Homework #1 Solution

Due 23:59, Oct. 27, 2022

Problem 2.10 **(5% + 5%)**

a. To show that the waveforms $f_n(t)$, $n = 1, \ldots, 3$ are orthogonal we have to prove that:

$$\int_{-\infty}^{\infty} f_m(t) f_n(t) dt = 0, \qquad m \neq n$$

Clearly:

$$\begin{aligned}
c_{12} &= \int_{-\infty}^{\infty} f_1(t) f_2(t) dt = \int_0^4 f_1(t) f_2(t) dt \\
&= \int_0^2 f_1(t) f_2(t) dt + \int_2^4 f_1(t) f_2(t) dt \\
&= \frac{1}{4} \int_0^2 dt - \frac{1}{4} \int_2^4 dt = \frac{1}{4} \times 2 - \frac{1}{4} \times (4-2) \\
&= 0
\end{aligned}$$

Similarly:

$$\begin{aligned}
c_{13} &= \int_{-\infty}^{\infty} f_1(t) f_3(t) dt = \int_0^4 f_1(t) f_3(t) dt \\
&= \frac{1}{4} \int_0^1 dt - \frac{1}{4} \int_1^2 dt - \frac{1}{4} \int_2^3 dt + \frac{1}{4} \int_3^4 dt \\
&= 0
\end{aligned}$$

and :

$$\begin{aligned}
c_{23} &= \int_{-\infty}^{\infty} f_2(t) f_3(t) dt = \int_0^4 f_2(t) f_3(t) dt \\
&= \frac{1}{4} \int_0^1 dt - \frac{1}{4} \int_1^2 dt + \frac{1}{4} \int_2^3 dt - \frac{1}{4} \int_3^4 dt \\
&= 0
\end{aligned}$$

Thus, the signals $f_n(t)$ are orthogonal. It is also straightforward to prove that the signals have unit energy :

$$\int_{-\infty}^{\infty} |f_i(t)|^2 dt = 1, \quad i = 1, 2, 3$$

Hence, they are orthonormal.

b. We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t) f_n(t) dt, \qquad n = 1, 2, 3$$

$$\begin{aligned}
x_1 &= \int_0^4 x(t)f_1(t)dt = -\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_1^2 dt - \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0 \\
x_2 &= \int_0^4 x(t)f_2(t)dt = \frac{1}{2}\int_0^4 x(t)dt = 0 \\
x_3 &= \int_0^4 x(t)f_3(t)dt = -\frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt + \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0
\end{aligned}$$

As it is observed, $x(t)$ is orthogonal to the signal wavaforms $f_n(t)$, $n = 1, 2, 3$ and thus it can not represented as a linear combination of these functions.

Problem 2.12 **(10%)**

As a set of orthonormal functions we consider the waveforms

$$f_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \quad f_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases} \quad f_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned}
\mathbf{s}_1 &= \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \\
\mathbf{s}_2 &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \\
\mathbf{s}_3 &= \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} \\
\mathbf{s}_4 &= \begin{bmatrix} 2 & 2 & 0 \end{bmatrix}
\end{aligned}$$

Note that $s_3(t) = s_2(t) - s_1(t)$ and that the dimensionality of the waveforms is 3.

Problem 2.15 **(10%)**

The relationship holds for $n = 2$ (2-1-34) : $p(x_1, x_2) = p(x_2|x_1)p(x_1)$
Suppose it holds for $n = k$, i.e : $p(x_1, x_2, ..., x_k) = p(x_k|x_{k-1}, ..., x_1)p(x_{k-1}|x_{k-2}, ..., x_1) \, ...p(x_1)$
Then for $n = k + 1$:

$$\begin{aligned}
p(x_1, x_2, ..., x_k, x_{k+1}) &= p(x_{k+1}|x_k, x_{k-1}, ..., x_1)p(x_k, x_{k-1}..., x_1) \\
&= p(x_{k+1}|x_k, x_{k-1}, ..., x_1)p(x_k|x_{k-1}, ..., x_1)p(x_{k-1}|x_{k-2}, ..., x_1) \, ...p(x_1)
\end{aligned}$$

Hence the relationship holds for $n = k + 1$, and by induction it holds for any n.

Problem 2.17 **(10%)**

Following the same procedure as in example 2-1-1, we prove :

$$p_Y(y) = \frac{1}{|a|} p_X\left(\frac{y-b}{a}\right)$$

Problem 2.20 **(5% + 5% + 5% + 5%)**

1) $y = g(x) = ax^2$. Assume without loss of generality that $a > 0$. Then, if $y < 0$ the equation $y = ax^2$ has no real solutions and $f_Y(y) = 0$. If $y > 0$ there are two solutions to the system, namely $x_{1,2} = \sqrt{y/a}$. Hence,

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \\ &= \frac{f_X(\sqrt{y/a})}{2a\sqrt{y/a}} + \frac{f_X(-\sqrt{y/a})}{2a\sqrt{y/a}} \\ &= \frac{1}{\sqrt{ay}\sqrt{2\pi\sigma^2}} e^{-\frac{y}{2a\sigma^2}} \end{aligned}$$

2) The equation $y = g(x)$ has no solutions if $y < -b$. Thus $F_Y(y)$ and $f_Y(y)$ are zero for $y < -b$. If $-b \leq y \leq b$, then for a fixed y, $g(x) < y$ if $x < y$; hence $F_Y(y) = F_X(y)$. If $y > b$ then $g(x) \leq b < y$ for every x; hence $F_Y(y) = 1$. At the points $y = \pm b$, $F_Y(y)$ is discontinuous and the discontinuities equal to

$$F_Y(-b^+) - F_Y(-b^-) = F_X(-b)$$

and

$$F_Y(b^+) - F_Y(b^-) = 1 - F_X(b)$$

The PDF of $y = g(x)$ is

$$\begin{aligned} f_Y(y) &= F_X(-b)\delta(y+b) + (1 - F_X(b))\delta(y-b) + f_X(y)[u_{-1}(y+b) - u_{-1}(y-b)] \\ &= Q\left(\frac{b}{\sigma}\right)(\delta(y+b) + \delta(y-b)) + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} [u_{-1}(y+b) - u_{-1}(y-b)] \end{aligned}$$

3) In the case of the hard limiter

$$\begin{aligned} p(Y = b) &= p(X < 0) = F_X(0) = \frac{1}{2} \\ p(Y = a) &= p(X > 0) = 1 - F_X(0) = \frac{1}{2} \end{aligned}$$

Thus $F_Y(y)$ is a staircase function and

$$f_Y(y) = F_X(0)\delta(y-b) + (1 - F_X(0))\delta(y-a)$$

4) The random variable $y = g(x)$ takes the values $y_n = x_n$ with probability

$$p(Y = y_n) = p(a_n \leq X \leq a_{n+1}) = F_X(a_{n+1}) - F_X(a_n)$$

Thus, $F_Y(y)$ is a staircase function with $F_Y(y) = 0$ if $y < x_1$ and $F_Y(y) = 1$ if $y > x_N$. The PDF is a sequence of impulse functions, that is

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^{N} [F_X(a_{i+1}) - F_X(a_i)]\,\delta(y - x_i) \\ &= \sum_{i=1}^{N} \left[Q\left(\frac{a_i}{\sigma}\right) - Q\left(\frac{a_{i+1}}{\sigma}\right)\right] \delta(y - x_i) \end{aligned}$$

Problem 2.39 **(10%)**

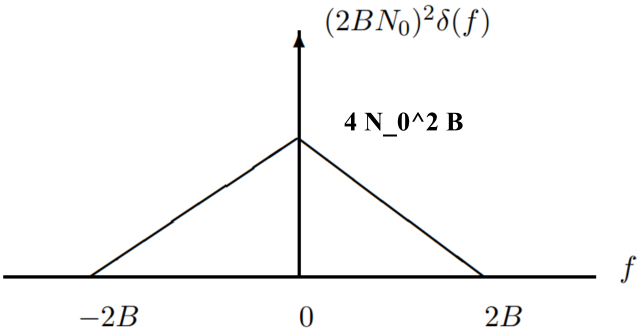
The power spectral density of $X(t)$ corresponds to : $R_{xx}(t) = 2BN_0 \frac{\sin 2\pi Bt}{2\pi Bt}$. From the result of Problem 2.14 :

$$R_{yy}(\tau) = R_{xx}^2(0) + 2R_{xx}^2(\tau) = (2BN_0)^2 + 8B^2N_0^2 \left(\frac{\sin 2\pi Bt}{2\pi Bt}\right)^2$$

Also :

$$\mathcal{S}_{yy}(f) = R_{xx}^2(0)\delta(f) + 2\mathcal{S}_{xx}(f) * \mathcal{S}_{xx}(f)$$

The following figure shows the power spectral density of $Y(t)$:



Problem 2.42 **(10%)**

$p_R(r) = \frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m r^{2m-1}e^{-mr^2/\Omega}, \quad X = \frac{1}{\sqrt{\Omega}}R$
We know that : $p_X(x) = \frac{1}{1/\sqrt{\Omega}} p_R\left(\frac{x}{1/\sqrt{\Omega}}\right)$.
Hence :

$$p_X(x) = \frac{1}{1/\sqrt{\Omega}}\frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m \left(x\sqrt{\Omega}\right)^{2m-1} e^{-m(x\sqrt{\Omega})^2/\Omega} = \frac{2}{\Gamma(m)} m^m x^{2m-1} e^{-mx^2}$$

Problem 2.46 **(10%)**

Since $H(0) = \sum_{-\infty}^{\infty} h(n) = 0 \Rightarrow m_y = m_x H(0) = 0$

The autocorrelation of the output sequence is

$$R_{yy}(k) = \sum_i \sum_j h(i)h(j)R_{xx}(k-j+i) = \sigma_x^2 \sum_{i=-\infty}^{\infty} h(i)h(k+i)$$

where the last equality stems from the autocorrelation function of $X(n)$:

$$R_{xx}(k-j+i) = \sigma_x^2\delta(k-j+i) = \left\{ \begin{array}{ll} \sigma_x^2, & j = k+i \\ 0, & o.w. \end{array} \right\}$$

Hence, $R_{yy}(0) = 6\sigma_x^2$, $R_{yy}(1) = R_{yy}(-1) = -4\sigma_x^2$, $R_{yy}(2) = R_{yy}(-2) = \sigma_x^2$, $R_{yy}(k) = 0$ otherwise.

Finally, the frequency response of the discrete-time system is :

$$\begin{aligned} H(f) &= \sum_{-\infty}^{\infty} h(n)e^{-j2\pi fn} \\ &= 1 - 2e^{-j2\pi f} + e^{-j4\pi f} \\ &= \left(1 - e^{-j2\pi f}\right)^2 \\ &= e^{-j2\pi f}\left(e^{j\pi f} - e^{-j\pi f}\right)^2 \\ &= -4e^{-j\pi f} \sin^2 \pi f \end{aligned}$$

which gives the power density spectrum of the output :

$$\mathcal{S}_{yy}(f) = \mathcal{S}_{xx}(f)|H(f)|^2 = \sigma_x^2\left[16 \sin^4 \pi f\right] = 16\sigma_x^2 \sin^4 \pi f$$

Problem 2.54 **(10%)**

$$\begin{aligned} E\left[x(t+\tau)x(t)\right] &= A^2 E\left[\sin\left(2\pi f_c(t+\tau)+\theta\right)\sin\left(2\pi f_c t+\theta\right)\right] \\ &= \tfrac{A^2}{2}\cos 2\pi f_c\tau - \tfrac{A^2}{2}E\left[\cos\left(2\pi f_c(2t+\tau)+2\theta\right)\right] \end{aligned}$$

where the last equality follows from the trigonometric identity :
$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$. But :

$$\begin{aligned} E\left[\cos\left(2\pi f_c(2t+\tau)+2\theta\right)\right] &= \int_0^{2\pi} \cos\left(2\pi f_c(2t+\tau)+2\theta\right) p(\theta) d\theta \\ &= \tfrac{1}{2\pi}\int_0^{2\pi} \cos\left(2\pi f_c(2t+\tau)+2\theta\right) d\theta = 0 \end{aligned}$$

Hence :

$$E\left[x(t+\tau)x(t)\right] = \frac{A^2}{2}\cos 2\pi f_c\tau$$

Appendix

Problem 2.39 is actually a follow-up question of **Problem 2.27** and **Problem 2.41**, so we also provide the solution of these two questions for you reference.

Problem 2.27

$$\psi(jv_1, jv_2, jv_3, jv_4) = E\left[e^{j(v_1x_1+v_2x_2+v_3x_3+v_4x_4)}\right]$$

$$E\left(X_1X_2X_3X_4\right) = (-j)^4 \frac{\partial^4 \psi(jv_1, jv_2, jv_3, jv_4)}{\partial v_1 \partial v_2 \partial v_3 \partial v_4}|_{v_1=v_2=v_3=v_4=0}$$

From (2-1-151) of the text, and the zero-mean property of the given rv's :

$$\psi(j\mathbf{v}) = e^{-\frac{1}{2}\mathbf{v}'\mathbf{M}\mathbf{v}}$$

where $\mathbf{v} = [v_1, v_2, v_3, v_4]'$, $\mathbf{M} = [\mu_{ij}]$.

We obtain the desired result by bringing the exponent to a scalar form and then performing quadruple differentiation. We can simplify the procedure by noting that :

$$\frac{\partial \psi(j\mathbf{v})}{\partial v_i} = -\mu_{\mathbf{i}}'\mathbf{v}e^{-\frac{1}{2}\mathbf{v}'\mathbf{M}\mathbf{v}}$$

where $\mu_{\mathbf{i}}' = [\mu_{i1}, \mu_{i2}, \mu_{i3}, \mu_{i4}]$. Also note that :

$$\frac{\partial \mu_{\mathbf{j}}'\mathbf{v}}{\partial v_i} = \mu_{ij} = \mu_{ji}$$

Hence :

$$\frac{\partial^4 \psi(jv_1, jv_2, jv_3, jv_4)}{\partial v_1 \partial v_2 \partial v_3 \partial v_4}|_{\mathbf{V}=\mathbf{0}} = \mu_{12}\mu_{34} + \mu_{23}\mu_{14} + \mu_{24}\mu_{13}$$

Problem 2.41

$Y(t) = X^2(t),\ R_{xx}(\tau) = E\left[x(t+\tau)x(t)\right]$

$$R_{yy}(\tau) = E\left[y(t+\tau)y(t)\right] = E\left[x^2(t+\tau)x^2(t)\right]$$

Let $X_1 = X_2 = x(t),\ X_3 = X_4 = x(t+\tau)$. Then, from problem 2.7 :

$$E\left(X_1X_2X_3X_4\right) = E\left(X_1X_2\right)E\left(X_3X_4\right) + E\left(X_1X_3\right)E\left(X_2X_4\right) + E\left(X_1X_4\right)E\left(X_2X_3\right)$$

Hence :

$$R_{yy}(\tau) = R_{xx}^2(0) + 2R_{xx}^2(\tau)$$