EE5630 Digital Signal Processing

HW#4

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Due Sunday Mar 29, 2020

(**Bilinear transformation and all pass systems in the Laplace domain**). The bilinear transformation $\mathcal{F}: \mathbb{C} \rightarrow \mathbb{C}$ is a mapping from the z-domain to the Laplace domain, defined as $s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$. Without loss of generality, let us assume that the scaling factor T_d is not important here, so we can choose $T_d = 2$ to simplify our discussions; hence, $s(z) = \frac{1-z^{-1}}{1+z^{-1}}$.

(a) Show that the transformation maps the unit circle in the z domain to the y-axis in the s domain; that is, $\mathrm{Re}\{s(e^{j\omega})\} = 0$ for any $\omega \in [-\pi, \pi]$.

(b) Find an expression for the inverse bilinear transform $z = z(s)$.

(c) Is $s(z)$ one-to-one and onto?

(d) Let $H(z) = \frac{z^{-1}-a^*}{1-az^{-1}}$ be the system function for a causal all-pass filter. Define $H_c(s) = H(z(s))$. Now, $H_c(s)$ can be thought of as the system function of a continuous-time linear system. Express $H_c(s)$ as $H_c(s) = B(s)/A(s)$.

(e) Show that $H_c(s)$ has a pair of pole s_p and zero s_z, with $\mathrm{Re}\{s_p\} < 0$ and $\mathrm{Re}\{s_z\} > 0$ and $\mathrm{Im}\{s_p\} = \mathrm{Im}\{s_z\}$.

(f) Calculate the phase response $\arg\{H_c(j\Omega)\}$ and argue that its slope is always negative; that is, $\frac{\partial}{\partial\Omega}\arg\{H_c(j\Omega)\} < 0$.

(g) Argue that, since $\frac{\partial}{\partial\Omega}\arg\{H_c(j\Omega)\} < 0$ for all $\Omega \in R$, we have $\frac{\partial}{\partial\omega}\arg\{H(e^{j\omega})\} < 0$ for any $\omega \in [-\pi, \pi]$. To conclude, this is a proof of group delay of an all-pass system being positive for all frequencies, whether the system is continuous time or discrete time. (*Hint*: you need to find a relation between Ω and ω and show that it is monotonically increasing.)