105021202 楊暐之 Intro. DSP

1.

let $x[n], \tilde{x}[n]$ be two input signals with corresponding output signals $y[n], \tilde{y}[n]$, respectively, and $c, \tilde{c} \in \mathbb{C}$, $n_0 \in \mathbb{Z}$

(a) $y[n] = x[-n]$ is linear, stable

but not time-invariant, causal

Linear: $c x[n] + \tilde{c}\tilde{x}[n] \longrightarrow c x[-n] + \tilde{c}\tilde{x}[-n] = c y[n] + \tilde{c}\tilde{y}[n]$

time-invariant: $x[n-n_0] \longrightarrow x[-n-n_0] \neq x[-(n-n_0)] = y[n-n_0]$ for $n_0 \neq 0$

causal: For negative $n$ (ex: $n=-1$, $y[-1] = x[1]$), $y[n]$ depends on future value of $x[n]$

stable: Assume $|x[n]| < \infty\ \forall n$, then $|y[n]| = |x[-n]| < \infty\ \forall n$ #

(b) $y[n] = \cos(\pi n)\, x[n]$ is linear, causal, stable.

but not time-invariant

linear: $c x[n] + \tilde{c}\tilde{x}[n] \longrightarrow \cos(\pi n)\left(c x[n] + \tilde{c}\tilde{x}[n]\right) = c \cos(\pi n) x[n] + \tilde{c}\cos(\pi n)\tilde{x}[n]$
$= c y[n] + \tilde{c}\tilde{y}[n]$

time-invariant: $x[n-n_0] \longrightarrow \cos(\pi n)\, x[n-n_0] \neq \cos(\pi(n-n_0))\, x[n-n_0] = y[n-n_0]$ for $n_0$ odd.

causal: Clearly, it's causal

stable: Assume $|x[n]| < \infty\ \forall n$ then $|y[n]| = |\cos(\pi n)\, x[n]| = |x[n]| < \infty\ \forall n$ #

(c) $y[n] = \sum_{k=n-1}^{\infty} x[k]$ is linear, time-invariant

but not causal, stable.

Linear: $c x[n] + \tilde{c}\tilde{x}[n] \longrightarrow \sum_{k=n-1}^{\infty}(c x[k] + \tilde{c}\tilde{x}[k]) = c\sum_{k=n-1}^{\infty} x[k] + \tilde{c}\sum_{k=n-1}^{\infty}\tilde{x}[k] = c y[n] + \tilde{c}\tilde{y}[n]$

time-invariant: $x[n-n_0] \rightarrow \sum_{k=n-1}^{\infty} x[k-n_0] = \sum_{l=n-n_0-1}^{\infty} x[l] = y[n-n_0]$

causal: Clearly, it's not causal.

stable: Assume $x[n] = 1\ \forall n$, then $|x[n]| < \infty\ \forall n$, but $|y[n]|$ is unbounded for all $n$

#

2. Suppose $L$ is time-invariant

Since $L$ is LTI, $L(\delta[n-2]) = L(x_1[n] - x_2[n+2]) = y_1[n] - y_2[n+2]$

Furthermore: $x_2[n] = \delta[n-3] + \delta[n-4]$

$\Rightarrow y_2[n] = L(x_2[n]) = L(\delta[n-3] + \delta[n-4]) = y_1[n-1] - y_2[n+1] + y_1[n-2] - y_2[n]$

$\neq y_2[n]$ $(\divideontimes)$

Thus, $L$ can't be time-invariant

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3. $x[n] \longleftrightarrow X(z) = \frac{1}{1-2z^{-1}}$ with ROC: $|z|<2$

(a) $y[n] = \left(\frac{1}{3}\right)^n x[n]$

$$\Rightarrow Y(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n](3z)^{-n} = X(3z) = \frac{1}{1-\frac{2}{3}z^{-1}} \text{ with ROC: } |z|<\frac{2}{3}$$

(b) $y[n] = x[n] * x[-n]$

(We have $x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z)$ with ROC containing $R_1 \cap R_2$)

Let $x_2[n] = x[-n]$, then $X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[-n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{z}\right)^{-n} =$

$$= X\left(\frac{1}{z}\right) = \frac{1}{1-2z} \text{ with ROC: } |z|>\frac{1}{2}$$

$$\Rightarrow Y(z) = X(z)X\left(\frac{1}{z}\right) = \frac{1}{(1-2z^{-1})(1-2z)} = \frac{1}{5-2z-2z^{-1}} \text{ with ROC: } \frac{1}{2}<|z|<2$$

(bounded by poles)

(c) $y[n] = n x[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \Rightarrow X'(z) = \sum_{n=-\infty}^{\infty} -n x[n] z^{-n-1} \Rightarrow -zX'(z) = \sum_{n=-\infty}^{\infty} n x[n]$$

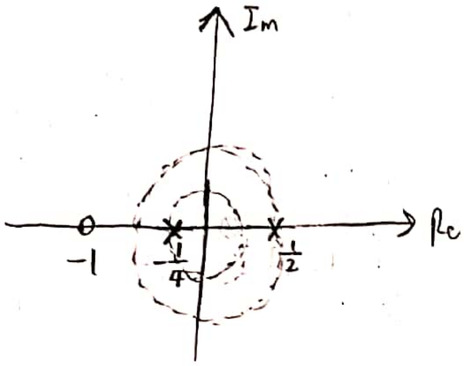
$$\Rightarrow y[n] = n x[n] \longleftrightarrow -zX'(z)$$

\#

1.

$$H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{2}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1+\frac{1}{4}z^{-1}}$$, causal LTI

(a)




x denotes pole, o denotes zero

Since the LTI system is causal,
$h[n]=0$ for $n<0$
$\Rightarrow h[n]$ is right sided
$\Rightarrow$ ROC of $H(z)$ is $|z|>\frac{1}{2}$

(b) A LTI system is stable iff $\sum_{n=-\infty}^{\infty}|h[n]|<\infty$.
Otherwise $\sum_{n=-\infty}^{\infty}|h[n]|<\infty$ iff $\sum_{n=-\infty}^{\infty}|h[n]z^{-n}|<\infty$ for $|z|=1$
$\Rightarrow$ ROC of $H(z)$ contains the unit circle.
Since ROC of $H(z)$ is $|z|>\frac{1}{2}$, unit circle is contained in ROC of $H(z)$

Thus, the system is stable.

(c) According to $H(z)$ and its ROC

$h[n]=2(\frac{1}{2})^n u[n]-(-\frac{1}{4})^n u[n]$ #

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5.

(a) $X(z)=\dfrac{z}{z^3+2z^2+\frac{5}{4}z+\frac{1}{4}}$ , $|z|>1$

$$=\frac{4z}{4z^3+8z^2+5z+1}=\frac{-4}{z+1}+\frac{-4}{(2z+1)^2}+\frac{8}{2z+1}$$

$$\frac{-4}{z+1}=\frac{-4z^{-1}}{1+z^{-1}}=z^{-1}\frac{-4}{1+z^{-1}},\ |z|>1 \longleftrightarrow -4\cdot(-1)^{n-1}u[n-1]$$

$$\frac{8}{2z+1}=z^{-1}\frac{4}{1+\frac{1}{2}z^{-1}},\ |z|>1 \longleftrightarrow 4\left(-\frac{1}{2}\right)^{n-1}u[n-1]$$

$$\frac{-4}{(2z+1)^2}=\frac{d}{dz}\left(\frac{2}{2z+1}\right)=-z^{-1}\left(-z\frac{d}{dz}\left(\frac{2}{2z+1}\right)\right),\ |z|>1 \longleftrightarrow -(n-1)\left(-\frac{1}{2}\right)^{n-2}u[n-2]$$

$$\Rightarrow X(z)\longleftrightarrow x[n]=\left[-4\cdot(-1)^{n-1}+4\left(-\frac{1}{2}\right)^{n-1}\right]u[n-1]-(n-1)\left(-\frac{1}{2}\right)^{n-2}u[n-2]$$

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(b) $X(z)=\dfrac{z}{(z^2-\frac{1}{3})^2}$ , $|z|<0.5$

$$=\frac{z}{(z+\frac{1}{\sqrt{3}})^2(z-\frac{1}{\sqrt{3}})^2}=\frac{0}{z+\frac{1}{\sqrt{3}}}+\frac{\frac{-\sqrt{3}}{4}}{(z+\frac{1}{\sqrt{3}})^2}+\frac{0}{(z-\frac{1}{\sqrt{3}})}+\frac{\frac{\sqrt{3}}{4}}{(z-\frac{1}{\sqrt{3}})^2}$$

$$\frac{\frac{-\sqrt{3}}{4}}{(z+\frac{1}{\sqrt{3}})^2}=\frac{d}{dz}\left(\frac{\frac{\sqrt{3}}{4}}{z+\frac{1}{\sqrt{3}}}\right)=-z^{-1}\left(-z\cdot\frac{d}{dz}\left(\frac{\frac{\sqrt{3}}{4}\cdot z^{-1}}{1+\frac{1}{\sqrt{3}}z^{-1}}\right)\right),\ |z|<0.5 \longleftrightarrow +\frac{\sqrt{3}(n-1)}{4}\left(\frac{-1}{\sqrt{3}}\right)^{n-2}u[-n+1]$$

$$\frac{\frac{\sqrt{3}}{4}}{(z-\frac{1}{\sqrt{3}})^2}=\frac{d}{dz}\left(\frac{\frac{-\sqrt{3}}{4}}{z-\frac{1}{\sqrt{3}}}\right)=-z^{-1}\left(-z\cdot\frac{d}{dz}\left(\frac{\frac{-\sqrt{3}}{4}z^{-1}}{1-\frac{1}{\sqrt{3}}z^{-1}}\right)\right),\ |z|<0.5 \longleftrightarrow \frac{-\sqrt{3}(n-1)}{4}\left(\frac{1}{\sqrt{3}}\right)^{n-2}u[n+1]$$

$$\Rightarrow X(z)\longleftrightarrow \frac{\sqrt{3}}{4}(n-1)\left[\left(\frac{-1}{\sqrt{3}}\right)^{n-2}-\left(\frac{1}{\sqrt{3}}\right)^{n-2}\right]u[n+1]$$

#

2. $r_{xx}[l] \equiv \sum_{n=-\infty}^{\infty} x[n]\,x[n-l]$

(a) Consider $x[l] * x[-l] = \sum_{n=-\infty}^{\infty} x[n]\,x[-(l-n)] = \sum_{n=-\infty}^{\infty} x[n]\,x[n-l]$

$\Rightarrow r_{xx}[l] = x[l] * x[-l]$

Otherwise $x[-n] \longleftrightarrow X(z^{-1})$

$\Rightarrow R_{xx}(z) = X(z)\,X(z^{-1})$

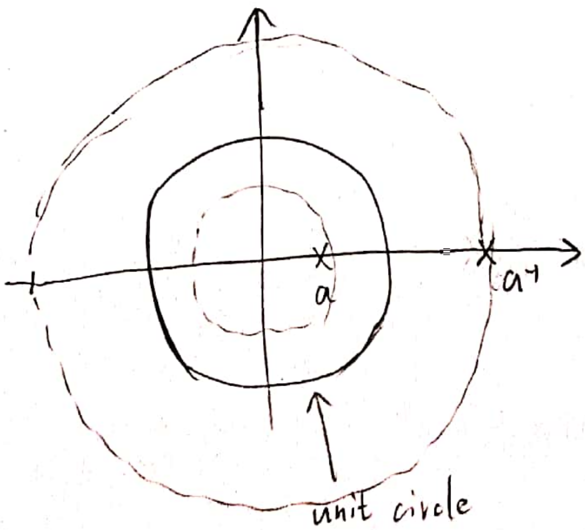
(b) $x[n] = a^n u[n]$, $|a|<1$

$\Rightarrow x[n] \longleftrightarrow X(z) = \frac{1}{1-az^{-1}}$ with ROC $|z|>|a|$

$\Rightarrow X(z^{-1}) = \frac{1}{1-az}$ with ROC $|z| < \left|\frac{1}{a}\right|$

$\Rightarrow R_{xx}(z) = \frac{1}{(1-az^{-1})(1-az)}$, with ROC containing $|a|<|z|<\left|\frac{1}{a}\right|$ (bounded by poles)

$R_{xx}(z)$ has no zero but has poles at $z=a$ and $z=a^{-1}$



7.

(a) $x[n] = \left(\frac{1}{4}\right)^n \cos\left(\frac{\pi n}{4}\right) u[n-2]$

Consider $\tilde{x}[n] = \left(\frac{1}{4}\right)^{n+2} u[n]$

$\Rightarrow \tilde{X}(e^{jw}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^{n+2} u[n] e^{-jwn}$

$= \frac{1}{16} \sum_{n=0}^{\infty} \left(\frac{1}{4} e^{-jw}\right)^n = \frac{1}{16\left(1-\frac{1}{4}e^{-jw}\right)}$

$\Rightarrow x[n] = \cos\left(\frac{\pi n}{4}\right) \tilde{x}[n-2] = \frac{1}{2}\left(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right) \tilde{x}[n-2]$

$\Rightarrow X(e^{jw}) = \frac{e^{-j2w}}{32} \left( \frac{1}{1-\frac{1}{4}e^{-j(w-\frac{\pi}{4})}} + \frac{1}{1-\frac{1}{4}e^{-j(w+\frac{\pi}{4})}} \right)$ #

(b) $x[n] = \sin\left(\frac{\pi n}{10}\right)\left(u[n] - u[n-10]\right)$

Consider $\tilde{x}[n] = u[n] - u[n-10]$

$\Rightarrow \tilde{X}(e^{jw}) = \sum_{n=-\infty}^{\infty} (u[n] - u[n-10]) e^{-jwn} = \sum_{n=0}^{9} e^{-jwn} = \frac{1-e^{-j10w}}{1-e^{-jw}}$

$\Rightarrow x[n] = \sin\left(\frac{\pi n}{10}\right) \tilde{x}[n] = \frac{1}{2j}\left(e^{j\frac{\pi}{10}n} - e^{-j\frac{\pi}{10}n}\right) \tilde{x}[n]$

$\Rightarrow X(e^{jw}) = \frac{1}{2j}\left[\tilde{X}(e^{j(w-\frac{\pi}{10})n}) - \tilde{X}(e^{j(w+\frac{\pi}{10})n})\right]$

$= \frac{1}{2j}\left[\frac{1-e^{-j10w}e^{j\pi}}{1-e^{-j(w-\frac{\pi}{10})}} - \frac{1-e^{-j10w}e^{-j\pi}}{1-e^{-j(w+\frac{\pi}{10})}}\right]$

$= \left(\frac{1+e^{-j10w}}{2j}\right)\left(\frac{1}{1-e^{-j(w-\frac{\pi}{10})}} - \frac{1}{1-e^{-j(w+\frac{\pi}{10})}}\right)$ #

(a) We have $x[n] \overset{F}{\longleftrightarrow} X(e^{jw})$

$y[n] \overset{F}{\longleftrightarrow} Y(e^{jw})$

Then $y^*[n] \overset{F}{\longleftrightarrow} Y^*(e^{-jw})$

$\Rightarrow y^*[-n] \overset{F}{\longleftrightarrow} Y^*(e^{jw})$

According to convolution property, we derive

$x[n] * y^*[-n] \overset{F}{\longleftrightarrow} X(e^{jw})Y^*(e^{jw})$

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(b)

Consider $z[n] = x[n]y^*[n]$

Then $Z(e^{jw}) = \sum_{n=-\infty}^{\infty} z[n] e^{-jwn} = \sum_{n=-\infty}^{\infty} x[n]y^*[n]e^{-jwn}$

$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y^*(e^{-j(w-\theta)})\, d\theta$

Set $w=0$, we derive

$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y^*(e^{j\theta})\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})Y^*(e^{jw})\, dw$

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(c) Let $x[n] = \frac{\sin(\pi n/5)}{2\pi n} \overset{F}{\longleftrightarrow} X(e^{jw}) = \begin{cases} \frac{1}{2} & \text{if } \frac{-\pi}{5} < w < \frac{\pi}{5} \\ 0 & \text{else} \end{cases}$

$y[n] = \frac{\sin(\pi n/3)}{7\pi n} \overset{F}{\longleftrightarrow} Y(e^{jw}) = \begin{cases} \frac{1}{7} & \text{if } \frac{-\pi}{3} < w < \frac{\pi}{3} \\ 0 & \text{else} \end{cases}$

Otherwise, since $y[n] = y^*[n]$, $Y(e^{jw}) = Y^*(e^{-jw})$

$\Rightarrow Y^*(e^{jw}) = Y(e^{-jw})$

Then according to the result in (b), we get

$\sum_{n=-\infty}^{\infty} \frac{\sin(\pi n/5)}{2\pi n}\frac{\sin(\pi n/3)}{7\pi n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})Y^*(e^{jw})\,dw = \frac{1}{2\pi}\int_{-\frac{\pi}{5}}^{\frac{\pi}{5}} \frac{1}{14}\,dw = \frac{1}{70}$

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