Convex Optimization for Communications and Signal Processing **Fall 2020**
Institute of Communications Engineering
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Homework #1 – Solution

Coverage: chapter 1–2

Due date: 28 October, 2020

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Problem 1. (5 points) Let $\mathcal{F} = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_1^2 \le x_2\}$. Is this a convex set? Justify your answer.

Solution:

Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathcal{F}$ and we have $x_1^2 \le x_2$ and $y_1^2 \le y_2$. Letting $\theta \in [0, 1]$ we have

$$(z_1, z_2) = \mathbf{z} = \theta\mathbf{x} + (1-\theta)\mathbf{y} = (\theta x_1 + (1-\theta)y_1, \theta x_2 + (1-\theta)y_2)\,.$$

Then,

$$\begin{aligned}
z_2^2 &= (\theta x_2 + (1-\theta)y_2)^2 \\
&= \theta^2 x_2^2 + 2\theta(1-\theta)x_2 y_2 + (1-\theta)^2 y_2^2 \\
&\le \theta^2 x_2^2 + \theta(1-\theta)(x_2^2 + y_2^2) + (1-\theta)^2 y_2^2 \qquad (\text{since } \ 2\mathrm{x}_2\mathrm{y}_2 \le \mathrm{x}_2^2 + \mathrm{y}_2^2) \\
&\le \theta^2 x_1 + \theta(1-\theta)(x_1 + y_1) + (1-\theta)^2 y_1 \qquad (\text{by definition of } \mathcal{F}) \\
&= \theta x_1 + (1-\theta)y_1 \\
&= z_1,
\end{aligned}$$

and hence $\mathbf{z} \in \mathcal{F}$ and this is a convex set.

Problem 2. (5 points) Let K_i for $i = 1, \ldots, n$ be cones.

(a) (2 points) Is $\mathcal{K}_1 \triangleq \cap_{i=1}^n K_i$ a cone? Justify your answer.

(b) (3 points) Let $\mathcal{K}_2 \triangleq \{\mathbf{x} = \sum_{i=1}^n \mathbf{x}_i | \mathbf{x}_i \in K_i\}$. Is $\mathcal{K}_2$ a cone? Justify your answer.

Solution:

(a) Let $\mathbf{x} \in \mathcal{K}_1$ implying that $\mathbf{x} \in K_i$ for all i. This implies $\theta\mathbf{x} \in K_i$ for all i where $\theta \in \mathbb{R}_+$, since K_i is a cone for all i. Clearly, $\theta\mathbf{x} \in \mathcal{K}_1$. Hence, $\mathcal{K}_1$ is a cone.

(b) Let $\mathbf{x} \in \mathcal{K}_2$. We have $\mathbf{x} = \sum_{i=1}^n \mathbf{x}_i$ where $\mathbf{x}_i \in K_i$. Since K_i for all i is a cone, we have $\theta\mathbf{x}_i \in K_i$ for all i where $\theta \in \mathbb{R}_+$. Clearly, $\theta\mathbf{x} \in \mathcal{K}_2$ and therefore $\mathcal{K}_2$ is a cone.

Problem 3. (10 points)

(a) (3 points) Represent the set $\left\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}_+^2 \,\middle|\, x_1 x_2 \ge 1\right\}$, as the intersection of some family of halfspaces.

(b) (3 points) Suppose that C and D are disjoint subsets of $\mathbb{R}^n$. Show that the set,

$$A = \left\{(\mathbf{a}, b) \in \mathbb{R}^{n+1} \,|\, \mathbf{a}^T\mathbf{x} \le b \,\forall \mathbf{x} \in C,\ \ \mathbf{a}^T\mathbf{x} \ge b,\ \forall \mathbf{x} \in D\right\},$$

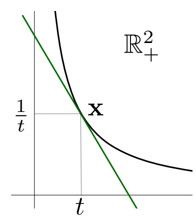
is a **convex cone**.

(c) (4 points) Consider two solid convex cones K_1 and K_2. Show that if $\mathbf{int}\, K_1 \cap \mathbf{int}\, K_2 = \emptyset$, then there is $\mathbf{y} \neq \mathbf{0}$ such that, $\mathbf{y} \in K_1^*$, $-\mathbf{y} \in K_2^*$.

Solution:

(a) The set is the intersection of all supporting halfspaces at points on its boundary, which is given by $\{\mathbf{x} \in \mathbb{R}^2_+ \,|\, x_1 x_2 = 1\}$. Using the basic calculus, the supporting hyperplane at the point $\mathbf{x} = (t, 1/t)$ is given by

$$x_1/t^2 + x_2 = 2/t,$$



so we can express the set as

$$\bigcap_{t>0} \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1/t^2 + x_2 \geq 2/t \right\}.$$

(b) The conditions $\mathbf{a}^T\mathbf{x} \leq b$, $\forall \mathbf{x} \in C$ and $\mathbf{a}^T\mathbf{x} \geq b$, $\forall \mathbf{x} \in D$ form a set of homogeneous linear inequalities in $(\mathbf{a}, b)$. Therefore A is the intersection of halfspaces that pass through the origin which implies a convex cone.

(c) Let $\mathbf{y} \neq \mathbf{0}$ be the normal vector of a separating hyperplane, which separates the interiors: $\mathbf{y}^T\mathbf{x} \geq \alpha$ for all $\mathbf{x} \in \mathbf{int}\, K_1$ and $\mathbf{y}^T\mathbf{x} \leq \alpha$ for all $\mathbf{x} \in \mathbf{int}\, K_2$. We must have $\alpha = 0$ because K_1 and K_2 are cones. So,

$$\mathbf{y} \in (\mathbf{int}\, K_1 \cup \{\mathbf{0}\})^* = K_1^*, \quad -\mathbf{y} \in (\mathbf{int}\, K_2 \cup \{\mathbf{0}\})^* = K_2^*.$$

Problem 4. (10 points) Consider the set of points, $X \triangleq \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and point $\mathbf{q}$, all in $\mathbb{R}^d$. Let $\mathcal{H}(\mathbf{q} - \mathbf{x}_i, \mathbf{x}_i)$ be the hyperplane that contains $\mathbf{x}_i$ and is perpendicular to the line segment between $\mathbf{q}$ and $\mathbf{x}_i$. Define $H_\mathbf{q}(\mathbf{x}_i)$ as the halfspace that does not contain the point $\mathbf{q} \in \mathbb{R}^d$ and bounded by hyperplane $\mathcal{H}(\mathbf{q} - \mathbf{x}_i, \mathbf{x}_i)$. Show that

$$\bigcap_{i=1}^{n} H_\mathbf{q}(\mathbf{x}_i) = \emptyset \quad \Leftrightarrow \quad \mathbf{q} \in \mathbf{conv}\, X.$$

Solution:

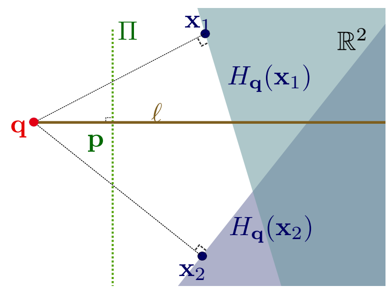
We prove both directions with contradiction.

- ($\Rightarrow$) Suppose that $\cap_{i=1}^n H_\mathbf{q}(\mathbf{x}_i) = \emptyset$ and $\mathbf{q} \notin \mathbf{conv}\, X$. Then there exists a hyperplane Π that separates $\mathbf{q}$ from $\mathbf{conv}\, X$. Let ℓ be the ray with tip $\mathbf{q}$, perpendicular to Π, and that intersects Π at the point $\mathbf{p} \neq \mathbf{q}$.

  Note that for each $\mathbf{x}_i$, the hyperplane $\mathcal{H}(\mathbf{q} - \mathbf{x}_i, \mathbf{x}_i)$ is not parallel to ℓ (by definition of hyperplane Π and the ray ℓ). Therefore, for each $\mathbf{x}_i$, there is a $\theta_i \in \mathbb{R}_{++}$ such that,

  $$\mathcal{L}_{\theta_i} \triangleq \left\{ \mathbf{q} + \theta(\mathbf{p} - \mathbf{q}) \,\middle|\, \theta \geq \theta_i,\, \theta \in \mathbb{R}_{++} \right\} \subset H_\mathbf{q}(\mathbf{x}_i).$$

  This implies that $\bigcap_{i=1}^n H_\mathbf{q}(\mathbf{x}_i)$ is nonempty which is a contradiction.



The above figure, illustrates the simplified case when $\mathbf{q} \notin \mathbf{conv}\, X$, $(X = \{\mathbf{x}_1, \mathbf{x}_2\})$ in $\mathbb{R}^2$.

- ($\Leftarrow$) Suppose that $\mathbf{q} \in \mathbf{conv}\, X$. Then there exists $\alpha_1, \ldots, \alpha_n$, such that every $\alpha_i \geq 0$,

$$\sum_{i=1}^{n} \alpha_i = 1 \quad \& \quad \mathbf{q} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i. \tag{1}$$

Note that for every $\mathbf{y} \in H_{\mathbf{q}}(\mathbf{x}_i)$,

$$(\mathbf{y} - \mathbf{q})^T (\mathbf{x}_i - \mathbf{q}) \geq |\mathbf{q} - \mathbf{x}_i|^2 > 0. \tag{2}$$

Suppose to the contrary that there exists a point $\mathbf{z} \in \bigcap_{i=1}^{n} H_{\mathbf{q}}(\mathbf{x}_i)$. Hence,

$$\begin{aligned}
\mathbf{0} &= (\mathbf{z} - \mathbf{q})^T (\mathbf{q} - \mathbf{q}) \\
&\underbrace{=}_{Eq.(1)} (\mathbf{z} - \mathbf{q})^T \Big(\sum_{i=1}^{n} \alpha_i (\mathbf{x}_i - \mathbf{q})\Big) \\
&= \sum_{i=1}^{n} \alpha_i (\mathbf{z} - \mathbf{q})^T (\mathbf{x}_i - \mathbf{q}),
\end{aligned}$$

but based on (1), at least one of the α_i is greater than zero and

$$\mathbf{0} = \sum_{i=1}^{n} \alpha_i (\mathbf{z} - \mathbf{q})^T (\mathbf{x}_i - \mathbf{q}) \underbrace{>}_{Eq.(1),(2)} \mathbf{0}$$

which is a contradiction.

Problem 5. (20 points) Which of the following sets S are polyhedra? If possible, express S in the form $S = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \preceq \mathbf{b}, \mathbf{F}\mathbf{x} = \mathbf{g}\}$.

(a) $S = \{y_1\mathbf{a}_1 + y_2\mathbf{a}_2 \mid -1 \leq y_1 \leq 1, \ -1 \leq y_2 \leq 1\}$, where $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^n$ are linearly independent.

(b) $S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \succeq \mathbf{0}, \ \mathbf{1}^T\mathbf{x} = 1, \ \sum_{i=1}^{n} x_i a_i = b_1, \ \sum_{i=1}^{n} x_i a_i^2 = b_2\}$, where $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.

(c) $S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \succeq \mathbf{0}, \ \mathbf{x}^T\mathbf{y} \leq 1$ for all $\mathbf{y}$ with $\|\mathbf{y}\|_2 = 1\}$.

(d) $S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \succeq \mathbf{0}, \ \mathbf{x}^T\mathbf{y} \leq 1$ for all $\mathbf{y}$ with $\sum_{i=1}^{n} |y_i| = 1\}$.

Solution:

(a) $\mathbf{x} = y_1\mathbf{a}_1 + y_2\mathbf{a}_2 \in \text{span}[\mathbf{a}_1, \mathbf{a}_2]$. Then for the basis $\mathbf{v}_k$ in the null space of matrix $[\mathbf{a}_1, \mathbf{a}_2]$, $k = 1, \ldots, n-2$, we have $\mathbf{v}_k^T\mathbf{x} = 0$. Choose $\mathbf{c}_1$, $\mathbf{c}_2$ which are orthogonal to $\mathbf{a}_2$ and $\mathbf{a}_1$, respectively, and lie in the subspace $\text{span}[\mathbf{a}_1, \mathbf{a}_2]$, and then we have

$$\begin{aligned}
-|\mathbf{c}_1^T\mathbf{a}_1| \leq \mathbf{c}_1^T\mathbf{x} \leq |\mathbf{c}_1^T\mathbf{a}_1|, \ (-1 \leq y_1 \leq 1) \\
-|\mathbf{c}_2^T\mathbf{a}_2| \leq \mathbf{c}_2^T\mathbf{x} \leq |\mathbf{c}_2^T\mathbf{a}_2|, \ (-1 \leq y_2 \leq 1)
\end{aligned}$$

so S is a polyhedron.

(b) Choose

$$\mathbf{A} = -\mathbf{I},\ \mathbf{b} = \mathbf{0},\ \mathbf{F} = \begin{bmatrix} \mathbf{1}^T \\ [a_1, a_2, \ldots, a_n] \\ [a_1^2, a_2^2, \ldots, a_n^2] \end{bmatrix}, \text{and } \mathbf{g} = \begin{bmatrix} 1 \\ b_1 \\ b_2 \end{bmatrix}.$$

Then S is a polyhedron.

(c) Since $\mathbf{x}$ lies in the intersection of unit ball and non-negative orthant, it can not be described by finite linear inequalities and equalities. Hence, S is not a polyhedron.

(d)

$$\begin{aligned} \mathbf{x}^T\mathbf{y} &\le \|\mathbf{x}\|_\infty \|\mathbf{y}\|_1 \le 1 \\ \Rightarrow \|\mathbf{x}\|_\infty &\le 1 \Rightarrow -\mathbf{1} \preceq \mathbf{x} \preceq \mathbf{1} \end{aligned}$$

and we have the constraint $\mathbf{x} \succeq \mathbf{0}$

$$\begin{aligned} &\Rightarrow \mathbf{0} \preceq \mathbf{x} \preceq \mathbf{1} \\ &\Rightarrow \{\mathbf{x} \mid \mathbf{x} \preceq \mathbf{1},\ \ -\mathbf{x} \preceq \mathbf{0}\} \\ &\Rightarrow S \text{ is a polyhedron.} \end{aligned}$$

Problem 6. (10 points) Consider the sets $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C} \subseteq \mathbb{R}^n$, where $\mathcal{A}$ is a closed set. Assume that $\mathcal{H}$ is a supporting hyperplane of $\mathcal{A}$ and $\mathcal{C}$. Prove $(\mathbf{bd}\ \mathcal{A}) \cap \mathcal{H} \subseteq \mathbf{bd}\ \mathcal{B}$.

Solution:

Let us prove by contradiction. Assume that there exists $\boldsymbol{x} \in (\mathbf{bd}\ \mathcal{A}) \cap \mathcal{H}$, but $\boldsymbol{x} \notin \mathbf{bd}\, \mathcal{B}$. Then

$$\mathbf{x} \in (\mathbf{bd}\ \mathcal{A}) \cap \mathcal{H} \subseteq \mathbf{bd}\ \mathcal{A} \subseteq \mathcal{A} \subseteq \mathcal{B}, \tag{3}$$

where the inequality "$\mathbf{bd}\ \mathcal{A} \subseteq \mathcal{A}$" is due to the fact that $\mathcal{A}$ is closed. By (3) and $\mathbf{x} \notin \mathbf{bd}\ \mathcal{B}$, we see that $\mathbf{x} \in \mathbf{int}\, \mathcal{B}$, which, together with the assumption of $\mathcal{B} \subseteq \mathcal{C}$, yields that $\mathbf{x} \in \mathbf{int}\ \mathcal{C}$. Hence, $\mathbf{x} \in \mathbf{int}\ \mathcal{C} \cap \mathcal{H}$, implying that $\mathcal{H}$ cannot be a supporting hyperplane of $\mathcal{C}$ which is a clear contradiction.

Problem 7. (10 points) A set C is *midpoint convex* if whenever two points a, b are in C, the average or midpoint $(a+b)/2$ is in C. Obviously a convex set is midpoint convex. Prove that if C is closed and midpoint convex, then C is convex.

Solution:

Let $\mathbf{x}, \mathbf{y} \in C$, and $\theta \in [0, 1]$. We prove $\theta\mathbf{x} + (1-\theta)\mathbf{y}$ through the following recursion:

$k = 1$:

By midpoint convexity, we have

$$\begin{aligned} \mathbf{z}_1^{(1)} &= \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} \in C \\ \Rightarrow \mathbf{z}_1^{(1)} &= \theta_1^{(1)}\mathbf{x} + (1-\theta_1^{(1)})\mathbf{y} \in C, \end{aligned}$$

where $\theta_1^{(1)} = 1/2$, and a 3-point sequence $\{\mathbf{x}, \mathbf{z}_1^{(1)},\ \mathbf{y}\}$ on the line segment between $\mathbf{x}$ and $\mathbf{y}$.

$k = 2$:

By midpoint convexity, we can have two more midpoints between each pair of consecutive points as follows

$$\begin{aligned} &\begin{cases} \mathbf{z}_1^{(2)} = \frac{1}{2}(\mathbf{x} + \mathbf{z}_1^{(1)}) \in C \\ \mathbf{z}_2^{(2)} = \frac{1}{2}(\mathbf{z}_1^{(1)} + \mathbf{y}) \in C \end{cases} \\ \Rightarrow &\begin{cases} \mathbf{z}_1^{(2)} = (2^{-1} + 2^{-2})\mathbf{x} + 2^{-2}\mathbf{y} \in C \\ \mathbf{z}_2^{(2)} = 2^{-2}\mathbf{x} + (2^{-1} + 2^{-2})\mathbf{y} \in C \end{cases} \\ \Rightarrow &\ \mathbf{z}_i^{(2)} = \theta_i^{(2)}\mathbf{x} + (1-\theta_i^{(2)})\mathbf{y} \in C, \end{aligned}$$

where $\theta_i^{(2)} = c_1 2^{-1} + 2^{-2}$, $i = 1, 2$, and $c_1 \in \{0, 1\}$, along with a 5-point sequence $\{\mathbf{x}, \mathbf{z}_1^{(2)}, \mathbf{z}_1^{(1)}, \mathbf{z}_2^{(2)}, \mathbf{y}\}$ on the line segment between $\mathbf{x}$ and $\mathbf{y}$. .

$k = 3$:

By midpoint convexity, we can obtain a 9-point sequence $\{\, \mathbf{x}, \mathbf{z}_1^{(3)}, \mathbf{z}_1^{(2)}, \mathbf{z}_2^{(3)}, \mathbf{z}_1^{(1)}, \mathbf{z}_3^{(3)}, \mathbf{z}_2^{(2)}, \mathbf{z}_4^{(3)}, \mathbf{y}\}$ on the line segment between $\mathbf{x}$ and $\mathbf{y}$, where

$$\begin{cases} \mathbf{z}_1^{(3)} = \frac{1}{2}(\mathbf{x} + \mathbf{z}_1^{(2)}) \in C \\ \mathbf{z}_2^{(3)} = \frac{1}{2}(\mathbf{z}_1^{(2)} + \mathbf{z}_1^{(1)}) \in C \\ \mathbf{z}_3^{(3)} = \frac{1}{2}(\mathbf{z}_1^{(1)} + \mathbf{z}_2^{(2)}) \in C \\ \mathbf{z}_4^{(3)} = \frac{1}{2}(\mathbf{z}_2^{(2)} + \mathbf{y}) \in C \end{cases}$$

$$\Rightarrow \begin{cases} \mathbf{z}_1^{(3)} = (2^{-1} + 2^{-2} + 2^{-3})\mathbf{x} + 2^{-3}\mathbf{y} \in C \\ \mathbf{z}_2^{(3)} = (2^{-1} + 2^{-3})\mathbf{x} + (2^{-2} + 2^{-3})\mathbf{y} \in C \\ \mathbf{z}_3^{(3)} = (2^{-2} + 2^{-3})\mathbf{x} + (2^{-1} + 2^{-3})\mathbf{y} \in C \\ \mathbf{z}_4^{(3)} = 2^{-3}\mathbf{x} + (2^{-1} + 2^{-2} + 2^{-3})\mathbf{y} \in C \end{cases}$$

$$\Rightarrow \mathbf{z}_i^{(3)} = \theta_i^{(3)}\mathbf{x} + (1 - \theta_i^{(3)})\mathbf{y} \in C,$$

where $\theta_i^{(3)} = c_1 2^{-1} + c_2 2^{-2} + 2^{-3}$, $i = 1, 2, 3, 4$, and $c_1, c_2 \in \{0, 1\}$.

For the recursion k, we come up with

$$\mathbf{z}_i^{(k)} = \theta_i^{(k)}\mathbf{x} + (1 - \theta_i^{(k)})\mathbf{y} \in C,\ i = 1, \ldots, 2^{k-1},$$

where the k-bit binary number $\theta_i^{(k)} = c_1 2^{-1} + c_2 2^{-2} + \cdots + c_{k-1} 2^{-(k-1)} + 2^{-k}$, $\forall i$, and $c_j \in \{0, 1\}$, $j = 1, \ldots, k-1$.
Since C is closed, there exists an unique i such that $\mathbf{z}_i^{(k)}$ converges to $\theta\mathbf{x} + (1-\theta)\mathbf{y}$ when $k \to \infty$, thereby yielding

$$\lim_{k\to\infty} \mathbf{z}_i^{(k)} = \lim_{k\to\infty} [\theta_i^{(k)}\mathbf{x} + (1 - \theta_i^{(k)})\mathbf{y}] = \theta\mathbf{x} + (1-\theta)\mathbf{y} \in C.$$

Therefore, C is convex.

Problem 8. (15 points) Define the *monotone nonnegative cone* as

$$K = \{\mathbf{x} = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}.$$

(a) Show that K is a proper cone.

(b) Find the dual cone K^*. *Hint.* Use the identity

$$\sum_{i=1}^{n} x_i y_i = (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)(y_1 + y_2 + y_3) \\ + \cdots + (x_{n-1} - x_n)(y_1 + \cdots + y_{n-1}) + x_n(y_1 + \cdots + y_n).$$

Solution:

(a) The set K is defined as n homogeneous linear inequalities, hence it is a closed (polyhedral) cone.
The interior of K is nonempty, because there are points that satisfy the inequalities with strict inequality, for example, $\mathbf{x} = (n, n-1, \ldots, 1)$.
To show that K is pointed, we note that if $\mathbf{x} \in K$, then $-\mathbf{x} \in K$ only if $\mathbf{x} = \mathbf{0}$. This implies that the cone does not contain an entire line.
To summarize the above, we prove that the set K satisfies the four conditions of proper cone.

(b) According to the definition, the dual cone of K is defined as

$$K^* = \{\mathbf{y} \mid \mathbf{y}^T\mathbf{x} \geq 0, \forall \mathbf{x} \in K\}.$$

Let $\mathbf{y} = [y_1, y_2, \ldots, y_n]$, we can prove that $\mathbf{y}^T\mathbf{x} \geq 0$ for all $\mathbf{x} \in K$ if and only if $\sum_{i=1}^k y_i \geq 0$ for all $k \in \{1, \ldots, n\}$. The following is the prove.

- ($\Rightarrow$) Let us prove this direction by contradiction. Suppose $\mathbf{y}^T\mathbf{x} \geq 0$ for all $\mathbf{x} \in K$ and there exist a number $m \in \{1, \ldots, n\}$, such that $\sum_{i=1}^m y_i < 0$. Then we can see that

$$\begin{aligned}
\mathbf{y}^T\mathbf{x} &= (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + \cdots + (x_{n-1} - x_n)(y_1 + \cdots + y_{n-1}) + x_n(y_1 + \cdots + y_n) \\
&= x_1[(\frac{x_1 - x_2}{x_1})y_1 + (\frac{x_2 - x_3}{x_1})(y_1 + y_2) + \cdots + (\frac{x_n}{x_1})(y_1 + \cdots + y_n)] \ (x_1 \neq 0) \\
&= x_1[\theta_1 y_1 + \theta_2(y_1 + y_2) + \cdots + \theta_n(y_1 + \cdots + y_n)] \ (\text{where } \theta_i \triangleq \frac{x_i - x_{i+1}}{x_1}) \\
&< 0 \ (\text{if } \theta_i = 0 \text{ for } i \in \{1, \ldots, n\} \setminus \{m\}),
\end{aligned}$$

  which is a contradiction to the assumption. Hence we conclude that if $\mathbf{y}^T\mathbf{x} \geq 0$ for all $\mathbf{x} \in K$, then $\sum_{i=1}^k y_i \geq 0$ for all $k \in \{1, \ldots, n\}$.

- ($\Leftarrow$) Since the vector $\mathbf{z} = [x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n, x_n] \succeq \mathbf{0}$ for all $\mathbf{x} \in K$ and $\sum_{i=1}^k y_i \geq 0$ for all $k \in \{1, \ldots, n\}$, we can easily known that $\mathbf{y}^T\mathbf{x} \geq 0$ for all $\mathbf{x} \in K$ by the hint.

Therefore,

$$K^* = \{\mathbf{y} \mid \sum_{i=1}^k y_i \geq 0, k = 1, \ldots, n\}.$$

Problem 9. (15 points) Let C be a set in $\mathbb{R}^n$. For any $\mathbf{x} \in \mathbf{conv}\ C$, prove that it can be represented as

$$\mathbf{x} = \sum_{i=1}^d \theta_i \mathbf{x}_i, \quad \theta_i \in [0, 1], \ \sum_{i=1}^d \theta_i = 1,$$

where $\mathbf{x}_i \in C$ such that $d \leq n + 1$.

Solution:

Let $\mathbf{x} \in \mathbf{conv}\ C$ and we have

$$\mathbf{x} = \sum_{i=1}^d \theta_i \mathbf{x}_i, \quad \mathbf{x}_i \in C \ \forall i, \ \theta_i \in [0, 1], \ \sum_{i=1}^d \theta_i = 1. \tag{4}$$

Assume d is the *minimum number of elements* of C for which $\mathbf{x}$ can be expressed by (4) and this implies θ_i is nonzero for all $i = 1, \ldots, d$. Suppose by contradiction $d > n + 1$. Then, the vectors $\mathbf{x}_i - \mathbf{x}_1$ for $i = 2, \ldots, d$ are linearly dependent. Hence, there exist real scalars $\alpha_2, \ldots, \alpha_d$ (not all zero) such that

$$\sum_{i=2}^d \alpha_i(\mathbf{x}_i - \mathbf{x}_1) = 0.$$

Let $\beta_1 \triangleq -\sum_{i=2}^d \alpha_i$ and $\beta_i \triangleq \alpha_i$ for $i = 2, \ldots, d$ (not all zero). Clearly, we have

$$\sum_{i=1}^d \beta_i \mathbf{x}_i = 0, \quad \sum_{i=1}^d \beta_i = 0, \quad \text{and at least one } \beta_i > 0.$$

Letting $\theta'_i \triangleq \theta_i - \hat{\mu}\beta_i$ for $i = 1, \ldots, d$ where $\hat{\mu} = \min_{i=1,\ldots,d}\{\theta_i/\beta_i \mid \beta_i > 0\}$. This implies $\theta_i - \hat{\mu}\beta_i \geq 0$, $\forall i$ and $\sum_{i=1}^{d} \theta'_i = 1$. Then we have

$$\sum_{i=1}^{d} \theta'_i \mathbf{x}_i = \sum_{i=1}^{d} (\theta_i - \hat{\mu}\beta_i)\mathbf{x}_i = \sum_{i=1}^{d} \theta_i \mathbf{x}_i - \hat{\mu} \sum_{i=1}^{d} \beta_i \mathbf{x}_i = \mathbf{x}.$$

Since at least one of θ'_i is zero, $\mathbf{x}$ is written as a convex combination of fewer than d elements of C which is a clear contradiction with the initial assumption of d and thereby the proof is completed.

Note:

This question actually refers to a well-known theory in convex geometry. *Carathéodory's theorem* states that: *if $C \subset \mathbb{R}^n$, then every point in* **conv** *C can be written as a convex combination of at the most of $n+1$ elements of C.*

Problem 10. (10 points) Suppose that C and D are closed convex cones in $\mathbb{R}^n$, and C^* and D^* are the associated dual cones. Show that

$$(C \cap D)^* = C^* + D^*.$$

Solution:

We first show that $C^* + D^* \subseteq (C \cap D)^*$. Suppose that $\mathbf{x} \in C^* + D^*$, i.e., there exist some $\mathbf{u} \in C^*$, $\mathbf{v} \in D^*$ such that $\mathbf{x} = \mathbf{u} + \mathbf{v}$. Then, for any $\mathbf{y} \in (C \cap D)$, we have $\mathbf{u}^T\mathbf{y} \geq 0$, $\mathbf{v}^T\mathbf{y} \geq 0$, and hence,

$$\mathbf{x}^T\mathbf{y} = \mathbf{u}^T\mathbf{y} + \mathbf{v}^T\mathbf{y} \geq 0,$$

which implies that $\mathbf{x} \in (C \cap D)^*$. Therefore, $C^* + D^* \subseteq (C \cap D)^*$.

Next, we show that $(C \cap D)^* \subseteq C^* + D^*$. Note that C^* and D^* are closed convex cones, and it can be proved by definition that $C^* + D^*$ is the conic hull of the closed set $C^* \cup D^*$. Hence, $C^* + D^*$ is also a closed convex cone, and $(C^* + D^*)^{**} = C^* + D^*$. By the property

$$C \subseteq D \Rightarrow D^* \subseteq C^*, \tag{5}$$

we have that

$$\begin{aligned} (C^* + D^*)^* &\subseteq (C \cap D) & (6)\\ \Rightarrow (C \cap D)^* &\subseteq C^* + D^* & (7) \end{aligned}$$

Therefore, it is sufficient to show (6). Since C and D are closed convex cones, we have $(C^*)^* = C$ and $(D^*)^* = D$. By the fact $\mathbf{0} \in C^*$, $\mathbf{0} \in D^*$, and by (5), it can be easily verified that

$$\begin{aligned} (C^* + D^*)^* &\subseteq (C^* + \{\mathbf{0}\})^* = C\\ (C^* + D^*)^* &\subseteq (D^* + \{\mathbf{0}\})^* = D. \end{aligned}$$

Therefore, we have proven (6), and thus (7) is true. The proof has been completed.