通訊系統 (II)

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Chapter 4
Frequency-Shift Keying Modulation

Introduction

Orthogonal Signaling

- Orthogonal signals are defined as a set of equal energy signals $s_m(t)$, $1 \le m \le M$, such that

$$\langle s_m(t), s_n(t) \rangle = 0, \quad m \ne n, 1 \le m, n \le M$$

- The signals are **linearly independent** and hence the number of **orthonormal basis functions** is $N = M$

$$\phi_j(t) = s_j(t) \big/ \sqrt{E}\,, 1 \le j \le N$$

  – where E is the symbol energy

- The signal vectors can be represented as

$$s_1 = \left[\sqrt{E}, 0, 0, \cdots, 0\right]$$
$$s_2 = \left[0, \sqrt{E}, 0, \cdots, 0\right]$$
$$\vdots$$
$$s_1 = \left[0, 0, 0, \cdots, \sqrt{E}\right]$$

Orthogonal Signaling (Cont.)

- The distance between two signal vectors $s_m(t)$ and $s_n(t)$ is

$$d_{mn} = \sqrt{2E}, \quad m \neq n, 1 \leq m, n \leq M$$

  - All signal points are **equally spaced**
  - The distance is the **minimum distance** $d_{\min}$
- Because each symbol contains $\log_2 M$ bits

$$E_b = E/\log_2 M$$

- Hence,

$$d_{\min} = \sqrt{2E_b \log_2 M}$$

Frequency-Shift Keying (FSK)

- **Frequency-Shift Keying (FSK)** is a special case of the construction of orthogonal signals
- Consider the construction of **orthogonal signal waveforms** that differ in **frequency**

$$s_m(t) = \mathrm{Re}\left[\tilde{s}_m(t) e^{j2\pi f_c t}\right] = \sqrt{\frac{2E}{T}} \cos\left(2\pi\left(f_c + m\Delta f\right)t\right), 0 \leq t \leq T$$

$$\tilde{s}_m(t) = \sqrt{\frac{2E}{T}} e^{j2\pi m\Delta f t}, 0 \leq t \leq T$$

- The messages are transmitted by a set of signals that only differ in **frequency**

Linear and Nonlinear Modulation

- In **QAM** signaling (**ASK** and **PSK** can be considered as special cases), the **lowpass equivalent** of the signal is of the form $A_m g(t)$ where A_m is a complex number
  - The **sum** of two lowpass equivalent signals is of the general form of the lowpass equivalent of a QAM signal
  - The sum of two QAM signals is **another QAM signal**
  - Hence, ASK, PSK, and QAM are sometimes called **linear modulation schemes**
- On the other hand, **FSK** signaling does not satisfy this property
  - Therefore it belongs to the class of **nonlinear modulation schemes**

Continuous-Phase Frequency-Shift Keying (CPFSK) Modulation

Continuous-Phase Binary FSK

Continuous-Phase Binary FSK

- In **binary FSK**, symbols 1 and 0 are distinguished from each other by transmitting one of two sinusoidal waves that **differ in frequency by a fixed amount**

$$s_i(t) = \begin{cases} \sqrt{2E_b/T_b}\cos(2\pi f_i t), & 0 \le t < T_b \\ 0, & \text{elsewhere} \end{cases}, i = 1, 2$$

  - where E_b is the signal **energy per bit** and the **transmitted frequency** is set at $f_i = (n_c + i)/T_b$ for some fixed integer n_c
  - Symbol 1 is represented by $s_1(t)$ and symbol 0 by $s_2(t)$
- The FSK signal described here is a **continuous-phase** signal
  - The phase continuity is always maintained, including the **inter-bit switching times**
  - $f_i = (n_c + i)/T_b \Rightarrow$ Zero-phase at $t = 0$ and $t = T_b$
  - **CPFSK**: Continuous-Phase Frequency-Shift Keying

Continuous-Phase Binary FSK (Cont.)

- Since $s_1(t)$ and $s_2(t)$ are orthogonal, we have the set of orthonormal basis functions described by

$$\phi_i(t) = \begin{cases} \sqrt{2/T_b}\cos(2\pi f_i t), & 0 \le t < T_b \\ 0, & \text{elsewhere} \end{cases}, \quad i = 1, 2$$

- Correspondingly, the coefficient s_{ij} for $i = 1, 2$ and $j = 1, 2$ is defined by

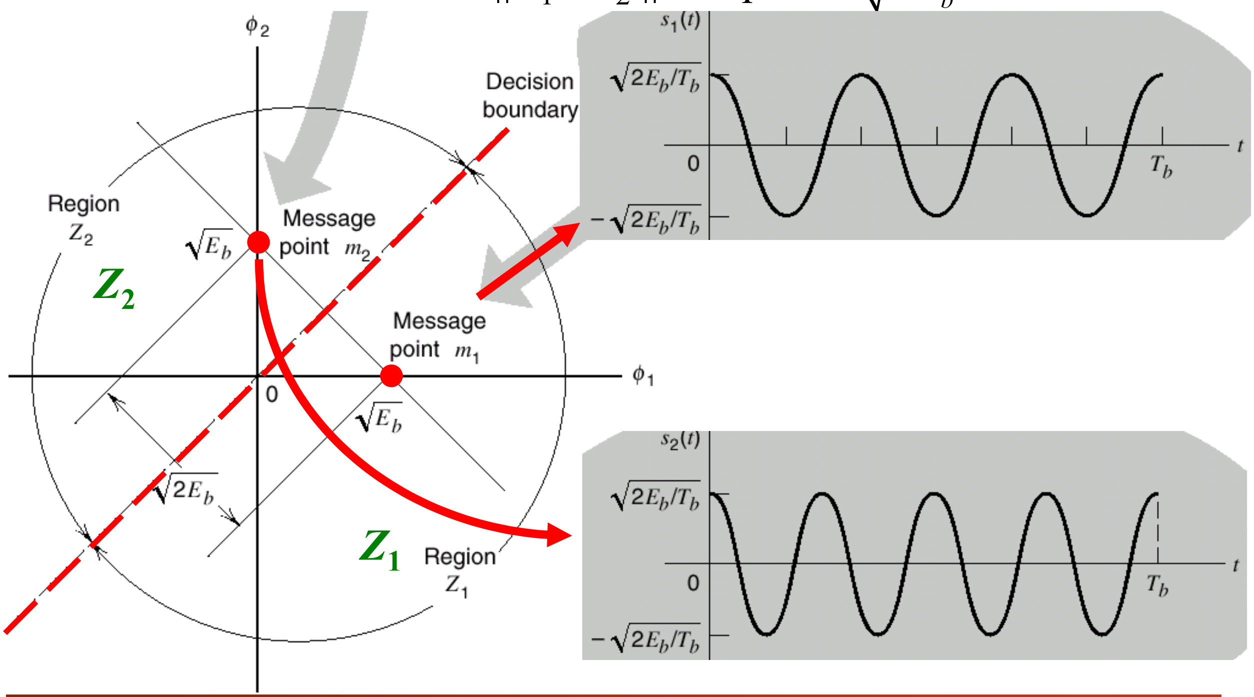
$$s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\,dt = \begin{cases} \sqrt{E_b}, & i = j \\ 0, & i \ne j \end{cases}, \quad i, j = 1, 2$$

- The two message points are defined by the signal vectors

$$\mathbf{s}_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}; \quad \mathbf{s}_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

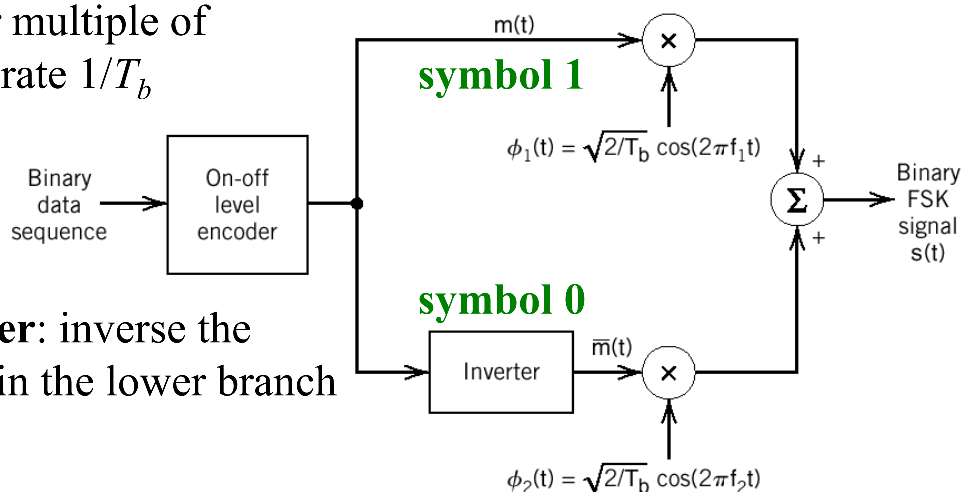
Continuous-Phase Binary FSK (Cont.)

- The Euclidean distance $\| \boldsymbol{s}_1 - \boldsymbol{s}_2 \|$ is equal to $\sqrt{2E_b}$



Generation of CP Binary FSK Signals

- A binary FSK signal generator consists of three components:
  - **On–off level encoder**: the output of which is a constant amplitude of $\sqrt{E_b}$ for **symbol 1** and zero for **symbol 0**
  - **Pair of oscillators**: whose frequencies f_1 and f_2 differ by an integer multiple of the bit rate $1/T_b$
  - **Inverter**: inverse the signal in the lower branch

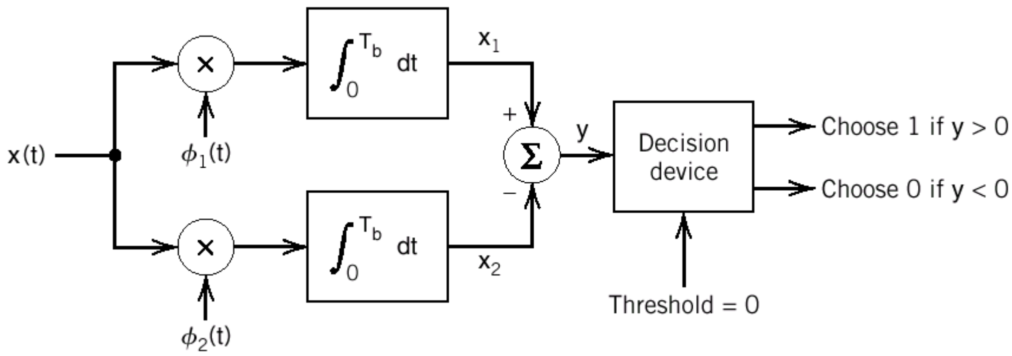


Generation of CP BFSK Signals (Cont.)

- To meet the **phase continuity** requirement
  - The two oscillators are **synchronized** with each other
- Alternatively, we may use a **voltage-controlled oscillator** (VCO), in which case phase continuity is automatically satisfied
  - Only **one** oscillator
  - With the output frequency controlled by the input signal

Detection of CP Binary FSK Signals

- A **coherent** detector consists of two **correlators** with a common input: the noisy received signal $x(t)$
  - The local coherent reference signals: $\phi_1(t)$ and $\phi_2(t)$
- The correlator outputs are then **subtracted**, one from the other
- The resulting difference is then compared with a threshold: **zero**
  - If $y > 0$: symbol 1; if $y < 0$: symbol 0; if $y = 0$: random guess



Error Probability of CP Binary FSK

- The observation vector **x** has two elements x_1 and x_2 that are defined by $$x_i = \int_0^{T_b} x(t)\phi_i(t)\,dt, \quad i = 1, 2$$
  - where $x(t)$ is the received signal, whose form depends on which symbol was transmitted
- Given that symbol i was transmitted, $x(t) = s_i(t) + w(t)$
  - where $w(t)$ is the sample function of a **white Gaussian noise process** of zero mean and power spectral density $N_0/2$
  - If $i = 1$, $\mathbb{E}[x_1] = \sqrt{E_b}$ and $\mathbb{E}[x_2] = 0$
  - If $i = 0$, $\mathbb{E}[x_1] = 0$ and $\mathbb{E}[x_2] = \sqrt{E_b}$
- We define a new Gaussian random variable Y with $y = x_1 - x_2$
  - $\mathbb{E}[y \mid 1] = +\sqrt{E_b}$ and $\mathbb{E}[y \mid 0] = -\sqrt{E_b}$
  - $\mathrm{Var}[Y] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] = N_0$

Error Probability of CP BFSK (Cont.)

- Suppose that symbol i was sent. The conditional probability density function of the random variable Y is given by

$$f_Y(y|1) = \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{1}{2N_0}\left(y - \sqrt{E_b}\right)^2\right]$$

$$f_Y(y|0) = \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{1}{2N_0}\left(y + \sqrt{E_b}\right)^2\right]$$

- The conditional probability of error given that symbol i was sent is

$$p_{10} = p_{01} = \frac{1}{2}\operatorname{erfc}\left(\sqrt{E_b/2N_0}\right)$$

- Finally, the BER for binary FSK using **coherent** detection is

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{E_b/2N_0}\right)$$

Power Spectra of Binary FSK Signals

- Consider the case that the two transmitted frequencies f_1 and f_2 differ by an amount **equal to the bit rate 1/T_b**
  - The arithmetic mean of f_1 and f_2: the carrier frequency f_c
  - $f_1 = f_c + 1/2T_b$ and $f_2 = f_c - 1/2T_b$
- The signal can be express as a frequency-modulated (FM) signal

$$s(t) = \sqrt{2E_b/T_b}\cos(2\pi f_c t \pm \pi t/T_b), \quad 0 \le t < T_b$$

$$= \sqrt{2E_b/T_b}\left[\underbrace{\cos(\pi t/T_b)}_{\text{In-phase}}\cos(2\pi f_c t) \mp \underbrace{\sin(\pi t/T_b)}_{\text{Quadrature}}\sin(2\pi f_c t)\right]$$

  - "–": symbol 1; "+": symbol 0

Power Spectra of Binary FSK Signals (Cont.)

$$s(t) = \sqrt{2E_b/T_b}\left[\cos(\pi t/T_b)\cos(2\pi f_c t) \mp \sin(\pi t/T_b)\sin(2\pi f_c t)\right]$$

- The **in-phase component**: completely independent of the input binary wave
  - The baseband PSD consists of **two delta functions** weighted by the factor $E_b/2T_b$ and occurring at $f = \pm 1/2T_b$
- The **quadrature component**: directly related to the input binary sequence
  - $-g(t)$ for symbol 1 and $+g(t)$ for symbol 0

$$g(t) = \sqrt{2E_b/T_b}\sin(\pi t/T_b), \quad 0 \le t < T_b$$

  - The baseband PSD of $g(t)$ is

$$S_g(f) = \frac{8E_b\cos^2(\pi T_b f)}{\pi^2\left(4T_b^2 f^2 - 1\right)^2}$$

Power Spectra of Binary FSK Signals (Cont.)

- Because the in-phase and quadrature components are **independent** of each other, the overall **baseband PSD** is

$$S_B(f) = \frac{E_b}{2T_b}\left[\delta\left(f - \frac{1}{2T_b}\right) + \delta\left(f + \frac{1}{2T_b}\right)\right] + \frac{8E_b\cos^2(\pi T_b f)}{\pi^2\left(4T_b^2 f^2 - 1\right)^2}$$

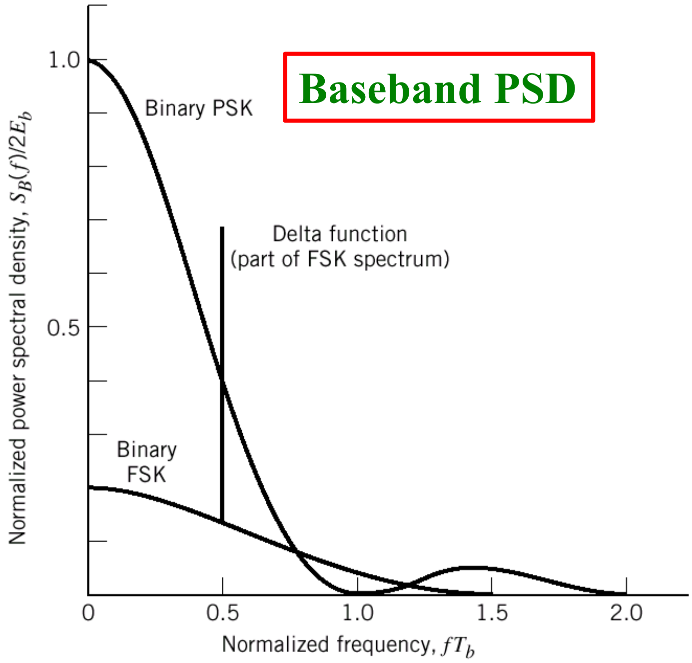
- The **passband** PSD becomes

$$S_P(f) = \frac{1}{4}\left[S_B\left(f - f_c\right) + S_B\left(f + f_c\right)\right]$$

- The PSD contains two **discrete frequency components** located at f_1 and f_2, with the sum power up to **1/2** the total signal power
  - The discrete frequency components provide a practical basis for **synchronizing** the receiver with the transmitter
  - The power is independent to data $\Rightarrow$ **low power efficiency**

Power Spectra of Binary FSK Signals (Cont.)

- The baseband power spectral density of a binary FSK signal with **continuous phase**
  - Ultimately **falls off** as the **inverse fourth power** of frequency f^{-4}



Minimum Shift Keying

Continuous-Phase Frequency-Shift Keying

- In the detection of binary FSK signal, the **phase information** contained in the received signal is **not fully exploited**
- By using the **continuous-phase property** in detection, it is possible to improve the noise performance at the receiver
  - This improvement is achieved at the expense of **increased system complexity**
- Consider a continuous-phase frequency-shift keying (**CPFSK**) signal

$$s(t) = \begin{cases} \sqrt{2E_b/T_b}\cos(2\pi f_1 t + \theta(0)), & \text{symbol 1} \\ \sqrt{2E_b/T_b}\cos(2\pi f_2 t + \theta(0)), & \text{symbol 0} \end{cases}$$

  - where $\theta(0)$ denotes the value of the phase **at time $t = 0$**

CPFSK (Cont.)

- Another way of representing the CPFSK signal $s(t)$ is to express it as a conventional **angle-modulated signal**

$$s(t) = \sqrt{2E_b/T_b}\cos\left[2\pi f_c t + \theta(t)\right]$$

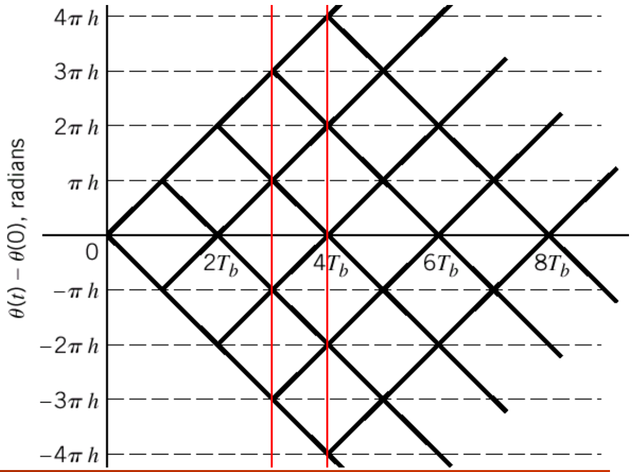
  - where $\theta(t)$ is the phase of $s(t)$ at time t
- The phase $\theta(t)$ of a CPFSK signal **increases** or **decreases linearly with time** during each bit duration of T_b
- That is, $\theta(t) = \theta(0) \pm (\pi h/T_b)t, \quad 0 \le t < T_b$
  - where "+": symbol 1; "–": symbol 0; and h: a dimensionless parameter referred to as the **deviation ratio**
- Because $2\pi f_c t + (\pi h/T_b)t = 2\pi f_1 t; \quad 2\pi f_c t - (\pi h/T_b)t = 2\pi f_2 t$
  - We deduce the relation: $h = (f_1 - f_2)T_b$
- h is normalized with respect to $1/T_b$: if $f_1 - f_2 = 1/T_b$, $h = 1$

CPFSK – Phase Trellis

- At time $t = T_b$,

$$\theta(T_b) - \theta(0) = \begin{cases} \pi h & \text{for symbol 1} \\ -\pi h & \text{for symbol 0} \end{cases}$$

- Sending symbol 1 (symbol 0) increases (decreases) the phase of a CPFSK signal $s(t)$ by $\boldsymbol{\pi h}$ radians
- **Phase tree**: A plot shows the transitions of phase across successive signaling intervals
- The phase of a CPFSK signal is an **odd or even multiple** of πh radians at odd or even multiples of T_b, respectively.



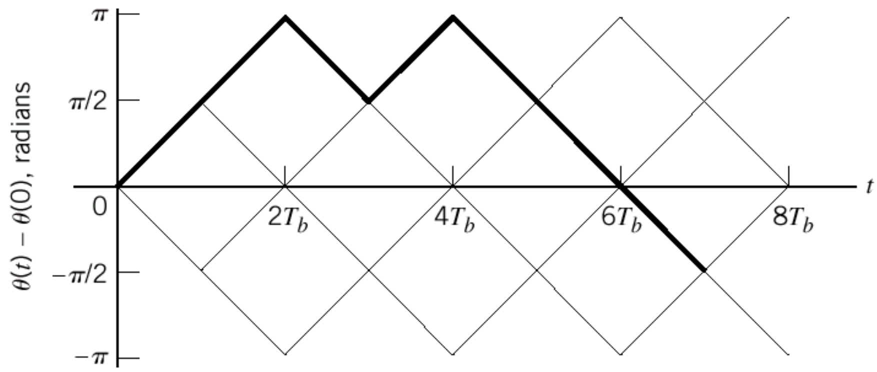
CPFSK – Phase Trellis (Cont.)

- The phase tree shows **phase continuity** of the CPFSK signal
- The coherent BFSK with $f_1 - f_2 = 1/T_b$ is also a CPFSK signal
  - $h = 1$
- Hence, the phase change **over one bit interval** is $\pm\pi$ radians
  - A change of $+\pi$ is **exactly the same** as a change of $-\pi$, modulo 2π
  - Therefore, there is **no memory** for this case
  - Knowing which particular change occurred in the **previous signaling interval** provides **no help** in the **current signaling interval**

CPFSK – Phase Trellis (Cont.)

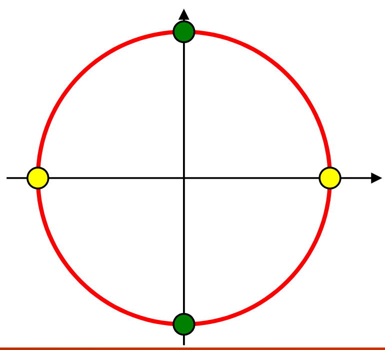
- In contrast, we have a completely different situation when the deviation ratio h is assigned the special value of **1/2**
  - The phase takes on $\pm\pi/2$ at **odd** multiples of T_b, and
  - The phase takes on 0 and π at **even** multiples of T_b

The data sequence: 1101000



Minimum Shift-Keying (MSK)

- With h = 1/2, symbol 1 and symbol 0 **do not interfere** with each another in the process of detection
  - The two signal points are **different**
  - The frequency deviation: $f_1 - f_2$ equals half the bit rate($1/2T_b$)
- The frequency deviation h = 1/2 is the **minimum frequency spacing** that allows the two FSK signals representing symbol 1 and symbol 0 to be **coherently orthogonal**
- The CPFSK signal with h = 1/2 is commonly referred to as **minimum shift-keying (MSK)**



Minimum Shift-Keying (Cont.)

- We may expand the CPFSK signal $s(t)$ in terms of its in-phase and quadrature components as

$$s(t)=\sqrt{2E_b/T_b}\cos\theta(t)\cos(2\pi f_c t)-\sqrt{2E_b/T_b}\sin\theta(t)\sin(2\pi f_c t)$$

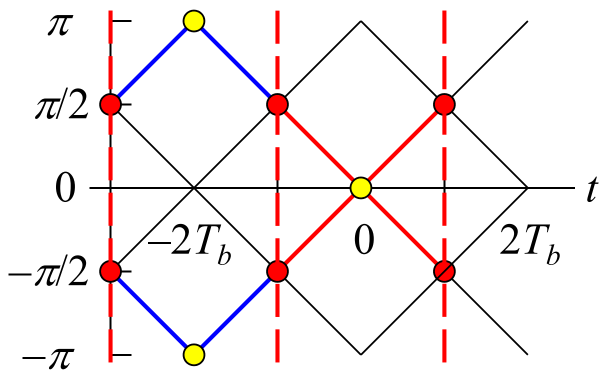
  - **In-phase**: $\cos\theta(t)$; **Quadrature**: $\sin\theta(t)$
- With $h = 1/2$,

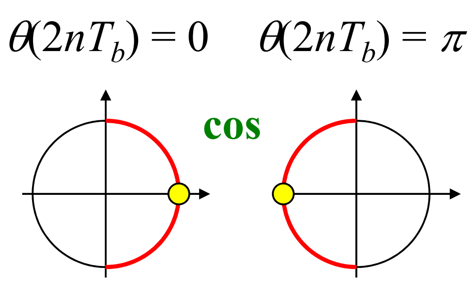
$$\theta(t)=\begin{cases}\theta(2nT_b)\pm(\pi/2T_b)t, (2n-1)T_b \le t < (2n+1)T_b \\ \theta((2n+1)T_b)\pm(\pi/2T_b)t, 2nT_b \le t < (2n+2)T_b\end{cases}$$

Minimum Shift-Keying (Cont.)

- Considering the **in-phase** component $\cos\theta(t)$
  - Note that $-\pi/2 \le (\pi/2T_b)t \le \pi/2$, for $(2n-1)T_b \le t \le (2n+1)T_b$
  - If $\theta(2nT_b) = 0$, $\cos\theta(t) = \cos(\pm\pi t/2T_b) = +\cos(\pi t/2T_b)$
  - If $\theta(2nT_b) = \pi$, $\cos\theta(t) = \cos(\pi\pm\pi t/2T_b) = -\cos(\pi t/2T_b)$

$$\boxed{\theta(t)=\theta(2nT_b)\pm(\pi/2T_b)t, (2n-1)T_b \le t < (2n+1)T_b}$$

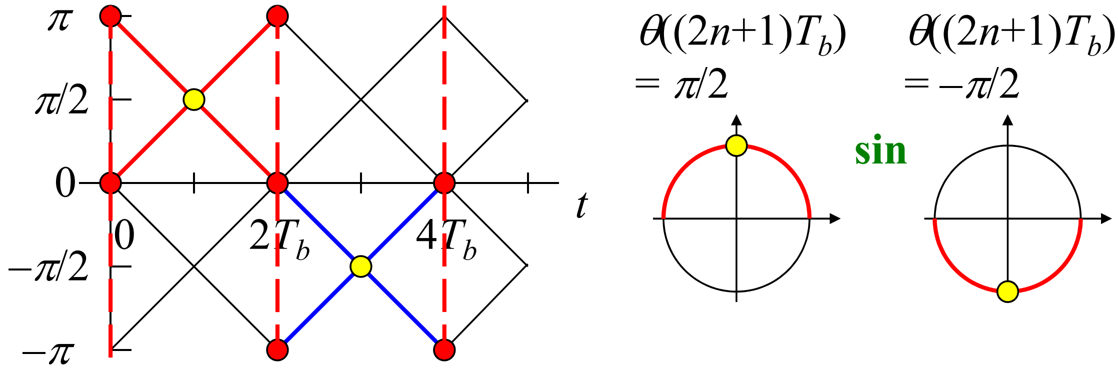




Minimum Shift-Keying (Cont.)

- Considering the **quadrature** component $\sin\theta(t)$
  - Note that $-\pi/2 \le (\pi/2T_b)t \le \pi/2$, for $2nT_b \le t \le (2n+2)T_b$
  - If $\theta((2n+1)T_b) = \pi/2$, $\sin\theta(t) = +\sin(\pi t/2T_b)$
  - If $\theta((2n+1)T_b) = -\pi/2$, $\sin\theta(t) = \sin(\pi + \pi t/2T_b) = -\sin(\pi t/2T_b)$

$$\boxed{\theta(t) = \theta\big((2n+1)T_b\big) \pm (\pi/2T_b)t,\ 2nT_b \le t < (2n+2)T_b}$$



Minimum Shift-Keying (Cont.)

- **In the interval $(2n-1)T_b \le t \le (2n+1)T_b$**, the polarity of $\cos\theta(t)$ depends only on $\theta(2nT_b)$
- The in-phase component consists of the **half-cycle cosine pulse**:

$$s_I(t) = \sqrt{2E_b/T_b}\cos\theta(t) = \sqrt{2E_b/T_b}\cos\left[\theta(2nT_b) \pm (\pi/2T_b)t\right]$$
$$= \pm\sqrt{2E_b/T_b}\cos(\pi t/2T_b)$$

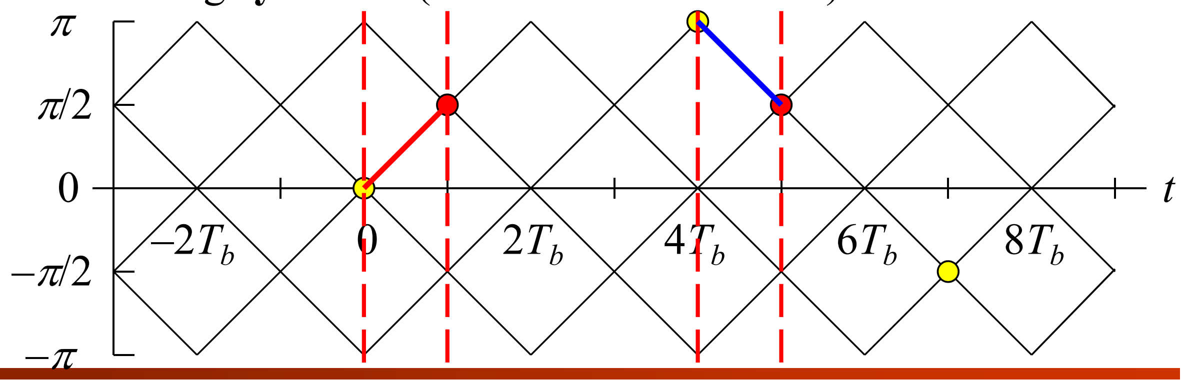
  - where "+": $\theta(2nT_b) = 0$; "–": $\theta(2nT_b) = \pi$
- **In the interval $2nT_b \le t \le (2n+2)T_b$,** the polarity of $\sin\theta(t)$ depends only on $\theta((2n+1)T_b)$
- The quadrature component consists of the **half-cycle sine pulse**:

$$s_Q(t) = \sqrt{2E_b/T_b}\sin\theta(t) = \pm\sqrt{2E_b/T_b}\sin(\pi t/2T_b)$$

  - where "+": $\theta((2n+1)T_b) = \pi/2$; "–": $\theta((2n+1)T_b) = -\pi/2$

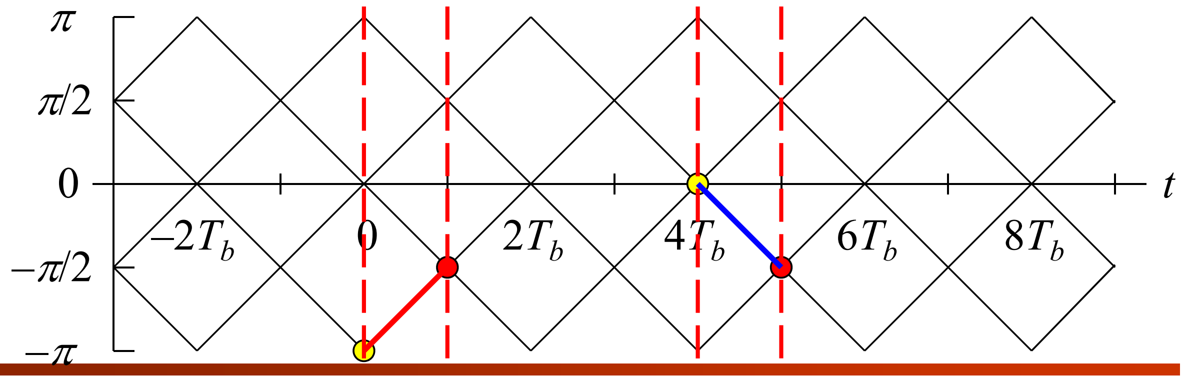
Minimum Shift-Keying (Cont.)

- Considering the symbol transmitted in $\mathbf{2}n\boldsymbol{T_b} \le \boldsymbol{t} \le \mathbf{(2}\boldsymbol{n}\mathbf{+1)}\boldsymbol{T_b}$, the phase states $\theta(2nT_b)$ and $\theta((2n+1)T_b)$ can each assume only one of two possible values, and one of **four** possibilities can arise:
  - $\theta(2nT_b) = 0$ and $\theta((2n+1)T_b) = \pi/2$, which occur when sending **symbol 1** (Phase transition: $+\pi/2$)
  - $\theta(2nT_b) = \pi$ and $\theta((2n+1)T_b) = \pi/2$, which occur when sending **symbol 0** (Phase transition: $-\pi/2$)



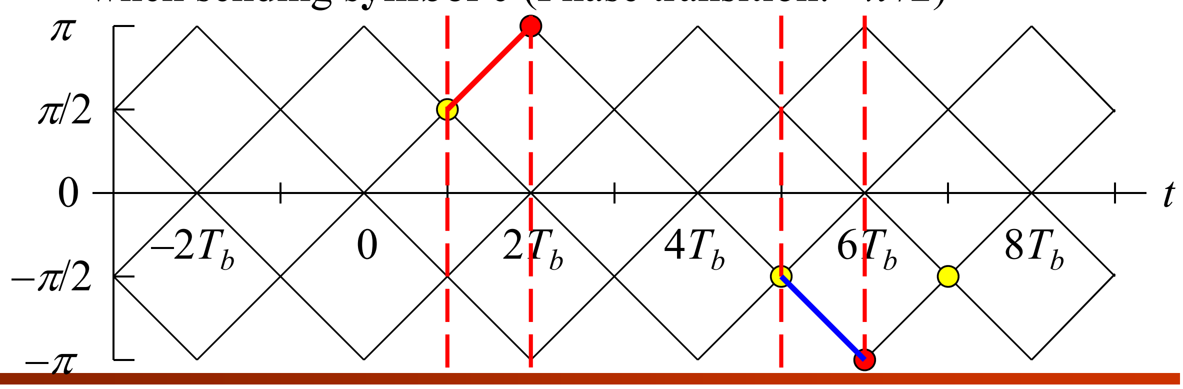
Minimum Shift-Keying (Cont.)

  - $\theta(2nT_b) = \pi$ and $\theta((2n+1)T_b) = -\pi/2$, which occur when sending **symbol 1** (Phase transition: $+\pi/2$)
  - $\theta(2nT_b) = 0$ and $\theta((2n+1)T_b) = -\pi/2$, which occur when sending **symbol 0** (Phase transition: $-\pi/2$)
- The transmitted symbol depends on the **phase-state pair** $\theta(2nT_b)$ and $\theta((2n+1)T_b)$, or equivalently, the **phase transition**



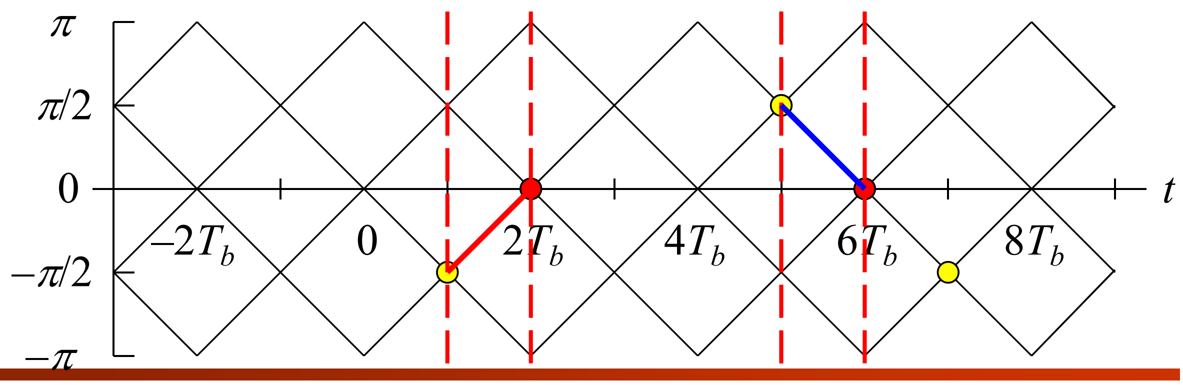
Minimum Shift-Keying (Cont.)

- Similarly, considering the symbol in $\boldsymbol{(2n+1)T_b \le t \le (2n+2)T_b}$, the phase states $\theta((2n+1)T_b)$ and $\theta((2n+2)T_b)$ can each be one of two possible values, and one of **four** possibilities can arise:
  - $\theta((2n+1)T_b) = \pi/2$ and $\theta((2n+2)T_b) = \pi$, which occur when sending **symbol 1** (Phase transition: $+\pi/2$)
  - $\theta((2n+1)T_b) = -\pi/2$ and $\theta((2n+2)T_b) = \pi$, which occur when sending **symbol 0** (Phase transition: $-\pi/2$)



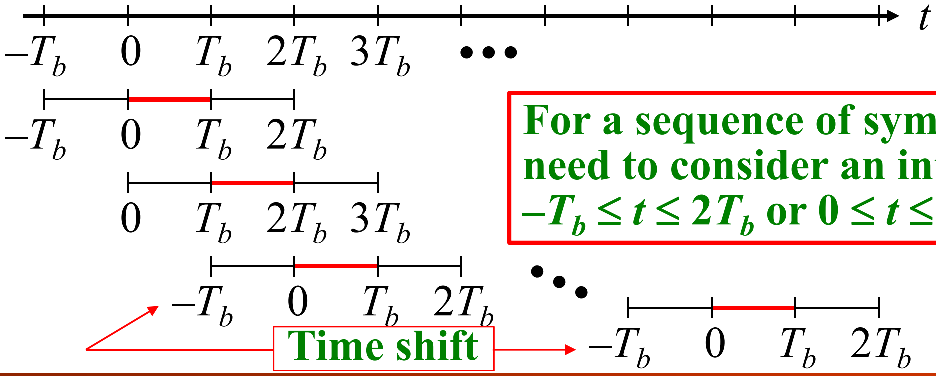
Minimum Shift-Keying (Cont.)

  - $\theta((2n+1)T_b) = -\pi/2$ and $\theta((2n+2)T_b) = 0$, which occur when sending **symbol 1** (Phase transition: $+\pi/2$)
  - $\theta((2n+1)T_b) = \pi/2$ and $\theta((2n+2)T_b) = 0$, which occur when sending **symbol 0** (Phase transition: $-\pi/2$)
- The symbol depends on the **phase-state pair** $\theta((2n+1)T_b)$ and $\theta((2n+2)T_b)$, or equivalently, the **phase transition**



Minimum Shift-Keying (Cont.)

- In the detection of the symbol transmitted in $0 \le t \le T_b$, we only need to consider the signal within $-T_b \le t \le 2T_b$
  - s_I within $-T_b \le t \le T_b$ and s_Q within $0 \le t \le 2T_b$
- In the detection of the symbol transmitted in $T_b \le t \le 2T_b$, we only need to consider the signal within $0 \le t \le 3T_b$
  - s_Q within $0 \le t \le 2T_b$ and s_I within $T_b \le t \le 3T_b$



Signal-Space Diagram of MSK

- The two orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$ characterizing the generation of MSK are

$$\phi_1(t) = \sqrt{2/T_b}\cos\left(\pi t/2T_b\right)\cos\left(2\pi f_c t\right), \quad 0 \le t \le T_b$$

$$\phi_2(t) = \sqrt{2/T_b}\sin\left(\pi t/2T_b\right)\sin\left(2\pi f_c t\right), \quad 0 \le t \le T_b$$

- The MSK signal is represented as

$$s(t) = s_1\phi_1(t) + s_2\phi_2(t), \quad 0 \le t \le T_b$$

  - where $s_1 = \int_{-T_b}^{T_b} s(t)\phi_1(t)\,dt = \sqrt{E_b}\cos\left[\theta(0)\right], \quad -T_b \le t \le T_b$

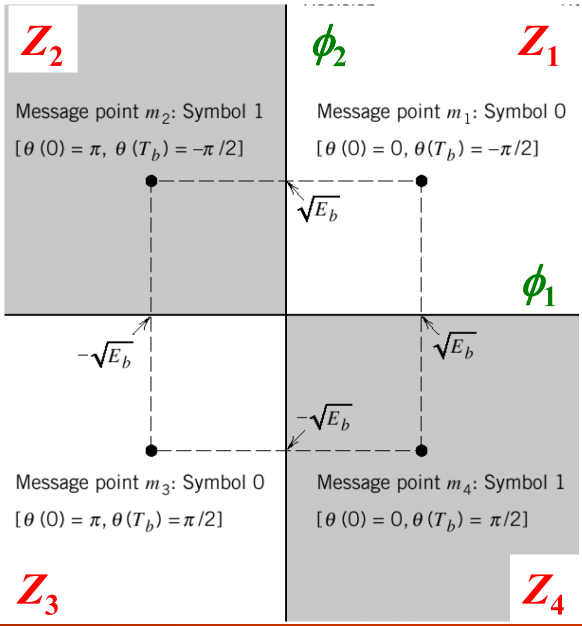
$$s_2 = \int_0^{2T_b} s(t)\phi_2(t)\,dt = -\sqrt{E_b}\sin\left[\theta(T_b)\right], \quad 0 \le t \le 2T_b$$

  - Both integrals are evaluated for a time interval equal to $2T_b$
  - In the time interval $0 \le t \le T_b$, the phase states $\theta(0)$ and $\theta(T_b)$ is common to both integrals

Signal-Space Diagram of MSK (Cont.)

- The signal constellation for an MSK signal is **two-dimensional** (i.e., $N = 2$), with **four possible message points** (i.e., $M = 4$)
- For the symbol transmitted in $0 \le t \le T_b$, we have ⇒
- Moving in a counterclockwise direction, the coordinates of the message points are:

$$\left(+\sqrt{E_b}, +\sqrt{E_b}\right): \text{Symbol } 0$$
$$\left(-\sqrt{E_b}, +\sqrt{E_b}\right): \text{Symbol } 1$$
$$\left(-\sqrt{E_b}, -\sqrt{E_b}\right): \text{Symbol } 0$$
$$\left(+\sqrt{E_b}, -\sqrt{E_b}\right): \text{Symbol } 1$$



Signal-Space Diagram of MSK (Cont.)

- Each symbol corresponds to **a binary symbol** and each symbol shows up in two opposite quadrants

$0 \le t \le T_b$

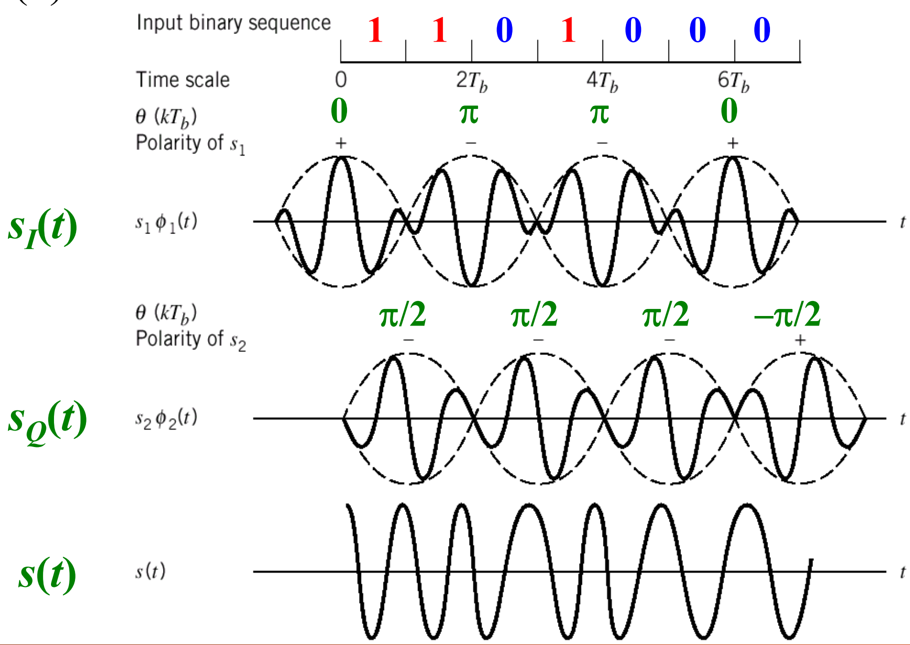
Symbol	$\theta(0)$	$\theta(T_b)$	s_1	s_2
0	0	$-\pi/2$	$+\sqrt{E_b}$	$+\sqrt{E_b}$
1	π	$-\pi/2$	$-\sqrt{E_b}$	$+\sqrt{E_b}$
0	π	$+\pi/2$	$-\sqrt{E_b}$	$-\sqrt{E_b}$
1	0	$+\pi/2$	$+\sqrt{E_b}$	$-\sqrt{E_b}$

$T_b \le t \le 2T_b$

Symbol	$\theta(T_b)$	$\theta(2T_b)$	s_1	s_2
0	$+\pi/2$	0	$+\sqrt{E_b}$	$-\sqrt{E_b}$
1	$-\pi/2$	0	$+\sqrt{E_b}$	$+\sqrt{E_b}$
0	$-\pi/2$	π	$-\sqrt{E_b}$	$+\sqrt{E_b}$
1	$+\pi/2$	π	$-\sqrt{E_b}$	$-\sqrt{E_b}$

MSK Waveforms

- The two modulation frequencies are $f_1 = 5/4T_b$ and $f_2 = 3/4T_b$ and $\theta(0)$ is zero at time $t = 0$



Error Probability of MSK

- In the case of an AWGN channel, the received signal is given by $x(t) = s(t) + w(t)$
  - where $s(t)$ is the transmitted MSK signal and $w(t)$ is the white Gaussian noise with zero mean and power spectral density $N_0/2$
- To decide whether symbol 1 or symbol 0 was sent in $0 \le t \le T_b$, we establish a procedure for the use of $x(t)$ to detect the phase states $\theta(0)$ and $\theta(T_b)$

$$x_1 = \int_{-T_b}^{T_b} x(t)\phi_1(t)\,dt = s_1 + w_1; \quad x_2 = \int_0^{2T_b} x(t)\phi_2(t)\,dt = s_2 + w_2$$

  - **$s_I(t)$:** If $x_1 > 0$, $\hat{\theta}(0) = 0$; if $x_1 < 0$, $\hat{\theta}(0) = \pi$
  - **$s_Q(t)$:** If $x_2 > 0$, $\hat{\theta}(T_b) = -\pi/2$; if $x_2 < 0$, $\hat{\theta}(T_b) = \pi/2$

Error Probability of MSK (Cont.)

- If estimates $\hat{\theta}(0) = 0$ and $\hat{\theta}(T_b) = -\pi/2$, or alternatively if $\hat{\theta}(0) = \pi$ and $\hat{\theta}(T_b) = \pi/2$, then the receiver decides in favor of **symbol 0**
- If $\hat{\theta}(0) = \pi$ and $\hat{\theta}(T_b) = -\pi/2$, or alternatively if $\hat{\theta}(0) = 0$ and $\hat{\theta}(T_b) = \pi/2$ then the receiver decides in favor of **symbol 1**
- The receiver makes an error when the I-channel assigns the wrong value to $\theta(0)$ **or** the Q-channel assigns the wrong value to $\theta(T_b)$
- It follows, therefore, that the BER for the **coherent detection** of MSK signals is given by $P_e = \frac{1}{2}\text{erfc}\left(\sqrt{E_b/N_0}\right)$
  - which is exactly the same as that for BPSK and QPSK
- This good performance is the result of **coherent detection** being performed on the basis of observations over $2T_b$ interval

Power Spectra of MSK Signals

- We assume that the input binary wave is random, with symbols 1 and 0 being equally likely and the symbols sent during adjacent time slots being statistically independent
- Depending on the value of phase state $\theta(0)$, the **in-phase** component equals $+g(t)$ or $-g(t)$, where the **pulse-shaping function**

$$g(t) = \begin{cases} \sqrt{2E_b/T_b}\cos\left(\pi t/2T_b\right), & -T_b \le t \le T_b \\ 0, & \text{otherwise} \end{cases}$$

- The power spectral density of the in-phase component equals

$$S_g(f) = \frac{16E_b}{\pi^2}\left[\frac{\cos(2\pi T_b f)}{16T_b^2 f^2 - 1}\right]^2$$

Power Spectra of MSK Signals (Cont.)

- Depending on the value of the phase state $\theta(T_b)$, the **quadrature** component equals $+g(t)$ or $-g(t)$, where

$$g(t) = \begin{cases} \sqrt{2E_b/T_b}\,\sin\left(\pi t/2T_b\right), & 0 \le t \le 2T_b \\ 0, & \text{otherwise} \end{cases}$$

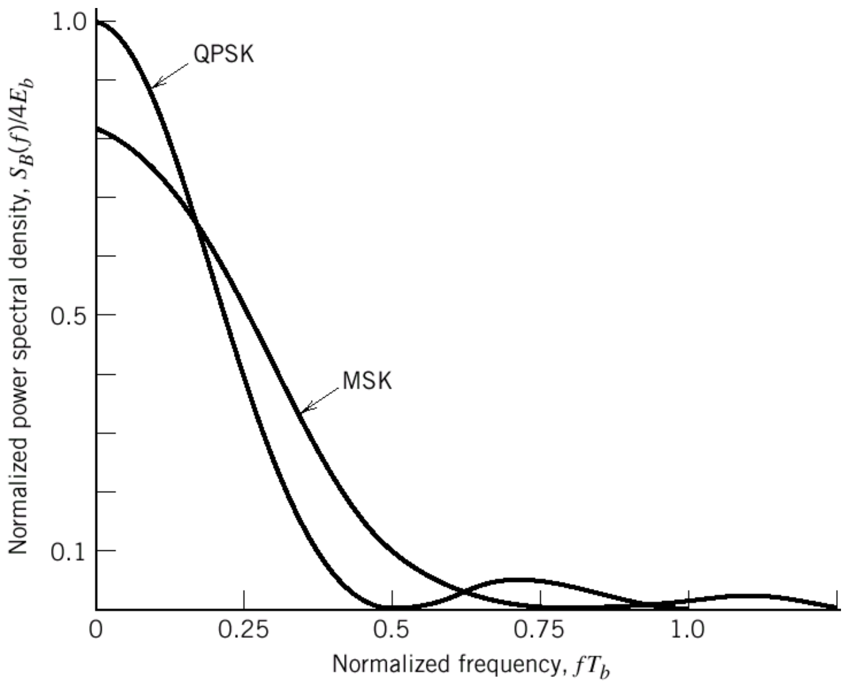
- The PSD is the same as that of the in-phase component
- The in-phase and quadrature components of the MSK signal are **statistically independent**
- The baseband power spectral density of $s(t)$ is given by

$$S_B(f) = 2S_g(f) = \frac{32E_b}{\pi^2}\left[\frac{\cos(2\pi T_b f)}{16T_b^2 f^2 - 1}\right]^2$$

- The baseband power spectral density of the MSK signal falls off as the **inverse fourth power** of frequency for $f >> 0$

Power Spectra of MSK Signals (Cont.)

- The QPSK signal it falls off as the **inverse square** of frequency
- MSK **does not** produce as much interference outside the signal band of interest as QPSK does



Gaussian-Filtered MSK (GMSK)

- Some desirable properties of MSK:
  - Modulated signal with **constant envelope**
  - Relatively **narrow-bandwidth** occupancy
  - Coherent detection performance equivalent to that of **QPSK**
- However, the **out-of-band** spectral characteristics of MSK signals may not satisfy some stringent requirements
  - At fT_b = **0.5**, the baseband PSD of the MSK signal drops by only $10 \log_{10} 9$ = **9.54 dB** below its midband value
  - If the transmission bandwidth is set as $\mathbf{1/T_b}$, the **adjacent channel interference** of using MSK is **not low enough** to satisfy the practical requirements of a wireless multiuser-communications environment

Gaussian-Filtered MSK (GMSK) (Cont.)

- We may **modify the power spectrum** of MSK into a more compact form while maintaining the constant-envelope property
- This modification can be achieved through the use of a **pre-modulation low-pass filter**,
  - A baseband **pulse-shaping filter**
- The pulse-shaping filter should satisfy the following conditions:
  - Frequency response with **narrow bandwidth** and **sharp cutoff** characteristics
  - Impulse response with relatively **low overshoot**
  - The carrier phase of the modulated signal assuming the two values $\pm\pi/2$ at **odd** multiples of T_b and the two values 0 and π at **even** multiples of T_b as in MSK

Gaussian-Filtered MSK (GMSK) (Cont.)

- These three conditions can be satisfied by using a baseband pulse-shaping filter whose **impulse response** (and, likewise, its **frequency response**) is defined by a **Gaussian function**
- The resulting method of binary FM is naturally referred to as **Gaussian-filtered minimum-shift keying (GMSK)**
- The **transfer function** $H(f)$ and **impulse response** $h(t)$ of the pulse-shaping filter

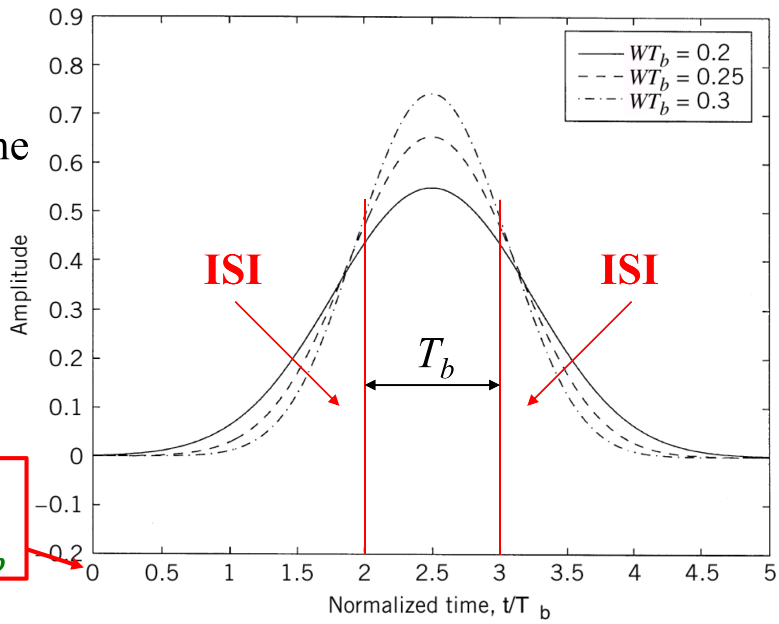
$$H(f) = \exp\left[-\frac{\ln 2}{2}\left(\frac{f}{W}\right)^2\right]; \quad h(t) = \sqrt{\frac{2\pi}{\ln 2}}W \exp\left(-\frac{2\pi^2}{\ln 2}W^2 t^2\right)$$

  - where W is the **3 dB baseband bandwidth** of the filter
- The **response** of this Gaussian filter to a **rectangular pulse** of unit amplitude and duration T_b is $g(t) = \int_{-T_b/2}^{T_b/2} h(t-\tau)\, d\tau$

Gaussian-Filtered MSK (GMSK) (Cont.)

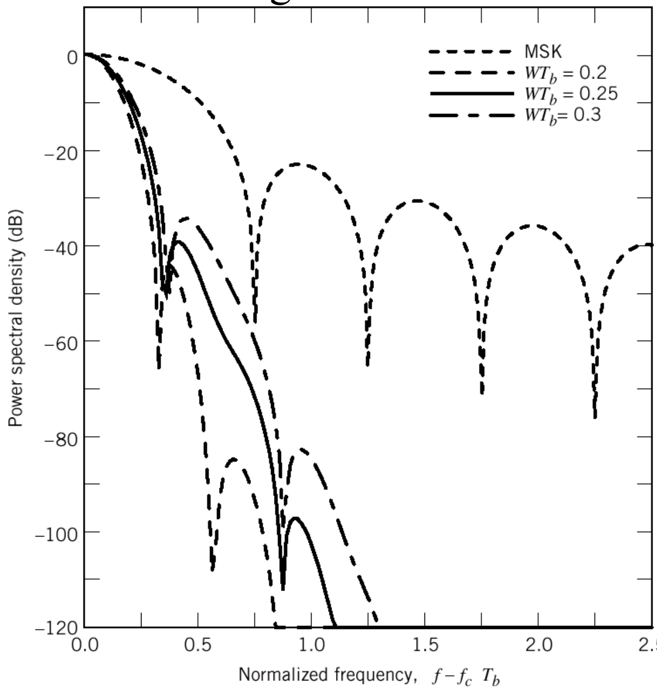
- $g(t)$ is **noncausal** and, therefore, **not physically realizable**
- For a causal response, $g(t)$ must be **truncated** and **shifted in time**
- As WT_b is **reduced**, the **time spread** of the frequency-shaping pulse is **increased**
- **Inter-symbol interference** (ISI) is introduced

Truncated at $t = \pm 2.5T_b$
Shifted in time by $2.5T_b$



Gaussian-Filtered MSK (GMSK) (Cont.)

- The power spectra of MSK and GMSK signals
- The condition of $WT_b = \infty$ corresponds to the case of the ordinary MSK

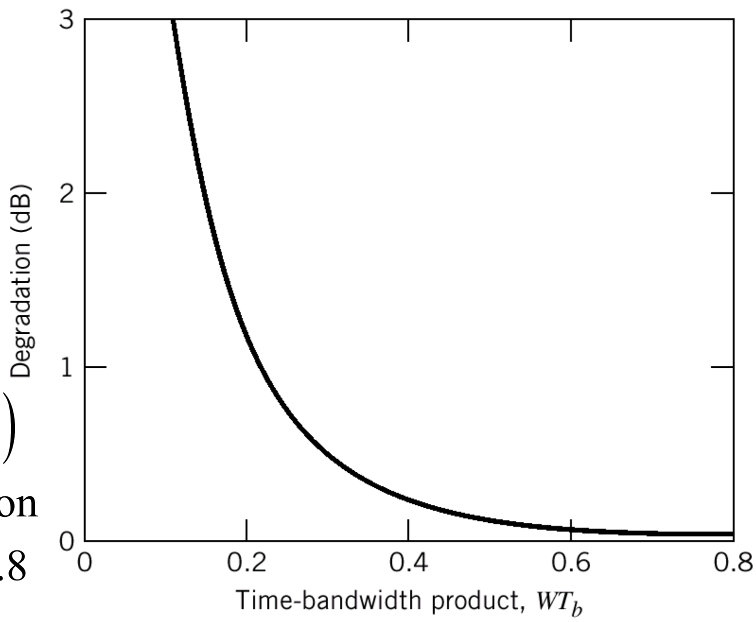


Gaussian-Filtered MSK (GMSK) (Cont.)

- The introduced **Inter-symbol interference** (ISI) degrades the symbol error performance at the receiver
- The time–bandwidth product WT_b offers a **tradeoff** between spectral compactness and performance loss
- The average symbol error rate is

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\alpha E_b / 2N_0}\right)$$

  - $\alpha = 2$: no degradation
  - $WT_b = 0.3 \Rightarrow \alpha = 1.8$



M-ary FSK

M-ary FSK

- For M-ary FSK, the transmitted signals are defined by

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left[\frac{\pi}{T}\left(n_c + i\right)t\right], \quad 0 \le t \le T$$

  - where $i = 1, 2, \cdots, M$; the carrier frequency: $f_c = n_c/(2T)$ for some fixed integer n_c; the symbol duration: T; the symbol energy E
- Since the individual signal frequencies are separated by $1/(2T)$ Hz, the M-ary FSK signals constitute an **orthogonal set**; that is,

$$\int_0^T s_i(t) s_j(t)\, dt = 0, \quad i \neq j$$

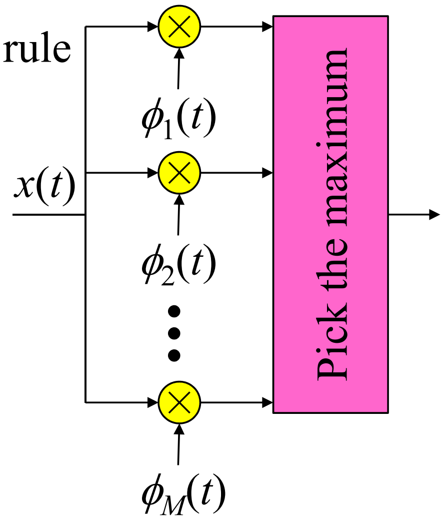
- A complete orthonormal set of **basis functions**, as shown by

$$\phi_i(t) = \frac{1}{\sqrt{E}} s_i(t), \quad 0 \le t \le T, i = 1, 2, \cdots, M$$

Error Probability of M-ary FSK

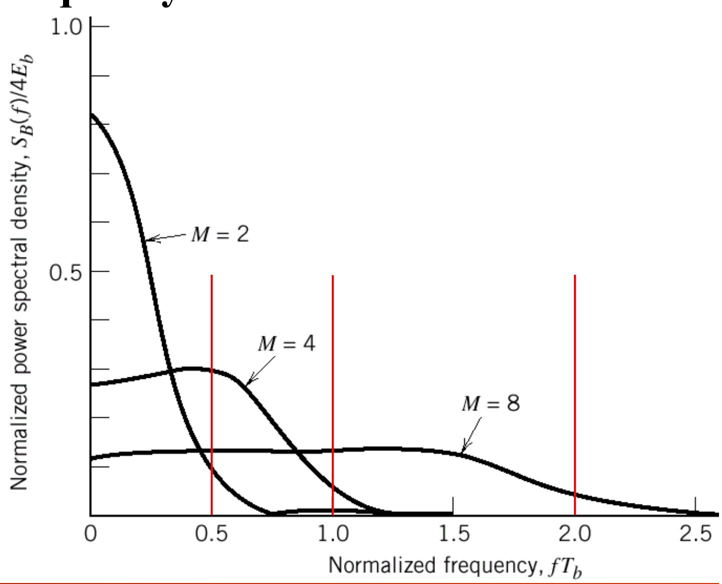
- For the coherent detection of M-ary FSK, the optimum receiver consists of **a bank of M correlators** or **matched filters**
- At the sampling times $t = kT$, the receiver makes decisions based on the **largest** matched filter output
  - The **maximum likelihood** decoding rule
- An exact formula for the probability of symbol error is difficult
- Since the minimum distance in M-ary FSK is $\sqrt{2E}$, an **upper bound** on the average probability of symbol error

$$P_e \le \frac{1}{2}(M-1)\,\mathrm{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)$$



Power Spectra of M-ary FSK Signals

- The spectral analysis of M-ary FSK signals is **complicated**
- A special case of assigning **uniformly spacing frequencies** to the multilevels with the **frequency deviation** h = 1/2
  - CPFSK
  - The M signal frequencies are separated by **1/2*T***, where T is the symbol duration
- The baseband power PSD of M-ary FSK signals for M = 2, 4, 8



Bandwidth Efficiency of *M*-ary FSK

- For **coherent detection**, the adjacent signals of M-ary FSK need only be separated from each other by a difference $1/2T$
- The **channel bandwidth** required to transmit M-ary FSK signals is $B = M / 2T$
  - The symbol period is equal to $T = T_b \log_2 M$
  - The bit rate is equal to $R_b = 1/T_b$
- Hence, we may redefine the channel bandwidth for M-ary FSK

$$B = R_b M / 2\log_2 M$$

- The **bandwidth efficiency** of M-ary FSK signals is therefore

$$\rho = \frac{R_b}{B} = \frac{2\log_2 M}{M}$$

Bandwidth Efficiency of *M*-ary FSK (Cont.)

- For ***M*-ary FSK**, the **increase** in the number of levels M tends to **decrease** the bandwidth efficiency

M	2	4	8	16	32	64
ρ (bits/s/Hz)	1	1	0.75	0.5	0.3125	0.1875

- By contrast, for ***M*-ary PSK**, the **increase** in the number of levels M tends to **increase** the bandwidth efficiency

M	2	4	8	16	32	64
ρ (bits/s/Hz)	1	1	1.5	2	2.5	3

- In other words, ***M*-ary PSK** signals are **spectrally efficient**, whereas ***M*-ary FSK** signals are **spectrally inefficient**

Discussion of Orthogonality

Binary FSK – Orthogonality

- Considering **binary FSK**, the transmitted signals are

$$s_i(t) = A\cos(2\pi f_i t + \theta_i), \quad 0 \le t < T_b, \quad i = 1, 2$$

  - where θ_i represents the carrier phase at the initial time
- To maintain the orthogonality between $s_1(t)$ and $s_2(t)$

$$\langle s_1(t), s_2(t) \rangle = \int_0^{T_b} s_1(t) s_2(t)\, dt = A^2 \int_0^{T_b} \cos(2\pi f_1 t + \theta_1)\cos(2\pi f_2 t + \theta_2)\, dt$$

$$= \frac{A^2}{2}\int_0^{T_b} \cos\left[2\pi(f_1 + f_2)t + \theta_1 + \theta_2\right]dt + \frac{A^2}{2}\int_0^{T_b} \cos\left[2\pi(f_1 - f_2)t + \theta_1 - \theta_2\right]dt$$

$$= A^2\left\{\sin\left[2\pi(f_1+f_2)T_b + \theta_1 + \theta_2\right] - \sin\left(\theta_1 + \theta_2\right)\right\} / 4\pi(f_1+f_2) \leftarrow \boxed{\approx 0}$$

$$+ A^2\left\{\sin\left[2\pi(f_1-f_2)T_b + \theta_1 - \theta_2\right] - \sin\left(\theta_1 - \theta_2\right)\right\} / 4\pi(f_1-f_2)$$

- Assuming $f_i >> 0$, the first term can be ignored

$$\boxed{\begin{array}{l} \sin x \sin y = \tfrac{1}{2}\left[\cos(x-y) - \cos(x+y)\right];\ \cos x \cos y = \tfrac{1}{2}\left[\cos(x-y) + \cos(x+y)\right] \\ \sin x \cos y = \tfrac{1}{2}\left[\sin(x+y) + \sin(x-y)\right];\ \cos x \sin y = \tfrac{1}{2}\left[\sin(x+y) - \sin(x-y)\right] \end{array}}$$

Binary FSK – Orthogonality (Cont.)

- **Continuous-phase** binary FSK: $\theta_1 = \theta_2$
  - If $f_1 - f_2 = m/2T_b$, for an integer $m > 0$

$$\langle s_1(t), s_2(t)\rangle \approx A^2 \left\{\sin\left[2\pi(f_1 - f_2)T_b + \theta_1 - \theta_2\right] - \sin\left(\theta_1 - \theta_2\right)\right\} \big/ 4\pi(f_1 - f_2)$$
$$= A^2 T_b \left\{\sin\left(m\pi\right)\right\} \big/ 2m\pi$$

  - The minimum value that makes $\langle s_1(t), s_2(t)\rangle = 0$ is $m = 1$
  - The minimum frequency spacing that maintains the orthogonality between $s_1(t)$ and $s_2(t)$ is $\Delta f = 1/2T_b$
- For continuous-phase FSK, the two sinusoidal carriers are said to be **coherently orthogonal**
  - Because the two phases are the same
  - The minimum frequency spacing is $\boldsymbol{\Delta f = 1/2T_b}$

Binary FSK – Orthogonality (Cont.)

- **Non-continuous-phase** binary FSK: $\theta_1 \neq \theta_2$
  - If $f_1 - f_2 = m/2T_b$, for an integer $m > 0$

$$\langle s_1(t), s_2(t)\rangle \approx A^2 \left\{\sin\left[2\pi(f_1 - f_2)T_b + \theta_1 - \theta_2\right] - \sin\left(\theta_1 - \theta_2\right)\right\} \big/ 4\pi(f_1 - f_2)$$
$$= A^2 \left\{\sin\left[\underline{m\pi} + \Delta\theta\right] - \sin\left(\Delta\theta\right)\right\} \big/ 4\pi(f_1 - f_2)$$

  - The minimum value that makes $\langle s_1(t), s_2(t)\rangle = 0$ is $m = 2$
  - The minimum frequency spacing that maintains the orthogonality between $s_1(t)$ and $s_2(t)$ is $\Delta f = 1/T_b$
- For non-continuous-phase FSK, the two sinusoidal carriers are said to be **noncoherently orthogonal**
  - Because there is **no relationship** between the two phases
  - The minimum frequency spacing is $\boldsymbol{\Delta f = 1/T_b}$
  - Which is **twice** as much as that of continuous-phase FSK

Homework

- **You must give detailed derivations or explanations, otherwise you get no points.**
- Communication Systems, Simon Haykin **(4^{th} Ed.)**
- 6.20;
- 6.22;
- 6.27;