COM 5232 Detection and Estimation Theory

Midterm Exam

April 19, 2022
13:20 $\sim$ 15:10

Note: There are 5 problems with total 100 points within 2 pages, please write your answer with detail in the answer sheet.

No credit without detail. No calculator. Closed books.

1. (20%) Find the matrix prewhitener $\mathbf{D}$ for the covariance matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$

2. (20%) We wish to detect a damped exponential $s[n] = Ar^n$, where A is unknown and r is known ($0 < r < 1$), in WGN with known variance σ^2. Under $\mathcal{H}_0$, $x[n] = w[n]$ and under $\mathcal{H}_1$, $x[n] = Ar^n + w[n]$ for $n = 0, 1, \ldots, N-1$. Show that the GLRT decides $\mathcal{H}_1$ if $\hat{A}^2 > \gamma'$, where $\hat{A}$ is the MLE of A, and equals to

$$\hat{A} = \frac{\sum_{n=0}^{N-1} x[n] r^n}{\sum_{n=0}^{N-1} r^{2n}}.$$

3. (20%) Consider the detection of a signal $s[n]$ embedded in WGN with variance σ^2 based on the observed samples $x[n],\ n = 0, 1, \ldots, 2N-1$. The signal is given by

$$s[n] = \begin{cases} -A & n = 0, 1, \ldots, N-1 \\ 0 & n = N, N+1, \ldots, 2N-1 \end{cases}$$

under $\mathcal{H}_0$; and by

$$s[n] = \begin{cases} A & n = 0, 1, \ldots, N-1 \\ 2A & n = N, N+1, \ldots, 2N-1 \end{cases}$$

under $\mathcal{H}_1$ and assume $A > 0$. Find the NP detector to the simplest form.

4. (20%) We wish to detect a random DC level A embedded in WGN with variance σ^2. Under $\mathcal{H}_0$, $x[n] = w[n]$ and under $\mathcal{H}_1$, $x[n] = A + w[n]$ for $n = 0, 1, \ldots, N-1$, where $A \sim \mathcal{N}(0, \sigma_A^2)$. Find the NP detector to the simplest form.

Hint : You can directly use the MMSE estimator of the signal, i.e.

$$\hat{\mathbf{s}} = \frac{\sigma_A^2}{\sigma_A^2 + \frac{\sigma^2}{N}} \bar{x} \mathbf{1}$$

where $\bar{x}$ is the sample mean and $\mathbf{1}$ is an $N \times 1$ vector of all ones. Or use the matrix inversion lemma

$$\left(\mathbf{A} + \mathbf{u}\mathbf{u}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{u}^T\mathbf{A}^{-1}}{1 + \mathbf{u}^T\mathbf{A}^{-1}\mathbf{u}}$$

to justify your answer.

5. (20%) Find the means and covariance for the random variables

$$\xi_1 = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \sin\left(2\pi f_0 n\right).$$
$$\xi_2 = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \cos\left(2\pi f_0 n\right)$$

The data are $x[n] = A\cos\left(2\pi f_0 n + \phi\right) + w[n]$, where $w[n]$ is WGN with variance σ^2. Assume that f_0 is not near 0 or 1/2 and N is large enough so that any "double-frequency" term can be approximated as zero.

i.e.

$$\sum_{n=0}^{N-1} \sin\left(4\pi f_0 n\right) \approx 0 \qquad \text{and} \qquad \sum_{n=0}^{N-1} \cos\left(4\pi f_0 n\right) \approx 0$$
