Convex Optimization for Communications and Signal Processing **Fall 2020**
Institute of Communications Engineering
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Homework #3 – Solutions
Coverage: Chapter 1–4
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Problem 1. (10 points)

(a) (3 points) Consider $f : \mathbb{R}^n \to \mathbb{R}$ as a convex function and denote $\mathbf{x}^\star$ as its global minimum. For all $\mathbf{y} \in \mathbb{R}^n$, the function $g(\alpha) \triangleq f(\mathbf{x}^\star + \alpha\mathbf{y})$ where $\alpha \in \mathbb{R}$. Prove that $\mathbf{x}^\star$ is the global minimum of f *if and only if* $\forall \mathbf{y} \in \mathbb{R}^n$, $\alpha^\star = 0$ is the global minimum of the function $g(\alpha)$.

(b) (7 points) Let's consider a case where the function f is nonconvex. Denote $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{x}^\star = (0, 0)$ and $p, q \in \mathbb{R}_{++}$, $p < q$. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = (x_2 - px_1^2)(x_2 - qx_1^2)$. Show that if $f(y, my^2) < 0$ for $y \neq 0$ and m satisfying $p < m < q$, then $\mathbf{x}^\star$ is not a local minimum of f even though it is a local minimum along every line passing through $\mathbf{x}^\star$.

Solution:

(a) ($\Rightarrow$) Obviously, if $\mathbf{x}^\star$ is a global minimum of convex function f, then it's also a global minimum of f restricted to any line passing through $\mathbf{x}^\star$.
($\Leftarrow$) Let's assume by contradiction that $\mathbf{x}^\star$ is not a global minimum of f. This implies that there exists some $\mathbf{z} \in \mathbb{R}^n$ such that $f(\mathbf{z}) < f(\mathbf{x}^\star)$. In view of the assumption that $g(\alpha)$ has $\alpha^\star = 0$ as its global minimum, it follows that $g(0) \leq g(0.5)$, or equivalently,

$$\begin{aligned} f(\mathbf{x}^\star) &\leq f(\mathbf{x}^\star + 0.5(\mathbf{z} - \mathbf{x}^\star)), \\ &= f(0.5\mathbf{z} + 0.5\mathbf{x}^\star), \\ &\leq 0.5f(\mathbf{z}) + 0.5f(\mathbf{x}^\star) \quad \text{(since } f \text{ is convex)} \\ &< f(\mathbf{x}^\star) \quad \text{(since } f(\mathbf{z}) < f(\mathbf{x}^\star)\text{)} \end{aligned}$$

which is a contradiction. Hence, it follows that $\mathbf{x}^\star$ is a global minimum of f. ■

(b) We first show that the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(\alpha) = f(\mathbf{x}^\star + \alpha\mathbf{d})$ has a local minimum at $\alpha = 0$ for all $\mathbf{d} \in \mathbb{R}^2$. We have

$$g(\alpha) = f(\mathbf{x}^\star + \alpha\mathbf{d}) = (\alpha d_2 - p\alpha^2 d_1^2)(\alpha d_2 - q\alpha^2 d_1^2) = \alpha^2(d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2).$$

Also,

$$g'(\alpha) = 2\alpha(d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2) + \alpha^2(-pd_1^2)(d_2 - q\alpha d_1^2) + \alpha^2(d_2 - p\alpha d_1^2)(-qd_1^2).$$

Thus $g'(0) = 0$. Furthermore,

$$\begin{aligned} g''(\alpha) &= 2(d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2) + 2\alpha(-pd_1^2)(d_2 - q\alpha d_1^2) \\ &+ 2\alpha(d_2 - p\alpha d_1^2)(-qd_1^2) + 2\alpha(-pd_1^2)(d_2 - q\alpha d_1^2) + \alpha^2(-pd_1^2)(-qd_1^2) \\ &+ 2\alpha(d_2 - p\alpha d_1^2)(-qd_1^2) + \alpha^2(-pd_1^2)(-qd_1^2) \end{aligned}$$

Thus $g''(0) = 2d_2^2$, which is positive if $d_2 \neq 0$. If $d_2 = 0, g(\alpha) = pq\alpha^4 d_1^4$, which is clearly minimized at $\alpha = 0$. Therefore, $(0,0)$ is a local minimum of f along every line that passes through $(0,0)$.

Based on definition, $f(y, my^2) = y^4(m-p)(m-q) < 0$ *if and only if* $p < m < q$ and $y \neq 0$. Note that $f(0,0) = 0$ and $f(y, my^2) < 0$, where $y \neq 0, m \in (p,q)$. Therefore, $f(y, my^2) < 0$, for $(y, my^2) \in B\big((0,0), r\big)$, where $B\big((0,0), r\big)$ is any norm ball with the center at the origin and radius $r > 0$. Hence, $\mathbf{x}^\star = (0,0)$ is not a local minimum of the function f (while it is the local minimum along any line passing through $\mathbf{x}^\star$).

Problem 2. (10 points) We consider the complex least ℓ_p-norm problem

$$\begin{aligned} &\min_{\mathbf{x}} \ \|\mathbf{x}\|_p \\ &\text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned}$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{b} \in \mathbb{C}^m$, and the variable is $\mathbf{x} \in \mathbb{C}^n$. Here $\|\cdot\|_p$ denotes the ℓ_p-norm on $\mathbb{C}^n$, defined as

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$$

for $p \geq 1$, and $\|\mathbf{x}\|_\infty = \max_{i=1,\ldots,n} |x_i|$. We assume that $\mathbf{A}$ is full rank, and $m < n$.

(a) (5 points) Formulate the complex least ℓ_2-norm problem as a least ℓ_2-norm problem with real problem data and variable.

(b) (5 points) Formulate the complex least ℓ_2-norm problem as an SOCP.

Solution:

(a) Define $\mathbf{z} = (\mathrm{Re}\{\mathbf{x}\}, \mathrm{Im}\{\mathbf{x}\}) \in \mathbb{R}^{2n}$, so $\|\mathbf{x}\|_2^2 = \|\mathbf{z}\|_2^2$. The complex linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ is the same as $\mathrm{Re}\{\mathbf{A}\mathbf{x}\} = \mathrm{Re}\{\mathbf{b}\}$, $\mathrm{Im}\{\mathbf{A}\mathbf{x}\} = \mathrm{Im}\{\mathbf{b}\}$, which in turn can be expressed as the set of linear equations

$$\begin{bmatrix} \mathrm{Re}\{\mathbf{A}\} & -\mathrm{Im}\{\mathbf{A}\} \\ \mathrm{Im}\{\mathbf{A}\} & \mathrm{Re}\{\mathbf{A}\} \end{bmatrix} \mathbf{z} = \begin{bmatrix} \mathrm{Re}\{\mathbf{b}\} \\ \mathrm{Im}\{\mathbf{b}\} \end{bmatrix}.$$

Thus, the complex least ℓ_2-norm problem can be expressed as

$$\begin{aligned} &\min_{\mathbf{z}\in\mathbb{R}^{2n}} \ \|\mathbf{z}\|_2 \\ &\text{s.t. } \begin{bmatrix} \mathrm{Re}\{\mathbf{A}\} & -\mathrm{Im}\{\mathbf{A}\} \\ \mathrm{Im}\{\mathbf{A}\} & \mathrm{Re}\{\mathbf{A}\} \end{bmatrix} \mathbf{z} = \begin{bmatrix} \mathrm{Re}\{\mathbf{b}\} \\ \mathrm{Im}\{\mathbf{b}\} \end{bmatrix}. \end{aligned}$$

(b) Using epigraph formulation, with new variable t, we write the problem as

$$\begin{aligned} \min_{t,\mathbf{z}} \ & t \\ \text{s.t.} \ & \left\| \begin{bmatrix} z_i \\ z_{n+i} \end{bmatrix} \right\|_2 \le t, \ i = 1, \dots, n, \\ & \begin{bmatrix} \mathrm{Re}\{\mathbf{A}\} & -\mathrm{Im}\{\mathbf{A}\} \\ \mathrm{Im}\{\mathbf{A}\} & \mathrm{Re}\{\mathbf{A}\} \end{bmatrix} \mathbf{z} = \begin{bmatrix} \mathrm{Re}\{\mathbf{b}\} \\ \mathrm{Im}\{\mathbf{b}\} \end{bmatrix}. \end{aligned}$$

This is an SOCP with n second-order cone constraints.

Problem 3. (15 points) Consider the following optimization problem:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^\mathbf{n}} \ & \sup_{\mathbf{P} \in \mathcal{D}} (1/2)\mathbf{x}^T \mathbf{P} \mathbf{x} \\ \text{s.t.} \ & \mathbf{A}\mathbf{x} \preceq \mathbf{b} \end{aligned}$$

where $\mathcal{D}$ is a subset of possible matrices $\mathbf{P}$.

(a) (5 points) Let $\mathcal{D} = \{\mathbf{P}_1, \dots, \mathbf{P}_K\} \subseteq \mathbb{S}^n_+$. Please reformulate the above problem into a QCQP using epigraph reformulation.

(b) (5 points) Let $\mathcal{D} = \{\mathbf{P} \in \mathbb{S}^n \mid -\gamma \mathbf{I} \preceq \mathbf{P} - \mathbf{P}_0 \preceq \gamma \mathbf{I}\}$, where $\gamma \in \mathbb{R}_{++}$ and $\mathbf{P}_0 \in \mathbb{S}^n_+$. Please reformulate the above problem into a QP.

(c) (5 points) Let $\mathcal{D} = \{\mathbf{P}_0 + \sum_{i=1}^K \mathbf{P}_i u_i \mid \|\mathbf{u}\|_2 \le 1\}$, where $\mathbf{P}_i \in \mathbb{S}^n_+, i = 0, \dots, K$. Please reformulate the above problem into a SOCP using epigraph reformulation.

Solution:

(a) Using epigraph reformulation, we can write the original problem as

$$\begin{aligned} \min \ & t \\ \text{s.t.} \ & (1/2)\mathbf{x}^T \mathbf{P}_i \mathbf{x} \le t, \ i = 1, \dots, K \\ & \mathbf{A}\mathbf{x} \preceq \mathbf{b} \end{aligned}$$

which is a QCQP with the variables $\mathbf{x}$ and t.

(b) Let $\Delta\mathbf{P} \triangleq \mathbf{P} - \mathbf{P}_0$. For a given $\mathbf{x}$, the supremum of $\mathbf{x}^T \Delta\mathbf{P} \mathbf{x}$ over $-\gamma\mathbf{I} \preceq \Delta\mathbf{P} \preceq \gamma\mathbf{I}$ is given by

$$\sup_{-\gamma\mathbf{I} \preceq \Delta\mathbf{P} \preceq \gamma\mathbf{I}} \mathbf{x}^T \Delta\mathbf{P} \mathbf{x} = \gamma \mathbf{x}^T \mathbf{x}$$

due to the fact that $\mathbf{A} \succeq \mathbf{B}$ implies $\mathbf{x}^T \mathbf{A} \mathbf{x} \ge \mathbf{x}^T \mathbf{B} \mathbf{x}$. Therefore, we can express the original problem as

$$\begin{aligned} \min \ & (1/2)\mathbf{x}^T (\mathbf{P}_0 + \gamma \mathbf{I}) \mathbf{x} \\ \text{s.t.} \ & \mathbf{A}\mathbf{x} \preceq \mathbf{b} \end{aligned}$$

which is a QP.

(c) For a given $\mathbf{x}$, the objective function is

$$\begin{aligned}
&\sup_{\|\mathbf{u}\|_2 \le 1} (1/2)\Big(\mathbf{x}^T(\mathbf{P}_0 + \sum_{i=1}^K \mathbf{P}_i u_i)\mathbf{x}\Big) \\
&= (1/2)\Big(\mathbf{x}^T\mathbf{P}_0\mathbf{x} + \sup_{\|\mathbf{u}\|_2 \le 1} \sum_{i=1}^K u_i(\mathbf{x}^T\mathbf{P}_i\mathbf{x})\Big) \\
&= (1/2)(\mathbf{x}^T\mathbf{P}_0\mathbf{x}) + \sup_{\|\mathbf{u}\|_2 \le 1} \sum_{i=1}^K u_i y_i \text{ (Define } y_i = (1/2)\mathbf{x}^T\mathbf{P}_i\mathbf{x}, \forall i) \\
&= (1/2)\mathbf{x}^T\mathbf{P}_0\mathbf{x} + \|\mathbf{y}\|_2. \text{ (by Cauchy Schwarz inequality)}
\end{aligned}$$

Therefore, the resulting problem can be expressed as

$$\begin{aligned}
\min\ & (1/2)\mathbf{x}^T\mathbf{P}_0\mathbf{x} + \|\mathbf{y}\|_2 \\
\text{s.t. } & (1/2)\mathbf{x}^T\mathbf{P}_i\mathbf{x} \le y_i,\ i = 1, \ldots, K \\
& \mathbf{A}\mathbf{x} \preceq \mathbf{b}.
\end{aligned}$$

Then by using the epigraph reformulation, the problem can be rewritten as

$$\begin{aligned}
\min\ & u + t && \text{(1a)} \\
\text{s.t. } & (1/2)\mathbf{x}^T\mathbf{P}_0\mathbf{x} \le u && \text{(1b)} \\
& \|\mathbf{y}\|_2 \le t && \text{(1c)} \\
& (1/2)\mathbf{x}^T\mathbf{P}_i\mathbf{x} \le y_i,\ i = 1, \ldots, K && \text{(1d)} \\
& \mathbf{A}\mathbf{x} \preceq \mathbf{b}. && \text{(1e)}
\end{aligned}$$

Since the inequality constraints (1b) and (1d) are not second-order constraints, we need to reformulate them. First, we change (1b) to $\mathbf{x}^T\mathbf{P}_0\mathbf{x} \le 2u$ and let $2u = c^2 - d^2$, and then it can be reformulated as the form of second-order constraint $\|[\mathbf{P}_0^{1/2}\mathbf{x}, d]\|_2 \le c$. Similarly, the constraint (1d) can follow the same way to reformulate it into a second-order constraint $\|[\mathbf{P}_0^{1/2}\mathbf{x}, f]\|_2 \le e$. Take $c = 2u + (1/4)$, $d = 2u - (1/4)$, $e = 2u + (1/4)$ and $f = 2u - (1/4)$, the problem can be expressed as an SOCP

$$\begin{aligned}
\min\ & u + t \\
\text{s.t. } & \left\| \begin{bmatrix} \mathbf{P}_0^{1/2}\mathbf{x} \\ 2u - (1/4) \end{bmatrix} \right\|_2 \le 2u + (1/4) \\
& \left\| \begin{bmatrix} \mathbf{P}_i^{1/2}\mathbf{x} \\ 2y_i - (1/4) \end{bmatrix} \right\|_2 \le 2y_i + (1/4),\ i = 1, \ldots, K, \\
& \|\mathbf{y}\|_2 \le t \\
& \mathbf{A}\mathbf{x} \preceq \mathbf{b}.
\end{aligned}$$

Problem 4. (10 points) The function $f : \mathbb{R}^n \to \mathbb{R}$ is a quasiconvex function. Consider the convex set $X \subseteq \mathbb{R}^n$ and denote $p^\star = \inf_{\mathbf{x} \in X} f(\mathbf{x})$. Assume that f is not constant on any line segment of X. Prove that every local minimum of f over the set X, is also the global minimum.

Solution:

Denote $\mathbf{x}^\star$ as a local minimum of f over X. We prove the statement by contradiction. Let's assume there exists a vector $\bar{\mathbf{x}} \in X$ such that $f(\bar{\mathbf{x}}) < f(\mathbf{x}^\star) = p^\star$. Then, $\bar{\mathbf{x}}$ and $\mathbf{x}^\star$ belong to the set $X \cap S_{p^\star}$, where $S_{p^\star} = \{\mathbf{x} \,|\, f(\mathbf{x}) \leq f(\mathbf{x}^\star) = p^\star\}$. Since f is quasiconvex (and X is a convex set), then the set $X \cap S_{p^\star}$ is also convex. Hence, the line segment connecting $\mathbf{x}^\star$ and $\bar{\mathbf{x}}$ belongs to the set, implying that

$$f(\theta\bar{\mathbf{x}} + (1-\theta)\mathbf{x}^\star) \leq f(\mathbf{x}^\star), \quad \forall \theta \in [0,1].$$

By assumption, f is not constant on a line segment connecting $\mathbf{x}^\star$ and $\bar{\mathbf{x}}$. Therefor, for each integer $k \geq 1$, there must exist a $\theta_k \in (0, \frac{1}{k}]$ such that

$$f(\theta_k\bar{\mathbf{x}} + (1-\theta_k)\mathbf{x}^\star) < f(\mathbf{x}^\star), \quad \exists \theta_k \in (0, 1/k].$$

Therefore, for any $r > 0$, there always exists a $k \geq 1$ such that $\mathbf{y}_k \triangleq \theta_k\bar{\mathbf{x}} + (1-\theta_k)\mathbf{x}^\star \in B\big(\mathbf{x}^\star, r\big)$ and $f(\mathbf{y}_k) < f(\mathbf{x}^\star)$, thus contradicting the local optimality of $\mathbf{x}^\star$ over X. Hence, there is no $\bar{\mathbf{x}} \in X$ such that $f(\bar{\mathbf{x}}) < f(\mathbf{x}^\star)$ and so $\mathbf{x}^\star$ is a global minimum of f. ■

Problem 5. (10 points) Formulate the following optimization problem as a semidefinite program (SDP).

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \ & \sup_{\|\mathbf{c}\|_2 \leq 1} \mathbf{c}^T \mathbf{F}(\mathbf{x})^{-1}\mathbf{c} \\ \text{s.t. } & \mathbf{F}(\mathbf{x}) \succ \mathbf{0}, \end{aligned}$$

where

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + x_1\mathbf{F}_1 + \cdots + x_n\mathbf{F}_n$$

and each $\mathbf{F}_i \in \mathbb{S}^m$.

Solution:

For a given $\mathbf{x} \in \mathbb{R}^n$, $\sup_{\|\mathbf{c}\|_2 \leq 1} \mathbf{c}^T\mathbf{F}(\mathbf{x})^{-1}\mathbf{c}$ is the maximum eigenvalue of $\mathbf{F}(\mathbf{x})^{-1}$, denoted by $\lambda_{\max}(\mathbf{F}(\mathbf{x})^{-1})$, which is the same as $1/\lambda_{\min}(\mathbf{F}(\mathbf{x}))$. Minimizing $1/\lambda_{\min}(\mathbf{F}(\mathbf{x}))$ is equivalent to maximize $\lambda_{\min}(\mathbf{F}(\mathbf{x}))$. Therefore, the problem can be written as

$$\max_{\mathbf{x}\in\mathbb{R}^n} \ \lambda_{\min}(\mathbf{F}(\mathbf{x})) \tag{2a}$$
$$\text{s.t. } \mathbf{F}(\mathbf{x}) \succ \mathbf{0}. \tag{2b}$$

Problem (2) can be equivalently formulated as

$$\max_{\mathbf{x}\in\mathbb{R}^n, \lambda\in\mathbb{R}} \ \lambda \tag{3a}$$
$$\text{s.t. } \mathbf{F}(\mathbf{x}) - \lambda\mathbf{I} \succeq \mathbf{0}, \tag{3b}$$
$$\mathbf{F}(\mathbf{x}) \succ \mathbf{0}, \tag{3c}$$

where $\lambda \in \mathbb{R}$ is a slack variable. We can relax problem (3) by allowing $\mathbf{F}(\mathbf{x}) \succeq \mathbf{0}$, leading to the following SDP:

$$\max_{\mathbf{x}\in\mathbb{R}^n, \lambda\in\mathbb{R}} \ \lambda \tag{4a}$$
$$\text{s.t. } \mathbf{F}(\mathbf{x}) - \lambda\mathbf{I} \succeq \mathbf{0}, \tag{4b}$$
$$\mathbf{F}(\mathbf{x}) \succeq \mathbf{0}. \tag{4c}$$

When the optimal solution $\lambda^\star$ to problem (4) is less than or equal to zero, i.e., $\lambda^\star \leq 0$, the original problem (2) is infeasible. When $\lambda^\star > 0$, an optimal solution $\mathbf{x}^\star$ to problem (4) is also an optimal solution of the original problem (2).

Problem 6. (10 points)

(a) (5 points) Verify that $\mathbf{x} \in \mathbb{R}^n$, $y, z, \in \mathbb{R}$ satisfy

$$\mathbf{x}^T\mathbf{x} \leq yz,\ \ y \geq 0,\ \ z \geq 0$$

if and only if

$$\left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \leq y + z,\ \ y \geq 0,\ \ z \geq 0.$$

(b) (5 points) Reformulate the following problem as an SOCP by (a).

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^n} \quad & \left(\sum_{i=1}^m \frac{1}{\mathbf{a}_i^T\mathbf{x} - b_i} \right)^{-1} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \succeq \mathbf{b}, \end{aligned}$$

where $\mathbf{a}_i^T$ is the ith row of $\mathbf{A} \in \mathbb{R}^{m\times n}$ and b_i is the ith element of $\mathbf{b} \in \mathbb{R}^m$.

Solution:

(a) Since $y \geq 0$, $z \geq 0$,

$$\left\| \begin{matrix} 2\mathbf{x} \\ (y - z) \end{matrix} \right\|_2 \leq (y + z)$$

if and only if

$$\left\| \begin{matrix} 2\mathbf{x} \\ (y - z) \end{matrix} \right\|_2^2 \leq (y + z)^2,$$

that is

$$4\mathbf{x}^T\mathbf{x} + y^2 - 2yz + z^2 \leq y^2 + 2yz + z^2,$$

which is equivalent to

$$\mathbf{x}^T\mathbf{x} \leq yz.$$

(b) We first reformulate the original problem to its epigraph form as

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & \sum_{i=1}^m t_i \\ \text{s.t.} \quad & \frac{1}{\mathbf{a}_i^T\mathbf{x} - b_i} \leq t_i,\ \ i = 1, \ldots, m \\ & \mathbf{A}\mathbf{x} \succeq \mathbf{b}. \end{aligned}$$

According to (a), the above problem is equivalent to

$$\begin{aligned}\min_{\mathbf{x}\in\mathbb{R}^n} \quad & \sum_{i=1}^{m} t_i \\ \text{s.t.} \quad & \left\| \begin{matrix} 2 \\ t_i - (\mathbf{a}_i^T\mathbf{x} - b_i) \end{matrix} \right\|_2 \le t_i + \mathbf{a}_i^T\mathbf{x} - b_i, \ i = 1, \ldots, m, \\ & t_i \ge 0, \ i = 1, \ldots, m, \\ & \mathbf{A}\mathbf{x} \succeq \mathbf{b},\end{aligned}$$

which is an SOCP.

Problem 7. (10 points) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Formulate the following problems as LPs. Explain the relation between the optimal solutions of the original problem and reformulated problem.

(a) (5 points)

$$\min_{\mathbf{x}\in\mathbb{R}^n} \ \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_\infty.$$

(b) (5 points)

$$\min_{\mathbf{x}\in\mathbb{R}^n} \ \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1.$$

Solution:

(a) By epigraph reformulation, we can convert the original problem to

$$\begin{aligned}\min_{t,\mathbf{x}} \ & t \\ \text{s.t.} \ & \mathbf{A}\mathbf{x} - \mathbf{b} \preceq t\mathbf{1}_m \\ & \mathbf{A}\mathbf{x} - \mathbf{b} \succeq -t\mathbf{1}_m.\end{aligned}$$

It can be seen that for any fixed $\mathbf{x}$, by minimizing t subject to the constraints, we obtain

$$t = \max_{i=1,\ldots,m} |\mathbf{a}_i^T\mathbf{x} - b_i| = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_\infty.$$

Therefore, $\mathbf{x}^\star$ is optimal to the original problem if and only if $(\mathbf{x}, t) = (\mathbf{x}^\star, \|\mathbf{A}\mathbf{x}^\star - \mathbf{b}\|_\infty)$ is optimal to the reformulated problem.

(b) Again, by epigraph reformulation, we can convert the original problem to

$$\begin{aligned}\min_{\mathbf{s},\mathbf{x}} \ & \mathbf{1}_m^T\mathbf{s} \\ \text{s.t.} \ & \mathbf{A}\mathbf{x} - \mathbf{b} \preceq \mathbf{s} \\ & \mathbf{A}\mathbf{x} - \mathbf{b} \succeq -\mathbf{s}.\end{aligned}$$

It can be seen that for any fixed $\mathbf{x}$, by minimizing $\mathbf{1}_m^T\mathbf{s}$ subject to the constraints, we obtain

$$s_i = |\mathbf{a}_i^T\mathbf{x} - b_i| \ \Rightarrow \ \mathbf{1}_m^T\mathbf{s} = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1.$$

Therefore, $\mathbf{x}^\star$ is optimal to the original problem if and only if $(\mathbf{x}, \mathbf{s}) = (\mathbf{x}^\star, [|\mathbf{a}_1^T\mathbf{x}^\star - b_1|, \ldots, |\mathbf{a}_m^T\mathbf{x}^\star - b_m|]^T)$ is optimal to the reformulated problem.

Problem 8. (10 points)

(a) (5 points) Show that $\mathbf{X} = \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$ solves the SDP

$$\begin{aligned} \min \quad & \mathrm{Tr}(\mathbf{X}) \\ \text{s.t.} \quad & \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{X} \end{bmatrix} \succeq \mathbf{0}, \end{aligned}$$

with variable $\mathbf{X} \in \mathbb{S}^n$, where $\mathbf{A} \in \mathbb{S}^m_{++}$ and $\mathbf{B} \in \mathbb{R}^{m\times n}$ are given.

(b) (5 points) Show that $\mathrm{Tr}\left(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)$ is a convex function of $(\mathbf{A}, \mathbf{B})$.

Solution:

By Schur complement, the original problem is equivalent to

$$\begin{aligned} \min \quad & \mathrm{Tr}(\mathbf{X}) \\ \text{s.t.} \quad & \mathbf{X} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \succeq \mathbf{0}. \end{aligned}$$

Under the constraint, $\mathbf{X} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \succeq \mathbf{0}$, $(\mathbf{X} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}) = \sum_{i=1}^n \lambda_i \geq \mathbf{0}$, where λ_i are the eigenvalues of $\mathbf{X} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$. Therefore, $\mathrm{Tr}(\mathbf{X}) \geq \mathrm{Tr}(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B})$, and the equality holds when $\mathbf{X} = \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} = \mathbf{X}^\star$.
Consider the following two SDPs,

$$\min \quad \mathrm{Tr}(\theta\mathbf{X}_1 + (1-\theta)\mathbf{X}_2) \tag{5a}$$

$$\text{s.t.} \quad \begin{bmatrix} \theta\mathbf{A}_1 + (1-\theta)\mathbf{A}_2 & \theta\mathbf{B}_1 + (1-\theta)\mathbf{B}_2 \\ \theta\mathbf{B}_1^T + (1-\theta)\mathbf{B}_2^T & \theta\mathbf{X}_1 + (1-\theta)\mathbf{X}_2 \end{bmatrix} \succeq \mathbf{0}, \tag{5b}$$

and

$$\min \quad \mathrm{Tr}(\theta\mathbf{X}_1 + (1-\theta)\mathbf{X}_2) \tag{6a}$$

$$\text{s.t.} \quad \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{B}_1^T & \mathbf{X}_1 \end{bmatrix} \succeq \mathbf{0} \tag{6b}$$

$$\begin{bmatrix} \mathbf{A}_2 & \mathbf{B}_2 \\ \mathbf{B}_2^T & \mathbf{X}_2 \end{bmatrix} \succeq \mathbf{0}, \tag{6c}$$

where $\theta \in [0,1]$, $\mathbf{X}_1 \in \mathbb{S}^n$, $\mathbf{X}_2 \in \mathbb{S}^n$, $\mathbf{A}_1 \in \mathbb{S}^m_{++}$, $\mathbf{A}_2 \in \mathbb{S}^m_{++}$, $\mathbf{B}_1 \in \mathbb{R}^{m\times n}$, and $\mathbf{B}_2 \in \mathbb{R}^{m\times n}$.
Since the linear combination of (6b) and (6c) implies (5b), the feasible set of problem (6) is included in the feasible set of problem (5). Therefore, the optimal value of problem (5) is less than or equal to that of problem (6). That is,

$$\mathrm{Tr}\left((\theta\mathbf{B}_1 + (1-\theta)\mathbf{B}_2)^T(\theta\mathbf{A}_1 + (1-\theta)\mathbf{A}_2)^{-1}(\theta\mathbf{B}_1 + (1-\theta)\mathbf{B}_2)\right) \leq \theta\mathrm{Tr}\left(\mathbf{B}_1^T\mathbf{A}_1^{-1}\mathbf{B}_1\right) + (1-\theta)\mathrm{Tr}\left(\mathbf{B}_2^T\mathbf{A}_2^{-1}\mathbf{B}_2\right).$$

Problem 9. (15 points) Consider the following optimization problem:

$$\begin{aligned} \min_{x\in\mathbb{R}, y\in\mathbb{R}} \quad & f_0(x,y) \triangleq (x-2)^2 + (y-3)^2 \\ \text{s.t.} \quad & x \geq 0 \\ & x \leq 2 \\ & y \geq 0 \\ & y \leq 3 \\ & 2x + y \leq 5. \end{aligned}$$

(a) (5 points) Show that problem is a convex optimization problem.

(b) (10 points) Find an optimal solution, $(x^\star, y^\star)$. (You need to write the details of derivations.)

Solution:

(a) Because $\nabla^2 f_0(x,y) = 2\mathbf{I} \succ \mathbf{0}$, f_0 is a strictly convex function. All the inequality constraints linear inequalities. Therefore, this problem is a convex QP. The feasible set C is shown in Figure 1 (the shaded region).

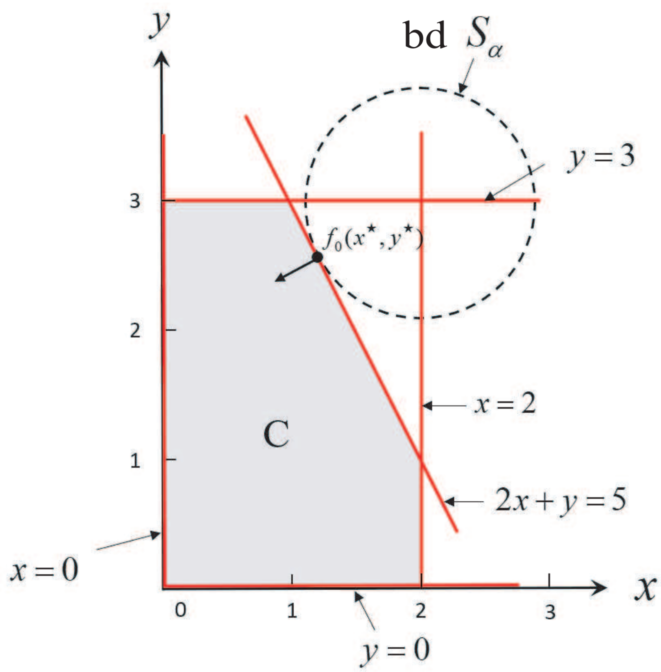

Figure 1

(b) The gradient of the objective function can be easily obtained as

$$\nabla f_0(x,y) = (2x-4, 2y-6).$$

According to the first-order optimality criterion, $(x^\star, y^\star)$ is optimal if and only if

$$\nabla f_0(x^\star, y^\star)^T (x - x^\star, y - y^\star) \geq 0, \text{ for all } (x,y) \in C. \tag{8}$$

Since $\nabla f_0(x,y) = \mathbf{0}$ holds true only for $(x,y) = (2,3) \notin C$, the optimum solution $(x^\star, y^\star) \in$ **bd** C. Note that each sublevel set $S_\alpha = \{(x,y) \mid f_0(x,y) \leq \alpha\}$ is closed with smooth boundary for all $\alpha > 0$ since $f_0(x,y)$ is differentiable. Suppose that $(x^\star, y^\star)$ is located on the boundary line segment between $(1,3)$ and $(2,1)$. Then $2x^\star + y^\star = 5$. Furthermore,

$$C \subset H_-(x^\star, y^\star) = \{(x,y) \mid h(x,y) = 2(x - x^\star) + (y - y^\star) \leq 0\}, \tag{9}$$

namely, $H(x^\star, y^\star) = \{(x,y) \mid h(x,y) = 0\}$ is a supporting hyperplane of C at $(x^\star, y^\star)$ with the normal vector $[2,1]^T$ (cf. Case 2 in Figure 4.2 of the textbook). Thus $\nabla f_0(x^\star, y^\star)$ must be parallel with $[2, 1]^T$ or perpendicular to the vector $[1, -2]^T$. Then we come up with the following two linear equations in $(x^\star, y^\star)$.

$$2x^\star + y^\star = 5$$
$$(2x^\star - 4) - 2(2y^\star - 6) = 0$$

Hence, the optimum solution can be obtained as $(x^\star, y^\star) = (6/5, 13/5)$, which is also the unique solution since f_0 is strictly convex. The $\textbf{bd}\ S_\alpha$ for $\alpha = f_0(x^\star, y^\star) = 4/5$ is shown in Figure 1.