1. (12%) Let X be a geometrically distributed random variable, i.e.,

$$P(X = k) = p(1-p)^{k-1}, k = 1, 2, 3, \ldots$$

(a) Find the entropy of X.

(b) Given that $X > K$, where K is a positive integer, what is the entropy of X?

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(a) $H(x) = -\sum_{k=1}^{\infty} p(1-p)^{k-1} \log_2 p(1-p)^{k-1}$

$= -\left[p\log_2 p \sum_{k=1}^{\infty}(1-p)^{k-1} + p\sum_{k=1}^{\infty}(1-p)^{k-1}\log_2(1-p)^{k-1} \right]$

$= -\left[p\cdot\left(\frac{1}{1-(1-p)}\right)\log_2 p + p\log_2(1-p)\underbrace{\sum_{k=1}^{\infty}(k-1)(1-p)^{k-1}}_{\sum_{k=0}^{\infty} k(1-p)^{k} = S_n} \right]$

$= -\left[\log_2 p + p\,\frac{1-p}{[1-(1-p)]^2}\log_2(1-p) \right]$

$= \underline{-\log_2 p - \frac{1-p}{p}\log_2(1-p)}$ # (6%)

(b)

$H(X \mid X > K) = ?$

$P(X=k \mid X>K) = \frac{P(X=k \mid X>K)}{P(X>K)} = \frac{p(1-p)^{k-1}}{P(X>K)}$

$P(X>K) = \sum_{k=K+1}^{\infty} p(1-p)^{k-1}$

$= p\left[\sum_{k=1}^{\infty}(1-p)^{k-1} - \sum_{k=1}^{K}(1-p)^{k-1} \right]$

$= p\left[\frac{1}{1-(1-p)} - \frac{1-(1-p)^{K}}{1-(1-p)} \right] = (1-p)^{K}$

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$= \frac{p(1-p)^{k-1}}{(1-p)^{K}} \overset{k=K+t}{=} p(1-p)^{t-1}$

$H(X \mid X>K) = -\sum_{t=1}^{\infty} p(1-p)^{t-1}\log_2 p(1-p)^{t-1}$

by (a)

$= \underline{-\log_2 p - \frac{1-p}{p}\log_2(1-p)}$ # (6%)

2. (12%) For the channel shown in Figure 1 and given that $P(A) = 1 - p$, $P(B) = P(C) = \frac{p}{2}$, find the channel capacity and the input distribution that achieves capacity.

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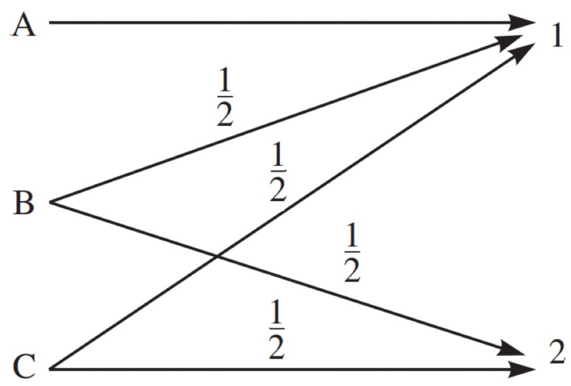

Figure 1: channel model

① $[P(X)] = [1-p \quad \frac{p}{2} \quad \frac{p}{2}]$ (1%)

③ $[P(Y)] = [P(X)][P(Y|X)] = [1-\frac{p}{2} \quad \frac{p}{2}]$ (1%)

② $[P(Y|X)] = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (1%)

④ $[P(X,Y)] = \begin{bmatrix} 1-p & 0 & 0 \\ 0 & \frac{p}{2} & 0 \\ 0 & 0 & \frac{p}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1-p & 0 \\ \frac{p}{4} & \frac{p}{4} \\ \frac{p}{4} & \frac{p}{4} \end{bmatrix}$ (1%)

$H(Y) = H_b(\frac{1}{2})$

$H(Y|X=A) = 0$

$H(Y|X=B) = H(Y|X=c) = 1$

$H(Y|X) = (1-p)H(Y|X=A) + \frac{p}{2}H(Y|X=B) + \frac{p}{2}H(Y|X=c) = p$ (2%)

$C = \max_p I(X,Y)$ $\quad (H_b(\frac{p}{2}) = \frac{p}{2}\log_2\frac{p}{2} + (1-\frac{p}{2})\log_2(1-\frac{p}{2}))$

$= \max H(Y) - H(Y|X) = \max_p H_b(\frac{p}{2}) - p$

$\frac{dc}{dp} = -\frac{1}{2}\log_2 e \ln\frac{\frac{p}{2}}{1-\frac{p}{2}} - 1 = 0$

$\Rightarrow p = 0.4$ # (3%)

$C = H_b(\frac{0.4}{2}) - 0.4 = 0.3219$ # (3%)

3. (12%) A voice-band telephone channel passes the frequencies in the band from 300 to 3300 Hz. It is desired to design a modem that transmits at a symbol rate of 2400 symbols/s, with the objective of achieving 9600 bits/s.
Select an appropriate QAM signal constellation, carrier frequency, and the roll-off factor of a pulse with a raised cosine spectrum that utilizes the entire frequency band.
Sketch the spectrum of the transmitted signal pulse and indicate the important frequencies.

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① $W = 3300 - 300 = 3000\ (Hz)$

② $R = \frac{1}{T} = 2400\ (symbol/s)$

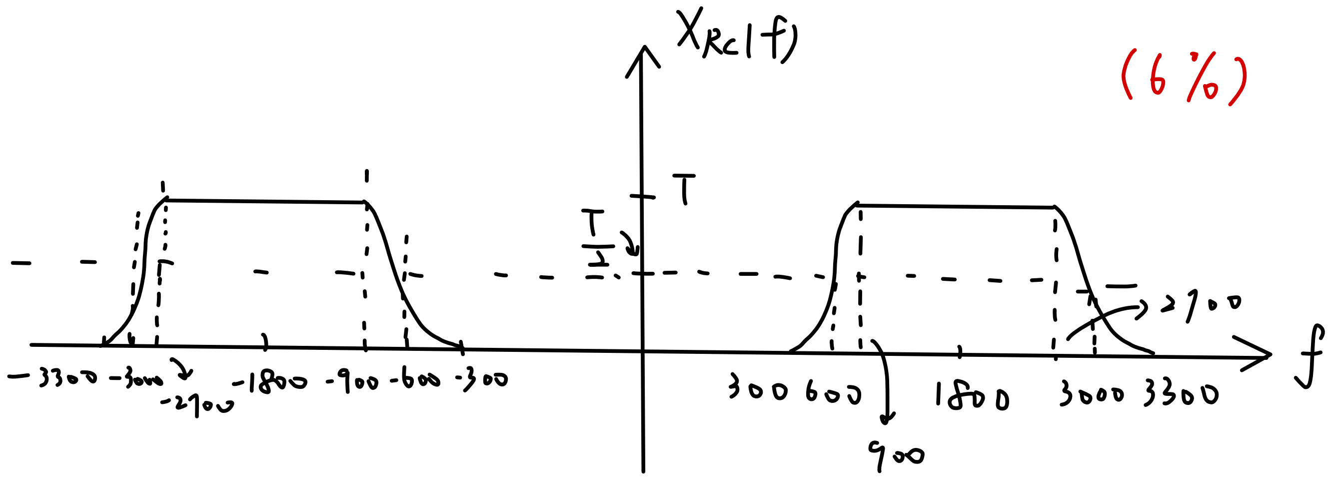
③ $k = \frac{9600}{2400} = 4\ (bit) \Rightarrow$ select $2^4 = 16$ QAM (2%)

④ $f_c = \frac{3300+300}{2} = 1800\ (Hz)$ (2%)

$B_T = \frac{1}{2T}(1+\beta) = \frac{W}{2} = 1500 \Rightarrow \beta = \frac{3}{12} = 0.25$ (2%)

sketch the spectrum of the transmitted signal pulse

(6%)



4. (24%) Three equiprobable messages m_1, m_2 and m_3 are to be transmitted over an AWGN channel with noise power spectral density $\frac{1}{2}N_0$. The messages are

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$$s_1(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad s_2(t) = -s_3(t) \begin{cases} 1 & 0 \le t \le \frac{1}{2}T \\ -1 & \frac{1}{2}T < t \le T \\ 0 & \text{otherwise} \end{cases}$$

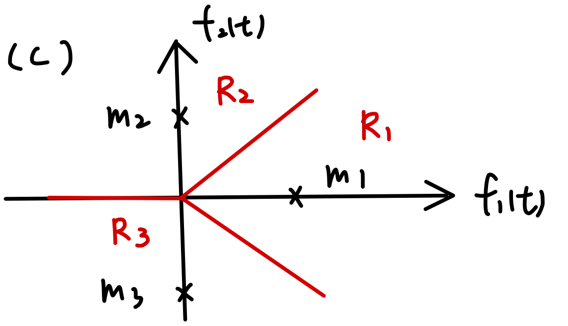
(a) What is the dimensionality of the signal space?
(b) Find an appropriate basis for the signal space.
(c) Please Draw the signal constellation of the messages m_1, m_2 and m_3 and sketch the optimal decision regions R_1, R_2 and R_3. for this problem.
(d) Which of the three messages is most vulnerable to errors and why? In other words, which of $P(\text{error} \mid m_i \text{ transmitted})$, $i = 1, 2, 3$, is largest?

(a) dimensionality = 2 (2%)

(b) $f_1(t) = \begin{cases} \frac{1}{\sqrt{T}}, & 0 \le t \le T \\ 0, & \text{else} \end{cases}$ $\quad f_2(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le \frac{1}{2}T \\ \frac{1}{\sqrt{T}} & \frac{1}{2}T \le t \le T \\ 0 & \text{else} \end{cases}$

$m_1 = [\sqrt{T}, 0]$ $\quad m_2 = [0, \sqrt{T}]$ $\quad m_3 = [0, -\sqrt{T}]$ (4%)

(c)



MAP → ML → MD

(8%)

(d) $P(e \mid m_1) = Q\left(\sqrt{\frac{2T}{2N_0}}\right) \cdot 2 = 2Q\left(\sqrt{\frac{T}{N_0}}\right)$

$P(e \mid m_2) = Q\left(\sqrt{\frac{2T}{2N_0}}\right) + Q\left(\sqrt{\frac{4T}{2N_0}}\right) = Q\left(\sqrt{\frac{T}{N_0}}\right) + Q\left(\sqrt{\frac{2T}{N_0}}\right)$

$= P(e \mid m_3)$

error rate (8%)

$\because Q\left(\sqrt{\frac{2T}{N_0}}\right) < Q\left(\sqrt{\frac{T}{N_0}}\right)$

$\therefore$ The probability $P(e \mid m_1)$ is large than $P(e \mid m_2)$ and $P(e \mid m_3)$

(2%)

5. (24%) Two discrete memoryless information sources X and Y each have an alphabet with six symbols, $\mathcal{X} = \mathcal{Y} = \{1, 2, 3, 4, 5, 6\}$. The probabilities of the letters for X are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$ and $\frac{1}{32}$. The source Y has a uniform distribution.
(a) Find the entropy of both X and Y.
(b) Design Huffman codes for each source. Which Huffman code is more efficient?
Hint: Efficiency of a Huffman code is defined as the ratio of the source entropy $H(X)$ to the average codeword length $\bar{R}$.
(c) If Huffman codes were designed for the second extension of these sources (i.e., two letters at a time), for which source would you expect a performance improvement compared to the single-letter Huffman code and why?

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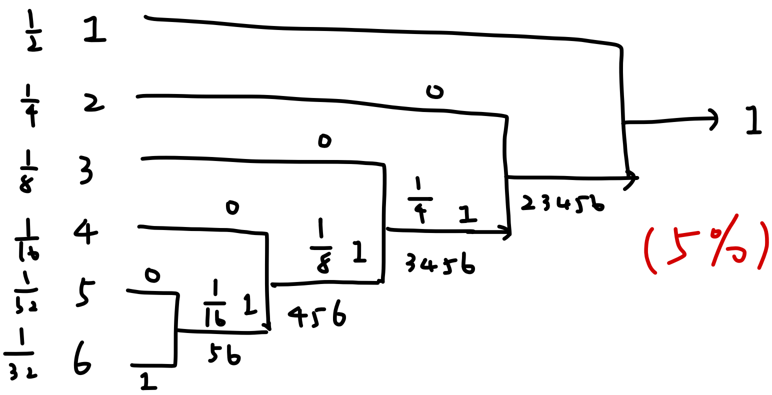
(a) $H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{8}\log_2\frac{1}{8} - \frac{1}{16}\log_2\frac{1}{16} - \frac{2}{32}\log_2\frac{1}{32} = \frac{31}{16} = 1.9375$ (2%)

$H(Y) = 6 \cdot -\frac{1}{6}\log_2\frac{1}{6} \doteq 2.5850$ (2%)

(b)

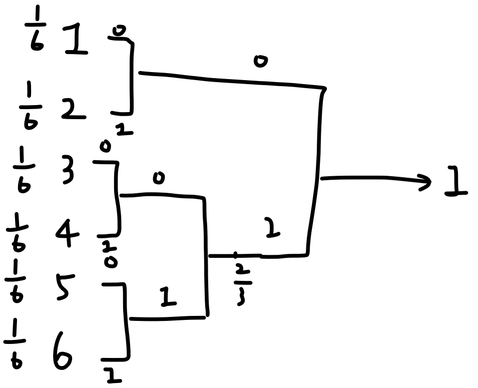


(5%)

1	0
2	1 0
3	1 1 0
4	1 1 1 0
5	1 1 1 1 0
6	1 1 1 1 1

$\overline{R}_X = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{16} + 5\cdot\frac{1}{32}\cdot 2 = \frac{31}{16} = 1.9375$ (2%)

$\eta_X = \frac{H(X)}{\overline{R}_X} = 1$ (2%)



(5%)

1	0 0
2	0 1
3	1 0 0
4	1 0 1
5	1 1 0
6	1 1 1

$\overline{R}_Y = \frac{1}{6}(2\cdot 2 + 3\cdot 4) = \frac{16}{6} \doteq 2.6667$ $\quad \eta_Y = \frac{H(Y)}{\overline{R}_Y} \doteq 0.9694$ (2%)

$\because \eta_X > \eta_Y \quad \therefore$ X is more efficient (2%)

(c) $\because$ Source X has achieved 100% efficiency.

$\therefore$ We expect an improvement for source Y (2%)

6. (16%) The transmission of a signal pulse with a raised cosine spectrum through a channel results in the following **(noise-free)** sampled output from the demodulator:

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$$x_m = \begin{cases} -0.5 & m = -2 \\ 0.1 & m = -1 \\ 1 & m = 0 \\ -0.2 & m = 1 \\ 0.05 & m = 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Design a three-tap zero-forcing linear equalizer so that the output is

$$q_m = \begin{cases} 1 & m = 0 \\ 0 & m = \pm 1 \end{cases}$$

(b) Determine q_m for $m = \pm 2, \pm 3$ by convolving the impulse response of the equalizer with the channel response.

(a)
$$\begin{bmatrix} 1 & 0.1 & -0.5 \\ -0.2 & 1 & 0.1 \\ 0.05 & -0.2 & 1 \end{bmatrix} \begin{bmatrix} C_{-1} \\ C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} C_{-1} \\ C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.9803 \\ 0.196 \end{bmatrix}$$ (8%)

(b)
$$q_m = \sum_{n=-1}^{1} C_n X_{m-n}$$

$q_{-3} = C_{-1} X_{-2} + C_0 X_{-3} + C_1 X_{-4} = 0$ (2%)

$q_{-2} = C_{-1} X_{-1} + C_0 X_{-2} + C_1 X_{-3} = -0.49015$ (2%)

$q_2 = C_{-1} X_3 + C_0 X_2 + C_1 X_1 = 0.009815$ (2%)

$q_3 = C_{-1} X_4 + C_0 X_3 + C_1 X_2 = 0.0098$ (2%)