COM 5232 Detection and Estimation Theory

Final Exam

June 7, 2022
13:20 $\sim$ 15:10

Note: There are 5 problems with total 100 points within 2 pages, please write your answer with detail in the answer sheet.

No credit without detail. No calculator. Closed books.

1. (20%)The data $\{x[0], x[1], \ldots, x[N-1]\}$ are observed where the $x[n]$'s are independent and identically distributed (IID) as $\mathcal{N}(0, \sigma^2)$. We wish to estimate the variance σ^2 as

$$\widehat{\sigma^2} = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n].$$

   (a) (10°%)Is this an unbiased estimator? (You need to show your derivation.)

   (b) (10%)Find the variance of $\widehat{\sigma^2}$.

   Hint : For a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, $\mathbb{E}\left[(X-\mu)^p\right] = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \sigma^4 \prod_{i=1}^{\frac{p}{2}} (2i-1) & \text{if } p \text{ is even} \end{cases}$

2. (20%)If $x[n] = A + w[n]$ for $n = 0, 1, \ldots, N-1$ are observed and $\mathbf{w} = [w[0]\ w[1]\ \ldots\ w[N-1]]^T \sim \mathcal{N}(0, \mathbf{C})$, find the CRLB for A.

3. (25%)We wish to estimate the amplitude of exponentials in noise. The observed data are

$$x[n] = \sum_{i=1}^{3} A_i r_i^n + w[n], \qquad n = 0, 1, 2, 3$$

   where $w[n]$ is WGN with variance σ^2 and $r_1 = 1, r_2 = -1, r_3 = 2$. Find the MVU estimator of the amplitudes.

   Hint : For an invertible matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, its inverse is given by

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

4. (15%)For the general linear model

$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{s} + \mathbf{w}$$

   where $\mathbf{s}$ is a known $N \times 1$ vector and $\mathbb{E}[\mathbf{w}] = \mathbf{0}, \mathbb{E}[\mathbf{w}\mathbf{w}^T] = \mathbf{C}$, find the BLUE of $\boldsymbol{\theta}$.

5. (20%)For the general LSE

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x},$$

(a) (10%)Find the PDF of the LSE if it is known that $\mathbf{x} \sim \mathcal{N}(\mathbf{H}\boldsymbol{\theta}, \sigma^2\mathbf{I})$.

(b) (10%)Is this an unbiased estimator?
