Practice #1

No need to turn it in, this is for practice purpose only.

Practice and get familiar with the questions are encouraged.

Some types of the questions might appear on the test.

3.2 Determine the signal space representation of the four signals $s_k(t)$, $k = 1, 2, 3, 4$, shown in Figure P3.2, by using as basis functions the orthonormal functions $\phi_1(t)$ and $\phi_2(t)$. Plot the signal space diagram, and show that this signal set is equivalent to that for a four-phase PSK signal.

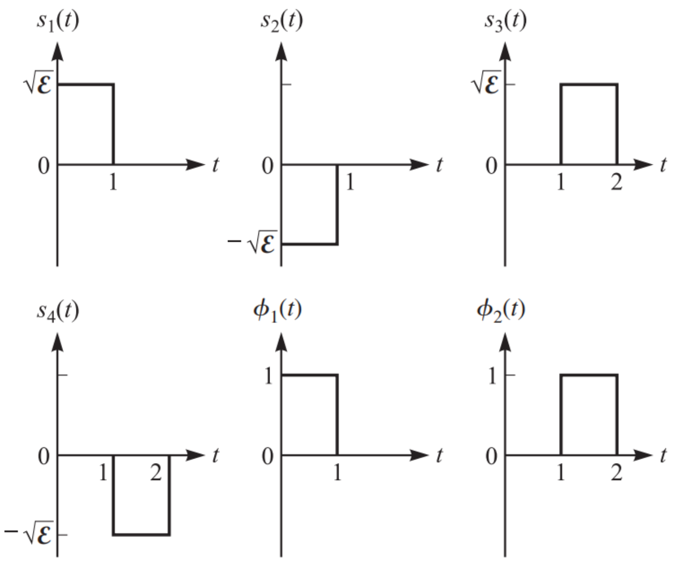
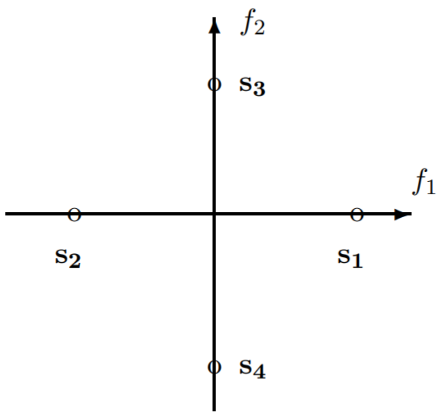


FIGURE P3.2

$$\mathbf{s}_1 = \left(\sqrt{\mathcal{E}}, 0\right)$$
$$\mathbf{s}_2 = \left(-\sqrt{\mathcal{E}}, 0\right)$$
$$\mathbf{s}_3 = \left(0, \sqrt{\mathcal{E}}\right)$$
$$\mathbf{s}_4 = \left(0, -\sqrt{\mathcal{E}}\right)$$

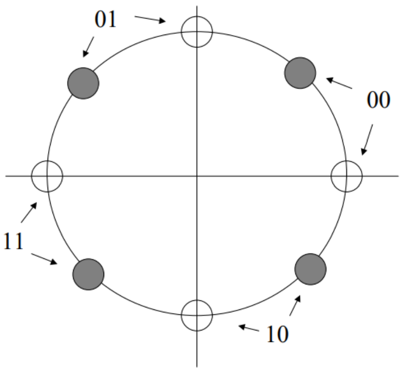


As we see, this signal set is indeed equivalent to a 4-phase PSK signal.

3.3 $\pi/4$-QPSK may be considered as two QPSK systems offset by $\pi/4$ rad.

1. Sketch the signal space diagram for a $\pi/4$-QPSK signal.
2. Using Gray encoding, label the signal points with the corresponding data bits.

1.2. The signal space diagram, together with the Gray encoding of each signal point is given in the following figure :



The signal points that may be transmitted at times $t = 2nT \quad n = 0, 1, \ldots$ are given with blank circles, while the ones that may be transmitted at times $t = 2nT + 1, \quad n = 0, 1, \ldots$ are given with filled circles.

3.4 Consider the octal signal point constellations in Figure P3.4.

1. The nearest-neighbor signal points in the 8-QAM signal constellation are separated in distance by A units. Determine the radii a and b of the inner and outer circles, respectively.
2. The adjacent signal points in the 8-PSK are separated by a distance of A units. Determine the radius r of the circle.

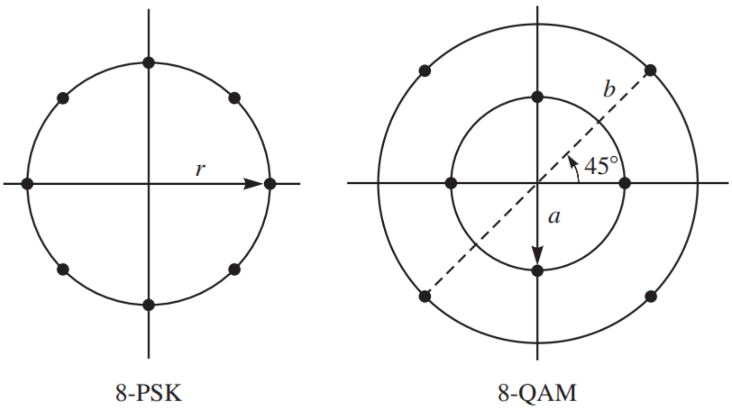


FIGURE P3.4

3. Determine the average transmitter powers for the two signal constellations, and compare the two powers. What is the relative power advantage of one constellation over the other? (Assume that all signal points are equally probable.)

1. Consider the QAM constellation of Fig. P3-4. Using the Pythagorean theorem we can find the radius of the inner circle as:

$$a^2 + a^2 = A^2 \Longrightarrow a = \frac{1}{\sqrt{2}} A$$

The radius of the outer circle can be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between then equal to $\theta = 75^o$, we obtain:

$$b^2 = a^2 + A^2 - 2aA\cos 75^o \Longrightarrow b = \frac{1+\sqrt{3}}{2} A$$

2. If we denote by r the radius of the circle, then using the cosine theorem we obtain:

$$A^2 = r^2 + r^2 - 2r\cos 45^o \Longrightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}$$

3. The average transmitted power of the PSK constellation is:

$$P_{\text{PSK}} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2-\sqrt{2}}}\right)^2 \Longrightarrow P_{\text{PSK}} = \frac{A^2}{2-\sqrt{2}}$$

whereas the average transmitted power of the QAM constellation:

$$P_{\text{QAM}} = \frac{1}{8}\left(4\frac{A^2}{2} + 4\frac{(1+\sqrt{3})^2}{4} A^2\right) \Longrightarrow P_{\text{QAM}} = \left[\frac{2+(1+\sqrt{3})^2}{8}\right] A^2$$

The relative power advantage of the PSK constellation over the QAM constellation is:

$$\text{gain} = \frac{P_{\text{PSK}}}{P_{\text{QAM}}} = \frac{8}{(2+(1+\sqrt{3})^2)(2-\sqrt{2})} = 1.5927 \text{ dB}$$

3.5 Consider the 8-point QAM signal constellation shown in Figure P3.4.

1. Is it possible to assign 3 data bits to each point of the signal constellation such that the nearest (adjacent) points differ in only 1 bit position?
2. Determine the symbol rate if the desired bit rate is 90 Mbits/s.

1. Although it is possible to assign three bits to each point of the 8-PSK signal constellation so that adjacent points differ in only one bit, (e.g. going in a clockwise direction : 000, 001, 011, 010, 110, 111, 101, 100). this is not the case for the 8-QAM constellation of Figure P3-4. This is because there are fully connected graphs consisted of three points. To see this consider an equilateral triangle with vertices A, B and C. If, without loss of generality, we assign the all zero sequence $\{0, 0, \ldots, 0\}$ to point A, then point B and C should have the form

$$\text{B} = \{0, \ldots, 0, 1, 0, \ldots, 0\} \qquad \text{C} = \{0, \ldots, 0, 1, 0, \ldots, 0\}$$

where the position of the 1 in the sequences is not the same, otherwise B=C. Thus, the sequences of B and C differ in two bits.

2. Since each symbol conveys 3 bits of information, the resulted symbol rate is :

$$R_s = \frac{90 \times 10^6}{3} = 30 \times 10^6 \text{ symbols/sec}$$

3.6 Consider the two 8-point QAM signal constellations shown in Figure P3.6. The minimum distance between adjacent points is $2A$. Determine the average transmitted power for each constellation, assuming that the signal points are equally probable. Which constellation is more power-efficient?

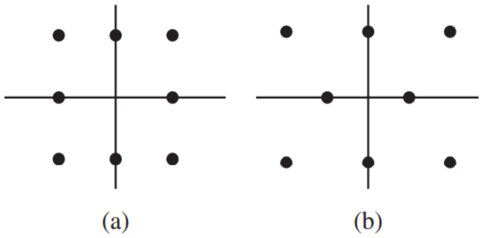


FIGURE P3.6

The constellation of Fig. P3-6(a) has four points at a distance $2A$ from the origin and four points at a distance $2\sqrt{2}A$. Thus, the average transmitted power of the constellation is:

$$P_a = \frac{1}{8}\left[4 \times (2A)^2 + 4 \times (2\sqrt{2}A)^2\right] = 6A^2$$

The second constellation has four points at a distance $\sqrt{7}A$ from the origin, two points at a distance $\sqrt{3}A$ and two points at a distance A. Thus, the average transmitted power of the second constellation is:

$$P_b = \frac{1}{8}\left[4 \times (\sqrt{7}A)^2 + 2 \times (\sqrt{3}A)^2 + 2A^2\right] = \frac{9}{2}A^2$$

Since $P_b < P_a$ the second constellation is more power efficient.

EXAMPLE 4.1–1. Consider two equiprobable message signals $\boldsymbol{s}_1 = (0, 0)$ and $\boldsymbol{s}_2 = (1, 1)$. The channel adds iid noise components n_1 and n_2 to the transmitted vector each with an exponential PDF of the form

$$p(n) = \begin{cases} e^{-n} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Since the messages are equiprobable, the MAP detector is equivalent to the ML detector, and the decision region D_1 is given by

$$D_1 = \left\{\boldsymbol{r} \in \mathbb{R}^2 : p(\boldsymbol{r}|\boldsymbol{s}_1) > p(\boldsymbol{r}|\boldsymbol{s}_2)\right\}$$

Noting that $p(\boldsymbol{r}|\boldsymbol{s} = (s_1, s_2)) = p(\boldsymbol{n} = \boldsymbol{r} - \boldsymbol{s})$, we have

$$D_1 = \left\{\boldsymbol{r} \in \mathbb{R}^2 : p_{\boldsymbol{n}}(r_1, r_2) > p_{\boldsymbol{n}}(r_1 - 1, r_2 - 1)\right\}$$

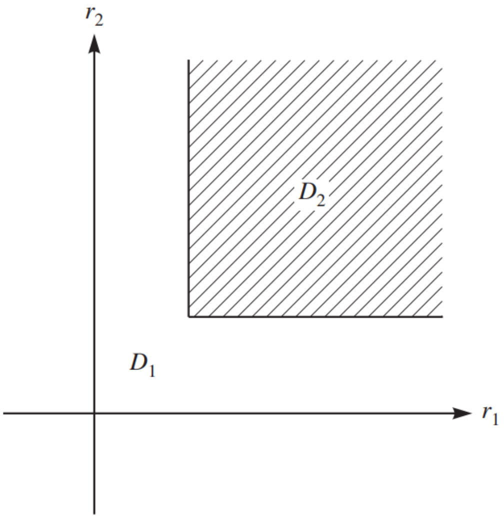


FIGURE 4.1–3
Decision regions D_1 and D_2.

where

$$p_{\boldsymbol{n}}(n_1, n_2) = \begin{cases} e^{-n_1-n_2} & n_1, n_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

From this relation we conclude that if either r_1 or r_2 is less than 1, then the point $\boldsymbol{r}$ belongs to D_1, and if both r_1 and r_2 are greater than 1, we have $e^{-r_1-r_2} < e^{-(r_1-1)-(r_2-1)}$ and $\boldsymbol{r}$ belongs to D_2.

Note that in this channel neither r_1 nor r_2 can be negative, because signal and noise are always nonnegative. Therefore,

$$D_2 = \left\{\boldsymbol{r} \in \mathbb{R}^2 : r_1 \geq 1, r_2 \geq 1\right\}$$

and

$$D_1 = \left\{\boldsymbol{r} \in \mathbb{R}^2 : r_1, r_2 \geq 0, \text{ either } 0 \leq r_1 < 1 \text{ or } 0 \leq r_2 < 1\right\}$$

The decision regions are shown in Figure 4.1–3. For this channel, when $\boldsymbol{s}_2$ is transmitted, regardless of the value of noise components, $\boldsymbol{r}$ will always be in D_2 and no error will occur.

Errors will occur only when $\boldsymbol{s}_1 = (0, 0)$ is transmitted and the received vector $\boldsymbol{r}$ belongs to D_2, i.e., when both noise components exceed 1. Therefore, the error probability is given by

$$\begin{aligned} P_e &= \frac{1}{2} \mathrm{P}\left[\boldsymbol{r} \in D_2 \,|\, \boldsymbol{s}_1 = (0, 0) \text{ sent}\right] \\ &= \frac{1}{2} \int_1^{\infty} e^{-n_1}\, dn_1 \int_1^{\infty} e^{-n_2}\, dn_2 \\ &= \frac{1}{2} e^{-2} \approx 0.0068 \end{aligned}$$

4.4 A binary digital communication system employs the signals

$$s_0(t) = 0 \qquad 0 \le t \le T$$
$$s_1(t) = A \qquad 0 \le t \le T$$

for transmitting the information. This is called *on-off signaling*. The demodulator cross-correlates the received signal $r(t)$ with $s(t)$ and samples the output of the correlator at $t + T$.

a. Determine the optimum detector for an AWGN channel and the optimum threshold, assuming that the signals are equally probable.

b. Determine the probability of error as a function of the SNR. How does on-off signaling compare with antipodal signaling?

a. The correlation type demodulator employes a filter :

$$f(t) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{T}} & 0 \le \mathrm{t} \le \mathrm{T} \\ 0 & \mathrm{o.w} \end{array} \right\}$$

as given in Example 5-1-1. Hence, the sampled outputs of the crosscorrelators are :

$$r = s_m + n, \qquad m = 0, 1$$

where $s_0 = 0$, $s_1 = A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable with variance :

$$\sigma_n^2 \frac{N_0}{2}$$

The probability density function for the sampled output is :

$$\begin{aligned} p(r|s_0) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \\ p(r|s_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} \end{aligned}$$

Since the signals are equally probable, the optimal detector decides in favor of s_0 if

$$\mathrm{PM}(\mathbf{r}, \mathbf{s}_0) = p(r|s_0) > p(r|s_1) = \mathrm{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of s_1. The decision rule may be expressed as:

$$\frac{\mathrm{PM}(\mathbf{r}, \mathbf{s}_0)}{\mathrm{PM}(\mathbf{r}, \mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

or equivalently :

$$r \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1}{2}A\sqrt{T}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.

b. The average probability of error is:

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} p(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} p(r|s_1)dr \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} dr \\
&= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{\text{SNR}}\right]
\end{aligned}$$

where

$$\text{SNR} = \frac{\frac{1}{2}A^2 T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.