Convex Optimization for Communications and Signal Processing **Fall 2020**
Institute of Communications Engineering
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Homework #2 – Solution

Coverage: chapter 1–3

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Problem 1. (10 points) Let $S \in \mathbb{R}^n$ be a compact set. Let $N : \mathbb{R}^n \to \mathbb{R}$ and $D : \mathbb{R}^n \to \mathbb{R}$ be continuous functions of $\mathbf{x}$, and $D(\mathbf{x}) > 0$ for all $\mathbf{x} \in S$. Show that the function F, defined as

$$F(t) = \max\left\{N(\mathbf{x}) - tD(\mathbf{x}) \mid \mathbf{x} \in S\right\}, \quad \textbf{dom } F = \mathbb{R},$$

is convex.

Solution:

Let $\mathbf{x}_0$ maximize $F(\theta t_1 + (1-\theta)t_2)$, where $t_1, t_2 \in \mathbb{R}$ and $\theta \in [0,1]$. Then

$$\begin{aligned}
F(\theta t_1 + (1-\theta)t_2) &= N(\mathbf{x}_0) - (\theta t_1 + (1-\theta)t_2)D(\mathbf{x}_0) \\
&= \theta(N(\mathbf{x}_0) - t_1 D(\mathbf{x}_0)) + (1-\theta)(N(\mathbf{x}_0) - t_2 D(\mathbf{x}_0)) \\
&\leq \theta \cdot \max\left\{N(\mathbf{x}) - t_1 D(\mathbf{x}) \mid \mathbf{x} \in S\right\} + (1-\theta) \cdot \max\left\{N(\mathbf{x}) - t_2 D(\mathbf{x}) \mid \mathbf{x} \in S\right\} \\
&= \theta F(t_1) + (1-\theta)F(t_2).
\end{aligned}$$

Therefore, we prove that function F is convex.

Problem 2. (10 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex and differentiable, with $\mathbb{R}_+ \subseteq \textbf{dom } f$. Show that the function F, defined as

$$F(x) = \frac{1}{x}\int_0^x f(t)dt, \quad \textbf{dom } F = \mathbb{R}_{++},$$

is convex.

Solution:

$$\begin{aligned}
F'(x) &= -\left(\frac{1}{x^2}\right)\int_0^x f(t)dt + \frac{f(x)}{x} \\
F''(x) &= \left(\frac{2}{x^3}\right)\int_0^x f(t)dt - 2\frac{f(x)}{x^2} + \frac{f'(x)}{x} \\
&= \left(\frac{2}{x^3}\right)\left(\int_0^x f(t)dt - xf(x) + \frac{x^2 f'(x)}{2}\right) \\
&= \left(\frac{2}{x^3}\right)\left(\int_0^x f(t)dt - xf(x) + x^2 f'(x) - \frac{1}{2}x^2 f'(x)\right) \\
&= \left(\frac{2}{x^3}\right)\int_0^x (f(t) - f(x) - f'(x)(t-x))dt.
\end{aligned}$$

Convexity now follow the fact that

$$f(t) \geq f(x) + f'(x)(t-x)$$

for all $x, t \in \textbf{dom } f$, which implies $F''(x) \geq 0$.

Problem 3. (10 points) Prove that a function $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex *if and only if* for all $\mathbf{x} \in \mathbf{dom}\, f$ and for all $\mathbf{v}$, the function $g(t) = f(\mathbf{x} + t\mathbf{v})$ is quasiconvex on its domain $\{t \mid \mathbf{x} + t\mathbf{v} \in \mathbf{dom}\, f\}$.

Solution:

Let us prove the necessity followed by sufficiency.

- **Necessity**: By contradiction, assume $g(t)$ is non-quasiconvex for some $\mathbf{x}$ and $\mathbf{v}$. There exist $t_1, t_2 \in \mathbf{dom}\, g$ (and $\mathbf{x} + t_1\mathbf{v}, \mathbf{x} + t_2\mathbf{v} \in \mathbf{dom}\, f$) such that

$$g(\theta t_1 + (1-\theta)t_2) > \max\{g(t_1), g(t_2)\}, \quad \theta \in [0,1],$$

by the modified Jensen's inequality, implying

$$f\big(\theta(\mathbf{x} + t_1\mathbf{v}) + (1-\theta)(\mathbf{x} + t_2\mathbf{v})\big) > \max\{f(\mathbf{x} + t_1\mathbf{v}), f(\mathbf{x} + t_2\mathbf{v})\}.$$

Therefore, f is not quasiconvex (contradiction with "f is quasiconvex"). Hence g is quasiconvex.

- **Sufficiency**: By contradiction, suppose that $f(\mathbf{x})$ is not quasiconvex. Then, there exist $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{dom}\, f$ and some $\theta \in (0,1)$ such that

$$f(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) > \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}.$$

Now $g(t) = f(\mathbf{x} + t\mathbf{v})$ must be a quasiconvex function. Let $\mathbf{x} = \mathbf{x}_1$ and $\mathbf{v} = \mathbf{x}_2 - \mathbf{x}_1$. Thus $0, 1 \in \mathbf{dom}\, g$ and also $[0,1] \subset \mathbf{dom}\, g$ since $\mathbf{dom}\, g$ is a convex set. Then we have

$$\begin{aligned} g(1-\theta) &= f(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) \\ &> \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\} \\ &= \max\{g(0), g(1)\} \end{aligned}$$

Therefore, $g(t)$ is not quasiconvex (contradiction with the premise of $g(t)$ is quasiconvex). Thus $f(\mathbf{x})$ must be quasiconvex.

■

Problem 4. (25 points) Show that the following functions are convex/concave:

(a) (5 points) $f(\mathbf{X}) = \sup_{\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{\|\mathbf{X}\mathbf{v}\|_2}{\|\mathbf{v}\|_2}$ on $\mathbb{R}^{m \times n}$.

(b) (5 points) $f(\mathbf{x}, \mathbf{Y}) = \mathbf{x}^T\mathbf{Y}^{-1}\mathbf{x}$ on $\mathbb{R}^n \times \mathbb{S}^n_{++}$.

(c) (7 points) $f(\mathbf{X}) = (\det(\mathbf{X}))^{1/m}$ with $\mathbf{dom}\ f = \mathbb{S}^m_{++}$.

(d) (8 points) $f(\mathbf{x}) = (\det(\mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n))^{1/m}$, on $\{\mathbf{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \mid \mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n \succ 0\}$, where $\mathbf{A}_i \in \mathbb{S}^m$.

Solution:

(a) Consider the function $g_{\mathbf{v}}(\mathbf{X}) = \frac{\|\mathbf{X}\mathbf{v}\|_2}{\|\mathbf{v}\|_2}$ with $\mathbf{dom}\ g_{\mathbf{v}} = \mathbb{R}^{m \times n}$ for any $\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. The domain of $g_{\mathbf{v}}(\cdot)$ is convex. Furthermore, for any $\theta \in [0,1]$ and $\mathbf{X}_1, \mathbf{X}_2 \in \mathbf{dom}\ g_{\mathbf{v}}$, we have

$$\begin{aligned} g_{\mathbf{v}}(\theta\mathbf{X}_1 + (1-\theta)\mathbf{X}_2) &= \frac{\|\theta\mathbf{X}_1\mathbf{v} + (1-\theta)\mathbf{X}_2\mathbf{v}\|_2}{\|\mathbf{v}\|_2} \\ &\le \frac{\|\theta\mathbf{X}_1\mathbf{v}\|_2}{\|\mathbf{v}\|_2} + \frac{\|(1-\theta)\mathbf{X}_2\mathbf{v}\|_2}{\|\mathbf{v}\|_2} \quad \text{(by triangle inequality)} \\ &= \theta g_{\mathbf{v}}(\mathbf{X}_1) + (1-\theta)g_{\mathbf{v}}(\mathbf{X}_2) \quad \text{(by the positive homogeneity of a norm).} \end{aligned}$$

Therefore, $g_{\mathbf{v}}$ is convex for any $\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Thus, the function $f(\mathbf{X}) = \sup_{\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} g_{\mathbf{v}}(\mathbf{X})$, which is the pointwise supremum of convex functions, is a convex function.

(b) The epigraph of $f(\mathbf{x}, \mathbf{Y})$ is given by

$$\begin{aligned}\textbf{epi } f &= \{(\mathbf{x}, \mathbf{Y}, t) |\ \mathbf{x}^T\mathbf{Y}^{-1}\mathbf{x} \leq t,\ (\mathbf{x}, \mathbf{Y}) \in \mathbb{R}^n \times \mathbb{S}^n_{++}\} \\ &= \left\{(\mathbf{x}, \mathbf{Y}, t) |\ \begin{bmatrix} \mathbf{Y} & \mathbf{x} \\ \mathbf{x}^T & t \end{bmatrix} \succeq \mathbf{0},\ (\mathbf{x}, \mathbf{Y}) \in \mathbb{R}^n \times \mathbb{S}^n_{++}\right\},\end{aligned}$$

which is convex. Therefore, $f(\mathbf{x}, \mathbf{Y})$ is convex.

(c) To see that $f(\mathbf{X})$ is concave on $\mathbb{S}^n_{++}$, we restrict $f(\mathbf{X})$ to a line and prove that $g(t) = (\det(\mathbf{Z} + t\mathbf{V}))^{1/m}$ is concave:

$$\begin{aligned}g(t) &= (\det(\mathbf{Z} + t\mathbf{V}))^{1/m} \\ &= (\det \mathbf{Z})^{1/m} \left(\det\left(\mathbf{I} + t\mathbf{Z}^{-1/2}\mathbf{V}\mathbf{Z}^{-1/2}\right)\right)^{1/m} \\ &= (\det \mathbf{Z})^{1/m} \left(\prod_{i=1}^{m} (1 + t\lambda_i)\right)^{1/m},\end{aligned}$$

where $\lambda_1, \ldots, \lambda_m$ denote the eigenvalues of $\mathbf{Z}^{-1/2}\mathbf{V}\mathbf{Z}^{-1/2}$. We have expressed g as the product of a positive constant and the geometric mean of $1 + t\lambda_i > 0, i = 1, \ldots, m$ (since: $\mathbf{Z} + t\mathbf{V} \in \mathbb{S}^n_{++}$). Therefore, g is concave.

(d) Suppose that $\mathbf{a} = [a_1, \ldots, a_n]^T \in \textbf{dom } f$, $\mathbf{b} = [b_1, \ldots, b_n]^T \in \textbf{dom } f$. Then, for $0 \leq \theta \leq 1$,

$$\begin{aligned}&\mathbf{A}_0 + (\theta a_1 + (1-\theta)b_1)\mathbf{A}_1 + \cdots + (\theta a_n + (1-\theta)b_n)\mathbf{A}_n \\ &= \theta(\mathbf{A}_0 + a_1\mathbf{A}_1 + \cdots + a_n\mathbf{A}_n) + (1-\theta)(\mathbf{A}_0 + b_1\mathbf{A}_1 + \cdots + b_n\mathbf{A}_n) \succ 0 \\ &\Rightarrow \theta\mathbf{a} + (1-\theta)\mathbf{b} \in \textbf{dom } f \\ &\Rightarrow \textbf{dom } f \text{ is convex.}\end{aligned}$$

Let,

$$\begin{aligned}\mathbf{A} &= \mathbf{A}_0 + a_1\mathbf{A}_1 + \cdots + a_n\mathbf{A}_n \\ \mathbf{B} &= \mathbf{A}_0 + b_1\mathbf{A}_1 + \cdots + b_n\mathbf{A}_n\end{aligned}$$

$$\begin{aligned}f(\theta\mathbf{a} + (1-\theta)\mathbf{b}) &= (\det(\theta\mathbf{A} + (1-\theta)\mathbf{B}))^{1/m} && \text{(2a)} \\ &\geq \theta(\det(\mathbf{A}))^{1/m} + (1-\theta)(\det(\mathbf{B}))^{1/m} && \text{(2b)} \\ &= \theta f(\mathbf{a}) + (1-\theta)f(\mathbf{b}). && \text{(2c)}\end{aligned}$$

The inequality (2b) is obtained from the fact that $(\det(\mathbf{X}))^{1/m} \triangleq h(g(\mathbf{X}))$ is a concave function on $\mathbb{S}^n_{++}$, because $g(\mathbf{X}) = \det(\mathbf{X})$ is concave on $\mathbb{S}^n_{++}$, $h(x) = x^{1/m}$ is non-decreasing and concave on $x > 0$. Therefore, $f(\mathbf{x})$ is a concave function.

Problem 5. (15 points) Let $f(\mathbf{X}) = \text{rank}(\mathbf{X})$, with $\textbf{dom } f \triangleq \mathcal{B} = \{\mathbf{X} \in \mathbb{R}^{M\times N} \mid \|\mathbf{X}\|_* = \|\mathbf{X}\|_{\mathcal{A}} \leq 1\} = \textbf{conv } \mathcal{A}$, where $\mathcal{A}$ is the set of rank-1 matrices. Prove that the convex envelope of f can be shown to be

$$g_f(\mathbf{X}) = \|\mathbf{X}\|_* = \sum_{i=1}^{\text{rank}(\mathbf{X})} \sigma_i(\mathbf{X}), \quad \mathbf{X} \in \mathcal{B}.$$

Solution:

According to the definition of epigraph, we have: $\mathbf{epi}\ f = \{(\mathbf{X}, t) \mid \text{rank}(\mathbf{X}) \le t, \mathbf{X} \in \mathcal{B}\}$. Let

$$\begin{aligned} B_0 &= \{(\mathbf{0}_{M\times N}, t_0) \mid t_0 \ge 0\} \subset \mathbf{epi}\ f, \\ B_1 &= \{(\mathbf{A}, t) \mid \mathbf{A} \in \mathcal{A}, t \ge 1\} \subset \mathbf{epi}\ f, \end{aligned}$$

then, for any $\theta_i \in [0,1]$ and $\sum_{i=0}^{k} \theta_i = 1, k \in \mathbb{Z}_{++}$, we have

$$\begin{aligned} \mathbf{conv}\ (B_0 \cup B_1) &= \{(\mathbf{X}, t) \mid \mathbf{X} = \sum_{i=1}^{k} \theta_i \mathbf{A}_i, t = \theta_0 t_0 + \sum_{i=1}^{k} \theta_i t_i, (\mathbf{A}_i, t_i) \in B_1\} \\ &= \{(\mathbf{X}, t) \mid \|\mathbf{X}\|_* \le t, \mathbf{X} \in \mathcal{B}\} = \mathbf{epi}\ \|\mathbf{X}\|_*,\ \mathbf{X} \in \mathcal{B} \subseteq \mathbf{conv}\ (\mathbf{epi}\ f), \end{aligned}$$

where the second equality is obtained due to

$$\begin{aligned} 0 &\le \|\sum_{i=1}^{k} \theta_i \mathbf{A}_i\|_* \le \sum_{i=1}^{k} \theta_i \|\mathbf{A}_i\|_* = 1 - \theta_0 (\text{i.e. } \mathbf{X} \in \mathcal{B}) \\ &\le \theta_0 t_0 + \sum_{i=1}^{k} \theta_i t_i = t. \end{aligned}$$

Furthermore, since $\|\mathbf{X}\|_* \le \text{rank}(\mathbf{X}) \le \text{t}$ for $\mathbf{X} \in \mathcal{B}$, we have $\mathbf{epi}\ f \subseteq \mathbf{epi}\ \|\mathbf{X}\|_* \implies \mathbf{conv}\ (\mathbf{epi}\ f) \subseteq \mathbf{conv}\ (\mathbf{epi}\ \|\mathbf{X}\|_*) = \mathbf{epi}\ \|\mathbf{X}\|_*$.

Thus we have $\mathbf{epi}\ \|\mathbf{X}\|_* = \mathbf{conv}\ (\mathbf{epi}\ f) = \mathbf{epi}\ g_f$, i.e., $g_f(\mathbf{X}) = \|\mathbf{X}\|_*$ for $\mathbf{X} \in \mathcal{B}$. ■

Problem 6. (15 points) Let $K \subseteq \mathbb{R}^m$ be a proper cone with associated generalized inequality $\preceq_K$, and let $f : \mathbb{R}^n \to \mathbb{R}^m$. For $\boldsymbol{\alpha} \in \mathbb{R}^m$, the $\boldsymbol{\alpha}$-sublevel set of f (with respect to $\preceq_K$) is defined as

$$C_{\boldsymbol{\alpha}} = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \preceq_K \boldsymbol{\alpha}\}.$$

The epigraph of f, with respect to $\preceq_K$, is defined as the set

$$\mathbf{epi}_K\ f = \{(\mathbf{x}, \mathbf{t}) \in \mathbb{R}^{n+m} \mid f(\mathbf{x}) \preceq_K \mathbf{t}\}.$$

Show the following:

(a) (4 points) If f is K-convex, then its sublevel sets $C_{\boldsymbol{\alpha}}$ are convex for all $\boldsymbol{\alpha}$.

(b) (5 points) f is K-convex if and only if $\mathbf{epi}_K\ f$ is a convex set.

(c) (6 points) $\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$ is $\mathbb{S}_+^k$-convex on $\{(\mathbf{A}, \mathbf{B}) \in \mathbb{S}_{++}^k \times \mathbb{R}^{k\times(n-k)}\}$.

Solution:

(a) For all $\mathbf{x},\ \mathbf{y} \in C_{\boldsymbol{\alpha}}$,

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \le \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}) \le \theta\boldsymbol{\alpha} + (1-\theta)\boldsymbol{\alpha} = \boldsymbol{\alpha}.$$

Hence $C_{\boldsymbol{\alpha}}$ is a convex set.

(b) ($\Rightarrow$) Suppose that $\boldsymbol{f}$ is K-convex, $(\mathbf{x}_1, \mathbf{t}_1) \in \mathbf{epi}_K\ f$, and $(\mathbf{x}_2, \mathbf{t}_2) \in \mathbf{epi}_K\ f$. Then,

$$\begin{aligned} f(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) &\preceq_K \theta f(\mathbf{x}_1) + (1-\theta) f(\mathbf{x}_2) \\ &\preceq_K \theta\mathbf{t}_1 + (1-\theta)\mathbf{t}_2 \\ &\theta(\mathbf{x}_1, \mathbf{t}_1) + (1-\theta)(\mathbf{x}_2, \mathbf{t}_2) = (\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2, \theta\mathbf{t}_1 + (1-\theta)\mathbf{t}_2) \in \mathbf{epi}_K\ f \\ &\Rightarrow \mathbf{epi}_K f \text{ is convex.} \end{aligned}$$

($\Leftarrow$) Suppose that $\mathbf{epi}_K\ f$ is convex. And, it is obvious that $(\mathbf{x}_1, f(\mathbf{x}_1)) \in \mathbf{epi}_K\ f$ and $(\mathbf{x}_2, f(\mathbf{x}_2)) \in \mathbf{epi}_K\ f$, for any $\mathbf{x}_1,\ \mathbf{x}_2 \in \mathbf{dom}\ f$. Therefore,

$$(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2, \theta f(\mathbf{x}_1) + (1-\theta) f(\mathbf{x}_2)) \in \mathbf{epi}_K\ f,\ \text{for } \theta \in [0,1].$$

By the definition of $\mathbf{epi}_K$,

$$f(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) \preceq_K \theta f(\mathbf{x}_1) + (1-\theta) f(\mathbf{x}_2) \text{ for } \theta \in [0,1],$$

therefore, f is K-convex.

(c) Let $K = \mathbb{S}^k_+$ Define $f(\mathbf{A},\mathbf{B}) = \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$ with dom $f = \mathbb{S}^k_{++} \times \mathbb{R}^{k\times(n-k)}$. Then, the epigraph of f is

$$\begin{aligned} \operatorname{epi}_K f &= \left\{ (\mathbf{A},\mathbf{B},\mathbf{T}) | \ \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \preceq_K \mathbf{T}, \ \mathbf{A} \in \mathbb{S}^k_{++}, \ \mathbf{B} \in \mathbb{R}^{k\times(n-k)} \right\} \\ &= \left\{ (\mathbf{A},\mathbf{B},\mathbf{T}) \middle| \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{T} \end{bmatrix} \succeq_K \mathbf{0}, \ \mathbf{A} \in \mathbb{S}^k_{++}, \ \mathbf{B} \in \mathbb{R}^{n\times(n-k)} \right\}, \end{aligned}$$

where the second equality comes from the property of *Schur complement*. It can be observed that $\operatorname{epi}_K f$ is convex. Thus, f is K-convex.

Problem 7. (15 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Prove the following.

(a) (10 points) If f and g are convex, both nonincreasing (or nondecreasing), and positive function on an interval, then the product of two functions fg is convex.

(b) (5 points) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then the ratio of two functions f/g is convex.

Solution:

(a) We prove the result by verifying Jensen's inequality. f and g are positive and convex, hence for $\theta \in [0,1]$,

$$\begin{aligned} f(\theta x + (1-\theta)y)g(\theta x + (1-\theta)y) &\le (\theta f(x) + (1-\theta)f(y))(\theta g(x) + (1-\theta)g(y)) \\ &= \theta f(x)g(x) + (1-\theta)f(y)g(y) + \theta(1-\theta)(f(y)-f(x))(g(x)-g(y)). \end{aligned}$$

Since f and g are both nonincreasing or both nondecreasing, the term $(f(y)-f(x))(g(x)-g(y))$ is less than or equal to zero. Therefore

$$f(\theta x + (1-\theta)y)g(\theta x + (1-\theta)y) \le \theta f(x)g(x) + (1-\theta)f(y)g(y).$$

(b) It suffice to note that $1/g$ is convex, positive and increasing, so the result follows from part (a).