COM 5120 Communication Theory

Midterm Exam Answer

November 10, 2022
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1. (13%) The answer is **(c)**, recall the definition of **sufficient statistic**, option (c) only contains r_1 and r_2 clearly not satisfied the definition.

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2. (15%)

$$P_r\{x[n],\ n=1,2,\cdots,N|p\} = p^{\sum x[n]}(1-p)^{N-\sum x[n]}$$

Denote $\overline{X} = \frac{1}{N}\sum_{n=1}^{N} x[n]$,

$$P_r\{x[n], n=1,2,\cdots,N|p\} = p^{N\overline{X}}(1-p)^{N-N\overline{X}}$$

To find

$$\max_p P_r\{x[n], n=1,2,\cdots,N|p\}$$
$$\frac{d\ln P_r\{x[n], n=1,2,\cdots,N|p\}}{dp} = \cdots$$
$$= \frac{N\overline{X}}{p} + \frac{N-N\overline{X}}{1-p}(-1) = 0$$

obtain $\widehat{p} = \overline{X} = \frac{1}{N}\sum_{n=1}^{N} x[n]$.

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3. (15%)
(a) $S_{\widehat{X}}(f) = |-j\text{sgn}(f)|^2 S_X(f) = S_X(f)$, hence $R_{\widehat{X}}(\tau) = R_X(\tau)$
(b) $S_{X\widehat{X}}(f) = S_X(f)(-j\text{sgn}(f))^* = j\text{sgn}(f)S_X(f)$, therefore, $R_{X\widehat{X}}(\tau) = -R_X(\tau)$
(c) $R_Z(\tau) = E\left[\left(X(t+\tau)+j\widehat{X}(t+\tau)\right)\left(X(t)-j\widehat{X}(t)\right)\right]$, expanding we have

$$R_Z(\tau) = R_X(\tau) + R_{\widehat{X}}(\tau) - j\left[R_{X\widehat{X}}(\tau) - R_{\widehat{X}X}(\tau)\right]$$

Using $R_{\widehat{X}}(\tau) = R_X(\tau)$, and the fact that $R_{X\widehat{X}}(\tau) = -\widehat{R}_X(\tau)$ is an odd function (since it is the HT of an even signal) we have $R_{\widehat{X}X}(\tau) = R_{X\widehat{X}}(-\tau) = -R_{X\widehat{X}}(\tau)$, we have

$$R_Z(\tau) = 2R_X(\tau) - jR_{X\widehat{X}}(\tau) = 2R_X(\tau) + j2\widehat{R}_X(\tau)$$

Taking FT of both sides we have

$$S_Z(f) = 2S_X(f) + j2(-j\text{sgn}(f)S_X(f)) = 2(1+\text{sgn}(f))S_X(f) = 4S_X(f)u_{-1}(f)$$

4. (14%)

(a) To show that the waveforms $f_n(t),\ n = 1, 2, 3$ are orthogonal we have to prove that:

$$\int_{-\infty}^{\infty} f_m(t) f_n(t)\, dt = 0, \qquad m \neq n$$

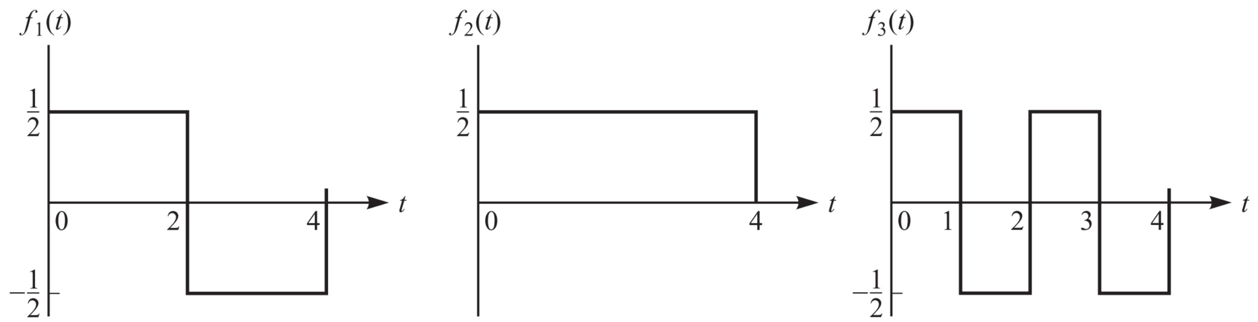

Figure 1: three waveforms $f_1(t),\ f_2(t),\ f_3(t)$

Clearly:

$$\begin{aligned}
c_{12} &= \int_{-\infty}^{\infty} f_1(t) f_2(t)\, dt = \int_0^4 f_1(t) f_2(t)\, dt = \int_0^2 f_1(t) f_2(t)\, dt + \int_2^4 f_1(t) f_2(t)\, dt \\
&= \int_0^2 \frac{1}{2} \cdot \frac{1}{2}\, dt + \int_2^4 \frac{-1}{2} \cdot \frac{1}{2}\, dt = \frac{1}{4} \int_0^2 1\, dt - \frac{1}{4} \int_2^4 1\, dt \\
&= \frac{1}{4} \cdot 2 - \frac{1}{4} \cdot (4 - 2) = 0
\end{aligned}$$

Similarly:

$$\begin{aligned}
c_{13} &= \int_{-\infty}^{\infty} f_1(t) f_3(t)\, dt = \int_0^4 f_1(t) f_3(t)\, dt \\
&= \int_0^1 f_1(t) f_3(t)\, dt + \int_1^2 f_1(t) f_3(t)\, dt + \int_2^3 f_1(t) f_3(t)\, dt + \int_3^4 f_1(t) f_3(t)\, dt \\
&= \int_0^1 \frac{1}{2} \cdot \frac{1}{2}\, dt + \int_1^2 \frac{1}{2} \cdot \frac{-1}{2}\, dt + \int_2^3 \frac{-1}{2} \cdot \frac{1}{2}\, dt + \int_3^4 \frac{-1}{2} \cdot \frac{-1}{2}\, dt \\
&= \frac{1}{4} \int_0^1 1\, dt - \frac{1}{4} \int_1^2 1\, dt - \frac{1}{4} \int_2^3 1\, dt + \frac{1}{4} \int_3^4 1\, dt = 0
\end{aligned}$$

and:

$$\begin{aligned}
c_{23} &= \int_{-\infty}^{\infty} f_2(t) f_3(t)\, dt = \int_0^4 f_2(t) f_3(t)\, dt \\
&= \int_0^1 f_2(t) f_3(t)\, dt + \int_1^2 f_2(t) f_3(t)\, dt + \int_2^3 f_2(t) f_3(t)\, dt + \int_3^4 f_2(t) f_3(t)\, dt \\
&= \int_0^1 \frac{1}{2} \cdot \frac{1}{2}\, dt + \int_1^2 \frac{1}{2} \cdot \frac{-1}{2}\, dt + \int_2^3 \frac{1}{2} \cdot \frac{1}{2}\, dt + \int_3^4 \frac{1}{2} \cdot \frac{-1}{2}\, dt \\
&= \frac{1}{4} \int_0^1 1\, dt - \frac{1}{4} \int_1^2 1\, dt + \frac{1}{4} \int_2^3 1\, dt - \frac{1}{4} \int_3^4 1\, dt = 0
\end{aligned}$$

Thus, the signals $f_n(t)$ are orthogonal. It is also straightforward to prove that the signals have unit energy:

$$\int_{-\infty}^{\infty} |f_i(t)|^2\, dt = 1,\ \ i = 1, 2, 3.$$

Hence, they are orthonormal.

(b) We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t) f_n(t)\, dt,\ \ n = 1, 2, 3, \quad \text{where } x = \begin{cases} -2, & 0 \leq t < 1 \\ 6, & 1 \leq t < 3 \\ 4, & 3 \leq t < 4 \end{cases}$$

$$\begin{aligned}
x_1 &= \int_0^4 x(t) f_1(t)\, dt \\
&= \int_0^1 x(t) f_1(t)\, dt + \int_1^2 x(t) f_1(t)\, dt + \int_2^3 x(t) f_1(t)\, dt + \int_3^4 x(t) f_1(t)\, dt \\
&= \int_0^1 -2 \cdot \frac{1}{2}\, dt + \int_1^2 6 \cdot \frac{1}{2}\, dt + \int_2^3 6 \cdot \frac{-1}{2}\, dt + \int_3^4 4 \cdot \frac{-1}{2}\, dt \\
&= -1 \int_0^1 1\, dt + 3 \int_1^2 1\, dt - 3 \int_2^3 1\, dt - 2 \int_3^4 1\, dt = -3 \\
x_2 &= \int_0^4 x(t) f_2(t)\, dt \\
&= \int_0^1 x(t) f_2(t)\, dt + \int_1^2 x(t) f_2(t)\, dt + \int_2^3 x(t) f_2(t)\, dt + \int_3^4 x(t) f_2(t)\, dt \\
&= \int_0^1 -2 \cdot \frac{1}{2}\, dt + \int_1^2 6 \cdot \frac{1}{2}\, dt + \int_2^3 6 \cdot \frac{1}{2}\, dt + \int_3^4 4 \cdot \frac{1}{2}\, dt \\
&= -1 + 3 + 3 + 2 = 7 \\
x_3 &= \int_0^4 x(t) f_3(t)\, dt \\
&= \int_0^1 x(t) f_3(t)\, dt + \int_1^2 x(t) f_3(t)\, dt + \int_2^3 x(t) f_3(t)\, dt + \int_3^4 x(t) f_3(t)\, dt \\
&= \int_0^1 -2 \cdot \frac{1}{2}\, dt + \int_1^2 6 \cdot \frac{-1}{2}\, dt + \int_2^3 6 \cdot \frac{1}{2}\, dt + \int_3^4 4 \cdot \frac{-1}{2}\, dt \\
&= -3
\end{aligned}$$

As it is observed, $x(t) \neq -3f_1(t) + 7f_2(t) - 3f_3(t)$ and thus it **can not** represented as a linear combination of these funtions.

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5. (14%)
(a) Consider the QAM constellation of Figure 2. Using the Pythagorean theorem we can find the radius of the inner circle as:

$$a^2 + a^2 = A^2 \Longrightarrow a = \frac{1}{\sqrt{2}} A$$

The radius of the outer circle an be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between then equal to $\theta = 105°$, we obtain:

$$b^2 = a^2 + A^2 - 2aA\cos 105° \Longrightarrow b = \frac{1+\sqrt{3}}{2}A$$

(b) If we denote by r the radius of the circle, then using the cosine theorem we obtain:

$$A^2 = r^2 + r^2 - 2r\cos 45° \Longrightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}$$

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6. (14%)
(a) The correlation type demodulator employes a filter:

$$f(t) = \begin{cases} \frac{1}{\sqrt{T}}, & 0 \le t < T \\ 0, & otherwise \end{cases}$$

Hence, the sampled outputs of the crosscorrelators are:

$$r = s_m + n, \ \ m = 0, 1$$

where $s_0 = 0, \ s_1 = A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable with variance:

$$\sigma_n^2 = \frac{N_0}{2}$$

The probability density function for the sampled output is:

$$p(r|s_0) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}}$$
$$p(r|s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}}$$

Since the signals are equally probable, the optimal detector decides in favor of s_0 if

$$PM(\mathbf{r}|\mathbf{s_0}) = p(r|s_0) > p(r|s_1) = PM(\mathbf{r}|\mathbf{s_1})$$

otherwise it decides in favor of s_1. The decision rule may be expressed as:

$$\frac{\mathrm{PM}(\mathbf{r}, \mathbf{s}_0)}{\mathrm{PM}(\mathbf{r}, \mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2 - r^2}{N_0}} = e^{-\frac{(2r - A\sqrt{T})A\sqrt{T}}{N_0}} \begin{array}{c} s_0 \\ > \\ < \\ s_1 \end{array} 1$$

or equivalently:

$$r \begin{array}{c} s_1 \\ > \\ < \\ s_0 \end{array} \frac{1}{2}A\sqrt{T}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.

(b) The average probability of error is:

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} p(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} p(r|s_0)dr \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{r^2}{N_0}}dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{(r-A\sqrt{T})^2}{N_0}}dr \\
&= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= Q[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}] = Q[\sqrt{\text{SNR}}], \text{where } \text{SNR} = \frac{\frac{1}{2}A^2T}{N_0}
\end{aligned}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

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7. (15%) The PDF of the noise n is:

$$p(n) = \frac{1}{\sqrt{2\sigma}}e^{-|n|\sqrt{\frac{2}{\sigma}}} = \frac{\lambda}{2}e^{-\lambda|n|}, \text{ where } \lambda = \sqrt{\frac{2}{\sigma}}$$

The optimal receiver uses the criterion:

$$\frac{p(r|A)}{p(r|-A)} = e^{-\lambda[|r-A|-|r+A|]} \begin{array}{c} A \\ > \\ < \\ -A \end{array} 1 \Longrightarrow r \begin{array}{c} A \\ > \\ < \\ -A \end{array} 0$$

The average probability of error is:

$$
\begin{aligned}
P(e) &= \frac{1}{2}P(e|A) + \frac{1}{2}P(e|-A) \\
&= \frac{1}{2}\int_{-\infty}^{0} f(r|A)dr + \frac{1}{2}\int_{0}^{\infty} f(r|-A)dr \\
&= \frac{1}{2}\int_{-\infty}^{0} \frac{\lambda}{2}e^{-\lambda|r-A|}dr + \frac{1}{2}\int_{0}^{\infty} \frac{\lambda}{2}e^{-\lambda|r+A|}dr \\
&= \frac{\lambda}{4}\int_{-\infty}^{A} e^{-\lambda|x|}dx + \frac{\lambda}{4}\int_{A}^{\infty} e^{-\lambda|x|}dx \\
&= 2\cdot\frac{\lambda}{4}\int_{A}^{\infty} e^{-\lambda|x|}dx \\
&= \frac{\lambda}{2}\cdot\frac{-1}{\lambda}\cdot(e^{-\lambda\infty} - e^{-\lambda A}) = \frac{-1}{2}\cdot(0 - e^{-\lambda A}) \\
&= \frac{1}{2}e^{-\lambda A} = \frac{1}{2}e^{-\sqrt{\frac{2}{\sigma}}A}
\end{aligned}
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