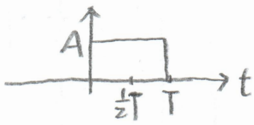
Communication System Homework 1 (Spring 2020)

2.3

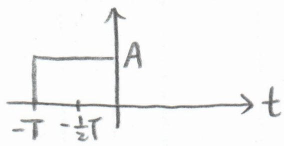
$$g(t) = g_e(t) + g_o(t)$$
$$g_e(t) = \frac{1}{2}[g(t) + g(-t)]$$
$$g_o(t) = \frac{1}{2}[g(t) - g(-t)]$$

(a)

$$g(t) = A\,\mathrm{rect}\left(\frac{t}{T} - \frac{1}{2}\right) \longrightarrow$$



$$g(-t) = A\,\mathrm{rect}\left(\frac{t}{T} + \frac{1}{2}\right) = A\,\mathrm{rect}\left(\frac{t-(-\frac{1}{2}T)}{T}\right)$$



$$g_e(t) = \frac{1}{2}[g(t) + g(-t)]$$
$$= \frac{A}{2}\mathrm{rect}\left(\frac{t}{2T}\right)$$

$$g_o(t) = \frac{1}{2}[g(t) - g(-t)]$$
$$= \frac{A}{2}\left[\mathrm{rect}\left(\frac{t-\frac{1}{2}T}{T}\right) - \mathrm{rect}\left(\frac{t+\frac{1}{2}T}{T}\right)\right]_{\#}$$

(b)

By the time-scaling property $g(-t) \rightleftharpoons G(-f)$

$$G_e(f) = \frac{1}{2}[G(f) + G(-f)]$$
$$= \frac{AT}{2}\left[\mathrm{sinc}(fT)e^{-j\pi fT} + \mathrm{sinc}(fT)e^{j\pi ft}\right]$$
$$= AT\,\mathrm{sinc}(fT)\cos(\pi fT) \text{ or } = AT\,\mathrm{sinc}(2fT)_{\divideontimes}$$

$$G_o(f) = \frac{1}{2}[G(f) - G(-f)]$$
$$= \frac{AT}{2}\left[\mathrm{sinc}(fT)e^{-j\pi fT} - \mathrm{sinc}(fT)e^{j\pi fT}\right]$$
$$= -jAT\,\mathrm{sinc}(fT)\sin(\pi fT)_{\divideontimes}$$

$$\begin{cases} \cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \\ \sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \end{cases}$$

2.6

(a)

If $g(t)$ is <u>even</u> and <u>real</u>, then

$g(t)=\frac{1}{2}[g(t)+g(-t)]$

and $g(t)=g^*(t) \Rightarrow G(f)=G^*(-f)$ — ①

$G^*(f)=\frac{1}{2}[G^*(f)+G^*(-f)]$

$\frac{1}{2}G^*(f)=\frac{1}{2}G^*(-f)$

$G^*(f)=G^*(-f)$ — ②

由①, ②得 $G^*(f)=G(f)$

$\therefore G(f)$ is all real #

If $g(t)$ is <u>odd</u> and <u>real</u>, then

$g(t)=\frac{1}{2}[g(t)-g(-t)]$

and $g(t)=g^*(t) \Rightarrow G(f)=G^*(-f)$ — ③

$G(f)=\frac{1}{2}[G(f)-G(-f)]$

$G^*(f)=\frac{1}{2}G^*(f)-\frac{1}{2}G^*(-f)$

$G^*(f)=-G^*(-f)$ — ④

由③, ④得 $G^*(f)=-G(f)$

$\therefore G(f)$ must be all imaginary ※

(b)

$(-j2\pi t)g(t) \rightleftharpoons \frac{d}{df}G(f)$

$t\cdot g(t) \rightleftharpoons (\frac{j}{2\pi})\frac{d}{df}G(f)$

The previous step can be repeated n times so:

$t^n g(t) \rightleftharpoons (\frac{j}{2\pi})^n G^{(n)}(f)$ 得証 #

(c)

Let $h(t)=t^n g(t)$

$H(f)=(\frac{j}{2\pi})^n G^{(n)}(f)$

$\therefore \int_{-\infty}^{\infty} h(t)dt = H(0)=(\frac{j}{2\pi})^n G^{(n)}(0)$ 得証 #

(d)

Let $g(t)=g_1(t)g_2^*(t) \rightleftharpoons G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$

$(G(0)=\int_{-\infty}^{\infty} g(t)dt)$

$\mathcal{F}\{g(t)\}=\mathcal{F}\{g_1(t)g_2^*(t)\}$

$=\int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda-f)d\lambda$

$\therefore g_1(t)g_2^*(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda-f)d\lambda$ 得証 #

(e)

由(d)可知 $g_1(t)g_2^*(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda-f)d\lambda$

$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)dt = G(0)$

$=\int_{-\infty}^{\infty} G_1(\lambda)G_2^*(\lambda-0)d\lambda$

replacing λ with f

$\Rightarrow \int_{-\infty}^{\infty} g_1(t)g_2^*(t)dt = \int_{-\infty}^{\infty} G_1(f)G_2^*(f)df$ 得証 #

2.10

$$Y(f)=\int_{-\infty}^{\infty} X(\nu)X(f-\nu)\,d\nu$$

$$|X(\nu)|\neq 0,\ \text{if } |\nu|\leq W$$

$$|X(f-\nu)|\neq 0,\ \text{if } |f-\nu|\leq W$$

$(f-\nu)\leq W$ for $f\leq W+\nu$ when $\nu\geq 0$ and $\nu\leq W$

$(f-\nu)\geq -W$ for $f\leq -W+\nu$ when $\nu\leq 0$ and $\nu\geq -W$

$\therefore$ $(f-\nu)\leq W$ for $0\leq\nu\leq W$ when $f\leq 2W$

$(f-\nu)\geq -W$ for $-W\leq\nu\leq 0$ when $f\geq -2W$

$\therefore$ Over the range of integration $[-W, W]$

the integral is non-zero if $|f|\leq 2W$ #

2.14

(a)

$$\frac{dg(t)}{dt}\rightleftharpoons j2\pi f G(f),\quad \text{Let } g'(t)=\frac{dg(t)}{dt}$$

By Rayleigh's theorem: $\int_{-\infty}^{\infty}|g(t)|^2dt=\int_{-\infty}^{\infty}|G(f)|^2df$

$$W_{rms}^2T_{rms}^2=\frac{\int_{-\infty}^{\infty}t^2|g(t)|^2dt\cdot\boxed{\int_{-\infty}^{\infty}f^2|G(f)|^2df}}{\left(\int_{-\infty}^{\infty}|g(t)|^2dt\right)^2}$$

$$=\frac{\int_{-\infty}^{\infty}t^2|g(t)|^2dt\cdot\int_{-\infty}^{\infty}g'(t)g'^*(t)dt}{4\pi^2\left(\int_{-\infty}^{\infty}|g(t)|^2dt\right)^2}$$

$$\left[\begin{aligned}\int_{-\infty}^{\infty}f^2|G(f)|^2df&=\int_{-\infty}^{\infty}|fG(f)|^2df\\&=\int_{-\infty}^{\infty}\left|\frac{1}{j2\pi}\frac{d}{dt}g(t)\right|^2dt\\&=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}|g'(t)|^2dt\\&=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}g'(t)g'^*(t)dt\end{aligned}\right]$$

$\because C+C^*=2\mathrm{Re}\{C\}\leq 2|C|,\ \forall C\in\mathbb{C}$

$$\left\{\int_{-\infty}^{\infty}[g_1^*(t)g_2(t)+g_1(t)g_2^*(t)dt]\right\}^2\leq\left[\int_{-\infty}^{\infty}2|g_1(t)g_2^*(t)|dt\right]^2$$

$$=4\left[\int_{-\infty}^{\infty}|g_1(t)g_2^*(t)|dt\right]^2$$

$$\leq 4\left[\int_{-\infty}^{\infty}|g_1(t)||g_2^*(t)|dt\right]^2$$

By C-S theorem

$$\leq 4\int_{-\infty}^{\infty}|g_1(t)|^2dt\int_{-\infty}^{\infty}|g_2(t)|^2dt\quad -①$$

Let $g_1(t) = t\,g(t)$, $g_2(t) = g'(t)$

$$\Rightarrow \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} |g'(t)|^2 dt \geq \frac{\left\{\int_{-\infty}^{\infty} t g^*(t) g'(t) + t g(t) g'^*(t) dt\right\}^2}{4} \quad (\text{By } ①)$$

$$= \frac{\left\{\int_{-\infty}^{\infty} t \frac{d}{dt}[g(t) g^*(t)] dt\right\}^2}{4}$$

By integration by parts　　令 $u = t$, $du = dt$, $dv = \frac{d}{dt}[g(t)g^*(t)]dt$, $V = |g(t)|^2$

$$[\overset{uv}{\underbrace{t|g(t)|^2\Big|_{-\infty}^{\infty}}_{=0}} - \overset{vdu}{\int_{-\infty}^{\infty} |g(t)|^2 dt}]^2 = \left[\int_{-\infty}^{\infty} |g(t)|^2 dt\right]^2$$

∵根據題目給的條件 $g(t)$ 比 $\frac{1}{\sqrt{t}}$ 快 converge 到 0 在 $t \to \infty$ 及 $t \to -\infty$

∴ $t|g(t)|^2\Big|_{-\infty}^{\infty} = 0$

$$= \frac{\left[\int_{-\infty}^{\infty} |g(t)|^2 dt\right]^2}{4} \quad —②$$

$$\therefore W_{rms}^2 T_{rms}^2 = \frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \cdot \int_{-\infty}^{\infty} g'(t) g'^*(t) dt}{4\pi^2 \left(\int_{-\infty}^{\infty} |g(t)|^2 dt\right)^2}$$

$$\geq \frac{\left(\int_{-\infty}^{\infty} |g(t)|^2 dt\right)^2}{16\pi^2 \left(\int_{-\infty}^{\infty} |g(t)|^2 dt\right)^2} \quad (\text{By } ②)$$

$$\geq \frac{1}{16\pi^2}$$

$\Rightarrow W_{rms} T_{rms} \geq \frac{1}{4\pi}$ 得証#

(b)

For $g(t) = \exp(-\pi t^2)$

$g(t) \rightleftharpoons \exp(-\pi f^2) = G(f)$

$$\therefore W_{rms}^2 T_{rms}^2 = \frac{\int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} f^2 |G(f)|^2 df}{\left(\int_{-\infty}^{\infty} |g(t)|^2 dt\right)^2}$$

$$= \frac{\int_{-\infty}^{\infty} t^2 \exp(-2\pi t^2) dt \int_{-\infty}^{\infty} f^2 \exp(-2\pi f^2) df}{\left(\int_{-\infty}^{\infty} \exp(-2\pi t^2) dt\right)^2}$$

$$= \frac{\left(\frac{1}{4\pi}\sqrt{\frac{1}{2}}\right)^2}{\frac{1}{2}} = \left(\frac{1}{4\pi}\right)^2$$

$\Rightarrow W_{rms} T_{rms} = \frac{1}{4\pi}$ 得証#

$\boxed{\text{Using a table of integrals: } \int_0^{\infty} x^2 \exp(-ax^2) dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}} \text{ for } a > 0}$

$$\therefore \int_{-\infty}^{\infty} t^2 \exp(-2\pi t^2) dt = \frac{1}{8\pi}\sqrt{\frac{\pi}{2\pi}} \cdot 2 = \frac{1}{4\pi}\sqrt{\frac{1}{2}}$$

$$\int_{-\infty}^{\infty} f^2 \exp(-2\pi f^2) df = \frac{1}{4\pi}\sqrt{\frac{1}{2}}$$

$\boxed{\text{Using a table of integrals: } \int_0^{\infty} \exp(-ax^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}}$

$$\therefore \int_{-\infty}^{\infty} \exp(-2\pi t^2) dt = \frac{1}{2}\sqrt{\frac{\pi}{2\pi}} \cdot 2 = \sqrt{\frac{1}{2}}$$

2.19

(a)

For the sake of convenience, let $h(t)$ be the filter time-shifted so that it is symmetric about the region $(t=0)$.

$$H(f)=\sum_{k=1}^{\frac{N-1}{2}} W_k \exp(-j2\pi f k)+\sum_{k=-1}^{-\frac{N-1}{2}} W_k \exp(-j2\pi f k)+W_0$$

$$=2\sum_{k=1}^{\frac{N-1}{2}} W_k \cos(2\pi f k)+W_0$$

Let $G(f)$ be the filter returned to its correct position. Then $G(f)=H(f)\exp(-j2\pi f(\frac{N-1}{2}))$, which is a time-shift of $(\frac{N-1}{2})$ samples.

$$\therefore G(f)=\exp(-j\pi f(N-1))\,2\sum_{k=1}^{\frac{N-1}{2}} W_k \cos(2\pi f k)+\exp(-j\pi f(N-1))\,W_0$$ #

(b)

By inspection, it is apparant that:

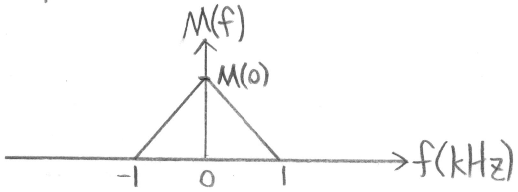
$\measuredangle G(f)=\measuredangle \exp(-j\pi f(N-1))$

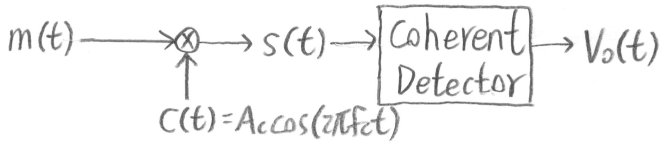
This meets the definition of linear phase. #

3.8

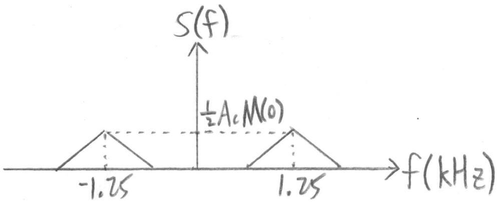
(a)

For $f_c=1.25\text{kHz}$, the spectra of the message signal $m(t)$, the product modulator output $s(t)$, and the coherent detector output $V(t)$ are as follows, respectively.





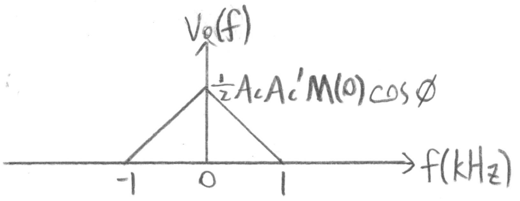
⇒



$S(t)=A_c\cos(2\pi f_c t)m(t)$

$S(f)=\frac{1}{2}A_c[M(f-f_c)+M(f+f_c)]$

⇒

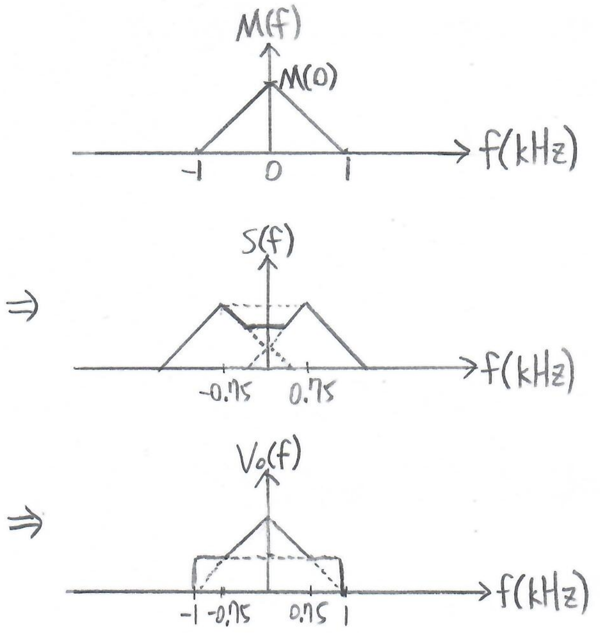


$V(t)=\frac{1}{2}A_cA_c'\cos(4\pi f_c t+\phi)m(t)+\frac{1}{2}A_cA_c'\cos\phi\, m(t)$

$V_o(t)=\frac{1}{2}A_cA_c'\cos\phi\, m(t)$ #

(b)

For the case when $f_c = 0.75$, the respective spectra are as follows:



To avoid sideband-overlap, the carrier frequency f_c must be greater than or equal to 1kHz. The lowest carrier frequency is therefore 1 kHz for each sideband of the modulated wave $S(t)$ to be inequality determined by $m(t)$.

3.9

The two AM modulator outputs are

$S_1(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$

$S_2(t) = A_c[1 - k_a m(t)]\cos(2\pi f_c t)$

Substracting $S_2(t)$ from $S_1(t)$:

$S(t) = S_1(t) - S_2(t)$

$= 2A_c k_a m(t)\cos(2\pi f_c t)$ which represents a DSB-SC modulated wave.

#

3.11

(a) Multiplying the signal by the local oscillator gives:

$$S_1(t) = A_c m(t)\cos(2\pi f_c t)\cos[2\pi(f_c+\Delta f)t]$$
$$= \frac{A_c}{2} m(t)\{\cos(2\pi\Delta f t) + \cos[2\pi(2f_c+\Delta f)t]\}$$

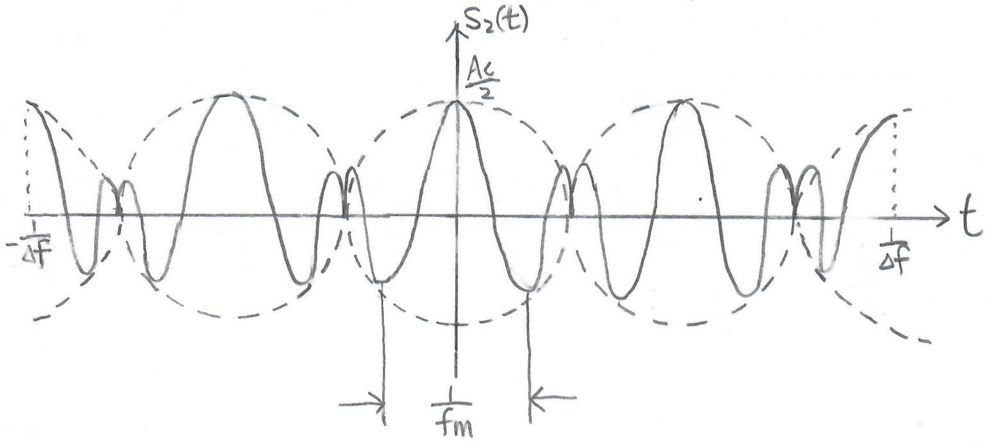
Low-pass filtering leaves:

$$S_2(t) = \frac{A_c}{2} m(t)\cos(2\pi\Delta f t)$$

Thus the output signal is the message signal modulated by a sinusoid of frequency Δf.

(b) If $m(t) = \cos(2\pi f_m t)$

then $S_2(t) = \frac{A_c}{2}\cos(2\pi f_m t)\cos(2\pi\Delta f t)$



3.14

When the local carriers have a phase error ϕ, we may write

$$\cos(2\pi f_c t+\phi) = \cos(2\pi f_c t)\cos\phi - \sin(2\pi f_c t)\sin\phi$$

In this case, we find that by multiplying the received signal $r(t)$ by $\cos(2\pi f_c t+\phi)$, and passing the resulting output through a low-pass filter, the corresponding low-pass filter output in the receiver has a spectrum equal to $\frac{A_c}{2}H(f-f_c)[\cos\phi M_1(f) - \sin\phi M_2(f)]$. This indicates that there is cross-talk at the demodulator outputs.

3.15

The transmitted signal is given by

$$s(t)=A_cm_1(t)\cos(2\pi f_ct)+A_cm_2(t)\sin(2\pi f_ct)$$
$$=A_c[V_0+m_\ell(t)+m_r(t)]\cos(2\pi f_ct)+A_c[m_\ell(t)-m_r(t)]\sin(2\pi f_ct)$$

(a)

$$s(t)\longrightarrow\boxed{\begin{array}{c}\text{Evelope}\\ \text{Detector}\end{array}}\longrightarrow|\tilde{s}(t)|=y(t)$$

$$s(t)=\mathrm{Re}\{\tilde{s}(t)e^{j2\pi f_ct}\}$$

$$\tilde{s}(t)=A_cm_1(t)-jA_cm_2(t)$$

$$y(t)=|\tilde{s}(t)|=\sqrt{A_c^2(V_0+m_\ell(t)+m_r(t))^2+A_c^2(m_\ell(t)-m_r(t))^2}$$
$$=A_c(V_0+m_\ell(t)+m_r(t))\sqrt{1+\left(\frac{m_\ell(t)-m_r(t)}{V_0+m_\ell(t)+m_r(t)}\right)^2}$$

To minimize the distortion in the envelope dector output due to the quadrature component, we choose the DC offset V_0 to be large. We may approximate $y(t)$ as

$$y(t)\approx A_c(V_0+m_\ell(t)+m_r(t))$$

which except for the DC component A_cV_0, is the proportional to the sum $m_\ell(t)+m_r(t)$.

(b)

$$V(t)=s(t)A_c'\sin(2\pi f_ct)$$
$$=A_cA_c'[m_1(t)\cos(2\pi f_ct)\sin(2\pi f_ct)+m_2(t)\sin(2\pi f_ct)\sin(2\pi f_ct)]$$
$$=\frac{1}{2}A_cA_c'[m_1(t)\sin(4\pi f_ct)+m_1(t)\overset{0}{\cancel{\sin(0)}}+m_2(t)\overset{1}{\cancel{\cos(0)}}-m_2(t)\cos(4\pi f_ct)]$$

Through Low-pass filter,

$$V_0(t)=\frac{1}{2}A_cA_c'm_2(t)$$

For coherent detection at the receiver, we need a replica of the carrier $A_c\cos(2\pi f_ct)$. This requirement can be satisfied by passing the received signal $s(t)$ through a narrow-band filter of mid-band frequency f_c. However, to extract the difference $m_\ell(t)-m_r(t)$, we need $\sin(2\pi f_ct)$, which is obtained by passing the narrow-band filter output through a $90°$-phase filter. Then multiplying $s(t)$ by $\sin(2\pi f_ct)$ and low-pass filtering, we can obtain a signal proportional to $m_\ell(t)-m_r(t)$.

(c)

① Equalize the outputs of the envelope detector and coherent detector.

② Pass the equalized outputs through an audio demixer to produce $m_\ell(t)$ and $m_r(t)$.

The output from the envelop detector : $V_{01} = A_c m_1(t)$

The output from the coherent detector : $V_{02} = A_c m_2(t)$

$$\Rightarrow \begin{cases} V_{01}' = \dfrac{V_{01}}{A_c} - V_0 \\ V_{02}' = \dfrac{V_{02}}{A_c} \end{cases} \Rightarrow \begin{cases} m_\ell(t) = \frac{1}{2}(V_{01}' + V_{02}') \\ m_r(t) = \frac{1}{2}(V_{01}' - V_{02}') \end{cases}$$ are obtained.