COM 5120 Communication Theory

Homework #1

Due:10/14/2021

1. (20%) Consider the rectangular pulse signal $p(t) = A\cdot\Pi(\frac{t}{\tau_0})$ and let the pulse train

$$x(t) = \sum_{n=-\infty}^{\infty} p(t - nT_0)$$

(1) (5%) Find the magnitude spectrum $|X(f)|$ of $x(t)$

(2) (7%) Find the power spectrum density $S_X(f)$ of $x(t)$

(3) (8%) Find the time -average autocorrelation function $R_X(\tau)$ of $x(t)$

Solutioln (1)

$$x(t) = p(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

$$\rightarrow X(f) = P(f)\cdot f_0 \sum_{m=-\infty}^{\infty} \delta(f - mf_0) \quad , \; f_0 = \frac{1}{T_0}$$

$$= \mathrm{A}\tau_0 \sin c(\tau_0 f)\cdot f_0 \sum_{m=-\infty}^{\infty} \delta(f - mf_0)$$

$$= \mathrm{A}\tau_0 f_0 \sum_{m=-\infty}^{\infty} \sin c(m\tau_0 f_0)\delta(f - mf_0)$$

$$|X(f)| = A\tau_0 f_0 \sum_{m=-\infty}^{\infty} |\sin c(m\tau_0 f_0)| \; \delta(f - mf_0)$$

Solution (2)

$$S_X(f) \sim |X(f)|^2 = \sum_{n=-\infty}^{\infty} | f_0 P(mf_0)|^2 \, \delta(f - mf_0)$$

$$= A^2\tau_0^2 f_0^2 \sum_{m=-\infty}^{\infty} \sin c^2(m\tau_0 f_0)\delta(f - mf_0)$$

Solution (3)

$$R_X(\tau) = F^{-1}\{S_X(f)\}$$

$$= F^{-1}\left\{ A^2\tau_0^2 f_0 \sin c^2(\tau_0 f)\cdot f_0 \sum_{m=-\infty}^{\infty} \delta(f - mf_0) \right\}$$

$$= \mathrm{A}^2\tau_0 f_0 tri(\frac{\tau}{\tau_0}) * \sum_{n=-\infty}^{\infty} \delta(\tau - nT_0)$$

2. (20%) Consider a finite periodic pulse signal $x(t)$ with period T , i.e.

$$x(t) = \sum_{m=-M}^{M} p(t-mT)$$

where p(t)=1 for $-d/2 \le t \le d/2$, otherwise p(t)=0, and d<T.

(1) (10%) Please determine $X(f)$, the Fourier transform of $x(t)$

(2) (10%) Please sketch $X(f)$ and mark the major null frequencies

Solution (1)

$$x(t) = P_I(t) \cdot P_{Td}(t)$$

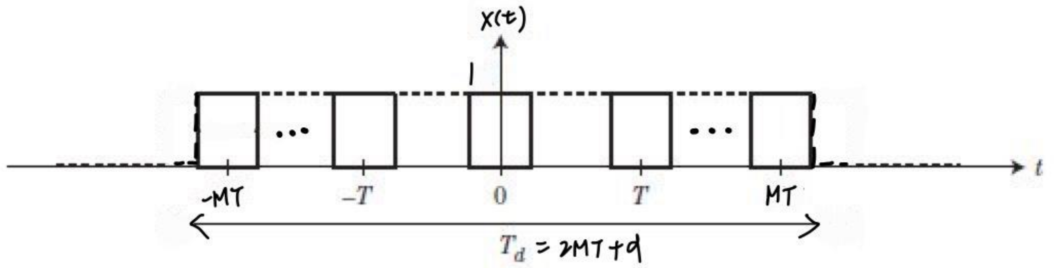
$$P_I(t) = \sum_{m=-\infty}^{\infty} p(t-mT) = p(t) * \sum_{m=-\infty}^{\infty} \delta(t-mT)$$

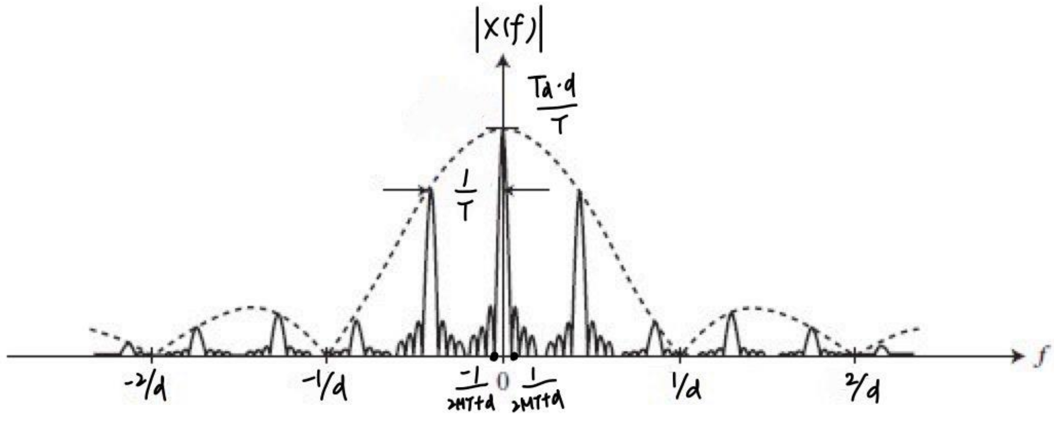
$$P_{Td}(t) = \begin{cases} 1, & -MT - \frac{d}{2} \le t \le MT + \frac{d}{2} \\ 0, & otherwise \end{cases}$$

$$\therefore X(f) = [\frac{d}{T} \sum_{m=-\infty}^{\infty} \sin c(dm \cdot \frac{1}{T}) \cdot \delta(f - m \cdot \frac{1}{T})] * T_d \sin c(fT_d)$$

$$= \frac{T_d \cdot d}{T} \sum_{m=-\infty}^{\infty} \sin c(dm \cdot \frac{1}{T}) \cdot \sin c[(f - m \cdot \frac{1}{T})T_d] \ , T_d = 2MT + d$$

Solution (2)





3. (20%) The Hilbert transform is given by $\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$. Prove the following

properties:

(1) (10%) If $x(t) = x(-t)$, then $\hat{x}(t) = -\hat{x}(-t)$

(2) (10%) If $x(t) = \cos\omega_0 t$,then $\hat{x}(t) = \sin\omega_0 t$

Solution (1)

$$
\begin{aligned}
& x(t) = x(-t) \text{ ; even function} \\
& \to X(f) \in R \text{ and } X(-f) = X(f) \text{ ; also even function} \\
& \quad \hat{X}(f) = -j\,\mathrm{sgn}(f)X(f) \\
& \to \hat{X}(-f) = -j\,\mathrm{sgn}(-f)X(-f) = j\,\mathrm{sgn}(f)X(f) = -\hat{X}(f) \\
& \quad \hat{x}(t) = -\hat{x}(-t) \text{ ; odd function}
\end{aligned}
$$

Solution (2)

$$
\begin{aligned}
& x(t) = \cos\omega_0 t \\
& \to X(f) = \frac{1}{2}\delta(f - \frac{\omega_0}{2\pi}) + \frac{1}{2}\delta(f + \frac{\omega_0}{2\pi}) \\
& \quad \hat{X}(f) = -j\,\mathrm{sgn}(f)X(f) = -\frac{j}{2}\delta(f - \frac{\omega_0}{2\pi}) + \frac{j}{2}\delta(f + \frac{\omega_0}{2\pi}) \\
& \to \hat{x}(t) = \sin\omega_0 t
\end{aligned}
$$

4.(20%) Consider a random process $x(t) = A\cos(2\pi f_0 t + \Theta)$,where A and f_0 are constants and Θ is a random variable with the pdf

$$
f_\Theta(\theta) = \begin{cases} \frac{1}{\pi}, |\theta| \leqq \frac{\pi}{2} \\ 0, otherwise \end{cases}
$$

(1) (10%) Is $x(t)$ a stationary random process? Explain your answer.

(2) (10%) Is $x(t)$ ergodic? Explain your answer.

Solution (1)

$$
\begin{aligned}
E[X(t)] &= AE[\cos(2\pi f_0 t + \theta)] \\
&= A\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos(2\pi f_0 t + \theta)\frac{1}{\pi}\, d\theta \\
&= \frac{A}{\pi}[\sin(2\pi f_0 t + \theta)\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}}] \\
&= \frac{A}{\pi}[\sin(2\pi f_0 t + \frac{\pi}{2}) - \sin(2\pi f_0 t - \frac{\pi}{2})] \\
&= \frac{A}{\pi}[\cos(2\pi f_0 t) + \cos(2\pi f_0 t)] = \frac{2\pi}{A}\cos(2\pi f_0 t)
\end{aligned}
$$

$\because E[X(t)] \neq$ constant

$\to x(t)$ is not a stationary random process (WSS)

Solution (2)

$X(t)$ is not stationary, so is not ergodic

5. (20)% Let X and Y be statistically independent Gaussian-distributed random variables, each with zero mean and unit variance. Define the Gaussian process

$$Z(t) = X\cos(2\pi t) + Y\sin(2\pi t)$$

Is the process Z(t) is WSS? Please prove it.

Solution

(1) $\mu_z = E[Z(t)] = E[X]\cos(2\pi t) + E[Y]\sin(2\pi t) = 0 + 0 = 0$

(2) $R_Z(t_1, t_2) = E[Z(t_1)Z(t_2)]$

$= E[(X\cos(2\pi t_1) + Y\sin(2\pi t_1))(X\cos(2\pi t_2) + Y\sin(2\pi t_2))]$

$= E[X^2]\cos(2\pi t_1)\cos(2\pi t_2) + E[Y^2]\sin(2\pi t_1)\sin(2\pi t_2)$

$= \cos(2\pi(t_1\text{-}t_2)) = \cos(2\pi\tau)$ ， $\tau = t_1 - t_2$

$\rightarrow Z(t)$ is WSS