COM 5232 Detection and Estimation Theory

Final Exam Solution

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13:20 $\sim$ 15:10

1. (a) Yes

$$
\begin{aligned}
\hat{\sigma}^2 &= \frac{1}{N}\sum_{n=0}^{N-1} x^2[n] \\
\implies \mathbb{E}[\hat{\sigma}^2] &= \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}\left[x^2[n]\right] \\
&= \frac{1}{N}\sum_{n=0}^{N-1} \sigma^2 \\
&= \sigma^2
\end{aligned}
$$

Thus, $\hat{\sigma}^2$ is an unbiased estimator.

(b)

$$
\begin{aligned}
Var(\hat{\sigma}^2) &= \mathbb{E}\left[(\hat{\sigma}^2)^2\right] - \left(\mathbb{E}\left[\hat{\sigma}^2\right]\right)^2 \\
&= \mathbb{E}\left[\hat{\sigma}^4\right] - \sigma^4 \\
&= \mathbb{E}\left[\frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{\substack{m=0\\ m\neq n}}^{N-1} x^2[n]x^2[m]\right] + \mathbb{E}\left[\frac{1}{N^2}\sum_{n=0}^{N-1} x^4[n]\right] - \sigma^4 \\
\text{(since } x[n]'\text{s are IID)} &= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{\substack{m=0\\ m\neq n}}^{N-1} \mathbb{E}\left[x^2[n]\right]\mathbb{E}\left[x^2[m]\right] + \frac{1}{N^2}\sum_{n=0}^{N-1} \underbrace{\mathbb{E}\left[x^4[n]\right]}_{3\sigma^4} - \sigma^4 \\
&= \frac{2}{N}\sigma^4
\end{aligned}
$$

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2.

$$
\begin{aligned}
p(\mathbf{x};A) &= \frac{1}{(2\pi)^{\frac{N}{2}}\det(\mathbf{C})^{\frac{-1}{2}}}\exp\left(\frac{-1}{2}(\mathbf{x}-A\mathbf{1})^T\mathbf{C}^{-1}(\mathbf{x}-A\mathbf{1})\right) \\
\implies \frac{\partial \ln p(\mathbf{x};A)}{\partial A} &= \mathbf{1}^T\mathbf{C}^{-1}\mathbf{x} - \mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}A \\
\implies \frac{\partial^2 \ln p(\mathbf{x};A)}{\partial A^2} &= -\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}
\end{aligned}
$$

Thus the CRLB for A is $\frac{1}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}}$

$\blacksquare$

3. Let

$$\mathbf{x} = [x[0]\ x[1]\ \dots\ x[N-1]]^T$$
$$\mathbf{w} = [w[0]\ w[1]\ \dots\ w[N-1]]^T$$
$$\mathbf{H} = \begin{bmatrix} r_1^0 & r_2^0 & r_3^0 \\ r_1^1 & r_2^1 & r_3^1 \\ \vdots & \vdots & \vdots \\ r_1^{N-1} & r_2^{N-1} & r_3^{N-1} \end{bmatrix},\ \mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

Then we have

$$\mathbf{x} = \mathbf{HA} + \mathbf{w}$$

and the MVU estimator is given by

$$\hat{\mathbf{A}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}$$

For $r_1 = 1, r_2 = -1, r_3 = 2$

$$\mathbf{H}^T\mathbf{H} = \begin{bmatrix} N & 0 & 2^N - 1 \\ 0 & N & \frac{1-(-2)^N}{3} \\ 2^N - 1 & \frac{1-(-2)^N}{3} & \frac{4^N-1}{3} \end{bmatrix}$$
$$\det(\mathbf{H}^T\mathbf{H}) = \frac{N^2(4^N-1)}{3} - N(2^N-1)^2 - \frac{N(1-2^N)^2}{9}$$
$$\Longrightarrow (\mathbf{H}^T\mathbf{H})^{-1} = \frac{1}{\det(\mathbf{H}^T\mathbf{H})} \begin{bmatrix} \frac{N}{3}(4^N-1) - \frac{1}{9}(2^N-1)^2 & -\frac{1}{3}(2^N-1)^2 & -N(2^N-1) \\ -\frac{1}{3}(2^N-1)^2 & \frac{N}{3}(4^N-1) - (2^N-1)^2 & \frac{N}{3}(2^N-1) \\ -N(2^N-1) & \frac{N}{3}(2^N-1) & N^2 \end{bmatrix}$$
$$(N=4) = \begin{bmatrix} \frac{7}{8} & -\frac{5}{24} & -\frac{1}{6} \\ -\frac{5}{24} & \frac{23}{72} & \frac{1}{18} \\ -\frac{1}{6} & \frac{1}{18} & \frac{2}{45} \end{bmatrix}$$

Thus, we derive the MVU estimator

$$\hat{\mathbf{A}} = \begin{bmatrix} \frac{7}{8} & -\frac{5}{24} & -\frac{1}{6} \\ -\frac{5}{24} & \frac{23}{72} & \frac{1}{18} \\ -\frac{1}{6} & \frac{1}{18} & \frac{2}{45} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & \frac{3}{4} & 0 & -\frac{1}{4} \\ \frac{1}{6} & -\frac{5}{12} & \frac{1}{3} & -\frac{1}{12} \\ -\frac{1}{15} & -\frac{2}{15} & \frac{1}{15} & \frac{2}{15} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$\blacksquare$

4.

$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \mathbf{s} + \mathbf{w}$$

Since $\mathbf{s}$ is known, consider

$$\begin{aligned}\mathbf{y} &= \mathbf{x} - \mathbf{s}\\ &= \mathbf{H}\boldsymbol{\theta} + \mathbf{w}\end{aligned}$$

Then the BLUE is given by

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{C}^{-1}\mathbf{y}\\ &= (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{C}^{-1}(\mathbf{x} - \mathbf{s})\end{aligned}$$

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5. (a)

$$\begin{aligned}&\hat{\boldsymbol{\theta}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}\\ &\mathbf{x} \sim \mathcal{N}(\mathbf{H}\boldsymbol{\theta}, \sigma^2\mathbf{I})\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\left[\hat{\boldsymbol{\theta}}\right] &= (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbb{E}\left[\mathbf{x}\right]\\ &= \boldsymbol{\theta}\\ Var(\hat{\boldsymbol{\theta}}) &= \mathbb{E}\left[\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}^T\right] - \boldsymbol{\theta}\boldsymbol{\theta}^T\\ &= (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbb{E}\left[\mathbf{x}\mathbf{x}^T\right]\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1} - \boldsymbol{\theta}\boldsymbol{\theta}^T\\ &= (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T(\sigma^2\mathbf{I} + \mathbf{H}\boldsymbol{\theta}\boldsymbol{\theta}^T\mathbf{H}^T)\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1} - \boldsymbol{\theta}\boldsymbol{\theta}^T\\ &= \sigma^2(\mathbf{H}^T\mathbf{H})^{-1}\end{aligned}$$

Thus $\hat{\boldsymbol{\theta}} \sim \mathcal{N}(\boldsymbol{\theta}, \sigma^2(\mathbf{H}^T\mathbf{H})^{-1})$

(b) Yes, it has been verified in (a).

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