COM 5120 Communication Theory

Homework #3

Reference solution

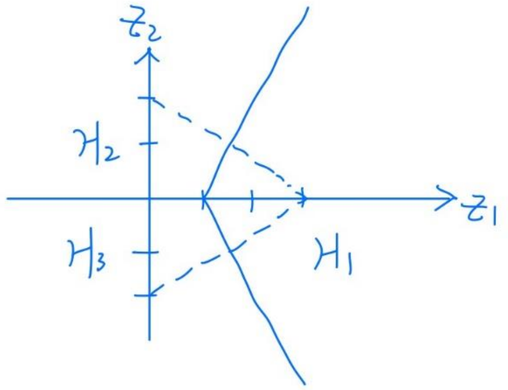
1.

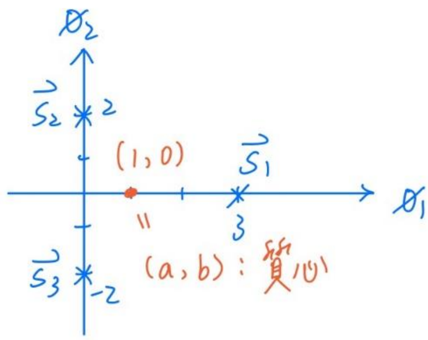
(a)

$\emptyset_1 = \sqrt{2}\cos(2\pi t), 0 \le t \le 1;\ \emptyset_2 = \sqrt{2}\sin(2\pi t), 0 \le t \le 1$

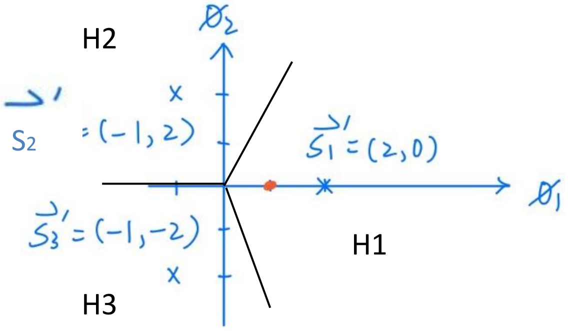
$s_1(t) = 3\emptyset_1;\ s_2(t) = 2\emptyset_2;\ s_3(t) = -2\emptyset_2$

(b)





(c)



$$E = [(-a)^2 + (2-b)^2] + [(3-a)^2 + (-b)^2] + [(-a)^2 + (-2-b)^2]$$
$$= 3a^2 - 6a + 3b^2 + 17 = f(a,b)$$

$$\frac{\partial f}{\partial a} = 6a - 6 = 0 \rightarrow a = 1\ ;\ \frac{\partial f}{\partial b} = 6b = 0 \rightarrow b = 0$$

Hence

S1’=S1-<1,0>=<2,0>

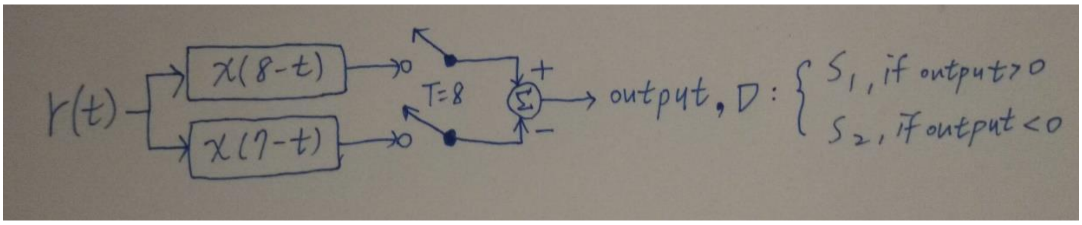
S2’=S2-<1,0>=<-1,2>

S3’=S3-<1,0>=<-1,-2>

S’ is an equivalent minimum-energy signal set.

2.

(a)



Note that the sampling for the match filter has to be done exactly at time t = T, where T is the arbitrary value used in the design of the matched filter. As long as this condition is satisfied, the choice of T is irrelevant; however from a practical point of view, T has to be selected in such a way that the resulting filters are causal; i.e, we must have h(t) = 0 for t < 0.

Thus, we choose T=8 in this case. The match filter to $s_1(t)$ is h(t)=s1(T-t)=x(8-t); The match filter to $s_2(t)$ is h(t)=s2(T-t)=x((8-t)-1)=x(7-t).

The sign of the output of above diagram determines the message:

D:{s1,if output>0;s2,if output<0}.

(b) This is binary, equiprobable signaling, hence $p_e = Q(\sqrt{\frac{d^2}{2N_0}})$ where

$$d^2 = \int_0^8 [s_1(t) - s_2(t)]^2 dt = \frac{1}{12} \times 4 + \frac{1}{4} \times 2 = \frac{5}{6}$$

Hence

$$p_e = Q(\sqrt{\frac{5}{12N_0}})$$

3.

(a)

$f(y|x=a) \sim N(a,\sigma_1^2) \quad H_1: x=a$

$f(y|x=-a) \sim N(-a,\sigma_2^2) \quad H_0: x=-a$

$$\frac{f(y|x=a)}{f(y|x=-a)} \gtrless \frac{1-p}{p} \rightarrow \frac{\frac{1}{\sigma_1}e^{-\frac{(y-a)^2}{2\sigma_1^2}}}{\frac{1}{\sigma_2}e^{-\frac{(y+a)^2}{2\sigma_2^2}}} \gtrless \frac{1-p}{p} \rightarrow \frac{(y+a)^2}{2\sigma_2^2} - \frac{(y-a)^2}{2\sigma_1^2} \gtrless \ln\left(\frac{(1-p)\sigma_1}{p\sigma_2}\right)$$

$$\rightarrow \frac{(y+a)^2}{\sigma_2^2} - \frac{(y-a)^2}{\sigma_1^2} \gtrless 2\ln\left(\frac{(1-p)\sigma_1}{p\sigma_2}\right)$$

(b)

$$\frac{(y+1)^2}{2} - \frac{(y-1)^2}{1} \gtrless 2\ln\left(\frac{\frac{1}{2}\times 1}{\frac{1}{2}\times\sqrt{2}}\right) \rightarrow (y^2+2y+1)-(2y^2-4y+2) \gtrless -2\ln(2)$$

$\rightarrow y^2-6y+1-2ln2 \gtrless 0$

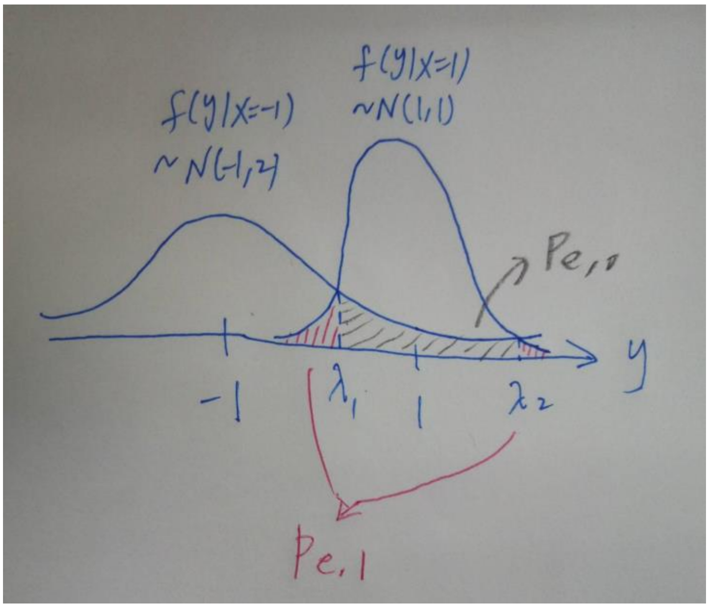
(Note that: Multiply -1 both sides, so $y^2-6y+1-2ln2>0$ indicates $H_0: x = -a$ now.)

$\rightarrow (y-\lambda_1)(y-\lambda_2) \gtrless 0$

$\rightarrow \lambda_1 = 3-\sqrt{8+2ln2}\ ;\ \lambda_2 = 3+\sqrt{8+2ln2}$

(c)

$$P_e = \frac{1}{2}P_{e_1} + \frac{1}{2}P_{e_0} = \frac{1}{2}[Q(1-\lambda_1)+Q(\lambda_2-1)] + \frac{1}{2}[Q\left(\frac{\lambda_1+1}{\sqrt{2}}\right) - Q(\frac{\lambda_2+1}{\sqrt{2}})]$$



4.

(a)

$$\begin{aligned}
Pe &= \frac{1}{2}P(e\,|+A) + \frac{1}{2}P(e\,|-A) \\
&= \frac{1}{2}\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(r-A)^2}{2\sigma^2}}dr + \frac{1}{2}\int_{0}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(r+A)^2}{2\sigma^2}}dr \\
&= \frac{1}{2}Q(\frac{A}{\sigma}) + \frac{1}{2}Q(\frac{A}{\sigma}) \\
&= Q(\frac{A}{\sigma})
\end{aligned}$$

(b)

$$\begin{aligned}
Pe &= \frac{1}{2}P(e\,|+A) + \frac{1}{2}P(e\,|-A) \\
&= \frac{1}{2}\int_{-\infty}^{0}\frac{1}{\sqrt{2}\sigma}e^{-\frac{\sqrt{2}}{\sigma}|r-A|}dr + \frac{1}{2}\int_{0}^{\infty}\frac{1}{\sqrt{2}\sigma}e^{-\frac{\sqrt{2}}{\sigma}|r+A|}dr \\
&= \frac{\sqrt{2}}{4\sigma}\int_{-\infty}^{-A}e^{-\frac{\sqrt{2}}{\sigma}|x|}dx + \frac{\sqrt{2}}{4\sigma}\int_{A}^{\infty}e^{-\frac{\sqrt{2}}{\sigma}|x|}dx \\
&= 2\times\frac{1}{4}e^{-\frac{\sqrt{2}}{\sigma}A} \\
&= \frac{1}{2}e^{-\frac{\sqrt{2}}{\sigma}A}
\end{aligned}$$

(c)

$$SNR = \frac{A^2}{\sigma_n^{\ 2}}$$

And the variance of the noise is :

$$\begin{aligned}
\sigma_n^{\ 2} &= \frac{\sqrt{2}}{2\sigma}\int_{-\infty}^{\infty}e^{-\frac{\sqrt{2}}{\sigma}|r|}r^2dr \\
&= \frac{\sqrt{2}}{\sigma}\int_{0}^{\infty}e^{-\frac{\sqrt{2}}{\sigma}r}r^2dr \\
&= \frac{\sqrt{2}}{\sigma}\times\frac{2!}{(\frac{\sqrt{2}}{\sigma})^3} \\
&= \sigma^2
\end{aligned}$$

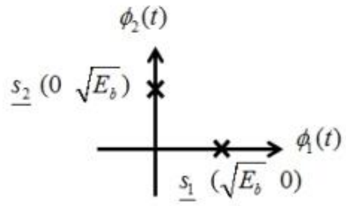
Therefore, $SNR = \frac{A^2}{\sigma^2}$

$$Pe = 10^{-5} = \frac{1}{2}e^{-\sqrt{2SNR}} \Rightarrow SNR = 58.534 = 17.67dB$$

5.

(a)

The BFSK constellation is:



The vector representation of BFSK signal is:

$\underline{s_1} = [\sqrt{E_b}\ \ 0]^T \quad \underline{s_2} = [0\ \ \sqrt{E_b}]^T$

The received signal is:

$$\underline{r} = \underline{s} + \underline{n} = \begin{bmatrix} n_1 \\ \sqrt{E_b} + n_2 \end{bmatrix} \text{(assume } s_2 \text{ is transmitted)}$$

The MAP detection is:

$$\hat{\underline{s}} = \underset{\underline{s}}{\text{argmax}}\ p(\underline{s} \mid \underline{r}) = \underset{\underline{s}}{\text{argmin}} \left|\underline{r} - \underline{s}\right|^2 = \underset{\underline{s}}{\text{argmax}}\ \ \underline{r} \cdot \underline{s}$$

$$\begin{aligned} P(\underline{s}_1 \mid \underline{s}_2 \text{ is transmitted}) &= P(\underline{r} \cdot \underline{s}_2 < \underline{r} \cdot \underline{s}_1 \mid \underline{s}_2) \\ &= P(E_b + \sqrt{E_b} n_2 < \sqrt{E_b} n_1 \mid \underline{s}_2) \\ &= P(n_1 - n_2 > \sqrt{E_b} \mid \underline{s}_2) \\ &= Q(\sqrt{\frac{E_b}{\mathrm{N}_0}}) \end{aligned}$$

$$E_b = \left\|s_2(\mathrm{t})\right\|^2 = \int_0^T A^2 \cos^2(2\pi(f_c - \frac{\Delta f}{2}))dt = \frac{A^2 T}{2}$$

$$\begin{aligned} P_e &= \frac{1}{2} P(\underline{s}_1 \mid \underline{s}_2 \text{ is transmitted}) + \frac{1}{2} P(\underline{s}_2 \mid \underline{s}_1 \text{ is transmitted}) \\ &= P(\underline{s}_1 \mid \underline{s}_2 \text{ is transmitted}) \\ &= Q(\sqrt{\frac{E_b}{\mathrm{N}_0}}) \end{aligned}$$

(b)

For non-coherent detection

$$\underline{r}_1 = e^{jf}\,\underline{s}_{11} + \underline{n}_1 \qquad \text{(assume } \underline{s}_{11} \text{ is sent)}$$

$$\begin{aligned}\hat{\underline{s}}_{m\ell} &= \arg\max_{\underline{s}_{m\ell}} \left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right| \\ &= \arg\max_{\underline{s}_{m\ell}} \{\sqrt{\mathrm{Re}(\left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right|)^2 + \mathrm{Im}(\left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right|)^2}\} \\ &= \arg\max_{\underline{s}_{m\ell}} R_m\end{aligned}$$

$$\begin{cases} R_1 = \left|\underline{r}_\ell \cdot \underline{s}_{1\ell}\right| = \left|2E_s e^{j\phi} + \underline{n}_\ell \cdot \underline{s}_{1\ell}\right| & m=1 \\ R_2 = \left|\underline{r}_\ell \cdot \underline{s}_{2\ell}\right| = \left|\underline{n}_\ell \cdot \underline{s}_{2\ell}\right| & m=2 \end{cases}$$

$$\begin{cases} \mathrm{Re}\{\left|\underline{r}_\ell \cdot \underline{s}_{1\ell}\right|\} \sim \mathcal{N}(2E_s\cos\phi, 2E_sN_0) \\ \mathrm{Im}\{\left|\underline{r}_\ell \cdot \underline{s}_{1\ell}\right|\} \sim \mathcal{N}(2E_s\sin\phi, 2E_sN_0) \end{cases} \qquad m=1$$

$$\begin{cases} \mathrm{Re}\{\left|\underline{r}_\ell \cdot \underline{s}_{1\ell}\right|\} \sim \mathcal{N}(0, 2E_sN_0) \\ \mathrm{Im}\{\left|\underline{r}_\ell \cdot \underline{s}_{1\ell}\right|\} \sim \mathcal{N}(0, 2E_sN_0) \end{cases} \qquad \mathrm{m}=2$$

R_1 is Rician distributed with mean $s = 2E_s \quad \sigma^2 = 2E_sN_0$

R_2 is *Rayleigh* distributed with mean $s = 0 \quad \sigma^2 = 2E_sN_0$

$$f_{R_1}(r_1) = \begin{cases} \dfrac{r_1}{\sigma^2} I_0\left(\dfrac{sr_1}{\sigma^2}\right)\exp\left(\dfrac{-r_1^2 + s^2}{2\sigma^2}\right) & r_1 > 0 \\ 0 & o.w \end{cases}$$

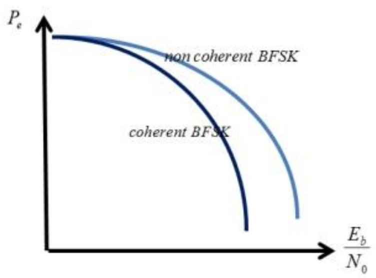
$$f_{R_2}(r_2) = \begin{cases} \dfrac{r_2}{\sigma^2}\exp\left(\dfrac{-r_2^2}{2\sigma^2}\right) & r_2 > 0 \\ 0 & o.w \end{cases}$$

$$\begin{aligned}P_c &= P\{R_2 < R_1\} \\ &= \int_0^\infty P(R_2 < r_1) f_{R_1}(r_1)\,\mathrm{d}r_1\end{aligned}$$

$$P(R_2 < r_1) = \int_0^{r_1} f_{R_2}(r_2)dr_2 = \int_0^{r_1} \frac{r_2^2}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} dr = -e^{\frac{-x^2}{2\sigma^2}}\Bigg|_0^{r_1} = 1 - e^{\frac{-r_1^2}{2\sigma^2}}$$

$$P_e = 1 - \int_0^\infty P(R_2 < r_1) f_{R_1}(r_1)\,\mathrm{d}r_1 = 1 - \int_0^\infty (1 - e^{\frac{-r_1^2}{2\sigma^2}}) \frac{r_1}{\sigma^2} I_0\left(\frac{sr_1}{\sigma^2}\right)\exp\left(\frac{-r_1^2 + s^2}{2\sigma^2}\right)\mathrm{d}r_1 = \frac{1}{2}\exp\left(\frac{-E_b}{2N_0}\right)$$

(c)



There is the bound $Q(x) \le \frac{1}{2}e^{\frac{-x^2}{2}}$(The bound is tighter than the Chernov bound, from p28, Ch2 Deterministic and Random Signal Analysis).We can substitute $\sqrt{\frac{E_b}{N_0}}$ for x,and then we get $Q\left(\sqrt{\frac{E_b}{N_0}}\right) \le \frac{1}{2}e^{\frac{-E_b}{2N_0}}$.