COM 5232 Detection and Estimation Theory

Midterm Exam Solution

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13:20 $\sim$ 15:10

1.

The matrix prewhitener $\mathbf{D}$ is given by $\mathbf{C}^{-1} = \mathbf{D}^T\mathbf{D}$.

$$\begin{aligned}
\mathbf{C}^{-1} &= \frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \\
&= \frac{1}{1-\rho^2}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \left(\frac{1}{\sqrt{1-\rho^2}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{1+\rho} & 0 \\ 0 & \sqrt{1-\rho} \end{bmatrix}\right)\left(\frac{1}{\sqrt{1-\rho^2}}\begin{bmatrix} \sqrt{1+\rho} & 0 \\ 0 & \sqrt{1-\rho} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\right) \\
\Rightarrow \text{Choose } \mathbf{D} &= \frac{1}{\sqrt{1-\rho^2}}\begin{bmatrix} \sqrt{1+\rho} & 0 \\ 0 & \sqrt{1-\rho} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
\end{aligned}$$

take $\rho = 0.5$ into matrix $\mathbf{D}$, we obtain

$$\begin{aligned}
\mathbf{D} &= \frac{1}{\sqrt{0.75}}\begin{bmatrix} \sqrt{1.5} & 0 \\ 0 & \sqrt{0.5} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \frac{1}{\sqrt{1.5}}\begin{bmatrix} \sqrt{1.5} & -\sqrt{1.5} \\ \sqrt{0.5} & \sqrt{0.5} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} \end{bmatrix}
\end{aligned}$$

■

2.

$$\begin{aligned}
&\mathcal{H}_0 : x[n] = w[n] && n = 0, 1, \ldots, N-1 \\
&\mathcal{H}_1 : x[n] = s[n] + w[n] && n = 0, 1, \ldots, N-1
\end{aligned}$$

where $s[n] = Ar^n$ with $0 < r < 1$ and $w[n]$ is WGN with variance σ^2. Let $\mathbf{x} = [x[0], x[1], \ldots, x[N-1]]^T$. Note that the MLE, $\hat{A}$, of A is given by

$$\begin{aligned}
\hat{A} &= \arg\max_A p(\mathbf{x}; A, \mathcal{H}_1) \\
&= \arg\min_A \sum_{n=0}^{N-1}(x[n] - Ar^n)^2 \\
&= \frac{\sum_{n=0}^{N-1} x[n]r^n}{\sum_{n=0}^{N-1} r^{2n}}
\end{aligned}$$

Then, the GLRT decide $\mathcal{H}_1$ if

$$\begin{aligned}
\frac{p(\mathbf{x};\hat{A},\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} &= \frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{-\frac{1}{2\sigma^2}\sum\limits_{n=0}^{N-1}(x[n]-\hat{A}r^n)^2}}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{-\frac{1}{2\sigma^2}\sum\limits_{n=0}^{N-1}x[n]^2}} \\
&= \frac{e^{-\frac{1}{2\sigma^2}\sum\limits_{n=0}^{N-1}(x[n]-\hat{A}r^n)^2}}{e^{-\frac{1}{2\sigma^2}\sum\limits_{n=0}^{N-1}x[n]^2}} > \gamma \\
(\text{take ln}) &\Rightarrow -\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-\hat{A}r^n)^2 + \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x[n]^2 > \ln(\gamma) \\
&\Rightarrow 2\hat{A}\sum_{n=0}^{N-1}x[n]r^n - \hat{A}^2\sum_{n=0}^{N-1}r^{2n} > 2\sigma^2\ln(\gamma) \\
&\Rightarrow 2\hat{A}\frac{\sum\limits_{n=0}^{N-1}x[n]r^n}{\sum\limits_{n=0}^{N-1}r^{2n}} - \hat{A}^2 > \frac{2\sigma^2\ln(\gamma)}{\sum\limits_{n=0}^{N-1}r^{2n}} \triangleq \gamma' \\
&\Rightarrow \hat{A}^2 > \gamma'
\end{aligned}$$

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3.

The observed sample $x[n]$ is

$$x[n] = s[n] + w[n], n = 0, 1, \ldots, , 2N-1$$

with $s[n]$ is mention before and $w[n] \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$. Now, let $\mathbf{x} = [x[0], x[1], \ldots, x[N-1]]^T$, the NP detector $L(\mathbf{x})$ decides $\mathcal{H}_1$ if

$$L(\mathbf{x}) = \frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} = \frac{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum\limits_{n=0}^{N-1}(x[n]-A)^2} \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum\limits_{n=N}^{2N-1}(x[n]-2A)^2}}{\frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum\limits_{n=0}^{N-1}(x[n]+A)^2} \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum\limits_{n=N}^{2N-1}x^2[n]}} > \gamma$$

which simplifies to

$$\ln L(\mathbf{x}) = \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}\left[(x[n]+A)^2 - (x[n]-A)^2\right] + \frac{1}{2\sigma^2}\sum_{n=N}^{2N-1}\left[x^2[n] - (x[n]-2A)^2\right] > \ln\gamma$$

we have finally

$$\begin{aligned}
\ln L(\mathbf{x}) &= \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}4Ax[n] + \frac{1}{2\sigma^2}\sum_{n=N}^{2N-1}4Ax[n] - \frac{2NA^2}{\sigma^2} > \ln\gamma \\
&\Rightarrow \frac{1}{2N}\sum_{n=0}^{2N-1}x[n] > \frac{\sigma^2}{4AN}\left(\ln\gamma + \frac{2NA^2}{\sigma^2}\right) \triangleq \gamma'
\end{aligned}$$

let the test statistic be $T(\mathbf{x}) = \frac{1}{2N}\sum\limits_{n=0}^{2N-1}x[n]$, the NP detector compares the sample mean $\bar{x} = T(\mathbf{x})$ to a threshold γ'.

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4.

If directly use $\hat{\mathbf{s}} = \frac{\sigma_A^2}{\sigma_A^2 + \frac{\sigma^2}{N}} \bar{x}\mathbf{1}$, then the NP detector decides $\mathcal{H}_1$ if

$$T(x) = \mathbf{x}^T \hat{\mathbf{s}} = \frac{N\sigma_A^2}{\sigma_A^2 + \frac{\sigma^2}{N}} \bar{x}^2 > \gamma''$$

Or, let $\mathbf{x} = [x[0], x[1], \ldots, x[N-1]]^T$, then we have

$$\begin{aligned} &\mathcal{H}_0 : \mathbf{x} \sim \mathcal{N}(0, \sigma^2 \mathbf{I}) \\ &\mathcal{H}_1 : \mathbf{x} \sim \mathcal{N}(0, \sigma_A^2 \mathbf{1}\mathbf{1}^T + \sigma^2 \mathbf{I}) \end{aligned}$$

Then the NP detector decides $\mathcal{H}_1$ if

$$\begin{aligned} \frac{p(\mathbf{x}; \mathcal{H}_1)}{p(\mathbf{x}; \mathcal{H}_0)} &= \frac{\frac{1}{(2\pi)^{\frac{N}{2}} \det^{\frac{1}{2}} (\sigma_A^2 \mathbf{1}\mathbf{1}^T + \sigma^2 \mathbf{I})} \exp\left(-\frac{1}{2}\mathbf{x}^T (\sigma_A^2 \mathbf{1}\mathbf{1}^T + \sigma^2 \mathbf{I})^{-1} \mathbf{x}\right)}{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left(-\frac{1}{2\sigma^2}\mathbf{x}^T\mathbf{x}\right)} > \gamma \\ \text{(take ln)} \Rightarrow -\frac{1}{2}\mathbf{x}^T & \left((\sigma_A^2 \mathbf{1}\mathbf{1}^T + \sigma^2 \mathbf{I})^{-1} - \frac{1}{\sigma^2}\mathbf{I} \right) \mathbf{x} > \ln(\gamma) + \frac{1}{2} \ln\left(\det(\sigma_A^2 \mathbf{1}\mathbf{1}^T + \sigma^2 \mathbf{I})\right) - \frac{N}{2} \ln \sigma^2 \triangleq \gamma' \\ \Rightarrow T(x) &\triangleq \mathbf{x}^T \underbrace{\sigma^2 \left(\frac{1}{\sigma^2}\mathbf{I} - (\sigma_A^2 \mathbf{1}\mathbf{1}^T + \sigma^2 \mathbf{I})^{-1} \right) \mathbf{x}}_{=\hat{\mathbf{s}}} > 2\sigma^2 \gamma' \end{aligned}$$

By matrix inversion lemma, we derive

$$\hat{\mathbf{s}} = \frac{\sigma_A^2}{\sigma_A^2 + \frac{\sigma^2}{N}} \bar{x}\mathbf{1}$$

$\Rightarrow$ The NP detector decides $\mathcal{H}_1$ if

$$T(x) = \mathbf{x}^T \hat{\mathbf{s}} = \frac{N\sigma_A^2}{\sigma_A^2 + \frac{\sigma^2}{N}} \bar{x}^2 > 2\sigma^2 \gamma'$$

$\blacksquare$

5.

Mean :

$$\begin{aligned} \mathbb{E}[\xi_1] &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \mathbb{E}[A\cos(2\pi f_0 n + \phi) + w[n]] \sin(2\pi f_0 n) \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (A\cos(2\pi f_0 n + \phi) \sin(2\pi f_0 n) + \mathbb{E}[w[n]] \sin(2\pi f_0 n)) \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left(\frac{A}{2} (\sin(4\pi f_0 n + \phi) - \sin(\phi)) \right) \\ &\approx -\frac{\sqrt{N}}{2} A \sin(\phi) \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left[\xi_2\right] &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \mathbb{E}\left[A\cos\left(2\pi f_0 n + \phi\right) + w[n]\right] \cos\left(2\pi f_0 n\right) \\
&= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left(A\cos\left(2\pi f_0 n + \phi\right)\cos\left(2\pi f_0 n\right) + \mathbb{E}\left[w[n]\right]\cos\left(2\pi f_0 n\right)\right) \\
&= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left(\frac{A}{2}(\cos\left(4\pi f_0 n + \phi\right) + \cos\left(\phi\right))\right) \\
&\approx \frac{\sqrt{N}}{2} A\cos\left(\phi\right)
\end{aligned}
$$

Covariance :

Use the fact $Cov(\underbrace{\xi_1 + a}_{\xi_1^{'}}, \underbrace{\xi_2 + b}_{\xi_2^{'}}) = Cov(\xi_1, \xi_2)$, where a, b are

$$
\begin{aligned}
a &= -\frac{A}{\sqrt{N}} \sum_{n=0}^{N-1} \cos\left(2\pi f_0 n + \phi\right) \sin\left(2\pi f_0 n\right) \\
b &= -\frac{A}{\sqrt{N}} \sum_{n=0}^{N-1} \cos\left(2\pi f_0 n + \phi\right) \cos\left(2\pi f_0 n\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
&Cov(\xi_1, \xi_2) = Cov(\xi_1^{'}, \xi_2^{'}) \\
&= \mathbb{E}\left[(\xi_1^{'} - \mathbb{E}\left[\xi_1^{'}\right])(\xi_2^{'} - \mathbb{E}\left[\xi_2^{'}\right])\right] \\
&= \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \mathbb{E}\left[w[m]w[n]\right] \cos\left(2\pi f_0 m\right) \sin\left(2\pi f_0 n\right) \\
&= \sigma^2 \sum_{n=0}^{N-1} \cos\left(2\pi f_0 n\right) \sin\left(2\pi f_0 n\right) \\
&= \frac{\sigma^2}{2} \sum_{n=0}^{N-1} \sin\left(4\pi f_0 n\right) \\
&\approx 0
\end{aligned}
$$

$\blacksquare$
