M} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

With average symbol energy

$$E_{av} = \frac{1}{M} \sum_{m=1}^M E_m = \frac{d_{\min}^2}{4M} \sum_{m=1}^M (2m-1-M)^2$$

$d_{\min} = \sqrt{2E_g}$

$$= \frac{d_{\min}^2}{4M} \sum_{m=1}^M [4m^2 + (M+1)^2 - 4m(M+1)] = \frac{d_{\min}^2}{12} (M^2 - 1) = \frac{E_g}{6} (M^2 - 1)$$

$$\Rightarrow \text{For M-PAM, } d_{\min} = \sqrt{\frac{12E_{av}}{M^2 - 1}} = \sqrt{\frac{12KE_b}{M^2 - 1}}, \text{ where } E_{av} = KE_b$$

$$\Rightarrow P_e = \frac{2(M-1)}{M} Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6KE_b}{(M^2-1)N_0}}\right) \approx 2Q\left(\sqrt{\frac{6(\log_2 M)E_b}{(M^2-1)N_0}}\right) \quad M \uparrow \quad P_e \uparrow$$

Q: If double the size,  $M' = 2M$ , how much more  $E_b$  is required to maintain the same  $P_e$ ?

$$\Rightarrow P_e' = 2Q\left(\sqrt{\frac{6\log_2^{M'} E_b'}{((M')^2 - 1)N_0}}\right) \approx 2Q\left(\sqrt{\frac{6(1 + \log_2^M)E_b'}{4(M^2 - 1)N_0}}\right) \Rightarrow E_b' \approx 4E_b$$

# Probability of Error for M-ary PSK

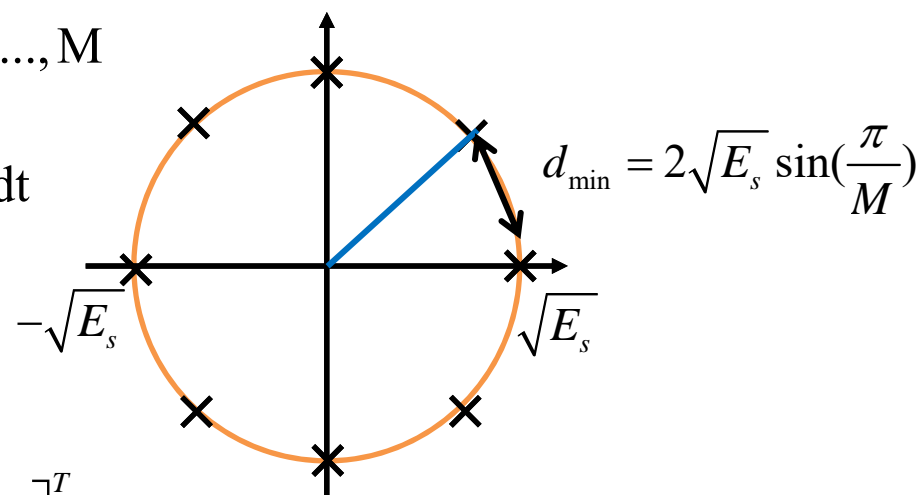
$$s_m(t) = g(t) \cos(2\pi f_c t + \frac{2\pi}{M}(m-1)) \quad m = 1, \dots, M$$

$$E_s = \|s_m(t)\|^2 = \frac{1}{2} E_g \quad \text{where } E_g = \int_0^T g^2(t) dt$$

$$\phi_1(t) = \sqrt{\frac{2}{E_g}} g(t) \cos(2\pi f_c t)$$

$$\phi_2(t) = -\sqrt{\frac{2}{E_g}} g(t) \sin(2\pi f_c t)$$

$$\begin{aligned} \underline{s}_m &= \left[ \sqrt{\frac{E_g}{2}} \cos\left(\frac{2\pi}{M}(m-1)\right), \sqrt{\frac{E_g}{2}} \sin\left(\frac{2\pi}{M}(m-1)\right) \right]^T \\ &= \left[ \sqrt{E_s} \cos\left(\frac{2\pi}{M}(m-1)\right), \sqrt{E_s} \sin\left(\frac{2\pi}{M}(m-1)\right) \right]^T, m = 1, 2, \dots, M \end{aligned}$$

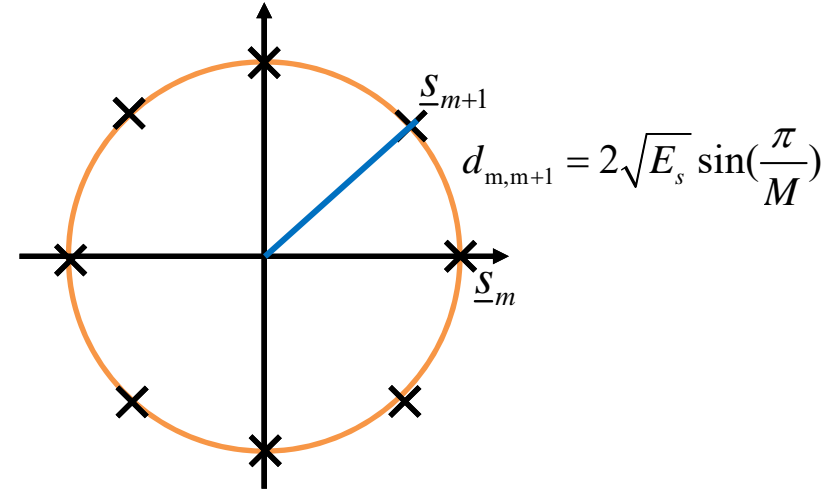


MAP detector (assume AWGN and equal prob. of symbol occurrence )

$$\hat{\underline{s}} = \underset{\underline{s}}{\operatorname{argmax}} p(\underline{s} | \underline{r}) = \underset{\underline{s}}{\operatorname{argmax}} \underline{r} \cdot \underline{s}$$

Choose  $\hat{\underline{s}} = \underline{s}_m$  where  $\underline{r}$  has the largest projection onto  $\underline{s}_m$

$$\begin{aligned}
P_e &= \sum_{i=1}^M P(\underline{s}_m) P(\bigcup_{\ell \neq m} \{\underline{r} \cdot \underline{s}_m < \underline{r} \cdot \underline{s}_\ell \mid \underline{s}_m\}) \\
&= P(\bigcup_{\ell \neq m} \{\underline{r} \cdot \underline{s}_m < \underline{r} \cdot \underline{s}_\ell \mid \underline{s}_m\}) \\
&\leq \sum_{\ell=2}^M P(\underline{r} \cdot \underline{s}_m < \underline{r} \cdot \underline{s}_\ell \mid \underline{s}_m) \\
&\cong P_e(\underline{s}_m, \underline{s}_{m+1}) + P_e(\underline{s}_m, \underline{s}_{m-1})
\end{aligned}$$



$P_e(\underline{s}_m, \underline{s}_\ell)$  represents the pairwise error probability when  $\underline{s}_m$  is sent and  $\underline{s}_\ell$  is detected.

$$P_e(\underline{s}_m, \underline{s}_{m+1}) = P(\mathcal{N}(0, \frac{N_0}{2}) > \frac{d_{m,m+1}}{2}) = Q(\frac{d_{\min}}{\sqrt{2N_0}})$$

$$P_e(\underline{s}_m, \underline{s}_{m-1}) = P(\mathcal{N}(0, \frac{N_0}{2}) > \frac{d_{m,m-1}}{2}) = Q(\frac{d_{\min}}{\sqrt{2N_0}})$$

$$\therefore \text{In MPSK } d_{m,m+1} = 2\sqrt{E_s} \sin(\frac{\pi}{M})$$

$$\therefore P_e \cong 2Q(\frac{2\sqrt{E_s} \sin(\frac{\pi}{M})}{\sqrt{2N_0}}) = 2Q(\sqrt{\frac{2E_s}{N_0}} \sin(\frac{\pi}{M}))$$

$$M \uparrow \quad R_b \uparrow \quad P_e \uparrow$$

Q: If double the size,  $M' = 2M$ , how much more  $\frac{E_b}{N_0}$  is required to maintain

the same  $P_e$ ?

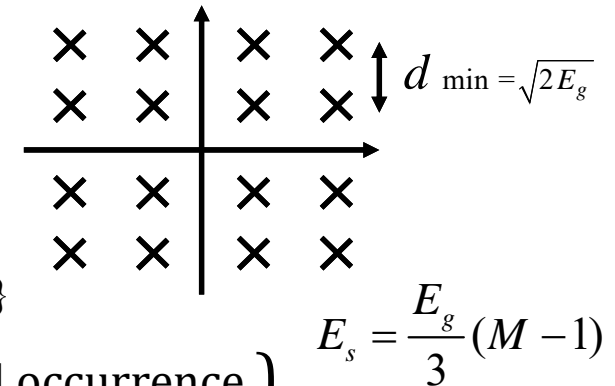
$$\text{For large } M, \quad \sin(\frac{\pi}{M}) \approx \frac{\pi}{M} \Rightarrow P_e' = 2Q(\frac{2\sqrt{\log_2^{M'} E_b'} \frac{\pi}{M'}}{\sqrt{2N_0}}) \approx 2Q(\frac{\sqrt{\log_2^M E_b'} \frac{\pi}{M}}{\sqrt{2N_0}}) \Rightarrow \frac{E_b'}{N_0} = 4(\frac{E_b}{N_0})$$

# Detection of M-ary QAM

$$s_m(t) = A_{mI}g(t)\cos(2\pi f_c t) - A_{mQ}g(t)\sin(2\pi f_c t)$$

$$\underline{s}_m = [s_{mI} \ s_{mQ}]^T = \left[ \sqrt{\frac{E_g}{2}} A_{mI} \ \sqrt{\frac{E_g}{2}} A_{mQ} \right]^T, \quad m = 1, 2, \dots, M$$

$$A_{mI}, A_{mQ} \in \{\pm 1, \pm 3, \dots, \pm(M-1)\}$$

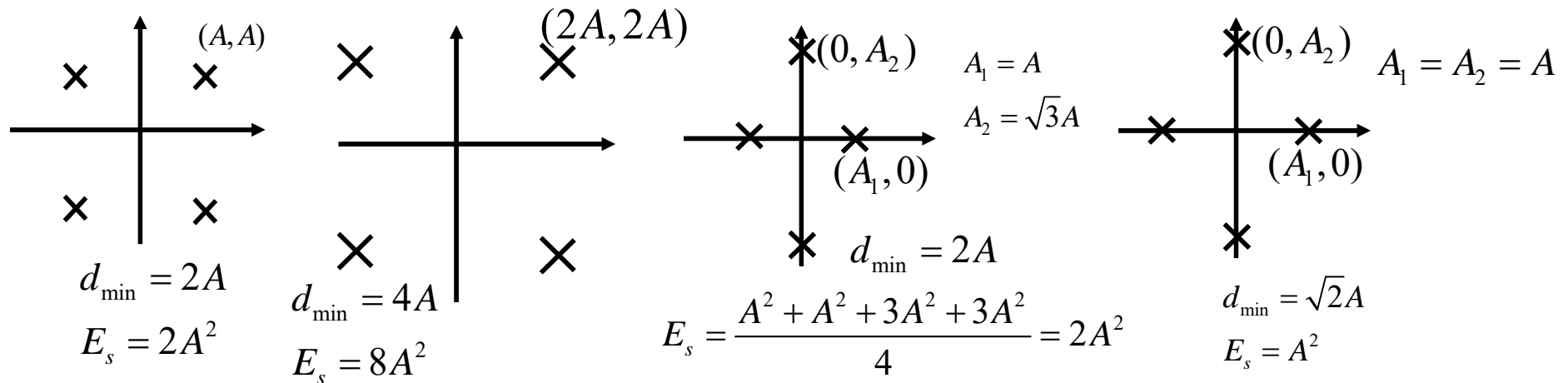


MAP detector (assume AWGN and equal prob. of symbol occurrence)

$$\hat{\underline{s}} = \arg \min_{\underline{s}} \|\underline{r} - \underline{s}\|^2$$

## ● Error performance and energy efficiency trade-off

The error probability is usually dominated by the minimum distance of constellation. For example, consider the following 4-QAM.

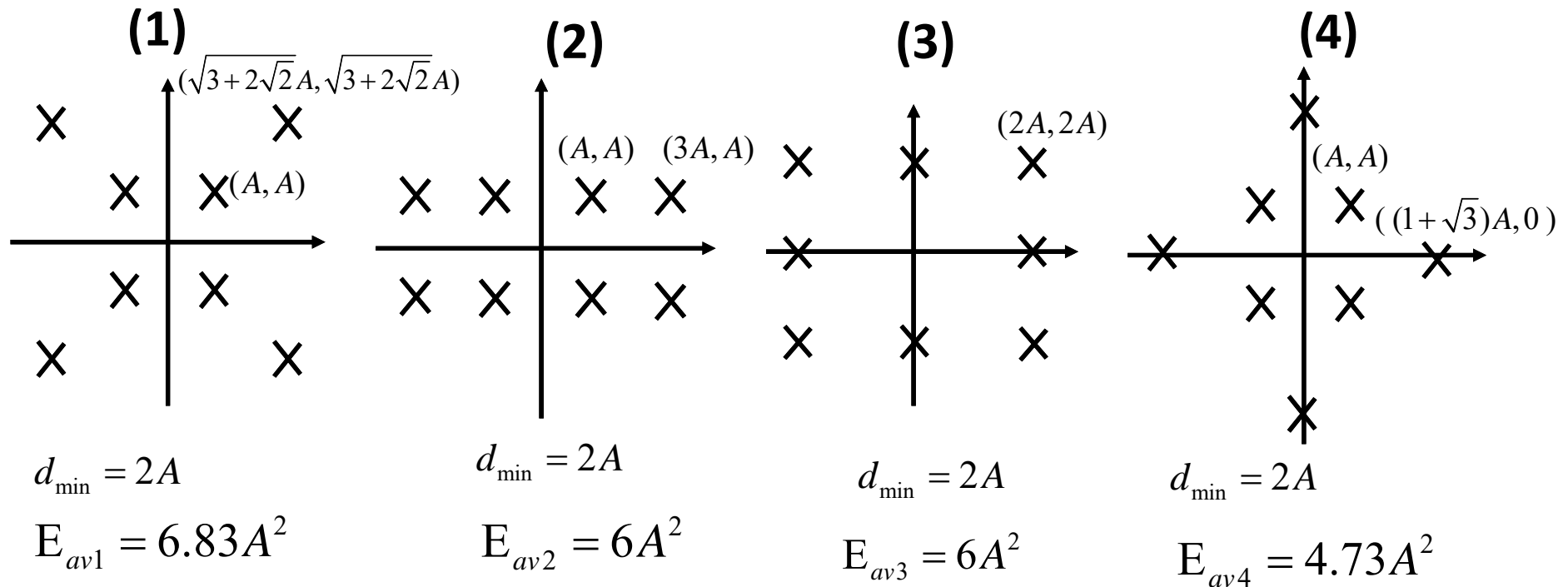


Q: What is the optimal distribution of the point on the signal space that yields the best performance in  $P_e$  and energy efficiency? 28

# Constellation Figure of Merit (CFM)

$$CFM = \frac{d_{\min}^2}{E_{av}} \quad \rightarrow \text{Yields quick evaluation among constellations.}$$

Example (M=8 8-QAM)



- ✓ The constellation (4) is known to be the best 8-QAM constellation in terms of CFM.      Q: How do you compare the 8-QAM in (2) and (3)?
- ✓ In general, the optimal constellation for M-QAM is difficult to proof.

# Error Probability of M-ary QAM

- In general, the error probability for M-ary QAM is

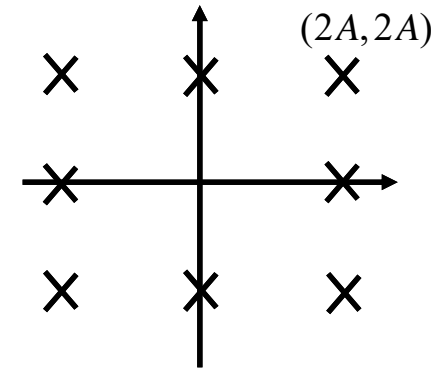
$$P_e = \sum_{m=1}^M P(s_m) P(\text{error} | s_m)$$

$$P(\text{error} | s_m) = P(s_m \rightarrow s_n | s_m) \quad \forall n \neq m$$

$$= \sum_{\substack{1 \leq n \leq M \\ n \neq m}} Q \left( \frac{\frac{d_{mn}}{2}}{\sqrt{\frac{N_0}{2}}} \right) \leq (M-1) Q \left( \frac{d_{\min}}{\sqrt{2N_0}} \right)$$

$$P_e = \sum_{m=1}^M P(s_m) P(\text{error} | s_m) \leq \sum_{m=1}^M \frac{M-1}{M} Q \left( \frac{d_{\min}}{\sqrt{2N_0}} \right)$$

$$\leq (M-1) Q \left( \frac{d_{\min}}{\sqrt{2N_0}} \right)$$



# Error Probability of M-ary QAM

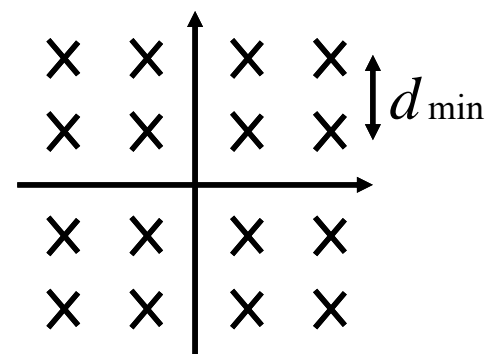
- For square M-QAM (e.g. M=4, 16, 64, ...,etc), they can be viewed as two  $\sqrt{M}$ -PAM in each dimension.

The correct probability  $P_c = (1 - P_{\sqrt{M}})^2$

The error probability for  $\sqrt{M}$ -PAM

$$P_{\sqrt{M}} = \frac{2(\sqrt{M}-1)}{\sqrt{M}} Q\left(\sqrt{\frac{6}{M-1} \frac{E_{\sqrt{M}-PAM}}{N_0}}\right) = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1} \frac{E_s}{N_0}}\right)$$

where  $E_s = 2E_{\sqrt{M}-PAM}$  Note:  $E_s = \frac{(M-1)d_{\min}^2}{6}$   $E_{\sqrt{M}-PAM} = \frac{(M-1)d_{\min}^2}{12}$



Average energy:

$$E_s = \frac{E_g}{3} (M-1)$$

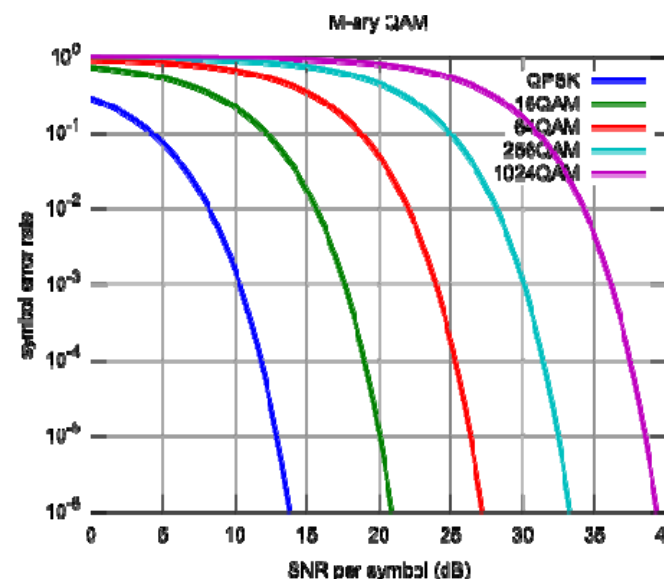
The error probability for square M-QAM

$$P_e = 1 - P_c = 1 - (1 - P_{\sqrt{M}})^2 = 2P_{\sqrt{M}} - P_{\sqrt{M}}^2$$

$$\rightarrow P_e \leq 2P_{\sqrt{M}} = 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{(M-1)} \frac{E_s}{N_0}}\right)$$

$$\leq 4Q\left(\sqrt{\frac{3}{(M-1)} \frac{KE_b}{N_0}}\right)$$

$$\rightarrow CFM = \frac{d_{\min}^2}{E_s} = \frac{6}{M-1} \quad M \uparrow P_e \uparrow CFM \downarrow$$



## Comparison of M-QAM and MPSK error probability

The error probability for M-QAM

$$P_{e,MQAM} \cong 4Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)$$

The error probability for MPSK

$$P_{e,MPSK} \cong 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin\left(\frac{\pi}{M}\right)\right)$$

}  $M \uparrow \quad P_e \uparrow$

Q: For fixed  $P_e$  which modulation is more power efficient?

i.e. when M increase which modulation requires less power?

➤ Compare the argument ratio in Q(x) for different M,  $R_M$ ,

$$R_M = \left(\frac{\arg P_{e,MQAM}}{\arg P_{e,MPSK}}\right)^2 = \frac{\frac{3}{M-1}}{2 \sin^2\left(\frac{\pi}{M}\right)} = \frac{3}{2(M-1) \sin^2\left(\frac{\pi}{M}\right)} \approx \frac{3M^2}{2(M-1)\pi^2}$$

$M \uparrow R_M \uparrow \Rightarrow$  MPSK requires more power to keep the same  $P_e$



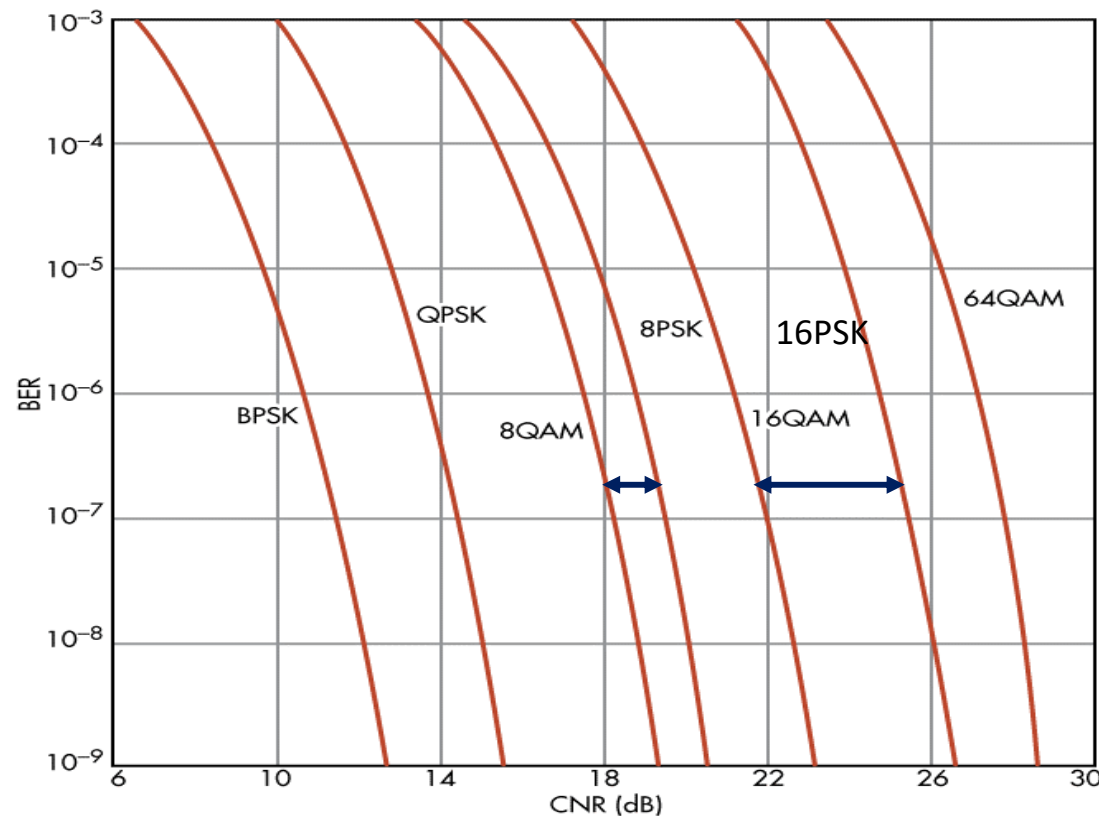
# Comparison of M-QAM and MPSK error probability

$$\text{MQAM} : P_e \leq 4Q\left(\sqrt{\frac{3}{(M-1)} \frac{E_s}{N_0}}\right)$$

$$M \uparrow \quad R_b \uparrow \quad P_e \uparrow$$

$$\text{MPSK} : P_e \cong 2Q\left(\frac{2\sqrt{E_s} \sin(\frac{\pi}{M})}{\sqrt{2N_0}}\right) = 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin\left(\frac{\pi}{M}\right)\right)$$

✓ MPSK error probability drops faster than MQAM as M increases.



## Comparison of Digital Modulation (Sec. 4.6)

- Bandwidth efficient modulation:

For **multiphase** signals(e.g. MPSK, MQAM) the bandwidth is governed by  $g(t)$  with symbol time  $T$ . The spectral efficiency

$$\rho = \frac{R}{BW} = \frac{\log_2 M / T}{2/T} = \frac{1}{2} \log_2 M \quad \text{bits/s/Hz}$$

$$M \uparrow \quad \rho \uparrow \quad P_e \uparrow$$

Disadvantage: As  $M \uparrow$ , require increased energy to maintain fixed  $P_e$

- Power efficient modulations:

For **orthogonal** signals(e.g. coherent MFSK  $\Delta f = \frac{1}{2T}$  )

$$M = 2^K \quad BW = M \Delta f = \frac{M}{2T}$$

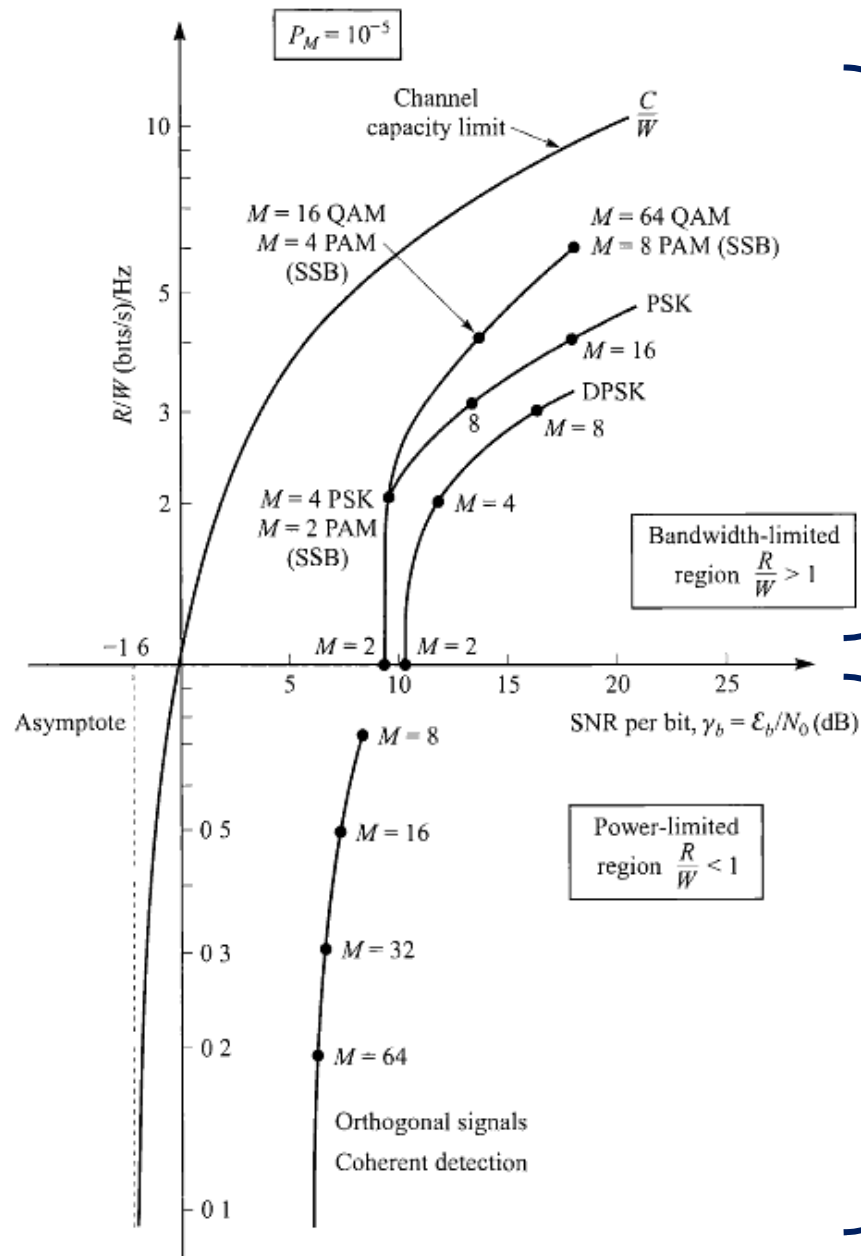
$$\rho = \frac{R}{BW} = \frac{\log_2 M / T}{M / 2T} = \frac{2 \log_2 M}{M} \quad \text{bits/s/Hz} \quad M \uparrow \quad R \uparrow \quad P_e \downarrow \quad \rho \downarrow$$

Advantage: As  $M \uparrow$ , less energy needed to maintain fixed  $P_e$ .

In general, there is no “best” modulation depending on the system requirement.

**FIGURE 4.6-1**

Comparison of several modulation schemes at  $P_e = 10^{-5}$  symbol error probability.



✓ Bandwidth efficient modulations  $\frac{R}{W} \geq 1$

- As  $M$  increases, MQAM/MPSK improves the spectral efficiency ( $R/W$ ). (BW efficient)
- But MFSK decreases in  $R/W$ , as  $M$  increases. (BW inefficient)

✓ Power efficient modulations  $\frac{R}{W} < 1$

- As  $M$  increases, MQAM/MPSK requires more energy (i.e.  $E_b$ ) to keep the  $P_e$ . (Power inefficient)
- But MFSK requires less  $E_b$  to keep the same  $P_e$ . So MFSK is power efficient.

## 4.5 Optimal Detection in Presence of Uncertainty

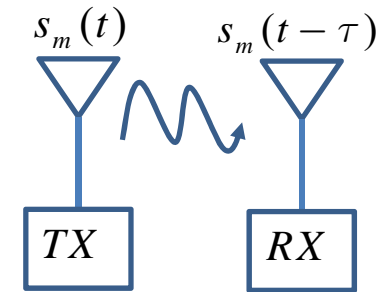
### ● Non-coherent detection of carrier modulated signals

- The Tx and Rx are not synchronized in phase.
- Time delay of transmission signal is unknown.

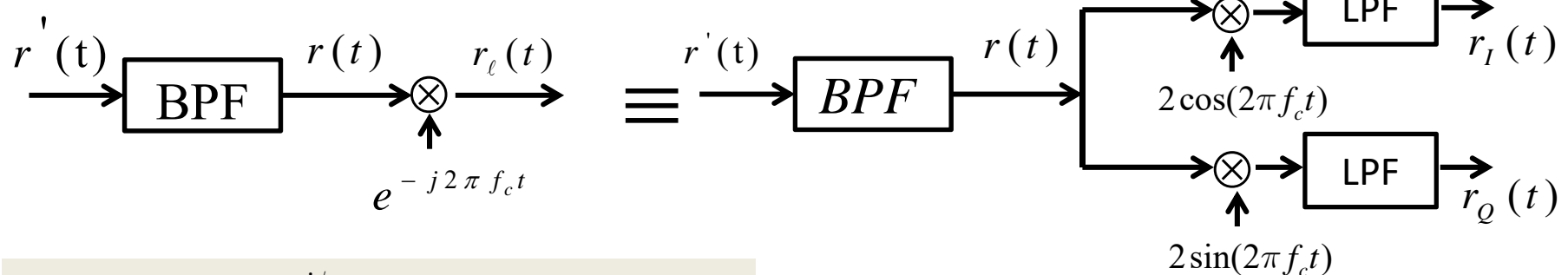
$$s_m(t) = s_{mI}(t) \cos(2\pi f_c t) - s_{mQ}(t) \sin(2\pi f_c t) = \text{Re} \{ s_{m\ell}(t) e^{j2\pi f_c t} \}$$

The received signals with AWGN noise

$$r(t) = s_m(t - \tau) + n(t) = \text{Re} \left\{ \underbrace{s_{m\ell}(t - \tau)}_{\approx s_{m\ell}(t) \text{ Why?}} \underbrace{e^{-j2\pi f_c \tau}}_{= \text{unknown random phase } \triangleq e^{j\phi}} e^{j2\pi f_c t} \right\} + n(t)$$



The baseband representation of received signal

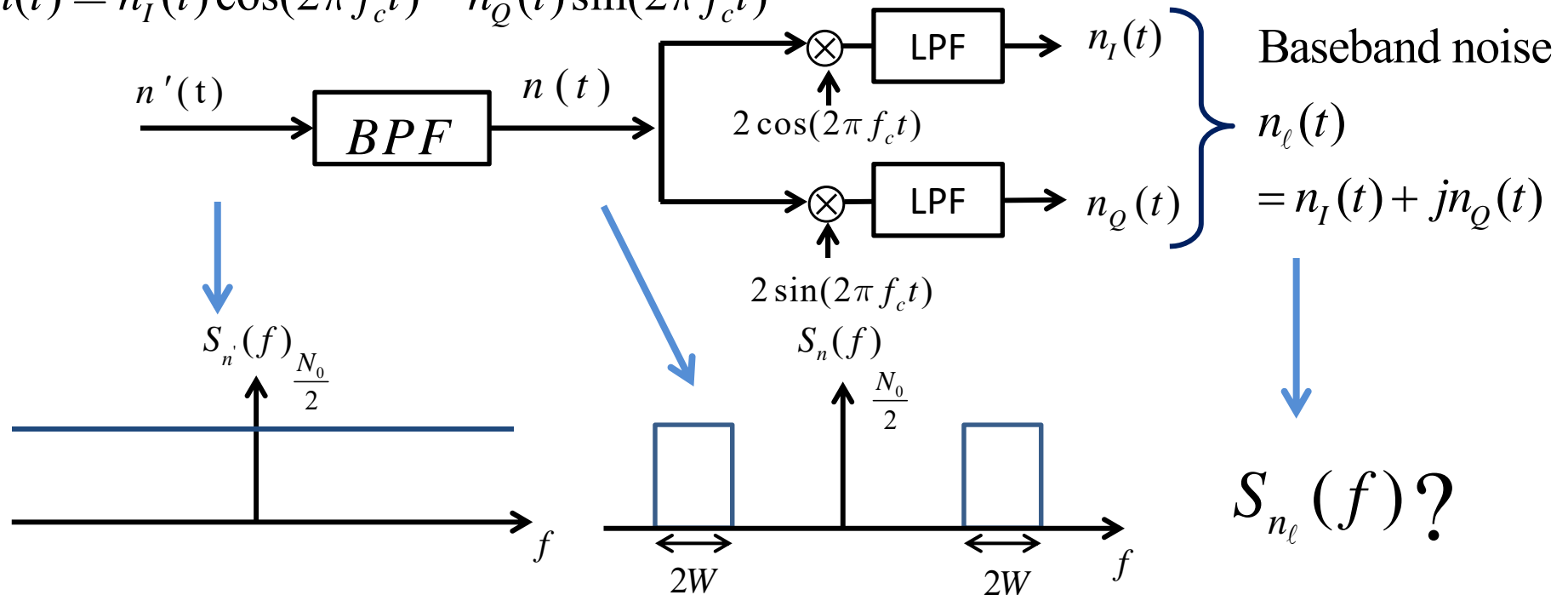


$$r_\ell(t) = s_{m\ell}(t) e^{j\phi} + n_\ell(t) = r_I(t) + j r_Q(t) \quad \text{where } n_\ell(t) = n_I(t) + j n_Q(t)$$

$n_I(t)$  and  $n_Q(t)$  are given by  $n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t)$

# Passband and Baseband Noises

$$n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t)$$



## ● The autocorrelation of baseband noise

$$\begin{aligned} R_{n_\ell}(\tau) &= E[n_\ell(t+\tau)n_\ell^*(t)] \\ &= E[(n_I(t+\tau) + jn_Q(t+\tau))(n_I(t) - jn_Q(t))] \\ &= E[n_I(t+\tau)n_I(t)] + E[n_Q(t+\tau)n_Q(t)] - jE[n_I(t+\tau)n_Q(t)] + jE[n_I(t)n_Q(t+\tau)] \\ &= R_{n_I}(\tau) + R_{n_Q}(\tau) \end{aligned}$$

# Passband and Baseband Noises

$$\begin{aligned}
 n_I(t) &= 2n(t) \cos(2\pi f_c t), & |f| &\leq W \\
 \rightarrow R_{n_I}(\tau) &= E[n_I(t+\tau)n_I(t)], & |f| &\leq W \\
 &= E[4n(t+\tau)n(t) \cos(2\pi f_c(t+\tau)) \cos(2\pi f_c t)], & |f| &\leq W \\
 &= 2R_n(\tau) \cos(2\pi f_c \tau), & |f| &\leq W
 \end{aligned}$$

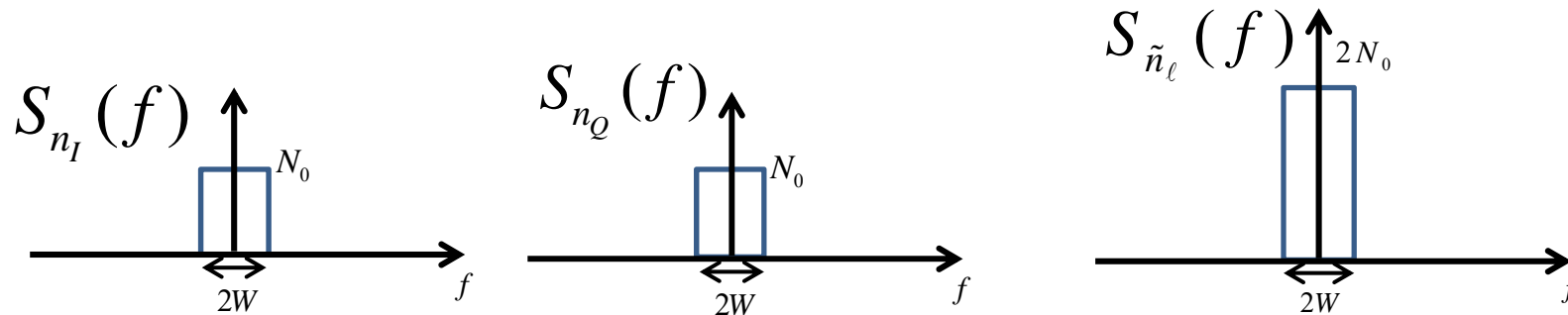
The power spectrum of baseband noise in I-channel is

$$S_{n_I}(f) = \mathbb{F}\{R_{n_I}(\tau)\} = S_n(f + f_c) + S_n(f - f_c) = N_0, \quad |f| \leq W$$

Similarly, we can also obtain baseband noise in Q-channel

$$S_{n_Q}(f) = N_0 \quad |f| \leq W$$

$$\rightarrow S_{n_\ell}(f) = S_{n_I}(f) + S_{n_Q}(f) = 2N_0, \quad |f| \leq W$$



# Detection of Non-coherent Carrier Modulated Signals

Assume  $\underline{s}_1$  was sent,

$$\underline{r}_\ell = e^{j\phi} \underline{s}_{1\ell} + \underline{n}_\ell \sim \mathcal{N}(e^{j\phi} \underline{s}_{m\ell}, 2N_0 \mathbf{I})$$

where  $\underline{s}_{1\ell} = [\sqrt{2E_s} \ 0 \ \dots \ 0 \ 0]^T$

$$\underline{s}_{2\ell} = [0 \ \sqrt{2E_s} \ 0 \ \dots \ 0]^T$$

$$\underline{s}_{M\ell} = [0 \ 0 \ \dots \ 0 \ \sqrt{2E_s}]^T$$

MAP detector

$$\hat{\underline{s}}_{m\ell} = \arg \max_{\underline{s}_{m\ell}} P(\underline{s}_{m\ell} | \underline{r}_\ell) = \arg \max_{\underline{s}_{m\ell}} P_m P(\underline{r}_\ell | \underline{s}_{m\ell}) \quad \text{Denote } P_m = P(\underline{s}_{m\ell})$$

Consider least favorable case, taking  $\phi$  as a uniformly distributed r.v.

$$f_\phi(\phi) = \frac{1}{2\pi}, \text{ where } \phi \in [0, 2\pi]$$

$$P(\underline{r}_\ell | \underline{s}_{m\ell}) = \int_0^{2\pi} f_\phi(\phi) \mathcal{N}(e^{j\phi} \underline{s}_{m\ell}, 2N_0 \mathbf{I}) d\phi$$

$$\hat{\underline{s}}_{m\ell} = \arg \max_{\underline{s}_{m\ell}} \frac{P_m}{2\pi} \left( \frac{1}{4\pi N_0} \right)^N \int_0^{2\pi} \exp\left(-\frac{\|\underline{r}_\ell - e^{j\phi} \underline{s}_{m\ell}\|^2}{4N_0}\right) d\phi$$

$\text{Note: } \|\underline{a} - \underline{b}\|^2 = (\underline{a} - \underline{b})^T (\underline{a} - \underline{b})^* = \|\underline{a}\|^2 + \|\underline{b}\|^2 - (\underline{a}^T \underline{b}^* + \underline{b}^T \underline{a}^*) = \|\underline{a}\|^2 + \|\underline{b}\|^2 - 2\text{Re}\{\underline{a}^T \underline{b}^*\}$

$$\|\tilde{\underline{s}}_{m\ell}\|^2 = 2E_s$$

$$= \arg \max_{\underline{s}_{m\ell}} \frac{P_m}{2\pi} \left( \frac{1}{4\pi N_0} \right)^N \exp\left(-\frac{E_s}{2N_0}\right) \int_0^{2\pi} \exp\left(\frac{1}{2N_0} \text{Re}\{\underline{r}_\ell^T (e^{j\phi} \underline{s}_{m\ell})^*\}\right) d\phi \quad \text{Why?}$$

Let  $\underline{r}_\ell \cdot \underline{s}_{m\ell} = \underline{r}_\ell^T \underline{s}_{m\ell}^* = \|\underline{r}_\ell\| \|\underline{s}_{m\ell}\| e^{j\theta}$

$$= \arg \max_{\underline{s}_{m\ell}} \frac{P_m}{2\pi} \left( \frac{1}{4\pi N_0} \right)^N \exp\left(-\frac{E_s}{2N_0}\right) \int_0^{2\pi} \exp\left(\frac{1}{2N_0} |\underline{r}_\ell \cdot \underline{s}_{m\ell}| \cos(\theta - \phi)\right) d\phi$$

Define  $I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(x \cos \phi) d\phi$

We have

$$\begin{aligned} \int_0^{2\pi} \exp\left(\frac{1}{2N_0} |\underline{r}_\ell \cdot \underline{s}_{m\ell}| \cos(\phi - \theta)\right) d\phi &= \int_0^{2\pi} \exp\left(\frac{1}{2N_0} |\underline{r}_\ell \cdot \underline{s}_{m\ell}| \cos(\phi')\right) d\phi' \\ &= 2\pi I_0\left(\frac{|\underline{r}_\ell \cdot \underline{s}_{m\ell}|}{2N_0}\right) \end{aligned}$$

$$\Rightarrow \hat{\underline{s}}_{m\ell} = \arg \max_{\underline{s}_{m\ell}} P_m \left(\frac{1}{4\pi N_0}\right)^N \exp\left(-\frac{E_s}{2N_0}\right) I_0\left(\frac{|\underline{r}_\ell \cdot \underline{s}_{m\ell}|}{2N_0}\right)$$

Since  $I_0(x)$  is a monotonically increasing function,

and assume  $P_m = \frac{1}{M} \forall m$ ,  $\forall m$ , the MAP detector becomes

$$\hat{\underline{s}}_{m\ell} = \arg \max_{\underline{s}_{m\ell}} |\underline{r}_\ell \cdot \underline{s}_{m\ell}| = \arg \max_m \sqrt{r_{mI}^2 + r_{mQ}^2} \quad \begin{array}{l} \text{where } r_{mI} = \text{Re}\{\underline{r}_\ell \cdot \underline{s}_{m\ell}\} \\ r_{mQ} = \text{Im}\{\underline{r}_\ell \cdot \underline{s}_{m\ell}\} \end{array}$$

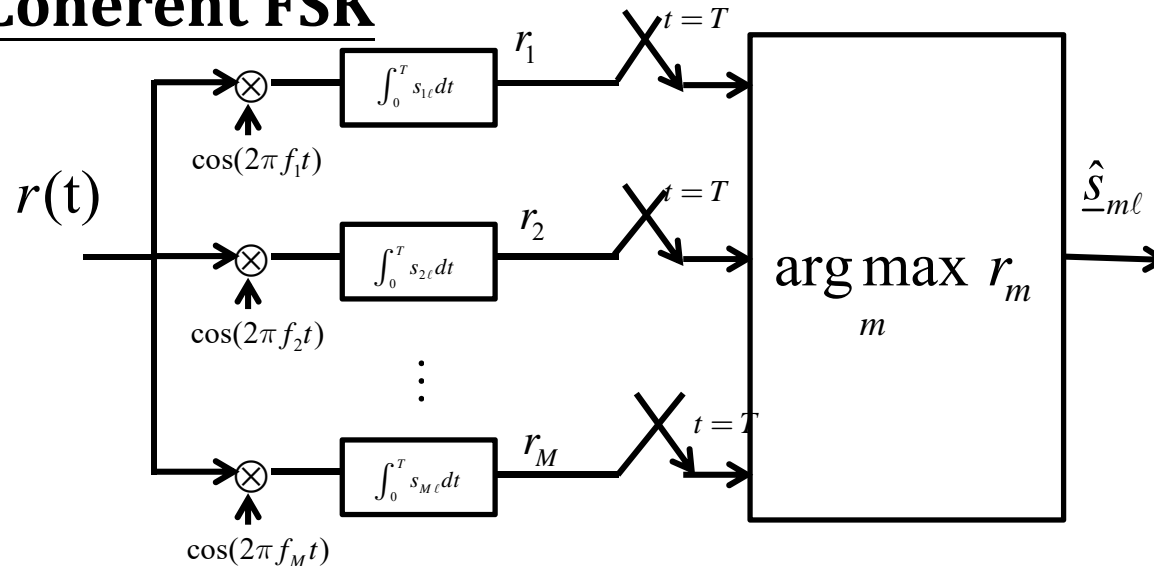
Note:  $\underline{r}_\ell = e^{j\phi} \underline{s}_{1\ell} + \underline{n}_\ell$

$$\rightarrow |\underline{r}_\ell \cdot \underline{s}_{m\ell}| = \sqrt{r_{mI}^2 + r_{mQ}^2}$$



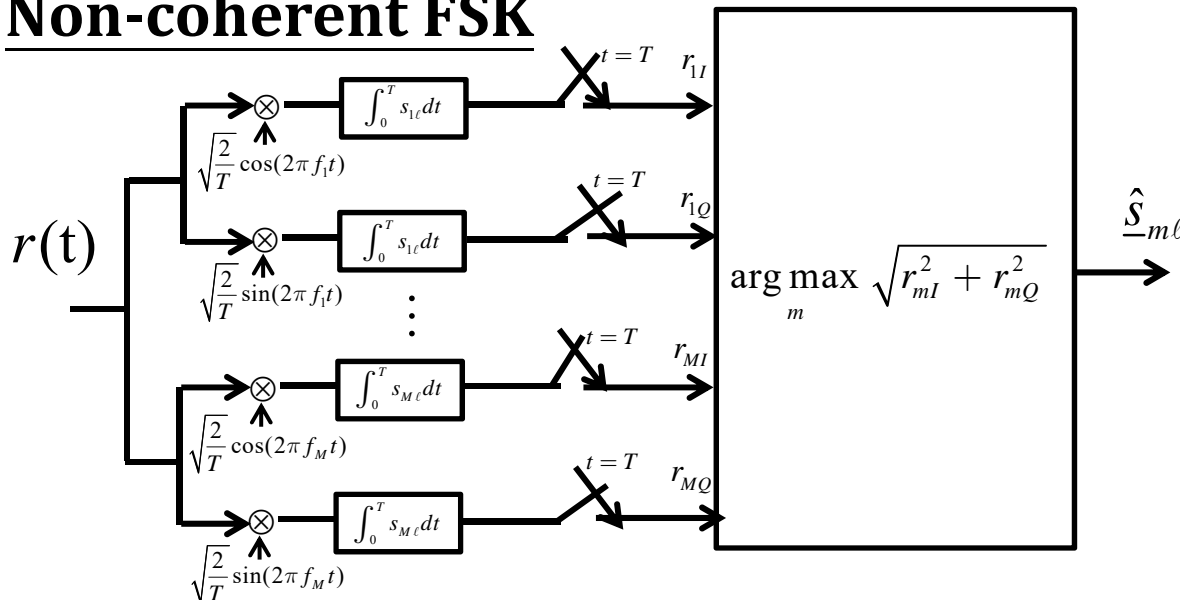
# Comparison of coherent and non-coherent MFSK architecture

## Coherent FSK



$$\begin{aligned}\hat{\underline{s}}_\ell &= \underset{\underline{s}_{m\ell}}{\operatorname{argmax}} p(\underline{s}_{m\ell} | \underline{r}_\ell) \\ &= \underset{\underline{s}_{m\ell}}{\operatorname{argmax}} \underline{r}_\ell \cdot \underline{s}_{m\ell} \\ &= \underset{\underline{s}_{m\ell}}{\operatorname{argmax}} r_m\end{aligned}$$

## Non-coherent FSK



$$\hat{\underline{s}}_\ell = \underset{\underline{s}_{m\ell}}{\operatorname{argmax}} |\underline{r}_\ell \cdot \underline{s}_{m\ell}|$$

$$\text{where } r_{mI} = \operatorname{Re}\{\underline{r}_\ell \cdot \underline{s}_{m\ell}\}$$

$$r_{mQ} = \operatorname{Im}\{\underline{r}_\ell \cdot \underline{s}_{m\ell}\}$$

$$\rightarrow |\underline{r}_\ell \cdot \underline{s}_{m\ell}| = \sqrt{r_{mI}^2 + r_{mQ}^2}$$

# Error probability of Orthogonal Signaling with Noncoherent Detection

Assume M equiprobable and equal energy carrier modulated signals, the signal constellation are

$$\underline{s}_{1\ell} = [\sqrt{2E_s} \quad 0 \quad \dots \quad 0 \quad 0]^T$$

$$\underline{s}_{2\ell} = [0 \quad \sqrt{2E_s} \quad 0 \quad \dots \quad 0]^T$$

$$\underline{s}_{M\ell} = [0 \quad 0 \quad \dots \quad 0 \quad \sqrt{2E_s}]^T$$

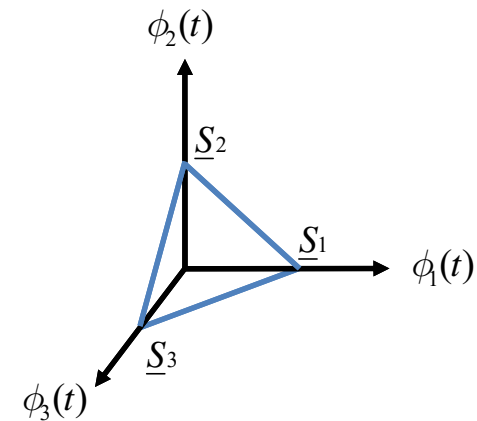
Assume  $\underline{s}_{1\ell}$  was sent, the received signal is

$$\underline{r}_\ell = e^{j\phi} \underline{s}_{1\ell} + \underline{n}_\ell$$

$$\text{where } \underline{n}_\ell = [n_{I1} + jn_{Q1}, \dots, n_{IM} + jn_{QM}]^T$$

MAP detector

$$\begin{aligned} \hat{\underline{s}}_{m\ell} &= \arg \max_{\underline{s}_{m\ell}} |\underline{r}_\ell \cdot \underline{s}_{m\ell}| \\ &= \arg \max_{\underline{s}_{m\ell}} \{ \sqrt{\text{Re}(\underline{r}_\ell \cdot \underline{s}_{m\ell})^2 + \text{Im}(\underline{r}_\ell \cdot \underline{s}_{m\ell})^2} \} \\ &= \arg \max_{\underline{s}_{m\ell}} R_m \end{aligned}$$



$$\begin{cases} R_1 = |\underline{r}_\ell \cdot \underline{s}_{1\ell}| = |2E_s e^{j\phi} + \underline{n}_\ell \cdot \underline{s}_{1\ell}| & m = 1 \\ R_m = |\underline{r}_\ell \cdot \underline{s}_{m\ell}| = |\underline{n}_\ell \cdot \underline{s}_{m\ell}| & m = 2 \sim M \end{cases}$$

$$\begin{cases} \text{Re}\{\underline{r}_\ell \cdot \underline{s}_{1\ell}\} \sim \mathcal{N}(2E_s \cos \phi, 2E_s N_0) \\ \text{Im}\{\underline{r}_\ell \cdot \underline{s}_{1\ell}\} \sim \mathcal{N}(2E_s \sin \phi, 2E_s N_0) \end{cases} \quad m = 1$$

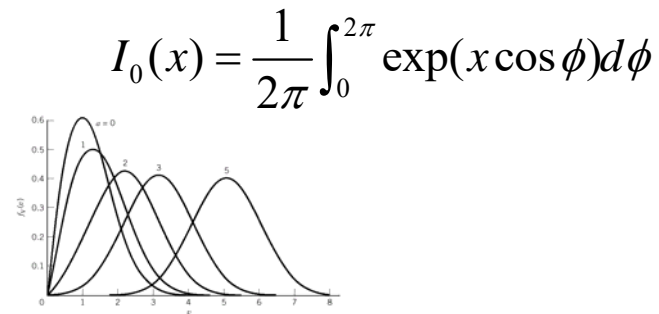
$$\begin{cases} \text{Re}\{\underline{r}_\ell \cdot \underline{s}_{1\ell}\} \sim \mathcal{N}(0, 2E_s N_0) \\ \text{Im}\{\underline{r}_\ell \cdot \underline{s}_{1\ell}\} \sim \mathcal{N}(0, 2E_s N_0) \end{cases} \quad m = 2, \dots, M$$

$R_1$  is Rician distributed with mean  $s = 2E_s$   $\sigma^2 = 2E_s N_0$

$R_m$  is *Rayleigh* distributed with mean  $s = 0$   $\sigma^2 = 2E_s N_0$

$$f_{R_1}(r_1) = \begin{cases} \frac{r_1}{\sigma^2} I_0\left(\frac{sr_1}{\sigma^2}\right) \exp\left(-\frac{r_1^2 + s^2}{2\sigma^2}\right) & r_1 > 0 \\ 0 & o.w \end{cases}$$

$$f_{R_m}(r_m) = \begin{cases} \frac{r_m}{\sigma^2} \exp\left(-\frac{r_m^2}{2\sigma^2}\right) & r_m > 0 \\ 0 & o.w \end{cases}$$



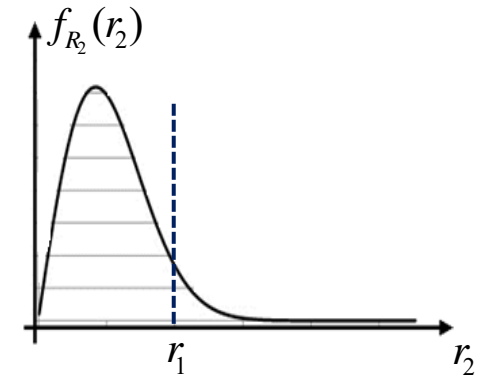
✓ With the system model and probability model, we can start to find  $P_e$  of MAP detector.

$$P_c = P\{R_2 < R_1, R_3 < R_1, \dots, R_M < R_1\}$$

$$= \int_0^\infty P(R_2 < r_1, R_3 < r_1, \dots, R_M < r_1 | r_1) f_{R_1}(r_1) dr_1$$

$$= \int_0^\infty [P(R_2 < r_1)]^{M-1} f_{R_1}(r_1) dr_1$$

$$P(R_2 < r_1) = \int_0^{r_1} f_{R_2}(r_2) dr_2 = \int_0^{r_1} \frac{r_2^2}{\sigma^2} e^{\frac{-r_2^2}{2\sigma^2}} dr_2 = -e^{\frac{-r_2^2}{2\sigma^2}} \Big|_0^{r_1} = 1 - e^{\frac{-r_1^2}{2\sigma^2}}$$



With binomial expansion,

$$(1-x)^{M-1} = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} x^n = 1 - \binom{M-1}{1} x + \binom{M-1}{2} x^2 - \dots \rightarrow \text{Let } x = e^{\frac{-r_1^2}{2\sigma^2}}$$

$$\therefore P_c = \int_0^\infty \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \exp\left(-\frac{nr_1^2}{2\sigma^2}\right) \cdot \left\{ \frac{r_1}{\sigma^2} I_0\left(\frac{sr_1}{\sigma^2}\right) \exp\left(-\frac{r_1^2 + s^2}{2\sigma^2}\right) \right\} dr_1$$

$$= \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \int_0^\infty \frac{r_1}{\sigma^2} I_0\left(\frac{sr_1}{\sigma^2}\right) \exp\left(-\frac{(n+1)r_1^2 + s^2}{2\sigma^2}\right) dr_1$$

$$= \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \exp\left(\frac{-ns^2}{2(n+1)\sigma^2}\right) \underbrace{\int_0^\infty \frac{r_1}{\sigma^2} I_0\left(\frac{sr_1}{\sigma^2}\right) \exp\left(-\frac{(n+1)r_1^2 + \frac{s^2}{n+1}}{2\sigma^2}\right) dr_1}_A$$

$$\text{where } A = \underbrace{\frac{1}{n+1} \int_0^\infty \frac{r'^2}{\sigma^2} I_0\left(\frac{s'r'}{\sigma^2}\right) \exp\left(-\frac{r'^2 + s'^2}{2\sigma^2}\right) dr'}_{=1} = \frac{1}{n+1} \quad \text{with } s' = \frac{s}{\sqrt{n+1}}, \quad r' = \sqrt{n+1} r_1$$

- Correct probability of non-coherent MFSK

$$P_c = \sum_{n=0}^{M-1} \frac{(-1)^n}{n+1} \binom{M-1}{n} \exp\left(\frac{-n}{n+1} \frac{E_s}{N_0}\right) \quad \text{where } s = 2E_s \quad \sigma^2 = 2E_s N_0$$

Error probability

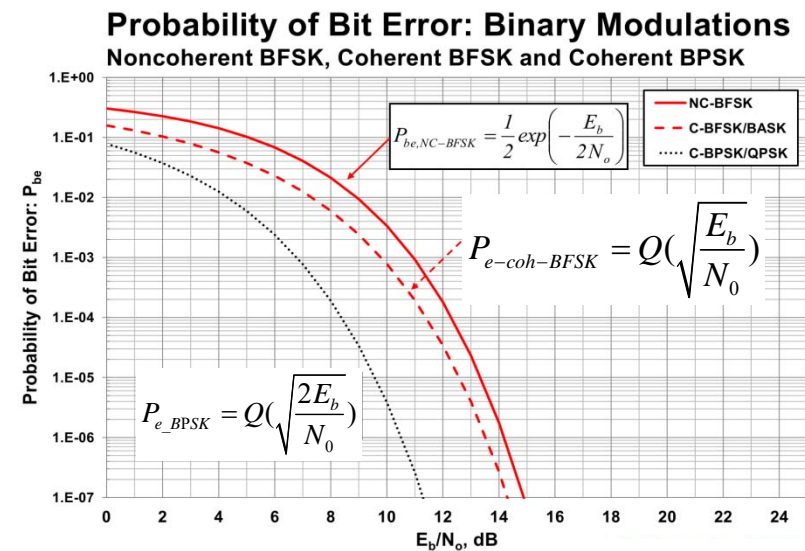
$$P_e = 1 - P_c = \sum_{n=1}^{M-1} \frac{(-1)^{n+1}}{n+1} \binom{M-1}{n} \exp\left(\frac{-n}{n+1} \frac{E_s}{N_0}\right)$$

- For noncoherent BFSK (i.e.  $M = 2$   $E_s = E_b$ )

$$P_{e-\text{noncoherent}} = \frac{1}{2} \exp\left(\frac{-E_b}{2N_0}\right)$$

- Comparing with coherent BFSK

$$P_{e-\text{coherent}} = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \leq \frac{1}{2} \exp\left(\frac{-E_b}{2N_0}\right)$$



## 4.8 Detection of Signaling Scheme with Memory

### Maximum Likelihood Sequence Detector(MLSD)

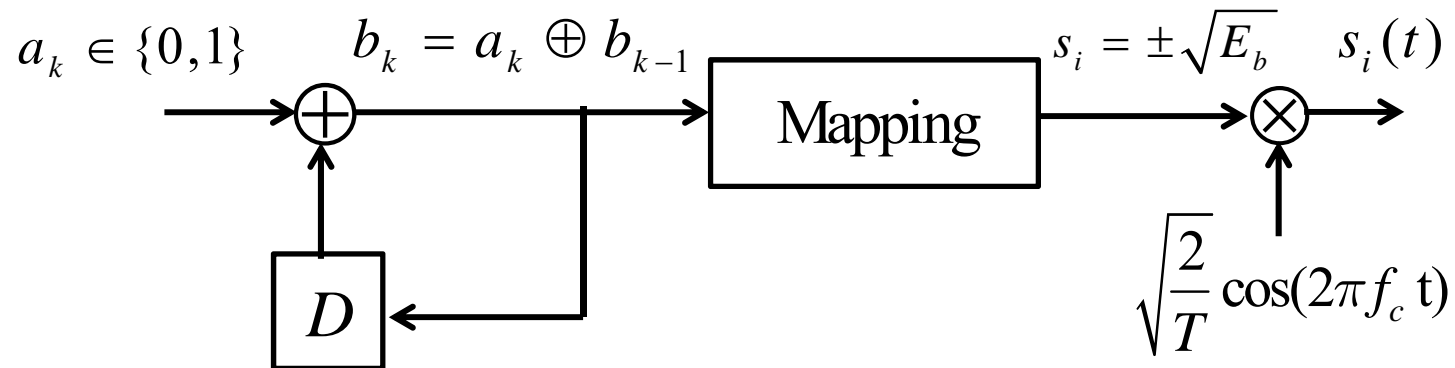
- The modulation with memory can be represented by a **trellis**.
- The **transmitted sequence** corresponds to **a path through the trellis**.

Assume transmitted sequence of K-symbols, the MLSD becomes

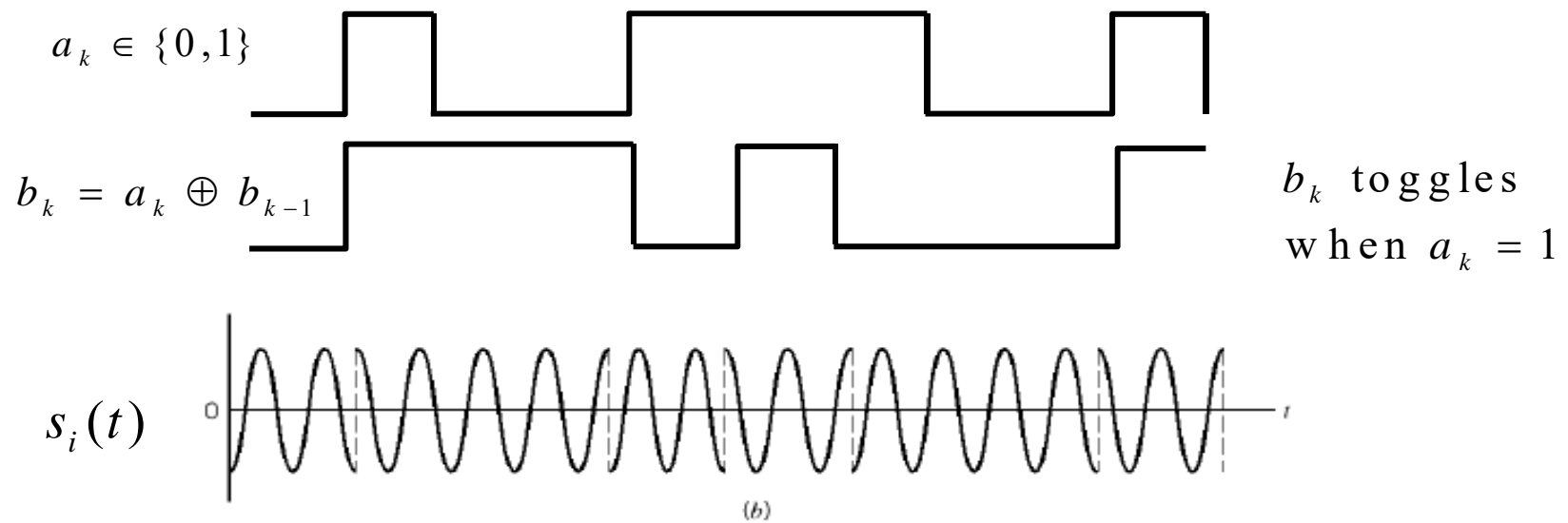
$$\begin{aligned}(\hat{\underline{s}}^{(1)}, \hat{\underline{s}}^{(2)}, \dots, \hat{\underline{s}}^{(K)}) &= \arg \min_{(\underline{s}^{(1)}, \underline{s}^{(2)}, \dots, \underline{s}^{(K)})} \sum_{k=1}^K \left\| \underline{r}^{(k)} - \underline{s}^{(k)} \right\|^2 \\ &= \arg \min_{(\underline{s}^{(1)}, \underline{s}^{(2)}, \dots, \underline{s}^{(K)})} \sum_{k=1}^K D(\underline{r}^{(k)}, \underline{s}^{(k)})\end{aligned}$$

The computation is complicated, to the order of  $O(M^K)$ .

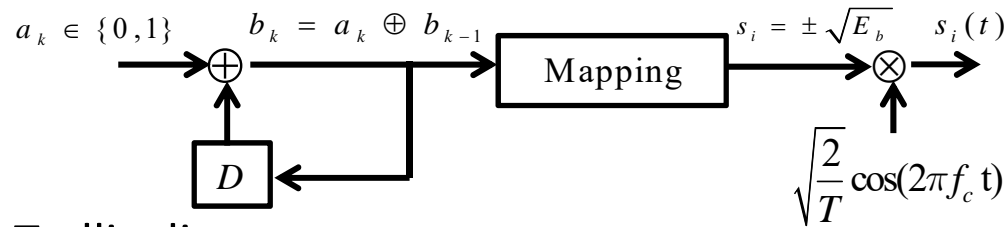
## Example(NRZI differential encoding DPSK)



Timing diagram



## Example(NRZI differential encoding DPSK)

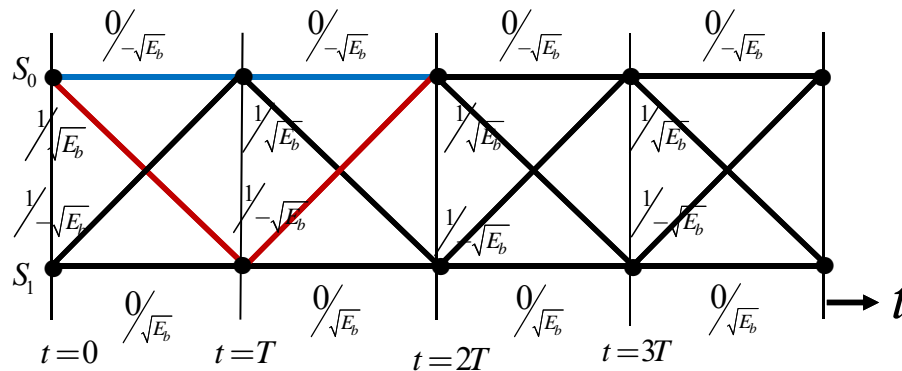


$$S_0 : b_k = 0, s_k = -\sqrt{E_b}$$

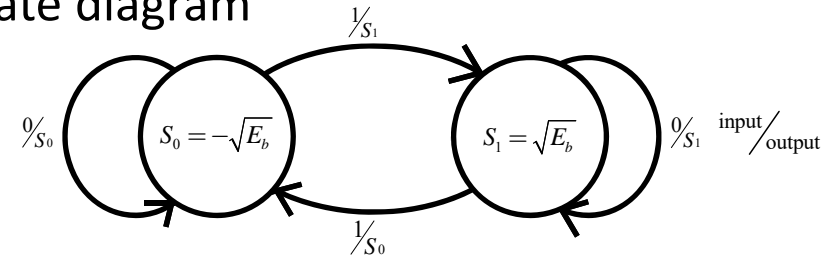
$$S_1 : b_k = 1, s_k = +\sqrt{E_b}$$

$$\text{input/output} = a_k/b_k$$

Trellis diagram



State diagram



Assume initial state at  $t=0$  is  $S_0$

At  $t = 2T$ , the two paths entering  $S_0$  are,

$$D_0(0,0) = (r_1 + \sqrt{E_b})^2 + (r_2 + \sqrt{E_b})^2 \quad (\text{blue line})$$

$$D_0(1,1) = (r_1 - \sqrt{E_b})^2 + (r_2 + \sqrt{E_b})^2 \quad (\text{red line})$$

Define the distance metric

$$D_{s_i}(a_{k-2}, a_{k-1})$$

Current state      Information bits at k-2 and k-1

Select one survival path entering  
 $S_0 = \min \{D_0(0,0), D_0(1,1)\}.$

At  $t = 2T$ , the two paths entering  $S_1$  are,

$$D_1(0,1) = (r_1 + \sqrt{E_b})^2 + (r_2 - \sqrt{E_b})^2 \quad \left. \begin{array}{l} \text{Select one survival path entering} \\ \end{array} \right\}$$

$$D_1(1,0) = (r_1 - \sqrt{E_b})^2 + (r_2 - \sqrt{E_b})^2 \quad \left. \begin{array}{l} \text{Select one survival path entering} \\ \end{array} \right\} S_1 = \min \{D_1(0,1), D_1(1,0)\}^{48}$$



## Example cont.

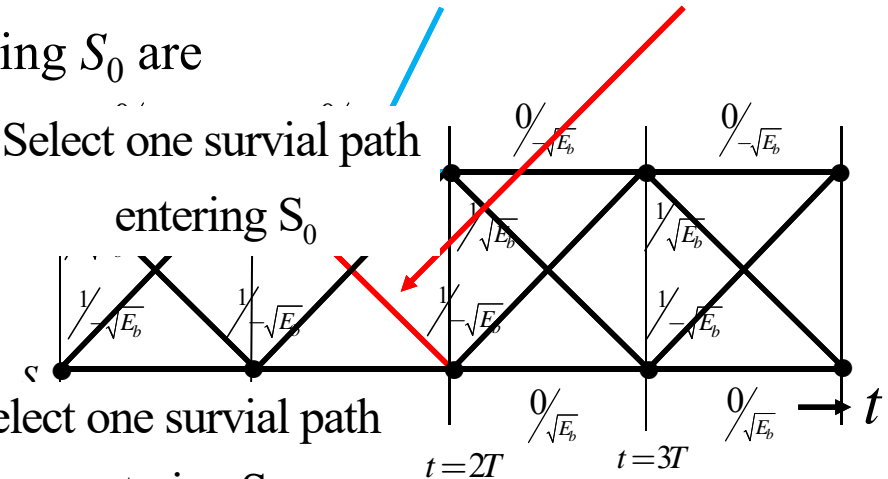
Assume the previous survivor paths at  $t=2T$  for  $S_0$  and  $S_1$  are  $D_0(0,0)$  and  $D_1(0,1)$ .

At  $t=3T$ , upon receipt of  $r_3$ , the paths entering  $S_0$  are

$$\left. \begin{aligned} D_0(0,0,0) &= D_0(0,0) + (r_3 + \sqrt{E_b})^2 \\ D_0(0,1,1) &= D_1(0,1) + (r_3 + \sqrt{E_b})^2 \end{aligned} \right\} \text{Select one survival path}$$

At  $t=3T$ , the paths entering  $S_1$  are

$$\left. \begin{aligned} D_1(0,0,1) &= D_0(0,0) + (r_3 - \sqrt{E_b})^2 \\ D_1(0,1,0) &= D_1(0,1) + (r_3 - \sqrt{E_b})^2 \end{aligned} \right\} \text{Select one survival path}$$



- At each stage,  $t=kT$ , the process keep two survivor paths (for  $S_0$  and  $S_1$  respectively), entering the next stage.
- The next stage,  $t=(k+1)T$ , can leverage on the previous survivor path to find MLSD result and save the computation complexity.
- Detection with memory: the process continued as each new signal sample is received. (The Viterbi algorithm)

## 4.9 Optimum Receiver for CPM signals

Continuous Phase Modulation (CPM) signal (Ex: CPFSK):

$$s(t) = \sqrt{\frac{2E}{T}} \cos[2\pi f_c t + \phi(t, \underline{I})]$$

For the carrier phase of a CPM, the phase trellis is

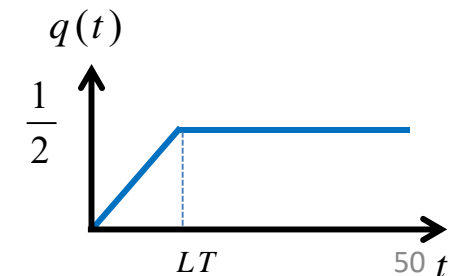
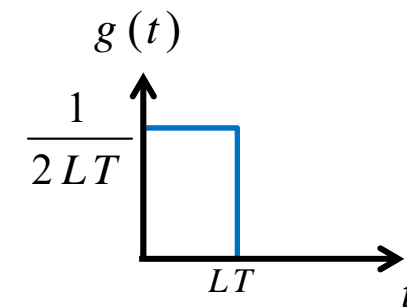
$$\varphi(t, \underline{I}) = 2\pi h \sum_{k=-\infty}^n I_k q(t - kT) \quad \text{where } h = T \cdot \Delta f = 2f_d T$$

$$= \pi h \sum_{k=-\infty}^{n-L} I_k + 2\pi h \sum_{k=n-L+1}^n I_k q(t - kT)$$

$$= \theta_n + \theta(t, \underline{I}) \quad nT \leq t \leq (n+1)T$$

$$\theta_{n+1} = \theta_n + \pi h I_n$$

$$\text{where } q(t) = \int_0^t g(\tau) d\tau \quad g(t) = \frac{1}{2LT} \text{rect}\left(\frac{t - \frac{LT}{2}}{LT}\right)$$



The modulation index  $h = T\Delta f = 2f_d T = \frac{m}{p}$   $m, p \in \text{relative prime integer}$

$L=1 \Rightarrow$  full response CPM: The phase trellis has  $\begin{cases} p & m \text{ even} \\ 2p & m \text{ odd} \end{cases}$  phase states

$L > 1 \Rightarrow$  a partial response CPM: The phase trellis has additional states due to the partial response of  $g(t)$

The phase trellis has (for  $L=1$ )

$$\begin{aligned} p \text{ phase states } \theta &= \left\{ 0, \frac{\pi m}{p}, \frac{2\pi m}{p} \dots \frac{(p-1)m\pi}{p} \right\} && \text{for even } m \\ 2p \text{ phase states } \theta &= \left\{ 0, \frac{\pi m}{p}, \frac{2\pi m}{p} \dots \frac{(2p-1)m\pi}{p} \right\} && \text{for odd } m \end{aligned}$$

### Example

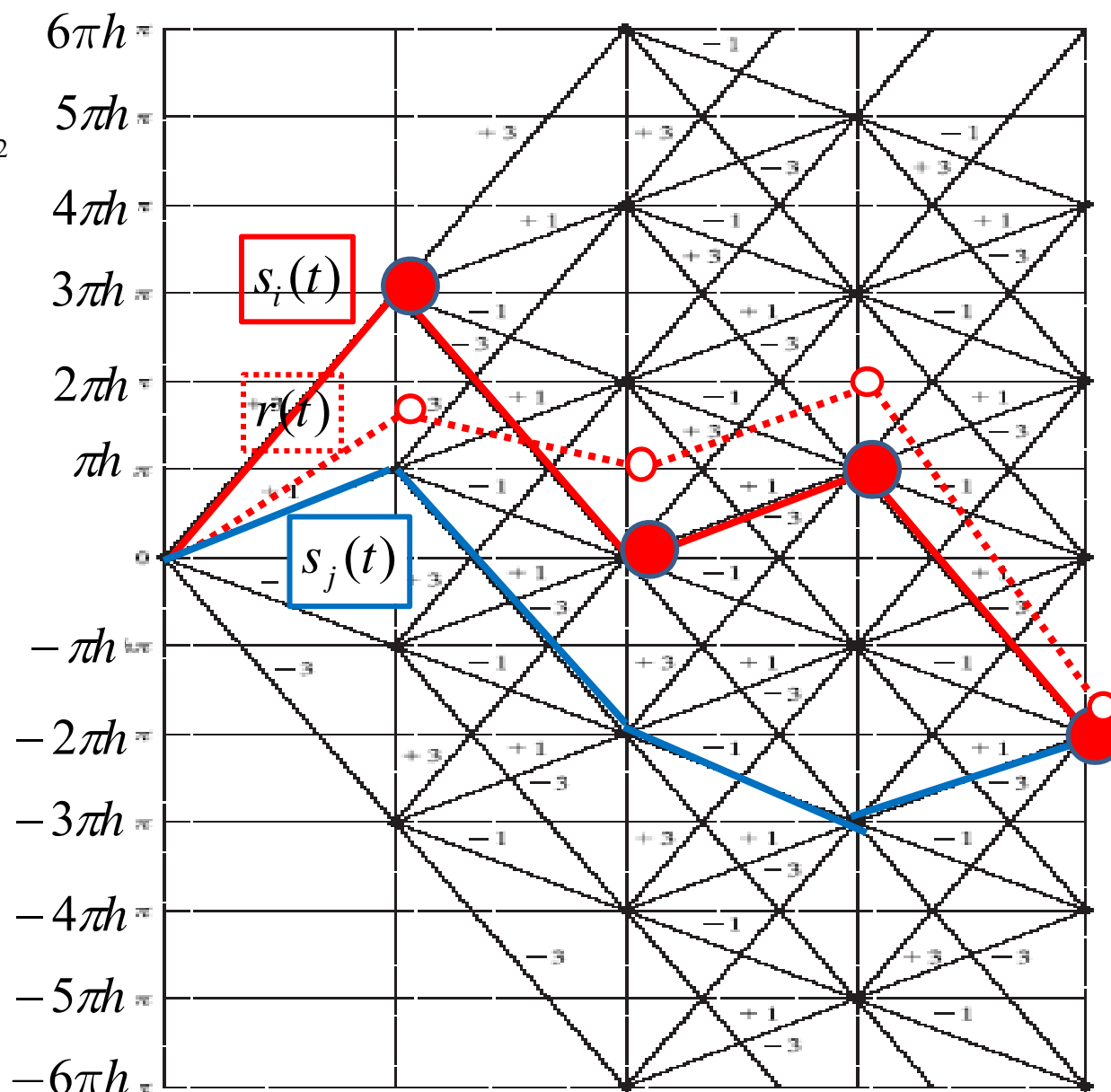
$$h = \frac{2}{3} \quad \theta \in \left\{ \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3} \right\} \quad h = \frac{1}{2} \quad \theta \in \left\{ \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \right\}$$

# Phase Trellis for M-ary CPFSK Signal (M=4)

$$(\hat{\underline{s}}^{(1)}, \hat{\underline{s}}^{(2)}, \dots, \hat{\underline{s}}^{(N)}) =$$

$$\arg \min_{(\underline{s}^{(1)}, \underline{s}^{(2)}, \dots, \underline{s}^{(N)})} \sum_{n=1}^N \left\| \underline{r}^{(N)} - \underline{s}^{(n)} \right\|^2$$

$$I_n \in \{\pm 1, \pm 3\}$$



Thus CPM can be decoded by Viterbi trellis decoding.

The state of the CPM signal at  $t=nT$  may be expressed as combined phase state and correlative states as a vector.

State vector at  $t = nT$  :

$$\underline{s}_n = (\theta_n, I_{n-1}, I_{n-2}, \dots, I_{n-L+1})$$

$\downarrow$   $\downarrow$   $\downarrow$   $\downarrow$   
 $p \text{ or } 2p$   $M$   $M$   $M$

$$N_s = \begin{cases} pM^{L-1} & m \in \text{even} \\ 2pM^{L-1} & m \in \text{odd} \end{cases} \quad (\text{The number of states})$$

At  $t=(n+1)T$  state vector becomes

$$\underline{s}_{n+1} = (\theta_{n+1}, I_n, I_{n-1}, \dots, I_{n-L+2})$$

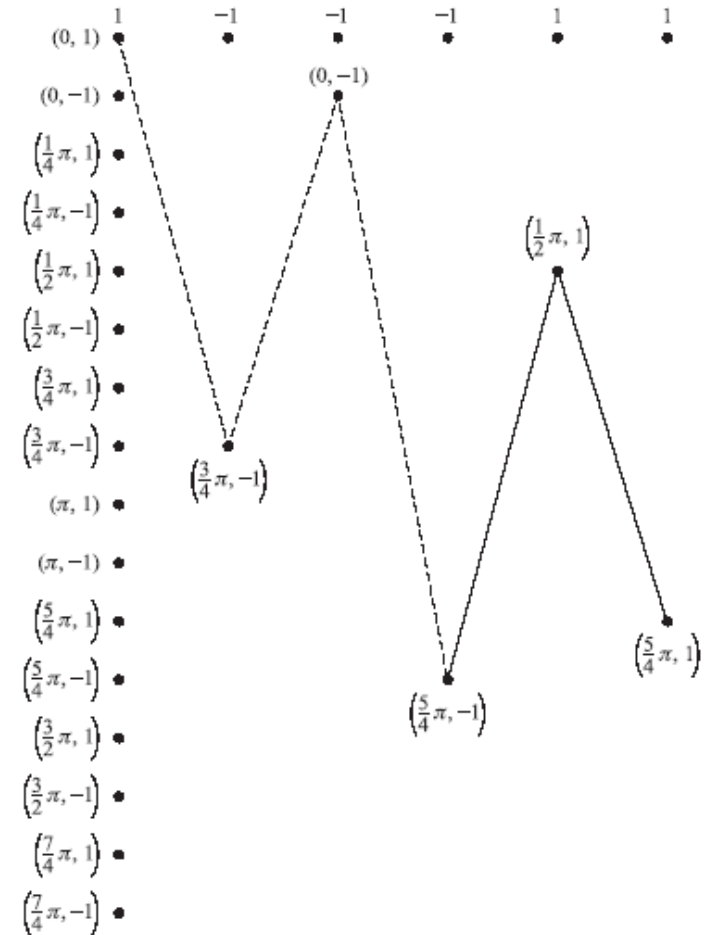


FIGURE 4.9-2  
A single signal path through the trellis.

## Performance of CPM

Let  $s_i(t)$  and  $s_j(t)$  be signalings that correspond to length  $N$  phase trajectories  $\phi(t, \underline{I}_i)$  and  $\phi(t, \underline{I}_j)$

$$s_i(t) = \sqrt{\frac{2E_s}{T}} \cos[2\pi f_c t + \phi(t, \underline{I}_i)]$$

The Euclidean distance between  $s_i(t)$  and  $s_j(t)$  over interval of  $NT$  is

$$\begin{aligned} d_{ij}^2 &= \int_0^{NT} (s_i(t) - s_j(t))^2 dt \\ &= \int_0^{NT} s_i^2(t) dt + \int_0^{NT} s_j^2(t) dt - 2 \int_0^{NT} s_i(t) s_j(t) dt \\ &= 2NE_s - 2 \frac{2E_s}{T} \int_0^{NT} \cos[2\pi f_c t + \phi(t, \underline{I}_i)] \cos[2\pi f_c t + \phi(t, \underline{I}_j)] dt \\ &= 2NE_s - \frac{2E_s}{T} \int_0^{NT} \cos[\phi(t, \underline{I}_i) - \phi(t, \underline{I}_j)] dt \\ &= \frac{2E_s}{T} \int_0^{NT} [1 - \cos[\phi(t, \underline{I}_i) - \phi(t, \underline{I}_j)]] dt \triangleq 2E_b \delta_{ij}^2 \end{aligned}$$

where  $\delta_{ij}^2 = \frac{\log_2^M}{T} \int_0^{NT} [1 - \cos[\phi(t, \underline{I}_i) - \phi(t, \underline{I}_j)]] dt = \frac{\log_2^M}{T} \int_0^{NT} [1 - \cos[\phi(t, \xi)]] dt$

$$\phi(t, \xi) = 2\pi h \sum_{i=-\infty}^n q(t - \xi T) \quad \xi \triangleq \underline{I}_i - \underline{I}_j$$

Since

$$\underline{I}_i, \underline{I}_j \in \{\pm 1, \pm 3, \dots, \pm(M-1)\}$$

$$\xi = \underline{I}_i - \underline{I}_j \in \{0, \pm 2, \pm 4, \dots, \pm 2(M-1)\}$$

The error rate performance of CPM can be measured by the minimum distance between any two phase trajectories of length  $N$ .

The pair-wise error probability is

$$P_{e,pair} = P\left(\mathcal{N}(0, N\sigma_n^2) > \frac{d_{ij}}{2}\right) = Q\left(\frac{\frac{d_{ij}}{2}}{\sqrt{N\sigma_n^2}}\right)$$

$$\text{Let } \sigma_n^2 = \frac{N_0}{2}$$

$$P_{e,pair} = Q\left(\frac{\sqrt{2E_b\delta_{ij}^2}}{2\sqrt{N\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{E_b\delta_{ij}^2}{NN_0}}\right)$$

The union error probability of CPM is

$$P_e = K_{\delta_{\min}} Q\left(\sqrt{\frac{E_b\delta_{\min}^2}{NN_0}}\right) \quad \delta_{\min}^2 = \min \delta_{i,j}^2 \quad \forall i, j$$

$K_{\delta_{\min}}$  is the number of paths having the minimum distance

# HW#3

Due: 11/xx/2021 (TBD)